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Bayesian switching multiple disorder problems

Pavel V. Gapeev*

The switching multiple disorder problem seeks to determine an ordered infinite sequence of times of alarms which are as close as possible to the unknown times of disorders, or change-points, at which the observable process changes its probability characteristics. We study a Bayesian formulation of this problem for an observable Brownian motion with switching constant drift rates. The method of proof is based on the reduction of the initial problem to an associated optimal switching problem for a three-dimensional diffusion posterior probability process and the analysis of the equivalent coupled parabolic-type free-boundary problem. We derive analytic-form estimates for the Bayesian risk function and the optimal switching boundaries for the components of the the posterior probability process.

1 Introduction.

Suppose that, at time $t = 0$, we begin to observe a sample path of some stochastic process $X = (X_t)_{t \ge 0}$, with probability characteristics changing their values at some unknown disorder times at which an unobservable two-state process $\Theta = (\Theta_t)_{t \ge 0}$ switches between one state and the other. The switching multiple disorder problem is to decide at which time instants $(\tau_n)_{n \in \mathbb{N}}$ one should give alarm signals to indicate the occurrence of changes in the current state of the process Θ, as close as possible to the initial disorder times. Such quickest disorder, or change-point, detection problems have originally arisen and still play a prominent role in quality control, where one observes the output of a production line and wishes to detect deviations from the acceptable levels. After the introduction of the original control charts by Shewhart [34], various modifications of the disorder problem have been recognised (see, e.g. Page [27]) and implemented in a number of applied sciences (see, e.g. Carlstein, Müller, and Siegmund [11]).

*London School of Economics, Department of Mathematics, Houghton Street, London WC2A 2AE, United Kingdom; e-mail: p.v.gapeev@lse.ac.uk

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The problem of detecting a single change in the constant drift rate of a Brownian motion (Wiener process) was formulated and explicitly solved by Shiryaev [35]-[36] and [39]-[40] (see also Shiryaev [41; Chapter IV] and Peskir and Shiryaev [29; Chapter VI, Section 22] for further references). The optimal time of alarm was sought as a stopping time minimising a linear combination of the false alarm probability and the expected delay time in the detection of the disorder. Shiryaev [35] and [37] proposed another formulation of the problem in which the occurrence of a single change should be preceded by a long period of observations, under which a stationary regime has been established. The resulting optimal *multi-stage* detection procedure consisted in searching for a sequence of stopping times minimising the average delay time given that the mean time between two false alarms is fixed. More recently, Feinberg and Shiryaev [15] derived an explicit solution of the quickest detection problem in the generalised Bayesian formulation and proved the asymptotic optimality of the associated detection procedure for the related minimax formulation. Extensive overviews of these and other related sequential quickest change-point detection methods were provided in Shiryaev [42] and Poor and Hadjiliadis [31].

In the present paper, we formulate and solve the switching multiple disorder problem for an observed Wiener process X changing its drift rate from μ_i to μ_{1-i}, when Θ changes its state from i to $1-i$, for every $i = 0, 1$. In contrast to the problem of detecting a *single* change, in the *switching multiple* disorder problem, one looks for an infinite non-decreasing sequence of the alarm times $(\tau_n)_{n \in \mathbb{N}}$ minimising a series of linear combinations of *discounted* average losses due to false alarms and delay penalty costs in the detection of the disorder times. We propose a formulation of the problem in which Θ is assumed to be a continuous time Markov chain of intensity λ, started at the state 0 or 1 with probabilities $1-\pi$ and π, respectively.

Apart from other possible areas of application, such a situation usually happens in models of illiquid financial markets, which have trading investors of different kinds. It is natural to assume that the *small* investors can only influence little fluctuations of the market prices of risky assets, while the *large* investors can affect the pricing trends as well, by means of either buying or selling substantial amounts of assets. More precisely, the pricing trends should either rise up or fall down at some random times, after essential amounts of assets are bought or sold, respectively. We can thus consider a model of such financial markets in which the dynamics (of logarithms) of the asset prices are described by a Brownian motion with switching drift rates. We may further assume that our model allows for an infinite number of free-of-charge transactions on the infinite time interval and use an exponential constant discounting rate r which can be chosen equal to the riskless short rate of a bank account. The problem of detecting a single change in the probability characteristics of the accessible financial data, which is associated with the appearance of arbitrage opportunities in the market, was considered by Shiryaev [42].

In the present paper, we reduce the initial Bayesian switching multiple disorder problem to an associated *optimal switching problem* for the posterior probability process, which is a filtering estimate of the current state of the unobservable drift rate of a Brownian motion. The use of exponential discounting makes our problem well-connected to the problem of single disorder detection with exponential delay penalty costs studied by Shiryaev [38], Poor [30], Beibel [6], and Bayraktar and Dayanik [1] (see also Bayraktar, Dayanik and Karatzas [2]-[3] for other important quickest detection problems for Poisson processes). We show that the optimal switching times can be expressed as the first times at which the appropriate posterior probability process exits certain connected regions restricted by boundaries, depending on the running states of some other conditional probability processes. We verify that the Bayesian risk functions and the optimal switching boundaries are uniquely characterised by means of the equivalent *coupled free-boundary problem* for a parabolic-type partial differential operator. We derive analytic form estimates for the resulting Bayesian risk functions and the optimal switching boundaries and formulate the appropriate explicit sequential switching multiple disorder detection procedure.

Optimal switching problems represent extensions of the corresponding optimal stopping problems and games in which one looks for an infinite sequence of optimal stopping times. A general approach for studying such problems was developed in Bensoussan and Friedman [7]-[8], and Friedman [16] (see also Friedman [17; Chapter XVI]). This investigation was continued by Brekke and Øksendal [10], Duckworth and Zervos [13], Yushkevich and Gordienko [46], and Hamadène and Jeanblanc [21] among others for the continuous time case, and by Yushkevich [44]-[45] for the discrete time case. Other optimal switching and impulse control problems, involving hidden Markov chains in the observable jump processes, were recently studied by Bayraktar and Ludkovski [4]-[5].

The paper is organised as follows. In Section 2, for the initial Bayesian quickest multiple disorder detection problem, we construct the associated optimal switching problem and reduce the latter to the appropriate three-dimensional coupled optimal stopping problem. In Section 3, we present the equivalent free-boundary problem and describe the structure of the optimal stopping boundaries. Applying the change-of-variable formula with local time on surfaces, obtained by Peskir [28], we prove that the solution of the coupled optimal stopping problem can be determined as a unique solution of the free-boundary problem, satisfying the appropriate smooth-fit conditions. In Section 4, we reduce the resulting parabolic-type partial differential operator to the normal form, which is amenable for further considerations. We derive closed form estimates for the Bayesian risk functions and the optimal switching boundaries, which are expressed in terms of Heun's double confluent functions, and describe the resulting sequential switching multiple disorder detection procedure. The main results are stated in Theorem 3.1 and Corollary 4.1. The optimal sequential detecting scheme is displayed more explicitly in Corollary 3.1.

2 Formulation of the problem.

In this section, we present a Bayesian formulation of the switching multiple disorder problem for an observable Brownian motion (see, e.g. [41; Chapter IV, Section 4] or [29; Chapter VI, Section 22] for the single disorder case). In these formulations, it is assumed that one observes a sample path of a Brownian motion X with the drift rate switching between μ_0 and μ_1 at some unobservable random times (we assume that $\mu_0 = 0$ and $\mu_1 = \mu$, without loss of generality).

2.1 The setting.

Let us assume that all the considerations take place on a probability space $(\Omega, \mathcal{G}, P_\pi)$ with a continuous-time Markov chain $\Theta = (\Theta_t)_{t \geq 0}$ with two states, 0 and 1, and an independent of Θ standard Brownian motion (Wiener process) $B = (B_t)_{t \geq 0}$ started at zero under P_π. Assume that Θ has the initial distribution $\{1-\pi, \pi\}$, the transition-probability matrix $\{(\lambda_0 e^{-(\lambda_0+\lambda_1)t} + \lambda_1)/(\lambda_0+\lambda_1), \lambda_0(1-e^{-(\lambda_0+\lambda_1)t})/(\lambda_0+\lambda_1); \lambda_1(1-e^{-(\lambda_0+\lambda_1)t})/(\lambda_0+\lambda_1), (\lambda_1 e^{-(\lambda_0+\lambda_1)t} + \lambda_0)/(\lambda_0 +$

$\lambda_1)\}$, so that the intensity-matrix $\{-\lambda_0, \lambda_0; \lambda_1, -\lambda_1\}$, for all $t \ge 0$ and some $\lambda_i > 0$, $i = 0, 1$, fixed. In other words, the Markov chain Θ changes its state from i to $1-i$ at exponentially distributed times of intensity λ_i, for every $i = 0, 1$, which are independent of the dynamics of the Brownian motion B. Such a process Θ is called *telegraphic signal* in the literature (see, e.g. [25; Chapter IX, Section 4] or [14; Chapter VIII]).

Suppose that we observe a continuous process $X = (X_t)_{t \ge 0}$ given by the expression:

$$X_t = \mu \int_0^t \Theta_s \, ds + \sigma \, B_t \tag{2.1}$$

where $\mu \neq 0$ and $\sigma > 0$ are some given constants. Being based upon the continuous observation of X, our task is to find among (non-decreasing) sequences of stopping times $(\tau_n)_{n \in \mathbb{N}}$ of X (i.e., stopping times with respect to the natural filtration $\mathcal{F}_t = \sigma(X_s \,|\, 0 \le s \le t)$ of the process X, for $t \ge 0$) an optimal sequence $(\tau_n^*)_{n \in \mathbb{N}}$ at which the alarms should be sounded *as close as possible* to the unknown switching times of the process Θ. More precisely, the *Bayesian switching multiple disorder problem* consists of computing the Bayesian risk function:

$$\begin{aligned} V^*(\pi) = \inf_{(\tau_n)_{n \in \mathbb{N}}} \sum_{k=1}^{\infty} E_\pi \Big[& e^{-r\tau_{2k-1}} \, I(\Theta_{\tau_{2k-1}} = \Theta_0) + e^{-r\tau_{2k}} \, I(\Theta_{\tau_{2k}} \neq \Theta_0) \\ & + c \int_{\tau_{2k-2}}^{\tau_{2k-1}} e^{-rt} \, I(\Theta_t \neq \Theta_0) \, dt + c \int_{\tau_{2k-1}}^{\tau_{2k}} e^{-rt} \, I(\Theta_t = \Theta_0) \, dt \Big) \Big] \end{aligned} \tag{2.2}$$

and finding the non-decreasing sequence of optimal stopping times $(\tau_n^*)_{n \in \mathbb{N}}$ with $\tau_0^* = 0$, at which the infimum is attained in (2.2), where $I(\cdot)$ denotes the indicator function. Note that the function $V^*(\pi)$ expresses the Bayesian risk of the whole sequence $(\tau_n)_{n \in \mathbb{N}}$ in the case in which the process Θ starts at Θ_0, which has the *prior* distribution $P_\pi(\Theta_0 = 1) = \pi$ and $P_\pi(\Theta_0 = 0) = 1 - \pi$, for all $\pi \in [0, 1]$. We therefore see that $E_\pi\big[e^{-r\tau_{2k-1}} \, I(\Theta_{\tau_{2k-1}} = \Theta_0)\big]$ and $E_\pi\big[e^{-r\tau_{2k}} \, I(\Theta_{\tau_{2k}} \neq \Theta_0)\big]$ expresses the average discounted loss due to a false alarm, and $E_\pi \int_{\tau_{2k-2}}^{\tau_{2k-1}} e^{-rt} \, I(\Theta_t \neq \Theta_0) \, dt$ and $E_\pi \int_{\tau_{2k-1}}^{\tau_{2k}} e^{-rt} \, I(\Theta_t = \Theta_0) \, dt$ expresses the average discounted loss due to a delay in detecting of the time at which Θ changes its state either from Θ_0 to $1 - \Theta_0$, or from $1 - \Theta_0$ to Θ_0, respectively, for any $k \in \mathbb{N}$. In this case, $c > 0$ is a cost rate due to a delay in detection, and $r > 0$ is a discounting rate.

Using the fact that $(\tau_n)_{n \in \mathbb{N}}$ is a non-decreasing sequence of stopping times with respect to the filtration $(\mathcal{F}_t)_{t \ge 0}$, by means of standard arguments, which are similar to those presented in [41; pages 195-197], we obtain:

$$\begin{aligned} E_\pi\Big[e^{-r\tau_n} \, I\Big(\Theta_{\tau_n} \lesseqqgtr \Theta_0\Big)\Big] &= E_\pi\Big[E_\pi\Big[e^{-r\tau_n} \, I\Big(\Theta_{\tau_n} \lesseqqgtr \Theta_0\Big) \,\Big|\, \mathcal{F}_{\tau_n}\Big]\Big] \\ &= E_\pi\Big[e^{-r\tau_n} \, P_\pi\Big(\Theta_{\tau_n} \lesseqqgtr \Theta_0 \,\Big|\, \mathcal{F}_{\tau_n}\Big)\Big] \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} & E_\pi \int_{\tau_{n-1}}^{\tau_n} e^{-rt} \, I\Big(\Theta_{\tau_n} \lesseqqgtr \Theta_0\Big) \, dt = E_\pi \int_0^\infty e^{-rt} \, I\Big(\tau_{n-1} \le t, \Theta_t \lesseqqgtr \Theta_0, t < \tau_n\Big) \, dt \\ & = E_\pi \int_0^\infty E_\pi\Big[e^{-rt} \, I\Big(\tau_{n-1} \le t, \Theta_t \lesseqqgtr \Theta_0, t < \tau_n\Big) \,\Big|\, \mathcal{F}_t\Big] \, dt = E_\pi \int_{\tau_{n-1}}^{\tau_n} e^{-rt} \, P_\pi\Big(\Theta_t \lesseqqgtr \Theta_0 \,\Big|\, \mathcal{F}_t\Big) \, dt \end{aligned} \tag{2.4}$$

holds for every $i = 0, 1$ and any $n \in \mathbb{N}$.

2.2 Sufficient statistics.

It is known from [25; Theorem 9.1] (see also [25; Chapter IX, Example 3]) that the *posterior probability* process $\Pi = (\Pi_t)_{t \ge 0}$ defined by $\Pi_t = P_\pi(\Theta_t = 1 \,|\, \mathcal{F}_t)$ solves the stochastic differential equation:

$$d\Pi_t = \big(\lambda_0 - (\lambda_0 + \lambda_1) \Pi_t \big) \, dt + \frac{\mu}{\sigma} \, \Pi_t (1 - \Pi_t) \, d\overline{B}_t \tag{2.5}$$

with $\Pi_0 = \pi$, where the innovation process $\overline{B} = (\overline{B}_t)_{t \ge 0}$ defined by:

$$\overline{B}_t = \frac{1}{\sigma} \bigg(X_t - \int_0^t \mu \, \Pi_s \, ds \bigg) \tag{2.6}$$

is a standard Brownian motion under the probability measure P_π, with respect to the filtration $(\mathcal{F}_t)_{t \ge 0}$, according to P. Lévy's characterisation theorem (see, e.g. [25; Chapter IV, Theorem 4.1]). It is also seen from (2.5) that Π is a (time-homogeneous strong) Markov process with respect to its natural filtration, which obviously coincides with $(\mathcal{F}_t)_{t \ge 0}$. It also follows from [25; Theorem 9.3] that the process $\Pi^i = (\Pi^i_t)_{t \ge 0}$ defined by $\Pi^i_t = P_\pi(\Theta_t = i, \Theta_0 = i \,|\, \mathcal{F}_t)$, for $i = 0, 1$, solves the stochastic differential equation:

$$d\Pi^i_t = \Big(\lambda_{1-i} \, (2i - 1) \, (i - \Pi_t) - \lambda_0 \, \Pi^0_t - \lambda_1 \, \Pi^1_t \Big) \, dt + \frac{\mu}{\sigma} \, \Pi^i_t (i - \Pi_t) \, d\overline{B}_t \tag{2.7}$$

with $\Pi^1_0 = 1 - \Pi^0_0 = \pi$, for any $\pi \in [0, 1]$. It follows from [26; Chapter VII, Theorem 7.2.4] that the (time-homogeneous) process $(\Pi, \Pi^0, \Pi^1) = (\Pi_t, \Pi^0_t, \Pi^1_t)_{t \ge 0}$ has the strong Markov property with respect to its natural filtration, which inherently coincides with $(\mathcal{F}_t)_{t \ge 0}$.

Taking into account the expressions in (2.3) and (2.4), and the fact that $\Pi^0_t + \Pi^1_t = P_\pi(\Theta_t = \Theta_0 \,|\, \mathcal{F}_t)$, we therefore conclude that the Bayesian risk function from (2.2) admits the representation:

$$\begin{aligned} V^*(\pi) = \inf_{(\tau_n)_{n \in \mathbb{N}}} \sum_{k=1}^\infty E_\pi \bigg[& e^{-r \tau_{2k-1}} \, (\Pi^0_{\tau_{2k-1}} + \Pi^1_{\tau_{2k-1}}) + e^{-r \tau_{2k}} \, (1 - \Pi^0_{\tau_{2k}} - \Pi^1_{\tau_{2k}}) \\ & + c \int_{\tau_{2k-2}}^{\tau_{2k-1}} e^{-rt} \, (1 - \Pi^0_t - \Pi^1_t) \, dt + c \int_{\tau_{2k-1}}^{\tau_{2k}} e^{-rt} \, (\Pi^0_t + \Pi^1_t) \, dt \bigg] \end{aligned} \tag{2.8}$$

where the infimum is taken over all non-decreasing sequences of stopping times $(\tau_n)_{n \in \mathbb{N}}$. By virtue of the strong Markov property of the process (Π, Π^0, Π^1), we can reduce the problem of (2.8) to the following *coupled optimal stopping problem*:

$$\begin{aligned} V^*_i(\pi, \pi_0, \pi_1) = \inf_{\zeta_i} E_{\pi, \pi_0, \pi_1} \bigg[& e^{-r\zeta_i} \, \big((2i - 1) \, (i - \Pi^0_{\zeta_i} - \Pi^1_{\zeta_i}) + V^*_{1-i}(\Pi_{\zeta_i}, \Pi^0_{\zeta_i}, \Pi^1_{\zeta_i}) \big) \\ & + c \int_0^{\zeta_i} e^{-rt} \, (1 - 2i) \, (1 - i - \Pi^0_t - \Pi^1_t) \, dt \bigg] \end{aligned} \tag{2.9}$$

where the infimum is taken over all stopping times ζ_i, $i = 0, 1$, of the process (Π, Π^0, Π^1), which starts at some $(\pi, \pi_0, \pi_1) \in [0, 1]^3$, under the probability measure P_{π, π_0, π_1}.

3 Main results and proofs.

In this section, we formulate and prove the main assertions of the paper, which are related to the coupled optimal stopping problem in (2.9), and thus to the quickest switching multiple disorder detection problem in (2.2) and (2.8).

3.1 The structure of the optimal stopping times.

In order to specify the structure of the optimal stopping times in the problem of (2.9), let us introduce the function:

$$F_i(\pi, \pi_0, \pi_1) = \frac{c\lambda_0}{r(\lambda_0 + \lambda_1 + r)} - \frac{c(1-i)}{r} + \frac{c(\lambda_1 - \lambda_0)}{r(\lambda_0 + \lambda_1 + r)}\, \pi + \sum_{j=0}^{1} \frac{c(\lambda_{1-j} - \lambda_j + r)}{r(\lambda_0 + \lambda_1 + r)}\, \pi_j \tag{3.1}$$

and use Itô's formula (see, e.g. [25; Theorem 4.4]) to obtain:

$$e^{-rt}\, F_i(\Pi_t, \Pi_t^0, \Pi_t^1) = F_i(\pi, \pi_0, \pi_1) + c \int_0^t e^{-rs} \left(1 - i - \Pi_s^0 - \Pi_s^1\right) ds + N_t^i \tag{3.2}$$

where the process $N^i = (N_t^i)_{t \ge 0}$ defined by:

$$N_t^i = \int_0^t e^{-rs}\, \frac{\mu}{\sigma} \left(\frac{c(\lambda_1 - \lambda_0)}{r(\lambda_0 + \lambda_1 + r)}\, \Pi_s(1 - \Pi_s) + \sum_{j=0}^{1} \frac{c(\lambda_{1-j} - \lambda_j + r)}{r(\lambda_0 + \lambda_1 + r)}\, \Pi_s^j (j - \Pi_s) \right) d\overline{B}_s \tag{3.3}$$

is a continuous square integrable martingale under P_{π,π_0,π_1}. Then, applying Doob's optional sampling theorem (see, e.g. [25; Theorem 3.6]), we get from the expression in (3.2) that:

$$E_{\pi,\pi_0,\pi_1}\big[e^{-r\zeta_i}\, F_i(\Pi_{\zeta_i}, \Pi_{\zeta_i}^0, \Pi_{\zeta_i}^1)\big] = F_i(\pi, \pi_0, \pi_1) + c\, E_{\pi,\pi_0,\pi_1} \int_0^{\zeta_i} e^{-rt} \left(1 - i - \Pi_t^0 - \Pi_t^1\right) dt \tag{3.4}$$

holds for all $(\pi, \pi_0, \pi_1) \in [0,1]^3$ and any stopping time ζ_i. Hence, inserting the expression of (3.4) into the one of (2.9), we see that the coupled optimal stopping problem takes the form:

$$U_i^*(\pi, \pi_0, \pi_1) = \inf_{\zeta_i} E_{\pi,\pi_0,\pi_1}\big[e^{-r\zeta_i} \big((1-2i)\, G_i(\Pi_{\zeta_i}, \Pi_{\zeta_i}^0, \Pi_{\zeta_i}^1) + U_{1-i}^*(\Pi_{\zeta_i}, \Pi_{\zeta_i}^0, \Pi_{\zeta_i}^1)\big)\big] \tag{3.5}$$

with $U_i^*(\pi, \pi_0, \pi_1) = V_i^*(\pi, \pi_0, \pi_1) + (1-2i) F_i(\pi, \pi_0, \pi_1)$ and

$$\begin{aligned} G_i(\pi, \pi_0, \pi_1) &= F_i(\pi, \pi_0, \pi_1) + F_{1-i}(\pi, \pi_0, \pi_1) + \pi_0 + \pi_1 - i \\ &= \frac{2c(\lambda_1 - \lambda_0)}{r(\lambda_0 + \lambda_1 + r)}\, \pi + \sum_{j=0}^{1} \left(\frac{2c(\lambda_{1-j} - \lambda_j + r)}{r(\lambda_0 + \lambda_1 + r)} + 1 \right) \pi_j + \frac{c(\lambda_0 - \lambda_1 - r)}{r(\lambda_0 + \lambda_1 + r)} - i \end{aligned} \tag{3.6}$$

for all $(\pi, \pi_0, \pi_1) \in [0,1]^3$ and $i = 0, 1$. Thus, by means of the results of general theory of optimal stopping problems (see, e.g. [41; Chapter III, Section 3] and [29; Chapter I, Section 2]), it follows from the structure of the reward in (3.5) that the optimal stopping times are given by:

$$\zeta_i^* = \inf\big\{ t \ge 0 \,\big|\, U_i^*(\Pi_t, \Pi_t^0, \Pi_t^1) = (1-2i)\, G_i(\Pi_t, \Pi_t^0, \Pi_t^1) + U_{1-i}^*(\Pi_t, \Pi_t^0, \Pi_t^1) \big\} \tag{3.7}$$

for $i = 0, 1$, whenever they exist. It is seen from the structure of the function $F_i(\pi, \pi_0, \pi_1)$ in (3.6) that if the point (π, π_0, π_1) belongs to the corresponding continuation region:

$$C_i^* = \{(\pi, \pi_0, \pi_1) \in [0,1]^3 \,|\, U_i^*(\pi, \pi_0, \pi_1) < (1-2i)\, G_i(\pi, \pi_0, \pi_1) + U_{1-i}^*(\pi, \pi_0, \pi_1)\} \tag{3.8}$$

then the points (π, π_0', π_1'), with either $\pi_j' \geq \pi_j$ or $\pi_j' \leq \pi_j$ and $\pi_{1-j}' = \pi_{1-j}$ when $\lambda_j \leq \lambda_{1-j}$ holds for some $j = 0, 1$, belong to either C_0^* or C_1^*, respectively. Then, taking into account the concavity of the functions $(1-2i)G_i(\pi, \pi_0, \pi_1) + U_{1-i}^*(\pi, \pi_0, \pi_1)$ in π_j on $[0,1]$, we may therefore conclude that there exist functions $0 < a_*(\pi, \pi_{1-j}), b_*(\pi, \pi_{1-j}) < 1$ such that the continuation regions in (3.8) for the coupled optimal stopping problems of (2.9) and (3.5) take the form:

$$C_0^* = \{(\pi, \pi_0, \pi_1) \in [0,1]^3 \,|\, \pi_j > a_*(\pi, \pi_{1-j})\} \;\; \text{and} \;\; C_1^* = \{(\pi, \pi_0, \pi_1) \in [0,1]^3 \,|\, \pi_j < b_*(\pi, \pi_{1-j})\} \tag{3.9}$$

so that the corresponding stopping regions are the closures of the sets:

$$D_0^* = \{(\pi, \pi_0, \pi_1) \in [0,1]^3 \,|\, \pi_j < a_*(\pi, \pi_{1-j})\} \;\; \text{and} \;\; D_1^* = \{(\pi, \pi_0, \pi_1) \in [0,1]^3 \,|\, \pi_j > b_*(\pi, \pi_{1-j})\} \tag{3.10}$$

when $\lambda_j \leq \lambda_{1-j}$ holds for some $j = 0, 1$. It follows from the facts that the gain function $G_i(\pi, \pi_0, \pi_1)$ in (3.6) is linear and the difference function $(U_i^* - U_{1-i}^*)(\pi, \pi_0, \pi_1)$ in (3.5) is concave in the closure of D_i^* from (3.10), for every $i = 0, 1$, that the boundaries $a_*(\pi, \pi_{1-j})$ and $b_*(\pi, \pi_{1-j})$ are continuous and of bounded variation.

Summarising the facts proved above, we are now ready to formulate the following assertion.

Lemma 3.1 *Let the process X be given by the equation in (2.1). Then, the optimal stopping times ζ_i^*, $i = 0, 1$, in the coupled optimal stopping problems of (2.9) and (3.5) take the form:*

$$\zeta_0^* = \inf\left\{t \geq 0 \,\middle|\, \Pi_t^j \leq a_*(\Pi_t, \Pi_t^{1-j})\right\} \;\; \textit{and} \;\; \zeta_1^* = \inf\left\{t \geq 0 \,\middle|\, \Pi_t^j \geq b_*(\Pi_t, \Pi_t^{1-j})\right\} \tag{3.11}$$

whenever they exist, for some continuous functions of bounded variation $0 < a_*(\pi, \pi_{1-j}), b_*(\pi, \pi_{1-j}) < 1$, when $\lambda_j \leq \lambda_{1-j}$ holds for some $j = 0, 1$. In that case, the optimal Bayesian times of alarms $(\tau_n^*)_{n \in \mathbb{N}}$ in the quickest switching multiple disorder detection problem of (2.8) are given by:

$$\tau_{2k-1}^* = \inf\left\{t \geq \tau_{2k-2}^* \,\middle|\, \Pi_t^j \leq a_*(\Pi_t, \Pi_t^{1-j})\right\} \textit{and} \tau_{2k}^* = \inf\left\{t \geq \tau_{2k-1}^* \,\middle|\, \Pi_t^j \geq b_*(\Pi_t, \Pi_t^{1-j})\right\} \tag{3.12}$$

for every $k \in \mathbb{N}$.

3.2 The coupled free-boundary problem.

By means of standard arguments based on the application of Itô's formula (see, e.g. [22; Chapter V, Section 5.1] or [26; Chapter VII, Section 7.3]), it is shown that the infinitesimal operator $\mathbb{L}_{(\Pi, \Pi^0, \Pi^1)}$ of the process (Π, Π^0, Π^1) from (2.5) and (2.7) has the structure:

$$\begin{aligned}
\mathbb{L}_{(\Pi, \Pi^0, \Pi^1)} &= (\lambda_0 - (\lambda_0 + \lambda_1)\, \pi)\, \partial_\pi + \frac{1}{2} \left(\frac{\mu}{\sigma}\right)^2 \pi^2 (1-\pi)^2\, \partial^2_{\pi\pi} - \left(\frac{\mu}{\sigma}\right)^2 \pi_0 \pi^2 (1-\pi)\, \partial^2_{\pi\pi_0} \\
&+ (\lambda_1 (\pi - \pi_1) - \lambda_0\, \pi_0)\, \partial_{\pi_0} + \frac{1}{2} \left(\frac{\mu}{\sigma}\right)^2 \pi_0^2 \pi^2\, \partial^2_{\pi_0\pi_0} + \left(\frac{\mu}{\sigma}\right)^2 \pi_1 \pi (1-\pi)^2\, \partial^2_{\pi\pi_1} \\
&+ (\lambda_0 (1 - \pi - \pi_0) - \lambda_1\, \pi_1)\, \partial_{\pi_1} + \frac{1}{2} \left(\frac{\mu}{\sigma}\right)^2 \pi_1^2 (1-\pi)^2\, \partial^2_{\pi_1\pi_1} - \left(\frac{\mu}{\sigma}\right)^2 \pi_0 \pi_1 \pi (1-\pi)\, \partial^2_{\pi_0\pi_1}
\end{aligned} \tag{3.13}$$

for all $(\pi, \pi_0, \pi_1) \in (0,1)^3$. We also note that the fact that the stochastic differential equations for the posterior probabilities in (2.5) and (2.7) are driven by the same (one-dimensional) innovation Brownian motion yields the property that the infinitesimal operator in (3.13) is of parabolic type.

In order to characterise the unknown value functions $U_i^*(\pi, \pi_0, \pi_1)$, $i = 0, 1$, from (3.5), as well as the unknown boundaries $a_*(\pi, \pi_{1-j})$ and $b_*(\pi, \pi_{1-j})$ from (3.11), we may use the results of the general theory of optimal stopping problems for continuous time Markov processes (see, e.g. [20], [41; Chapter III, Section 8] and [29; Chapter IV, Section 8]). More precisely, we formulate the associated *coupled free-boundary problem*:

$$(\mathbb{L}_{(\Pi,\Pi^0,\Pi^1)} U_i - rU_i)(\pi, \pi_0, \pi_1) = 0 \quad \text{for} \quad (\pi, \pi_0, \pi_1) \in C_i \tag{3.14}$$

$$(U_0 - U_1 - G_0)(\pi, \pi_0, \pi_1)\big|_{\pi_j = a(\pi, \pi_{1-j})+} = (U_1 - U_0 + G_1)(\pi, \pi_0, \pi_1)\big|_{\pi_j = b(\pi_0, \pi_{1-j})-} = 0 \tag{3.15}$$

$$(U_i - U_{1-i})(\pi, \pi_0, \pi_1) = (1 - 2i)\, G_i(\pi, \pi_0, \pi_1) \quad \text{for} \quad (\pi, \pi_0, \pi_1) \in D_i \tag{3.16}$$

$$(U_i - U_{1-i})(\pi, \pi_0, \pi_1) < (1 - 2i)\, G_i(\pi, \pi_0, \pi_1) \quad \text{for} \quad (\pi, \pi_0, \pi_1) \in C_i \tag{3.17}$$

$$(\mathbb{L}_{(\Pi,\Pi^0,\Pi^1)} U_i - rU_i)(\pi, \pi_0, \pi_1) > 0 \quad \text{for} \quad (\pi, \pi_0, \pi_1) \in D_i \tag{3.18}$$

with $0 < a(\pi, \pi_{1-j}), b(\pi, \pi_{1-j}) < 1$, where the *instantaneous-stopping conditions* of (3.15) are satisfied at $a_*(\pi, \pi_{1-j})$ and $b_*(\pi, \pi_{1-j})$ for all $(\pi, \pi_{1-j}) \in [0,1]^2$, when $\lambda_j \leq \lambda_{1-j}$ for some $j = 0, 1$. Note that the superharmonic characterisation of the value function (see, e.g. [41; Chapter III, Section 8] and [29; Chapter IV, Section 9]) implies that $U_i^*(\pi, \pi_0, \pi_1)$, $i = 0, 1$, from (3.5) are the largest functions satisfying the expressions in (3.14)-(3.18) with the boundaries $a_*(\pi, \pi_{1-j})$ and $b_*(\pi, \pi_{1-j})$, respectively. Moreover, since the system in (3.14)-(3.18) admits multiple solutions, we need to use certain additional conditions which would specify the appropriate solution providing the value function and the optimal switching boundaries for the initial problem of (3.5). For this, let us assume that the following *smooth-fit conditions*:

$$(U_0 - U_1 - G_0)_{\pi_j}(\pi, \pi_0, \pi_1)\big|_{\pi_j = a(\pi, \pi_{1-j})+} = (U_1 - U_0 + G_1)_{\pi_j}(\pi, \pi_0, \pi_1)\big|_{\pi_j = b(\pi, \pi_{1-j})-} = 0 \tag{3.19}$$

hold for all $(\pi, \pi_{1-j}) \in (0,1)^2$, when $\lambda_j \leq \lambda_{1-j}$ for some $j = 0, 1$.

We further provide an analysis of the parabolic-type free-boundary problem in (3.14)-(3.17), satisfying the inequality in (3.18) and the conditions of (3.19), and such that the resulting boundaries are continuous and of bounded variation. Since such free-boundary problems cannot normally be solved explicitly, the existence and uniqueness of classical as well as viscosity solutions of the variational inequalities, arising in the context of optimal stopping problems, have been extensively studied in the literature (see, e.g. Friedman [17], Bensoussan and Lions [9], Krylov [24], or Øksendal [26]). Although the necessary conditions for existence and uniqueness of such solutions in [17; Chapter XVI, Theorem 11.1], [24; Chapter V, Section 3, Theorem 14] with [24; Chapter VI, Section 4, Theorem 12], and [26; Chapter X, Theorem 10.4.1] can be verified by virtue of the regularity of the coefficients of the three-dimensional diffusion process, the application of these classical results would still have rather inexplicit character.

We therefore continue with the following verification assertion related to the free-boundary problem formulated above.

Lemma 3.2 *Assume that the optimal stopping times ζ_i^*, $i = 0, 1$, in the problem of (3.5) have a form of (3.11) with the continuous boundaries of bounded variation* $0 < a_*(\pi, \pi_{1-j}), b_*(\pi, \pi_{1-j}) <$

1, when $\lambda_j \le \lambda_{1-j}$ holds for some $j = 0, 1$. Then, the value functions from (3.5) admit the representations:

$$U_0^*(\pi, \pi_0, \pi_1) = \begin{cases} U_0(\pi, \pi_0, \pi_1; a_*(\pi, \pi_{1-j})), & \text{for } \pi_j > a_*(\pi, \pi_{1-j}) \\ G_0(\pi, \pi_0, \pi_1) + U_1^*(\pi, \pi_0, \pi_1), & \text{for } \pi_j \le a_*(\pi, \pi_{1-j}) \end{cases} \tag{3.20}$$

and

$$U_1^*(\pi, \pi_0, \pi_1) = \begin{cases} U_1(\pi, \pi_0, \pi_1; b_*(\pi_0, \pi_1)), & \text{for } \pi_j < b_*(\pi, \pi_{1-j}) \\ -G_1(\pi, \pi_0, \pi_1) + U_0^*(\pi, \pi_0, \pi_1), & \text{for } \pi_j \ge b_*(\pi, \pi_{1-j}) \end{cases} \tag{3.21}$$

with

$$U_0(\pi, \pi_0, \pi_1; a_*(\pi, \pi_{1-j})) = E_{\pi, \pi_0, \pi_1}\big[e^{-r\zeta_0^*}\big(G_0(\Pi_{\zeta_0^*}, \Pi^0_{\zeta_0^*}, \Pi^1_{\zeta_0^*}) + U_1^*(\Pi_{\zeta_0^*}, \Pi^0_{\zeta_0^*}, \Pi^1_{\zeta_0^*})\big)\big] \tag{3.22}$$

and

$$U_1(\pi, \pi_0, \pi_1; b_*(\pi_0, \pi_{1-j})) = E_{\pi, \pi_0, \pi_1}\big[e^{-r\zeta_1^*}\big(-G_1(\Pi_{\zeta_1^*}, \Pi^0_{\zeta_1^*}, \Pi^1_{\zeta_1^*}) + U_0^*(\Pi_{\zeta_1^*}, \Pi^0_{\zeta_1^*}, \Pi^1_{\zeta_1^*})\big)\big] \tag{3.23}$$

whenever the inequalities of (3.18) hold in the regions from (3.10) for every $i = 0, 1$, where the boundaries $a_*(\pi, \pi_{1-j})$ and $b_*(\pi, \pi_{1-j})$ are uniquely determined by the conditions of (3.19).

Proof. In order to verify the assertions stated above, let us denote by $U_i(\pi, \pi_0, \pi_1)$, $i = 0, 1$, the right-hand sides of the expressions in (3.22) and (3.23), respectively. It follows by the strong Markov property of the process (Π, Π^0, Π^1) that the functions $U_0(\pi, \pi_0, \pi_1; a_*(\pi, \pi_{1-j}))$ in (3.22) and $U_1(\pi, \pi_0, \pi_1; b_*(\pi, \pi_{1-j}))$ in (3.23) solve the partial differential equation of (3.14) and satisfy the instantaneous-stopping conditions of (3.15). Then, using the fact that the function $U_i(\pi, \pi_0, \pi_1)$ satisfies the conditions of (3.16)-(3.17) by construction, we can apply the local time-space formula from Peskir [28] (see also [29; Chapter II, Section 3.5] for a summary of the related results and further references) to obtain:

$$\begin{aligned} e^{-rt}\, U_i(\Pi_t, \Pi_t^0, \Pi_t^1) &= U_i(\pi, \pi_0, \pi_1) + M_t^i + L_t^i \\ &+ \int_0^t e^{-rs}\, (\mathbb{L}_{(\Pi, \Pi^0, \Pi^1)} U_i - rU_i)(\Pi_s, \Pi_s^0, \Pi_s^1)\, I\big((\Pi_s, \Pi_s^0, \Pi_s^1) \notin C_i^*\big)\, ds \end{aligned} \tag{3.24}$$

where the process $M^i = (M_t^i)_{t \ge 0}$ defined by:

$$\begin{aligned} M_t^i = \int_0^t e^{-rs} \Big(&(U_i)_\pi(\Pi_s, \Pi_s^0, \Pi_s^1)\, \frac{\mu}{\sigma}\, \Pi_s(1 - \Pi_s) + \sum_{j=0}^{1} (U_i)_{\pi_j}(\Pi_s, \Pi_s^0, \Pi_s^1)\, \frac{\mu}{\sigma}\, \Pi_s^j(j - \Pi_s) \Big) \\ &\times I\big(\Pi_s^j \ne a_*(\Pi_s, \Pi_s^{1-j}), \Pi_s^j \ne b_*(\Pi_s, \Pi_s^{1-j})\big)\, d\overline{B}_s \end{aligned} \tag{3.25}$$

is a continuous local martingale under the probability measure P_{π, π_0, π_1} with respect to the filtration $(\mathcal{F}_t)_{t \ge 0}$, for every $i = 0, 1$. Here, the process $L^i = (L_t^i)_{t \ge 0}$ is given by:

$$L_t^i = \frac{1}{2} \int_0^t e^{-rs}\, \Delta_{\pi_j} U_i(\Pi_s, \Pi_s^0, \Pi_s^1)\, I\big(\Pi_s^j = c_i(\Pi_s^0, \Pi_s^{1-j})\big)\, d\ell_s^i \tag{3.26}$$

where we set $\Delta_{\pi_j} U_i(\pi, c_i(\pi, \pi_{1-j}), \pi_{1-j}) = (U_i)_{\pi_j}(\pi, c_i(\pi, \pi_{1-j})+, \pi_{1-j}) - (U_i)_{\pi_j}(\pi, c_i(\pi, \pi_{1-j})-, \pi_{1-j})$ with $c_0(\pi, \pi_{1-j}) = a(\pi, \pi_{1-j})$ and $c_1(\pi, \pi_{1-j}) = b(\pi, \pi_{1-j})$, and the process $\ell^i = (\ell^i_t)_{t \ge 0}$ defined by:

$$\ell^i_t = P_{\pi,\pi_0,\pi_1} - \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t I\big(-\varepsilon < \Pi^j_s - c_i(\Pi_s, \Pi^{1-j}_s) < \varepsilon \big) \left(\frac{\mu}{\sigma} \right)^2 \langle \Pi^j - c_i(\Pi, \Pi^{1-j}) \rangle_s \tag{3.27}$$

is the local time of Π^j at the surface $c_i(\Pi, \Pi^{1-j})$ at which the partial derivative $(U_i)_{\pi_j}(\pi, \pi_0, \pi_1)$ may not exist, and $\langle \Pi^j - c_i(\Pi, \Pi^{1-j}) \rangle$ is the quadratic variation of the process $\Pi^j - c_i(\Pi, \Pi^{1-j})$. It follows from the structure of the gain function $(1-2i)G_i(\pi, \pi_0, \pi_1) + U^*_{1-i}(\pi, \pi_0, \pi_1)$ in (3.5), and the optimal stopping times ζ^*_i in (3.11), that the inequalities $\Delta_{\pi_j} U_i(\pi, c_i(\pi, \pi_{1-j}), \pi_{1-j}) \le 0$ should hold for all $(\pi, \pi_{1-j}) \in [0,1]^2$, when $\lambda_j \le \lambda_{1-j}$ for some $j = 0, 1$, so that the continuous process L^i defined in (3.26) is non-increasing. We may therefore conclude that $L^i_t = 0$, $i = 0, 1$, can hold for all $t \ge 0$ if and only if the smooth-fit conditions of (3.19) are satisfied.

Using the assumption that the inequality in (3.18) holds with the boundaries $a_*(\pi, \pi_{1-j})$ and $b_*(\pi, \pi_{1-j})$, we conclude from the conditions in (3.15)-(3.17) that $(\mathbb{L}_{(\Pi,\Pi^0,\Pi^1)} U_i - r U_i)(\pi, \pi_0, \pi_1) \ge 0$ holds for any $\pi_j \ne a_*(\pi, \pi_{1-j})$ and $\pi_j \ne b_*(\pi, \pi_{1-j})$, when $\lambda_j \le \lambda_{1-j}$ for some $j = 0, 1$. Moreover, by virtue of the fact that ζ^*_i is an optimal stopping time, the inequality $(U_i - U_{1-i})(\pi, \pi_0, \pi_1) \le (1-2i) G_i(\pi, \pi_0, \pi_1)$ holds for all $(\pi, \pi_0, \pi_1) \in [0,1]^3$ and every $i = 0, 1$. Since the time spent by Π^j at the surfaces $a_*(\Pi, \Pi^{1-j})$ and $b_*(\Pi, \Pi^{1-j})$ of bounded variation is of Lebesgue measure zero, the indicators which appear in the integrals in the second lines of (3.24) and in (3.25) can be ignored. Thus, the expression in (3.24) yields that the inequalities:

$$\begin{aligned} & e^{-r\zeta_i} \big((1-2i)\, G_i(\Pi_{\zeta_i}, \Pi^0_{\zeta_i}, \Pi^1_{\zeta_i}) + U_{1-i}(\Pi_{\zeta_i}, \Pi^0_{\zeta_i}, \Pi^1_{\zeta_i}) \big) + L^i_{\zeta_i} \\ & \ge e^{-r\zeta_i}\, U_i(\Pi_{\zeta_i}, \Pi^0_{\zeta_i}, \Pi^1_{\zeta_i}) + L^i_{\zeta_i} \ge U_i(\pi, \pi_0, \pi_1) + M^i_{\zeta_i} \end{aligned} \tag{3.28}$$

hold for any stopping time ζ_i and every $i = 0, 1$. Let $(\nu^n_i)_{n \in \mathbb{N}}$ be an arbitrary localising sequence of stopping times for the processes M^i. Then, taking the expectations with respect to P_{π,π_0,π_1} in (3.28), by means of the optional sampling theorem (see, e.g. [25; Theorem 3.6]), we get that the inequalities:

$$\begin{aligned} & E_{\pi,\pi_0,\pi_1} \big[e^{-r(\zeta_i \wedge \nu^n_i)} \big((1-2i)\, G_i(\Pi_{\zeta_i \wedge \nu^n_i}, \Pi^0_{\zeta_i \wedge \nu^n_i}, \Pi^1_{\zeta_i \wedge \nu^n_i}) + U_{1-i}(\Pi_{\zeta_i \wedge \nu^n_i}, \Pi^0_{\zeta_i \wedge \nu^n_i}, \Pi^1_{\zeta_i \wedge \nu^n_i}) \big) + L^i_{\zeta_i \wedge \nu^n_i} \big] \\ & \ge E_{\pi,\pi_0,\pi_1} \big[e^{-r(\zeta_i \wedge \nu^n_i)}\, U_i(\Pi_{\zeta_i \wedge \nu^n_i}, \Pi^0_{\zeta_i \wedge \nu^n_i}, \Pi^1_{\zeta_i \wedge \nu^n_i}) + L^i_{\zeta_i \wedge \nu^n_i} \big] \\ & \ge U_i(\pi, \pi_0, \pi_1) + E_{\pi,\pi_0,\pi_1} M^i_{\zeta_i \wedge \nu^n_i} = U_i(\pi, \pi_0, \pi_1) \end{aligned} \tag{3.29}$$

hold. Hence, letting n go to infinity and using Fatou's lemma, we obtain:

$$\begin{aligned} & E_{\pi,\pi_0,\pi_1} \big[e^{-r\zeta_i} \big((1-2i)\, G_i(\Pi_{\zeta_i}, \Pi^0_{\zeta_i}, \Pi^1_{\zeta_i}) + U_{1-i}(\Pi_{\zeta_i}, \Pi^0_{\zeta_i}, \Pi^1_{\zeta_i}) \big) + L^i_{\zeta_i} \big] \\ & \ge E_{\pi,\pi_0,\pi_1} \big[e^{-r\zeta_i}\, U_i(\Pi_{\zeta_i}, \Pi^0_{\zeta_i}, \Pi^1_{\zeta_i}) + L^i_{\zeta_i} \big] \ge U_i(\pi, \pi_0, \pi_1) \end{aligned} \tag{3.30}$$

for any stopping time ζ_i such that $E_{\pi,\pi_0,\pi_1} L^i_{\zeta_i} > -\infty$ and all $(\pi, \pi_0, \pi_1) \in [0,1]^3$, where $L^i_{\zeta_i} = 0$ holds whenever the conditions of (3.19) are satisfied. By virtue of the structure of the stopping times in (3.11), it is readily seen that the equalities in (3.30) hold with ζ^*_i instead of ζ_i when $(\pi, \pi_0, \pi_1) \in D^*_i$, for every $i = 0, 1$.

Let us now show that the equalities are attained in (3.30) when ζ^*_i replaces ζ_i when $(\pi, \pi_0, \pi_1) \in C^*_i$, for every $i = 0, 1$, and the smooth-fit conditions of (3.19) hold. By virtue of the fact that the function $U_i(\pi, \pi_0, \pi_1)$ and the continuous boundaries of bounded variation $a_*(\pi, \pi_{1-j})$ and $b_*(\pi, \pi_{1-j})$ solve the partial differential equation in (3.14) and satisfy the conditions of (3.15) and (3.19), it follows from the expression in (3.24) and the structure of the stopping times in (3.11) that the equalities:

$$\begin{aligned} & e^{-r(\zeta_i^* \wedge \nu_i^n)} \big((1-2i) \, G_i(\Pi_{\zeta_i^* \wedge \nu_i^n}, \Pi^0_{\zeta_i^* \wedge \nu_i^n}, \Pi^1_{\zeta_i^* \wedge \nu_i^n}) + U_{1-i}(\Pi_{\zeta_i^* \wedge \nu_i^n}, \Pi^0_{\zeta_i^* \wedge \nu_i^n}, \Pi^1_{\zeta_i^* \wedge \nu_i^n}) \big) \\ & = e^{-r(\zeta_i^* \wedge \nu_i^n)} \, U_i(\Pi_{\zeta_i^* \wedge \nu_i^n}, \Pi^0_{\zeta_i^* \wedge \nu_i^n}, \Pi^1_{\zeta_i^* \wedge \nu_i^n}) + L^i_{\zeta_i^* \wedge \nu_i^n} = U_i(\pi, \pi_0, \pi_1) + M^i_{\zeta_i^* \wedge \nu_i^n} \end{aligned} \tag{3.31}$$

hold for $(\pi, \pi_0, \pi_1) \in C_i^*$ and any localising sequence $(\nu_i^n)_{n \in \mathbb{N}}$ of M^i. Hence, taking expectations and letting n go to infinity in (3.31), and using the fact that $G_i(\pi, \pi_0, \pi_1)$ and $U_i(\pi, \pi_0, \pi_1)$, $i = 0, 1$, are bounded functions, we apply the Lebesgue dominated convergence theorem to obtain the equalities:

$$E_{\pi, \pi_0, \pi_1} \big[e^{-r\zeta_i^*} \big((1-2i) \, G_i(\Pi_{\zeta_i^*}, \Pi^0_{\zeta_i^*}, \Pi^1_{\zeta_i^*}) + U_{1-i}(\Pi_{\zeta_i^*}, \Pi^0_{\zeta_i^*}, \Pi^1_{\zeta_i^*}) \big) \big] = U_i(\pi, \pi_0, \pi_1) \tag{3.32}$$

for all $(\pi, \pi_0, \pi_1) \in [0, 1]^3$. We may therefore conclude that the function $U_i(\pi, \pi_0, \pi_1)$ coincides with the value function $U_i^*(\pi, \pi_0, \pi_1)$ of the optimal stopping problem in (3.5), for $i = 0, 1$, whenever the smooth-fit conditions of (3.19) hold.

In order to prove uniqueness of the value functions $U_i^*(\pi, \pi_0, \pi_1)$, $i = 0, 1$, and the boundaries $a_*(\pi, \pi_{1-j})$ and $b_*(\pi, \pi_{1-j})$ as solutions of the free-boundary problem in (3.14)-(3.17) with the smooth-fit conditions of (3.19), let us assume that there exist other continuous boundaries of bounded variation $a'(\pi, \pi_{1-j})$ and $b'(\pi, \pi_{1-j})$ such that the inequality in (3.18) is satisfied. Then, define the functions $U_i'(\pi, \pi_0, \pi_1)$, $i = 0, 1$, as in (3.20) and (3.21) with $U_0'(\pi, \pi_0, \pi_1; a'(\pi, \pi_{1-j}))$ and $U_1'(\pi, \pi_0, \pi_1; b'(\pi, \pi_{1-j}))$ as in (3.22) and (3.23), and the stopping times ζ_i', $i = 0, 1$, as in (3.11) with $a'(\pi, \pi_{1-j})$ and $b'(\pi, \pi_{1-j})$ instead of $a_*(\pi, \pi_{1-j})$ and $b_*(\pi, \pi_{1-j})$, respectively. Following the arguments from the previous part of the proof and using the fact that the functions $U_i'(\pi, \pi_0, \pi_1)$, $i = 0, 1$, solve the partial differential equation in (3.14) and satisfies the conditions of (3.15) and (3.19) with $a'(\pi, \pi_{1-j})$ and $b'(\pi, \pi_{1-j})$ instead of $a(\pi, \pi_{1-j})$ and $b(\pi, \pi_{1-j})$ by construction, we apply the change-of-variable formula from [28] to get:

$$\begin{aligned} e^{-rt} \, U_i'(\Pi_t, \Pi^0_t, \Pi^1_t) &= U_i'(\pi, \pi_0, \pi_1) + M^{i\prime}_t \\ &+ \int_0^t e^{-rs} \, (\mathbb{L}_{(\Pi, \Pi^0, \Pi^1)} U_i' - r U_i')(\Pi_s, \Pi^0_s, \Pi^1_s) \, I\big((\Pi_s, \Pi^0_s, \Pi^1_s) \notin C_i' \big) \, ds \end{aligned} \tag{3.33}$$

where the process $M^{i\prime} = (M^{i\prime}_t)_{t \geq 0}$ defined as in (3.25) with $(U_i')_\pi(\pi, \pi_0, \pi_1)$ and $(U_i')_{\pi_j}(\pi, \pi_0, \pi_1)$ instead of $(U_i')_\pi(\pi, \pi_0, \pi_1)$ and $(U_i')_{\pi_j}(\pi, \pi_0, \pi_1)$ is a continuous local martingale with respect to the probability measure P_{π, π_0, π_1}, and C_i' is defined as in (3.9) with $a_*(\pi, \pi_{1-j})$ and $b_*(\pi, \pi_{1-j})$ instead of $a'(\pi, \pi_{1-j})$ and $b'(\pi, \pi_{1-j})$. Thus, taking into account the structure of the stopping times ζ_i', $i = 0, 1$, we obtain from (3.33) that:

$$\begin{aligned} & e^{-r(\zeta_i' \wedge \nu_i^{n\prime})} \big((1-2i) \, G_i(\Pi_{\zeta_i' \wedge \nu_i^{n\prime}}, \Pi^0_{\zeta_i' \wedge \nu_i^{n\prime}}, \Pi^1_{\zeta_i' \wedge \nu_i^{n\prime}}) + U_{1-i}'(\Pi_{\zeta_i' \wedge \nu_i^{n\prime}}, \Pi^0_{\zeta_i' \wedge \nu_i^{n\prime}}, \Pi^1_{\zeta_i' \wedge \nu_i^{n\prime}}) \big) \\ & = e^{-r(\zeta_i' \wedge \nu_i^{n\prime})} \, U_i'(\Pi_{\zeta_i' \wedge \nu_i^{n\prime}}, \Pi^0_{\zeta_i' \wedge \nu_i^{n\prime}}, \Pi^1_{\zeta_i' \wedge \nu_i^{n\prime}}) = U_i'(\pi, \pi_0, \pi_1) + M^{i\prime}_{\zeta_i' \wedge \nu_i^{n\prime}} \end{aligned} \tag{3.34}$$

holds for $(\pi, \pi_0, \pi_1) \in C_i'$ and any localising sequence $(\nu_i^{n\prime})_{n \in \mathbb{N}}$ of $M^{i\prime}$. Hence, taking expectations and letting n go to infinity in (3.34) and using the fact that $G_i(\pi, \pi_0, \pi_1)$ and $U_i(\pi, \pi_0, \pi_1)$,

$i=0,1$, are bounded functions, by means of the Lebesgue dominated convergence theorem, we have that the equality:

$$E_{\pi,\pi_0,\pi_1}\big[e^{-r\zeta_i'}\,\big((1-2i)\,G_i(\Pi_{\zeta_i'},\Pi^0_{\zeta_i'},\Pi^1_{\zeta_i'})+U'_{1-i}(\Pi_{\zeta_i'},\Pi^0_{\zeta_i'},\Pi^1_{\zeta_i'})\big)\big]=U_i'(\pi,\pi_0,\pi_1) \tag{3.35}$$

is satisfied. Therefore, recalling the fact that ζ_i^*, $i=0,1$, are the optimal stopping times in (3.5) and comparing the expressions in (3.32) and (3.35), we see that the inequality $U_i'(\pi,\pi_0,\pi_1)\ge U_i(\pi,\pi_0,\pi_1)$ should hold for all $(\pi,\pi_0,\pi_1)\in[0,1]^3$.

To prove the fact that $a_*(\pi,\pi_{1-j})\le a'(\pi,\pi_{1-j})$ and $b'(\pi,\pi_{1-j})\le b_*(\pi,\pi_{1-j})$ holds, let us take a point $\pi_j<a_*(\pi,\pi_{1-j})\wedge a'(\pi,\pi_{1-j})$ or $\pi_j>b_*(\pi,\pi_{1-j})\vee b'(\pi,\pi_{1-j})$, for which we have $U_i'(\pi,\pi_0,\pi_1)=U_i(\pi,\pi_0,\pi_1)=G_i(\pi,\pi_0,\pi_1)$, $i=0,1$. For this, we consider the stopping times:

$$\varkappa_0^*=\inf\big\{t\ge 0\,\big|\,\Pi_t^j\ge a_*(\Pi_t,\Pi_t^{1-j})\big\}\quad\text{and}\quad \varkappa_1^*=\inf\big\{t\ge 0\,\big|\,\Pi_t^j\le b_*(\Pi_t,\Pi_t^{1-j})\big\}. \tag{3.36}$$

Then, inserting $\varkappa_i^*\wedge\nu_i^n$ and $\varkappa_i^*\wedge\nu_i^{n\prime}$ into (3.24) and (3.33) in place of t, and using the arguments similar to the ones above, we obtain:

$$\begin{aligned}&E_{\pi,\pi_0,\pi_1}\big[e^{-r\varkappa_i^*}\,U_i(\Pi_{\varkappa_i^*},\Pi^0_{\varkappa_i^*},\Pi^1_{\varkappa_i^*})\big]=U_i(\pi,\pi_0,\pi_1)\\&+E_{\pi,\pi_0,\pi_1}\int_0^{\varkappa_i^*}e^{-rs}\,(\mathbb{L}_{(\Pi,\Pi^0,\Pi^1)}U_i-rU_i)(\Pi_s,\Pi_s^0,\Pi_s^1)\,I\big((\Pi_s,\Pi_s^0,\Pi_s^1)\notin C_i^*\big)\,ds\end{aligned} \tag{3.37}$$

and

$$\begin{aligned}&E_{\pi,\pi_0,\pi_1}\big[e^{-r\varkappa_i^*}\,U_i'(\Pi_{\varkappa_i^*},\Pi^0_{\varkappa_i^*},\Pi^1_{\varkappa_i^*})\big]=U_i'(\pi,\pi_0,\pi_1)\\&+E_{\pi,\pi_0,\pi_1}\int_0^{\varkappa_i^*}e^{-rs}\,(\mathbb{L}_{(\Pi,\Pi^0,\Pi^1)}U_i'-rU_i')(\Pi_s,\Pi_s^0,\Pi_s^1)\,I\big((\Pi_s,\Pi_s^0,\Pi_s^1)\notin C_i'\big)\,ds\end{aligned} \tag{3.38}$$

for all $(\pi,\pi_0,\pi_1)\in[0,1]^3$. Hence, taking into account the fact that $U_i'(\pi,a_*(\pi,\pi_{1-j}),\pi_{1-j})\ge U_i(\pi,a_*(\pi,\pi_{1-j}),\pi_{1-j})$ and $U_i'(\pi,b_*(\pi,\pi_{1-j}),\pi_{1-j})\ge U_i(\pi,b_*(\pi,\pi_{1-j}),\pi_{1-j})$ holds for every $i=0,1$, we get from (3.37) and (3.38) that the inequality:

$$\begin{aligned}&E_{\pi,\pi_0,\pi_1}\int_0^{\varkappa_i^*}e^{-rs}\,(\mathbb{L}_{(\Pi,\Pi^0,\Pi^1)}U_i-rU_i)(\Pi_s,\Pi_s^0,\Pi_s^1)\,I\big((\Pi_s,\Pi_s^0,\Pi_s^1)\notin C_i^*\big)\,ds\\&\le E_{\pi,\pi_0,\pi_1}\int_0^{\varkappa_i^*}e^{-rs}\,(\mathbb{L}_{(\Pi,\Pi^0,\Pi^1)}U_i'-rU_i')(\Pi_s,\Pi_s^0,\Pi_s^1)\,I\big((\Pi_s,\Pi_s^0,\Pi_s^1)\notin C_i'\big)\,ds\end{aligned} \tag{3.39}$$

is satisfied. Thus, by virtue of the assumption of continuity of $a'(\pi,\pi_{1-j})$ and $b'(\pi,\pi_{1-j})$, we see from (3.39) that $a_*(\pi,\pi_{1-j})\le a'(\pi,\pi_{1-j})$ and $b'(\pi,\pi_{1-j})\le b_*(\pi,\pi_{1-j})$ holds for all $(\pi,\pi_0,\pi_1)\in[0,1]^3$.

We finally show that $a'(\pi,\pi_{1-j})$ and $b'(\pi,\pi_{1-j})$ should coincide with $a_*(\pi,\pi_{1-j})$ and $b_*(\pi,\pi_{1-j})$. For this, we take $\pi_j\in(a_*(\pi,\pi_{1-j}),a'(\pi,\pi_{1-j}))$ or $\pi_j\in(b'(\pi,\pi_{1-j}),b_*(\pi,\pi_{1-j}))$ for some $(\pi,\pi_{1-j})\in[0,1]^2$ for such it exists. Hence, inserting $\zeta_i^*\wedge\nu_i^{n\prime}$ into (3.33) in place of t and using the arguments similar to the ones above, we obtain:

$$\begin{aligned}&E_{\pi,\pi_0,\pi_1}\big[e^{-r\zeta_i^*}\,U_i'(\Pi_{\zeta_i^*},\Pi^0_{\zeta_i^*},\Pi^1_{\zeta_i^*})\big]=U_i'(\pi,\pi_0,\pi_1)\\&+E_{\pi,\pi_0,\pi_1}\int_0^{\zeta_i^*}e^{-rs}\,(\mathbb{L}_{(\Pi,\Pi^0,\Pi^1)}U_i'-rU_i')(\Pi_s,\Pi_s^0,\Pi_s^1)\,I\big((\Pi_s,\Pi_s^0,\Pi_s^1)\notin C_i'\big)\,ds\end{aligned} \tag{3.40}$$

for all $(\pi, \pi_0, \pi_1) \in [0,1]^3$. Thus, since we have $U_i'(\pi, \pi_0, \pi_1) = U_i(\pi, \pi_0, \pi_1) = G_i(\pi, \pi_0, \pi_1)$ for $\pi_j = a_*(\pi, \pi_{1-j})$ and $\pi_j = b_*(\pi, \pi_{1-j})$, and $U_i'(\pi, \pi_0, \pi_1) \geq U_i(\pi, \pi_0, \pi_1)$, we see from the expressions in (3.32) and (3.40) that the inequality:

$$E^*_{\pi,\pi_0,\pi_1} \int_0^{\zeta_i^*} e^{-rs} \, (\mathbb{L}_{(\Pi,\Pi^0,\Pi^1)} U_i' - rU_i')(\Pi_s, \Pi_s^0, \Pi_s^1) \, I\big((\Pi_s, \Pi_s^0, \Pi_s^1) \notin C_i'\big) \, ds \leq 0 \tag{3.41}$$

should hold for every $i = 0, 1$, but that is impossible due to the assumption of continuity of $a_*(\pi, \pi_{1-j})$ and $b_*(\pi, \pi_{1-j})$. We may therefore conclude that $a_*(\pi, \pi_{1-j}) = a'(\pi, \pi_{1-j})$ and $b_*(\pi, \pi_{1-j}) = b'(\pi, \pi_{1-j})$, so that $U_i'(\pi, \pi_0, \pi_1)$ coincides with $U_i(\pi, \pi_0, \pi_1)$ for all $(\pi, \pi_0, \pi_1) \in [0,1]^3$ and every $i = 0, 1$. $\square$

3.3 The location of the optimal stopping boundaries.

Suppose that the inequality $a_*(\pi, \pi_{1-j}) < b_*(\pi, \pi_{1-j})$ holds for all $(\pi, \pi_{1-j}) \in [0,1]^2$ when $\lambda_j \leq \lambda_{1-j}$ for some $j = 0, 1$. This property means that the solution of the coupled optimal stopping problem and the corresponding optimal quickest switching multiple disorder detection procedure is nontrivial. In this case, the equalities in (3.14) and (3.16) directly imply that the inequality in (3.18) takes the form:

$$(1 - 2i) \, H_i(\pi, \pi_0, \pi_1) > 0 \quad \text{for} \quad (\pi, \pi_0, \pi_1) \in D_i \tag{3.42}$$

with

$$H_i(\pi, \pi_0, \pi_1) = \lambda_0 + c + ir + (\lambda_1 - \lambda_0) \, \pi - (2(\lambda_0 + c) + r) \, \pi_0 - (2(\lambda_1 + c) + r) \, \pi_1 \tag{3.43}$$

for every $i = 0, 1$. Observe that the expressions in (3.42)-(3.43) are equivalent to the fact that the sets:

$$R_0 = \{(\pi, \pi_0, \pi_1) \in [0,1]^3 \,|\, (2(\lambda_0 + c) + r) \, \pi_0 + (2(\lambda_1 + c) + r) \, \pi_1 > \lambda_0 + c + (\lambda_1 - \lambda_0) \, \pi\} \tag{3.44}$$

and

$$R_1 = \{(\pi, \pi_0, \pi_1) \in [0,1]^3 \,|\, (2(\lambda_0 + c) + r) \, \pi_0 + (2(\lambda_1 + c) + r) \, \pi_1 < r + \lambda_0 + c + (\lambda_1 - \lambda_0) \, \pi\} \tag{3.45}$$

belong to the continuation regions C_0^* and C_1^* from (3.9), which means that the inequalities:

$$a_*(\pi, \pi_{1-j}) < \overline{a}(\pi, \pi_{1-j}) \equiv \frac{\lambda_0 + c + (\lambda_1 - \lambda_0)\pi - (2(\lambda_{1-j} + c) + r)\pi_{1-j}}{2(\lambda_j + c) + r} \tag{3.46}$$

and

$$b_*(\pi, \pi_{1-j}) > \overline{b}(\pi, \pi_{1-j}) \equiv \frac{r + \lambda_0 + c + (\lambda_1 - \lambda_0)\pi - (2(\lambda_{1-j} + c) + r)\pi_{1-j}}{2(\lambda_j + c) + r} \tag{3.47}$$

are satisfied, so that $0 < a_*(\pi, \pi_{1-j}) < \overline{a}(\pi, \pi_{1-j}) < \overline{b}(\pi, \pi_{1-j}) < b_*(\pi, \pi_{1-j}) < 1$ holds for all $(\pi, \pi_{1-j}) \in (0,1)^2$. It is therefore natural to call the parameters of the model *admissible* when the inequalities in (3.46)-(3.47) are satisfied, since otherwise, the optimal stopping times in the problem of (3.5) do not have the structure of (3.11) whenever they exist.

3.4 The structure of the optimal stopping boundaries.

Applying Itô's formula to the expression in (3.6), we get:

$$e^{-rt}\, G_i(\Pi_t, \Pi_t^0, \Pi_t^1) = G_i(\pi, \pi_0, \pi_1) + \int_0^t e^{-rs}\, H_i(\Pi_s, \Pi_s^0, \Pi_s^1)\, ds + N_t^{*i} \tag{3.48}$$

where the function $H_i(\pi, \pi_0, \pi_1)$ is given by (3.43), the process $N^{*i} = (N_t^{*i})_{t\geq 0}$ defined by:

$$N_t^{*i} = N_t^i + N_t^{1-i} + \sum_{j=0}^{1} \int_0^t e^{-rs}\, \frac{\mu}{\sigma}\, \Pi_s^j (j - \Pi_s)\, d\overline{B}_s \tag{3.49}$$

is a continuous square integrable martingale under the probability measure P_{π,π_0,π_1}, and the processes $N^i = (N_t^i)_{t\geq 0}$, $i = 0, 1$, are defined in (3.3). Then, applying Doob's optional sampling theorem, we get from the expression in (3.48) that:

$$E_{\pi,\pi_0,\pi_1}\big[e^{-r\zeta_i^*}\, G_i(\Pi_{\zeta_i}, \Pi_{\zeta_i}^0, \Pi_{\zeta_i}^1)\big] = G_i(\pi, \pi_0, \pi_1) + E_{\pi,\pi_0,\pi_1} \int_0^{\zeta_i} e^{-rt}\, H_i(\Pi_t, \Pi_t^0, \Pi_t^1)\, dt \tag{3.50}$$

holds for all $(\pi, \pi_0, \pi_1) \in [0,1]^3$ and any stopping time ζ_i. Moreover, we can observe from the application of the change-of-variable formula in (3.24) that the expression:

$$\begin{aligned} e^{-rt}\, U_{1-i}^*(\Pi_t, \Pi_t^0, \Pi_t^1) &= U_{1-i}^*(\pi, \pi_0, \pi_1) + M_t^{*(1-i)} \\ &+ \int_0^t e^{-rs}\, (\mathbb{L}_{(\Pi,\Pi^0,\Pi^1)} U_{1-i}^* - rU_{1-i}^*)(\Pi_s, \Pi_s^0, \Pi_s^1)\, I\big((\Pi_s, \Pi_s^0, \Pi_s^1) \in D_{1-i}^*\big)\, ds \end{aligned} \tag{3.51}$$

holds, where $M^{*(1-i)} = (M_t^{*(1-i)})_{t\geq 0}$ defined by:

$$\begin{aligned} M_t^{*(1-i)} &= \int_0^t e^{-rs}\, (U_{1-i}^*)_\pi(\Pi_s, \Pi_s^0, \Pi_s^1)\, \frac{\mu}{\sigma}\, \Pi_s(1 - \Pi_s)\, d\overline{B}_s \\ &+ \sum_{j=0}^{1} \int_0^t e^{-rs}\, (U_{1-i}^*)_{\pi_j}(\Pi_s, \Pi_s^0, \Pi_s^1)\, \frac{\mu}{\sigma}\, \Pi_s^j (j - \Pi_s)\, d\overline{B}_s \end{aligned} \tag{3.52}$$

is a continuous local martingale under P_{π,π_0,π_1}, for every $i = 0, 1$. By virtue of the concavity of the value functions $U_i^*(\pi, \pi_0, \pi_1)$, $i = 0, 1$, it follows that the derivatives in (3.52) are bounded, so that the process $(M_{\zeta_i^*\wedge t}^{*(1-i)})_{t\geq 0}$ is a square integrable integrable martingale under P_{π,π_0,π_1}. Hence, applying Doob's optional sampling theorem, we get from the expressions in (3.51) that:

$$\begin{aligned} E_{\pi,\pi_0,\pi_1}\big[e^{-r\zeta_i^*}\, U_{1-i}^*(\Pi_{\zeta_i^*}, \Pi_{\zeta_i^*}^0, \Pi_{\zeta_i^*}^1)\big] &= U_{1-i}^*(\pi, \pi_0, \pi_1) \\ + E_{\pi,\pi_0,\pi_1} \int_0^{\zeta_i^*} e^{-rt}\, (\mathbb{L}_{(\Pi,\Pi^0,\Pi^1)} U_{1-i}^* &- rU_{1-i}^*)(\Pi_t, \Pi_t^0, \Pi_t^1)\, I\big((\Pi_t, \Pi_t^0, \Pi_t^1) \in D_{1-i}^*\big)\, dt \end{aligned} \tag{3.53}$$

holds for all $(\pi, \pi_0, \pi_1) \in [0,1]^3$ and every $i = 0, 1$. Thus, getting the expressions in (3.50) and (3.53) together, we obtain from the definition of the optimal stopping times in (3.5) that:

$$\begin{aligned} (U_i^* - U_{1-i}^* - (1-2i)\, G_i)(\pi, \pi_0, \pi_1) &= E_{\pi,\pi_0,\pi_1} \int_0^{\zeta_i^*} e^{-rt}\, (1-2i)\, H_i(\Pi_t, \Pi_t^0, \Pi_t^1)\, dt \\ + E_{\pi,\pi_0,\pi_1} \int_0^{\zeta_i^*} e^{-rt}\, (\mathbb{L}_{(\Pi,\Pi^0,\Pi^1)} U_{1-i}^* &- rU_{1-i}^*)(\Pi_t, \Pi_t^0, \Pi_t^1)\, I\big((\Pi_t, \Pi_t^0, \Pi_t^1) \in D_{1-i}^*\big)\, dt \end{aligned} \tag{3.54}$$

holds for all $(\pi, \pi_0, \pi_1) \in [0,1]^3$ and every $i = 0, 1$.

Let us now fix some $(\pi, \pi_0, \pi_1) \in C_i^*$ and denote by $\zeta_i^* = \zeta_i^*(\pi, \pi_0, \pi_1)$ the optimal stopping time in the problem of (3.5). In this case, it follows from (3.54) and the structure of the optimal stopping times in (3.7) that the inequality:

$$(U_i^* - U_{1-i}^* - (1-2i)\, G_i)(\pi, \pi_0, \pi_1) \le E_{\pi, \pi_0, \pi_1} \int_0^{\zeta_i^* \wedge \zeta_{1-i}^*} e^{-rt}\, (1-2i)\, H_i(\Pi_t, \Pi_t^0, \Pi_t^1)\, dt < 0 \quad (3.55)$$

holds, where the function $H_i(\pi, \pi_0, \pi_1)$ defined in (3.43) admits the representation:

$$\begin{aligned} H_i(\pi, \pi_0, \pi_1) &= \lambda_0 + c + \frac{2r(c(\lambda_{1-j} - \lambda_j - \lambda_0 - \lambda_1) - \lambda_j(\lambda_0 + \lambda_1 + r))}{2c(\lambda_{1-j} - \lambda_j + r) + r(\lambda_0 + \lambda_1 + r)}\, i \\ &+ \frac{c(2(\lambda_j + c) + r)(\lambda_0 - \lambda_1 - r)}{2c(\lambda_{1-j} - \lambda_j + r) + r(\lambda_0 + \lambda_1 + r)} - \frac{r(2(\lambda_j + c) + r)(\lambda_0 + \lambda_1 + r)}{2c(\lambda_{1-j} - \lambda_j + r) + r(\lambda_0 + \lambda_1 + r)}\, G_i(\pi, \pi_0, \pi_1) \\ &+ (\lambda_{1-j} - \lambda_j)\, \frac{2c(\lambda_0 + \lambda_1 + 2(c + r)) + r(\lambda_0 + \lambda_1 + r)}{2c(\lambda_{1-j} - \lambda_j + r) + r(\lambda_0 + \lambda_1 + r)} \big((1-2j)\, \pi - \pi_{1-j}\big) \end{aligned} \quad (3.56)$$

for every $i = 0, 1$, when $\lambda_j \le \lambda_{1-j}$ for some $j = 0, 1$. Let us then take $(\pi', \pi_0', \pi_1') \in [0,1]^3$ such that $G_i(\pi, \pi_0, \pi_1) = G_i(\pi', \pi_0', \pi_1')$ holds with $\pi_{1-j} \le \pi_{1-j}'$ in case $i = 0$ and $\pi_{1-j}' \le \pi_{1-j}$ in case $i = 1$, as well as $\pi' \le \pi$ in case $i = j$ and $\pi \le \pi'$ in case $i \ne j$. Hence, using the facts that (Π, Π^0, Π^1) is a time-homogeneous strong Markov process and $\zeta_i^* = \zeta_i^*(\pi, \pi_0, \pi_1)$ does not depend on (π', π_0', π_1'), taking into account the comparison results for solutions of stochastic differential equations in Veretennikov [43], we obtain:

$$\begin{aligned} (U_i^* - U_{1-i}^* - (1-2i)\, G_i)(\pi', \pi_0', \pi_1') &\le E_{\pi', \pi_0', \pi_1'} \int_0^{\zeta_i^* \wedge \zeta_{1-i}^*} e^{-rt}\, (1-2i)\, H_i(\Pi_t, \Pi_t^0, \Pi_t^1)\, dt \\ &\le E_{\pi, \pi_0, \pi_1} \int_0^{\zeta_i^* \wedge \zeta_{1-i}^*} e^{-rt}\, (1-2i)\, H_i(\Pi_t, \Pi_t^0, \Pi_t^1)\, dt \end{aligned} \quad (3.57)$$

holds for every $i = 0, 1$. By virtue of the inequality in (3.55) and the expression in (3.56), we may therefore conclude that $(\pi', \pi_0', \pi_1') \in C_i^*$, so that the functions:

$$a_*(\pi, \pi_{1-j}) + \frac{2c(1-2j)(\lambda_{1-j} - \lambda_j)\pi + (2c(\lambda_j - \lambda_{1-j} + r) + r(\lambda_0 + \lambda_1 + r))\pi_{1-j}}{2c(\lambda_{1-j} - \lambda_j + r) + r(\lambda_0 + \lambda_1 + r)} \quad (3.58)$$

and

$$b_*(\pi, \pi_{1-j}) + \frac{2c(1-2j)(\lambda_{1-j} - \lambda_j)\pi + (2c(\lambda_j - \lambda_{1-j} + r) + r(\lambda_0 + \lambda_1 + r))\pi_{1-j}}{2c(\lambda_{1-j} - \lambda_j + r) + r(\lambda_0 + \lambda_1 + r)} \quad (3.59)$$

are decreasing in π_{1-j} and increasing or decreasing in π on $[0,1]$ in case $j = 0$ or $j = 1$, respectively.

We are now in a position to formulate the main assertion of the paper, which follows from a straightforward combination of Lemmata 3.1 and 3.2 together with the arguments above.

Theorem 3.1 *Suppose that the assumptions of Lemmata 3.1 and 3.2 hold for admissible parameters of the model. Then, the value functions* $V_i^*(\pi, \pi_0, \pi_1)$, $i = 0, 1$, *in the coupled optimal* *stopping problem of (2.9) are given by* $V_i^*(\pi, \pi_0, \pi_1) = U_i^*(\pi, \pi_0, \pi_1) - (1-2i) F_i(\pi, \pi_0, \pi_1)$ *with* $F_i(\pi, \pi_0, \pi_1)$ *defined in (3.1). Here, the value functions* $U_i^*(\pi, \pi_0, \pi_1)$, $i = 0, 1$, *in (3.5) have the form of (3.20)-(3.21) with (3.22)-(3.23), and the continuous optimal stopping boundaries* $a_*(\pi, \pi_{1-j})$ *and* $b_*(\pi, \pi_{1-j})$ *in (3.11) are uniquely specified by the smooth-fit conditions of (3.19) and satisfy the properties proved above, when* $\lambda_j \le \lambda_{1-j}$ *holds for some* $j = 0, 1$. *Moreover, the boundaries* $a_*(\pi, \pi_{1-j})$ *and* $b_*(\pi, \pi_{1-j})$ *satisfy the inequalities in (3.46)-(3.47) and are such that the functions in (3.58)-(3.59) are decreasing in* π_{1-j}, *and either increasing or decreasing in* π *on* $[0, 1]$ *in case of either* $j = 0$ *or* $j = 1$, *respectively.*

Based on the result proved above, let us finally formulate the following explicit optimal sequential procedure for the Bayesian switching multiple disorder detection.

Corollary 3.1 *Suppose that the assumptions of Theorem 3.3 hold. Then, in the quickest switching multiple disorder detection problem of (2.8) for the observation process* X *from (2.1), the Bayesian risk function takes the form* $V^*(\pi) = V_0^*(\pi, 1-\pi, \pi)$, *for all* $\pi \in [0, 1]$, *and the optimal switching times* $(\tau_n^*)_{n \in \mathbb{N}}$ *have the form of (3.12). Moreover, the following quickest multiple disorder detection procedure is optimal for every* $k \in \mathbb{N}$:

(i) stop the observations at time τ_{2k-1}^* *from (3.12), that is, as soon as the process* Π^j *from (2.7) exits the region* $(a_*(\Pi, \Pi^{1-j}), 1]$, *conclude that the process* Θ *has switched from the state* Θ_0 *to* $1 - \Theta_0$, *when* $\lambda_j \le \lambda_{1-j}$ *holds for some* $j = 0, 1$, *and then, continue with step (ii);*

(ii) stop the observations at time τ_{2k}^* *from (3.12), that is, as soon as the process* Π^j *exits the region* $[0, b_*(\Pi, \Pi^{1-j}))$, *conclude that the process* Θ *has switched from the state* $1 - \Theta_0$ *to* Θ_0, *when* $\lambda_j \le \lambda_{1-j}$ *holds for some* $j = 0, 1$, *and then, continue with the step (i).*

4 Some analytic-form estimates.

In this section, we provide analytic-form estimates for the value functions of the coupled optimal stopping problems of (2.9) and (3.5), and thus, for the Bayesian risk function in (2.8) as well as for the optimal stopping boundaries from (3.11) and (3.12).

4.1 The change of variables.

In order to derive such estimates, we shall reduce the operator in (3.13) to the normal form, by means of the one-to-one correspondence transformation process proposed by A.N. Kolmogorov in [23] (see also [18]-[19]). For this, let us define the processes $Y = (Y_t)_{t \ge 0}$ and $Z = (Z_t)_{t \ge 0}$ by:

$$Y_t = \Pi_t^0 / (1 - \Pi_t) \equiv P_\pi(\Theta_0 = 0 \,|\, \mathcal{F}_t, \Theta_t = 0) \quad \text{and} \quad Z_t = \Pi_t^1 / \Pi_t \equiv P_\pi(\Theta_0 = 1 \,|\, \mathcal{F}_t, \Theta_t = 1) \tag{4.1}$$

for all $t \ge 0$. Then, by means of Itô's formula, we get that the processes Y and Z admit the representations:

$$dY_t = \lambda_1 \, \frac{(\Pi_t - \Pi_t^1)(1 - \Pi_t) - \Pi_t^0 \Pi_t}{(1 - \Pi_t)^2} \, dt = \lambda_1 \, \frac{\Pi_t}{1 - \Pi_t} \, (1 - Y_t - Z_t) \, dt \tag{4.2}$$

and

$$dZ_t = \lambda_0 \, \frac{(\Pi_t - \Pi_t^1)(1 - \Pi_t) - \Pi_t^0 \Pi_t}{\Pi_t^2} \, dt = \lambda_0 \, \frac{1 - \Pi_t}{\Pi_t} \, (1 - Y_t - Z_t) \, dt \tag{4.3}$$

with $Y_0 = Z_0 = 1$. It is seen from the equations in (4.2)-(4.3) that the processes Y and Z are of bounded variation on their state space $[0,1]$.

It follows from the expressions in (4.1) that there exists a one-to-one correspondence between the processes (Π, Π^0, Π^1) and (Π, Y, Z). Hence, the function $U_i^*(\pi, \pi_0, \pi_1)$ from (2.9) is equal to the one of the coupled optimal stopping problem:

$$W_i^*(\pi, y, z) = \inf_{\zeta_i} E_{\pi,y,z}\big[e^{-r\zeta_i}\left((1-2i)\,\widehat{G}_i(\Pi_{\zeta_i}, Y_{\zeta_i}, Z_{\zeta_i}) + W_{1-i}^*(\Pi_{\zeta_i}, Y_{\zeta_i}, Z_{\zeta_i})\right)\big] \tag{4.4}$$

where the infimum is taken over all stopping times ζ_i, for every $i = 0, 1$, and the function $\widehat{G}_i(\pi, y, z) = G_i(\pi, y(1-\pi), z\pi)$ admits the representation:

$$\widehat{G}_i(\pi, y, z) = A(y,z)\,\pi + \left(\frac{2c(\lambda_1 - \lambda_0 + r)}{r(\lambda_0 + \lambda_1 + r)} + 1\right) y + \frac{c(\lambda_0 - \lambda_1 - r)}{r(\lambda_0 + \lambda_1 + r)} - i \tag{4.5}$$

with

$$A(y,z) = \frac{2c(\lambda_1 - \lambda_0)}{r(\lambda_0 + \lambda_1 + r)} + \left(\frac{2c(\lambda_0 - \lambda_1 + r)}{r(\lambda_0 + \lambda_1 + r)} + 1\right) z - \left(\frac{2c(\lambda_1 - \lambda_0 + r)}{r(\lambda_0 + \lambda_1 + r)} + 1\right) y \tag{4.6}$$

for all $(\pi, y, z) \in [0,1]^3$. Here $P_{\pi,y,z}$ is a probability measure under which the diffusion process $(\Pi, Y, Z) = (\Pi_t, Y_t, Z_t)_{t \ge 0}$ starts at some $(\pi, y, z) \in [0,1]^3$ and solves the equations of (2.5) and (4.2)-(4.3). It thus follows from (3.11) that there exist functions $g_*(y,z)$ and $h_*(y,z)$ such that $0 < g_*(y,z) \lesseqgtr h_*(y,z) < 1$ when $A(y,z) \gtreqless 0$, and the optimal stopping times in the problem of (4.4) have the structure:

$$\zeta_0^* = \inf\big\{t \ge 0 \,\big|\, \Pi_t \lesseqgtr g_*(Y_t, Z_t) \text{ when } A(Y_t, Z_t) \gtreqless 0\big\} \tag{4.7}$$

and

$$\zeta_1^* = \inf\big\{t \ge 0 \,\big|\, \Pi_t \gtreqless h_*(Y_t, Z_t) \text{ when } A(Y_t, Z_t) \gtreqless 0\big\}. \tag{4.8}$$

In this case, the continuation regions from (3.9) take the form:

$$C_0^* = \big\{(\pi, y, z) \in [0,1]^3 \,\big|\, \pi \gtrless g_*(y,z) \text{ when } A(y,z) \gtreqless 0\big\} \tag{4.9}$$

and

$$C_1^* = \big\{(\pi, y, z) \in [0,1]^3 \,\big|\, \pi \lessgtr h_*(y,z) \text{ when } A(y,z) \gtreqless 0\big\} \tag{4.10}$$

so that the corresponding regions from (3.10) are given by:

$$D_0^* = \big\{(\pi, y, z) \in [0,1]^3 \,\big|\, \pi \lesseqgtr g_*(y,z) \text{ when } A(y,z) \gtreqless 0\big\} \tag{4.11}$$

and

$$D_1^* = \big\{(\pi, y, z) \in [0,1]^3 \,\big|\, \pi \gtreqless h_*(y,z) \text{ when } A(y,z) \gtreqless 0\big\} \tag{4.12}$$

respectively. Here, due to the monotonicity of the functions in (3.58)-(3.59), the boundaries $g_*(y,z)$ and $h_*(y,z)$ are uniquely determined from the equations $zg(y,z) = a_*(g(y,z), y(1-g(y,z)))$ and $zh(y,z) = b_*(h(y,z), y(1-h(y,z)))$ in case $\lambda_1 \le \lambda_0$, and $y(1-g(y,z)) = a_*(g(y,z), zg(y,z))$ and $y(1-h(y,z)) = b_*(h(y,z), zh(y,z))$ in case $\lambda_0 \le \lambda_1$, respectively.

4.2 The coupled free-boundary problem.

Standard arguments then show that the infinitesimal operator $\mathbb{L}_{(\Pi,Y,Z)}$ of the process (Π, Y, Z) from (2.5) and (4.2)-(4.3) has the structure:

$$\mathbb{L}_{(\Pi,Y,Z)} = (\lambda_0 - (\lambda_0 + \lambda_1)\,\pi)\,\partial_\pi + \frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2 \pi^2(1-\pi)^2\,\partial^2_{\pi\pi} \tag{4.13}$$

$$+\,\lambda_1\,\frac{\pi}{1-\pi}\,(1-y-z)\,\partial_y + \lambda_0\,\frac{1-\pi}{\pi}\,(1-y-z)\,\partial_z \tag{4.14}$$

for all $(\pi, y, z) \in (0,1)^3$. It can be shown by means of the same arguments as in the proof of Theorem 3.3 above that the value functions $U_i^*(\pi, y, z)$, $i = 0, 1$, from (4.4) and the boundaries $g_*(y, z)$ and $h_*(y, z)$ from (4.7)-(4.8) solve the free-boundary problem:

$$(\mathbb{L}_{(\Pi,Y,Z)}W_i - rW_i)(\pi, y, z) = 0 \quad \text{for} \quad (\pi, y, z) \in C_i \tag{4.15}$$

$$(W_0 - W_1 - \widehat{G}_0)(\pi, y, z)\big|_{\pi=g(y,z)\pm} = 0 \quad \text{when} \quad A(y,z) \gtrless 0 \tag{4.16}$$

$$(W_1 - W_0 + \widehat{G}_1)(\pi, y, z)\big|_{\pi=h(y,z)\mp} = 0 \quad \text{when} \quad A(y,z) \lesseqgtr 0 \tag{4.17}$$

$$(W_i - W_{1-i})(\pi, y, z) = (1-2i)\,\widehat{G}_i(\pi, y, z) \quad \text{for} \quad (\pi, y, z) \in D_i \tag{4.18}$$

$$(W_i - W_{1-i})(\pi, y, z) < (1-2i)\,\widehat{G}_i(\pi, y, z) \quad \text{for} \quad (\pi, y, z) \in C_i \tag{4.19}$$

$$(\mathbb{L}_{(\Pi,Y,Z)}W_i - rW_i)(\pi, y, z) > 0 \quad \text{for} \quad (\pi, y, z) \in D_i \tag{4.20}$$

with

$$(W_0 - W_1 - \widehat{G}_0)_\pi(\pi, y, z)\big|_{\pi=g(y,z)\pm} = 0 \quad \text{when} \quad A(y,z) \gtrless 0 \tag{4.21}$$

$$(W_1 - W_0 + \widehat{G}_1)_\pi(\pi, y, z)\big|_{\pi=h(y,z)\mp} = 0 \quad \text{when} \quad A(y,z) \lesseqgtr 0 \tag{4.22}$$

where the *instantaneous-stopping conditions* in (4.16)-(4.17) and the *smooth-fit conditions* (4.21)-(4.22) hold for all $(y, z) \in (0,1)^2$.

Since the solution of the free-boundary problem of (4.15)-(4.20) with (4.21)-(4.22) cannot be found in an explicit form, let us introduce the functions $\widehat{W}_i(\pi, y, z)$ and the boundaries $\widehat{g}(y, z)$ and $\widehat{h}(y, z)$ satisfying the boundary conditions of (4.16)-(4.19) and (4.21)-(4.22) and the expressions:

$$(\mathbb{L}_{(\Pi,Y,Z)}W_i - rW_i)(\pi, y, z) \tag{4.23}$$
$$= \lambda_1\,\frac{\pi}{1-\pi}\,(1-y-z)\,(W_i)_y(\pi, y, z) + \lambda_0\,\frac{1-\pi}{\pi}\,(1-y-z)\,(W_i)_z(\pi, y, z) \quad \text{for} \quad (\pi, y, z) \in C_i$$

$$(\mathbb{L}_{(\Pi,Y,Z)}W_i - rW_i)(\pi, y, z) \tag{4.24}$$
$$> \lambda_1\,\frac{\pi}{1-\pi}\,(1-y-z)\,(W_i)_y(\pi, y, z) + \lambda_0\,\frac{1-\pi}{\pi}\,(1-y-z)\,(W_i)_z(\pi, y, z) \quad \text{for} \quad (\pi, y, z) \in D_i$$

for $\widehat{C}_i$ and $\widehat{D}_i$ defined as in (4.9)-(4.12) with $\widehat{g}(y, z)$ and $\widehat{h}(y, z)$ instead of $g_*(y, z)$ and $h_*(y, z)$, respectively. Observe that the equalities in (4.23) and (4.18) directly imply that the inequality in (4.24) takes the form:

$$(1-2i)\,\widehat{H}_i(\pi, y, z) > 0 \quad \text{for} \quad (\pi, y, z) \in D_i \tag{4.25}$$

with

$$\widehat{H}_i(\pi, y, z) = c + ir + \lambda_0\,(1-\pi)\,(1-2y) + \lambda_1\,\pi\,(1-2z) - (2c+r)\,(y + (z-y)\,\pi) \tag{4.26}$$

for every $i = 0, 1$. We further look for functions which solve the resulting ordinary differential coupled free-boundary problem of (4.23)+(4.16)-(4.19)+(4.21)-(4.22)+(4.24) in which the variables y and z are parameters.

4.3 The existence of solution of the ordinary free-boundary problem.

The general solution of the second-order ordinary differential equation in (4.23) has the form:

$$W_i(\pi, y, z) = \sum_{j=0}^{1} K_{ij}(y, z)\, Q_j(\pi) \tag{4.27}$$

where $K_{ij}(y, z)$, $i, j = 0, 1$, are some continuously differentiable functions, and the functions $Q_j(\pi)$, $j = 0, 1$, are given by:

$$Q_j(\pi) = \sqrt{\pi(1-\pi)}\left(\frac{\pi}{1-\pi}\right)^{(\lambda_1-\lambda_0)/\rho} \exp\left(\frac{2\lambda_0 + (\lambda_1-\lambda_0)(j+\pi)}{\rho(j+(1-2j)\pi)}\right) \times S_j\!\left((-1)^{j+1}\varphi, \psi_0, \xi, \psi_1; \frac{\sqrt{\lambda_0}(1-\pi)+\sqrt{\lambda_1}\pi}{\sqrt{\lambda_0}(1-\pi)-\sqrt{\lambda_1}\pi}\right) \tag{4.28}$$

for all $\pi \in (0, 1)$ with

$$\rho = \left(\frac{\mu}{\sigma}\right)^2, \quad \varphi = \frac{8\sqrt{\lambda_0\lambda_1}}{\rho}, \quad \xi = \frac{32\sqrt{\lambda_0\lambda_1}(\lambda_0-\lambda_1)}{\rho^2} \tag{4.29}$$

and

$$\psi_j = \frac{(-1)^{j+1}}{\rho^2}\big(\rho^2 + 4(2r+\lambda_0+\lambda_1) + 4(\lambda_1-\lambda_0)^2 - 16(\lambda_0\lambda_1 + (-1)^j\rho\sqrt{\lambda_0\lambda_1})\big). \tag{4.30}$$

Here, the functions $S_j(\alpha, \beta, \gamma, \delta; x)$, $j = 0, 1$, are two positive fundamental solutions (i.e. nontrivial linearly independent particular solutions) of Heun's double confluent ordinary differential equation:

$$S''(x) + \frac{2x^5 - \alpha x^4 - 4x^3 + 2x + \alpha}{(x-1)^3(x+1)^3}\, S'(x) + \frac{\beta x^2 + (2\alpha+\gamma)x + \delta}{(x-1)^3(x+1)^3}\, S(x) = 0 \tag{4.31}$$

with the boundary conditions $S(0) = 1$ and $S'(0) = 0$. Note that the series expansion of the solution of the equation in (4.31) converges under all $-1 < x < 1$, and the appropriate analytic continuation can be obtained through the identity $S(\alpha, \beta, \gamma, \delta; x) = S(-\alpha, -\delta, -\gamma, -\beta; 1/x)$. The (irregular) singularities at -1 and 1 of the equation in (4.31) are of unit rank and can be transformed into that of a confluent hypergeometric equation (see, e.g. Decarreau et al. [12] and Ronveaux [33] for an extensive overview and further details). According to the results from Rogers and Williams [32; Chapter V, Section 50], we can specify the positive (strictly) convex functions $Q_j(\pi)$, $j=0,1$, as (strictly) decreasing and increasing on the interval $(0,1)$ and having singularities at 0 and 1, respectively.

Taking into account the fact that the function $\pi \mapsto W_0(\pi,y,z)$ should be bounded as $\pi \uparrow 1$ when $A(y,z)>0$ and as $\pi \downarrow 0$ when $A(y,z)<0$, while the function $\pi \mapsto W_1(\pi,y,z)$ should be bounded as $\pi \downarrow 0$ when $A(y,z)>0$ and as $\pi \uparrow 1$ when $A(y,z)<0$, we get that the solution in (4.27) should be of the form:

$$W_i(\pi,y,z) = \sum_{j=0}^{1} K_{ij}(y,z)\, Q_j(\pi)\, I\big((-1)^{i+j} A(y,z) > 0\big) \tag{4.32}$$

for $i=0,1$. Then, applying the instantaneous-stopping and smooth-fit conditions from (4.16)-(4.17) and (4.21)-(4.22) to the function in (4.32), we obtain that the equalities:

$$(W_0 - W_1 - \widehat{G}_0)(g(y,z),y,z) = (W_1 - W_0 + \widehat{G}_1)(h(y,z),y,z) = 0 \tag{4.33}$$

$$(W_0 - W_1 - \widehat{G}_0)_\pi(g(y,z),y,z) = (W_1 - W_0 + \widehat{G}_1)_\pi(h(y,z),y,z) = 0 \tag{4.34}$$

hold for some $0 < g(y,z) \lesseqgtr h(y,z) < 1$ fixed when $A(y,z) \gtreqless 0$, respectively. It thus follows that the functions:

$$W_0(\pi,y,z;g(y,z)) = \sum_{j=0}^{1} K_{0j}(y,z;g(y,z))\, Q_j(\pi)\, I\big((-1)^{j} A(y,z) > 0\big) \tag{4.35}$$

and

$$W_1(\pi,y,z;h(y,z)) = \sum_{j=0}^{1} K_{1j}(y,z;h(y,z))\, Q_j(\pi)\, I\big((-1)^{j} A(y,z) < 0\big) \tag{4.36}$$

provide a solution of the system in (4.23)+(4.16)-(4.19)+(4.21)-(4.22)+(4.24), for any $0 < g(y,z) \lesseqgtr h(y,z) < 1$ fixed when $A(y,z) \gtreqless 0$. Here, the functions $K_{0j}(y,z;g(y,z))$ and $K_{1j}(y,z;h(y,z))$, for every $j=0,1$, are determined as solutions of the linear system of (4.33)-(4.34), for all $(y,z) \in [0,1]^2$.

4.4 The uniqueness of solution of the ordinary free-boundary problem.

Using the standard comparison arguments for solutions of the second order ordinary differential equations in (4.23), we conclude that the resulting curves $\pi \mapsto W_0(\pi,y,z;g(y,z))$ and $\pi \mapsto W_1(\pi,y,z;h(y,z))$ from (4.35)-(4.36) do not intersect each other on the intervals $(g(y,z),1]$ and $[0,h(y,z))$, respectively, for different $0<g(y,z)<h(y,z)<1$ fixed when $A(y,z)>0$ and on the intervals $[0,g(y,z))$ and $(h(y,z),1]$, respectively, for different $0<h(y,z)<g(y,z)<1$ fixed when $A(y,z)<0$. We also observe by virtue of the properties of the functions $Q_j(\pi)$, $j=0,1$, in (4.28) that $W_0(\pi,y,z;g(y,z))$ and $W_1(\pi,y,z;h(y,z))$ are bounded and concave on $(g(y,z),1]$ and $[0,h(y,z))$ when $A(y,z)>0$ and on $[0,g(y,z))$ and $(h(y,z),1]$ when $A(y,z)<0$, respectively. Moreover, we see that $(W_0)_\pi(\pi,y,z;g(y,z)) \to \infty$ and $(W_1)_\pi(\pi,y,z;h(y,z)) \to -0$ as $\pi \downarrow 0$, as well as $(W_0)_\pi(\pi,y,z;g(y,z)) \to +0$ and $(W_1)_\pi(\pi,y,z;h(y,z)) \to -\infty$ as $\pi \uparrow 1$

when $A(y,z) > 0$, while $(W_0)_\pi(\pi,y,z;g(y,z)) \to -0$ and $(W_1)_\pi(\pi,y,z;h(y,z)) \to \infty$ as $\pi \downarrow 0$, as well as $(W_0)_\pi(\pi,y,z;g(y,z)) \to -\infty$ and $(W_1)_\pi(\pi,y,z;h(y,z)) \to +0$ as $\pi \uparrow 1$ when $A(y,z) < 0$, respectively. It thus follows from the structure of the gain functions $\widehat{G}_i(\pi,y,z)$, $i=0,1$, in (4.5) that system in (4.33)-(4.34) with (4.32) admits a unique solution $\widehat{g}(y,z)$ and $\widehat{h}(y,z)$.

On the other hand, it follows from the structure of the regions from (4.11)-(4.12) that the inequalities in (4.25) with (4.26) are equivalent to:

$$c + \lambda_0\,(1-2y) - (2c+r)\,y \gtreqless \widehat{g}(y,z)\,(\lambda_0\,(1-2y) - \lambda_1\,(1-2z) + (2c+r)\,(z-y)) \tag{4.37}$$

and

$$r + c + \lambda_0\,(1-2y) - (2c+r)\,y \lesseqgtr \widehat{h}(y,z)\,(\lambda_0\,(1-2y) - \lambda_1\,(1-2z) + (2c+r)\,(z-y)) \tag{4.38}$$

when $A(y,z) \gtrless 0$ for all $(y,z) \in (0,1)^2$, respectively.

Summarising these facts above and taking into account the arguments of Subsection 3.3, let us formulate the following assertion.

Corollary 4.1 *Assume that the unique solution $\widehat{g}(y,z)$ and $\widehat{h}(y,z)$ of the system in (4.33)-(4.34) with (4.35)-(4.36) satisfies the inequalities in (4.37)-(4.38) when $A(y,z) \gtrless 0$, respectively. Then, the functions:*

$$\widehat{W}_0(\pi,y,z) = \begin{cases} W_0(\pi,y,z;\widehat{g}(y,z)), & \text{for } \pi \gtrless \widehat{g}(y,z) \text{ if } A(y,z) \gtrless 0 \\ \widehat{G}_0(\pi,y,z) + \widehat{W}_1(\pi,y,z), & \text{for } \pi \lesseqgtr \widehat{g}(y,z) \text{ if } A(y,z) \gtrless 0 \end{cases} \tag{4.39}$$

and

$$\widehat{W}_1(\pi,y,z) = \begin{cases} W_1(\pi,y,z;\widehat{h}(y,z)), & \text{for } \pi \lessgtr \widehat{h}(y,z) \text{ if } A(y,z) \gtrless 0 \\ -\widehat{G}_1(\pi,y,z) + \widehat{W}_0(\pi,y,z), & \text{for } \pi \gtrless \widehat{h}(y,z) \text{ if } A(y,z) \gtrless 0 \end{cases} \tag{4.40}$$

where the functions $W_0(\pi,y,z;g(y,z))$ and $W_1(\pi,y,z;h(y,z))$ are given by (4.35)-(4.36), coincide with the value functions of the coupled optimal stopping problem:

$$\widehat{W}_i(\pi,y,z) = \inf_{\zeta_i} E_{\pi,y,z}\bigg[e^{-r\zeta_i}\left((1-2i)\,\widehat{G}_i(\Pi_{\zeta_i},Y_{\zeta_i},Z_{\zeta_i}) + \widehat{W}_{1-i}(\Pi_{\zeta_i},Y_{\zeta_i},Z_{\zeta_i})\right) \tag{4.41}$$
$$- \int_0^{\zeta_i} e^{-rt}\,(1-Y_t-Z_t)\left(\frac{\lambda_1\Pi_t}{1-\Pi_t}\,(\widehat{W}_i)_y(\Pi_t,Y_t,Z_t) + \frac{\lambda_0(1-\Pi_t)}{\Pi_t}\,(\widehat{W}_i)_z(\Pi_t,Y_t,Z_t)\right)dt\bigg]$$

with $\widehat{G}_i(\pi,y,z)$ given by (4.5), and the set $\widehat{C}_i$ is defined as C_i^* in (4.9)-(4.10) with $\widehat{g}(y,z)$ and $\widehat{h}(y,z)$ instead of $g_*(y,z)$ and $h_*(y,z)$, respectively, for $i=0,1$. Moreover, the functions $\widehat{g}(y,z)$ and $\widehat{h}(y,z)$ provide hitting boundaries for the stopping times:

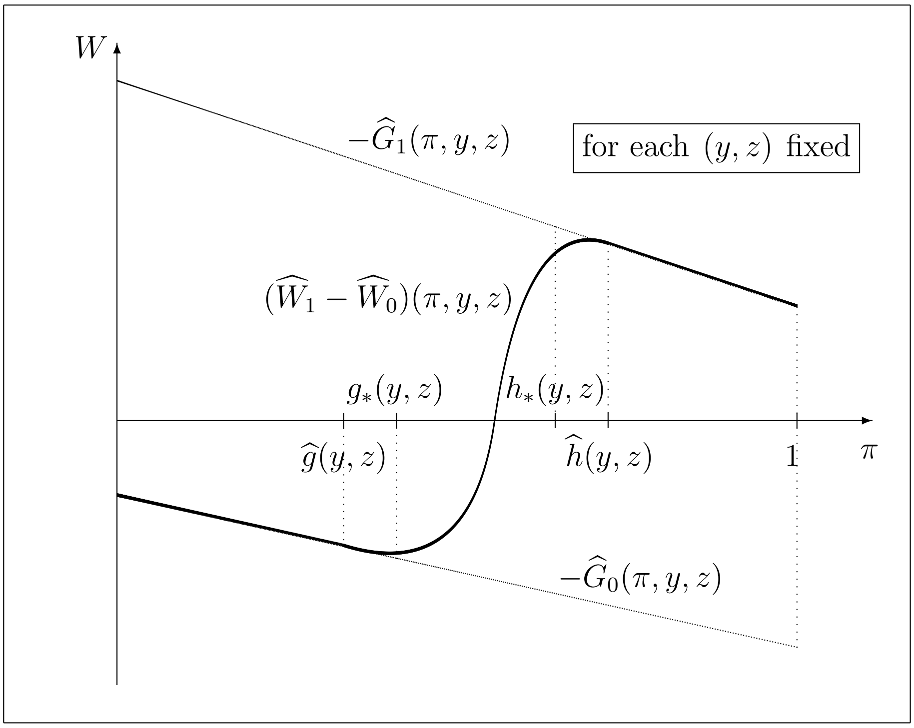
$$\widehat{\zeta}_0 = \inf\left\{t \geq 0 \,\middle|\, \Pi_t \lesseqgtr \widehat{g}(Y_t,Z_t) \text{ when } A(Y_t,Z_t) \gtrless 0\right\} \tag{4.42}$$

and

$$\widehat{\zeta}_1 = \inf\left\{t \geq 0 \,\middle|\, \Pi_t \gtreqless \widehat{h}(Y_t,Z_t) \text{ when } A(Y_t,Z_t) \gtrless 0\right\} \tag{4.43}$$

which turn out to be optimal in (4.41), whenever the integral there is of finite expectation, for any $(y,z) \in [0,1]^2$ fixed.

The proof of this assertion follows the arguments of the proof of Theorem 3.1 and is based on the verification Lemma 3.2.



Remark 4.1 Note that the functions $\widehat{W}_i(\pi, y, z)$ in (4.41) provide lower (upper) estimates for the initial value functions $W_i^*(\pi, y, z)$ from (4.4) whenever the both partial derivatives $(\widehat{W}_i)_y(\pi, y, z)$ and $(\widehat{W}_i)_z(\pi, y, z)$ are negative (positive), for $i = 0, 1$, and all $(\pi, y, z) \in [0, 1]^3$. By virtue of the structure of the value functions, this fact implies that the boundaries $\widehat{g}(y, z)$ and $\widehat{h}(y, z)$ in (4.42)-(4.43) provide lower (upper) and upper (lower) estimates for the initial optimal switching boundaries $g_*(y, z)$ and $h_*(y, z)$ in (4.7)-(4.8) whenever $A(y, z) > 0$, and upper (lower) and lower (upper) estimates whenever $A(y, z) < 0$, for any $(y, z) \in [0, 1]^2$ fixed.

The figure above represents a computer drawing of the function $(\widehat{W}_1 - \widehat{W}_0)(\pi, y, z)$ with the optimal switching boundaries $\widehat{g}(y, z)$ and $\widehat{h}(y, z)$ satisfying the conditions of (4.37)-(4.38), in the case in which the both partial derivatives $(\widehat{W}_i)_y(\pi, y, z)$ and $(\widehat{W}_i)_z(\pi, y, z)$ are negative, for $i = 0, 1$, and $A(y, z) > 0$ holds, for any $(y, z) \in [0, 1]^2$ fixed. The same picture also corresponds to the case in which the parameters of the model are admissible, so that the inequalities in (3.46)-(3.47) hold for the original optimal switching boundaries $a_*(\pi, \pi_{1-j})$ and $b_*(\pi, \pi_{1-j})$, for any $(\pi, \pi_{1-j}) \in [0, 1]^2$ fixed, when $\lambda_j \leq \lambda_{1-j}$ holds for some $j = 0, 1$.

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