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Homomorpisms from Functional Equations:
The Goldie Equation
by
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To Nick Bingham on the occasion of his 70th birthday

Abstract. The theory of regular variation, in its Karamata and Bojanić-Karamata/de Haan forms, is long established and makes essential use of the Cauchy functional equation. Both forms are subsumed within the recent theory of Beurling regular variation, developed elsewhere. Various generalizations of the Cauchy equation, including the Gołąb-Schinzel functional equation (GS) and Goldie's equation (GBE) below, are prominent there. Here we unify their treatment by 'algebraicization': extensive use of group structures introduced by Popa and Javor in the 1960s turn all the various (known) solutions into homomorphisms, in fact identifying them 'en passant', and show that (GS) is present everywhere, even if in a thick disguise.

Key words: Beurling regular variation, Beurling's equation, self-neglecting functions, Cauchy equation, Gołąb-Schinzel equation, circle group, Popa group.

Mathematics Subject Classification (2000): 26A03; 33B99, 39B22, 34D05; 39A20

1 Introduction

We are concerned with an 'algebraic conversion' of two functional equations, so that their solution functions may be viewed as homomorphisms between appropriate group structures on $\mathbb{R}$; see §2.2 for the motivation. Both are known in the functional equations literature in connection originally with problems arising in utility theory and go back to Lundberg [Lun] and Aczél [Acz] (see below for more recent studies); there, however, they were studied in order to classify their solutions, cf. [AczD]. Our purposes here, which are algebraic, are different and arise for us in the context of the classical Karamata theory of Regular Variation (briefly, RV – see [BinGT], henceforth BGT, the standard text, and [BinO3] for updates) and of the recently developed theory of Beurling RV, as in [BinO4,6], which includes the Karamata theory. The nearest to our theme of homomorphy is the paper of Kahlig and Schwaiger [KahS], which studies a sequence of deformations taking the equation (GS) below in the limit to the classical Cauchy functional equation (CFE) of additivity (and so of homomorphy).

The first functional equation, arising in Beurling RV, is the *generalized Goldie-Beurling equation* on $\mathbb{R}_+ := [0,\infty)$:

$$K(x + y\eta(x)) - K(y) = \psi(y)K(x) \qquad (x, y \in \mathbb{R}_+) \qquad (GBE)$$

(in the two unknowns K and ψ), where for some $\rho \in \mathbb{R}_+$

$$\eta(x) = \eta_\rho(x) \equiv 1 + \rho x \qquad (x \in \mathbb{R}),$$

the classical Karamata case being $\rho = 0$ and the general Beurling case $\rho > 0$. In the RV literature this equation appears in [BinG], in work inspired by Bojanić and Karamata [BojK], and is due principally to Goldie ('Goldie's equation'). In both these cases the solution K describes a function derived from the limiting behaviour of some *regularly varying* function (see §2.2 below). For this reason one may expect (by analogy with the various derivatives encountered in functional analysis) that K should be an analogue of a linear function. Indeed, for $\rho = 0$ and specializing to $\psi \equiv 1$, the earliest classical case, K is *additive*; hence a search for homomorphism, when $\rho > 0$, dictates our agenda here as an algebraic complement and companion piece to the analytic argument of [BinO5,6].

We denote by GS the family of functions η above, which satisfy the *Gołąb-Schinzel functional equation*

$$\eta(x + y\eta(x)) = \eta(x)\eta(y) \qquad (x, y \in \mathbb{R}), \tag{GS}$$

although its 'conditional form' arising from the restriction $x, y \geq 0$ might be regarded as more appropriate. (Actually, the more correct and natural domain is $\{x : \eta(x) > 0\}$, as in [BinO6], but see the comment below on *involutory extension*, [BrzM] and also §3.) For their significance to RV see the recent [BinO6] and for their significance elsewhere, especially to the theory of functional equations, [Brz5].

The second functional equation of interest substitutes for the K on the right of (GBE) a third unknown function κ, yielding a natural 'Pexiderized' generalization[1]

$$K(x + y\eta(x)) - K(y) = \psi(y)\kappa(x) \qquad (x, y \in \mathbb{R}_+), \tag{GBE-P}$$

considered also in [ChuT]. Passage to this more general format is motivated by a desire to include a further equation of Goldie (see [BinO5,(GFE)]), and also the equation (GS) – as the case $K = \psi = \eta$ and $\kappa = \eta - 1$ – which turns out to be highly thematic (see [Ost2], and Theorem $1'$). Note that in this specialization the corresponding derivatives are identical: $K' = \kappa'$; cf. §5.

2 Popa circle groups and asymptotic analysis

We review in §2.1 algebraic background relevant to our study of the functional equations of §1, and in §2.2 the asymptotic analysis which leads to these functional equations and associated functional inequalities.

[1] As in Pexider's equation : $f(xy) = g(x) + h(y)$ (in the three unknowns f, g, h) and its generalizations – cf. [Brz1, 3], and the recent [Jab1].

2.1 Circle groups

Recall that any ring R equipped with its *circle product* $x \circ y := x + y + xy$ (see [Jac2, II.3], [Jac3]) is a monoid ([Coh1, 3.1]), the *circle monoid*, with neutral element 0. (This format is preferable below to the alternative: $x + y - xy$, isomorphic under the negation $x \mapsto -x$.) When $x \circ y = y \circ x = 0$, the elements x, y are *quasi-inverses* of each other in the ring, and the *quasi-units* (those having quasi-inverses) form the *circle group* of R. See e.g. [ColE] for recent advances on circle groups and historical background. There is an intimate connection with the *Jacobson radical* of a ring (see [Coh2, 5.4], or Jacobson [Jac1] in 1945), characterized as the maximal ideal of quasi-units after Perlis' introduction of the circle operation in 1942. The corresponding notion in a Banach algebra is that of (left and right) *adverses*, similarly defined – see [Loo, 20C, 21C]. For the connection of adverses and the Jacobson radical with the automatic continuity of homomorphisms, thematic here, see [Dal, Prop. 3.1] (specific to characters), and [Dal, §4] (more general).

The *operation*

$$x \circ_\eta y := x + y\eta(x),$$

with $\eta : \mathbb{R} \to \mathbb{R}$ arbitrary, was introduced in 1965 for the study of equation (GS) by Popa [Pop], and later Javor [Jav] (in the broader context of $\eta : \mathbb{E} \to \mathbb{F}$, with $\mathbb{E}$ a vector space over a commutative field $\mathbb{F}$), who observed that this equation is equivalent to the operation $\circ_\eta$ being associative on $\mathbb{R}$, and that $\circ_\eta$ confers a group structure on $\mathbb{G}_\eta := \{g : \eta(g) \neq 0\}$ – see [Pop, Prop. 2], [Jav, Lemma 1.2]. Below we term this a *Popa circle group*, or *Popa group* for short (see §2,3), as the case $\eta_1(x) = 1 + x$ (i.e. with $\rho = 1$ above, so a 'shift') yields precisely the circle group of the ring $\mathbb{R}$.

As $\circ_\eta$ turns η into a homomorphism from $\mathbb{G}_\eta$ to $(\mathbb{R}\backslash\{0\}, \times)$:

$$\eta(x \circ_\eta y) = \eta(x)\eta(y) \qquad (x, y \in \mathbb{G}_\eta)$$

– and of $\mathbb{G}_\eta^+ := \{g : \eta(g) > 0\}$ to $(\mathbb{R}_+\backslash\{0\}, \times)$ – given the group-theoretic framework of RV which leads to the equations (GBE) and $(GBE\text{-}P)$, it is natural to seek further group structures in order to algebraicize (GBE) as a property of K that expresses homomorphism between Popa groups:

$$K(x \circ_\eta y) = K(y) \circ_\sigma K(x) \text{ for some } \sigma \in GS, \qquad (CBE)$$

with $\sigma(K(y)) \equiv \psi(y)$, this to be termed the *Cauchy-Beurling equation* (CBE).

Given its origin in RV, (CBE) quite naturally calls for the domain variables to range over $\mathbb{R}_+$; this domain is a semigroup rather than a subgroup under $\circ_\eta$, as it omits the interval $(-1/\rho, 0) \subseteq \mathbb{G}_\eta^+$. The missing interval, however, comprises the $\circ_\eta$-inverses $x_\eta^{-1} := -x/\eta(x)$ for $x \in \mathbb{R}_+$, so throughout the paper we persistently refer to 'homomorphisms' justifiably so, if only because of the implicit *involutory extension* of the domain of K to $\mathbb{G}_\eta^+$ obtained by taking $K(x_\eta^{-1}) := K(x)_\sigma^{-1}$ for $x_\eta^{-1} := -x/\eta(x) \in (-1/\rho, 0)$. Here y_σ^{-1} analogously denotes the inverse in the range group $\circ_\sigma$.

The case $\rho = 0$ of (GBE), rewritten (with x, y interchanged) as the difference equation

$$\Delta_y K(x) - K(y)\psi(x) = 0, \qquad \Delta_y K(x) := K(x+y) - K(x),$$

already suggests that $K(y)$ should induce some form of 'shear' or shift. This difference theme is exploited in §5 on flows, and linked with integration.

Theorem 1 in §3 gives necessary and sufficent conditions for (GBE) to be algebraicized (as above), yielding 'en passant' the form of such a K directly from classical results concerning (CFE). Likewise Theorem 1′ in §4 gives necessary and sufficent conditions for GBE-P to be algebraicized; this builds on the technique of Theorem 1, and is similar but more involved. Again this yields en passant the form of such a K directly from classical results concerning (CFE).

Theorem 2 in §5 reduces $(GBE$-$P)$ more directly to the context of Theorem 1 for a *differentiable* auxiliary ψ. (The differentiability assumption is again motivated by regular variation.) Interpreting $\circ_\eta$ as a group action, or flow, the underlying homomorphy is now expressed not by K but by the *relative flow-velocity* $f(x) := \eta(x)/\psi(x)$: under mild regularity assumptions, if K solves $(GBE$-$P)$, then f satisfies

$$f(x \circ_\eta y) = f(x)f(y) \qquad (x, y \in \mathbb{R}_+).$$

There is a converse for $\psi := \eta/f$ – see Prop. D in §5.

Our quest for algebraicization links with results not only of Aczél but also of Chudziak [Chu1], who in 2006 considered the problem of identifying pairs (f, g) satisfying the functional equation

$$f(x + yg(x)) = f(x) \circ f(y) \qquad (ChE)$$

for $f : \mathbb{R} \to (S, \circ)$ with $(S, \circ)$ a semigroup, and $g : \mathbb{R} \to \mathbb{R}$ continuous.

We note three related recent papers: [Chu2], where $\mathbb{R}$ is replaced by a vector space over the field of real or complex numbers; [ChuK], where f, g are both assumed continuous and $(S, \circ)$ is the group of multiplicative reals; [Jab2], where the functions f, g are assumed only to be bounded above locally.

2.2 Connection with regular variation

The functions in GS have their origin for RV in the asymptotic analysis of *self-equivarying functions* φ, briefly $\varphi \in SE$ [Ost2], which for some function $\eta > 0$ satisfy

$$\varphi(x + t\varphi(x))/\varphi(x) \to \eta(t) \qquad (x \to \infty,\ \forall t \in \mathbb{R}_+), \qquad (SE)$$

locally uniformly in t. For $\eta \equiv 1$, these specialize to the *self-neglecting* functions of Beurling (BGT 2.3.1, [Kor, IV.11]; cf. [BinO4]). For $\varphi \in SE$ the limit $\eta = \eta^\varphi$ is necessarily in GS [Ost2]. Only (CFE) visibly identifies its solution K as a *homomorphism* – of the additive group $(\mathbb{R}, +)$ – whereas homomorphy is a central feature in the recent topological development of the theory of regular variation [BinO1,2], [Ost1]. The role of homomorphy is new in this context, and is one of our principal contributions here.

At its simplest, a functional equation as above arises when taking limits

$$K_F(t) := \lim_{x\to\infty}[F(x + t\varphi(x)) - F(x)], \tag{BK}$$

for $\varphi \in SE$; then, with η the associated limit as in (SE) above, for s, t ranging over the set $\mathbb{A}$ on which the limit function K_F (*Beurling kernel*) exists as a locally uniform limit,

$$K_F(s + t) = K_F(s/\eta(t)) + K_F(t) : \qquad K_F(t + s\eta(t)) = K_F(s) + K_F(t).$$

As we shall see, both $\mathbb{A}$ and $K_F(\mathbb{A})$ carry group structures under which K_F is a homomorphism. Thus, even in the classical context, (GS) plays a significant role albeit disguised and previously unnoticed, despite its finger-print, namely the terms +1 or -1, appearing in the formulas for K_F (cf. Th. 1(iv) below). See [BinO2] for a deeper analysis of the connection between asymptotics of the form (BK) in a general topological setting involving group homomorphisms, and [BinO6] and [Ost3] for the broader context here.

Previously, in [BinO5], the equations $(GBE\text{-}P)$ above were analyzed using only Riemann sums and associated Riemann integrals, introduced there as a means of extending Goldie's initial approach (via geometric series). Below we offer an approach to all of the above equations that is new to the regular variation literature, and partly familiar, albeit in a different setting, in the GS-literature of 'addition formulae' – see [Brz3, 4, 6] and [Mur] (this goes back to Aczél and Gołąb [AczG]): here we intertwine Popa groups and integration.

Corresponding to a less restrictive asymptotic analysis (BGT Ch. 3), the functional equations above give way to functional inequalities. For instance,

$$F(x + y) \leq e^y F(x) + F(y), \tag{GFI}$$

the *Goldie functional inequality* (see [BinO5] for background and references; cf. end of §3) becomes group-subadditivity:

$$G(x + y) \leq G(x) \circ_k G(y).$$

Our analysis lends new clarification, via the language of homomorphisms, to the 'classical relation' in RV, connecting K and the auxiliary function ψ, which says that $K = c(\psi - 1)$ and $\psi \equiv e^{\cdot}$ (cf. [BGT Lemma 3.2.1], [BinO5, Th. 1]); in particular, we point below to the implicit role of GS. Also, we explain and extend the result of [BinO5, Th. 9] that the solution (on $\mathbb{R}_+$) in K, subject to $K(0) = 0$, assuming positivity of κ (i.e. to the right of 0), and continuity and positivity of ψ, satisfies for some $c \geq 0$

$$K(x) = c \cdot \tau_f(x), \text{ for } \tau_f(x) := \int_0^x \mathrm{d}u/f(u), \text{ with } f := \eta/\psi.$$

For an interpretation of τ_f, inspired by Beck [Bec], as the *occupation time measure* (of $[0, x]$) of the continuous f-flow: $dx/dt = f(x)$, see [BinO6] (and [BinO4]).

3 Algebraicization of Goldie's equation

3.1 Preliminaries

We return to Popa's contribution [Pop], recalling again from Javor [Jav] that $\circ_\eta$ is associative iff η satisfies the Gołąb-Schinzel equation (GS) above. Then for $\eta \neq 0$ $(\mathbb{G}_\eta, \circ_\eta)$ is a group ([Pop, Prop. 2], [Jav, Lemma 1.2]), and (GS) asserts that η is a homomorphism from $\mathbb{G}_\eta$ to $(\mathbb{R}^*, \cdot) := (\mathbb{R}\backslash\{0\}, \times)$:

$$\eta(x \circ_\eta y) = \eta(x)\eta(y).$$

If η is injective on $\mathbb{G}_\eta$, then $\circ_\eta$ is *commutative*, as (GS) is symmetric on the right-hand side. Continuous solutions of (GS), positive on $\mathbb{R}_+$, are given by $\eta_\rho(x)$ as above (see e.g. [Brz5] or the more recent [BinO5]). Whenever context permits, if $\eta \equiv \eta_\rho$, write the group operation and the Popa group as

$$a \circ_\rho b, \qquad (\mathbb{G}_\rho, \circ_\rho);$$

here $\mathbb{G}_\rho = \mathbb{R}\backslash\{-\rho^{-1}\}$ and $\mathbb{G}_0 = \mathbb{R}$; we are also concerned with the *subgroup* $\mathbb{G}_\rho^+ = \{x : x > -\rho^{-1}\}$.

As $(x \circ_\rho y)/\rho \to xy$ as $\rho \to \infty$, write also $\mathbb{G}_\infty := \mathbb{R}\backslash\{0\} = \mathbb{R}^*$, and $\circ_\infty \equiv \cdot$ (multiplication); then $\mathbb{G}_\rho$ takes in the additive reals at one end ($\rho = 0$), and the multiplicative reals at the other; indeed

$$a \circ_0 b := a + b.$$

For the intermediate values of $\rho \in (0, \infty)$, $\eta_\rho : \mathbb{G}_\rho \to \mathbb{R}^*$ is an isomorphism, as

$$\eta_\rho(x \circ_\rho y) = \eta_\rho(x)\eta_\rho(y).$$

Rescaling its domain, $\mathbb{G}_\rho$ is typified by the case $\rho = 1$, where

$$a \circ_1 b = a + b + ab = (1 + a)(1 + b) - 1 : \qquad (\mathbb{G}_1, \circ_1) = (\mathbb{R}^*, \cdot) - 1$$

(so the circle group of $\mathbb{R}$) and the isomorphism η_1 is a translation (cf. [Pop, §3]), and so $\mathbb{G}_1$ and $\mathbb{R}^*$ are conjugates.

Before considering homomorphisms between the groups above we formulate a result that has two useful variants, relying on *commutativity* or *associativity*, whence the subscripts. Below *positive* means positive on $\mathbb{R}_+$.

Lemma 1_{com}. *If (CBE) holds for some injective K, σ with $\circ_\sigma$ commutative, and $\eta : \mathbb{R}_+ \to \mathbb{R}$ – then $\eta(u) \equiv 1 + \rho u$, for some constant ρ.*

Proof. Here $K(u + v\eta(u)) = K(u) \circ_\sigma K(v) = K(v) \circ_\sigma K(u) = K(v + u\eta(v))$, as $\circ_\sigma$ is commutative. By injectivity, for all $u, v \geq 0$

$$u + v\eta(u) = v + u\eta(v) : \quad u(1 - \eta(v)) = v(1 - \eta(u)),$$

so $(\eta(u)-1)/u \equiv \rho = \text{const.}$ for $u>0$; taking $v=1$, $\eta(u) \equiv 1+\rho u$ for all $u \geq 0$. $\square$

Lemma 1_{assoc}. *If (CBE) holds for some injective K, σ with $\circ_\sigma$ associative, and positive continuous $\eta : \mathbb{R} \to \mathbb{R}$ – then $\eta(u) = 1+\rho u$ $(u \geq 0)$, for some constant ρ.*

Proof. This follows e.g. from Javor's observation above connecting associativity with (GS) ([Jav, p. 235]). Here $K(u \circ_\eta (v \circ_\eta w)) = K(u) \circ_\sigma K(v) \circ_\sigma K(w) = K((u \circ_\eta v) \circ_\eta w)$, so from injectivity:

$$u \circ_\eta (v \circ_\eta w) = (u \circ_\eta v) \circ_\eta w,$$

i.e. $\circ_\eta$ is associative, so satisfies (GS). By results in [Brz2] and [BrzM] (cf.[Ost2, §6]0, positivity and continuity imply $\eta \in GS$. $\square$

For $\circ_\eta = \circ_0$ and $\circ_\sigma = \circ_\infty$, the equation (CBE) reduces to the exponential format of (CFE) ([Kuc, §13.1]; cf. [Jab1]). The critical case for Beurling regular variation is for $\rho \in (0,\infty)$, with positive continuous solutions described as follows. In the table below the four corner formulas correspond to classical variants of (CFE).

Proposition A (cf. [Chu1]). *For $\circ_\eta = \circ_r, \circ_\sigma = \circ_s$, and f Baire/measurable satisfying (CBE), there is $\gamma \in \mathbb{R}$ so that $f(t)$ is given for $t \geq 0$ by:*

Popa parameter	$s=0$	$s \in (0,\infty)$	$s=\infty$
$r=0$	γt	$(e^{\gamma t}-1)/s$	$e^{\gamma t}$
$r \in (0,\infty)$	$\gamma \log(1+rt)$	$[(1+rt)^\gamma - 1]/s$	$(1+rt)^\gamma$
$r=\infty$	$\gamma \log t$	$(t^\gamma - 1)/s$	t^γ

Proof. Each case reduces to (CFE) on $\mathbb{R}_+$, or a classical variant by an appropriate shift and rescaling. For instance, given f, for $r,s>0$ set

$$F(t) := 1 + sf((t-1)/r): \qquad f(\tau) = (F(1+r\tau)-1)/s.$$

Then with $u = 1+rx, v = 1+ry$, as $(uv-1)/r = x \circ_r y$,

$$F(uv) = 1 + sf(x \circ_r y) = 1 + sf(x) + sf(y) + s^2 f(x)f(y) = F(u)F(v),$$

for $u,v \geq 0$. So, as F is Baire/measurable (see again [Kuc, §13]), $F(t) = t^\gamma$ and so $f(t) = [(1+rt)^\gamma - 1]/s$. The remaining cases are similar. $\square$

3.2 Main result for (GBE)

Theorem 1 is our main result, relating solubility of (GBE) and the existence of a homomorphism, as in condition (iii). Here condition (ii) identifies the connection $K(u) \equiv (\psi(u)-1)/s$, which is no surprise in view of [BojK, (2.2)] and BGT Lemma 3.2.1, and the recent [BinO5, Th. 1] – cf. §2. This, however, is the nub, as $\psi(u) \equiv 1 + sK(u) = \eta_s(K(u))$.

Note that (iv) covers the classical Cauchy case, provided that for $\gamma = 0$ we interpret both $c(e^{\gamma x}-1)/\gamma$ and $c[(1+\rho x)^\gamma - 1]/\rho\gamma$ via 'l'Hospital's Rule' as cx.

Convention. Below and elsewhere a function is *non-trivial* if it not identically zero and not identically 1; it *positive* if it is positive on $(0,\infty)$.

Theorem 1. *For $\eta \in GS$ in the setting above, (GBE) holds for positive ψ, K with K non-trivial iff*
(i) *K is injective;*
(ii) *$\sigma =: \psi K^{-1} \in GS$, equivalently, either $\psi \equiv 1$, or for some $s > 0$*

$$K(u) \equiv (\psi(u)-1)/s \text{ and } \psi(0) = 1, \text{ so } K(0) = 0;$$

(iii)

$$K(x \circ_\eta y) = K(x) \circ_\sigma K(y). \tag{Hom-1}$$

Then
(iv) *for some constants c, γ and all $x \geq 0$*

$$K(x) \equiv c \cdot [(1+\rho x)^\gamma - 1]/\rho\gamma, \text{ or } \quad K(t) \equiv \gamma \log(1+\rho t) \quad (\rho_\eta > 0),$$
$$\text{or} \quad K(x) \equiv c \cdot (e^{\gamma x} - 1)/\gamma \quad (\rho_\eta = 0).$$

Proof. Consider any non-zero K; this is strictly monotone and so injective, as

$$K(x+y) - K(y) = K(x)\psi(y/\eta(x)) \qquad (x, y \in \mathbb{G}_\eta^+),$$

and so continuous, by [BinO5, Th. 9, or Lemma]. So ψ is continuous, since

$$\psi(y) \equiv [K(\xi \circ_\eta y) - K(y)]/K(\xi),$$

for any ξ with $K(\xi) \neq 0$. For convenience, write $k := K^{-1}$ and $\sigma(t) := \psi(k(t))$, i.e. a composition so continuous. Then

$$K(y) \circ_\sigma K(x) = K(y) + \psi(k(K(y))K(x) = K(x)\psi(y) + K(y), \tag{*}$$

so with $u = K(x), v = K(y)$, (GBE) becomes

$$K(k(u) \circ_\eta k(v)) = v + u\psi(k(v)) = v \circ_\sigma u : \qquad k(u \circ_\sigma v) = k(u) \circ_\eta k(v),$$

as $\circ_\eta$ is commutative ($\eta \in GS$). So (Hom-1) follows from (*). Lemma 1_{com} now applies to k, as $\circ_\eta$ is commutative. So $\sigma \in GS$ (as σ is positive and continuous). So for some $s, \rho \geq 0$

$$\eta(t) \equiv 1 + \rho t \text{ and } \sigma(t) \equiv 1 + st \qquad (t \geq 0).$$

That is, $\psi(K^{-1}(t)) \equiv \sigma(t) \equiv 1 + st$, so $\psi(x) = 1 + sK(x)$, on substituting $t = K(x)$. So if $s > 0$

$$K(x) = (\psi(x) - 1)/s \qquad (x \geq 0).$$

If $s = 0$, then $\psi(t) \equiv 1$. In any case $\psi(0) = 1$, since setting $y = 0$ in (GBE) gives $(1 - \psi(0))K(x) \equiv K(0) = 0$, but K is injective, so non-trivial. Substituting into (GBE) yields (as in [BinO5, Th. 1] for the case $\circ_\eta = +$)

$$\psi(x \circ_\eta y) = \psi(y)(\psi(x) - 1) + \psi(y) = \psi(x)\psi(y),$$

so $\psi : \mathbb{G}_\rho \to \mathbb{G}_\infty$ is a continuous homomorphism, and Prop. A applies. If $\rho = \rho_\eta = 0$, then $\psi(t) \equiv 1$ or $\psi(t) \equiv e^{\gamma t}$ with $\gamma \neq 0$, and for $c = \gamma/s$

$$K(t) \equiv c(e^{\gamma t} - 1)/\gamma, \qquad (s > 0), \text{ or } \qquad K(t) \equiv \gamma \log(1 + \rho t) \qquad (s = 0).$$

Otherwise, $\psi \equiv (1 + \rho x)^\gamma$ with $\gamma \neq 0$, and then for $c = \rho\gamma/a$

$$K(x) \equiv [(1 + \rho x)^\gamma - 1]/a = c[(1 + \rho x)^\gamma - 1]/\rho\gamma,$$

with $\gamma = 0$ yielding linear K by our 'L'Hospital convention'. The converse is similar but simpler. $\square$

Remarks. 1. For (iv) see [Acz] and [Chu1], and note from the comparison that all positive solutions arise as homomorphisms.
2. Since $0 = 1_{\mathbb{G}}$ for $\mathbb{G}$ a Popa group, (Hom-1) implies $K(0) = 0$.

3.3 Functional Inequalities

The following Goldie functional inequality, for $\eta \in GS$ continuous, also arises (in Beurling regular variation) for $K : \mathbb{G}_\eta \to \mathbb{R}$:

$$K(x \circ_\eta y) \leq \psi(y)K(x) + K(y) \qquad (x, y \in \mathbb{G}_\eta); \qquad (GBFI)$$

the case $\eta \equiv 1$ arises in RV (BGT Ch. 3; see also [BinO5]). With $\sigma(x) := \psi(K^{-1}(x))$ this is

$$K(x \circ_\eta y) \leq K(x) \circ_\sigma K(y) \qquad (x, y \in \mathbb{G}_\eta)$$

with $\circ_\sigma$ commutative as $\sigma \in GS$. The inequality (GFI) above has (via logarithmic transformation) the equivalent form

$$F(xy) \leq yF(x) + F(y) \qquad (x, y \in \mathbb{R}_+), \qquad (GFI_+)$$

for $F : \mathbb{R}_+ \to \mathbb{R}_+$. The Popa approach with $\sigma = F^{-1}$ here similarly yields

$$F(xy) \leq F(y) \circ_\sigma F(x) \qquad (x, y \in \mathbb{R}_+),$$

i.e. group-theoretic subadditivity (cf. BGT Ch. 3). The Goldie functional inequality may also be transformed (setting $F = -f^{-1}$) to a 'Gołąb-Schinzel functional inequality'

$$f(u + vf(u)) \leq f(u)f(v), \qquad (GSI)$$

recently studied in [Jab3].

4 Algebraicization of the Pexiderized equation

In this section, as in Th. 1 (§3), we characterize circumstances when solubility of (GBE-P) is equivalent to a homomorphy (under K). The focus will be on the function κ – see below. A word of warning: the roles of the functions ψ and κ are *complementary* rather than equivalent: their interchange forces an interchange of x and y significant for the subtracted term (on the left). In the subsequent section (§5) the focus is on ψ via ψ/η (though κ is used).

In (GBE) the value of $\psi(0)$, if non-zero, has no significant role, and may without loss of generality be scaled to unity; in (GBE-P) the value $\psi(0)$ has a more significant role, both in relation to $\sigma := \psi K^{-1}$ (cf. Th. 1) and in controlling whether K and κ are identical. This is clarified by Proposition B in §4.1. Throughout this section

$$\eta(x) \equiv 1 + \rho x \qquad (\rho \geq 0).$$

4.1 Preliminaries

Our first step is Proposition B below. We begin with some useful

Observations. 1. *The value of* $K(0)$ *may be arbitrary; if* $\psi(0) = 0$, *then* K *is constant.*
The solubility of (GBE-P) is unaffected by the choice of $K(0)$, since $K(x)$ may be replaced by $K(x) - K(0)$. If $\psi(0) = 0$, then $K(x) \equiv K(0)$ – take $y = 0$ in (GBE-P).
2. *If* K *satisfies* (GBE-P), *then*
(i) $K(x) \equiv \psi(0)\kappa(x) + K(0)$;
(ii) *provided* $\psi(0) \neq 0$, $\psi(0)\kappa$ *satisfies* (GBE) *and* $\kappa(0) = 0$.
Taking $y = 0$ gives (i). Substitution, for $\psi(0) \neq 0$, into (GBE-P) yields

$$\kappa(x + y\eta(x)) = \kappa(y) + \kappa(x)\psi(y)/\psi(0).$$

In particular, $\kappa(0) = 0$ (put $x = y = 0$).
3. *For* ψ *and* κ *positive, both* K *and* κ *are continuous and invertible.*
This follows from [BinO5, Lemma] as κ here is strictly monotone; hence so is $K(x)$ by 2(ii). From here we have the following extension of [BinO5, Th. 1]:

Lemma 2. *For* ψ *and* κ *positive, there is* $s \geq 0$ *such that*

$$\tilde{\psi}(x) := \psi(x)/\psi(0) = 1 + s\kappa(x).$$

So $\kappa : \mathbb{G}_\rho \to \mathbb{G}_s$ *is a homomorphism, and*

$$\text{either } \tilde{\psi} \equiv 1, \text{ or } \kappa(x) = (\tilde{\psi}(x) - 1)/s \text{ with } s > 0,$$

equivalently for $s > 0$, $\tilde{\psi} : \mathbb{G}_\rho \to \mathbb{G}_\infty$ *is a homomorphism:*

$$\tilde{\psi}(x + y\eta(x)) = \tilde{\psi}(x)\tilde{\psi}(y).$$

Proof. Since $K(x \circ_\eta y) = K(y \circ_\eta x)$ and $K(x) = \psi(0)\kappa(x) + K(0)$,

$$\psi(0)\kappa(y) + K(0) + \psi(y)\kappa(x) = \psi(0)\kappa(x) + K(0) + \psi(x)\kappa(y).$$

So

$$\kappa(y)[1 - \tilde{\psi}(x)] = \kappa(x)[1 - \tilde{\psi}(y)] : \qquad [\tilde{\psi}(x) - 1]/\kappa(x) = [\tilde{\psi}(y) - 1]/\kappa(y) = s,$$

say, the latter for $x, y > 0$ since κ is positive. So, for $y > 0$ and $x \geq 0$ arbitrary, $\tilde{\psi}(x) - 1 = \kappa(x)[\tilde{\psi}(y) - 1]/\kappa(y)$. Substituting $\tilde{\psi}(x) \equiv 1 + s\kappa(x)$ in (GBE-P),

$$\kappa(x + y\eta(x)) = \kappa(y) + \kappa(x)(1 + s\kappa(y)) \qquad (x, y \in \mathbb{R}).$$

For $s > 0$, writing κ in terms of $\tilde{\psi}$ and cancelling s,

$$[\tilde{\psi}(x + y\eta(x)) - 1] = [\tilde{\psi}(y) - 1] + [\tilde{\psi}(x) - 1]\tilde{\psi}(y). \qquad \square$$

4. *If* $K = \kappa$*, then* $\psi(0) = 1$ *and* $\sigma(t) := \psi(K^{-1}(t)) = 1 + st$.
Immediate from 2(ii) above and $\psi(x)/\psi(0) = 1 + s\kappa(x)$. Theorem 1 in §3 above motivates the interest in ψK^{-1}.

Proposition B extends this last observation, and helps clarify Theorem 1′ in §4.3, the main result of this section.

Proposition B. *If* $\sigma := \psi K^{-1} \in GS$ *and* $\psi(t)/\psi(0) = 1 + s\kappa(t)$*, then* $\sigma(t) \equiv 1 + st$ *and one of the following two conditions holds:*
(i) $\psi(t) \equiv \psi(0)$ *and* $K(x) \equiv \psi(0)\kappa(x) + K(0)$*;*
(ii) $s > 0$ *and* $K = \kappa$ *iff* $\psi(0) = 1$.

Proof. Put $\sigma(t) := 1 + ct$, with $c \geq 0$. Since $\psi(K^{-1}(t)) = \psi(0)(1 + s\kappa(K^{-1}(t)))$,

$$1 + ct = \psi(0)[1 + s\kappa(K^{-1}(t))] : \qquad [1 + s\kappa(x)]\psi(0) = (1 + cK(x)).$$

From the latter, $c = 0$ iff $s = 0$, as K and κ are non-constant. If $c = s = 0$, then, again as K is non-constant, $\psi(t) \equiv \psi(0)$, and so Observation 2(ii) applies. Suppose next that $c > 0$. Then

$$K(x) = \psi(0)\kappa(x)s/c + (\psi(0) - 1)/c.$$

So, again since $K(x) = \psi(0)\kappa(x) + K(0)$ and $\kappa(0) = 0$,

$$K(0) = (\psi(0) - 1)/c, \text{ and } s = c > 0.$$

So if $\psi(0) = 1$, then $K(0) = 0$ and $K(x) = \kappa(x)$. Conversely, if $K = \kappa$, then $K(0) = 0$, and so $\psi(0) = 1$. $\square$

4.2 A generalized circle operation

In identifying when κ is a homomorphism, Theorem 1 yields no similar information about K unless $K(0) = 0$, i.e. unless the constant $K(0)$ is 'subtracted' (translated away) from K. So a fresh idea is needed to accommodate on the *domain* of K the 'constant' in the *range* of K. That leads to an apparently more general circle operation below; however, Proposition C reduces it back to a Popa operation, albeit 'disguised' by another translation in the domain of K. This yields a group structure on $\mathbb{R}$ with a possibly *non-zero neutral element* (a pre-image under κ, masquarading as the 'subtracted' constant of K).

The strategy now is this. Suppose that (GBE-P) is soluble with ψ, κ positive; as K and κ are strictly monotone (cf. observation 3 above), put $y = K^{-1}(v)$, $x = \kappa^{-1}(u)$. Then

$$K(\kappa^{-1}(u) + K^{-1}(v)\eta(\kappa^{-1}(u))) = v + \psi(K^{-1}(v))u = v \circ_\sigma u,$$

where $\sigma(t) := \psi(K^{-1}(t))$. Apply K^{-1} :

$$\kappa^{-1}(u) \circ_\eta K^{-1}(v) = \kappa^{-1}(u) + K^{-1}(v)\eta(\kappa^{-1}(u)) = K^{-1}(v \circ_\sigma u).$$

Writing

$$\alpha(t) := \kappa^{-1}(K(t)), \qquad \beta(t) := \eta(\kappa^{-1}(K(t))), \qquad u \equiv K(K^{-1}(u)), \qquad (\alpha\text{-}\beta)$$

this says

$$\alpha(K^{-1}(u)) + K^{-1}(v)\beta(K^{-1}(u)) = K^{-1}(v \circ_\sigma u).$$

This motivates a *generalized Popa operation*:

$$u \circ v \text{ or } u \circ_{\alpha\beta} v := \alpha(u) + v\beta(u),$$

with α, β continuous, positive functions and α invertible.

Supposing this to yield a group structure (see Prop. C), and assuming $\sigma \in GS$ (so that $\circ_\sigma$ is commutative), we arrive at a homomorphism

$$K^{-1}(u \circ_\sigma v) = K^{-1}(u) \circ K^{-1}(v). \qquad \text{(Hom-2)}$$

We need to note the example $\alpha(x) = x+b$ with $\beta(x) \equiv 1$. Here $x \circ y = x+y+b$, so that $x \circ y \circ z = x + y + z + 2b$, and the neutral element e satisfies

$$x + e + b = x \text{ iff } e = -b, \qquad \text{and then } x^{-1} = -x - 2b.$$

We write $+_b$ for this operation and call this group the b-*shifted additive reals*.[2] We will see that b is responsible for the 'constant': $K(0) = \kappa(b)$. Note that $+_0 = + = \circ_0$.

Proposition C. *The operation $\circ = \circ_{\alpha\beta}$ is a group operation on a subset $\mathbb{A} \subseteq \mathbb{R}$ containing 0 and dense in $\mathbb{R}_+$ iff the subset is closed under $\circ$ and for some constants b, c with $bc = 0$*

$$\alpha(x) \equiv x + b \text{ and } \beta(x) \equiv 1 + c(x + b).$$

[2] The multiplicative analogue $x \circ y := xy/b$ comes from the format $x \circ y := \alpha(x) + x\beta(y)$.

That is:

$$\alpha(x) \equiv x \text{ and } \beta(x) \equiv 1 + cx, \text{ OR } \alpha(x) \equiv x + b \text{ and } \beta(x) \equiv 1.$$

So this is either a Popa group $x \circ_c y := x + y(1 + cx)$, *or the* b*-shifted additive reals with the operation* $x +_b y := x + y + b$.

Proof. Suppose that $\circ$ defines a group. In the application later we assume that α is injective, but here for x an element of the group, $x \circ 0 = \alpha(x)$, and then α must be injective (by the assumed group properties of $\circ$ including $0 \in \mathbb{A}$). By associativity,

$$(x \circ y) \circ z = \alpha(x \circ y) + z\beta(x \circ y) \qquad (x, y, z \in \mathbb{A}),$$

and

$$x \circ (y \circ z) = \alpha(x) + (y \circ z)\beta(x) = \alpha(x) + (\alpha(y) + z\beta(y))\beta(x) \qquad (x, y, z \in \mathbb{A}).$$

Comparing the z terms,

$$\beta(x \circ y) = \beta(x)\beta(y) \qquad (x, y \in \mathbb{A}),$$

and so

$$\alpha(x \circ y) = \alpha(x) + \alpha(y)\beta(x) \qquad (x, y \in \mathbb{A}). \qquad (**)$$

So, as α is injective, the preceding two equations imply

$$\beta\alpha^{-1}(\alpha(x) + \alpha(y)\beta(x)) = \beta(x)\beta(y) \qquad (x, y \in \mathbb{A}).$$

Put $u := \alpha(x)$ and $v = \alpha(y)$:

$$\beta\alpha^{-1}(u + v\beta\alpha^{-1}(u)) = \beta\alpha^{-1}(u)\beta\alpha^{-1}(v) \qquad (u, v \in \mathbb{A}),$$

so that $\beta\alpha^{-1} \in GS$, by the assumed positivity, continuity and density. So for some $c \geq 0$

$$\beta\alpha^{-1}(u) \equiv 1 + cu : \qquad \beta(v) \equiv 1 + c\alpha(v) \qquad (u, v \in \mathbb{A}).$$

So

$$x \circ y = \alpha(x) + y(1 + c\alpha(x)) \qquad (x, y \in \mathbb{A}).$$

So by (**)

$$\alpha(\alpha(x) + y(1 + c\alpha(x))) = \alpha(x) + \alpha(y)(1 + c\alpha(x)) \qquad (x, y \in \mathbb{A}).$$

Recalling that $\beta\alpha^{-1}(u) \equiv 1 + cu$, and writing $u = \alpha(x)$ and v for y, this is

$$\alpha(u + v(1 + cu)) = u + \alpha(v)(1 + cu) = u(1 + c\alpha(v)) + \alpha(v) \qquad (u, v \in \mathbb{A}).$$

Now set $v = 0 \in \mathbb{A}$ to obtain, with $a := (1 + c\alpha(0))$ and $b := \alpha(0)$,

$$\alpha(u) = au + b \qquad (u \in \mathbb{A}).$$

As α is injective $0 \neq a \in \mathbb{A}$. If e is the neutral element, then

$$y = e \circ y = \alpha(e) + y\beta(e) \qquad (y \in \mathbb{A}),$$

so $\alpha(e) = 0$ (taking $y = 0$) and $\beta(e) = 1$ (taking $0 \neq y \in \mathbb{A}$). So $\alpha(e) = ae + b = 0$, and so $e = -b/a$. Right-sided neutrality requires that

$$x = x \circ e = \alpha(x) + e\beta(x) = ax + b + e(1 + cax) = ax - bcx + b + e \qquad (x \in \mathbb{A}).$$

So $e = -b = -b/a$, so $a = 1$ and $bc = 0$.

One possibility is $b = 0 = e$, i.e. $\alpha(x) \equiv x$ and $\beta(x) \equiv 1 + cx$. (Indeed, $e = 1_c = 0$.) The other possibility is $c = 0$, in which case $\beta(x) \equiv 1$, $\alpha(x) \equiv x + b$, and $e = -b$. $\square$

4.3 Main result for $(GBE\text{-}P)$

Applying Prop. C we deduce the circumstances when $(GBE\text{-}P)$ may be transformed to a homomorphism between (usually, Popa) groups. We then read off the form of the solution function from Prop. A. In the theorem below we see that $K(x) \equiv (\psi(y) - 1)/s$ only in the cases (i) and (iii), but not in (ii) – compare Th. 1. Indeed, in (ii) K is affinely related to κ, unless $K(0) = 0$ (and then iff $b = 0$ and $\kappa \equiv K$). Section 5 pursues the affine relation.

Note that in all cases κ is a homomorphism between Popa groups.

Theorem 1′ *If $(GBE\text{-}P)$ is soluble for ψ positive, κ positive and invertible, $\eta(x) \equiv 1 + \rho x$ (with $\rho \geq 0$), then, for α, β selected as in (α-β) above, $\circ = \circ_{\alpha\beta}$ is a group operation and K^{-1} is a homomorphism under $\circ$:*

$$K^{-1}(u \circ_\sigma v) = K^{-1}(u) \circ K^{-1}(v) \qquad (u, v \in \mathbb{R}_+),$$

iff $\sigma := \psi K^{-1} \in GS$ and one of the following three conditions holds:
(i) $\rho = 0$, $\circ = \circ_0$ *and* $\circ_\sigma = \circ_s$ *for some* $s > 0$*; then for some* $\gamma \in \mathbb{R}$

$$K(t) \equiv \kappa(t) \equiv (e^{\gamma t} - 1)/s\ , \qquad \psi(t) \equiv e^{\gamma t};$$

(ii) $\rho = 0$, $\circ_\sigma = \circ_0$ *and* $\circ = +_b$ *for some* $b \in \mathbb{R}$; *then*

$$K(t) \equiv \kappa(t + b) = \kappa(t) + \kappa(b), \qquad \psi(t) \equiv 1 \qquad (t \in \mathbb{R}),$$

and $\kappa : \mathbb{G}_0 \to \mathbb{G}_0$ *is linear*;
(iii) $\rho > 0$, $\circ = \circ_\rho$ *and* $\circ_\sigma = \circ_s$ *for some* $s \geq 0$*; then for some* $\gamma \in \mathbb{R}$ *and* $t \geq 0$

$$\begin{aligned} K(t) &\equiv \kappa(t) \equiv [(1 + \rho t)^\gamma - 1]/s, \quad (s > 0) \quad \text{, or } \gamma \log(1 + rt) \quad (s = 0), \\ \psi(t) &\equiv (1 + \rho t)^\gamma \quad (s > 0) \quad \text{, or } \psi(t) \equiv 1 \quad (s = 0). \end{aligned}$$

Proof. We suppose that $\circ$ is a group operation. As above

$$K^{-1}(v \circ_\sigma u) = K^{-1}(u) \circ K^{-1}(v);$$

using this and associativity of $\circ$, Lemma 1_{assoc} (with $k = K^{-1}$ for K, $\circ$ for $\circ_\sigma$ and $\circ_\sigma$ for $\circ_\eta$) entails $\sigma \in GS$, as σ is positive and continuous: so

$$\sigma(t) = \psi(K^{-1}(t)) = 1 + st : \qquad \psi(t) = 1 + sK(t) \quad (t \in \mathbb{R}),$$

for some $s \geq 0$, (see Prop. B). So $\circ_\sigma$ is commutative, (Hom-2) holds, and $\psi(0) = 1$.

By Prop. C, K^{-1} is a homomorphism iff one of the two cases below arises. *Case (i): Popa case* $\circ = \circ_c$. For some $c \geq 0$

$$\kappa^{-1}(K(x)) = \alpha(x) \equiv x, \text{ and } \beta(y) = \eta(\kappa^{-1}(K(y))) \equiv 1 + cy.$$

So

$$K(t) = \kappa(t) \text{ and } 1 + \rho\kappa^{-1}(K(t)) = 1 + ct \quad (t \in \mathbb{R}).$$

For $\rho > 0$, on rearranging $\kappa^{-1}(K(y)) \equiv cy/\rho$, combining and using injectivity:

$$K(t) = \kappa(t) = \kappa(ct/\rho) : \qquad c = \rho \quad (t \in \mathbb{R}).$$

So $\circ = \circ_\rho$ and by (Hom-2),

$$K^{-1}(u \circ_s v) = K^{-1}(u) \circ_\rho K^{-1}(v) \quad (u, v \in \mathbb{R}). \tag{Hom-3}$$

So $K : \mathbb{G}_\rho \to \mathbb{G}_s$ is a homomorphism. By Prop. A and Observation 3 above, for some γ

$$K(t) \equiv ((1 + \rho t)^\gamma - 1)/s, \quad \text{or } \gamma \log(1 + \rho t) \quad (s = 0).$$

If $\rho = 0$, then $c = 0$, i.e. $\eta \equiv \beta \equiv 1$, and so again (Hom-3) holds but with $\rho = 0$:

$$K(t) = (e^{\gamma t} - 1)/s \quad (s > 0), \text{ or } \gamma t \quad (s = 0).$$

Case (ii): Shifted case. For some b

$$\kappa^{-1}(K(x)) = \alpha(x) \equiv x + b, \text{ and } \beta(y) = \eta(\kappa^{-1}(K(y))) = 1 + \rho\kappa^{-1}(K(y)) \equiv 1.$$

So $\rho = 0$, as $\kappa^{-1}(K(y)) \equiv y + b$ is non-zero. Furthermore, as $K(x) \equiv \kappa(x + b)$, writing K and ψ in terms of κ in (*GBE-P*),

$$K(x + y) = \kappa(x + y + b) = \kappa(y + b) + \kappa(x)(1 + s\kappa(y + b)) \quad (x, y \in \mathbb{R}).$$

Putting $z = y + b$,

$$\kappa(x + z) = \kappa(z) + \kappa(x)(1 + s\kappa(z)) = \kappa(x) \circ_s \kappa(z) \quad (x, z \in \mathbb{R}).$$

So[3] $\kappa : \mathbb{G}_0 \to \mathbb{G}_s$ is a homomorphism (and $\kappa(0) = 0$). Two subcases arise. *Subcase* $s = 0$: then $\psi \equiv 1$ and κ is linear: $K(x) = \kappa(x + b) = \kappa(x) + \kappa(b)$.

[3] Alternatively, apply Prop. A to $F(t) := K^{-1}(t) + b$, as $F : \mathbb{G}_s \to \mathbb{G}_0$, since $K^{-1}(u \circ_\eta v) = K^{-1}(u) + K^{-1}(v) + b$.

Subcase $s > 0$: then, as $\psi(0) = 1$, $K(x) = \kappa(x) + K(0)$ with $K(0) = \kappa(b)$; but by Prop. A and Observation 3

$$\kappa(x) = (e^{\gamma x} - 1)/s : \qquad K(x) = (e^{\gamma(x+b)} - 1)/s = e^{\gamma b}\kappa(x) + \kappa(b).$$

So here $b = 0$, and $K = \kappa$, which is included as $\circ = \circ_0 = +_0$.

The converse is similar and simpler. □

Remarks. 1. The implications (i)-(iii) are new here, but for their conclusions see also [Acz] and [Chu1]; as with Th. 1 in §3 above, a comparison shows that all positive solutions arise as homomorphisms.
2. The transformations used to obtain a homomorphism in fact simplify (*GBE-P*) to the case where $\kappa(u) = u$ and $\psi(v) = 1 + c\alpha(v)$.

5 Flows

Using Riemann sums and their limits [BinO5, Th. 9] gives conditions[4] such that if (*GBE-P*) is soluble, then the solution function K and the auxiliary κ are differentiable, and $K' = c \cdot \psi/\eta$ for some constant c. We give a new proof which also extends our understanding of (*GBE-P*) by reference to the underlying flow velocity $f := \eta/\psi$.

Indeed, this section focuses via f on the auxiliary ψ rather than on κ, though κ continues to play a part. We assume below that $\psi(0) \neq 0$, in order to pursue the consequences of the affine relation (cf. Observation 2(ii) of §4)

$$K(x) \equiv \psi(0)\kappa(x) + K(0),$$

reducing the quest to finding κ – in terms of τ_f, as defined in §2. To link results below to earlier ones, note that if $K(0) = 0$, then $K \equiv \kappa$ iff $\psi(0) = 1$ (cf. also Prop. B); however, unlike in (*GBE*) – where the Popa homomorphy of Th. 1 entails $K(0) = 0$ – in (*GBE-P*) the value $K(0)$ *need not be zero*, since the homomorphy of Th. 1$'$ refers to a *generalized* Popa group on $\mathbb{R}$ as in §4.2 (so possibly with a non-zero neutral element).

5.1 Main result for flows

The setting in Theorem 2 below differs slightly from [BinO5, Th. 9] – we do not assume non-negativity of κ, ψ, but instead that ψ is differentiable, since in applications ψ is such (in view of Prop. A). From this, continuity of K will be shown to imply *automatic differentiability* – for a textbook treatment of such matters see Járai [Jar]. We could just as easily have assumed ψ monotone (also implied by Prop. A), since a monotone, continuous real function is differentiable almost everywhere [Rud, §8.15] (and is absolutely continuous iff it is the integral of its derivative).

[4] Specializing to the present context: κ positive to the right near 0 and ψ continuous.

Theorem 2. *For $\kappa, \eta \in GS$ continuous and ψ not identically zero and differentiable: if the solution K to (GBE-P) is continuous, then either K is constant or:*
(i) *K is differentiable and $K'(x) \equiv \kappa'(0)/f(x)$ for $f(x) := \eta(x)/\psi(x)$;*
(ii) *$\kappa'(x)/\kappa'(0) \equiv K'(x)/K'(0)$ and $\kappa(x) = c \cdot \tau_f(x)$ for some $c \in \mathbb{R}$;*
(iii) *$K_0' := K/K'(0) : (\mathbb{R}, \circ_\sigma) \to (\mathbb{R}, \cdot)$ is a homomorphism:*

$$K_0'(x + y\eta(x)) = K_0'(x)K_0'(y).$$

We defer the proof to the end of §5.3, but note the immediate

Corollary 1. *In the setting of Theorem 2*
(i) *$K(x) \equiv \kappa'(0)\tau_f(x) + K(0)$, where $f(x) := \eta(x)/\psi(x)$ is the relative flow-velocity;*
(ii) *$\kappa(x) \equiv aK(x) + b$ for some $a, b \in \mathbb{R}$;*
(iii) *provided $\psi(0) = 1$, the flow-velocity $f : (\mathbb{R}, \circ_\eta) \to (\mathbb{R}, \cdot)$ is a homomorphism, equivalently f solves Chudziak's functional equation (ChE). So $\psi(x) \equiv \eta(x)/f(x)$, where f satisfies (ChE).*

Remarks. 1. As (GS) corresponds to $K = \psi = \eta$, $\kappa(u) = \eta(u) - 1$, here $K(0) = \psi(0) = \eta(0) = 1$ and $f = \eta/\psi = 1$; so $\tau_f(x) = x$ and $\kappa(x) = cx$, so $\eta(x) = K(x) = \kappa(x) + K(0) = cx + \eta(0) = 1 + cx$.
2. The classical (GBE) case corresponds to $\eta \equiv 1$ (i.e. $\rho = 0$) and $\kappa = K$, so $\psi(x) = 1/f(x) = e^{\gamma x}$, as f is a Popa-homomorphism by Prop. A. So $K(x) = \kappa(x) = c \cdot \tau_f(x)$ with $\tau_f(x) \equiv (e^{\gamma x} - 1)/\gamma$.

5.2 Sufficiency

We begin with a Proposition which, taken together with Theorem 2 above, characterizes the solutions to (*GBE-P*) in terms of f.

Proposition D. *If f satisfies (CBE), then subject to $K(0) = 0$, $K \equiv \tau_f(x)$ solves (GBE-P) for the auxiliaries $\psi(x) := \eta(x)/f(x)$ and $\kappa \equiv \tau_f(x)$.*

Proof. Substituting for K in (*GBE-P*), and using $u + \eta\sigma(u) = v + u\eta(v)$, as $\eta \in GS$, we are to prove that

$$K(v + u\eta(v)) - K(v) = \int_v^{v+u\eta(v)} \mathrm{d}t/f(t) = \psi(v)\kappa(u) = \psi(v)\int_0^u \mathrm{d}t/f(t).$$

This follows from

$$\begin{aligned} \psi(v)\int_0^u \mathrm{d}t/f(t) &= \eta(v)/f(v)\int_0^u \mathrm{d}t/f(t) = \eta(v)\int_0^u \mathrm{d}t/f(v)f(t) \\ &= \eta(v)\int_0^u \mathrm{d}t/f(v + t\sigma(v)) \text{ (put } w = v + t\eta(v)) \\ &= \int_v^{v+u\eta(v)} \mathrm{d}w/f(w). \qquad \square \end{aligned}$$

Corollary 2. *In the setting of Prop. D the solution $K \equiv \tau_f$ of (GBE-P) takes one of the forms (for $x \geq 0$):*

$$\tau_f(x) \equiv \int_0^x e^{\gamma t}\mathrm{d}t = (e^{\gamma x} - 1)/\gamma, \qquad (\rho = 0, \gamma \neq 0),$$

$$\tau_f(x) \equiv \int_0^x (1+\rho t)^\gamma \mathrm{d}t = ((1+\rho x)^{\gamma+1} - 1)/\rho(\gamma+1), \qquad (\rho \in (0,\infty), \gamma \neq -1),$$

$$\tau_f(x) \equiv x, \qquad (\rho \in [0,\infty]).$$

Proof. Apply Prop. A, writing γ for $-\gamma$. The final formula is the limit of the cases $\rho = 0$ and $\rho > 0$ as γ approaches 0 or -1, respectively. $\square$

5.3 Necessity

The proof of Theorem 2 (converse to Prop. *D* above) rests on three results, the first a 'smoothness result'. (For continuity and differentiability of integrals with respect to a parameter, see [Jar, §§ 3 and 11].) Recall that for a Popa group $\mathbb{G} = \mathbb{G}_\eta$, $1_{\mathbb{G}} = 0$ and ${}_\circ^{-1}$ denotes its inverse.

Proposition E (Convolution Formula). *For differentiable $\eta \in GS$,*

$$\mathrm{d}x_\circ^{-1} = -\eta(s)^{-2}\mathrm{d}s,$$

so for $x \geq 0$

$$a * b(x) := \int_0^x a(x \circ_\eta t_\circ^{-1})b(t)\mathrm{d}t = \eta(x)\int_0^x a(s)b(x \circ_\eta s_\circ^{-1})\frac{\mathrm{d}s}{\eta(s)^2},$$

for a, b continuous; in particular, if b is differentiable/$\mathcal{C}^\infty$, then so is the convolution function $a * b$, and

$$(a*b)'(x) = \eta'(x)a * b(x) + b(0)a(x)/\eta(x) + \int_0^x a(s)b'(x \circ_\eta s_\circ^{-1})\frac{\mathrm{d}s}{\eta(s)^3}.$$

Proof. Noting $\eta_\rho'(x) = \rho$, differentiation of $\eta(s_\circ^{-1}) = 1/\eta(s)$ gives

$$\rho\mathrm{d}(s_\circ^{-1}) = -\eta(s)^{-2}\rho\mathrm{d}s.$$

Put $s = x \circ_\eta t_\circ^{-1}$; then $t = x \circ_\eta s_\circ^{-1} = x + s_\circ^{-1}\eta(x)$. Finally,

$$b(x \circ_\eta s_\circ^{-1}) = b(x + s_\circ^{-1}\eta(x)),$$

which is differentiable in x; so $\mathrm{d}b(x \circ_\eta s_\circ^{-1})/\mathrm{d}x = b'(x \circ_\eta s_\circ^{-1})/\eta(s)$, since $1 + \rho s_\circ^{-1} = \eta(s_\circ^{-1}) = \eta(s)^{-1}$. $\square$

The following two lemmas prepare the final ground-work for a proof of Theorem 2.

Lemma 4. *For continuous κ, a non-trivial (i.e. non-zero) differentiable function ψ, and continuous $\eta \in GS$: if the solution K to (GBE-P) is continuous, then K satisfies the difference equation*

$$K(x+u) - K(x) = \kappa(u/\eta(x))\psi(x),$$

so K has the flow representation

$$xK(x) = \int_0^x K(t)\mathrm{d}t + \int_0^t \kappa((x-t)/\eta(t))\psi(t)\mathrm{d}t,$$

and so is differentiable on $\mathbb{R}_+$.

Proof. For $u, v \geq 0$ take $w := u + v\eta(u) = v + u\eta(v)$, then $u = (w-v)/\eta(v)$, so

$$K(w) = \kappa((w-v)/\eta(v))\psi(v) + K(v). \tag{***}$$

Now write x for v and u for $(w-v)$ to obtain

$$K(x+u) - K(x) = \kappa(u/\eta(x))\psi(x).$$

In (***) integrate w.r.t. v from 0 to w; then

$$wK(w) = \int_0^w K(v)\mathrm{d}v + \int_0^w \kappa((w-v)/\eta(v))\psi(v)\mathrm{d}v.$$

The second term, being a Beurling convolution, is differentiable by Prop. E. $\square$

Lemma 5 (Flow Homomorphism). *If K is a differentiable solution to (GBE-P), normalized so that $K(0) = 0$, then either $K \equiv 0$, or:*
(i) $K'(x) \equiv \kappa'(0) \cdot \psi(x)/\eta(x) = \kappa'(0)/f(x)$, *for $f(x)$ the flow-velocity* (of §2);
(ii) $K'(x)/K'(0) \equiv \kappa'(x)/\kappa'(0)$, *so $\kappa(x) = c\tau_f(x)$ for some $c \in \mathbb{R}$;*
(iii) $K_0' := K'/K'(0) : (\mathbb{R}_+, \circ_\sigma) \to (\mathbb{R}_+, \cdot)$ *satisfies*

$$K_0'(x + y\eta(x)) = K_0'(x)K_0'(y).$$

In particular, if $\psi(0) = 1$, then

$$f(x + y\eta(x)) = f(x)f(y).$$

Proof. Fixing y with $\psi(y) \neq 0$, it follows from (*GBE-P*) that $\kappa(x)$ is differentiable everywhere. Differentiating with respect to x and using $x \circ_\eta y = y \circ_\eta x$

$$K'(x + y\eta(x))\eta(y) = \psi(y)\kappa'(x).$$

As $\eta(0) = 1$, substituting 0 alternately for one of x and y, and then for both:

$$K'(y)\eta(y) = \psi(y)\kappa'(0), \qquad K'(x) = \psi(0)\kappa'(x), \qquad K'(0) = \psi(0)\kappa'(0).$$

So if $\psi(0)\kappa'(0) = 0$, then $K \equiv 0$. Otherwise, combining,

$$\kappa'(x)/\kappa'(0) = K'(x)/\kappa'(0)\psi(0) = K'(x)/K'(0),$$

and in particular $\kappa'(x) = c\psi(x)/\eta(x) = c/f$, with $c = \kappa'(0)/\psi(0)$. So $\kappa(x) = c\tau_f(x)$, as $\kappa(0) = 0$ (from (GBE-P) for $x = y = 0$), giving (i) and (ii). So

$$\frac{K'(x + y\eta(x))}{K'(0)} = \frac{1}{K'(0)}\frac{\psi(y)}{\eta(y)}\kappa'(x) = \frac{1}{K'(0)} \cdot \frac{K'(y)}{\kappa'(0)} \cdot \frac{K'(x)}{\psi(0)} = \frac{K'(x)K'(y)}{K'(0)K'(0)},$$

equivalently, if $\psi(0) = 1$, $K'(0) = \kappa'(0)$, $f(x) \equiv \kappa'(0)/K'(x)$ is a homomorphism. □

Proof of Th. 2. Assuming K non-constant, rescaling if necessary, without loss of generality $K(0) = 0$ and $K'(0) = 1$. Now combine Lemmas 4 and 5. □

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