The Dependence of the Richardson Number on Scale Length

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Technical Paper No. 111
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This report was prepared with support
under Grant WBG-59 from the
National Environmental Satellite Center, ESSA

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July 1967

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ABSTRACT

It can be shown theoretically that the Richardson number depends on the thickness L of the layer over which it is computed. The relationship has the form $Ri \propto L^p$ where $0 < p < 4/3$. Experimentally, FPS-16 radar wind measurements and detailed radiosonde observations show that p may also be a function of L and that with actual wind profiles even negative values of p may be encountered.

From this study it appears that until accurate observations of the state of the atmospheric mesostructure are available, no unique correlation between Ri and clear air turbulence (CAT) should be expected to exist.

1. Introduction

In his derivation of a turbulence criterion, Richardson (1920) considered the balance between the turbulence generating forces caused by shearing stresses and the alleviating forces produced by a stable stratification of the atmosphere. If the ratio between these two forces, expressed by the non-dimensional Richardson number (Ri), is smaller than a certain critical value, laminar flow will break down into turbulence. The critical Richardson number, thus, considers a state of "just-no-turbulence" (see, for instance, Brunt, 1952; Sutton, 1953; and Hess, 1959).

A rather wide range of such critical values of Richardson number (Ri) has been established by laboratory experiments, and by measurements in the free atmosphere. It is generally believed that for average atmospheric conditions the critical value should not lie far from 1, even though several simplifying assumptions have been made in the derivation of this turbulence criterion (Calder, 1949; Dugstad, 1956).

Richardson's criterion may be written as

$$Ri = \frac{K_H}{K_M} \frac{\frac{g}{\overline{T}} \left(\frac{\partial T}{\partial z} + \Gamma \right)}{\left(\frac{\partial \bar{u}}{\partial z} \right)^2} \tag{1}$$

where K_H is the eddy diffusivity of heat, K_M is the eddy viscosity, g the acceleration of gravity, $\overline{T}$ the mean temperature, $\frac{\partial T}{\partial z}$ the observed vertical temperature lapse rate, Γ the dry adiabatic lapse rate and $\frac{\partial \underline{u}}{\partial z}$ the vertical (vector) wind shear.

The ratio $\frac{K_H}{K_M}$ depends on stability, whereby K_H may exceed K_M in unstable air, and vice versa in very stable air (see Lumley and Panofsky, 1964). Laboratory measurements suggest values close to 1 for this ratio; however, values as large as 3 have been reported in unstable air. The most appropriate value to be assumed for this ratio is still under dispute and, as pointed out by Lumley and Panofsky, conditions in the free atmosphere may not necessarily rely on values found satisfactory in the laboratory. Petterssen and Swinbank (1947) found $\frac{K_H}{K_M} = 0.65$ in the free atmosphere over England.

In addition to these uncertainties, the viscous dissipation of kinetic energy and the work done by fluctuating static pressure forces have been neglected in Richardson's original derivation (Calder, 1949). These effects may reduce the value of the critical Richardson number by a certain quantity which is difficult to evaluate and, therefore, is usually ignored.

2. Richardson's Number and CAT

Richardson's number has been applied frequently in correlating the occurrence of clear air turbulence (CAT) with atmospheric stability and vertical wind shear. A short summary of the rather divergent findings is given in Table I.

A certain amount of discrepancy should be expected in such correlations because of the following short comings of the measurements on which estimates of Ri are based.

(i) Lack of resolution: The time lag in the temperature elements of present radiosonde systems and the two-minute overlapping averaging which is applied to radar wind measurements set a severe limitation to the detail with which stability and vertical wind shear may be determined from routine aerological observations. At stratospheric levels, and especially in the vicinity of strong jet streams, the details of vertical wind profiles measured by standard equipment become rather unreliable (Reiter, 1958, 1961, 1963). There is little hope, therefore, to find perfect correlations between CAT and small details in such sounding measurements.

TABLE I

Empirical Correlations Between Ri and CAT

Investigator	Δz	Relationship between CAT and Ri
Anderson (1957)	300 m	50% probability of CAT with Ri < 1.06, 82% with Ri < 6
Bannon (1951)		30% probability of CAT with Ri < 3. No relation between CAT and Ri in stratosphere
Berenger and Heissat (1959)		70% probability of CAT with Ri ≤ 1
Briggs (1961)		80% of CAT forecasts successful for Ri < 5 or horizontal shear > 0.3 hr^{-1}
Briggs and Roach (1963)		Significant increase in turbulence for a decrease in Ri. For Ri ≤ 5, 99 cases had no turbulence and 50 had slight to moderate turbulence
Colson (1963)	50-100 mb	Fair correlation between CAT and Ri for flights below 29,000'
Endlich (1964)		Ri_c = 1 generally delineated regions that were larger than (but included) actual CAT regions
Endlich and McLean (1965)	2000'	50% increase in frequency of occurrence of all classes of turbulence with Ri ≤ 0.7

Table I Continued.

Investigator	Δz	Relationship between CAT and Ri
Endlich and Mancuso (1964)	50, 25 mb	Ri = 0.6 correctly identified 28% of turbulent cases and 92% of non-turbulent cases. $Ri_c = 1$ "over forecasts" CAT
Jaffe (1963)		Ri criterion verified as CAT indicator with $Ri_c = 1.5$
Kao and Woods (1964)		Agreement between CAT and low Ri (~ 1.0)
Klemin and Pinus (1953)*		80-90% probability of CAT with $Ri \leq 0.5$. 50% probability of CAT with $0.5 < Ri < 4.0$
Korilova (1958)*		79% probability of CAT with $Ri < 10$
Kronebach (1964)	between standard reporting levels	Ri useful for 12-hour CAT forecast ($Ri_c = 1.0$)
Lake (1956)		No clear relationship between CAT and Ri
Panofsky and McLean (1964)		All CAT reports occurred in regions of low Ri. Uncertainty of wind shears leads to overestimate of Ri
P'chelko (1960)*		31% probability of CAT with $Ri < 10$
Petterssen and Swinbank (1947)	50 mb	Found $Ri_c = 1.54$ for the free atmosphere
Pinus and Shmeter (1962)		85% probability of CAT with $Ri \leq 4$

Table I Continued.

Investigator	Δz	Relationship between CAT and Ri
Pinus and Shmeter (1965)		Concluded: Ri does not give a necessary and sufficient condition for the occurrence of turbulence but the smaller the value, the greater the probability of turbulence
Rustenbeck (1963)	2000'	In general, CAT frequency 64%, $Ri \leq 5$, although in the region 4000' above to 10,000' below the level of maximum winds 79% of CAT with $Ri \leq 5$. Poor correlation in the stratosphere
Scoggins (1963)		No correlation
Scorer (1957)		CAT probability approaches 100% with $Ri \leq 0.01$
Stinson et al. (1964)	250 m	Layers with $Ri < 1$ were common and persisted for many hours
Weinstein et al. (1966)	250 m	Utilized Ri criterion (e.g., $Ri < 0.5$ turbulent, $Ri > 1.0$ non-turbulent) to show that the strong shears can be maintained in a stable stratosphere by quasi-inertial oscillations
Zavarina and Yudin (1960)	1000 m	Found good correlation with $Ri_c = 1$

* Studies summarized by Pinus and Shmeter (1962)

(ii) Lack of synchronization: If CAT occurrence has to be compared with Richardson numbers computed from soundings taken at several hundred kilometers distance, poor correlation has to be expected (Zavarina, 1958; Zavarina and Yudin, 1960). The same holds for non-synchronous measurements at different heights, such as they may be obtained from aircraft cross-sectional flights (Reiter, 1960). Substituting from the equations of motion in Eq. (1), one arrives at (Radok and Clarke, 1958)

$$Ri = \frac{f^2 \theta}{g \frac{\partial \theta}{\partial z} \left[\left(\frac{\partial z}{\partial n}\right)_\theta - \left(\frac{\partial z}{\partial n}\right)_p - \frac{\theta}{g} \left(\frac{\partial \dot{V}}{\partial \theta}\right) \right]^2} \qquad (2)$$

$(\partial z/\partial n)_\theta$ and $(\partial z/\partial n)_p$ are the slopes of isentropic and isobaric surfaces, respectively. It appears from Eq. (2) that Richardson's number is very sensitive to changes in the vertical wind shear with time, expressed by $\partial \dot{V}/\partial \theta$. Such changes are very difficult to measure with present techniques.

(iii) Improper choice of scale length: For practical applications of Richardson's number, the differentials in Eq. (1) will have to be replaced by differences. Since vertical shear and thermal stability may not be considered constant in the free atmosphere, Ri thus computed will be a function of Δz. Lumley and Panofsky (1964) remark that "the more detailed the measurements, the better is the relation (between Ri and CAT). It is quite possible that, if wind measurements were more closely spaced, and accurate local Richardson numbers could be computed, the correlation would be perfect."

It is quite obvious that Richardson numbers computed between the 500 and 300 mb surface leave much to be desired if correlations with CAT in the "jet stream front" are sought. On the other hand, computation of Ri over very thin layers may become equally meaningless because a large portion of the spectrum of eddy sizes may be filtered out by too detailed a resolution in the "spot" measurements of a sounding.

3. The Scale Dependence of Richardson's Number

Zavarina and Yudin (1960) were led to the conclusion that Richardson numbers computed over layers of finite extent are directly dependent on the layer thickness. We may write

$$\mathrm{Ri} = \frac{\frac{g}{\overline{T}}(\Gamma - \gamma)}{(\frac{\partial \bar{u}}{\partial z})^2 + (\frac{\partial \bar{v}}{\partial z})^2} \tag{3}$$

where Γ is the dry-adiabatic lapse rate and $\gamma = -\frac{\partial T}{\partial z}$ is the actual lapse rate, $\overline{T}$ is the mean absolute temperature of the layer. Furthermore,

$$\Gamma - \gamma = \Gamma - \gamma_{st} - \frac{\delta T'}{L} \tag{4}$$

γ_{st} is the lapse rate in the standard atmosphere, L is the thickness of the layer over which Richardson's number is computed, and T' is the departure of the actual temperature from the standard temperature. δ signifies differences between the top and bottom of the layer L.

We may also write

$$(\frac{\partial \bar{u}}{\partial z})^2 + (\frac{\partial \bar{v}}{\partial z})^2 \simeq \frac{(\delta \bar{u})^2 + (\delta \bar{v})^2}{L^2} \tag{5}$$

$(\delta \bar{u})^2$ and $(\delta \bar{v})^2$ assume the role of transverse structure functions (Tatarski, 1961)

$$D_{trans} = \overline{(u_1 - u_2)^2} \quad , \tag{6}$$

where subscripts 1 and 2 refer to values measured at the top and bottom of the layer L, respectively.

Kulik (1957) and Mahlman (1965) find that for large scales of motions, the structure function of the wind is proportional to the scale of averaging

$$(\delta \bar{u})^2 + (\delta \bar{v})^2 \propto L \quad . \tag{7}$$

For such large scales one may also assume that

$$\frac{\delta T'}{L} << \Gamma - \gamma_{st} \tag{8}$$

where, for a given standard lapse rate, the quantity $\Gamma - \gamma_{st}$ assumes a constant value. Therefore,

$$Ri \propto \frac{L^2}{L} = L \quad . \tag{9}$$

The proportionality (7) also underlies Taylor's (1952) earlier observations from which he concluded that for velocity differences measured time intervals t apart

$$\overline{(u - u_t)^2} \propto t \quad . \tag{10}$$

From later investigations (Taylor, 1955) he found a relationship

$$\overline{(u - u_t)^2} \propto t^{2/3} \tag{11}$$

to be more valid (see Pasquill, 1962).

For shallow layers L, Zavarina and Yudin (1960) propose the proportionalities

$$[\,(\delta \bar{u})^2 + (\delta \bar{v})^2\,] \propto L^{2/3}$$

$$\delta T' \propto L^{1/3} \qquad (12)$$

in agreement with turbulence theory: According to Obukhov and Yaglom (1958), and to Tatarski (1961), the transverse structure function for the wind field may be written as

$$D_{trans}(L) = C\epsilon^{2/3} L^{2/3} \; . \qquad (13)$$

Obukhov (1958) arrives at a structure function of the temperature field of turbulent flow, having the form

$$\overline{H}(L) = a^2 \frac{\overline{N}}{\epsilon^{1/3}} L^{2/3} \qquad (14)$$

where $\overline{N} = \kappa \overline{(\text{grad } T)^2}$, κ being the thermal conductivity with a magnitude of 0.19 cm^2/sec for air, and a^2 is assumed to be a constant (see also Stephens and Reiter, 1966). Temperature in this derivation has been considered a conservative passive parameter.

The RMS values of the temperature fluctuation, which carry the same dimensions as $\delta T'$ in (12), may be written as

$$\sigma_{\Delta T}(L) = \sqrt{\overline{(T_1 - T_2)^2}} = a \frac{\sqrt{N}}{\sqrt[6]{\epsilon}} L^{1/3} \qquad (15)$$

hence the assumed proportionality $\delta T' \propto L^{1/3}$. Subscripts 1 and 2, again, refer to values at the top and bottom of the layer L.

From (12) it appears that for shallow layers

$$\mathrm{Ri} \propto \frac{L^2 \cdot L^{1/3}}{L^{2/3} \cdot L} = L^{2/3} . \tag{16}$$

For practical purposes one should, therefore, expect that

$$\mathrm{Ri} \propto L^p \tag{17}$$

where $2/3 < p < 1$. The lower boundary of this range rests on the same assumptions as those on which turbulent conditions in the inertial subrange were derived. The upper boundary assumption, however, is based on empirical evidence of the behavior of the structure function.

One may take yet another theoretical approach: According to Tatarski (1961), the one-dimensional spectral density

$$E(k) = \frac{\Gamma(p+1)}{2\pi} \sin\frac{\pi p}{2} \, c^2 k^{-(p+1)} \tag{18}$$

corresponds to the structure function

$$D(L) = c^2 L^p \quad \text{for} \quad 0 < p < 2 \quad . \tag{19}$$

In this notation Γ stands for the gamma function. Bolgiano (1959, 1962) suggested that under stable conditions a buoyant subrange may be established in which

$$E(k) \propto k^{-11/5} \quad . \tag{20}$$

This would, according to (18) and (19), yield a structure function

$$D(L) \propto L^{6/5} \tag{21}$$

and a Richardson number (assuming no effect of temperature fluctuations)

$$Ri \propto L^{4/5} \quad . \tag{22}$$

On the other hand, a buoyant subrange with

$$E(k) \propto K^{-3}$$

as suggested by Shur (1962), Vinnichenko, Pinus and Shur (1965) and by Lumley (1965) would yield

$$D \propto L^{2} \tag{23}$$

and hence

$$Ri \propto L^{o} \tag{24}$$

i. e., Ri is independent of layer thickness.

The temperature spectrum in Bolgiano's assumptions on the buoyant subrange is $\propto k^{-7/5}$. This would yield $\frac{\delta T'}{L} \propto L^{-4/5}$ and again

$$Ri \propto L^{o} \tag{25}$$

Zavarina's and Yudin's conclusions, given by (17), and the expressions (22), (24), and (25) are mainly based on theoretical reasoning. From standard radiosonde equipment and from aircraft measurements of present accuracy, it will be difficult, if not impossible, to check the validity of the proportionality (17). The very accurate wind measurements with FPS-16 radar (Scoggins, 1962) and wind computations from rocket response (Reisig, 1956), however, have the required resolution to make such an investigation possible.

Using the latter, Essenwanger (1963) (see also Essenwanger and Billions, 1965 and Essenwanger, 1965) concluded that mean vertical vector wind shears, ν (in sec^{-1}), are related to layer thicknesses L over which they are computed, by a power law of the form

$$\overline{\nu}_L = a_o L^{a_1} \qquad (26)$$

where a_o and a_1 are "constants" which may depend on altitude, location and time. The rocket data used by Essenwanger suggest a value $a_1 \cong -0.5$ for <u>mean</u> shears, $\overline{\nu}_L$, using layer thicknesses from 48 m to 960 m. For observed <u>extreme</u> shears a similar exponential dependence on layer thickness seems to hold, with an exponent $a_1 \cong -2/3$. Belmont and Shen (1966) find an exponent of -1/3 using the notation of Eq. (26) for average shears measured by jimspheres. The latter data have been smoothed slightly before subjecting them to the statistical investigation.

Assuming with Zavarina and Yudin that for sufficiently large L the Richardson number becomes independent of $\delta T'$ (see inequality (8)), one may write

$$Ri \propto L \qquad (27)$$

for mean conditions prevailing in Essenwanger's data sample. This is in agreement with Zavarina's and Yudin's deductions. An exponent of 1/3 may be assumed in this proportionality if, according to (12), $\frac{\delta T'}{L} \propto L^{-2/3}$ is adopted.

Considering that extreme shears ($a_1 \cong -2/3$) are probably characteristic of layers with a near-critical Richardson number, one may write

$$Ri \propto L^{4/3} \text{ or } L^{+2/3} \qquad (28)$$

depending on the condition $\frac{\delta T'}{L} \cong 0$ or $\frac{\delta T'}{L} \propto L^{-2/3}$. The latter of the two exponents in expression (28) again falls within the range specified by Zavarina and Yudin.

Armendariz and Rider (1966) found that the relationship

$$(\Delta V)_L = b_o L^{b_1} \tag{29}$$

satisfied balloon data obtained over White Sands, New Mexico very well. $(\Delta V)_L$ is the magnitude of the wind vector difference (in $ft \cdot sec^{-1}$) measured over the layer L. According to their results $b_1 \cong 1/3$, slightly larger than this value for mean wind differences and slightly smaller for maximum wind differences. This is in agreement with Essenwanger's results, since $(\Delta V)_L = \nu \cdot L$, hence $b_1 = (a_1 + 1)$. The same relationship between Richardson number and layer thickness as given in (28), therefore, should be applicable to the White Sands data.

These results are summarized in Table II.

From the theoretical considerations outlined above, we may arrive at the following conclusions:

(i) In general, the functional relationship between Richardson's number and the thickness of the layer over which this number has been estimated may be expressed by a power-law of the form

$$Ri \propto L^p \tag{30}$$

(ii) From turbulence theory it appears that the exponent p should be positive, and larger (ca. 1) for thick layers than for thin ones (ca. 2/3).

(iii) From the relationship (16) it appears that for $p = 2/3$ the shear contribution to Richardson's number would be proportional to $L^{-4/3}$, meaning that the shear should decrease with increasing thickness L. This calls for the presence of a convex ("blunt") wind profile. One might argue, therefore, that a convex wind profile is a necessary condition for isotropic turbulence to develop, such as it prevails in the inertial subrange. A concave wind profile, on the other hand, might indicate that perturbation kinetic energy in the layer under consideration is not yet in inertial equilibrium.

TABLE II

Theoretical and Previous Empirical Relationships
Between Ri and Layer Thickness L

Author	Scale	Structure Function or Shear	Stability Influence	Ri
Zavarina	Large	$D \propto L$	None	L^1
Zavarina	Small (Inertial)	$D \propto L^{2/3}$	$\delta T \propto L^{1/3}$	$L^{2/3}$
Bolgiano	Buoyant Subrange	$D \propto L^{6/5}$	None (stable)	$L^{4/5}$
Shur	Buoyant Subrange	$D \propto L^2$	None (stable)	L^o
Bolgiano	Buoyant Subrange	$D \propto L^{6/5}$	$\delta T \propto L^{1/5}$	L^o
Essenwanger	Unspecified	$\nu = a_o L^{-1/2}$		L^1
Essenwanger	Unspecified	$\nu = a_o L^{-1/2}$	$\delta T' \propto L^{1/3}$	$L^{1/3}$
Essenwanger	Unspecified	$\nu = a_o L^{-2/3}$		$L^{4/3}$
Armendariz	Unspecified	$\nu = a_o L^{-2/3}$	$\delta T' \propto L^{1/3}$	$L^{2/3}$
Belmont	Smoothed Profile	$\nu = a_o L^{-1/3}$	None	$L^{2/3}$

Total range of proportionality for $Ri \propto L^p$, $0 \leq p \leq L^{4/3}$

4. Experimental Results from FPS-16 Soundings

A number of detailed wind soundings, taken at Cape Kennedy on 29 December 1964 and on 10 February and 27 April 1965 were investigated. Detailed analysis efforts were concentrated on the first of these days, because wind measurements conveniently fell between radiosonde observation times so that vertical temperature profiles are available for this period. Table III contains a list of sounding runs used in this study.

TABLE III

Release Times of Soundings

	FPS-16 Radar Spherical Balloon	Radiosonde
29 December 1964		1115 GMT
	1306 GMT	
	1600 GMT	
	1731 GMT	1715 GMT
	1900 GMT	
	2031 GMT	
	2200 GMT	
		2315 GMT
10 February 1965		1046 GMT
	1305 GMT	
	1530 GMT	
	2305 GMT	
27 April 1965		1144 GMT
	1600 GMT	
	2314 GMT	

A typical FPS-16 wind sounding from this period is shown in Fig. 1. The vertical temperature profiles mentioned in Table III are shown in Fig. 2. The tropopause during this period is located at approximately 12000 m.

In order to synchronize the temperature data with the wind data, the former were linearly interpolated to give values corresponding to the release time of the FPS-16 soundings. Since changes in the vertical temperature structure were relatively small, no correction was attempted for varying ascent rates of wind- and temperature-sounding balloons. For each level z of any wind sounding listed in Table III the temperature was obtained by programming the equation

$$T_t(z) = T_i(z) + \frac{T_{i+1}(z) - T_i(z)}{\Delta t} t \qquad (31)$$

where subscripts i and i+1 refer to values measured at height z by the i^{th} and $(i+1)^{st}$ radiosonde in Table III. Δt is the time interval between soundings, and t, the time of the wind sounding, is counted from the release time of the i^{th} radiosonde.

Since wind data were reported for every 25 m, temperature data, however, for every 250 m along the vertical profiles, another linear interpolation had to be made of the form

$$T_t(z_i) = T_t(z_o) + \frac{T_t(z_n) - T_t(z_o)}{\Delta z} z_i \qquad (32)$$

where $i = 0, 1, 2 \ldots n$, $\Delta z = z_n = 250$ m, and $z_i = 25 \cdot i$ meters.

Using the actual wind data and the interpolated temperature data obtained from Eqs. (31) and (32), Richardson's number may be computed for increasing layer thicknesses L. $\overline{T}$ in Eq. (1) was approximated by

$$\overline{T} = \frac{T_1 + T_2}{2} \qquad (33)$$

where subscripts 1 and 2 refer to values at the top and the bottom of layer L. Tests showed that this simplified expression for mean temperatures introduced negligibly small errors when compared with the more accurate expression

$$T = \frac{1}{n} \sum_{i=0}^{n} T_i \qquad (34)$$

where values T_i are available at 25 m intervals.

A computer program delivered values of Ri, using Eq. (1) and the interpolation schemes mentioned above, for layers L which were centered at

$$z_i = (2250 + 250 \cdot i) \text{ meters} \qquad (35)$$

for i = 0, 1, 2, . . . 47. The highest layers under consideration, thus, were centered at an altitude of 14,000 m in the stratosphere.

For each level z_i Richardson numbers were computed for layers L = 50 m, 100 m, . . . 4000 m, centered at level z_i.

Fig. 3 shows the distribution of Richardson's number with height for the 1731 GMT sounding on 29 December 1964. Due to the variation of Ri values over nearly four orders of magnitude, Ri has been plotted on a logarithmic scale. Vertical Ri-profiles have been entered in this diagram for various layer thicknesses L. One finds on the average an increase of Ri with increasing L. This is in qualitative agreement with the calculations by Zavarina and Yudin.

One finds, furthermore, from Fig. 3 that by increasing the layer thickness L, details in the vertical distribution of Ri not only become obscure, but also misrepresented. The secondary maximum in Ri, for instance, which appears at 5750 m for L = 250 m, becomes the dominant feature with L = 4000 m.

Regions with Ri < 1 become almost completely obliterated when increasing L to 500 m and beyond. Only the low values of Ri between 4750 and 5500 m are reflected even by larger layer thicknesses. This would mean that Ri = 1 should not be considered a critical Richardson number for the onset of turbulence in the free atmosphere if sizeable layer thicknesses have to be used for its computation.

In order to study the effect of layer thickness on computed values of Richardson's number, results of the computations outlined above were arranged into characteristic groups. Naturally, not all results of the calculations can be presented here. The following examples have been chosen for the typical features which they reveal.

Fig. 4 contains a composite of curves Ri (L) obtained from the region of low Richardson number near 4750 to 5000 m for observation times as indicated. From Fig. 1 one may see that the layer of strong shear extends over a depth of approximately 1 km and is located near the top of a stable layer (Fig. 2). In agreement with this we find a discontinuity in the slopes of the Ri (L) curves near L = 1000 m (Fig. 4). Within the shearing layer (L < 1000 m), the exponent p in the proportionality $Ri \propto L^p$ ranges from approximately 1/3 to 1, with an average value close to p = 2/3.[1] This agrees with the theoretical derivations by Zavarina and Yudin (1960) and with the empirical findings on vertical wind shears by Essenwanger, Armendariz and Rider (see Table II). The wind profile is slightly convex, hence the positive exponent p.

As the computations are extended beyond the shearing layer (L > 1000 m), an exponent $p \cong 5$ seems to prevail. Such values for p are not predicted by any of the theoretical approaches outlined in Chapter 3. It should be pointed out, though, that none of the previous derivations take into account any predominant mesoscale structure that may appear in vertical wind profiles.

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[1] Since the plots of Ri versus L are logarithmic, p is merely the slope of the average trend and is easily determined.

The abrupt change in slope that appears in Fig. 4 near $L \cong 1000$ m is the manifestation of such a predominant vertical scale characterizing the wind profiles under investigation.

Another layer with relatively low Richardson number is located near 9250 m (Fig. 3). Curves of Ri (L) for this layer are shown in Fig. 5. The average value for p seems to lie between 2/3 and 1. The center of the layer lies close to a minimum of wind speed in the vertical wind profile (Fig. 1). This brings about the positive exponent p. Thermal stability is low in this region (Fig. 2).

A gradual increase in layer thickness to 4000 m remains within the general characteristics of the wind or temperature profile. Hence, no abrupt changes in slope are found in Fig. 5.

Fig. 6 contains curves Ri (L) for layers centered at z = 5500 m. According to Fig. 1 this level characterizes a peak (maximum) in the vertical wind profiles. (Curves were not plotted beyond 1900 GMT because the wind speed maximum deteriorated beyond this observation time.) These curves appear to be rather similar to those presented in Fig. 4, except for the larger slope values, p, for layer thicknesses $L > 1100$ m.

Fig. 7 shows data for layers centered at 6250 m. The vertical wind profiles in this region reveal a layer of approximately 500 m thickness in which $\Delta\bar{u}$ is close to zero. On either side of this layer wind shears of the same sign prevail. A range of layer thicknesses results in which p becomes negative. More examples of a similar nature may be found in the available data sample, occurring with layers of nearly constant wind speed. Over the extent of such a layer, the Richardson number becomes independent of layer thickness. This fact is portrayed by the "hump" in Fig. 7. As L is increased beyond the thickness of this layer, the adjacent wind shears of equal sign tend to reduce the value of Ri, thus rendering $p < 0$ for a limited range of thicknesses L.

From these data it is easily seen that no general rules for the dependence of Richardson's number on layer thickness, such as proposed, for instance, by Zavarina and Yudin (1960) can be adopted. This is not too surprising because as the layer thickness is increased beyond a few hundred meters, the time-persistent mesoscale structure of the atmosphere becomes prominent in the form of irregularly spaced stable and less stable layers with positive or negative vertical wind shears. This mesoscale structure, however, does not follow turbulence theory derived for the inertial subrange (Essenwanger, 1965).

The theoretical derivations were mainly based on a structure function of the form

$$D_{trans} = \overline{\Delta u^2} \qquad (36)$$

The averaging process, indicated by the "bar", relates to time, assuming that mean wind velocity components are known for two levels separated by the distance L. Figs. 4 to 7, however, represent instantaneous conditions. Certain discrepancies, therefore, should be expected between the results shown in these diagrams and the predictions made from theory.

On the other hand, one should recognize the fact that the meso-structure in the vertical wind profiles change only slowly. This may be recognized from the systematic behavior of the Ri = f (L) curves plotted for various observation times in Figs. 4 to 7. Thus, for time averaging over several hours one should not expect a drastic improvement in the agreement between theory and observations because of the prevalence of this mesostructure.

This structure may be more effectively removed by a space-averaging process rather than a time average. One might define a vertically space-averaged Richardson number

$$\widetilde{Ri\,(\Delta z)} = \frac{1}{n+1} \sum_{i=0}^{n} Ri_i\,(\Delta z) \qquad (37)$$

where the subscript i refers to the number of the layer given by Eq. (35). Such average Richardson numbers are plotted in Fig. 8 for the sounding at 1731 GMT on 29 December 1964. For comparison, slope lines for $p = 1$ and $p = 2/3$ have been entered into this diagram.

It should be pointed out that $\widetilde{Ri}$ computed from Eq. (37) will be different from an average Richardson number computed from space-averaged structure functions based upon values of $\widetilde{\Delta u}^2$ and $\widetilde{\Delta T}$. However, since we are interested only in the exponent of the proportionality $Ri \propto L^p$, the error resulting from an evaluation of Ri instead of $\widetilde{\Delta u}^2$ and $\widetilde{\Delta T}$ will not enter into our consideration.

From Fig. 8 we see that the curve Ri $(\widetilde{L})$ is rather much disturbed by irregular "peaks". These are the result of a few dominant layers with high Richardson numbers. In spite of these, the average slope of the curve seems to lie between $p = 1$ and $p = 2/3$, as predicted by Zavarina and Yudin. The minima in this curve, however, seem to align with a slope of $p \cong 4/5$ for layers of 100 to 1000 m thickness, and with $p \cong 3/2$ for thicker layers. The latter value falls close to the results of Essenwanger, Armendariz and Rider, whereas the former value would result by applying Bolgiano's buoyant subrange.

Even though Fig. 8 lends some encouragement to the theoretical reasoning advanced earlier, it will be of little practical value. Hydrodynamic instabilities of vertical shear and stratification, which may result in CAT (Reiter, 1966) are not so much controlled by the average Richardson number $\widetilde{Ri}$ measured throughout the troposphere, but by flow processes and instabilities within relatively thin layers. The Richardson number within these, however, shows a very unpredictable yet strong, dependence on layer thickness L, as evident from Figs. 4 to 7.

Conclusions

From the foregoing discussion we may arrive at the following rather general conclusions on the behavior of Richardson's number in the free atmosphere:

(1) There usually is a strong dependence of Ri on layer thickness L. The functional relationship, however, varies with height, and from case to case. The same holds for a "critical" Richardson number, under which CAT might be expected. Hence, it does not appear fruitful to specify "critical" Ri numbers for CAT occurrence unless the dependence Ri = f (L) is known for the time and the vicinity of the atmospheric level in question.

(2) Even if L was specified together with values of Ri_c, not much would be gained in CAT prediction, in view of the variability of p in the possible relationship $Ri \propto L^p$. The detailed characteristics of the vertical wind and temperature profiles would have to be known, in order to provide an estimate of the physical causes for a certain value of p.

(3) For sets of curves Ri = f (L) obtained at (short) time intervals for the same height z of the center point of the layer, a simple relationship of the form $Ri \propto L^p$ rarely ever holds for $50 \text{ m} \leq L \leq 4000 \text{ m}$ (see Figs. 4 to 7). The theoretical approach taken by Zavarina and Yudin also predicts different values of p to hold for different ranges of L. The experimental results presented here reveal, however, that instead of p = const for certain sub-ranges of L, a functional relationship p = f (L) has to be expected (approximated by the heavy dashed lines in Figs. 4 and 5). Since this function, again, varies from case to case, and from level to level, no attempt has been made to express it explicitly for any of the cases shown in Figs. 4 to 7. That such a functional relationship exists may be

shown as follows: We may rewrite the Richardson number in finite difference form as a function of scale length L , i.e.,

$$Ri = \frac{\frac{g}{\bar{\theta}} \frac{\Delta \theta}{L}}{(\Delta u/L)^2} \qquad (38)$$

Logarithic differentiation of (38) leads to

$$\frac{\partial \ln Ri}{\partial L} = \frac{\partial \ln \Delta \theta}{\partial L} + \frac{1}{L} - 2 \frac{\partial \ln \Delta u}{\partial L} \quad . \qquad (39)$$

Eq. (17) may be written as

$$Ri = A L^p \quad . \qquad (40)$$

where A is a constant of proportionality. Logarithmic differentiation of (40) yields

$$\frac{\partial \ln Ri}{\partial L} = \frac{\partial p \ln L}{\partial L} = \frac{p}{L} + \frac{\partial p}{\partial L} \ln L \qquad (41)$$

Thus we may arrive at

$$\frac{\partial p}{\partial L} = \frac{1-p}{L \ln L} + \frac{1}{\ln L} \left(\frac{\partial \ln \Delta \theta}{\partial L} - 2 \frac{\partial \ln \Delta u}{\partial L} \right) \quad . \qquad (42)$$

It appears from (42), that the variation of the exponent, p , with scale length is dependent on scale length L and on the shape of the temperature and wind profiles.

(4) The data presented in these diagrams have been collected at Cape Kennedy mainly under conditions of weak anticyclonic flow

aloft. It would be of interest to explore the general relationship Ri (L) as shown in Fig. 8 for different flow regimes, especially those prevailing near the jet stream, and for different locations.

(5) In order to establish a more meaningful relationship between a critical Richardson number and CAT it will be necessary to measure detailed vertical wind and temperature profiles in regions where and at times when, CAT is actually experienced. This would necessitate simultaneous radiosonde ascents, FPS-16 rawinsonde measurements, and aircraft measurements of CAT--preferably of turbulence spectra. Such an experimental field program should take advantage of geographic areas over which CAT is experienced relatively frequently. The region over, and to the lee of, the Rocky Mountains would offer such advantageous locations (Foltz, 1967; Reiter and Foltz, 1967).

(6) In spite of the discouraging outlook on a generally valid relationship between Ri_c , L , and the occurrence of CAT which appears to emerge from the present investigation, one might be able to establish certain threshold values of ΔV observed over certain minimum layer thicknesses L_{min} together with certain degrees of thermal stability, under which CAT is likely to occur. Since the phenomenon of CAT is tied to a rather narrow range of "wavelengths" in the spectrum of atmospheric perturbations (ca. 20-300 m, see Reiter and Burns, 1966), one should expect that a shearing layer would have to attain a certain minimum thickness L_{min} before eddies of a size and energy to be felt as CAT could be generated. A wavelength dependence Richardson number (Reiter, 1961, 1963) might offer a means of estimating characteristic eddy sizes developing out of unstable flow conditions. More detailed information from a well-planned and well-executed field program will be necessary, however, before such possible relationships can be investigated.

Acknowledgements

The authors are indebted to Dr. O. Essenwanger for stimulating discussions on the statistical relationships between vertical wind shears and layer thickness. This study was carried out under sponsorship of National Environmental Satellite Center, Environmental Science Services Administration, WBG-59 and is a continuation of an investigation initiated by H. P. Foltz. The Computer Facility of the National Center for Atmospheric Research has made available free computer time to this research project.

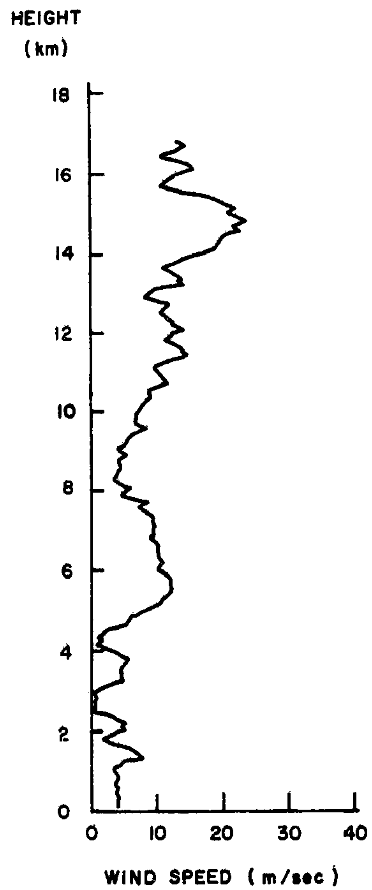

FIG. 1. Scalar wind profile from spherical balloon sounding taken at Cape Kennedy, Florida, 1731 GMT, 29 December 1964.

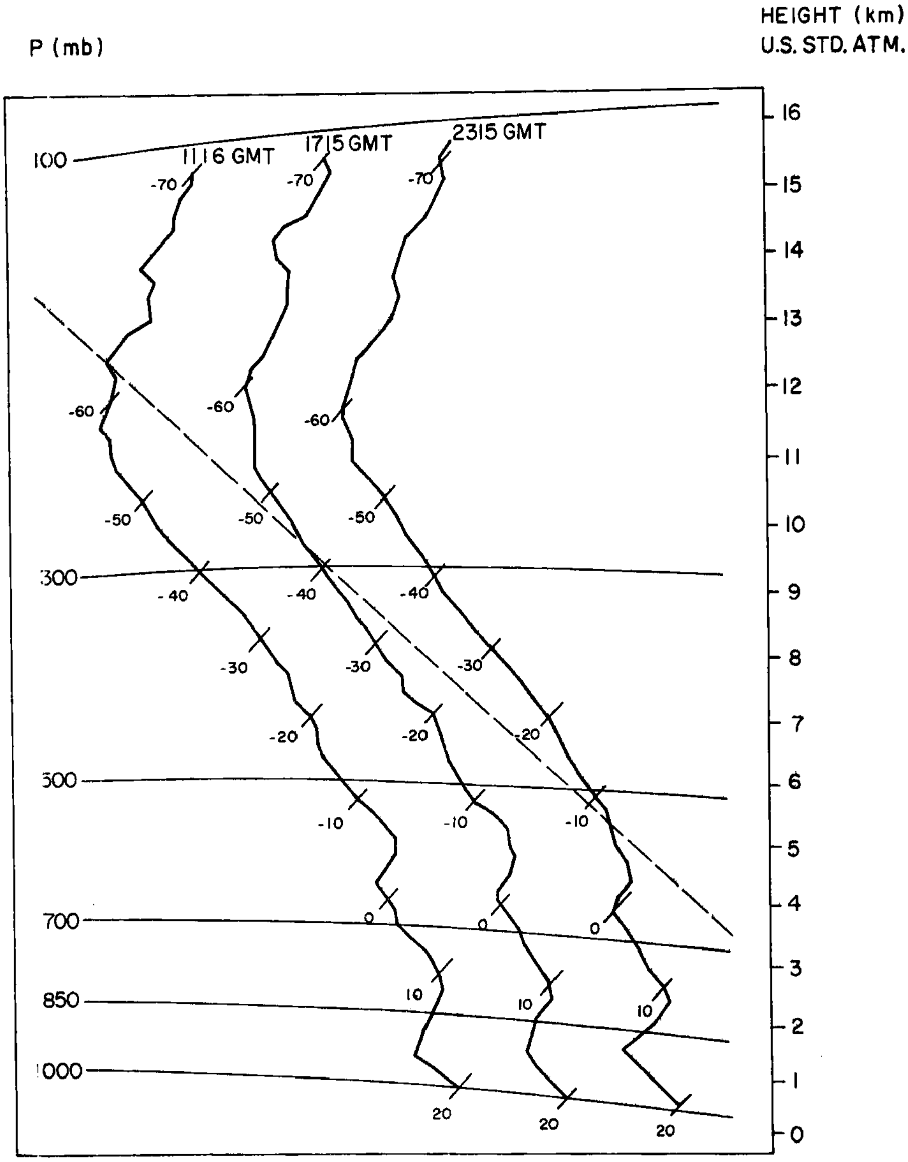

FIG. 2. Temperature profiles (heavy lines) from radiosonde observations at Cape Kennedy, Florida, 1116, 1715, 2315 GMT, 29 December 1964. Curved, quasi-horizontal lines indicate pressure levels; sloping dashed line is a dry adiabat. Profiles are labelled in temperature increments of 10 C.

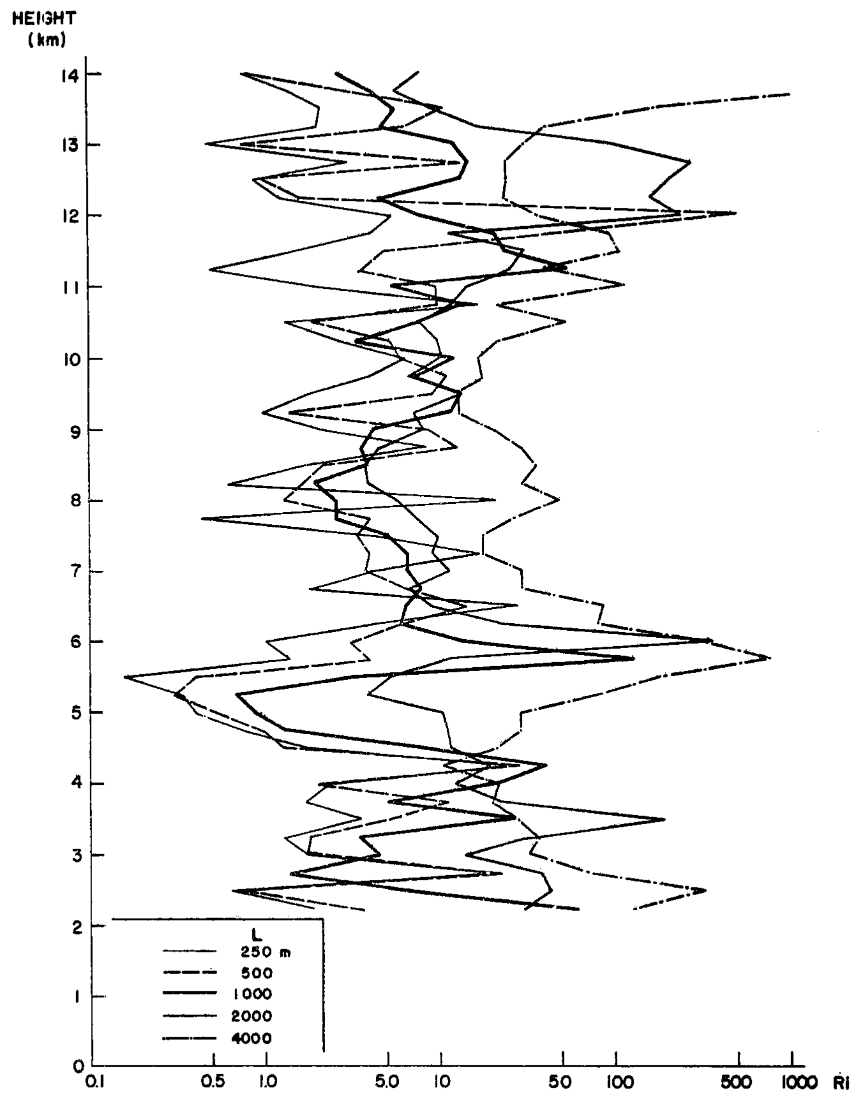

FIG. 3. Profile of the logarithm of Richardson number for various layer thicknesses (L) as indicated in the diagram.

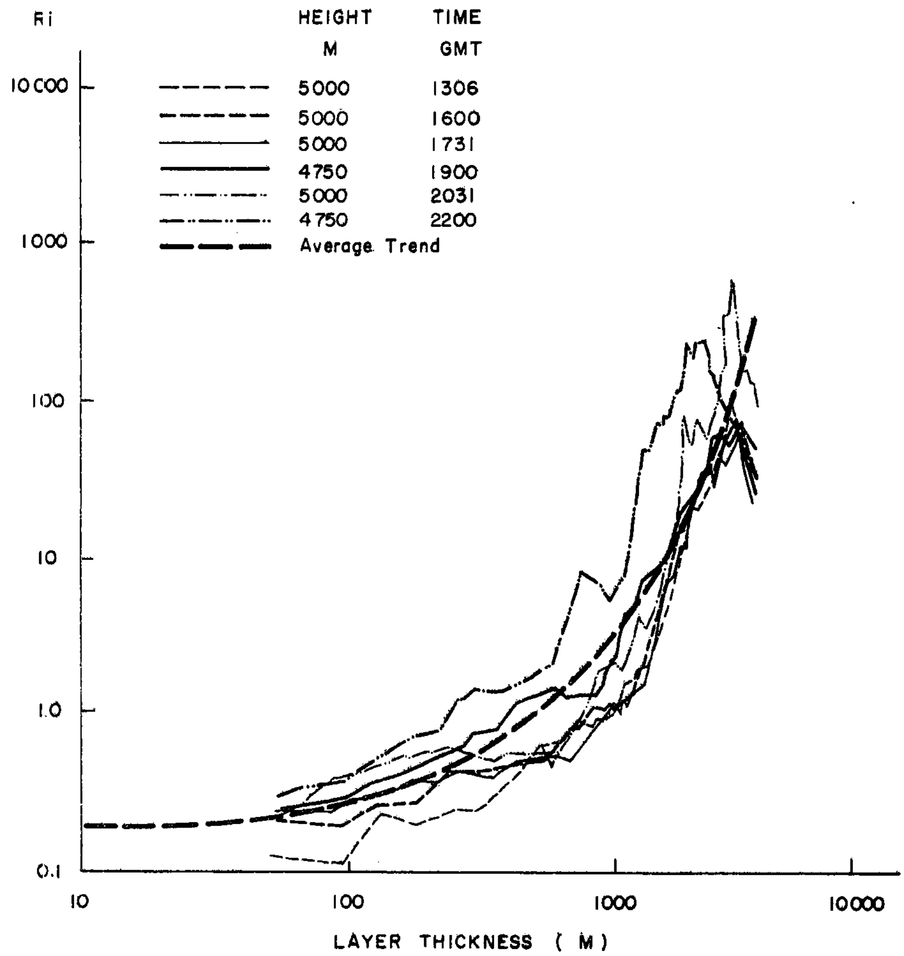

FIG. 4. Logarithmic plot of Richardson number (Ri) versus scale length (L). Layer center point is near 5 km. Actual heights and times of curves are indicated in the upper left-hand corner of the diagram. Cape Kennedy, Florida, 29 December 1964.

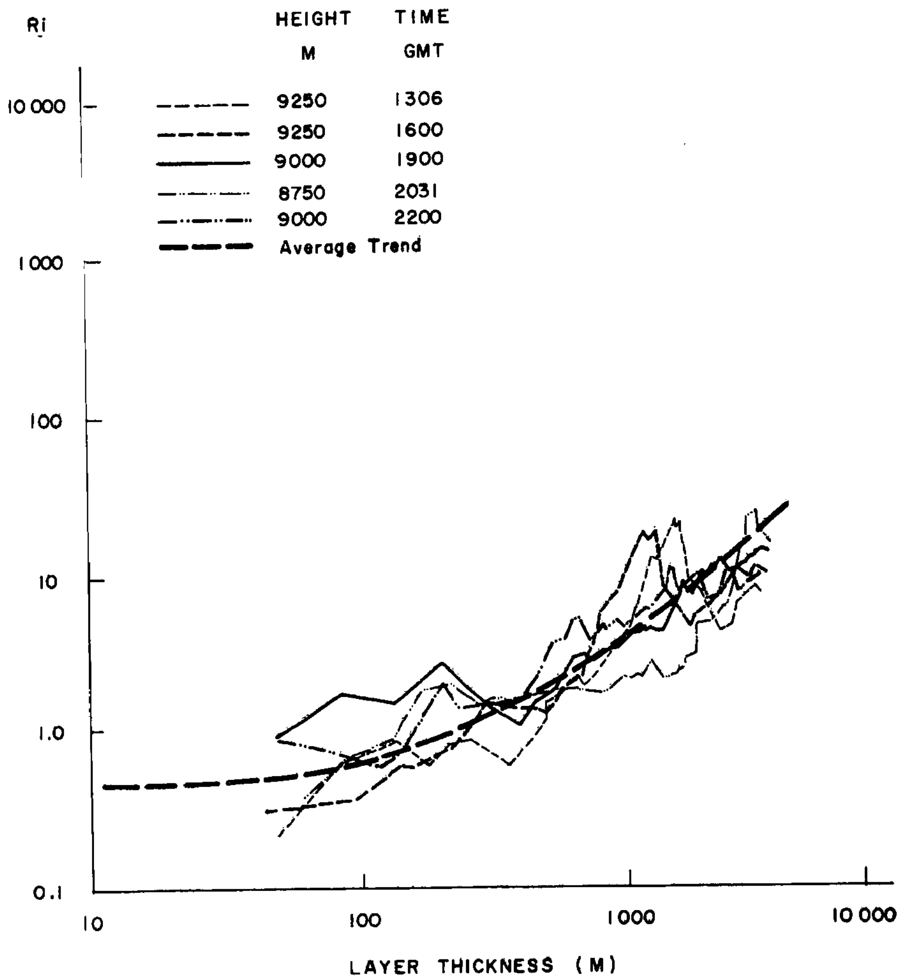

FIG. 5. Same as Fig. 3 for layer center point near 9 km.

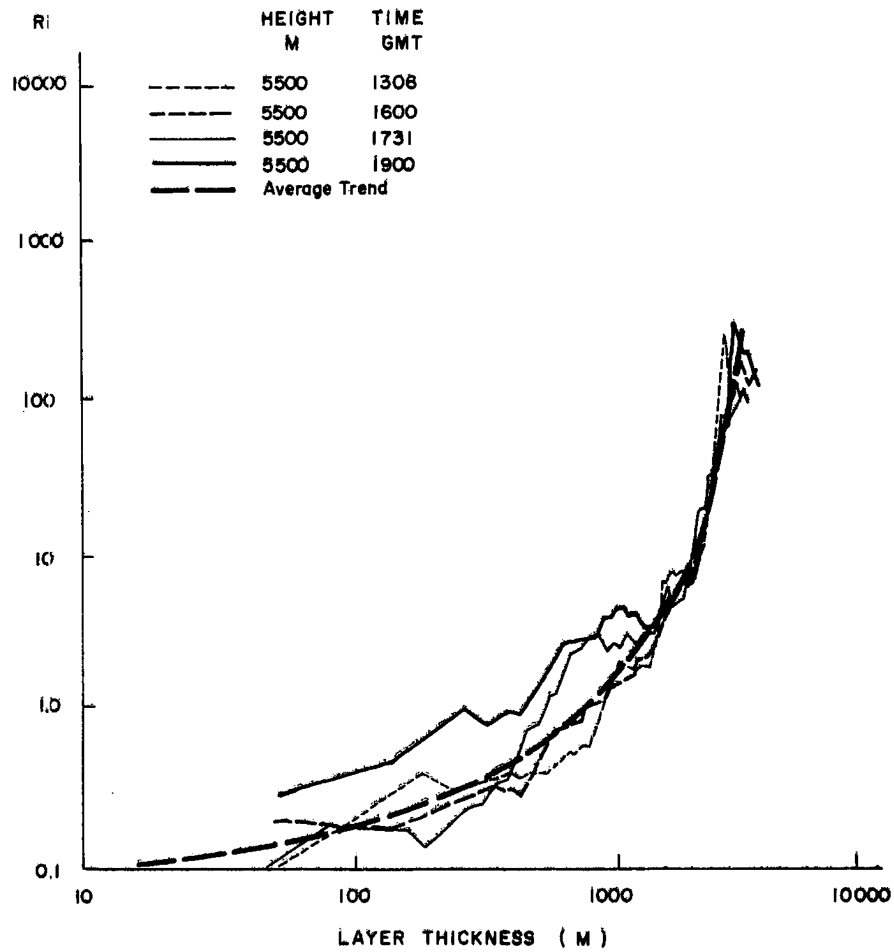

FIG. 6. Same as Fig. 3 for layer center point near 5.5 km.

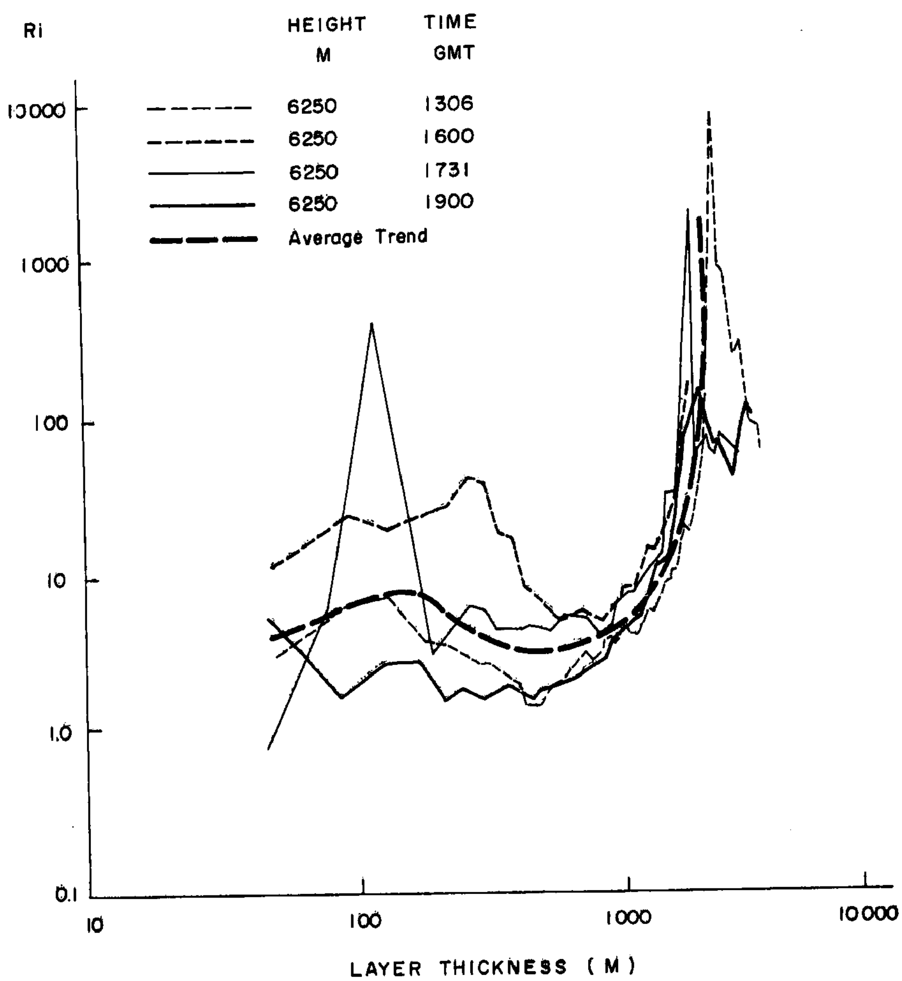

FIG. 7. Same as Fig. 3 for layer center point near 6.25 km.

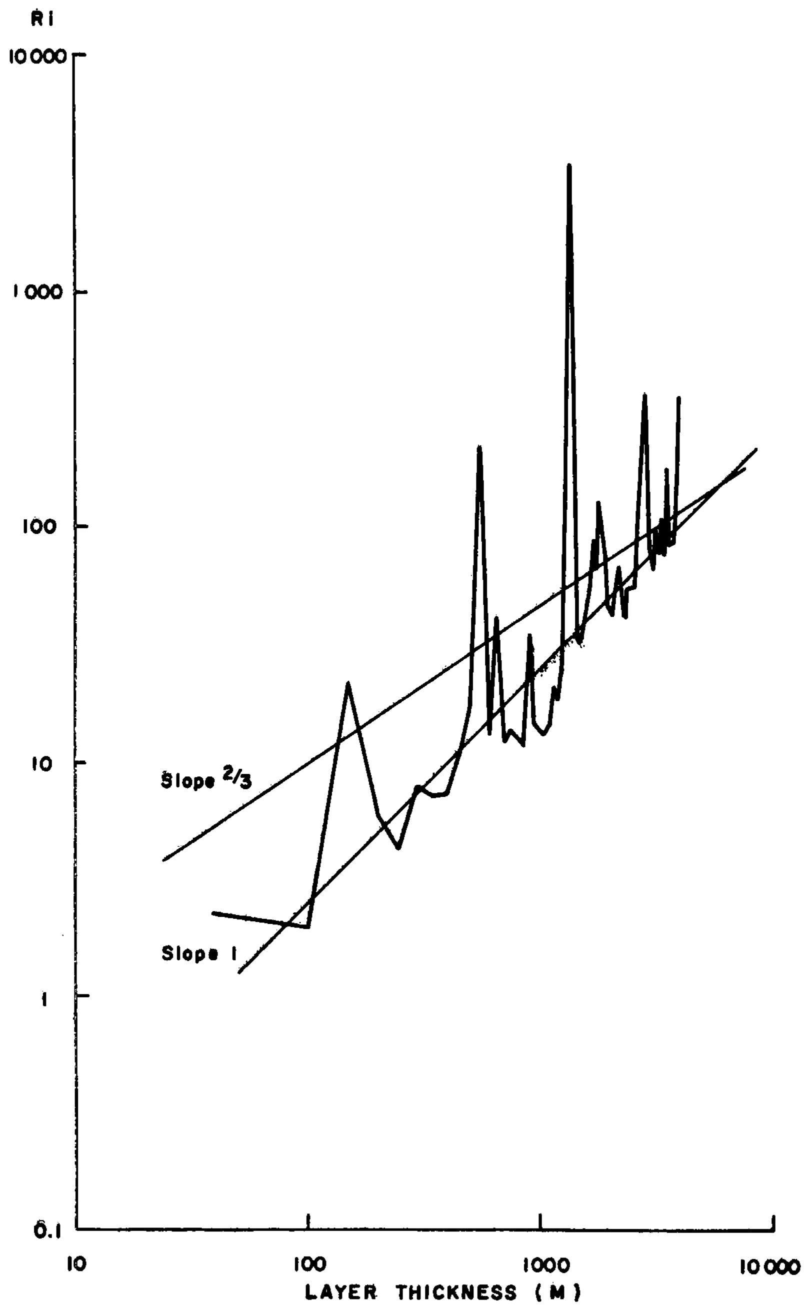

FIG. 8. Logarithmic plot of vertically-averaged Richardson number (Ri) versus scale length (L) for 1731 GMT, 29 December 1964. Thin lines represent slopes of 2/3 and 1, respectively.

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