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When to sell Apple and the NASDAQ? Trading bubbles with a stochastic disorder model

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Abstract
In this paper, the authors apply a continuous time stochastic process model developed by Shiryaev and Zhutlukhin for optimal stopping of random price processes that appear to be bubbles. By a bubble we mean the rising price is largely based on the expectation of higher and higher future prices. Futures traders such as George Soros attempt to trade such markets. The idea is to exit near the peak from a starting long position. The model applies equally well on the short side, that is when to enter and exit a short position. In this paper we test the model in two technology markets. These include the price of Apple computer stock AAPL from various times in 2009-2012 after the local low of March 6, 2009; plus a market where it is known that the generally very successful bubble trader George Soros lost money by shorting the NASDAQ-100 stock index too soon in 2000. The Shiryaev-Zhitlukhin model provides good exit points in both situations that would have been profitable to speculators following the model.

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Trading bubbles is difficult and even the best traders like George Soros sometimes lose a lot of money by shorting too soon. The finance and economics literature has little on timing bubbles. There is however some interest by inefficient market types, see for example Stiglitz [1990] and Evanoff et al [2012)]. What we mean by a bubble is a price that is going up just because it is going up! In this paper we present a model developed by Shiryaev and Zhitlukhin [2012ab] that seems to work well timing when to exit a long position or when to exit a short position. To keep the exposition simple, we just apply the model here in two very interesting technology situations namely, Apple Computer stock (AAPL) in the past few years and the internet technology bubble around 2000 measured by the Nasdaq (NDX100) which has futures contracts sold on it. In both cases, the results are good. The mathematics of the model is sketched in the appendix and is an application of modern mathematical finance stochastic calculus. Shirayev has worked on such models for many years and Shiryaev and Zhitlukhin [2012b] present the model in a form that is useful to trade bubbles.

The basic idea is that there is a fast rate of growth in prices, then a peak and then a fast decline. The model tries to exit near the peak in prices or valley of its short position. Usually financial markets fall faster than they rise. But we have found that in these two markets and others that the rate of increase and decrease are very similar and different speeds add no value. The paper shows entries and exits. For readers interested in how the model works, the appendix should be helpful. But it is not important to read the appendix to understand the results of the model which are in tables and graphs in the exhibits.

AAPL rises and falls

AAPL had a spectacular run since the bottom of the 2007-2009 crash in March 2009, see Exhibit 1 which shows the price history from September 1984 to the end of 2012; Exhibit 2 shows the more recent period, from the beginning of 2009 to the end of 2012.

A sequence of valuable and easy to use products created huge interest and sales around the world. These include the iPod, the iPhone, and the iPad. All of these products had high margins for the company which accumulated large cash levels. In November 2012 they had \$121 billion in cash or \$128 per share of the 941 million shares outstanding. The company has generated cash faster than any corporation in history. The stock was never at a high price earnings ratio and was a favorite of hedge funds, open and closed mutual funds, ETFs and various small and large investors. And indeed it was traded as a proxy for the market with high liquidity. Its forward price earnings ratio in November 2012 was 10.17 with estimated earnings per share of \$49.28. The company has a quarterly dividend of \$2.65 per share and a buy back of about \$10 million in stock. An increased or special dividend could also occur as well as increased buy back of stock.

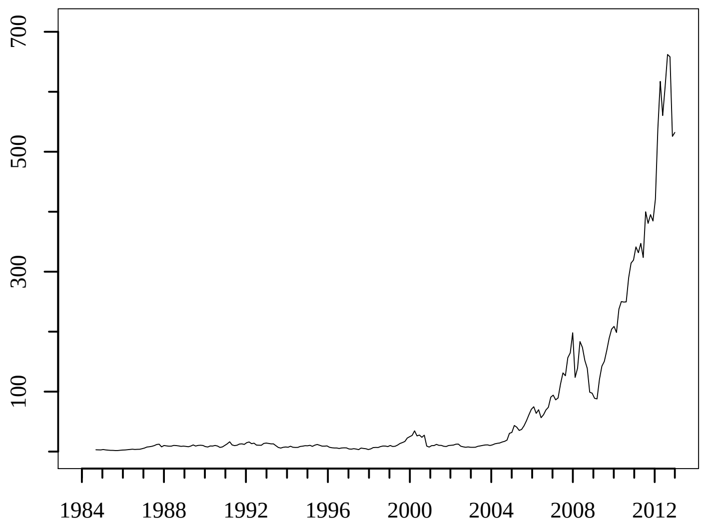

Exhibit 1: The history of AAPL stock price from September 1984 to the end of 2012 (adjusted for dividends and splits).

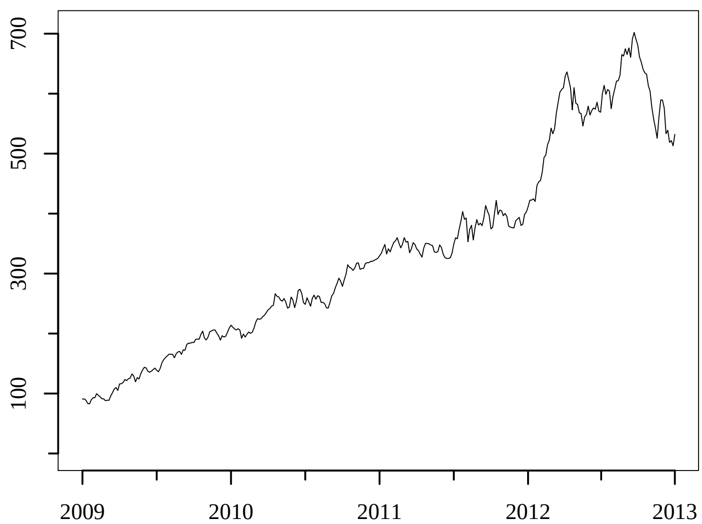

Exhibit 2: AAPL stock price from the beginning of 2009 to the end of 2012.

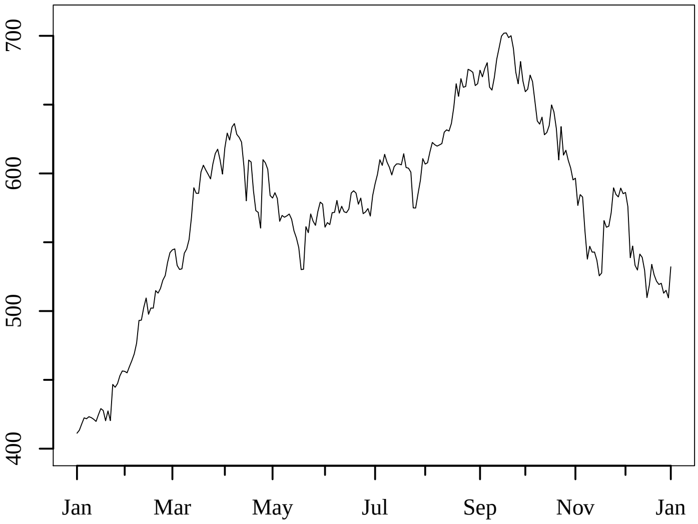

Exhibit 3: AAPL stock price in 2012.

Steve Jobs left Apple in 1985 because of a power struggle with John Sculley who he brought over from Pepsi asking ”do you want to sell sugared water all your life or change the world”. Sculley came to Apple but he and Jobs had a disagreement on strategy and marketing which stagnated the company. The board favored the marketer over the genius. Jobs sold all but one of his AAPL shares. The company languished while he continued developing ideas at NeXT and Pixar. When Jobs returned to Apple in 1996, he brought the new NeXT platform and ideas for user friendly products that had not yet been imagined by the market. He transformed the company into a winner. He held a lot of AAPL stock but more of Pixar which merged with Disney. After his death on October 5, 2011, many feared that the sequence of great products would cease and that the pace of innovation could not be maintained, that the market cap of about $500 billion, various lawsuits for patent infringement, competition and labour and supply chain issues might slow it down. Some thought it was a bubble and others thought it would continue rallying because it was not expensive not feeding on itself as in a typical bubble. Nonetheless, the stock peaked at 705.07 on September 21, 2012 and then fell dramatically to the local low of 505.75 on November 16, 2012. Later, in pre market trading on December 17, 2012, it fell to 499. On December 31, 2012, AAPL closed the year at 532.17; see Exhibit 3 for the price action in 2012.

The concentration of ownership by mutual funds (see Exhibit 4) creates conundrum for Apple as regulations prohibit ownership to exceed a percentage of a fund’s assets, so as AAPL rises relative to other stocks, funds often must sell shares. Some of the selling was tax loss selling in 2012 before expected higher capital gains and dividend rates in 2013

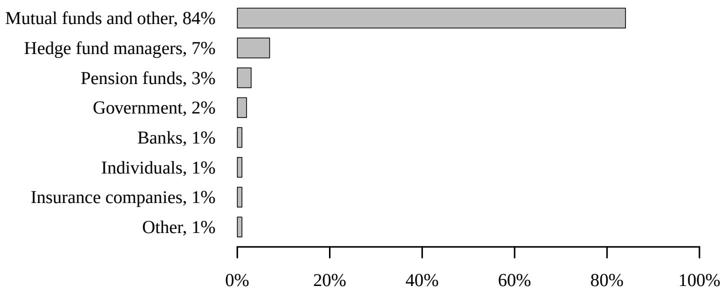

Exhibit 4: Holders of Apple, April 17, 2012. Source: Bloomberg via Eric Jackson.

since more gains are in AAPL than in any other stock. Despite the large decline in the latter part of 2012 the stock increased 30% in 2012.

Application of the model to the AAPL bubble

We apply the model to the Apple price bubble starting at the local low of 82.33 on March 6, 2009 and considering eight different entering dates for opening a long position: June 30, 2009; December 31, 2009; June 30, 2010; December 31, 2010; June 30, 2011; December 30, 2011; June 29, 2012; and July 31, 2012. It is assumed that the trend reversal will happen before the end of 2012. Higher tax rates on dividends and capital gains are expected in 2013, thus a sale in 2012 is suggested.

To apply the model, we identify the sequence of prices $P_0, \dots, P_N$ with the daily closing prices between March 6, 2009 and December 31, 2012. There are 962 trading days in this time interval, so $N = 961$.

Exhibit 5 lists the results for the eight entering dates and four different choices of the parameter μ_2. The names of the first seven columns are self-explanatory. Column "% of max." gives the ratio of the closing price on the exiting date to the highest closing price ($702.10 on September 19, 2012). Column "Return rate, %" contains the annual rates of return, if one buys Apple shares on each of the entering dates and sells on the date suggested by the model. The rate is computed by the compound interest formula $r = \log(S_\tau/S_0) \cdot (\tau/0.252)^{-1}$, where 252 is the average number of trading days in a year, so one year has the length of 0.252 in t-time.

Tests varying $\mu_2 = -\alpha\mu_1$ for $\alpha = 0.5, 1, 2, 3$ indicate that the choice $\mu_2 = -\mu_1$ is the optimal one, and works equally well both for early and late entering dates giving nearly 90% of the maximum price.

Exhibit 6 presents the graph of AAPL prices with entering dates marked by the dots,

Entering date	Entering price	μ_1	σ	T	Exit date	Exit price	% of max.	Return rate, %
				$\mu_2 = -\mu_1$				
2009-06-30	142.43	0.666	0.224	8.81	2012-10-11	628.10	89.46	45.11
2009-12-31	210.73	0.452	0.186	7.53	2012-10-11	628.10	89.46	39.26
2010-06-30	251.53	0.344	0.192	6.29	2012-10-09	635.85	90.56	40.64
2010-12-31	322.56	0.305	0.177	5.01	2012-10-08	638.17	90.89	38.55
2011-06-30	335.67	0.249	0.169	3.76	2012-10-08	638.17	90.89	50.44
2011-12-30	405.00	0.234	0.173	2.49	2012-10-08	638.17	90.89	59.07
2012-06-29	584.00	0.245	0.175	1.24	2012-10-09	635.85	90.56	30.62
2012-07-31	610.76	0.245	0.174	1.03	2012-10-11	628.10	89.46	13.83
				$\mu_2 = -0.5\mu_1$				
2009-06-30	142.43	0.666	0.224	8.81	2011-11-18	374.94	53.40	40.38
2009-12-31	210.73	0.452	0.186	7.53	2012-10-08	638.17	90.89	40.00
2010-06-30	251.53	0.344	0.192	6.29	2012-10-09	635.85	90.56	40.64
2010-12-31	322.56	0.305	0.177	5.01	2012-10-09	635.85	90.56	38.26
2011-06-30	335.67	0.249	0.169	3.76	2012-10-11	628.10	89.46	48.73
2011-12-30	405.00	0.234	0.173	2.49	2012-10-11	628.10	89.46	56.13
2012-06-29	584.00	0.245	0.175	1.24	2012-10-19	609.84	86.86	13.99
2012-07-31	610.76	0.245	0.174	1.03	2012-10-19	609.84	86.86	-0.67
				$\mu_2 = -2\mu_1$				
2009-06-30	142.43	0.666	0.224	8.81	2012-10-31	595.32	84.79	42.86
2009-12-31	210.73	0.452	0.186	7.53	2012-10-11	628.10	89.46	39.26
2010-06-30	251.53	0.344	0.192	6.29	2012-10-09	635.85	90.56	40.64
2010-12-31	322.56	0.305	0.177	5.01	2012-10-08	638.17	90.89	38.55
2011-06-30	335.67	0.249	0.169	3.76	2012-05-17	530.12	75.50	51.87
2011-12-30	405.00	0.234	0.173	2.49	2012-05-17	530.12	75.50	71.41
2012-06-29	584.00	0.245	0.175	1.24	2012-10-08	638.17	90.89	32.40
2012-07-31	610.76	0.245	0.174	1.03	2012-10-08	638.17	90.89	23.05
				$\mu_2 = -3\mu_1$				
2009-06-30	142.43	0.666	0.224	8.81	2012-11-07	558.00	79.48	40.67
2009-12-31	210.73	0.452	0.186	7.53	2012-10-19	609.84	86.86	37.88
2010-06-30	251.53	0.344	0.192	6.29	2012-10-11	628.10	89.46	39.97
2010-12-31	322.56	0.305	0.177	5.01	2012-10-09	635.85	90.56	38.26
2011-06-30	335.67	0.249	0.169	3.76	2012-10-08	638.17	90.89	50.44
2011-12-30	405.00	0.234	0.173	2.49	2012-10-08	638.17	90.89	59.07
2012-06-29	584.00	0.245	0.175	1.24	2012-10-08	638.17	90.89	32.40
2012-07-31	610.76	0.245	0.174	1.03	2012-10-08	638.17	90.89	23.05

Exhibit 5: Results of applying the model to AAPL stock with various entry dates and values of μ_2.

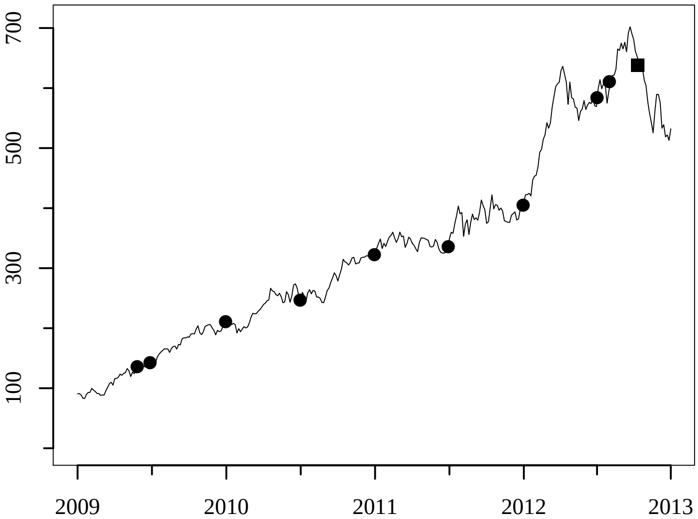

Exhibit 6: Buying and selling dates for AAPL when $\mu_2 = -\mu_1$. The dots indicate the eight entering dates, and the square indicates the exit date on October 8, 2012.

and the date October 8, 2012 (one of the exit dates) marked by the square. Exhibits 7-8 present the graphs of the exiting process ψ_t and the optimal stopping boundaries $a(t)$ for the entering dates December 30, 2011 and June 29, 2012 with $\mu_2 = -\mu_1$. By comparing with Exhibit 3, it is interesting to see how the process ψ_t reacts on changes in the price process S_t: for example, the increase of ψ_t in May 2012 on Exhibit 7 was caused by the corresponding fall of the AAPL price as can be seen from Exhibit 3.

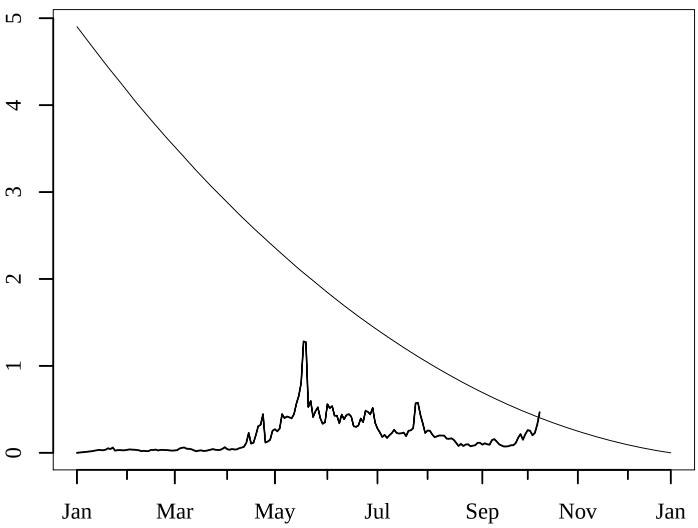

Exhibit 7: The process ψ_t and the function $a(t)$ for AAPL when buying long on December 30, 2011; $\mu_2 = -\mu_1$.

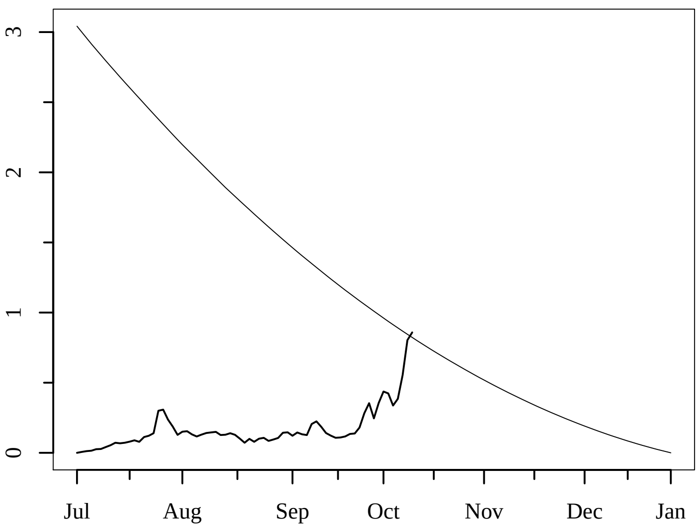

Exhibit 8: The process ψ_t and the function $a(t)$ for AAPL when buying long on June 29, 2012; $\mu_2 = -\mu_1$.

The Internet bubble crash during 2000-2002

Alan Greenspan, the chairman of the US Federal Reserve System (Fed), began in 1994 a low interest rate policy that dropped short term rates continuously over a multiyear period. This led to an increase in the S&P500 stock index from 470.42 in January 1995 to 1469.25 at the end of 1999, as shown in Exhibits 9 and 10. The price earnings ratios were high and Shiller used these to predict the crash starting in 1996, see Campbell and Shiller [1998] and Shiller [1996, 2000, 2009]. It is known that stock price rises usually start with low price earnings ratios and end with high price earnings ratios, see Exhibit 11. But predicting when the market will crash using just price earnings ratios is problematic.

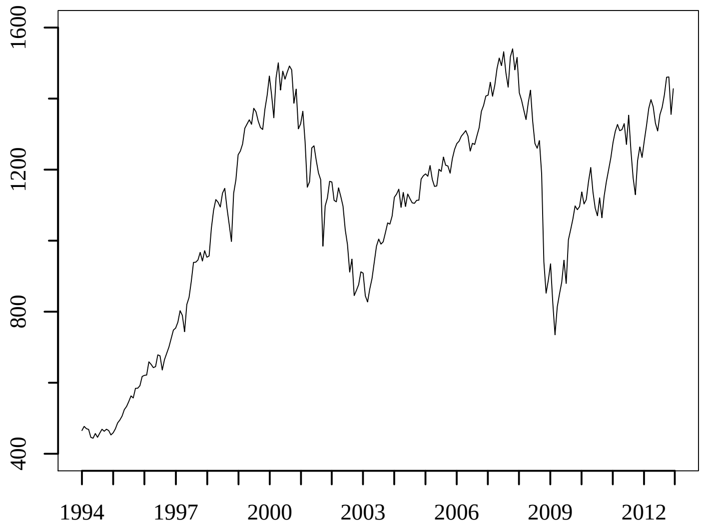

Exhibit 9: S&P500 index, 1994-2012.

However, Ziemba has found in many markets over many years that the *bond-stock earnings yield differential (BSEYD) model*[1] predicts crashes better than just high price-earnings ratios, see Ziemba and Schwartz [1991], Ziemba [2003] and Lleo and Ziemba [2012]. The model signaled a crash in the S&P500 in April 1999. It was in the danger zone all of 1999 starting in April and it got deeper in the danger zone as the year progressed, see Exhibit 10. The S&P500 rose from 1229.23 at the end of December 1998 to 1469.25 at the end of December 1999. The PE ratio was flat, increasing only from 32.34 to 33.29 while long bond yield rose from 5.47% to 6.69%. The lowest value of S&P500 in April 1999 was 1282.56 on April 1, and the highest value was 1371.56 on April 27. The signal did work but the real decline was not until September 2000 with a temporary fall from the March 24, 2000 high of 1552.87 and a recovery into the September 1, 2000 peak of 1530.09. By October 10, 2002 S&P500 fell to 768.63 having two temporary recoveries from the local lows of 1091.99

Month	S&P500	(a) P/E	(b) 30 yr. bond	(c=1/a) Stocks return	(b-c) Crash signal	Month	S&P500	(a) P/E	(b) 30 yr. bond	(c=1/a) Stocks return	(b-c) Crash signal
95 Jan	470.42	17.10	8.02	5.85	2.17	98 Jan	980.28	24.05	6.01	4.16	1.85
Mar	500.71	16.42	7.68	6.09	1.59	Mar	1101.75	27.71	6.11	3.61	2.50
May	533.40	16.39	7.29	6.10	1.19	May	1090.82	27.62	6.10	3.62	2.48
Jul	562.06	17.23	6.90	5.80	1.10	Jul	1120.67	28.46	5.83	3.51	2.32
Sep	584.41	16.88	6.74	5.92	0.82	Sep	1017.01	26.10	5.47	3.83	1.64
Nov	605.37	17.29	6.36	5.78	0.58	Nov	1163.63	31.15	5.54	3.21	2.33
96 Jan	636.02	18.09	6.18	5.53	0.65	99 Jan	1279.64	32.64	5.49	3.06	2.43
Mar	645.50	19.09	6.82	5.24	1.58	Feb	1238.33	32.91	5.66	3.04	2.62
May	669.12	19.62	7.21	5.10	2.11	Mar	1286.37	34.11	5.87	2.93	2.94
Jul	639.96	18.80	7.23	5.32	1.91	Apr	1335.18	35.82	5.82	2.79	3.03
Sep	687.31	19.65	7.26	5.09	2.17	May	1301.84	34.60	6.08	2.89	3.19
Nov	757.02	20.92	6.79	4.78	2.01	Jun	1372.71	35.77	6.36	2.80	3.56
97 Jan	786.16	21.46	6.95	4.66	2.29	Jul	1328.72	35.58	6.34	2.81	3.53
Mar	757.12	20.45	7.11	4.89	2.22	Aug	1320.41	36.00	6.35	2.78	3.57
May	848.28	21.25	7.08	4.71	2.37	Sep	1282.70	30.92	6.50	3.23	3.27
Jul	954.29	23.67	6.78	4.22	2.56	Oct	1362.92	31.61	6.66	3.16	3.50
Sep	947.28	23.29	6.70	4.29	2.41	Nov	1388.91	32.24	6.48	3.10	3.38
Nov	955.40	23.45	6.27	4.26	2.01	Dec	1469.25	33.29	6.69	3.00	3.69

Exhibit 10: BSEYD model for the S&P500, 1995-1999. Source: Ziemba [2003].

on April 4, 2001 and 944.75 on September 21, 2001. There were other signals:

> History shows that a period of shrinking breadth is usually followed by a sharp decline in stock values of the small group of leaders. Then broader market takes a more modest tumble. Paul Bagnell in late November 1999 in the *Globe and Mail.*

Ziemba [2003, Chapter 2] describes this episode in stock market history. There was considerable mean-reversion in the eventual crash in 2000 the September 11, 2001 attack and in the subsequent 2002 decline of 22%. This decline was similar to previous crashes.

The concentration of stock market gains into very few stocks with momentum and size being the key variables predicting performance was increasing before 1997 in Europe and North America. Table 2.6 in Ziemba [2003] shows that in 1998, the largest cap stocks had the highest return in North America and Europe but small cap stocks outperformed in Asia and Japan. The situation was similar from 1995 to 1999 with 1998 and 1999 the most exaggerated.

Fully 41% of the stocks in the S&P500 did not fall or actually rose during this period and an additional 19% declined by 10% or less annualized. These were small cap stocks with market values of $10 billion of less. The fall in the S&P500 was mainly in three areas:

Begin Year	End Year	Geometric Mean, %	Beg PE	End PE	Begin Year	End Year	Geometric Mean, %	Beg PE	End PE
1975	1994	9.6	10.9	20.5	1981	2000	12.8	8.8	41.7
1977	1996	9.7	11.5	25.9	1979	1998	12.9	9.4	36.0
1942	1961	9.9	12.2	20.5	1982	2001	13.0	8.5	32.1
1983	2002	10.9	7.3	25.9	1980	1999	14.0	8.9	42.1
1978	1997	11.9	10.4	31.0					

Exhibit 11: Nine 20-year periods of gains beginning low PE and ending high PE. Source: Bertocchi, Schwartz and Ziemba [2010].

information technology, telecommunications and large cap stocks. Information technology stocks in the S&P500 fell 64% and telecom stocks fell 60% from January 1 to October 31, 2002. The largest cap stocks (with market caps of $50 billion plus) lost 37%. But most other stocks either lost only a little or actually gained. Materials fell 10% but consumer discretionary gained 4.5%, consumer staples gained 21%, energy gained 12%, financial services gained 19%, health care gained 29%, industrials gained 7% and utilities gained 2%. Equally weighted, the S&P500 index lost only 3%. So there was a strong small cap effect. The stocks that gained were the very small cap stocks with market caps below $10 billion. Some 138 companies with market caps between $5-10 billion gained 4% on average and 157 companies with market caps below $5 billion gained on average 23%.

While the BSEYD model has been shown to be useful in predicting S&P500 declines, it is silent on the NASDAQ technology index of the largest 100 stocks by market capitalization, the NDX100, see Exhibit 12. This index with a major Internet component had a spectacular increase during a period where many thought the Internet companies would prosper despite price earnings ratios of 100 plus and many with no earnings at all. Valuation attempts were made to justify these high prices; see Schwartz and Moon [2000] for one such example. Predicting the top of this bubble was not easy as the Internet index (not shown) fell 17% one day and then proceeded to reach new highs. For example, the noted investor George Soros lost some $5 billion of the $12 billion in the Quantum hedge fund during this crash.

The NDX100 peaked at 4816.35 on March 24, 2000 starting from 398.26 in 1994. In the decline it fell to 795.25 on October 8, 2002. Below we apply the Shiryaev and Zhitlukhin model to both the questions when to close a long and a short positions on NDX100 for various entering dates. The results appear in Exhibits 13–16. For a long position we assume that the bubble bursts by the end of 2000, and for a short position we assume that the market recovery starts by the end of 2003.

Depending upon the long position entry, the exit yielded about 75% of the maximum price with investor gains of about 40-60% a year. Again, like with AAPL, the speed of decrease $\mu_2 = -\mu_1$ provides optimal results. The shorting analysis was also successful for

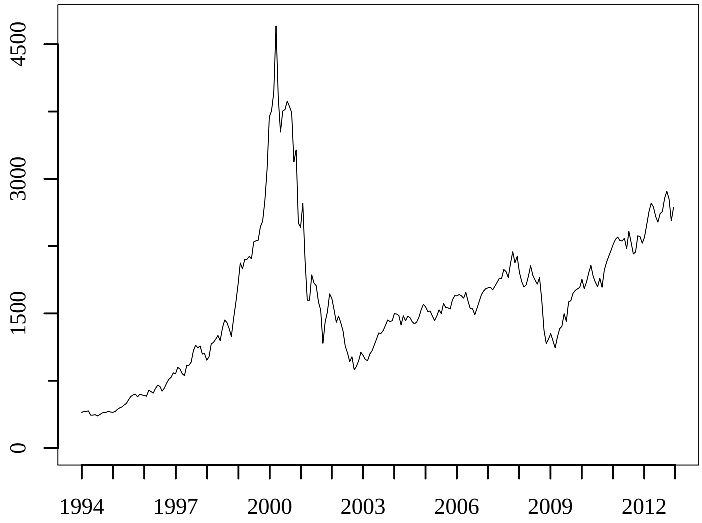

Exhibit 12: NDX100 index, 1994-2012.

the model with the exits gaining about 25-45% a year (for $\mu_2 = -\mu_1$) and getting close to the minimum over the time period considered.

Entering date	Entering price	μ_1	σ	T	Exit date	Exit price	% of max.	Growth rate, %
				$\mu_2 = -\mu_1$				
1994-12-30	404.27	0.014	0.105	15.15	2000-04-12	3633.63	77.23	41.48
1995-12-29	576.23	0.082	0.124	12.63	2000-04-12	3633.63	77.23	42.89
1996-12-31	821.36	0.105	0.130	10.09	2000-04-12	3633.63	77.23	45.26
1997-06-30	957.30	0.109	0.134	8.84	2000-04-12	3633.63	77.23	47.81
1997-12-31	990.83	0.101	0.141	7.56	2000-04-13	3553.81	75.54	55.88
1998-06-30	1337.34	0.117	0.142	6.32	2000-04-12	3633.63	77.23	55.85
1998-12-31	1836.01	0.134	0.156	5.04	2000-04-13	3553.81	75.54	51.37
1999-06-30	2296.77	0.140	0.164	3.80	2000-04-13	3553.81	75.54	55.00
				$\mu_2 = -2\mu_1$				
1996-12-31	821.36	0.105	0.130	10.09	2000-04-11	3909.21	83.09	47.54
1997-06-30	957.30	0.109	0.134	8.84	2000-04-11	3909.21	83.09	50.51
1997-12-31	990.83	0.101	0.141	7.56	2000-04-12	3633.63	77.23	56.95
1998-06-30	1337.34	0.117	0.142	6.32	2000-04-12	3633.63	77.23	55.85
1998-12-31	1836.01	0.134	0.156	5.04	2000-04-12	3633.63	77.23	53.26
1999-06-30	2296.77	0.140	0.164	3.80	2000-04-12	3633.63	77.23	58.09
				$\mu_2 = -3\mu_1$				
1996-12-31	821.36	0.105	0.130	10.09	1998-08-31	1140.34	24.24	19.69
1997-06-30	957.30	0.109	0.134	8.84	1998-08-31	1140.34	24.24	14.95
1997-12-31	990.83	0.101	0.141	7.56	2000-04-10	3998.26	84.98	61.35
1998-06-30	1337.34	0.117	0.142	6.32	2000-04-10	3998.26	84.98	61.47
1998-12-31	1836.01	0.134	0.156	5.04	2000-04-11	3909.21	83.09	59.14
1999-06-30	2296.77	0.140	0.164	3.80	2000-04-11	3909.21	83.09	67.69

Exhibit 13: Results of applying the model to a long position on NDX-100 index.

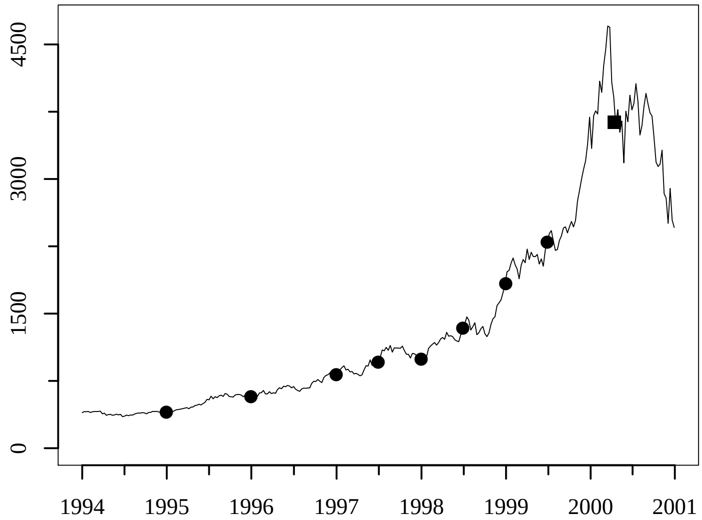

Exhibit 14: Entering and exit dates for a long position on the NDX100 when $\mu_2 = -\mu_1$. The dots indicate the eight entering dates, and the square indicates the exit date on April 12, 2000.

Entering date	Entering price	μ_1	σ	T	Exit date	Exit price	% of min.	Growth rate, %
				$\mu_2 = -\mu_1$				
2001-01-31	2593.00	-0.309	0.412	7.31	2002-11-27	1125.67	139.90	46.01
2001-06-29	1830.19	-0.287	0.395	6.27	2003-01-09	1076.05	133.73	35.13
2002-02-28	1359.22	-0.226	0.345	4.64	2003-03-13	1029.79	127.98	26.80
				$\mu_2 = -0.25\mu_1$				
2001-01-31	2593.00	-0.309	0.412	7.31	2003-03-17	1077.01	133.85	41.78
2001-06-29	1830.19	-0.287	0.395	6.27	2003-04-22	1102.44	137.01	28.32
2002-02-28	1359.22	-0.226	0.345	4.64	2003-06-03	1198.57	148.96	10.00
				$\mu_2 = -0.5\mu_1$				
2001-01-31	2593.00	-0.309	0.412	7.31	2003-01-09	1076.05	133.73	45.70
2001-06-29	1830.19	-0.287	0.395	6.27	2003-03-13	1029.79	127.98	34.18
2002-02-28	1359.22	-0.226	0.345	4.64	2003-04-17	1083.56	134.66	19.97
				$\mu_2 = -2\mu_1$				
2001-01-31	2593.00	-0.309	0.412	7.31	2003-03-17	1077.01	133.85	41.78
2001-06-29	1830.19	-0.287	0.395	6.27	2003-03-17	1077.01	133.85	31.37
2002-02-28	1359.22	-0.226	0.345	4.64	2003-01-14	1094.87	136.07	24.66

Exhibit 15: Results of applying the model to a short position on NDX-100 index.

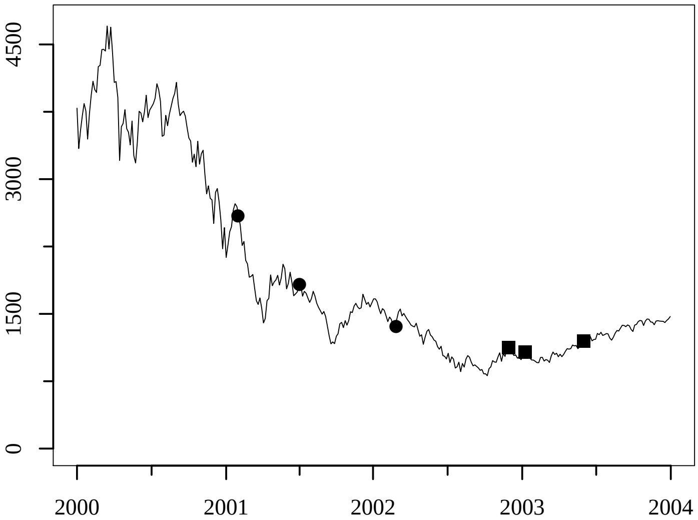

Exhibit 16: Entering and exit dates for a short position on the NDX100 when $\mu_2 = -\mu_1$. The dots indicate the three entering dates, and the squares indicate the three exit dates.

Appendix

The model of Shiryaev and Zhitlukhin [2012b][2] assumes that the prices are modeled by *geometric Brownian motion with a disorder* $(S_t)_{t\geq 0}$, which is a stochastic process defined by the differential:

$$dS_t = [\mu_1 \mathbb{I}(t < \theta) + \mu_2 \mathbb{I}(t \geqslant \theta)]S_t dt + \sigma dB_t,$$

where $\mu_1 > 0 > \mu_2$ or $\mu_1 < 0 < \mu_2$, $\sigma > 0$ are constant parameters, $B = (B_t)_{t\geq 0}$ is a standard Brownian motion, and θ is an unknown moment of trend reversal[3], when the drift coefficient of the process S changes from value μ_1 to value μ_2.

We observe a sequence of asset prices $P_0, P_1, \ldots, P_N$, which initially has a positive trend and at some unknown moment of time the trend reverses. It is assumed that the trend will definitely reverse before the final time, N. Choosing an entering time $n < N$ for opening a long position (i.e. buying some amount of the asset), we want to find the optimal moment of time to close the position and sell the assets while sequentially observing the prices $P_n, P_{n+1}, \ldots, P_N$. Let P_k be the *daily* closing values (of AAPL or NDX100), although other time scales can be considered as well. The model also applies to the case when the prices have initially a negative trend and one opens a short position (i.e. sells assets that he or she does not hold with the objective to return them later after buying for a lower price).

The process S_t runs in continuous time $t \geq 0$, and we choose the time scale where each trading day has length $\Delta t = 0.01$ (for convenience), and $t = 0$ represents the entering date n, while $t = T$ represents the final date N, where $T = (N - n)\Delta t$. Thus, the observed sequence of prices P_k represents the values of the process S_t at the moments of time $t = (k - n)\Delta t$.

Adopting the *Bayesian approach*, we assume that θ is a random variable taking values in $[0, T]$ and independent of B. Since in practice it is difficult to determine the actual structure of the distribution of θ, we consider "the worst" case – when θ is *uniformly distributed* on $[0, T]$ (as the uniform distribution has the maximum entropy on a finite interval).

Mathematically, the moment when one closes the position is represented by a *stopping time*[4] τ of the observable process S. If a long position is opened on date n, the problem consists in finding the stopping time $\tau^*_{\text{long}} \leq T$ that maximizes the mean price at τ^*_{long}; if a short position is open, we seek for the $\tau^*_{\text{short}} \leq T$ which minimizes the mean closing price. In other words

$$\mathsf{E}S_{\tau^*_{\text{long}}} = \sup_{\tau \leq T} \mathsf{E}S_\tau, \qquad \mathsf{E}S_{\tau^*_{\text{short}}} = \inf_{\tau \leq T} \mathsf{E}S_\tau,$$

where E denotes mathematical expectation, and $\sup_{t\leq T}$, and $\inf_{t\leq T}$ denote the supremum and the infimum over all stopping times $\tau \leq T$.

The Shiryaev and Zhitlukhin model for finding the optimal τ^* is based on the observation of the process $\psi = (\psi_t)_{t\geq 0}$, called the *Shiryaev–Roberts statistic* (see e. g. Poor and Hadjiliadis, 2009), on the time interval $[0, T]$, specified by

$$\psi_t = \frac{1}{T}\exp(-\mu X_t - \mu^2 t/2)\int_0^t \exp(\mu X_s + \mu^2 s/2)ds,$$

where $X_t = \sigma^{-1}(\log(S_t/S_0) - (\mu_1 - \sigma^2/2)t)$, and $\mu = (\mu_1 - \mu_2)/\sigma$. The method closes a position (a long position as well as a short one) at the first time τ^* when the process ψ_t crosses some time-dependent level $a(t)$:

$$\tau^* = \inf\{t \geq 0 : \psi_t \geq a(t)\}.$$

The function $a(t)$ depends on the parameters μ_1, μ_2, σ, T and can be found from a certain integral equation, see (Shiryaev and Zhitlukhin, 2012b) for details. This function is decreasing and $a(T) = 0$, so ψ_t always crosses it by time T.

To apply the method, we must estimate the parameters μ_1, μ_2, and σ. The values of μ_1 and σ are found using the data $P_0, \ldots, P_n$. Under the assumption of geometric Brownian motion, the sequence $\{\xi_k\}_{k=1}^n$, $\xi_k = \log(P_k/P_{k-1})$, consists of independent normal random variables with mean $(\mu_1 - \sigma^2/2)\Delta t$ and standard deviation $\sigma\Delta t$. So we apply the standard formulae

$$\sigma = \sqrt{\frac{1}{(n-1)\Delta t}\sum_{k=1}^n (\xi_k - \bar{\xi})^2}, \qquad \mu_1 = \bar{\xi}/\Delta t + \sigma^2/2, \qquad \text{where } \bar{\xi} = \frac{1}{n}\sum_{k=1}^n \xi_k.$$

The choice of μ_2 is subjective. In our applications we mainly use $\mu_2 = -\mu_1$, so that, in the model, the decrease of the price has the same "speed" as the increase. We know that prices of financial assets generally fall faster than they rise but in a bubble both the increase and decrease can be similar as the calculations below show. We also consider $\mu_2 = -0.5\mu_1$, $\mu_2 = -2\mu_1$, and $\mu_2 = -3\mu_1$, which however do not give any significant improvement of $\mu_2 = -\mu_1$.

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Notes

[1]The BSEYD model relates the yield on stocks, measured by the ratio of earnings to stock prices, to the yield on nominal Treasury bonds. When the bond yield is too high, there is a shift out of stocks into bonds. If the adjustment is large, it causes an equity market correction (a decline of 10% within one year). See e. g. (Ziemba, 2003) for details.

[2]The model extends the previous result by Novikov and Shiryaev [2009]. Other papers that consider similar models related to detecting changes in price processes include Beibel and Lerche [1997], Gapeev and Peskir [2006], and Ekström and Lindberg [2013].

[3]The moment θ of trend reversal in the model is commonly called the *moment of disorder*. This terminology comes from the theory of quality control, where similar models were first applied.

[4]A *stopping time* τ of a process X defined on some probability space $(\Omega, \mathcal{F}, \mathsf{P})$ is a mapping $\tau \colon \Omega \to [0, \infty)$ such that the set $\{\omega : \tau(\omega) \le t\}$ belongs to the σ-algebra $\sigma(X_s; s \le t)$ for any $t \ge 0$, see e. g. (Liptser and Shiryaev, 2000). It represents the idea that a decision to stop at a time t should be based only on the information obtained from the paths of the process X up to time t.

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