

[Georgy Chabakauri](#), [Kathy Yuan](#),  
[Konstantinos E. Zachariadis](#)

## Multi-asset noisy rational expectations equilibrium with contingent claims

### Working paper

**Original citation:**

Chabakauri, Georgy, Yuan, Kathy and Zachariadis, Konstantinos (2014) *Multi-asset noisy rational expectations equilibrium with contingent claims*. Working papers, Social Science Research Network (SSRN), Rochester, USA.

Originally available from the [Social Science Research Network \(SSRN\)](#)

Financial support from the Paul Woolley Centre, LSE

This version available at: <http://eprints.lse.ac.uk/60736/>

Available in LSE Research Online: January 2015

© 2014 The Authors

LSE has developed LSE Research Online so that users may access research output of the School. Copyright © and Moral Rights for the papers on this site are retained by the individual authors and/or other copyright owners. Users may download and/or print one copy of any article(s) in LSE Research Online to facilitate their private study or for non-commercial research. You may not engage in further distribution of the material or use it for any profit-making activities or any commercial gain. You may freely distribute the URL (<http://eprints.lse.ac.uk>) of the LSE Research Online website.

# Multi-Asset Noisy Rational Expectations Equilibrium with Contingent Claims\*

Georgy Chabakauri<sup>†</sup>  
LSE

Kathy Yuan<sup>‡</sup>  
LSE

Konstantinos E. Zachariadis<sup>§</sup>  
LSE

June 2014

## Abstract

We consider a noisy rational expectations equilibrium in a multi-asset economy populated by informed and uninformed investors, and noise traders. Informed investors privately observe an aggregate risk factor affecting the probabilities of different states of the economy. Uninformed investors attempt to extract that information from asset prices, but full revelation is prevented by noise traders. We relax the usual assumption of normally distributed asset payoffs and allow for assets with more general payoff distributions, including contingent claims, such as options and other derivatives. We show that assets reveal information about the risk factor only if they help span the exposure of probabilities of states to the risk factor. When the market is complete, we provide equilibrium asset prices and optimal portfolios of investors in closed form. In incomplete markets, we derive prices and portfolios in terms of easily computable inverse functions.

*Journal of Economic Literature Classification Numbers:* D82, G12, G14.

*Keywords:* asymmetric information, rational expectations, learning from prices, contingent claims, derivative securities.

---

\*We are grateful to Suleyman Basak, Margaret Bray, Christian Julliard, Igor Makarov and Dimitri Vayanos for helpful comments. We also thank the Paul Woolley Centre at the LSE for financial support. All errors are our own.

<sup>†</sup>E-mail address: G.Chabakauri@lse.ac.uk.

<sup>‡</sup>E-mail address: K.Yuan@lse.ac.uk.

<sup>§</sup>E-mail address: K.Zachariadis@lse.ac.uk.

# 1. Introduction

The informational role of prices has been in the forefront of the economic literature since the seminal work of Hayek (1945). Investors in financial markets use their private information to extract gains from trading financial securities. Their trades impound information into the prices of assets, from which the information can be partially recovered by other investors. Informed investors often trade in a multitude of correlated securities, which creates a diffusion of information across securities and makes their prices interdependent because the price of each security can assist in inferring the payoff distribution of any other. The economic literature typically studies the informational role of prices in restrictive settings with normally distributed asset payoffs, which do not allow studying markets for assets with positive payoffs and for derivative securities with contingent payoffs, such as stock options. In this paper, we propose a multi-asset noisy rational expectations equilibrium (REE) model where private information can be contained in the prices of all securities and where payoffs of securities can be positive and contingent on the payoffs of other securities.

We consider a single-period multi-asset economy with a finite but arbitrary number of discrete states. The assets can have positive payoffs, and can be derivative securities, such as options. The probabilities of states are functions of an aggregate risk factor. The economy is populated by three groups of investors, informed and uninformed investors with constant relative risk aversion (CARA) preferences over terminal wealth, and noise traders with exogenous random asset demands drawn from a multivariate normal distribution. The informed investors observe the realization of the risk factor, whereas the uninformed investors use asset prices to update their initial prior on the risk factor. The presence of noise traders is a friction that prevents prices from being fully revealing. In this economy, we solve for equilibrium asset prices and investors' portfolios, and establish conditions under which asset prices reveal information about the risk factor. Our solution approach differs from the long-standing tradition of solving noisy REE models by 'guessing and verifying'. Instead, we employ a direct computation of equilibria.

The tractability of our analysis stems from two innovations. First, we define the probability measure directly over the states of the economy rather than over asset payoffs. Second, we use a specific structure for the probabilities of states conditional on observing the risk factor. This structure is inspired by multinomial logit models, widely employed in

econometrics, and is such that the log-likelihood ratios of different states are linear functions of the risk factor. The loadings on the risk factor in the structure of probabilities are interpreted as the economy's exposure to risk, and the distributions of asset payoffs are implied by the probabilities of states. The logit-like conditional probabilities can be chosen in such a way that our conditional multinomial payoff distributions converge to various continuous-state distributions from the exponential family, including the normal distribution and distributions with positive support, as the number of states increases. We also allow for non-redundant derivative securities with contingent payoffs that have even more complex payoff distributions, which are implied by the payoff distributions of underlying securities. Furthermore, for a fixed realization of aggregate risk, the conditional probabilities can be calibrated to approximate any continuous-space distribution.

The structure of probabilities of states makes the informed investor's portfolio a linear function of the risk factor. This linear function is then combined with noise traders' demands in the market clearing conditions. As a result, the prices of assets reveal a linear combination of the risk factor and noise traders' demands, which allows us to solve the information filtering problem of the uninformed investor. Consequently, when the number of traded non-redundant securities equals the number of states, that is, the financial market is complete, we find equilibrium prices and portfolios in terms of elementary functions for some tractable probability density functions of the aggregate risk factor. We also extend our baseline analysis to the case of general probabilities of states and general distributions of the risk factor and noise trader demands, and obtain the equilibrium in closed form in terms of easily computable integrals.

When the number of traded assets is less than the number of states, so that the financial market is incomplete, we obtain the equilibrium in terms of easily computable inverse functions. We solve the incomplete market model under the assumption that the traded assets span the exposure of the probabilities of states to the risk factor. We show that this assumption is satisfied in many realistic economies, such as CARA-normal economies with one risky asset, CARA-normal economies with options, and in economies with risky corporate debt and equity. Our spanning condition has an important economic interpretation. It reveals that the tractability of our model stems not from market completeness but from the ability to use private information to span the exposure to the risk factor.

We find a surprising result: only those assets that help span the economy's exposure to the aggregate risk factor have trading volumes that depend on the realization of that factor. All other securities are *informationally redundant* in the sense that their trading volumes and prices do not reveal new information despite the fact that they are non-redundant from the perspective of completing the market. Using our condition for informational redundancy, we demonstrate that adding derivatives to standard CARA-normal models with one risky asset, such as the model of Grossman and Stiglitz (1980), does not reveal more information about the realization of the risk factor.

Our intuition is as follows. The informed investors have incentives to allocate more wealth to states with higher probabilities. Therefore, they invest part of their wealth in a portfolio that replicates the exposure to the aggregate risk because this risk determines the probabilities of states. Moreover, asset holdings in the replicating portfolio are proportional to the size of the observed risk factor due to the logit-like structure of probabilities. Therefore, in general, trading volumes reveal private information to uninformed investors. The assets that do not span the exposure to the risk factor are still held by the investors because they are non-redundant from the perspective of completing the market. However, the latter assets do not have demand components that depend on the risk factor, and hence their trading volumes do not reveal new information.

The conditions for informational non-redundancy of assets allow us to study price discovery in derivatives markets. One standard textbook narrative is that call options are preferred by traders who expect the price of the underlying to go up while put options by those who expect it to drop. This narrative is however done in the context of standard derivatives pricing models in which information is symmetric and options are redundant [e.g., Black and Scholes (1973)]. In contrast, our model offers a rationale for information driven trades on derivatives and demonstrates that if derivatives help span the exposure to risk, then their prices and trading volumes reveal new information in addition to that revealed by the underlying asset, consistent with the empirical literature on price discovery in option markets [e.g., Easley, O'Hara, and Srinivas (1998); Chan, Chung, and Fong (2002); Chakravarty, Gulen, and Mayhew (2004); Pan and Poteshman (2006)].

In the case of complete markets we derive comparative statics for asset prices and investors' portfolios, which help us disentangle the information and substitution effects.

The comparative statics are derived in closed form in terms of risk-neutral covariances of asset payoffs with the exposure to risk, and the risk-neutral variance-covariance matrix of asset payoffs. The presence of asymmetric information makes asset prices more sensitive to the variations in the risk factor and noise traders' demand. We demonstrate that informed investor's demand for an asset is a downward sloping function of that asset's own price, holding the prices of all other assets fixed. In contrast, the demand of the uninformed investor can be upward sloping in an asset's own price when the information effect dominates. The latter effect arises because high asset prices may signal positive information received by the informed investor, which may increase asset demands.

Our paper is related to large literature on noisy REE models, which were pioneered by Grossman (1976), Grossman and Stiglitz (1980) and Hellwig (1980). These works and their various extensions typically consider economies with CARA investors and one risky asset with normally distributed payoffs. Admati (1985) extends these CARA-normal models to the case of multiple securities and shows that many insights from the single asset model cannot be extrapolated to the multi-asset case. The latter work also provides comparative statics and demonstrates that asset demands can be upward sloping, similarly to this paper. Brennan and Cao (1996) consider an economy with a risky asset with normally distributed payoff and a power derivative written on it. Other related works include Diamond and Verrecchia (1981), Marín and Rahi (2000), Vives (2008), and Kurlat and Veldkamp (2013). In contrast to the above literature, we allow for assets with more general payoff distributions, including derivative securities, and provide new results regarding price discovery in securities markets.

There is a growing literature that departs from CARA-normal frameworks. Yuan (2005) studies a two-state non-linear REE where cash flows are normally distributed but the states are endogenously determined by prices, leading to truncated normal payoffs. Breon-Drish (2010) considers an economy with non-normal payoffs and demonstrates that observing the trading volume of informed and noise traders provides valuable information, and non-normality may give rise to jumps in asset prices. Breon-Drish (2012) provides closed-form solutions and proves existence and uniqueness of REE in economies with asset payoffs that have distributions from the exponential family. Other related works include Vanden (2008), Bernardo and Judd (2000), and Barlevy and Veronesi (2000). The main difference of our paper from the above literature is that we allow for multiple assets,

general payoff distributions, and contingent claims. Albagli, Hellwig, and Tsyvinski (2013) consider an REE with risk-neutral investors and limits to arbitrage in the form of position limits, and hence their work is different from our no-arbitrage model.

Our paper is also related to the literature that studies the informational role of non-redundant derivatives. Back (1993) provides a micro-foundation on stochastic volatility based on asymmetric information in a dynamic Kyle (1985) model where a single informed investor trades in the stock and a single call option. Biais and Hillion (1994) have a static model with a stock and show that the introduction of a single option can have ambiguous effects on the dissemination of information. Malamud (2014) studies an REE with options in a continuous-space complete-markets economy in a paper concurrent with ours. He characterizes REE in terms of fixed points of operators and finds conditions for price discovery under general preferences.

Our paper differs from the above literature in that we consider multi-asset economies both with complete and incomplete markets, provide closed-form solutions not only for options but also for general contingent claims, and our framework is easily extendable to economies with multiple risk factors. Furthermore, considering incomplete markets allows us to disentangle two effects of adding assets to the economy: the effects of completing the markets and revealing information. We also provide new conditions for informational redundancy of assets in terms of the exposures of the probabilities of states to risk factors.

The structure of the paper is as follows. Section 2 describes the model, investors' optimizations, and distributional assumptions. Section 3 solves for equilibria both in economies with complete and incomplete financial markets. Section 4 provides the analysis of equilibrium. Section 5 extends the model to the case of general distributions and probabilities of states. Section 6 concludes. Appendix A provides the proofs for all results reported in the main text. Appendix B contains some benchmark cases. Appendix C contains auxiliary results.

## 2. Model

### 2.1. Securities Markets and Information Structure

We consider a single-period exchange economy with two dates  $t = 0$  and  $t = T$ , and  $N$  states  $\omega_1, \dots, \omega_N$  at the terminal date, where  $N \geq 2$ . The economy is populated by three representative investors, informed and uninformed investors, labeled  $I$  and  $U$ , and noise traders. Each representative investor stands for a group of a continuum of identical investors with unit mass. Investors  $I$  and  $U$  have CARA preferences over terminal wealth and risk aversions  $\gamma_I$  and  $\gamma_U$ . The investors can trade  $M \geq 2$  zero net supply securities: one riskless bond paying \$1 at  $T$ , and  $M - 1$  risky assets with state-contingent terminal payoff  $C_m(\omega_n)$  in state  $\omega_n$ , where  $m = 1, \dots, M - 1$  and  $n = 1, \dots, N$ . These assets can be Arrow-Debreu securities, options, or other derivative securities, and are assumed to be non-redundant in the sense that no asset has payoffs that can be replicated by trading other assets. The investors are competitive and do not have impact on prices.

The probabilities of states  $\omega_n$  are functions of a shock  $\varepsilon \in \mathbb{R}$ , and are denoted by  $\pi_n(\varepsilon)$ . Shock  $\varepsilon$  has a prior probability density function (PDF)  $\varphi_\varepsilon(x)$ . We think of  $\varepsilon$  as an aggregate risk factor that affects the probabilities of the states of the economy and hence the payoff distributions of all securities in the economy. Before the markets open (i.e., at time  $t = -1$ ), the informed investors observe  $\varepsilon$ . The uninformed investors have only public information. Noise traders have exogenous random demands  $\nu = (\nu_1, \dots, \nu_{M-1})^\top$  with joint normal distribution  $N(0, \Sigma_\nu)$ , where  $\Sigma_\nu$  is a  $(M - 1) \times (M - 1)$  symmetric positive-definite matrix. Random demands  $\nu$  prevent asset prices from being fully revealing.

We denote the vector of observed time  $t = 0$  prices of the risky assets by  $p = (p_1, \dots, p_{M-1})^\top$ , the vector of risky assets' payoffs in state  $\omega_n$  by  $\Pi_n = (C_1(\omega_n), \dots, C_{M-1}(\omega_n))^\top$ , and the vector of asset  $m$ 's payoffs in different states by  $C_m = (C_m(\omega_1), \dots, C_m(\omega_N))^\top$ . The price of the riskless asset is set to  $p_0 = e^{-rT}$ , where  $r$  is an exogenously set risk-free rate of return.<sup>1</sup> The prices of risky assets are endogenously determined in equilibrium. Finally, by  $P(\varepsilon, \nu) \in \mathbb{R}^{M-1}$  we denote the vector of equilibrium prices as functions of shock  $\varepsilon$  and noise  $\nu$ .

---

<sup>1</sup>In models with utility over terminal wealth risk-free rate  $r$  is indeterminate and is set exogenously.



## 2.2. Investors' Optimization and Definition of Equilibrium

Each investor  $i = I, U$  is endowed with initial wealth  $W_{i,0}$ , and allocates it to buy  $\alpha_i$  units of the riskless asset and  $\theta_{i,m}$  units of risky asset  $m$ . By  $\theta_i = (\theta_{i,1}, \dots, \theta_{i,M-1})^\top$  we denote the vector of units of risky assets purchased by investor  $i$ . The budget constraints of investors  $I$  and  $U$  at time  $t = 0$  are given by  $W_{i,0} = \alpha_i p_0 + p^\top \theta_i$ . Investor  $i$ 's wealth at time  $t = T$  and state  $n$  is then given by  $W_{i,T,n} = \alpha_i + \Pi_n^\top \theta_i$ . In what follows, the subscript  $n$  will be dropped to denote a random variable in an uncertain state. Substituting out  $\alpha_i$  we obtain the budget constraint in the following form:  $W_{i,T} = W_{i,0}e^{rT} + (\Pi - e^{rT}p)^\top \theta_i$ . The informed and the uninformed investors solve the following utility maximization problems

$$\max_{\theta_I} \mathbb{E} \left[ -e^{-\gamma_I W_{I,T}} \mid \varepsilon, p \right], \quad (1)$$

$$\max_{\theta_U} \mathbb{E} \left[ -e^{-\gamma_U W_{U,T}} \mid P(\varepsilon, \nu) = p, p \right], \quad (2)$$

respectively, subject to each investors' self-financing budget constraint

$$W_{i,T} = W_{i,0}e^{rT} + (\Pi - e^{rT}p)^\top \theta_i, \quad i = I, U. \quad (3)$$

The solutions to the above optimization problems give investors' optimal portfolios of risky assets  $\theta_I^*(p; \varepsilon)$  and  $\theta_U^*(p)$ . The prices  $p$  should be such that all the markets for the risky securities clear. More formally, the definition of equilibrium is as follows.

**Definition 1.** *A competitive noisy rational expectations equilibrium is a set of asset prices  $P(\varepsilon; \nu)$  and investor asset holdings  $\theta_I^*(p; \varepsilon)$  and  $\theta_U^*(p)$  such that  $\theta_I^*$  and  $\theta_U^*$  solve optimization problems (1) and (2) subject to self-financing budget constraints (3), taking asset prices as given, and the market clearing conditions are satisfied:*

$$\theta_I^*(P(\varepsilon, \nu); \varepsilon) + \theta_U^*(P(\varepsilon, \nu)) + \nu = 0. \quad (4)$$

## 2.3. Probability Distributions

To solve the model in closed form, we consider probabilities of states  $\pi_n(\varepsilon)$  given by:

$$\pi_n(\varepsilon) = \frac{e^{a_n + b_n \varepsilon}}{\sum_{k=1}^N e^{a_k + b_k \varepsilon}}, \quad n = 1, \dots, N. \quad (5)$$

The structure of probabilities is similar to that of probabilities in multinomial logit models, widely used in econometrics. When  $\varepsilon = 0$ , by properly choosing parameters  $a_n$ , states  $\omega_n$  can have a multinomial distribution that approximates a particular desired continuous distribution in the limit as  $N \rightarrow \infty$ . We label vector  $b = (b_1, \dots, b_N)^\top$  as *economy's exposure to the aggregate risk factor*, because it determines the deviations of probabilities  $\pi_n(\varepsilon)$  from benchmark probabilities  $\pi_n(0)$  in response to shock  $\varepsilon$ . We choose to work with a discrete state-space because it gives rise to easily invertible matrix operators, rather than less tractable integral operators when the state-space is continuous.

Shock  $\varepsilon$  is a scalar random variable with generalized normal distribution  $\hat{N}(\mu_\varepsilon, \sigma_\varepsilon^2)$ , mean  $\mathbb{E}[\varepsilon] = \mu_\varepsilon$  and variance  $\text{var}[\varepsilon] = \sigma_\varepsilon^2$ , which has PDF  $\varphi_\varepsilon(x)$  given by:

$$\varphi_\varepsilon(x) = \frac{\left(\sum_{k=1}^N e^{a_k + b_k x}\right) e^{-0.5(x-\mu_0)^2/\sigma_0^2}}{\int_{-\infty}^{\infty} \left(\sum_{k=1}^N e^{a_k + b_k x}\right) e^{-0.5(x-\mu_0)^2/\sigma_0^2} dx}. \quad (6)$$

PDF (6) allows us to obtain the equilibrium in terms of elementary functions. Distribution (6) is given in terms of vectors  $a = (a_1, \dots, a_N)^\top$  and  $b = (b_1, \dots, b_N)^\top$ , and scalars  $\mu_0$  and  $\sigma_0^2$ . For fixed  $a$  and  $b$  we can pick  $\mu_0$  and  $\sigma_0^2$  so that  $\varepsilon$  has any desired mean  $\mu_\varepsilon$  and variance  $\sigma_\varepsilon^2$ . The relationship between  $(\mu_0, \sigma_0^2)$  and  $(\mu_\varepsilon, \sigma_\varepsilon^2)$  is given by Equations (C.1) and (C.2) in Appendix C. In Section 5 we extend the analysis to general probabilities  $\pi_n(\varepsilon)$  and PDFs  $\varphi_\varepsilon(x)$  and  $\varphi_\nu(x)$  for shock  $\varepsilon$  and noisy demands  $\nu$ .

By varying vectors  $a$  and  $b$ ,  $\pi_n(\varepsilon)$  have flexible shapes. Figure 1 shows probabilities  $\pi_n(\varepsilon)$  and PDF function  $\varphi_\varepsilon(x)$  for an example with  $N = 100$ . The probabilities are plotted against the risky asset payoff  $C(\omega_n) = 300(n-1)/(N-1)$ . Vectors  $a$  and  $b$  are calibrated in such a way that  $\pi_n(1)$  and  $\pi_n(-1)$  are discrete approximations of gamma distributions with shape and scale parameter pairs (1,2) and (5,1), respectively. Panel (b) shows function  $\varphi_\varepsilon(x)$  along with the PDF of a normal distribution  $N(\mu_\varepsilon, \sigma_\varepsilon^2)$  for  $\mu_\varepsilon = 0$ ,  $\sigma_\varepsilon = 1$ . We note that the two PDFs are very close to each other.

An important special limiting case (i.e., when  $N \rightarrow \infty$ ) of our model is the standard CARA-normal model with one risky asset with random payoff  $C(\omega) \sim N(\varepsilon, \sigma_C^2)$ , an informed investor who observes mean  $\mathbb{E}[C(\omega)] = \varepsilon$ , and an uninformed investor with prior distribution  $\varepsilon \sim N(\mu_\varepsilon, \sigma_\varepsilon^2)$ . This is the Grossman and Stiglitz (1980) model without costs of information acquisition.

Such an economy can be approximated in our framework as follows. Let  $M = 2$  and

consider the following form for the payoff of the single risky asset and parameters  $a$  and  $b$ :  $C(\omega_n) = -A + nh$ , where  $h = 2A/N$ ,  $a_n = -0.5C(\omega_n)^2/\sigma_C^2$  and  $b_n = C(\omega_n)/\sigma_C^2$ , all for  $n = 1, \dots, N$ . For large  $A$  and  $N$  we observe that  $(\sum_{k=1}^N e^{a_k + b_k \varepsilon} h) e^{-0.5\varepsilon^2/\sigma_C^2} \approx \int_{-\infty}^{\infty} e^{-0.5(C-\varepsilon)^2/\sigma_C^2} dC = \sqrt{2\pi\sigma_C^2}$ . Therefore, as  $A, N \rightarrow \infty$ , using Equations (5) and (6) and some algebra, we obtain point-wise convergences

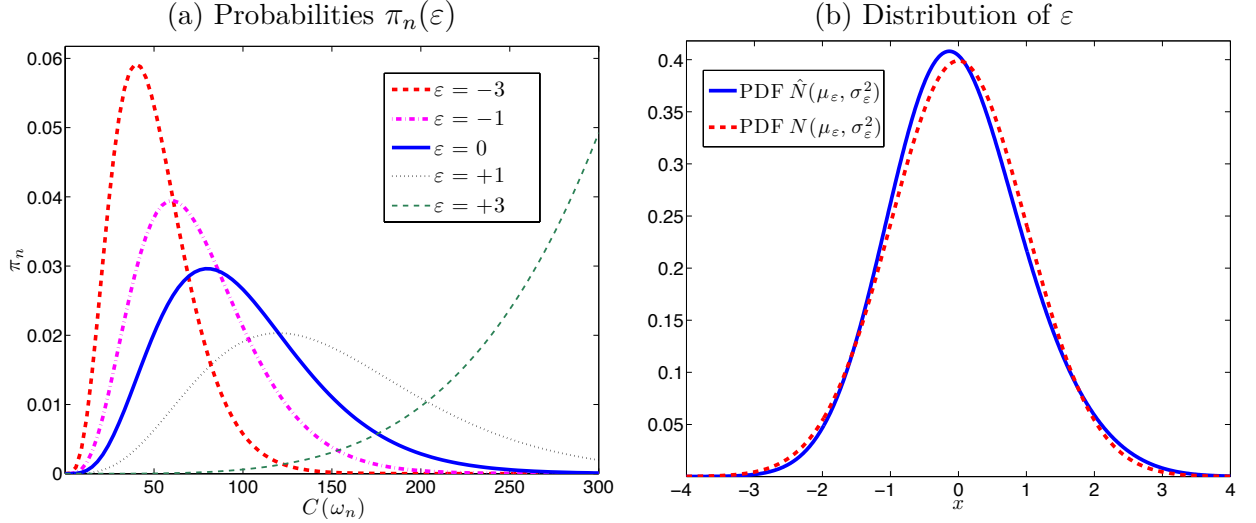
$$\pi_n(\varepsilon) = \frac{e^{-0.5(C(\omega_n)-\varepsilon)^2/\sigma_C^2} h}{\sum_{k=1}^N e^{-0.5(C(\omega_k)-\varepsilon)^2/\sigma_C^2} h} \rightarrow \frac{e^{-0.5(C-\varepsilon)^2/\sigma_C^2} dC}{\sqrt{2\pi\sigma_C^2}}, \quad \varphi_\varepsilon(x) \rightarrow \frac{e^{-0.5(x-\mu_\varepsilon)^2/\sigma_\varepsilon^2}}{\sqrt{2\pi\sigma_\varepsilon^2}}. \quad (7)$$

Consequently, our multinomial model is approximately CARA-normal for large  $A$  and  $N$ .

More generally, our model includes as special limiting cases economies with an asset with payoff  $C(\omega)$ , which has PDF conditional on observing  $\varepsilon$  given by  $\exp\{a(C) + b(C)\varepsilon + c(\varepsilon)\}$ . The latter function can be obtained in the limit as in the case of the CARA-normal model. We can solve such models also with added options and other derivatives written on payoff  $C(\omega)$ . We note that conditional PDF  $\exp\{a(C) + b(C)\varepsilon + c(\varepsilon)\}$  is more general than the one in Breon-Drish (2012), where coefficient  $b(C)$  is linear. Furthermore, the distribution of asset payoffs such as  $(C - K)^+$  is even more complex than the latter PDF. Our paper is the first to allow for such complex distributions in asymmetric information economies.

One of the disadvantages of CARA-normal models, popular in the literature, is that asset payoffs can be negative. Moreover, it is very difficult to include assets with nonlinear payoffs, such as put and call options. Our model is free from these disadvantages, and allows extra flexibility in modeling probability distributions and asset payoffs. To the best of our knowledge ours is the first noisy REE model that admits closed form solutions in the multi-asset case and where joint normality of assets' payoffs is not required.

**Remark 1 (Multi-dimensional Shock  $\varepsilon$ ).** Our model can be easily generalized to the case of multi-dimensional shocks  $\varepsilon$ . In this case, the probabilities of states  $\omega_n$  are given by  $\pi_n(\varepsilon) = \exp(a_n + b_n^\top \varepsilon) / \sum_{k=1}^N \exp(a_k + b_k^\top \varepsilon)$ , where  $b_n$  are now vectors. This model includes a CARA-normal model with multiple correlated assets as a special case, which can be demonstrated similarly to the case of a scalar shock  $\varepsilon$ . Therefore, the model with multi-dimensional shock  $\varepsilon$  can approximate the multi-asset CARA-normal model with asset payoffs as in Admati (1985).



**Figure 1: Probabilities  $\pi_n(\varepsilon)$  and Distribution of  $\varepsilon$**

Panel (a) shows probabilities  $\pi_n(\varepsilon)$  for different  $\varepsilon$  for asset payoffs  $C(\omega_n)$ . Panel (b) shows the PDF functions of distributions  $\hat{N}(\mu_\varepsilon, \sigma_\varepsilon^2)$  and  $N(\mu_\varepsilon, \sigma_\varepsilon^2)$  with  $\mu_\varepsilon = 0$  and  $\sigma_\varepsilon = 1$ . In this example,  $N = 100$ ,  $C(\omega_n) = 300(n - 1)/(N - 1)$  and vectors  $a$  and  $b$  are calibrated so that  $\pi_n(-1)$  and  $\pi_n(1)$  are discrete approximations of gamma distributions with shape and scale parameter pairs (1,2) and (5,1), respectively.

### 3. Characterization of Equilibrium

In this section, we first consider an economy with  $M = N$  securities, which we call a complete-markets economy, and characterize the equilibrium in closed form. Then, we consider a general economy with  $M \leq N$  securities, which includes the case of incomplete markets, and under an additional assumption characterize asset prices in terms of easily computable multivariate inverse functions.

#### 3.1. Complete-Market Economy with $M = N$ Securities

We start with a complete-markets economy with  $M = N$ . For example, the market can be completed by issuing a sufficient number of non-redundant derivative securities, as demonstrated in Ross (1976). In our model, derivative securities can reveal additional information about the underlying asset, which in turn can be used for more accurate pricing of derivatives. Therefore, prices  $p$  of all risky assets should be found simultaneously. Due

to market completeness, we look for equilibrium prices  $p$  in the following form:

$$p_m = \left[ \pi_1^{\text{RN}} C_m(\omega_1) + \pi_2^{\text{RN}} C_m(\omega_2) + \dots + \pi_N^{\text{RN}} C_m(\omega_N) \right] e^{-rT}, \quad (8)$$

where  $m = 1, \dots, N - 1$ , and  $\pi_n^{\text{RN}}$  is the *risk-neutral* probability of state  $\omega_n$ . The risk-neutral valuation in our model is possible because any arising arbitrage opportunities will be eliminated by investors  $I$  and  $U$ .

All investors agree on risk-neutral probabilities, because they are uniquely determined from Equations (8) as functions of prices  $p$ . However, because of asymmetric information, investors  $I$  and  $U$  have different real probabilities of states  $\omega_n$ . In particular, investor  $I$ 's probabilities of states  $\omega_n$  are given by  $\pi_n(\varepsilon)$  because investor  $I$  observes  $\varepsilon$ . Investor  $U$  observes only prices, and filters out shock  $\varepsilon$  from the market clearing condition (4). Consequently, investor  $U$ 's real probabilities of states  $\omega_n$  are given by conditional expectations. To see this, we rewrite the expected utility of investor  $U$  as

$$\mathbb{E} \left[ -e^{-\gamma_U W_{U,T}} | P(\varepsilon, \nu) = p, p \right] = - \sum_{n=1}^N \left( \mathbb{E} \left[ \pi_n(\varepsilon) | P(\varepsilon, \nu) = p, p \right] e^{-\gamma_U W_{U,T,n}} \right). \quad (9)$$

Then, by  $\pi_n^U(p; \theta_U^*(p)) = \mathbb{E} \left[ \pi_n(\varepsilon) | P(\varepsilon, \nu) = p, p \right]$  we denote investor  $U$ 's posterior probabilities of states  $\omega_n$ . The probabilities, in general, depend on equilibrium portfolios  $\theta_U^*(p)$  through the market clearing conditions, as demonstrated below.

Investor  $U$  faces a continuum of states of the economy because of the uncertainty about shock  $\varepsilon$ . However, objective function (9) demonstrates that because asset payoffs can take only  $N$  values that do not depend on shock  $\varepsilon$  and noise  $\nu$  investor  $U$ 's optimization can be solved as a complete-markets problem with  $N$  states and  $N$  securities, in which real probabilities  $\pi_n^U(p; \theta_U^*(p))$  are taken as given. Because investors have different real probabilities, the model is similar to models with heterogeneous beliefs in which investors are endowed with different probability measures [e.g., Basak (2000), Basak (2005)]. In contrast to the latter models, in our model the beliefs are endogenous and investor  $U$  does not observe probabilities  $\pi_n(\varepsilon)$  of investor  $I$ .

We obtain investors' portfolios  $\theta_i^*$  from the first order conditions (FOC), which equate marginal utilities and state price densities (SPD). Because investors have different probabilities of states  $\omega_n$ , they also have different SPDs, given by [e.g., Duffie (2001)]:

$$\xi_I(\omega_n) = \frac{\pi_n^{\text{RN}} e^{-rT}}{\pi_n(\varepsilon)}, \quad \xi_U(\omega_n) = \frac{\pi_n^{\text{RN}} e^{-rT}}{\pi_n^U(p; \theta_U^*(p))}. \quad (10)$$

Consequently, the FOCs equating marginal utilities and SPDs are given by:

$$\gamma_I e^{-\gamma_I W_{I,T,n}} = \ell_I \frac{\pi_n^{\text{RN}} e^{-rT}}{\pi_n(\varepsilon)}, \quad \gamma_U e^{-\gamma_U W_{U,T,n}} = \ell_U \frac{\pi_n^{\text{RN}} e^{-rT}}{\pi_n^U(p; \theta_U^*(p))}, \quad (11)$$

where  $\ell_i$  denote Lagrange multipliers for investors' budget constraints. From the FOCs in (11) and Equation (3) for wealth  $W_{i,T}$ , after some algebra, we obtain investors' portfolios, which are reported in Lemma 1 below.

**Lemma 1 (Investors' optimal portfolios).**

1) *Suppose, probabilities  $\pi_n(\varepsilon)$  and PDF  $\varphi_\varepsilon(x)$  are general functions (not necessarily as in Section 2.3) such that the equilibrium exists. Then, optimal portfolios of informed and uninformed investors,  $\theta_I^*(p; \varepsilon)$  and  $\theta_U^*(p)$ , are given by:*

$$\theta_I^*(p; \varepsilon) = \frac{1}{\gamma_I} \Omega^{-1} \left\{ \left( \ln\left(\frac{\pi_1(\varepsilon)}{\pi_N(\varepsilon)}\right), \dots, \ln\left(\frac{\pi_{N-1}(\varepsilon)}{\pi_N(\varepsilon)}\right) \right)^\top - v \right\}, \quad (12)$$

$$\theta_U^*(p) = \frac{1}{\gamma_U} \Omega^{-1} \left\{ \left( \ln\left(\frac{\pi_1^U(p; \theta_U^*(p))}{\pi_N^U(p; \theta_U^*(p))}\right), \dots, \ln\left(\frac{\pi_{N-1}^U(p; \theta_U^*(p))}{\pi_N^U(p; \theta_U^*(p))}\right) \right)^\top - v \right\}, \quad (13)$$

where the uninformed investor's posterior probabilities are given by

$$\pi_n^U(p; \theta_U^*(p)) = \mathbb{E}[\pi_n(\varepsilon) | P(\varepsilon, \nu) = p, p], \quad n = 1, \dots, N, \quad (14)$$

$\Omega$  is a  $(N-1) \times (N-1)$  matrix with rows  $(\Pi_n - \Pi_N)^\top$  and elements  $\Omega_{n,k} = C_k(\omega_n) - C_k(\omega_N)$ , where  $k, n = 1, \dots, N-1$ , and  $v = \left( \ln\left(\frac{\pi_1^{\text{RN}}}{\pi_N^{\text{RN}}}\right), \dots, \ln\left(\frac{\pi_{N-1}^{\text{RN}}}{\pi_N^{\text{RN}}}\right) \right)^\top$ .

2) *If probabilities  $\pi_n(\varepsilon)$  are given by Equation (5) and  $\varphi_\varepsilon(x)$  is an arbitrary PDF, then investor  $I$ 's optimal portfolio is a linear function of shock  $\varepsilon$ , given by:*

$$\theta_I^*(p; \varepsilon) = \frac{\lambda \varepsilon}{\gamma_I} - \frac{1}{\gamma_I} \Omega^{-1} (v - \tilde{a}), \quad (15)$$

where  $\tilde{a} = (a_1 - a_N, \dots, a_{N-1} - a_N)^\top \in \mathbb{R}^{N-1}$  and  $\lambda = \Omega^{-1} (b_1 - b_N, \dots, b_{N-1} - b_N)^\top \in \mathbb{R}^{N-1}$ .

Lemma 1 determines optimal portfolios of investors in terms of real and risk-neutral probabilities due to market completeness, which guarantees that matrix  $\Omega$  is invertible. It also provides further insight into the filtering problem of investor  $U$ . In particular, it demonstrates that investor  $U$ 's portfolio  $\theta_U^*(p)$  solves a fixed-point problem. This is because on one hand, the demand for assets affects prices  $p$  and the filtering problem via the market clearing condition, and on the other hand, prices  $p$  determine the demand for assets.

Lemma 1 also demonstrates that portfolio  $\theta_I^*(p; \varepsilon)$  is a linear function of shock  $\varepsilon$  when probabilities  $\pi_n(\varepsilon)$  are given by Equation (5) and  $\varphi_\varepsilon(x)$  is an arbitrary PDF. This result can be easily demonstrated by substituting probabilities (5) into portfolio (12).

The intuition for the linear term in Equation (15) is as follows. FOC of investor  $I$  in (11) demonstrates that, holding risk-neutral probabilities fixed, investor  $I$  has incentive to allocate more wealth to states with higher probabilities  $\pi_n(\varepsilon)$ , that is, to states with higher  $a_n + b_n\varepsilon$ . From the definition of  $\lambda$  in Lemma 1 it can be easily observed that it solves a system of equations  $b_n = \lambda_0 + \Pi_n^\top \lambda$ , where  $\Pi_n$  is the vector of risky asset payoffs in state  $\omega_n$ , and  $\lambda_0$  is a constant. Consequently, portfolio  $\lambda$  replicates risk exposures  $b_n$  in states  $\omega_n$ , up to a constant  $\lambda_0$ , and hence  $\lambda\varepsilon$  replicates  $b\varepsilon$ . Similarly, portfolio  $\Omega^{-1}\tilde{a}$  replicates exposures  $a_n$ . Investing in the latter two replicating portfolios gives investor  $I$  more wealth in states with high  $\pi_n(\varepsilon)$ . The demand for these portfolios is inversely related to risk aversion  $\gamma_I$  because more risk averse investors are less willing to shift wealth from bad states to good states, which gives rise to terms  $\lambda\varepsilon/\gamma_I$  and  $\Omega^{-1}\tilde{a}/\gamma_I$  in portfolio (15).

The intuition for term  $\Omega^{-1}v/\gamma_I$  in investor  $I$ 's portfolio (15) is as follows. FOCs (11) demonstrate that investors have incentive to allocate less wealth to states with high risk-neutral probabilities  $\pi_n^{\text{RN}}$ . This incentive reflects the price level effect because  $\pi_n^{\text{RN}}e^{-rT}$  can be interpreted as the value of \$1 in state  $\omega_n$ . The price effect gives rise to term  $\Omega^{-1}v/\gamma_I$  in portfolio (15), which can be demonstrated along the same lines as above. Portfolio (15) then reflects the relative strength of the effects of probabilities  $\pi_n(\varepsilon)$  and  $\pi_n^{\text{RN}}$ .

The linearity of portfolio (15) simplifies the filtering problem of investor  $U$ . In particular, substituting  $\theta_I^*(p; \varepsilon)$  and  $\theta_U^*(p)$  into the market clearing condition (4), we find that

$$\frac{\lambda\varepsilon}{\gamma_I} + \nu + H(p) = 0, \quad (16)$$

where  $H(p)$  is a function of prices  $p$ , which is given by

$$H(p) = \theta_U^*(p) - \frac{1}{\gamma_I}\Omega^{-1}(v - \tilde{a}), \quad (17)$$

where  $\tilde{a}$  and  $v$  are defined in Lemma 1. Vector  $v$  is a function of risk-neutral probabilities  $\pi_n^{\text{RN}}$ , which in turn are functions of asset prices  $p$ , as argued above. Therefore, vector  $v$ , and hence also  $H(\cdot)$ , are functions of  $p$ .

Equation (16) demonstrates that observing prices  $p$  allows investor  $U$  to infer a linear combination of shocks  $\lambda\varepsilon/\gamma_I + \nu$ . We restrict attention to equilibria in which asset prices

are continuous functions of shock  $\varepsilon$  and noisy demand  $\nu$ . In such equilibria,  $\lambda\varepsilon/\gamma_I + \nu$  is the only information that will be revealed by asset prices, which can be demonstrated similarly to Breon-Drish (2012). The posterior distribution of  $\varepsilon$  after observing  $\lambda\varepsilon/\gamma_I + \nu$  is available in closed form when  $\varepsilon \sim \hat{N}(\mu_\varepsilon, \sigma_\varepsilon^2)$ , which allows us to compute investor  $U$ 's posterior probabilities  $\pi_n^U(p; \theta_v^*(p))$  also in closed form. Lemma 2 reports the results.

**Lemma 2 (Conditional distributions).** *Suppose, probabilities  $\pi_n(\varepsilon)$  of states  $\omega_n$  are given by Equation (5), and shock  $\varepsilon$  has generalized normal distribution  $\hat{N}(\mu_\varepsilon, \sigma_\varepsilon^2)$  with PDF function given by (6). Let  $\tilde{\varepsilon} = \lambda\varepsilon/\gamma_I + \nu + H(p)$ , i.e. the left-hand side of (16). Then, the posterior PDF  $\varphi_{\varepsilon|\tilde{\varepsilon}}(x|y)$  of shock  $\varepsilon$ , conditional on observing vector  $\tilde{\varepsilon}$ , and the probabilities  $\pi_n^U(p; \theta_v^*(p))$  of investor  $U$  are given by:*

$$\varphi_{\varepsilon|\tilde{\varepsilon}}(x|y) = \frac{\exp\left\{-0.5\left(y - \lambda x/\gamma_I - H(p)\right)^\top \Sigma_\nu^{-1}\left(y - \lambda x/\gamma_I - H(p)\right)\right\} \varphi_\varepsilon(x)}{G_1(y; p)}, \quad (18)$$

$$\pi_n^U(p; \theta_v^*(p)) = \frac{1}{G_2(p)} \exp\left\{a_n + \frac{1}{2} \frac{b_n^2 - 2b_n\left(\lambda^\top \Sigma_\nu^{-1} H(p)/\gamma_I - \mu_0/\sigma_0^2\right)}{\lambda^\top \Sigma_\nu^{-1} \lambda/\gamma_I^2 + 1/\sigma_0^2}\right\}, \quad (19)$$

where function  $H(p)$  is given by Equation (17), and  $G_1(y; p)$  and  $G_2(p)$  are some functions that do not depend on state  $\omega_n$ .

From Lemma 2 we observe that  $\ln\left(\pi_n^U(p; \theta_v^*(p))/\pi_n^U(p; \theta_v^*(p))\right)$  is a linear function of  $\theta_v^*(p)$ , which allows us to solve the fixed point problem in Equation (13) in closed form and to obtain  $\theta_v^*(p)$  as a function of vector  $v$ . Then, we find vector  $v$ , which itself is a function of the risk-neutral probabilities  $\pi_n^{\text{RN}}$ , from the market clearing condition (16). Next, we find  $\pi_n^{\text{RN}}$  in terms of elements of vector  $v$ . Finally, we obtain the equilibrium prices in terms of risk neutral-probabilities using Equation (8). Proposition 1 reports the equilibrium.

**Proposition 1 (Unique equilibrium with  $M = N$  assets).** *Suppose, probabilities  $\pi_n(\varepsilon)$  of states  $\omega_n$  are given by Equation (5), and shock  $\varepsilon$  has generalized normal distribution  $\hat{N}(\mu_\varepsilon, \sigma_\varepsilon^2)$  with PDF function given by (6). Then, there exists unique equilibrium, in which investors' portfolios  $\theta_I^*(p; \varepsilon)$  and  $\theta_U^*(p)$ , risk-neutral probabilities  $\pi^{\text{RN}}$ , and asset prices  $P(\varepsilon, \nu)$  are given by:*

$$\theta_I^*(p; \varepsilon) = \frac{\lambda\varepsilon}{\gamma_I} - \frac{1}{\gamma_I} \Omega^{-1}(v - \tilde{a}), \quad (20)$$

$$\theta_U^*(p) = (E + Q)^{-1} \left( \frac{1}{\gamma_I} Q \Omega^{-1}(v - \tilde{a}) - \frac{1}{\gamma_U} \Omega^{-1}(v - \hat{a}) + \frac{(\mu_0/\sigma_0^2)\lambda}{\gamma_U(\lambda^\top \Sigma_\nu^{-1} \lambda/\gamma_I^2 + 1/\sigma_0^2)} \right), \quad (21)$$



$$\pi_n^{\text{RN}} = \frac{e^{v_n}}{1 + \sum_{k=1}^{N-1} e^{v_k}}, \quad \pi_N^{\text{RN}} = \frac{1}{1 + \sum_{k=1}^{N-1} e^{v_k}}, \quad (22)$$

$$P_m(\varepsilon, \nu) = \left[ \pi_1^{\text{RN}} C_m(\omega_1) + \pi_2^{\text{RN}} C_m(\omega_2) + \dots + \pi_N^{\text{RN}} C_m(\omega_N) \right] e^{-rT}, \quad (23)$$

where  $m = 1, \dots, M - 1$ ;  $v = (v_1, \dots, v_{N-1})^\top \equiv \left( \ln(\pi_1^{\text{RN}}/\pi_N^{\text{RN}}), \dots, \ln(\pi_{N-1}^{\text{RN}}/\pi_N^{\text{RN}}) \right)^\top \in \mathbb{R}^{N-1}$  and  $Q \in \mathbb{R}^{(N-1) \times (N-1)}$  are given by

$$v = \tilde{a} + \frac{1}{2} \frac{\gamma_I}{\gamma_I + \gamma_U} \frac{\tilde{b}^{(2)} + 2(\mu_0/\sigma_0^2)\Omega\lambda}{\lambda^\top \Sigma_\nu^{-1} \lambda / \gamma_I^2 + 1/\sigma_0^2} + \frac{\gamma_I \gamma_U}{\gamma_I + \gamma_U} \Omega (E + Q) \left( \frac{\lambda \varepsilon}{\gamma_I} + \nu \right), \quad (24)$$

$$Q = \frac{\lambda \lambda^\top \Sigma_\nu^{-1}}{\gamma_U \gamma_I \left( \lambda^\top \Sigma_\nu^{-1} \lambda / \gamma_I^2 + 1/\sigma_0^2 \right)}, \quad (25)$$

$E$  is the  $(N - 1) \times (N - 1)$  identity matrix,  $\Omega$  is a  $(N - 1) \times (N - 1)$  matrix with rows  $(\Pi_n - \Pi_N)^\top$ ,  $\tilde{a} = (a_1 - a_N, \dots, a_{N-1} - a_N)^\top$ ,  $\hat{a} = \tilde{a} + 0.5\tilde{b}^{(2)}/(\lambda^\top \Sigma_\nu^{-1} \lambda / \gamma_I^2 + 1/\sigma_0^2)$ ,  $\tilde{b}^{(2)} = (b_1^2 - b_N^2, \dots, b_{N-1}^2 - b_N^2)^\top$ ,  $\lambda = \Omega^{-1}(b_1 - b_N, \dots, b_{N-1} - b_N)^\top$ .<sup>2</sup>

Proposition 1 provides a fully closed-form characterization of equilibrium with multiple assets in terms of elementary functions. From the results in Proposition 1 we observe that prices are non-linear functions of shock  $\varepsilon$  and noise  $\nu$ , in contrast to previous noisy REE models [e.g., Grossman and Stiglitz (1980); Admati (1985), among others]. However, linearity is preserved for vector  $v$ , which determines the risk-neutral probabilities and hence also the prices. In particular,  $v$  is a linear function of  $\lambda\varepsilon/\gamma_I + \nu$ , which summarizes the information impounded into the market prices from  $I$ 's trading strategy.<sup>3</sup> Furthermore, the tractability of our analysis allows us to study comparative statics for asset prices and investors' portfolios. These comparative statics are reported in Proposition 2 below.

**Proposition 2 (Comparative statics).** *The comparative statics for price  $P_m(\varepsilon, \nu)$  of asset  $m$  with respect to shock  $\varepsilon$  and noisy demands  $\nu$  are as follows:*

$$\frac{\partial P_m(\varepsilon, \nu)}{\partial \varepsilon} = \frac{\gamma_U}{\gamma_U + \gamma_I} \left( 1 + \frac{\lambda^\top \Sigma_\nu^{-1} \lambda / (\gamma_U \gamma_I)}{\lambda^\top \Sigma_\nu^{-1} \lambda / \gamma_I^2 + 1/\sigma_0^2} \right) \text{cov}^{\text{RN}}(b, C_m) e^{-rT}, \quad (26)$$

<sup>2</sup>Inverse matrix  $(E + Q)^{-1}$  in Equation (21) can be computed in closed form, and is given by:

$$(E + Q)^{-1} = E - \frac{\lambda \lambda^\top \Sigma_\nu^{-1}}{\gamma_U \gamma_I \left( \lambda^\top \Sigma_\nu^{-1} \lambda / \gamma_I^2 + 1/\sigma_0^2 \right) + \lambda^\top \Sigma_\nu^{-1} \lambda}.$$

The latter result can be either directly verified or obtained from Lemma A.1 in Appendix A in which we set  $u = \lambda$  and  $w = \Sigma_\nu^{-1} \lambda / (\gamma_U \gamma_I (\lambda^\top \Sigma_\nu^{-1} \lambda / \gamma_I^2 + 1/\sigma_0^2))$ .

<sup>3</sup>For comparison, in Appendix B we additionally provide equilibrium prices and portfolios in three benchmark cases: 1) when both investors are fully informed; 2) when both investors are fully uninformed; 3) when investor  $U$  is uninformed and naive, i.e., does not learn from prices.

$$\frac{\partial P_m(\varepsilon, \nu)}{\partial \nu_l} = \frac{\gamma_U \gamma_I}{\gamma_U + \gamma_I} \left( \text{cov}^{\text{RN}}(C_l, C_m) + \frac{\lambda^\top \Sigma_\nu^{-1} \lambda / (\gamma_U \gamma_I)}{\lambda^\top \Sigma_\nu^{-1} \lambda / \gamma_I^2 + 1/\sigma_0^2} \text{cov}^{\text{RN}}(b, C_m) \right) e^{-rT}. \quad (27)$$

The comparative statics for investors' portfolios with respect to prices  $p$  are as follows:

$$\frac{\partial \theta_I^*(p; \varepsilon)}{\partial p} = -\frac{1}{\gamma_I} \left( \text{var}^{\text{RN}}[\Pi] \right)^{-1} e^{rT}, \quad (28)$$

$$\frac{\partial \theta_U^*(p)}{\partial p} = \left( -\frac{1}{\gamma_U} E + \frac{\gamma_U + \gamma_I}{\gamma_U \gamma_I} \frac{\lambda \lambda^\top \Sigma_\nu^{-1}}{\lambda^\top \Sigma_\nu^{-1} \lambda / \gamma_I^2 + 1/\sigma_0^2} \right) \left( \text{var}^{\text{RN}}[\Pi] \right)^{-1} e^{rT}, \quad (29)$$

where  $\text{cov}^{\text{RN}}(\cdot, \cdot)$  and  $\text{var}^{\text{RN}}(\cdot)$  are covariance and variance-covariance matrices under the risk-neutral probability measure, and  $\Pi$  is the vector of risky assets payoffs in random state  $\omega$ . Furthermore, informed investor's demand for risky asset  $m$  is a downward-sloping function of that asset's price  $p_m$ , holding the prices of other assets fixed.

Proposition 2 provides closed-form comparative statics for asset prices and investors' portfolios. The comparative statics for prices are reminiscent of demand-pressure effects in Gârleanu, Pedersen and Poteshman (2008). These demand pressures in our setting arise due to the fact that all assets are non-redundant, and are driven by the covariances of the economy's risk exposure  $b$  and asset payoffs  $C_m$ . The covariances arise because, as demonstrated above, investor  $I$  has an incentive to replicate risk exposure  $b\varepsilon$  in order to shift wealth to states with higher probability  $\pi_n(\varepsilon)$ .

Proposition 2 decomposes derivatives of prices and portfolio  $\theta_U^*(p)$  into two types of terms, which are shown inside the brackets. First terms capture classic substitution effects and are present even without asymmetric information. Second terms, which depend on vector  $\lambda$ , capture the information effects. The latter terms are absent in Equation (28) for  $\partial \theta_I^*(p; \varepsilon) / \partial p$  because the informed investor has perfect information.

Importantly, we find that investor  $I$ 's demand for risky asset  $m$  is a downward sloping function of price  $p_m$  of that security. However, the latter result cannot be guaranteed for investor  $U$ . The reason is that, from investor  $U$ 's perspective, high asset prices might convey positive information about shock  $\varepsilon$ , in which case the demand for asset  $m$  may go up despite high price  $p_m$ . To demonstrate this, assume for simplicity that noisy demands are i.i.d., and hence  $\Sigma_\nu = \sigma_\nu^2 E$ . Consequently, matrix  $\lambda \lambda^\top \Sigma_\nu^{-1}$  is positive semi-definite, and hence has non-negative elements on the main diagonal.<sup>4</sup> If these elements are positive

<sup>4</sup>Element  $i$  on the diagonal of matrix  $A$  is given by  $e_i^\top A e_i$ , where  $e_i = (0, 0, \dots, 1, \dots, 0)^\top$  is a vector with 1 on  $i$ 's place and all other components equal to zero. If  $A$  is positive semi-definite, then  $e_i^\top A e_i \geq 0$ , and hence the diagonal elements are non-negative.

and large, the matrix on the right-hand side of Equation (29) may have positive elements on the diagonal.

We verify the above intuition for substitution and information effects in a simple economy with two risky assets in which we set  $r = 0$ ,  $T = 1$ ,  $a = (-5.37, -4.11, -5.7)^\top$  and  $b = (1.19, 2.44, 3.69)^\top$ , where  $a$  and  $b$  are calibrated from gamma distributions, similarly to the example in Section 2. The risk aversions of investors are given by  $\gamma_I = 0.004$  and  $\gamma_U = 0.04$  [see Paravisini, Rappoport, and Ravina (2010)]. We consider two risky assets with payoffs  $C_1 = (0, 75, 300)^\top$  and  $C_2 = (0, 0, 225)^\top$ . In this economy, investor  $U$ 's demand for the first asset increases with the increase in its price, whereas the demand for the second security decreases with the increase in its price, holding other prices fixed.

### 3.2. General Economy with $M \leq N$ Securities

In this section, we study a general economy with  $M$  securities, where  $M \leq N$ , which subsumes complete and incomplete market economies as special cases. For tractability, we impose the following assumption.

**Assumption 1.** *Factor  $b$  is spanned by the traded assets in the economy. That is, there exist unique constant  $\lambda_0$  and vector  $\lambda = (\lambda_1, \dots, \lambda_{M-1})^\top \in \mathbb{R}^{M-1}$  such that*

$$b = \lambda_0 \mathbb{I}_N + \lambda_1 C_1 + \dots + \lambda_{M-1} C_{M-1}, \quad (30)$$

or equivalently,  $b_n = \lambda_0 + \Pi_n^\top \lambda$ , where  $\mathbb{I}_N \in \mathbb{R}^N$  is a vector of ones,  $C_m$  are payoffs of the risky assets, and  $\Pi_n$  is the vector of risky asset payoffs in state  $n$ .

Assumption 1 implies that the financial market is *informationally complete* in the sense that the information contained in asset payoffs is sufficient for replicating the risk exposure  $b$ , which determines the shifts in probabilities  $\pi_n(\varepsilon)$ . We provide several realistic economies that satisfy this assumption. First example, is the complete market economy, where  $M = N$ , and hence there always exist constant  $\lambda_0$  and vector  $\lambda$  satisfying Equation (30). Second example is an incomplete-market economy with only one risky asset with payoff  $C_1 = b$ . As discussed in Section 2.3, a CARA-normal model with one risky asset is a special case of the latter economy. Third example is an economy with asset  $C_1 = b$ , and call options with payoffs  $C_2 = (b - K_2)^+, \dots, C_{M-1} = (b - K_{M-1})^+$  written on asset 1, in which case  $\lambda_0 = 0$ ,  $\lambda_1 = 1$ ,  $\lambda_2 = 0, \dots, \lambda_{M-1} = 0$ . A CARA-normal model with a risky asset  $C_1$  and

several call options written on it, is a special case of the latter economy. Fourth example is an economy with firms that have cash flows  $b$  and issue risky debt and equity with payoffs  $\min(b, K)$  and  $(b - K)^+$ , where  $K$  is the face value of debt.

Next, we demonstrate that if Assumption 1 is satisfied, then the optimal portfolio of investor  $I$  is a linear function of signal  $\varepsilon$ . From Assumption 1 we have that  $b_n = \lambda_0 + \Pi_n^\top \lambda$ , which we substitute into the objective function (1) of the informed investor. After some algebra, we rewrite investor  $I$ 's objective function as follows:

$$\begin{aligned}
\mathbb{E}\left[-e^{-\gamma_I W_{I,T}}|\varepsilon, p\right] &= -\frac{\sum_{k=1}^N \exp\{a_k + b_k \varepsilon - \gamma_I (\Pi_k - e^{rT} p)^\top \theta\}}{\sum_{k=1}^N \exp\{a_k + b_k \varepsilon\}} \\
&= -\exp\{(\lambda_0 + e^{rT} p^\top \lambda) \varepsilon - e^{rT} p^\top (\lambda \varepsilon - \gamma_I \theta)\} \\
&\quad \times \frac{\sum_{k=1}^N \exp\{a_k + \Pi_k^\top (\lambda \varepsilon - \gamma_I \theta)\}}{\sum_{k=1}^N \exp\{a_k + b_k \varepsilon\}} \\
&= -\exp\{(\lambda_0 + e^{rT} p^\top \lambda) \varepsilon - e^{rT} p^\top \hat{\theta}\} \frac{\sum_{k=1}^N \exp\{a_k + \Pi_k^\top \hat{\theta}\}}{\sum_{k=1}^N \exp\{a_k + b_k \varepsilon\}},
\end{aligned} \tag{31}$$

where  $\hat{\theta} = \lambda \varepsilon - \gamma_I \theta$ . From the last line in (31) we observe that finding optimal portfolio  $\theta_I^*(p; \varepsilon)$  reduces to finding optimal  $\hat{\theta}_I^*$ , which solves the optimization problem

$$\max_{\hat{\theta}} e^{rT} p^\top \hat{\theta} - g_I(\hat{\theta}), \tag{32}$$

where  $g_I(\hat{\theta}) = \ln\left(\sum_{i=1}^N \exp\{a_i + \Pi_i^\top \hat{\theta}\}\right)$ . From the optimization problem (32), we see that  $\hat{\theta}_I^*$  does not depend on shock  $\varepsilon$ . Therefore, investor  $I$ 's optimal portfolio is given by

$$\theta_I^*(p; \varepsilon) = \frac{\lambda \varepsilon}{\gamma_I} - \frac{\hat{\theta}_I^*(p)}{\gamma_I}. \tag{33}$$

From Equation (33) we conclude that the optimal portfolio of investor  $I$  is a linear function of shock  $\varepsilon$ , as in the complete-markets case in Section 3.2. Furthermore, similarly to Lemma 1, we point out that linearity holds for general PDF  $\varphi_\varepsilon(x)$ . Substituting portfolio  $\theta_I^*(p; \varepsilon)$  into the market clearing condition we obtain equation

$$\frac{\lambda \varepsilon}{\gamma_I} + \nu + \hat{H}(p) = 0, \tag{34}$$

where  $\hat{H}(p) = \theta_U^*(p) - \hat{\theta}_I^*(p)/\gamma_I$ , which has similar structure to the market clearing condition (16) in the complete-markets economy. The probabilities  $\pi_n^U$  of the uninformed investor are then found as in Lemma 2, assuming that shock  $\varepsilon$  is distributed according to  $\hat{N}(\mu_\varepsilon, \sigma_\varepsilon^2)$ .

The optimal portfolio  $\theta_v^*(p)$  of the uninformed investor is then found similarly to that of investor  $I$ . The vector of prices is found from the market clearing conditions, as in the complete-markets economy. Proposition 3 summarizes the main results.

**Proposition 3 (Unique equilibrium with  $M \leq N$  assets).** *Suppose probabilities  $\pi_n(\varepsilon)$  are given by Equation (5), shock  $\varepsilon$  has generalized normal distribution  $\hat{N}(\mu_\varepsilon, \sigma_\varepsilon^2)$  with PDF function given by (6), and Assumption 1 is satisfied. If there exists an equilibrium in the economy, then the investors' optimal portfolios are given by*

$$\theta_I^*(p; \varepsilon) = \frac{\lambda \varepsilon}{\gamma_I} - \frac{1}{\gamma_I} f_I^{-1}(e^{rT} p), \quad (35)$$

$$\theta_v^*(p) = (E + Q)^{-1} \left( \frac{1}{\gamma_I} Q f_I^{-1}(e^{rT} p) - \frac{1}{\gamma_v} f_v^{-1}(e^{rT} p) + \frac{(\mu_0/\sigma_0^2)\lambda}{\gamma_v(\lambda^\top \Sigma_v^{-1} \lambda / \gamma_I^2 + 1/\sigma_0^2)} \right). \quad (36)$$

Price vector  $P(\varepsilon, \nu)$  is a function of  $\lambda \varepsilon / \gamma_I + \nu$  and is the unique solution of equations

$$\frac{1}{\gamma_v} f_v^{-1}(e^{rT} P(\varepsilon, \nu)) + \frac{1}{\gamma_I} f_I^{-1}(e^{rT} P(\varepsilon, \nu)) = (E + Q) \left( \frac{\lambda \varepsilon}{\gamma_I} + \nu \right) + \frac{(\mu_0/\sigma_0^2)\lambda}{\gamma_v(\lambda^\top \Sigma_v^{-1} \lambda / \gamma_I^2 + 1/\sigma_0^2)}, \quad (37)$$

where  $E$  is the  $(M-1) \times (M-1)$  identity matrix; the  $(M-1) \times (M-1)$  matrix  $Q$  is given by Equation (25). Equation (37) has at most one solution, such that  $\lambda \varepsilon_1 / \gamma_I + \nu_1 \neq \lambda \varepsilon_2 / \gamma_I + \nu_2$  implies  $P(\varepsilon_1, \nu_1) \neq P(\varepsilon_2, \nu_2)$ . Functions  $f_I, f_v : \mathbb{R}^{M-1} \rightarrow \mathbb{R}^{M-1}$  are uniquely invertible on their ranges, and for  $x \in \mathbb{R}^{M-1}$  are given by

$$f_I(x) = \frac{\sum_{n=1}^N \Pi_n \exp \{a_n + \Pi_n^\top x\}}{\sum_{n=1}^N \exp \{a_n + \Pi_n^\top x\}}, \quad (38)$$

$$f_v(x) = \frac{\sum_{n=1}^N \Pi_n \exp \{a_n + \frac{1}{2} \frac{b_n^2}{\lambda^\top \Sigma_v^{-1} \lambda / \gamma_I^2 + 1/\sigma_0^2} + \Pi_n^\top x\}}{\sum_{n=1}^N \exp \{a_n + \frac{1}{2} \frac{b_n^2}{\lambda^\top \Sigma_v^{-1} \lambda / \gamma_I^2 + 1/\sigma_0^2} + \Pi_n^\top x\}}, \quad (39)$$

and  $f_I^{-1}$  and  $f_v^{-1}$  denote the corresponding inverse functions.

Optimal portfolios (35) and (36) have the same structure as portfolios (20) and (21) in the complete-markets economy. However, risky asset prices are no longer available in closed form, and solve a system of non-linear algebraic equations (37). The latter system of equations reveals that prices are functions of a linear combination of shocks,  $\lambda \varepsilon / \gamma_I + \nu$ , similarly to the complete-markets case. Although the function on the left-hand side of Equation (37) is invertible on its range, we do not have a proof that its range coincides with  $\mathbb{R}^{M-1}$ , which is required for the existence of price  $P(\varepsilon, \nu)$ . However,

we note, that in the economy with a single risky asset it is easy to verify that function  $f_U^{-1}(e^{rT}P(\varepsilon, \nu))/\gamma_U + f_I^{-1}(e^{rT}P(\varepsilon, \nu))/\gamma_U$  is strictly monotone and has range  $\mathbb{R}$ , which guarantees the existence and uniqueness of price  $P(\varepsilon, \nu)$ .

The inverse functions  $x = f_i^{-1}(y)$  can be found by solving  $M - 1$  equations with  $M - 1$  unknowns  $y = f_i(x)$ . When the markets are complete the latter equation can be solved in closed form. For example, when  $M = N$  solving equation  $y = f_I(x)$  reduces to solving a system with  $M - 1$  linear equations with  $M - 1$  unknowns  $y_n = \exp\{a_n - a_N + (\Pi_n - \Pi_N)^\top x\}$ . Then, the vector of unknowns  $x$  can be found by solving another system of  $M - 1$  linear equations and unknowns, given by  $\ln(y_n) = a_n - a_N + (\Pi_n - \Pi_N)^\top x$ . Note that the equilibrium in Proposition 1 can be derived as a special case of that in Proposition 3.

To find equilibrium prices, we note that solving Equation (37) is equivalent to solving the following system of equations for  $x_I$  and  $x_U$ , which does not involve inverse functions:

$$\frac{x_I}{\gamma_I} + \frac{x_U}{\gamma_U} = (E + Q) \left( \frac{\lambda \varepsilon}{\gamma_I} + \nu \right) + \frac{(\mu_0/\sigma_0^2)\lambda}{\gamma_U(\lambda^\top \Sigma_\nu^{-1} \lambda / \gamma_I^2 + 1/\sigma_0^2)}, \quad (40)$$

$$e^{rT} p = f_I(x_I), \quad e^{rT} p = f_U(x_U). \quad (41)$$

Furthermore, solving the above system reduces to finding  $x_I$ , which satisfies equation  $f_I(x_I) = f_U(\gamma_U R(\varepsilon, \nu) - (\gamma_U/\gamma_I)x_I)$ , where  $R(\varepsilon, \nu)$  denotes the right-hand side of Equation (40). The latter equation can be solved using Newton's method [e.g., Judd (1998)], and then the equilibrium prices can be found from Equations (41).

## 4. Information Revelation and Market Transparency

The empirical literature demonstrates that derivatives markets play important role in the process of *price discovery* by revealing new information about asset prices [e.g., Chan, Chung, and Fong (2002); Chakravarty, Gulen, and Mayhew (2004); Pan and Poteshman (2006), among others]. Our results provide new insights about the informational role of derivatives. As demonstrated in Section 2 both for complete- and incomplete-markets economies, investor  $I$ 's optimal portfolio is a linear function of shock  $\varepsilon$  given by  $\theta_I^*(p; \varepsilon) = \lambda \varepsilon / \gamma_I - \hat{\theta}_I^*(p) / \gamma_I$ , where  $\hat{\theta}_I^*(p)$  is some function of prices  $p$ . Moreover, as demonstrated in Lemma 1, the latter result does not depend on the distributional assumptions about shock  $\varepsilon$ . Thus, the trading volume of the informed investor reveals information about shock  $\varepsilon$ .

We find a surprising result that not all derivatives play role in price discovery. Whether or not the trading volume of asset  $m$  releases new information about  $\varepsilon$  depends on whether this asset has a corresponding non-zero element  $\lambda_m$  in vector  $\lambda$ , which is the portfolio of risky assets that replicates the risk exposure  $b$ . If  $\lambda_m = 0$ , then Equation (30) for portfolio  $\theta_I^*(p; \varepsilon)$  reveals that the trading volume of asset  $m$  does not depend on  $\varepsilon$ . We call such assets *informationally redundant*. Hence, derivatives may not be redundant from a spanning perspective but can be redundant from an informational perspective.

The trading volume of asset  $m$  reveals new information about  $\varepsilon$  iff this asset helps replicate risk exposure  $b$ , that is,  $\lambda_m \neq 0$ , as can be seen from Equation (30) for vector  $b$ . The intuition is that, as demonstrated in Section 3.1, investor  $I$  has incentives to shift wealth to states with higher probabilities  $\pi_n(\varepsilon)$ , which gives rise to linear term  $\lambda\varepsilon/\gamma_I$  in investor  $I$ 's portfolio, where  $\lambda\varepsilon$  is a portfolio that replicates risk exposures  $b\varepsilon$ . As a result, only assets that help replicate exposure  $b$  have demands which are sensitive to shock  $\varepsilon$ , and hence transmit the information to the financial market.

As demonstrated in Section 3.2, a CARA-normal model with risky asset  $C_1 = b$  and options  $(b - K_2)^+, \dots, (b - K_{M-1})^+$  is a special case of our model. Moreover, it can be easily observed that only the underlying asset  $C_1$  suffices for replicating vector  $b$ , so that  $\lambda = (1, 0, \dots, 0)^\top$ . Therefore, we arrive at the second surprising conclusion that in a standard CARA-normal setting options or other derivatives do not play any role in price discovery. For derivatives to release information about  $\varepsilon$  the underlying asset should not span vector  $b$ , so that  $b \neq \lambda_0 + \lambda_1 C_1$  for any  $\lambda_0$  and  $\lambda_1$ .

When the market is complete, a condition for informational redundancy can be derived for general probability functions  $\pi_n(\varepsilon)$ . In particular, Equation (12) for the portfolio  $\theta_I^*(p; \varepsilon)$  demonstrates that investor  $I$ 's holding of asset  $m$  does not depend on shock  $\varepsilon$  iff vector  $\Omega^{-1} \left( \ln \left( (\pi_1(\varepsilon)/\pi_N(\varepsilon)), \dots, \ln(\pi_{N-1}(\varepsilon)/\pi_N(\varepsilon)) \right) \right)^\top$  has zero  $m^{\text{th}}$  element.

To quantify the informational contents and transparency of asset markets we suggest looking at the posterior precision of shock  $\varepsilon$ , as estimated by investor  $U$ , which is given by  $1/\hat{\sigma}_\varepsilon^2$ , where  $\hat{\sigma}_\varepsilon^2 = \text{var}[\varepsilon|p]$ . To reflect the important economic role of precision  $1/\hat{\sigma}_\varepsilon^2$  we call it the *transparency index*. For simplicity, we derive the posterior variance  $\hat{\sigma}_\varepsilon^2$  assuming that shock  $\varepsilon$  has prior distribution  $\varepsilon \sim N(\mu_\varepsilon, \sigma_\varepsilon^2)$ , which yields a more tractable expression for  $\hat{\sigma}_\varepsilon^2$  than the generalized normal distribution (6). As demonstrated on Figure 1, the

normal distribution  $\varepsilon \sim N(\mu_\varepsilon, \sigma_\varepsilon^2)$  is a good approximation for distribution (6), and hence the posterior variances of  $\varepsilon$  implied by the two distributions are also close. Lemma 3 below provides a closed-form expression for the transparency index.

**Lemma 3 (Market transparency).** *Let shock  $\varepsilon$  be normally distributed,  $\varepsilon \sim N(\mu_\varepsilon, \sigma_\varepsilon^2)$ . Then, the transparency index  $1/\hat{\sigma}_\varepsilon^2$ , where  $\hat{\sigma}_\varepsilon^2 = \text{var}[\varepsilon|p]$ , is given by*

$$\frac{1}{\hat{\sigma}_\varepsilon^2} = \frac{1}{\sigma_\varepsilon^2} + \frac{\lambda^\top \Sigma_\nu^{-1} \lambda}{\gamma_I^2}. \quad (42)$$

Equation (42) implies that  $1/\hat{\sigma}_\varepsilon^2 > 1/\sigma_\varepsilon^2$ . Intuitively, the presence of informed traders reduces the uncertainty about  $\varepsilon$  by releasing new information via asset prices. More formally, the quantity of the released information can be measured as the difference between the entropies of posterior and prior distributions of  $\varepsilon$ .<sup>5</sup> In the case of normally distributed  $\varepsilon$  the latter measure is given by  $\ln(\hat{\sigma}_\varepsilon) - \ln(\sigma_\varepsilon)$ , and hence is a monotone function of  $\hat{\sigma}_\varepsilon$ .

Furthermore, transparency does not depend on the number of traded assets even when all the assets are correlated. Consistent with the above intuition on price discovery,  $1/\hat{\sigma}_\varepsilon^2$  is determined only by assets that replicate the risk exposure  $b$ . Assume, for simplicity, that  $\Sigma_\nu = \sigma_\nu^2 E$ , where  $E$  is the identity matrix. If asset  $C_1 = b$  is traded, we obtain that  $\lambda = (1, 0, \dots, 0)^\top$ , and hence  $1/\hat{\sigma}_\varepsilon^2 = 1/\sigma_\varepsilon^2 + 1/(\gamma_I^2 \sigma_\nu^2)$ . Therefore,  $1/\hat{\sigma}_\varepsilon^2$  does not depend on the number of traded derivatives because they do not help span risk exposure  $b$ . If there are two risky assets with payoffs  $(b - K)^+$  and  $\min(b, K)$ , interpreted as equity and debt, then  $\lambda = (1, 1, 0, \dots, 0)^\top$ . Therefore,  $1/\hat{\sigma}_\varepsilon^2 = 1/\sigma_\varepsilon^2 + 2/(\gamma_I^2 \sigma_\nu^2)$ , and transparency increases.<sup>6</sup>

Transparency also depends on the variance-covariance matrix of noise trader demands  $\nu$ . In particular, higher correlations make the market more transparent by allowing inferring more information by comparing the market clearing conditions across securities. For example, consider a model with two risky assets and assume that  $\nu_1 = \nu_2$ , so that noisy demands are perfectly correlated. Taking the difference of the market clearing conditions (16) for the two markets we find that  $(\lambda_1 - \lambda_2)\varepsilon/\gamma_I + (\nu_1 - \nu_2) + (1, -1)^\top H(p) = 0$ , and hence shock  $\varepsilon$  can be perfectly learned from prices if  $\lambda_1 \neq \lambda_2$ . More formally, matrix  $\Sigma_\nu$

<sup>5</sup>Entropy of variable  $\varepsilon$  is defined as  $-\int_{-\infty}^{\infty} \varphi_\varepsilon(x) \ln \varphi_\varepsilon(x) dx$ . When  $\varepsilon$  is normally distributed, its entropy is given by  $0.5 \ln(2\pi e \sigma_\varepsilon^2)$ .

<sup>6</sup>The results on the informational role of derivatives explain why in Brennan and Cao (1996) investors do not learn from the derivative asset. In particular, they consider a CARA-normal framework with a stock and a power derivative. In our terminology, the stock's payoff linearly spans  $b$ . Therefore,  $\lambda = (1, 0)^\top$ , and hence the derivative does not reveal any useful information.



becomes close to singular when noisy demands cross-correlations become closer to one. Therefore, the determinant of  $\Sigma_\nu^{-1}$  becomes large, and hence transparency  $1/\hat{\sigma}_\varepsilon^2$  increases.

Finally, transparency  $1/\hat{\sigma}_\varepsilon^2$  is a decreasing function of the informed investor's risk aversion  $\gamma_I$ . Intuitively, investors with higher risk aversions have smaller demand for risky assets. Therefore, their private information is more difficult to filter out from the market clearing conditions, and hence the market becomes less transparent.

## 5. Extension to Economies with General Probabilities and Distributions

In this section, we consider a complete-markets economy with  $M = N$  assets and extend our analysis to general probabilities  $\pi_n(\varepsilon)$  and probability densities  $\varphi_\varepsilon(x)$  of signal  $\varepsilon$  and  $\varphi_\nu(x)$  of noise  $\nu$ . We obtain prices  $P(\varepsilon, \nu)$  and investor  $I$ 's portfolio  $\theta_I^*(p; \varepsilon)$  in closed form, and characterize investor  $U$ 's portfolio  $\theta_U^*(p)$  as a solution to a fixed-point problem.

In the economy with general distributions, portfolio  $\theta_I^*(p; \varepsilon)$  is no longer a linear function of  $\varepsilon$ . However, this portfolio remains separable in risk  $\varepsilon$  and prices  $p$ . In particular, from Equation (12) we find that portfolio  $\theta_I^*(p; \varepsilon)$  is given by  $\theta_I^*(p; \varepsilon) = \eta(\varepsilon)/\gamma_I - (1/\gamma_I)\Omega^{-1}v$ , where  $v$  is a function of risk-neutral probabilities, and  $\eta(\varepsilon)$  is defined as

$$\eta(\varepsilon) = \Omega^{-1} \left( \ln\left(\frac{\pi_1(\varepsilon)}{\pi_N(\varepsilon)}\right), \dots, \ln\left(\frac{\pi_{N-1}(\varepsilon)}{\pi_N(\varepsilon)}\right) \right)^\top. \quad (43)$$

Substituting investors' portfolios into market clearing conditions we obtain:

$$\frac{\eta(\varepsilon)}{\gamma_I} + \nu + \tilde{H}(p) = 0, \quad (44)$$

where  $\tilde{H}(p) = \theta_U^*(p) - \Omega^{-1}v/\gamma_I$ . The derivation of the equilibrium follows the same steps as in Section 2. First, we calculate investor  $U$ 's posterior probabilities  $\pi_n^U$  conditional on observing prices  $p$ . Then, we use  $\pi_n^U$  to derive the optimal portfolio of investor  $U$  from Equation (12) in Lemma 1. Next, we demonstrate that portfolio  $\theta_U^*(p)$  solves a fixed point problem. Finally, we obtain closed-form prices  $P(\varepsilon, \nu)$  from the market clearing condition (4). Proposition 4 reports the equilibrium.

**Proposition 4 (Equilibrium with  $M = N$  and general probabilities and distributions).** *Let probabilities  $\pi_n(\varepsilon)$  and probability density functions  $\varphi_\varepsilon(x)$  and  $\psi_\nu(x)$  of*

risk  $\varepsilon$  and noise  $\nu$  be general smooth functions defined on  $\mathbb{R}$  and  $\mathbb{R}^{M-1}$ , respectively, and  $\pi_n(\varepsilon) > 0$  for  $n = 1, \dots, N$ . We show that:

1) If there exists an REE then investors' portfolios  $\theta_I^*(p; \varepsilon)$  and  $\theta_U^*(p)$ , risk-neutral probabilities  $\pi_n^{\text{RN}}$  and asset prices  $P(\varepsilon, \nu)$  are given by:

$$\theta_I^*(p; \varepsilon) = \frac{\eta(\varepsilon)}{\gamma_I} - \frac{1}{\gamma_I} \Omega^{-1} \nu, \quad (45)$$

$$\theta_U^*(p) = -\frac{1}{\gamma_U} \Omega^{-1} \left( \nu - \Psi(\tilde{H}(p)) \right), \quad (46)$$

$$\pi_n^{\text{RN}} = \frac{e^{v_n}}{1 + \sum_{k=1}^{N-1} e^{v_k}}, \quad \pi_N^{\text{RN}} = \frac{1}{1 + \sum_{k=1}^{N-1} e^{v_k}}, \quad (47)$$

$$P_m(\varepsilon, \nu) = \left[ \pi_1^{\text{RN}} C_m(\omega_1) + \pi_2^{\text{RN}} C_m(\omega_2) + \dots + \pi_N^{\text{RN}} C_m(\omega_N) \right] e^{-rT}, \quad (48)$$

where  $m = 1, \dots, M-1$ ,  $\eta(\varepsilon)$  is given by (43), matrix  $\Omega$  is as in Proposition 1, and  $v = (v_1, \dots, v_{N-1})^\top \equiv \left( \ln(\pi_1^{\text{RN}}/\pi_N^{\text{RN}}), \dots, \ln(\pi_{N-1}^{\text{RN}}/\pi_N^{\text{RN}}) \right)^\top$  is a function of prices  $p$ , which in equilibrium is given by

$$v = \frac{\gamma_U \gamma_I}{\gamma_U + \gamma_I} \left( \frac{1}{\gamma_U} \Psi \left( -\frac{\eta(\varepsilon)}{\gamma_I} - \nu \right) + \Omega \left( \frac{\eta(\varepsilon)}{\gamma_I} + \nu \right) \right), \quad (49)$$

function  $\tilde{H}(p) : \mathbb{R}^{M-1} \rightarrow \mathbb{R}^{M-1}$  for each  $p$  solves a system of equations

$$\frac{1}{\gamma_U} \Psi(\tilde{H}(p)) - \Omega \tilde{H}(p) = \frac{\gamma_U + \gamma_I}{\gamma_U \gamma_I} \nu, \quad (50)$$

and vector-valued function  $\Psi(\cdot) : \mathbb{R} \rightarrow \mathbb{R}^{M-1}$  is given by

$$\Psi(z) = \left( \Psi_1(z) - \Psi_N(z), \dots, \Psi_{N-1}(z) - \Psi_N(z) \right)^\top. \quad (51)$$

$$\Psi_n(z) = \ln \left( \int_{-\infty}^{+\infty} \varphi_\nu \left( -\frac{\eta(x)}{\gamma_I} - z \right) \pi_n(x) \varphi_\varepsilon(x) dx \right). \quad (52)$$

2) There exists an REE iff system (50) has a unique solution belonging to the support of the distribution of random variable  $\eta(\varepsilon)/\gamma_I + \nu$ . If it exists, the REE is unique.

Proposition 4 provides a fully closed-form characterization of asset prices as functions of  $\eta(\varepsilon)/\gamma_I + \nu$  for general probabilities and distributions. Our equilibrium asset prices  $P_m(\varepsilon, \nu)$  do not involve inverse functions, in contrast to the incomplete-markets economies in Section 3.2 and in Breon-Drish (2012). The equilibrium is derived in terms of functions  $\Psi_n(z)$ , which are related to posterior probabilities. In particular, as shown in Appendix A,

$\ln(\pi_n^U/\pi_N^U) = \Psi_n(\tilde{H}(p)) - \Psi_N(\tilde{H}(p))$ , where  $\tilde{H}(p) = \theta_U^*(p) - \Omega^{-1}v/\gamma_I$ . Therefore, the components of vector  $\Psi(z)$  can be interpreted as log-likelihood ratios of posterior probabilities. We note, that Proposition 1 is a special case of Proposition 4.

By definition of vector  $v$  in Proposition 4, it depends on observed prices  $p$  via the risk-neutral probabilities, which can be obtained in terms of prices  $p$  from Equation (8) for the risk-neutral valuation of securities. Consequently, optimal portfolios (45) and (46) are functions of prices via vector  $v$  and have similar structure to those in Proposition 1.

The existence and uniqueness of the solution of Equation (50) are essential for the existence of the REE. As demonstrated in the proof of Proposition 4 in Appendix A, if Equation (50) has multiple solutions, then there exist pairs  $(\varepsilon_1, \nu_1)$  and  $(\varepsilon_2, \nu_2)$  such that  $\eta(\varepsilon_1)/\gamma_I + \nu_1 \neq \eta(\varepsilon_2)/\gamma_I + \nu_2$ , and yet  $P(\varepsilon_1, \nu_1) = P(\varepsilon_2, \nu_2)$ . Therefore, observing price  $P$  does not reveal the realization of  $\eta(\varepsilon)/\gamma_I + \nu$ , which contradicts the market clearing condition (44), and hence cannot happen in equilibrium.

Note that if we allow the uninformed investor to observe both prices  $p$  and aggregate demand of noise traders and informed investors  $\theta_I^*(p; \varepsilon) + \nu$ , as in Breon-Drish (2010, Sec. 4.2), then there exists unique REE even if Equation (50) has multiple solutions. The reason is that investor  $U$  can now directly infer  $\eta(\varepsilon)/\gamma_I + \nu$  because  $\theta_I^*(p; \varepsilon) + \nu = \eta(\varepsilon)/\gamma_I + \nu - \Omega^{-1}v/\gamma_I$ . Therefore, observing prices  $p$  and demand  $\theta_I^*(p; \varepsilon) + \nu$  allows inferring  $\eta(\varepsilon)/\gamma_I + \nu$ . In the new REE equilibrium with conditioning on demands, portfolio  $\theta_I^*(p; \varepsilon)$  and price  $P(\varepsilon, \nu)$  remain the same as in Proposition 4. The only difference is that investor  $U$ 's portfolio now directly depends on  $\eta(\varepsilon)/\gamma_I + \nu$  and is given by:

$$\theta_U^*(p; \frac{\eta(\varepsilon)}{\gamma_I} + \nu) = -\frac{1}{\gamma_U} \Omega^{-1} \left( v - \Psi \left( -\frac{\eta(\varepsilon)}{\gamma_I} - \nu \right) \right). \quad (53)$$

The latter equation can be derived similarly to Equation (46). Consequently, under the new concept of equilibrium, the REE exists, is unique, and is available in closed form.

**Remark 2 (Normal  $\nu$ ).** If  $\nu \sim N(0, \Sigma_\nu)$ , probabilities  $\pi_n(\varepsilon)$  are given by Equation (5) and  $\varphi_\varepsilon(x)$  remains general, then solving Equation (50) reduces to solving one equation with one unknown. In particular, it can be shown that  $\Psi_n(z) = F_n(\lambda^\top \Sigma_\nu^{-1} z)$ , where  $\lambda$  is the same as in Section 3, and  $F_n(w)$  is a function of one variable, given by:

$$F_n(w) = \ln \left( \int_{-\infty}^{+\infty} \exp \left\{ -\frac{\lambda^\top \Sigma_\nu^{-1} \lambda}{2\gamma_I^2} x^2 - \frac{w \cdot x}{\gamma_I} \right\} \pi_n(x) \varphi_\varepsilon(x) dx \right).$$

Then, multiplying both sides of Equation (50) by  $\lambda^\top \Sigma_\nu^{-1} \Omega^{-1}$  we obtain that  $w^*(p) = \lambda^\top \Sigma_\nu^{-1} \tilde{H}(p)$  satisfies the following one equation with one unknown:

$$\frac{1}{\gamma_U} \lambda^\top \Sigma_\nu^{-1} \Omega^{-1} F(w^*) - w^* = \frac{\gamma_U + \gamma_I}{\gamma_U \gamma_I} \lambda^\top \Sigma_\nu^{-1} \Omega^{-1} v. \quad (54)$$

It can be shown that Equation (50) has a unique solution iff Equation (54) has unique solution  $w^*$  for any  $v$ . From the latter equation we obtain that  $\tilde{H}(p) = (1/\gamma_U) \Omega^{-1} F(w^*(p)) - (\gamma_U + \gamma_I)/(\gamma_U \gamma_I) \Omega^{-1} v$ , where  $F(w) = (F_1(w) - F_N(w), \dots, F_{N-1}(w) - F_N(w))^\top$ .

## 6. Conclusion

This paper studies an REE equilibrium in a multi-asset economy with asymmetric information. In contrast to previous works, our model allows for general payoffs of assets, which do not need to be normally distributed. We provide a tractable closed-form characterization of equilibrium for a large class of probabilities of states of the economy and probability density functions of signals. We derive exact conditions under which the trading volumes for risky assets reveal information about the signal. The tractability of the model allows us to obtain simple comparative statics for optimal portfolios and asset prices.

## Appendix A: Proofs

**Proof of Lemma 1.** We derive portfolio (12) of the informed investor, whereas the proof for the uninformed is similar. Taking log on both sides of investor  $I$ 's FOC (11), and substituting wealth  $W_{I,T,n}$  from the budget constraint (3), we obtain:

$$(\theta_I^*)^\top (\Pi_n - e^{rT}p) = \frac{1}{\gamma_I} \left( \ln(\pi_n(\varepsilon)) - \ln(\pi_n^{\text{RN}}) \right) + \text{const}, \quad n = 1, \dots, N, \quad (\text{A.1})$$

where  $\text{const}$  is a constant that does not depend on  $n$ . Writing down Equation (A.1) for  $n = N$  and subtracting it from the other equations in (A.1), we obtain the following system of  $N - 1$  equations with  $N - 1$  unknown components of vector  $\theta_I^*$ :

$$(\theta_I^*)^\top (\Pi_n - \Pi_N) = \frac{1}{\gamma_I} \left( \ln\left(\frac{\pi_n(\varepsilon)}{\pi_N(\varepsilon)}\right) - \ln\left(\frac{\pi_n^{\text{RN}}}{\pi_N^{\text{RN}}}\right) \right), \quad n = 1, \dots, N, \quad (\text{A.2})$$

where  $\Pi_n - \Pi_N = (C_1(\omega_n) - C_1(\omega_N), \dots, C_{N-1}(\omega_n) - C_{N-1}(\omega_N))^\top$ . Solving the system of equations (A.2), we obtain investor  $I$ 's optimal portfolio

$$\theta_I^*(p; \varepsilon) = \frac{1}{\gamma_I} \Omega^{-1} \left\{ \left( \ln\left(\frac{\pi_1(\varepsilon)}{\pi_N(\varepsilon)}\right), \dots, \ln\left(\frac{\pi_{N-1}(\varepsilon)}{\pi_N(\varepsilon)}\right) \right)^\top - \left( \ln\left(\frac{\pi_1^{\text{RN}}}{\pi_N^{\text{RN}}}\right), \dots, \ln\left(\frac{\pi_{N-1}^{\text{RN}}}{\pi_N^{\text{RN}}}\right) \right)^\top \right\}.$$

Finally, substituting probabilities  $\pi_n(\varepsilon)$  from Equation (5) into the above equation, we obtain investor  $I$ 's portfolio weight (15). ■

**Proof of Lemma 2.** From Bayes rule we have that

$$\varphi_{\varepsilon|\tilde{\varepsilon}}(x|y) = \frac{\varphi_{\tilde{\varepsilon}|\varepsilon}(y|x)\varphi_\varepsilon(x)}{\int_{-\infty}^{\infty} \varphi_{\tilde{\varepsilon}|\varepsilon}(y|x)\varphi_\varepsilon(x)dx}.$$

Note that, since  $\nu \sim N(0, \Sigma_\nu)$ ,  $\tilde{\varepsilon} = \lambda\varepsilon/\gamma_I + \nu + H(p)$  conditional on  $\varepsilon$  has multivariate normal distribution  $N(\lambda\varepsilon/\gamma_I + H(p), \Sigma_\nu)$ . Hence substituting for  $\varphi_{\tilde{\varepsilon}|\varepsilon}$  above, we have

$$\varphi_{\varepsilon|\tilde{\varepsilon}}(x|y) = \frac{\exp\left\{-0.5\left(y - \lambda x/\gamma_I - H(p)\right)^\top \Sigma_\nu^{-1} \left(y - \lambda x/\gamma_I - H(p)\right)\right\} \varphi_\varepsilon(x)}{G_1(y; p)}, \quad (\text{A.3})$$

where  $G_1(y; p)$  is a function that does not depend on state  $\omega_n$  and normalizes the density. Next, to find probability  $\pi_n^U$ , from the market clearing condition (16), we note that by observing price  $p$  the uninformed investor can only learn that shock  $\varepsilon$  and noise trader demand  $\nu$  satisfy Equation (16). Therefore, from Equation (14) for  $\pi_n^U$  we obtain:

$$\begin{aligned} \pi_n^U &= \mathbb{E}[\pi_n(\varepsilon) | \lambda\varepsilon/\gamma_I + \nu + H(p) = 0] \\ &= \int_{-\infty}^{\infty} \frac{e^{a_n + b_n x}}{\sum_{i=1}^N e^{a_i + b_i x}} \varphi_{\varepsilon|\tilde{\varepsilon}}(x|0) dx = \frac{1}{G_1(y; p)} \int_{-\infty}^{\infty} e^{k(x)} dx, \end{aligned} \quad (\text{A.4})$$

where  $k(x)$  is a quadratic function of  $x$  given by:

$$\begin{aligned}
k(x) &= a_n + b_n x - 0.5 \left( \lambda x / \gamma_I + H(p) \right)^\top \Sigma_\nu^{-1} \left( \lambda x / \gamma_I + H(p) \right) - 0.5 (x - \mu_0)^2 / \sigma_0^2 \\
&= - \frac{\lambda^\top \Sigma_\nu^{-1} \lambda / \gamma_I^2 + 1 / \sigma_0^2}{2} \left( x - \frac{\mu_0 / \sigma_0^2 + b_n - \lambda^\top \Sigma_\nu^{-1} H(p) / \gamma_I}{\lambda^\top \Sigma_\nu^{-1} \lambda / \gamma_I^2 + 1 / \sigma_0^2} \right)^2 \\
&\quad + a_n + \frac{1}{2} \frac{b_n^2 - 2b_n \left( \lambda^\top \Sigma_\nu^{-1} H(p) / \gamma_I - \mu_0 / \sigma_0^2 \right)}{\lambda^\top \Sigma_\nu^{-1} \lambda / \gamma_I^2 + 1 / \sigma_0^2} + g(p),
\end{aligned} \tag{A.5}$$

where  $g(p)$  is some function which only depends on  $p$ , and not on  $x$  or  $n$ . Substituting Equation (A.5) back into integral (A.4), after integrating, we obtain Equation (19) for  $\pi_n^U$ ,

$$\pi_n^U(p; \theta_U^*(p)) = \frac{1}{G_2(p)} \exp \left\{ a_n + \frac{1}{2} \frac{b_n^2 - 2b_n \left( \lambda^\top \Sigma_\nu^{-1} H(p) / \gamma_I - \mu_0 / \sigma_0^2 \right)}{\lambda^\top \Sigma_\nu^{-1} \lambda / \gamma_I^2 + 1 / \sigma_0^2} \right\},$$

where  $G_2(p)$  is a function, which does not depend on  $\omega_n$  and is not needed later. ■

**Proof of Proposition 1.** Let  $v = \left( \ln(\pi_1^{\text{RN}} / \pi_N^{\text{RN}}), \dots, \ln(\pi_{N-1}^{\text{RN}} / \pi_N^{\text{RN}}) \right)^\top$ . Then, the risk-neutral probabilities are given by  $\pi_n^{\text{RN}} = e^{v_n} / (1 + \sum_{i=1}^{N-1} e^{v_i})$  for  $n = 1, \dots, N-1$  and  $\pi_N^{\text{RN}} = 1 / (1 + \sum_{i=1}^{N-1} e^{v_i})$ . Therefore, from Equation (8) for prices  $p$  we obtain that the prices are given by Equation (23).

Investor  $I$ 's portfolio (20) is the same as in Equation (15) in Lemma 1. To find investor  $U$ 's portfolio  $\theta_U^*(p)$ , we use Equation (13) in Lemma 1, which gives  $\theta_U^*(p)$  in terms of investor  $U$ 's probabilities  $\pi_n^U(p; \theta_U^*(p))$ . Substituting probabilities  $\pi_n^U(p; \theta_U^*(p))$  from (19) into portfolio (13) we obtain:

$$\begin{aligned}
\theta_U^*(p) &= \frac{1}{\gamma_U} \Omega^{-1} \left\{ \left( \ln \left( \frac{\pi_1^U(p; \theta_U^*(p))}{\pi_N^U(p; \theta_U^*(p))} \right), \dots, \ln \left( \frac{\pi_{N-1}^U(p; \theta_U^*(p))}{\pi_N^U(p; \theta_U^*(p))} \right) \right)^\top - v \right\} \\
&= \frac{1}{\gamma_U} \Omega^{-1} \left\{ \tilde{a} + \frac{1}{2} \frac{\tilde{b}^{(2)} - 2\tilde{b} \left( \lambda^\top \Sigma_\nu^{-1} H(p) / \gamma_I - \mu_0 / \sigma_0^2 \right)}{\lambda^\top \Sigma_\nu^{-1} \lambda / \gamma_I^2 + 1 / \sigma_0^2} - v \right\},
\end{aligned} \tag{A.6}$$

where  $\tilde{b}^{(2)} = (b_1^2 - b_N^2, \dots, b_{N-1}^2 - b_N^2)$  and  $\tilde{b} = (b_1 - b_N, \dots, b_{N-1} - b_N)$ . Recalling that  $\lambda = \Omega^{-1} \tilde{b}$ , and rearranging terms in Equation (A.6), we obtain:

$$\theta_U^*(p) = \frac{1}{\gamma_U} \Omega^{-1} \hat{a} + \frac{1}{\gamma_U} \frac{(\mu_0 / \sigma_0^2) \lambda}{\lambda^\top \Sigma_\nu^{-1} \lambda / \gamma_I^2 + 1 / \sigma_0^2} - Q H(p) - \frac{1}{\gamma_U} \Omega^{-1} v, \tag{A.7}$$

where  $\hat{a}$  and matrix  $Q$  are given by:

$$\hat{a} = \tilde{a} + \frac{1}{2} \frac{\tilde{b}^{(2)}}{(\lambda^\top \Sigma_\nu^{-1} \lambda / \gamma_I^2 + 1 / \sigma_0^2)}, \quad Q = \frac{\lambda \lambda^\top \Sigma_\nu^{-1}}{\gamma_U \gamma_I (\lambda^\top \Sigma_\nu^{-1} \lambda / \gamma_I^2 + 1 / \sigma_0^2)}. \tag{A.8}$$

Next, we substitute  $H(p) = \theta_v^* - \Omega^{-1}(v - \tilde{a})/\gamma_I$  from Equation (17) into Equation (A.7), and after some algebra, we obtain a system of linear equations for portfolio  $\theta_v^*(p)$ :

$$\theta_v^*(p) = \frac{1}{\gamma_U} \Omega^{-1} \hat{a} + \frac{1}{\gamma_U} \frac{(\mu_0/\sigma_0^2)\lambda}{\lambda^\top \Sigma_\nu^{-1} \lambda / \gamma_I^2 + 1/\sigma_0^2} - Q\theta_v^* + \frac{1}{\gamma_I} Q\Omega^{-1}(v - \tilde{a}) - \frac{1}{\gamma_U} \Omega^{-1} v.$$

Solving this system of equations, we obtain  $\theta_v^*(p)$  in Proposition 1, given by

$$\theta_v^*(p) = (E + Q)^{-1} \left( \frac{1}{\gamma_I} Q\Omega^{-1}(v - \tilde{a}) - \frac{1}{\gamma_U} \Omega^{-1}(v - \hat{a}) + \frac{(\mu_0/\sigma_0^2)\lambda}{\gamma_U(\lambda^\top \Sigma_\nu^{-1} \lambda / \gamma_I^2 + 1/\sigma_0^2)} \right).$$

Next, we find equilibrium prices. Substituting optimal portfolios  $\theta_I^*(p; \varepsilon)$  and  $\theta_v^*(p)$  from Equations (20) and (21) into the market clearing condition  $\theta_I^*(p; \varepsilon) + \theta_v^*(p) + \nu = 0$ , after rearranging terms, we obtain the following equation for vector  $v$ :

$$(E + Q)^{-1} \left( \frac{1}{\gamma_I} Q\Omega^{-1}(v - \tilde{a}) - \frac{1}{\gamma_U} \Omega^{-1}(v - \hat{a}) + \frac{(\mu_0/\sigma_0^2)\lambda}{\gamma_U(\lambda^\top \Sigma_\nu^{-1} \lambda / \gamma_I^2 + 1/\sigma_0^2)} \right) - \frac{1}{\gamma_I} \Omega^{-1}(v - \tilde{a}) + \frac{\lambda\varepsilon}{\gamma_I} + \nu = 0. \quad (\text{A.9})$$

We observe that the above equation can be further simplified by noting that

$$\begin{aligned} (E + Q)^{-1} \frac{1}{\gamma_I} Q\Omega^{-1}(v - \tilde{a}) &= (E + Q)^{-1} (E + Q - E) \frac{1}{\gamma_I} \Omega^{-1}(v - \tilde{a}) \\ &= \frac{1}{\gamma_I} \Omega^{-1}(v - \tilde{a}) - (E + Q)^{-1} \frac{1}{\gamma_I} \Omega^{-1}(v - \tilde{a}). \end{aligned}$$

Substituting the latter expression into Equation (A.9), canceling like terms, substituting  $\hat{a}$  from Equation (A.8) into Equation (A.9), and solving it for  $v - \tilde{a}$  we obtain

$$v = \tilde{a} + \frac{1}{2} \frac{\gamma_I}{\gamma_I + \gamma_U} \frac{\tilde{b}^{(2)} + 2(\mu_0/\sigma_0^2)\Omega\lambda}{\lambda^\top \Sigma_\nu^{-1} \lambda / \gamma_I^2 + 1/\sigma_0^2} + \frac{\gamma_I \gamma_U}{\gamma_I + \gamma_U} \Omega (E + Q) \left( \frac{\lambda\varepsilon}{\gamma_I} + \nu \right),$$

which gives  $v$  in Equation (24). The equilibrium asset prices are then given by Equation (23) in terms of vector  $v$ . Because  $v$  is a linear function of  $\lambda\varepsilon/\gamma_I + \nu$ , function  $P(\varepsilon, \nu)$  is a one-to-one mapping between  $\lambda\varepsilon/\gamma_I + \nu$  and prices  $p$ . Therefore, observing asset prices indeed reveals  $\lambda\varepsilon/\gamma_I + \nu$ , which completes the proof. ■

**Proof of Proposition 2.** Although vector  $v$  in Proposition 1 is  $(N - 1)$ -dimensional, for convenience we set  $v_N = 0$ . First, we find comparative statics for prices. Differentiating

risk-neutral probability  $\pi_n^{\text{RN}}$  given by (22) with respect to  $\varepsilon$  we obtain:

$$\begin{aligned}\frac{\partial \pi_n^{\text{RN}}}{\partial \varepsilon} &= \pi_n^{\text{RN}} \frac{\partial v_n}{\partial \varepsilon} - \frac{\pi_n^{\text{RN}}}{\sum_{k=1}^N e^{v_k}} \sum_{k=1}^N \frac{\partial v_k}{\partial \varepsilon} e^{v_k} \\ &= \pi_n^{\text{RN}} \frac{\partial v_n}{\partial \varepsilon} - \pi_n^{\text{RN}} \mathbb{E}^{\text{RN}} \left[ \frac{\partial v(\omega)}{\partial \varepsilon} \right],\end{aligned}\tag{A.10}$$

where  $v(\omega)$  now denotes a random variable that takes value  $v_n$  in state  $\omega_n$ . Next, differentiating price (23) with respect to  $\varepsilon$ , and using Equation (A.10), we obtain:

$$\begin{aligned}\frac{\partial P_m(\varepsilon, \nu)}{\partial \varepsilon} &= \mathbb{E}^{\text{RN}} \left[ \frac{\partial v(\omega)}{\partial \varepsilon} C_m(\omega) \right] - \mathbb{E}^{\text{RN}} \left[ \frac{\partial v(\omega)}{\partial \varepsilon} \right] \mathbb{E}^{\text{RN}} [C_m(\omega)] \\ &= \text{cov}^{\text{RN}} \left( \frac{\partial v(\omega)}{\partial \varepsilon}, C_m(\omega) \right).\end{aligned}\tag{A.11}$$

Differentiating Equation (24) for vector  $v$ , substituting matrix  $Q$  from Equation (25), and denoting by  $e_k = (0, \dots, 1, \dots, 0)^\top \in \mathbb{R}^{N-1}$  a vector with  $k^{\text{th}}$  element equal to 1 and other elements equal to 0, we obtain:

$$\begin{aligned}\frac{\partial v_k}{\partial \varepsilon} &= \frac{\gamma_U}{\gamma_U + \gamma_I} e_k^\top \left( \Omega \lambda + \frac{\Omega \lambda \lambda^\top \Sigma_\nu^{-1} \lambda / (\gamma_U \gamma_I)}{\lambda^\top \Sigma_\nu^{-1} \lambda / \gamma_I^2 + 1 / \sigma_0^2} \right) \\ &= \frac{\gamma_U}{\gamma_U + \gamma_I} (b_k - b_N) \left( 1 + \frac{\lambda^\top \Sigma_\nu^{-1} \lambda / (\gamma_U \gamma_I)}{\lambda^\top \Sigma_\nu^{-1} \lambda / \gamma_I^2 + 1 / \sigma_0^2} \right),\end{aligned}\tag{A.12}$$

where to derive the second line we used the fact that  $e_k^\top \Omega \lambda = e_k^\top (b_1 - b_N, \dots, b_{N-1} - b_N)^\top = b_k - b_N$ , when  $k < N$ . Equation (A.12) also holds for  $k = N$ , in which case  $\partial v_N / \partial \varepsilon = 0$  because  $b_k - b_N = 0$ . Therefore, using Equation (A.12) we compute the covariance in Equation (A.11), and obtain:

$$\frac{\partial P_m(\varepsilon, \nu)}{\partial \varepsilon} = \frac{\gamma_U}{\gamma_U + \gamma_I} \left( 1 + \frac{\lambda^\top \Sigma_\nu^{-1} \lambda / (\gamma_U \gamma_I)}{\lambda^\top \Sigma_\nu^{-1} \lambda / \gamma_I^2 + 1 / \sigma_0^2} \right) \text{cov}^{\text{RN}}(b, C_m) e^{-rT},\tag{A.13}$$

where we eliminated  $b_N$  because subtracting a constant does not affect covariances.

To find the derivative with respect to  $\nu_l$ , following the same steps as above, we obtain:

$$\frac{\partial P_m(\varepsilon, \nu)}{\partial \nu_l} = \text{cov}^{\text{RN}} \left( \frac{\partial v(\omega)}{\partial \nu_l}, C_m(\omega) \right),\tag{A.14}$$

where  $l = 1, \dots, M-1$ . Then, differentiating Equation (24) for vector  $v$  and recalling that



we additionally set  $v_N = 0$ , similarly to Equation (A.12) we obtain:

$$\begin{aligned}
\frac{\partial v_k}{\partial v_l} &= \frac{\gamma_U}{\gamma_U + \gamma_I} e_k^\top \Omega (E + Q) e_l \\
&= \frac{\gamma_U}{\gamma_U + \gamma_I} \left( e_k^\top \Omega e_l + \frac{e_k^\top \Omega \lambda \lambda^\top \Sigma_\nu^{-1} e_l / (\gamma_U \gamma_I)}{\lambda^\top \Sigma_\nu^{-1} \lambda / \gamma_I^2 + 1 / \sigma_0^2} \right), \\
&= \frac{\gamma_U}{\gamma_U + \gamma_I} \left( C_k(\omega_l) - C_k(\omega_N) + \frac{(b_k - b_N) \lambda^\top \Sigma_\nu^{-1} e_l / (\gamma_U \gamma_I)}{\lambda^\top \Sigma_\nu^{-1} \lambda / \gamma_I^2 + 1 / \sigma_0^2} \right),
\end{aligned} \tag{A.15}$$

where to derive the last line we used the fact that by the definition of matrix  $\Omega$  and vector  $\lambda$ ,  $e_k^\top \Omega e_l = C_k(\omega_l) - C_k(\omega_N)$  and  $e_k^\top \Omega \lambda = b_k - b_N$ . Clearly, Equation (A.15) also holds for  $k = N$ , because then it implies that  $\partial v_k / \partial v_l = 0$ . Therefore, from Equations (A.15) and (A.14), we obtain the result in Proposition 2:

$$\frac{\partial P_m(\varepsilon, \nu)}{\partial v_l} = \frac{\gamma_U \gamma_I}{\gamma_U + \gamma_I} \left( \text{cov}^{\text{RN}}(C_l, C_m) + \frac{\lambda^\top \Sigma_\nu^{-1} e_l / (\gamma_U \gamma_I)}{\lambda^\top \Sigma_\nu^{-1} \lambda / \gamma_I^2 + 1 / \sigma_0^2} (b, C_m) \right) e^{-rT}.$$

Now, we find the derivatives of optimal portfolios with respect to prices  $p$ . First, we need to compute  $\partial v / \partial p$ . To do this, we find the Jacobian  $J_p = \partial p / \partial v$  and then by the inverse function theorem we have  $\partial v / \partial p = J_p^{-1}$ . Substituting  $\pi_N^{\text{RN}} = 1 - \pi_1^{\text{RN}} - \dots - \pi_{N-1}^{\text{RN}}$  into Equation (8) for prices  $p$  in terms of risk-neutral probabilities, we obtain:

$$p_m = \left[ \pi_1^{\text{RN}} (C_m(\omega_1) - C_m(\omega_N)) + \dots + \pi_{N-1}^{\text{RN}} (C_m(\omega_{N-1}) - C_m(\omega_N)) + C_m(\omega_N) \right] e^{-rT}, \tag{A.16}$$

where  $m = 1, \dots, N-1$ . Let  $J_\pi$  be the Jacobian of vector  $(\pi_1^{\text{RN}}, \dots, \pi_{N-1}^{\text{RN}})^\top$ , that is, a matrix with  $(n, k)$  element given by  $\partial \pi_n^{\text{RN}} / \partial v_k$ . Differentiating Equation (A.16) we find that  $J_p = \Omega^\top J_\pi e^{-rT}$ , and hence

$$J_p \Omega e^{rT} = \Omega^\top J_\pi \Omega. \tag{A.17}$$

To find  $J_\pi$  we first calculate  $\partial \pi_n^{\text{RN}} / \partial v_k$ , where  $\pi_n^{\text{RN}}$  is given by the first equation in (22):

$$\frac{\partial \pi_n^{\text{RN}}}{\partial v_k} = \begin{cases} -\pi_n^{\text{RN}} \pi_k^{\text{RN}}, & \text{if } n \neq k, \\ \pi_n^{\text{RN}} - (\pi_n^{\text{RN}})^2, & \text{if } n = k. \end{cases} \tag{A.18}$$

From Equation (A.18) we find  $J_\pi = \text{diag}\{\pi_1^{\text{RN}}, \dots, \pi_{N-1}^{\text{RN}}\} - (\pi_1^{\text{RN}}, \dots, \pi_{N-1}^{\text{RN}})^\top (\pi_1^{\text{RN}}, \dots, \pi_{N-1}^{\text{RN}})$ , where  $\text{diag}\{\dots\}$  is a diagonal matrix. Substituting  $J_\pi$  into Equation (A.17) we obtain:

$$J_p \Omega e^{rT} = \Omega^\top \left( \text{diag}\{\pi_1^{\text{RN}}, \dots, \pi_{N-1}^{\text{RN}}\} - (\pi_1^{\text{RN}}, \dots, \pi_{N-1}^{\text{RN}})^\top (\pi_1^{\text{RN}}, \dots, \pi_{N-1}^{\text{RN}}) \right) \Omega. \tag{A.19}$$

Recalling that  $\Omega$  is a matrix with rows  $(\Pi_n - \Pi_N)^\top$ , where  $\Pi_n = (C_1(\omega_n), \dots, C_{M-1}(\omega_n))^\top$  and denoting  $\tilde{C}_n = (C_n(\omega_1) - C_n(\omega_N), \dots, C_n(\omega_{N-1}) - C_n(\omega_N))^\top$ , we find that the  $(n, k)$  element of matrix  $J_p \Omega e^{rT}$  is given by:

$$\begin{aligned} \{J_p \Omega e^{rT}\}_{n,k} &= \tilde{C}_n^\top \text{diag}\{\pi_1^{\text{RN}}, \dots, \pi_{N-1}^{\text{RN}}\} \tilde{C}_k - \tilde{C}_n^\top (\pi_1^{\text{RN}}, \dots, \pi_{N-1}^{\text{RN}})^\top (\pi_1^{\text{RN}}, \dots, \pi_{N-1}^{\text{RN}}) \tilde{C}_k \\ &= \sum_{i=1}^N (C_n(\omega_i) - C_n(\omega_N)) (C_k(\omega_i) - C_k(\omega_N)) \pi_i^{\text{RN}} \\ &\quad - \left( \sum_{i=1}^N (C_n(\omega_i) - C_n(\omega_N)) \pi_i^{\text{RN}} \right) \left( \sum_{i=1}^N (C_k(\omega_i) - C_k(\omega_N)) \pi_i^{\text{RN}} \right) \\ &= \text{cov}^{\text{RN}}(C_n, C_k), \end{aligned}$$

where to derive the second equality we added zero terms  $(C_n(\omega_N) - C_n(\omega_N))(C_k(\omega_N) - C_k(\omega_N))\pi_N^{\text{RN}}$ ,  $(C_n(\omega_N) - C_n(\omega_N))\pi_N^{\text{RN}}$  and  $(C_k(\omega_N) - C_k(\omega_N))\pi_N^{\text{RN}}$  to summations, and then removed constants  $C_n(\omega_N)$  and  $C_k(\omega_N)$ , because they do not affect covariances.

Therefore, we conclude that  $J_p \Omega e^{rT} = \text{var}^{\text{RN}}[\Pi]$ . Then, by the inverse function theorem, we now find that  $\Omega^{-1} \partial v / \partial p = (\text{var}^{\text{RN}}[\Pi])^{-1} e^{rT}$ . Using the latter equality and differentiating optimal portfolios (20) and (21) with respect to  $p$  we obtain that the first of these two partial derivatives is given by (28) and the second is given by:

$$\frac{\partial \theta_u^*(p)}{\partial p} = \left( \frac{1}{\gamma_I} E - \frac{\gamma_I + \gamma_U}{\gamma_I \gamma_U} (E + Q)^{-1} \right) (\text{var}^{\text{RN}}[\Pi])^{-1} e^{rT}. \quad (\text{A.20})$$

Using Lemma A.1 below, we find that

$$(E + Q)^{-1} = E - \frac{\lambda \lambda^\top \Sigma_\nu^{-1}}{\gamma_U \gamma_I (\lambda^\top \Sigma_\nu^{-1} \lambda / \gamma_I^2 + 1 / \sigma_0^2) + \lambda^\top \Sigma_\nu^{-1} \lambda}.$$

Substituting  $(E+Q)^{-1}$  above into Equation (A.20), we obtain Equation (29) for  $\partial \theta_u^*(p) / \partial p^\top$ .

Finally, we demonstrate that  $\partial \theta_{I,m}^*(p; \varepsilon) / \partial p_m < 0$ , i.e. investor  $I$ 's demand for asset  $m$  is downward sloping in asset  $m^{\text{th}}$  price. This result follows from the fact that matrix  $(\text{var}^{\text{RN}}[\Pi])^{-1}$  is positive-definite (as the inverse of a positive-definite matrix), and its element  $m$  of the diagonal is given by  $e_m^\top (\text{var}^{\text{RN}}[\Pi])^{-1} e_m > 0$ , where  $e_m = (0, 0, \dots, 1, \dots, 0)^\top$  is a vector with  $m^{\text{th}}$  element equal to 1 and other elements equal to zero. Then, from Equation (28) it follows that  $\partial \theta_{I,m}^*(p; \varepsilon) / \partial p_m < 0$ . ■

**Proof of Proposition 3.** Investor  $I$ 's optimization problem (32) yields the FOC for the optimal  $\hat{\theta}_I^* = \lambda \varepsilon - \gamma_I \theta_I^*$ :

$$g_I'(\hat{\theta}_I^*) = e^{rT} p, \quad (\text{A.21})$$

where  $g_I(x) = \ln\left(\sum_{n=1}^N \exp\{a_n + \Pi_n^\top x\}\right)$  and  $g'_I(x) = \partial g_I(x)/\partial x^\top$  is a column vector for  $x \in \mathbb{R}^{M-1}$ . Assuming that  $g'_I(\cdot)$  is invertible (which we prove below), we find that  $\hat{\theta}_I^* = f_I^{-1}(e^{rT} p)$ . Then, from equation  $\hat{\theta}_I^* = \lambda\varepsilon - \gamma_I \theta_I^*$ , which defines  $\hat{\theta}_I^*$ , we obtain portfolio  $\theta_I^*(p, \varepsilon)$  in Equation (35).

Now, we find the portfolio of investor  $U$ . Let  $\tilde{\varepsilon} = \lambda\varepsilon/\gamma_I + \nu + \hat{H}(p)$ , i.e., the left hand side of the market clearing condition (34), where  $\hat{H}(p) = -\hat{\theta}_I^*(p)/\gamma_I + \theta_U^*(p)$ . The inference problem of investor  $U$  is similar to that in the complete-markets economy. Following exactly the same steps as in Lemma 1, we obtain:

$$\begin{aligned} \varphi_{\varepsilon|\tilde{\varepsilon}}(x|y) &= \exp\left\{-0.5\left(y - \lambda x/\gamma_I - \hat{H}(p)\right)^\top \Sigma_\nu^{-1} \left(y - \lambda x/\gamma_I - \hat{H}(p)\right)\right\} \frac{\varphi_\varepsilon(x)}{G_1(y; p)}, \\ \pi_n^U(p; \theta_U^*(p)) &= \exp\left\{a_n + \frac{1}{2} \frac{b_n^2 - 2b_n(\lambda^\top \Sigma_\nu^{-1} \hat{H}(p)/\gamma_I - \mu_0/\sigma_0^2)}{\lambda^\top \Sigma_\nu^{-1} \lambda/\gamma_I^2 + 1/\sigma_0^2}\right\} \frac{1}{G_2(p)}, \end{aligned}$$

where  $G_1(y; p)$  and  $G_2(p)$  are some functions, irrelevant for subsequent derivations. Moreover, using that, by Assumption 1,  $b_n = \lambda_0 + \Pi_n^\top \lambda$ , from the last equation we obtain:

$$\begin{aligned} \pi_n^U(p; \theta_U^*(p)) &= \exp\left\{a_n + \frac{1}{2} \frac{b_n^2 + 2\Pi_n^\top (\lambda\mu_0/\sigma_0^2 - \lambda\lambda^\top \Sigma_\nu^{-1} \hat{H}(p)/\gamma_I)}{\lambda^\top \Sigma_\nu^{-1} \lambda/\gamma_I^2 + 1/\sigma_0^2}\right\} \times \\ &\quad \times \exp\left\{\lambda_0 \frac{(\mu_0/\sigma_0^2 - \lambda^\top \Sigma_\nu^{-1} \hat{H}(p)/\gamma_I)}{\lambda^\top \Sigma_\nu^{-1} \lambda/\gamma_I^2 + 1/\sigma_0^2}\right\} \frac{1}{G_2(p)} \\ &= \exp\left\{a_n + \frac{1}{2} \frac{b_n^2 + 2\Pi_n^\top (\lambda\mu_0/\sigma_0^2 - \lambda\lambda^\top \Sigma_\nu^{-1} \hat{H}(p)/\gamma_I)}{\lambda^\top \Sigma_\nu^{-1} \lambda/\gamma_I^2 + 1/\sigma_0^2}\right\} \frac{1}{G_3(p)}, \end{aligned}$$

where  $G_3(p)$  is a function that does not depend on  $n$  and is not needed for the proofs.

Using probabilities  $\pi_n^U$  we rewrite investor  $U$ 's objective function (9) as follows:

$$\begin{aligned} -\sum_{n=1}^N \pi_n^U(p; \theta_U) \exp\{-\gamma_U (W_{U,0} e^{rT} + \theta_U^\top (\Pi_n - e^{rT} p))\} &= -\frac{\exp\{-\gamma_U W_{U,0} e^{rT}\}}{G_3(p)} \times \\ \exp\{\gamma_U e^{rT} p^\top \theta_U\} \sum_{n=1}^N \exp\left\{a_n + \frac{1}{2} \frac{b_n^2 + 2\Pi_n^\top (\lambda\mu_0/\sigma_0^2 - \lambda\lambda^\top \Sigma_\nu^{-1} \hat{H}(p)/\gamma_I)}{\lambda^\top \Sigma_\nu^{-1} \lambda/\gamma_I^2 + 1/\sigma_0^2} - \gamma_U \Pi_n^\top \theta_U\right\}. \end{aligned} \tag{A.22}$$

Factoring out  $\Pi_n^\top$  in the term in the curly brackets in the last line above we have

$$\begin{aligned} a_n + \frac{1}{2} \frac{b_n^2 + 2\Pi_n^\top (\lambda\mu_0/\sigma_0^2 - \lambda\lambda^\top \Sigma_\nu^{-1} \hat{H}(p)/\gamma_I)}{\lambda^\top \Sigma_\nu^{-1} \lambda/\gamma_I^2 + 1/\sigma_0^2} - \gamma_U \Pi_n^\top \theta_U &= \\ a_n + \frac{1}{2} \frac{b_n^2}{\lambda^\top \Sigma_\nu^{-1} \lambda/\gamma_I^2 + 1/\sigma_0^2} + \Pi_n^\top \left( \frac{\lambda\mu_0/\sigma_0^2 - \lambda\lambda^\top \Sigma_\nu^{-1} \hat{H}(p)/\gamma_I}{\lambda^\top \Sigma_\nu^{-1} \lambda/\gamma_I^2 + 1/\sigma_0^2} - \gamma_U \theta_U \right). \end{aligned}$$

Now, similarly to  $g_I(\cdot)$ , we define function  $g_U : \mathbb{R}^{M-1} \rightarrow \mathbb{R}$  for  $x \in \mathbb{R}^{M-1}$ :

$$g_U(x) = \ln\left(\sum_{n=1}^N \exp\left\{a_n + \frac{1}{2} \frac{b_n^2}{\lambda^\top \Sigma_\nu^{-1} \lambda / \gamma_I^2 + 1 / \sigma_0^2} + \Pi_n^\top x\right\}\right).$$

Then, investor  $U$ 's optimization problem from Equation (A.22), becomes

$$\min_{\theta_U} \gamma_U e^{rT} p^\top \theta_U + g_U\left(\frac{\lambda \mu_0 / \sigma_0^2 - \lambda \lambda^\top \Sigma_\nu^{-1} \hat{H}(p) / \gamma_I}{\lambda^\top \Sigma_\nu^{-1} \lambda / \gamma_I^2 + 1 / \sigma_0^2} - \gamma_U \theta_U\right).$$

Let  $f_U = g'_U$ , then the FOC for the uniformed's optimal portfolio,  $\theta_U^*$  is,

$$f_U\left(\frac{\lambda \mu_0 / \sigma_0^2 - \lambda \lambda^\top \Sigma_\nu^{-1} \hat{H}(p) / \gamma_I}{\lambda^\top \Sigma_\nu^{-1} \lambda / \gamma_I^2 + 1 / \sigma_0^2} - \gamma_U \theta_U^*\right) = e^{rT} p.$$

Assuming that  $f_U$  is invertible, as shown below, and  $e^{rT} p$  belongs to its range, we obtain

$$\frac{\lambda \mu_0 / \sigma_0^2 - \lambda \lambda^\top \Sigma_\nu^{-1} \hat{H}(p) / \gamma_I}{\lambda^\top \Sigma_\nu^{-1} \lambda / \gamma_I^2 + 1 / \sigma_0^2} - \gamma_U \theta_U^* = f_U^{-1}(e^{rT} p).$$

Substituting for  $\hat{H}(p) = -f_I^{-1}(e^{rT}) / \gamma_I + \theta_U^*$  and factoring out  $\gamma_U \theta_U^*$  we have

$$\frac{\lambda \mu_0 / \sigma_0^2 + \lambda \lambda^\top \Sigma_\nu^{-1} f_I^{-1}(e^{rT} p) / \gamma_I^2}{\lambda^\top \Sigma_\nu^{-1} \lambda / \gamma_I^2 + 1 / \sigma_0^2} - \gamma_U \theta_U^* (E + Q) = f_U^{-1}(e^{rT} p),$$

where, as before,  $E$  is the  $(M-1) \times (M-1)$  identity matrix and matrix  $Q$  is given by

$$Q = \frac{\lambda \lambda^\top \Sigma_\nu^{-1}}{\gamma_U \gamma_I (\lambda^\top \Sigma_\nu^{-1} \lambda / \gamma_I^2 + 1 / \sigma_0^2)},$$

as in Proposition 1. Solving for  $\theta_U^*$  yields

$$\begin{aligned} \theta_U^*(p) &= \frac{1}{\gamma_U} (E + Q)^{-1} \left( \frac{\lambda \mu_0 / \sigma_0^2 + \lambda \lambda^\top \Sigma_\nu^{-1} f_I^{-1}(e^{rT} p) / \gamma_I^2}{\lambda^\top \Sigma_\nu^{-1} \lambda / \gamma_I^2 + 1 / \sigma_0^2} - f_U^{-1}(e^{rT} p) \right) \\ &= (E + Q)^{-1} \left( \frac{1}{\gamma_I} Q f_I^{-1}(e^{rT} p) - \frac{1}{\gamma_U} f_U^{-1}(e^{rT} p) + \frac{(\mu_0 / \sigma_0^2) \lambda}{\gamma_U (\lambda^\top \Sigma_\nu^{-1} \lambda / \gamma_I^2 + 1 / \sigma_0^2)} \right). \end{aligned} \quad (\text{A.23})$$

Now, we verify that function  $f_I(x)$  is invertible. The invertibility of  $f_U(x)$  is demonstrated along the same lines. Recalling that by definition  $\exp(g_I(x)) = \sum_{n=1}^N \exp\{a_n + \Pi_n^\top x\}$  and then differentiating both sides of the latter equation twice, we obtain:

$$\frac{\partial g_I(x)}{\partial x^\top} \exp(g_I(x)) = \sum_{n=1}^N \Pi_n \exp\{a_n + \Pi_n^\top x\}, \quad (\text{A.24})$$

$$\left( \frac{\partial^2 g_I(x)}{\partial x^\top \partial x} + \frac{\partial g_I(x)}{\partial x^\top} \frac{\partial g_I(x)}{\partial x} \right) \exp(g_I(x)) = \sum_{n=1}^N \Pi_n \Pi_n^\top \exp\{a_n + \Pi_n^\top x\}. \quad (\text{A.25})$$

Next, we introduce a new probability measure  $q_n(x) = \exp \{a_n + \Pi_n^\top x\} / \sum_{k=1}^N \exp \{a_k + \Pi_k^\top x\}$ . Dividing both sides of Equations (A.24) and (A.25) by  $\exp(g_I(x)) = \sum_{n=1}^N \exp \{a_n + \Pi_n^\top x\}$ , respectively, we observe that the derivatives can be rewritten as

$$\frac{\partial g_I(x)}{\partial x^\top} = \mathbb{E}^{q(x)}[\Pi], \quad \frac{\partial^2 g_I(x)}{\partial x^\top \partial x} = \text{var}^{q(x)}[\Pi],$$

where  $\mathbb{E}^{q(x)}[\Pi]$  and  $\text{var}^{q(x)}[\Pi]$  are the mean and the variance of the risky assets payoffs vector  $\Pi$  under probability measure  $q(x)$ . Because all assets are non-redundant, matrix  $\text{var}^{q(x)}[\Pi]$  is positive definite, and hence is invertible. Then, function  $f_I(x) = \partial g_I(x) / \partial x^\top$  is invertible on its range by Theorem 6 in Gale and Nikaidô (1965).

Finally, we derive the equation for prices. Substituting  $\theta_I^*$  and  $\theta_U^*$  from Equations (35) and (36) into the market clearing condition  $\theta_I^*(p; \varepsilon) + \theta_U^*(p) + \nu = 0$  yields, after some algebra, the following system of nonlinear algebraic equations for prices,

$$\frac{1}{\gamma_U} f_U^{-1}(e^{r^\top} P(\varepsilon, \nu)) + \frac{1}{\gamma_I} f_I^{-1}(e^{r^\top} P(\varepsilon, \nu)) = (E + Q) \left( \frac{\lambda \varepsilon}{\gamma_I} + \nu \right) + \frac{(\mu_0 / \sigma_0^2) \lambda}{\gamma_U (\lambda^\top \Sigma_\nu^{-1} \lambda / \gamma_I^2 + 1 / \sigma_0^2)}.$$

The function of the left-hand side of the latter equation has a positive-definite Jacobian in each point, which is the sum of positive definite Jacobians of functions  $f_U^{-1}(e^{r^\top} p) / \gamma_U$  and  $f_I^{-1}(e^{r^\top} p) / \gamma_I$ . Then, by Theorem 6 in Gale and Nikaidô (1965), the function on the left-hand side is univalent, and hence invertible on its range. If the range of this function coincides with  $\mathbb{R}^{M-1}$  then price  $P(\varepsilon, \nu)$  exists and is unique. ■

**Proof of Lemma 3.** The proof is similar to that of Lemma 2. Assume that  $\varepsilon \sim N(\mu_\varepsilon, \sigma_\varepsilon^2)$ , and we observe vector  $\tilde{\varepsilon} = \lambda \varepsilon + \nu + H(p)$ . From Bayes' rule

$$\varphi_{\varepsilon|\tilde{\varepsilon}}(x|y) = \frac{\varphi_{\tilde{\varepsilon}|\varepsilon}(y|x) \varphi_\varepsilon(x)}{\int_{-\infty}^{\infty} \varphi_{\tilde{\varepsilon}|\varepsilon}(y|x) \varphi_\varepsilon(x) dx}, \quad (\text{A.26})$$

where now  $\varphi_\varepsilon(x) = (1/\sqrt{2\pi\sigma_\varepsilon^2}) \exp(-0.5(x - \mu_\varepsilon)^2/\sigma_\varepsilon^2)$ . Since  $\nu \sim N(0, \Sigma_\nu)$ ,  $\tilde{\varepsilon} = \lambda \varepsilon / \gamma_I + \nu + H(p)$  conditional on  $\varepsilon$  has a multivariate normal distribution  $N(\lambda \varepsilon / \gamma_I + H(p), \Sigma_\nu)$ . Substituting for  $\varphi_{\tilde{\varepsilon}|\varepsilon}$  and  $\varphi_\varepsilon$  in the numerator above, we have

$$\begin{aligned} \varphi_\varepsilon(x) &= \exp\left\{-0.5(y - \lambda x / \gamma_I - H(p))^\top \Sigma_\nu^{-1} (y - \lambda x / \gamma_I - H(p)) - 0.5(x - \mu_\varepsilon)^2 / \sigma_\varepsilon^2\right\} \frac{1}{G_1(y, p)}, \\ &= \exp\left\{-0.5\left(1/\sigma_\varepsilon^2 + \lambda^\top \Sigma_\nu^{-1} \lambda / \gamma_I^2\right) \left(x - \frac{\mu_\varepsilon / \sigma_\varepsilon^2 + (y - H(p))^\top \Sigma_\nu^{-1} \lambda / \gamma_I}{1/\sigma_\varepsilon^2 + \lambda^\top \Sigma_\nu^{-1} \lambda / \gamma_I^2}\right)^2\right\} \frac{1}{G_2(y; p)} \end{aligned}$$

where  $G_1(y; p)$  and  $G_2(y; p)$  are some functions that do not depend on  $x$ . We observe that the above equation gives the PDF of a standard normal distribution with mean and precision parameters given by

$$\hat{\mu}_\varepsilon = \frac{\mu_\varepsilon/\sigma_\varepsilon^2 + (y - H(p))^\top \Sigma_\nu^{-1} \lambda / \gamma_I}{1/\sigma_\varepsilon^2 + \lambda^\top \Sigma_\nu^{-1} \lambda / \gamma_I^2}, \quad \frac{1}{\hat{\sigma}_\varepsilon^2} = \frac{1}{\sigma_\varepsilon^2} + \frac{\lambda^\top \Sigma_\nu^{-1} \lambda}{\gamma_I^2},$$

which completes the proof. ■

**Proof of Proposition 4.** First, we find posterior probabilities. Let  $\tilde{\varepsilon} = \eta(\varepsilon)/\gamma_I + \nu + \tilde{H}(p)$ , where  $\tilde{H}(p) = \theta_U^*(p) - (1/\gamma_I)\Omega^{-1}v$ . Then, conditional density  $\varphi_{\varepsilon|\tilde{\varepsilon}}(x|y)$  is given by:

$$\varphi_{\varepsilon|\tilde{\varepsilon}}(x|y) = \frac{\varphi_\nu(y - \eta(x)/\gamma_I - \tilde{H}(p))\varphi_\varepsilon(x)}{\int_{-\infty}^{+\infty} \varphi_\nu(y - \eta(x)/\gamma_I - \tilde{H}(p))\varphi_\varepsilon(x)dx}. \quad (\text{A.27})$$

Suppose the equilibrium exists. Portfolio (45) of investor  $I$  remains exactly the same as in Equation (12) in Lemma 1. Now, we find investor  $U$ 's portfolio. Similarly to Equation (A.4) in the Proof of Lemma 2, we note from the market clearing condition (44) that in equilibrium  $\tilde{\varepsilon} = 0$ , and then find investor  $U$ 's probabilities  $\pi_n^U$  as follows:

$$\begin{aligned} \pi_n^U &= \int_{-\infty}^{+\infty} \pi_n(x)\varphi_{\varepsilon|\tilde{\varepsilon}}(x|0)dx \\ &= \frac{1}{G_1(p)} \int_{-\infty}^{+\infty} \varphi_\nu\left(-\frac{\eta(x)}{\gamma_I} - \tilde{H}(p)\right)\pi_n(x)\varphi_\varepsilon(x)dx, \\ &= \frac{\exp\left\{\Psi_n\left(\tilde{H}(p)\right)\right\}}{G_2(p)}, \end{aligned} \quad (\text{A.28})$$

where  $G_1(p)$  and  $G_2(p)$  are some functions that do not depend on  $n$ . From Equation (A.28) we obtain that  $\ln(\pi_n^U/\pi_N^U) = \Psi_n(\tilde{H}(p)) - \Psi_N(\tilde{H}(p))$ . Substituting the latter expression into Equation (13) for investor  $U$ 's portfolio we obtain portfolio  $\theta_U^*(p)$  in (46).

Subtracting  $(1/\gamma_I)\Omega^{-1}v$  from both sides of investor  $U$ 's portfolio (46), multiplying both sides by  $\Omega$ , using the definition of  $\tilde{H}(p)$ , and rearranging terms we obtain Equation (50):

$$\frac{1}{\gamma_U}\Psi(\tilde{H}(p)) - \Omega\tilde{H}(p) = \frac{\gamma_U + \gamma_I}{\gamma_U\gamma_I}v. \quad (\text{A.29})$$

Next, we derive vector  $v$ . From the market clearing condition (44) we observe that in equilibrium  $\tilde{H}(p) = -\eta(\varepsilon)/\gamma_I - \nu$ . Substituting the latter expression into Equation (A.29) and solving it for  $v$ , we obtain Equation (49) for vector  $v$ . Then, risk-neutral probabilities

and asset prices are given by Equations (47) and (48), respectively, which can be shown exactly in the same way as in Proposition 1. This completes the derivation of equilibrium portfolios, risk-neutral probabilities, and prices.

It remains to verify that price  $P(\varepsilon, \nu)$  uniquely reveals combination  $\eta(\varepsilon)/\gamma_I + \nu$  when Equation (50) has unique solution. Suppose, there exist pairs  $(\varepsilon_1, \nu_1)$  and  $(\varepsilon_2, \nu_2)$  such that  $\eta(\varepsilon_1)/\gamma_I + \nu_1 \neq \eta(\varepsilon_2)/\gamma_I + \nu_2$  and  $P(\varepsilon_1, \nu_1) = P(\varepsilon_2, \nu_2)$ . We note, that the equation for  $P(\varepsilon, \nu)$  is derived from Equation (A.29) by setting  $\tilde{H}(p) = -\eta(\varepsilon)/\gamma_I - \nu$ , as discussed above. Therefore, for fixed  $p = P(\varepsilon_1, \nu_1) = P(\varepsilon_2, \nu_2)$  Equation (A.29) has two solutions:  $\tilde{H}_1(p) = -\eta(\varepsilon_1)/\gamma_I - \nu_1$  and  $\tilde{H}_2(p) = -\eta(\varepsilon_2)/\gamma_I - \nu_2$ , which leads to contradiction. Therefore, the prices uniquely reveal  $\eta(\varepsilon)/\gamma_I + \nu$ , and hence we have the REE.

Suppose, there exists an REE but Equation (A.29) has multiple solutions. Note that because  $v$  is derived by setting  $\tilde{H}(p) = -\eta(\varepsilon)/\gamma_I - \nu$  in Equation (A.29), if the latter equation has multiple solutions then there exist shocks and noisy demands such that  $\eta(\varepsilon_1)/\gamma_I + \nu_1 \neq \eta(\varepsilon_2)/\gamma_I + \nu_2$  and  $P(\varepsilon_1, \nu_1) = P(\varepsilon_2, \nu_2)$ . However, market clearing condition (44) implies that if  $P(\varepsilon_1, \nu_1) = P(\varepsilon_2, \nu_2)$  then  $\eta(\varepsilon_1)/\gamma_I + \nu_1 = \eta(\varepsilon_2)/\gamma_I + \nu_2$ , which leads to contradiction. Therefore, REE exists iff Equation (50) has unique solution. ■

**Lemma A.1 (Sherman-Morrison formula).** *Let  $A \in \mathbb{R}^{(M-1) \times (M-1)}$  be an invertible matrix and  $u, w \in \mathbb{R}^{M-1}$  be two column vectors. Then, matrix  $(A + uw^\top)$  is invertible, and its inverse is given by:*

$$(A + uw^\top)^{-1} = A^{-1} - \frac{A^{-1}uw^\top A^{-1}}{1 + w^\top A^{-1}u}. \quad (\text{A.30})$$

**Proof.** This Lemma is a special case of binomial inversion theorem and can be directly verified by multiplying the right-hand side of Equation (A.30) by  $(A + uw^\top)$ . ■

## Appendix B: Three Benchmark Economies

Below, we present three benchmark cases: 1) a full information economy where both investors observe shock  $\varepsilon$ ; 2) an economy where both investors are uninformed; 3) and an economy with naive uninformed investors that do not learn anything from prices. Proposition B.1 below provides closed-form characterizations of equilibria in all three cases.

### Proposition B.1 (Fully Informed, Fully Uninformed and Naive Equilibria).

1) If both investors  $i = I, U$  have full information, then optimal portfolios  $\theta_i^*$  are given by

$$\theta_i^*(p; \varepsilon) = \frac{\lambda \varepsilon}{\gamma_i} - \frac{1}{\gamma_i} \Omega^{-1}(v^{FI} - \tilde{a}), \quad (\text{B.1})$$

and asset prices are given by Equation (23), in which  $v$  is replaced with

$$v^{FI} = \tilde{a} + \Omega \lambda \varepsilon + \frac{\gamma_I \gamma_U}{\gamma_I + \gamma_U} \Omega \nu. \quad (\text{B.2})$$

2) If both investors  $i = I, U$  are uninformed and have prior distribution  $\varepsilon \sim \hat{N}(\mu_\varepsilon, \sigma_\varepsilon^2)$ , then their optimal portfolios are given by

$$\theta_i^*(p) = -\frac{1}{\gamma_i} \Omega^{-1}(v^{FU} - \hat{a}^{FU}) + \frac{1}{\gamma_i} \mu_0 \lambda, \quad (\text{B.3})$$

where  $\hat{a}^{FU} = \tilde{a} + 0.5 \tilde{b}^{(2)} \sigma_0^2$ , and asset prices are given by Equation (23), in which  $v$  is replaced with

$$v^{FU} = \tilde{a} + \frac{1}{2} \frac{\tilde{b}^{(2)} + 2(\mu_0/\sigma_0^2)\Omega\lambda}{1/\sigma_0^2} + \frac{\gamma_I \gamma_U}{\gamma_I + \gamma_U} \Omega \nu. \quad (\text{B.4})$$

3) If investor  $U$  naively ignores the information provided by prices, then the optimal portfolios of investors  $I$  and  $U$  are given by

$$\theta_I^*(p; \varepsilon) = \frac{\lambda \varepsilon}{\gamma_I} - \frac{1}{\gamma_I} \Omega^{-1}(v^{NV} - \tilde{a}), \quad (\text{B.5})$$

$$\theta_U^*(p) = \frac{\lambda \mu_0}{\gamma_U} - \frac{1}{\gamma_U} \Omega^{-1}(v^{NV} - \hat{a}^{NV}), \quad (\text{B.6})$$

where  $\hat{a}^{NV} = \tilde{a} + 0.5 \tilde{b}^{(2)} \sigma_0^2$ , and asset prices are given by Equation (23), in which  $v$  is replaced with

$$v^w = \tilde{a} + \frac{1}{2} \frac{\gamma_I}{\gamma_I + \gamma_U} \frac{\tilde{b}^{(2)} + 2(\mu_0/\sigma_0^2)\Omega\lambda}{1/\sigma_0^2} + \frac{\gamma_I \gamma_U}{\gamma_I + \gamma_U} \Omega \left( \frac{\lambda \varepsilon}{\gamma_I} + \nu \right), \quad (\text{B.7})$$



In all cases vectors  $\tilde{a}$  and  $\tilde{b}^{(2)}$  and matrix  $\Omega$  are the same as in Proposition 1.

**Proof of Proposition B.1.** In all cases  $v$ ,  $\tilde{a}$ ,  $\tilde{b}^{(2)}$  are as in Proposition 1 and  $\tilde{b} = (b_1 - b_N, \dots, b_{N-1} - b_N)$ .

- 1) In the *fully informed* case both investors observe  $\varepsilon$  at  $t = -1$ . Similarly to Equation (15), we can write Equation (B.1) for the optimal portfolio of investors

$$\theta_i^*(p; \varepsilon) = \frac{\lambda \varepsilon}{\gamma_i} - \frac{1}{\gamma_i} \Omega^{-1} (v - \tilde{a}), \quad (\text{B.8})$$

where  $\lambda = \Omega^{-1} \tilde{b}$ . Substituting Equation (B.8) in the market clearing condition  $\theta_I^*(p; \varepsilon) + \theta_U^*(p; \varepsilon) + \nu = 0$  and solving for  $v$  produces Equation (B.2).

- 2) In the *fully uniformed* case both  $U$  and  $I$  investors just have a prior over  $\varepsilon$  at  $t = -1$  but do not observe  $\varepsilon$ . Consequently, they do not learn from prices. Hence, their posteriors over  $\varepsilon$  are identical to their common prior. From Equation (C.3) in Appendix C,  $\pi_n^i = \exp(a_n + b_n^2 \sigma_0^2 / 2 + b_n \mu_0) / G_2$  ( $i \in \{I, U\}$ ,  $n = 1, \dots, N$ ), where  $G_2$  is a constant irrelevant for subsequent derivations. Similarly to Equation (13) we can write Equation (B.3) for their optimal portfolios

$$\theta_i^*(p) = \frac{1}{\gamma_i} \Omega^{-1} (\tilde{a} + \tilde{b}^{(2)} \sigma_0^2 / 2 + \tilde{b} \mu_0 - v) = \frac{\lambda \mu_0}{\gamma_i} - \frac{1}{\gamma_i} \Omega^{-1} (v - \hat{a}), \quad (\text{B.9})$$

where in the second equality we substituted  $\hat{a} = \tilde{a} + \tilde{b}^{(2)} \sigma_0^2 / 2$ , and  $\lambda = \Omega^{-1} \tilde{b}$ . Substituting Equation (B.9) in the market clearing condition  $\theta_I^*(p) + \theta_U^*(p) + \nu = 0$  and solving for  $v$  produces Equation (B.4).

- 3) In the *naive* case, investor  $U$  ignores information provided by prices and only uses the prior. Investor  $I$  knows  $\varepsilon$  as before. Hence, similarly to Equations (12) and (13), and using the prior from Equation (C.3) in Appendix C, we can write Equations (B.5) and (B.6) for investors' optimal portfolios

$$\theta_I^*(p; \varepsilon) = \frac{1}{\gamma_I} \Omega^{-1} (\tilde{a} + \tilde{b} \varepsilon - v) = \frac{\lambda \varepsilon}{\gamma_I} - \frac{1}{\gamma_I} \Omega^{-1} (v - \tilde{a}), \quad (\text{B.10})$$

$$\theta_U^*(p) = \frac{1}{\gamma_U} \Omega^{-1} (\tilde{a} + \tilde{b}^{(2)} \sigma_0^2 / 2 + \tilde{b} \mu_0 - v) = \frac{\lambda \mu_0}{\gamma_U} - \frac{1}{\gamma_U} \Omega^{-1} (v - \hat{a}), \quad (\text{B.11})$$

where in the second equalities of both equations we used that  $\lambda = \Omega^{-1} \tilde{b}$ , and in the second equality of the second equation we also substituted for  $\hat{a} = \tilde{a} + \tilde{b}^{(2)} \sigma_0^2 / 2$ . Substituting Equations (B.10) and (B.11) in the market clearing condition  $\theta_I^*(p; \varepsilon) + \theta_U^*(p) + \nu = 0$  and solving for  $v$  produces Equation (B.7). ■

## Appendix C: Mean and Variance of $\varepsilon$

**Lemma C.1 (Prior mean and prior variance of  $\varepsilon$  and prior probabilities).** *Assume that  $\varepsilon$  follows Distribution (6), then its mean  $\mu_\varepsilon$  and variance  $\sigma_\varepsilon^2$  are given by the following expressions in terms of the parameters  $(\mu_0, \sigma_0^2)$  and the vectors  $(a, b)$ :*

$$\mu_\varepsilon = \frac{\sum_{i=1}^N \exp\left(a_i + \frac{\mu_i^2}{2\sigma_0^2}\right) \mu_i}{\sum_{i=1}^N \exp\left(a_i + \frac{\mu_i^2}{2\sigma_0^2}\right)}, \quad (\text{C.1})$$

$$\sigma_\varepsilon^2 = \sigma_0^2 + \left[ \frac{\sum_{i=1}^N \exp\left(a_i + \frac{\mu_i^2}{2\sigma_0^2}\right) \mu_i^2}{\sum_{i=1}^N \exp\left(a_i + \frac{\mu_i^2}{2\sigma_0^2}\right)} - \frac{\left(\sum_{i=1}^N \exp\left(a_i + \frac{\mu_i^2}{2\sigma_0^2}\right) \mu_i\right)^2}{\left(\sum_{i=1}^N \exp\left(a_i + \frac{\mu_i^2}{2\sigma_0^2}\right)\right)^2} \right], \quad (\text{C.2})$$

where  $\mu_i = (b_i + \mu_0/\sigma_0^2)/(1/\sigma_0^2)$ . Furthermore, the prior unconditional probabilities of states  $\omega_1, \dots, \omega_N$ , defined as  $\pi_n^{\text{prior}} = \mathbb{E}[\pi_n(\varepsilon)]$ , are given by:

$$\pi_n^{\text{prior}} = \frac{\exp(a_n + b_n \mu_0 + \sigma_0^2 b_n^2 / 2)}{\sum_{k=1}^N \exp(a_k + b_k \mu_0 + \sigma_0^2 b_k^2 / 2)}, \quad n = 1, \dots, N. \quad (\text{C.3})$$

**Proof of Lemma C.1.** The PDF of  $\varepsilon$  is given by

$$\varphi_\varepsilon(x) = \frac{\sum_{i=1}^N e^{a_i + b_i x} \phi(x; \mu_0, \sigma_0)}{\int_{-\infty}^{\infty} \sum_{i=1}^N e^{a_i + b_i x} \phi(x; \mu_0, \sigma_0) dx},$$

where  $\phi(x; \mu, \sigma)$  denotes the pdf of a random variable distributed  $N(\mu, \sigma^2)$ ,

$$\phi(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}.$$

After some algebra, we rewrite  $\varphi_\varepsilon(x)$  as follows:

$$\begin{aligned} \varphi_\varepsilon(x) &= \frac{\sum_{i=1}^N e^{a_i + b_i \mu_0 + \sigma_0^2 b_i^2 / 2} \phi(x; \sigma_0^2 b_i + \mu_0, \sigma_0)}{\sum_{i=1}^N e^{a_i + b_i \mu_0 + \sigma_0^2 b_i^2 / 2}}, \\ &= \frac{\sum_{i=1}^N \exp\left(a_i + \frac{\mu_i^2}{2\sigma_0^2}\right) \phi(x; \mu_i, \sigma_0)}{\sum_{i=1}^N \exp\left(a_i + \frac{\mu_i^2}{2\sigma_0^2}\right)}, \end{aligned}$$

where by definition we set  $\mu_i = \sigma_0^2 b_i + \mu_0 = (b_i + \mu_0/\sigma_0^2)/(1/\sigma_0^2)$ . Computing  $\mu_\varepsilon$  and  $\sigma_\varepsilon^2$  with distribution  $\varphi_\varepsilon(x)$ , after straightforward algebra, we obtain Equations (C.1) and (C.2).

For the prior probabilities, employing similar algebra, we have:

$$\begin{aligned}
\pi_n^{\text{prior}} &= \int_{-\infty}^{\infty} \pi_n(x) \varphi_\varepsilon(x) dx = \frac{\int_{-\infty}^{\infty} e^{a_n + b_n x} e^{-0.5(x - \mu_0)^2 / \sigma_0^2} dx}{\sum_{k=1}^N \int_{-\infty}^{\infty} e^{a_k + b_k x} e^{-0.5(x - \mu_0)^2 / \sigma_0^2} dx} \\
&= \frac{\int_{-\infty}^{\infty} e^{a_n + b_n \mu_0 + \sigma_0^2 b_n^2 / 2} \phi(x; \sigma_0^2 b_n + \mu_0, \sigma_0) dx}{\sum_{k=1}^N \int_{-\infty}^{\infty} e^{a_k + b_k \mu_0 + \sigma_0^2 b_k^2 / 2} \phi(x; \sigma_0^2 b_k + \mu_0, \sigma_0) dx} \\
&= \frac{e^{a_n + b_n \mu_0 + \sigma_0^2 b_n^2 / 2}}{\sum_{k=1}^N e^{a_k + b_k \mu_0 + \sigma_0^2 b_k^2 / 2}},
\end{aligned}$$

which completes the proof. ■

## References

- Admati, A. R., 1985, “A Noisy Rational Expectations Equilibrium for Multi-Asset Securities Markets,” *Econometrica*, 53, pp. 629–658.
- Albagli, E., C. Hellwig, and A. Tsyvinski, 2013, “A Theory of Asset Prices based on Heterogeneous Information,” *Working Paper*.
- Back, K., 1993, “Asymmetric Information and Options,” *Review of Financial Studies*, 6, pp. 435–472.
- Barlevy, G., and P. Veronesi, 2000, “Information Acquisition in Financial Markets,” *Review of Economic Studies*, 67, pp. 79–90.
- Basak, S., 2000, “A Model of Dynamic Equilibrium Asset Pricing with Heterogeneous Beliefs and Extraneous Risk,” *Journal of Economic Dynamics & Control*, 24, pp. 63–95.
- Basak, S., 2005, “Asset Pricing with Heterogeneous Beliefs,” *Journal of Banking and Finance*, 29, pp. 2849–2881.
- Bernardo, A. E., and K. L. Judd, 2000, “Asset Market Equilibrium with General Tastes, Returns, and Informational Asymmetries,” *Journal of Financial Markets*, 3, pp. 629–658.
- Biais, B., and P. Hillion, 1994, “Insider and Liquidity Trading in Stock and Options Markets,” *Review of Financial Studies*, 7, pp. 743–780.
- Black, F., and M. Scholes, 1973, “The Pricing of Options and Corporate Liabilities,” *Journal of Political Economy*, 81, pp. 637–654.
- Brennan, M. J., and H. H. Cao, 1996, “Information, Trade, and Derivative Securities,” *Review of Financial Studies*, 9, pp. 163–208.
- Breon-Drish, B., 2010, “Asymmetric Information in Financial Markets: Anything Goes,” *Working Paper*.

- Breon-Drish, B., 2012, “On Existence and Uniqueness of Equilibrium in a Class of Noisy Rational Expectations Models,” *Working Paper*.
- Chakravarty, S., H. Gulen, and S. Mayhew, 2004, “Informed Trading in Stock and Option Markets,” *Journal of Finance*, 59, pp. 1235–1257.
- Chan, K., Y. P. Chung, and W.-M. Fong, 2002, “The Informational Role of Stock and Option Volume,” *Review of Financial Studies*, 15, pp. 1049–1075.
- Diamond, D. W., and R. E. Verrecchia, 1981, “Information Aggregation in a Noisy Rational Expectations Economy,” *Journal of Financial Economics*, 9, pp. 221–235.
- Duffie, D., 2001, *Dynamic Asset Pricing Theory, 3rd Edition*, Princeton University Press.
- Easley, D., M. O’Hara, and P. S. Srinivas, 1998, “Option Volume and Stock Prices: Evidence on Where Informed Traders Trade,” *Journal of Finance*, 53, pp. 431–465.
- Gale, D., and H. Nikaidô, 1965, “The Jacobian Matrix and Global Univalence of Mappings,” *Math. Annalen*, 159, pp. 81–93.
- Grossman, S. J., 1976, “On the Efficiency of Competitive Stock Markets Where Trades Have Diverse Information,” *Journal of Finance*, 31, pp. 573–585.
- Grossman, S. J., and J. E. Stiglitz, 1980, “On the Impossibility of Informationally Efficient Markets,” *American Economic Review*, 70, pp. 393–408.
- Hayek, F., 1945, “The Use of Knowledge in Society,” *American Economic Review*, 35, 519–530.
- Hellwig, M. R., 1980, “On the Aggregation of Information in Competitive Markets,” *Journal of Economic Theory*, 22, 477–498.
- Judd, K. L., 1998, *Numerical Methods in Economics*, MIT Press.
- Kurlat, P., and L. Veldkamp, 2013, “Should We Regulate Financial Information?,” *Working Paper*.
- Kyle, A. S., 1985, “Continuous Auctions and Insider Trading,” *Econometrica*, 53, 1316–1336.

- Malamud, S., 2014, “Price Discovery through Options,” *Working Paper*.
- Marín, J. M., and R. Rahi, 2000, “Information Revelation and Market Incompleteness,” *Review of Economic Studies*, 67, pp. 455–481.
- Pan, J., and A. M. Poteshman, 2006, “The Information in Option Volume for Future Stock Prices,” *Review of Financial Studies*, 19, pp. 871–908.
- Paravisini, D., V. Rappoport, and E. Ravina, 2010, “Risk Aversion and Wealth: Evidence from Person-to-Person Lending Portfolios,” *Working Paper 16063, National Bureau of Economic Research*.
- Ross, S. A., 1976, “The arbitrage theory of capital asset pricing,” *Journal of Economic Theory*, 13, pp. 341–360.
- Vanden, J. M., 2008, “Information Quality and Options,” *Review of Financial Studies*, 21, pp. 2635–2676.
- Vives, X., 2008, *Information and Learning in Markets*, Princeton University Press.
- Yuan, K., 2005, “Asymmetric Price Movements and Borrowing Constraints: A Rational Expectations Equilibrium Model of Crisis, Contagion, and Confusion,” *Journal of Finance*, 60, pp. 379–411.