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LARGE TIME BEHAVIOR OF SOLUTIONS TO SEMI-LINEAR EQUATIONS WITH QUADRATIC GROWTH IN THE GRADIENT

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Abstract. This paper studies the large time behavior of solutions to semi-linear Cauchy problems with quadratic nonlinearity in gradients. The Cauchy problem considered has a general state space and may degenerate on the boundary of the state space. Two types of large time behavior are obtained: i) pointwise convergence of the solution and its gradient; ii) convergence of solutions to associated backward stochastic differential equations. When the state space is $\mathbb{R}^d$ or the space of positive definite matrices, both types of convergence are obtained under growth conditions on coefficients. These large time convergence results have direct applications in risk sensitive control and long term portfolio choice problems.

1. Introduction

Given an open domain $E \subseteq \mathbb{R}^d$ and functions A_{ij}, $\overline{A}_{ij}$, B_i, V, $i, j = 1, \cdots, d$, from E to $\mathbb{R}$, define the differential operator

$$\mathfrak{F} := \frac{1}{2}\sum_{i,j=1}^{d} A_{ij} D_{ij} + \frac{1}{2}\sum_{i,j=1}^{d} \overline{A}_{ij} D_i D_j + \sum_{i=1}^{d} B_i D_i + V, \tag{1.1}$$

where $D_i = \partial_{x^i}$ and $D_{ij} = \partial^2_{x^i x^j}$. We consider the following Cauchy problem:

$$\partial_t v = \mathfrak{F}[v], \quad (t, x) \in (0, \infty) \times E, \quad v(0, x) = v_0(x). \tag{1.2}$$

Precise conditions on E, the coefficients, and the initial condition v_0 will be presented later. In particular, these conditions allow for general domains E and for $A = (A_{ij})_{1 \leq i,j \leq d}$ to be both unbounded and degenerate on the boundary of E. Our goal is to study the large time asymptotic behavior of solutions $v(t, \cdot)$ to (1.2).

The asymptotic behavior of $v(t, \cdot)$ is closely related to the following ergodic analogue of (1.2):

$$\lambda = \mathfrak{F}[v], \quad x \in E, \tag{1.3}$$

whose solution is a pair (v, λ) with $\lambda \in \mathbb{R}$. In our main result, we prove the existence of $(\hat{v}, \hat{\lambda})$ solving (1.3) such that $h(t, x) := v(t, x) - \hat{\lambda} t - \hat{v}(x)$, $x \in E$, satisfies

$$h(t, \cdot) \to C \quad \text{and} \quad \nabla h(t, \cdot) \to 0 \quad \text{in } C(E) \quad \text{as } t \to \infty. \tag{1.4}$$

Here C is a constant, $\nabla = (D_1, \ldots, D_d)$ is the gradient, and convergence in $C(E)$ stands for locally uniform convergence in E. In addition to the previous pointwise convergence, we also obtain the

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following probabilistic type of convergence: for any fixed $t \geq 0$, as functions of $x \in E$,

$$\mathbb{E}^{\mathbb{P}^{\hat{v},x}}\left[\int_0^t \nabla h' \overline{A} \nabla h(T-s, X_s)ds\right] \to 0 \quad \text{and} \quad \mathbb{E}^{\mathbb{P}^{\hat{v},x}}\left[\sup_{0\leq s\leq t} |h(T,x) - h(T-s, X_s)|\right] \to 0, \tag{1.5}$$

in $C(E)$ as $T \to \infty$. Here, $\nabla h'$ is the transpose of ∇h and $(\mathbb{P}^{\hat{v},x})_{x\in E}$ are probability measures under which the coordinate process X is ergodic (cf. Proposition 2.3 below).

The Cauchy problem (1.2) and its ergodic analog (1.3) are closely related to *risk sensitive* control problems of both finite and infinite horizon: see [12, 1, 31, 27] among others. Indeed, consider

$$\max_{z\in\mathcal{Z}} \frac{1}{\theta} \log\left(\mathbb{E}\left[\exp\left(\theta\left(v_0(X_T) + \int_0^T c(X_s, z_s)ds\right)\right)\right]\right), \tag{1.6}$$

where $T > 0$ represents the horizon, $\theta \in \mathbb{R} \setminus \{0\}$ is the risk-sensitivity parameter, and $\mathcal{Z}$ is a set of acceptable control processes. For a given $z \in \mathcal{Z}$, X is an E-valued diffusion with dynamics $dX_t = b(X_t, z_t)dt + a(X_t)dW_t, X_0 = x$, where W is a d-dimensional Brownian motion and a is a matrix such that $aa' = A$. With v denoting the value function, the standard dynamical programming argument yields the following Hamilton-Jacobi-Bellman (HJB) equation for v:

$$\partial_t v = \frac{1}{2}\sum_{i,j=1}^d A_{ij}(x)D_{ij}v + \sup_z \left\{\frac{\theta}{2}\sum_{i,j=1}^d A_{ij}(x)D_i v D_j v + \sum_{i=1}^d b_i(x,z)D_i v + c(x,z)\right\}. \tag{1.7}$$

When $z \mapsto b(x,z)$ is linear and $z \mapsto c(x,z)$ is quadratic the risk-sensitive control problem is called the *linear exponential quadratic problem* and the HJB equation reduces to a semilinear equation of type (1.2), where the pointwise optimizer z in (1.7) is a linear function of ∇v and is expected to yield an optimal control. The long-run analog to (1.6) is obtained by maximizing the growth rate:

$$\max_{z\in\mathcal{Z}} \liminf_{T\to\infty} \frac{1}{\theta T} \log\left(\mathbb{E}\left[\exp\left(\theta \int_0^T c(X_s, z_s)ds\right)\right]\right). \tag{1.8}$$

Here, in the linear exponential quadratic case, the solution $(\hat{v}, \hat{\lambda})$ from (1.3) governs both the long-run optimal control and maximal growth rate for (1.8), while the long-run optimal control is again a linear function of $\nabla\hat{v}$. Thus, the convergence in (1.4) implies that the optimal control for the finite horizon problem converges to its long-run analog as the horizon goes to infinity.

The convergence in (1.4) and (1.5) also has direct applications to long-term portfolio choice problems from Mathematical Finance (cf. [3, 4, 2, 13, 14, 28, 11, 32, 9] amongst many others). In particular, solutions to (1.2) and (1.3) are the value functions for the *Merton* problem where the goal is to maximize expected utility from terminal wealth (finite horizon) or the expected utility growth rate (infinite horizon) for the constant relative risk aversion (CRRA) utility investor in a Markovian factor model. As in the risk-sensitive control problem, optimal investment policies are governed by ∇v and $\nabla\hat{v}$ respectively and hence (1.4) implies convergence of the optimal trading strategies as the horizon becomes large. In fact, through the lens of *portfolio turnpikes* (see [19] and references therein), which state that as the horizon T becomes large, the optimal polices for a *generic* utility function over any finite window $[0,t]$ converge to that of a CRRA utility, the convergence in (1.4) identifies optimal policies for a wide class of utilities in the presence of a long horizon. Here, however, the validity of turnpike results rely upon the convergence in (1.5) instead of (1.4) (cf. [19]). As such, (1.5) is essential for proving turnpike results.

In addition to portfolio turnpikes, the convergence in (1.5) implies convergence of solutions to backwards stochastic differential equations (BSDE) associated to (1.2) and (1.3). This connection is made precise in Remark 2.10, but the basic idea is that given solutions v to (1.2) and $(\hat{v}, \hat{\lambda})$ to (1.3), for any $T > 0$, one can construct BSDE solutions (Y^T, Z^T) and $(\hat{Y}, \hat{Z})$ to (2.14) and (2.15) below, respectively. Then, with $\mathcal{Y}^T := Y^T - \hat{Y} - \hat{\lambda}(T - \cdot)$ and $\mathcal{Z}^T := Z^T - \hat{Z}$, (1.5) implies

$$\lim_{T\to\infty} \mathbb{E}^{\mathbb{P}^{\hat{v},x}}\left[\int_0^t \|\mathcal{Z}^T_s\|^2 ds\right] = 0 \quad \text{and} \quad \lim_{T\to\infty} \mathbb{E}^{\mathbb{P}^{\hat{v},x}}\left[\sup_{0\le s\le t}\left|\mathcal{Y}^T_s - \mathcal{Y}^T_0\right|\right] = 0, \qquad \text{for any } t > 0.$$

In the aforementioned applications, several models for X are widely used. In particular, the *Wishart process* (cf. [5] and Example 3.8 below) has been used for option pricing (cf. [17, 18, 7, 8]) and portfolio optimization (cf. [6, 22]) in multi-variate stochastic volatility models. Wishart processes, taking values in the space of positive definite matrices $\mathbb{S}^d_{++}$, are multivariate generalizations of the square root Bessel diffusion. They offer modeling flexibility, by allowing stochastic correlations between factors, while still maintaining analytical tractability, by keeping the affine structure. However, the volatility of the Wishart process degenerates on the boundary of $\mathbb{S}^d_{++}$. Therefore, to include this case, our convergence results need to treat domains other than $\mathbb{R}^d$ and diffusions with coefficients degenerating on the boundary of the state space.

The convergence (1.4) has been obtained via stochastic analysis techniques. [31] and [32] study large time asymptotics when the state space is $\mathbb{R}^d$ and A may degenerate for large $|x|$, proving a weak form of the convergence in (1.4), i.e., $\lim_{t\to\infty} h(t, \cdot)/t = 0$. In [26], the convergence in (1.4) has been obtained when the state space is $\mathbb{R}^d$ and A is the identity matrix. Even though [26] considers uniformly parabolic equations, by appropriately localizing their arguments, we are able to treat degeneracy on the boundary and replace $\mathbb{R}^d$ by a general domain E. This allows us, in Section 2, to develop a general framework to study the large time asymptotics in (1.4) and (1.5). One crucial difference between our treatment and [26] lies in proving the comparison result for solutions to (1.2). The uniform parabolic assumption is explicitly used in [26], and their arguments cannot be extended to the locally parabolic case. We replace the uniform parabolic assumption with an assumption on the Lyapunov function (cf. Assumption 2.6 below) used to construct solutions $\hat{v}$ to (1.3). Additionally, while existing results focused on convergence (1.4), the convergence of type (1.5) was missing in the literature, and in general, does not follow from (1.4) directly without imposing cumbersome integrability assumptions which are hard to check in general settings.

The general framework presented in Section 2 gives conditions for convergence in terms of two functions ϕ_0 and ψ_0. Once these two functions satisfy appropriate properties, convergence results in Theorems 2.9 and 2.11 follow. When the state space is specified, ϕ_0 and ψ_0 provide a channel to explicit convergence results with assumptions only depending upon the model coefficients. Indeed, when the state space is $\mathbb{R}^d$ or $\mathbb{S}^d_{++}$, growth assumptions on model coefficients are presented which imply the existence of ϕ_0 and ψ_0, hence the main results (cf. Theorems 3.3 and 3.9) readily follow. Though the choice of ϕ_0 and ψ_0 depends upon the state space and model coefficients, the procedures to verify their properties are similar. Therefore the general framework developed in Section 2 could be applied to other domains as well.

In the rest of the paper, Section 4 proves convergence results in Section 2. Section 5 verifies results specific to $\mathbb{R}^d$ and $\mathbb{S}^d_{++}$. Lastly, Appendix A identifies $\mathbb{S}^d_{++}$ as a subset of $\mathbb{R}^{d(d+1)/2}$ which allows us to consider equations with $\mathbb{S}^d_{++}$-valued spatial variables as special cases of (1.2) and (1.3).

Finally, we summarize several notations used throughout the paper:

- $\mathbb{M}^d$: the space of $d \times d$ real matrices. For $x \in \mathbb{M}^d$, let x' be the transpose of x, $\text{Tr}(x)$ be the trace of x, and $\|x\| = \sqrt{\text{Tr}(x'x)}$. For $M, N \in \mathbb{M}^d$, the Kronecker product of M and N is denoted by $M \otimes N \in \mathbb{M}^{d^2}$.

- $\mathbb{S}^d$: the space of $d\times d$ symmetric matrices. $\mathbb{S}^d_{++}$: the space of $d\times d$ strictly positive definite symmetric matrices. For $M,N\in\mathbb{S}^d_{++}$, $M\geq N$ when $M-N$ is positive semi-definite. Given $M\in\mathbb{S}^d_{++}$, denote by $\sqrt{M}$ the unique $m\in\mathbb{S}^d_{++}$ such that $m^2=M$.
- For regions $E\subseteq\mathbb{R}^d$ and $F\subseteq\mathbb{R}^k$ and $\gamma\in(0,1]$ denote by $C^{k,\gamma}(E,F)$ the space of k times differentiable functions whose k^{th} derivative is locally Hölder continuous with exponent γ. Write $C^{k,\gamma}(E)$ for $C^{k,\gamma}(E;\mathbb{R})$.

2. Main results

2.1. **Setup.** We begin by precisely stating assumptions on the region E as well as the regularity of the coefficients in (1.1). As for E, assume i) $E\subseteq\mathbb{R}^d$ is an open connected domain star shaped with respect to some $x_0\in E$ [1]; ii) there exist a sequence $(E_n)_{n\in\mathbb{N}}$ of open, bounded, connected domains, each star shaped with respect to x_0 and with $C^{2,\gamma}$ boundary for some $\gamma\in(0,1]$ such that $\bar{E}_n\subset E_{n+1}$ for each n; and iii) $E=\cup_n E_n$.

Regarding regularity, for $A_{ij},\bar{A}_{ij},B_i,V,\ i,j=1,...,d$, in (1.1), set $A:=(A_{ij})_{i,j=1,\dots,d}$, $\bar{A}:=(\bar{A}_{ij})_{i,j=1,\dots,d}$, and $B:=(B_i)_{i=1,\dots,d}$. Assume $A,\bar{A}\in C^{2,\gamma}(E,\mathbb{S}^d)$, $B\in C^{1,\gamma}(E,\mathbb{R}^d)$ and $V\in C^{1,\gamma}(E)$ for some $\gamma\in(0,1]$.

A *classical* solution to (1.2) is a function $v\in C^{1,2}((0,\infty)\times E)\cap C([0,\infty)\times E)$ which satisfies (1.2). A *classical* solution to (1.3) is a pair (v,λ) such that $v\in C^2(E)$, $\lambda\in\mathbb{R}$, and (1.3) is satisfied.[2]

The following *local* ellipticity assumptions are imposed on (1.2) and (1.3):

Assumption 2.1. The functions A and $\overline{A}$ satisfy

i) For any $n\in\mathbb{N}$, $x\in E_n$, and $\xi\in\mathbb{R}^d$, $\xi' A(x)\xi\geq c_n|\xi|^2$, for some constant $c_n>0$;
ii) There exist constants $\overline{\kappa}\geq\underline{\kappa}>0$ such that

$$\underline{\kappa}A(x)\leq\overline{A}(x)\leq\overline{\kappa}A(x),\quad \text{for all}\quad x\in E.$$

Let us introduce some more notation which will be used throughout the article. For a fixed $\phi\in C^{2,\gamma}(E)$, under the aforementioned domain, regularity and ellipticity assumptions, the *generalized* martingale problem (cf. [36]) on E for

$$\mathcal{L}^\phi:=\frac{1}{2}\sum_{i,j=1}^d A_{ij}D_{ij}+\sum_{i=1}^d\left(B_i+\sum_{j=1}^d\overline{A}_{ij}D_j\phi\right)D_i, \tag{2.1}$$

has a unique solution, denoted by $(\mathbb{P}^{\phi,x})_{x\in E}$. Here, the probability space is the continuous path space $\Omega=C([0,\infty);E)$. The coordinate process is denoted by X so that $X(\omega)_t=\omega_t$ for $\omega\in\Omega$. The filtration $\mathbb{F}=(\mathcal{F}_t)_{t\geq0}$ is the right-continuous enlargement of the filtration generated by X. When $\phi\equiv0$ denote $\mathcal{L}$ for $\mathcal{L}^0$. Additionally, as a slight abuse of notation, for a given function $v\in C^{1,2}((0,\infty)\times E)$ and $T>0$, define

$$\mathcal{L}^{v,T-t}:=\frac{1}{2}\sum_{i,j=1}^d A_{ij}D_{ij}+\sum_{i=1}^d\left(B_i+\sum_{j=1}^d\overline{A}_{ij}D_j v(T-t,\cdot)\right)D_i,\quad 0\leq t\leq T. \tag{2.2}$$

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[1] A domain $E\subset\mathbb{R}^d$ is star shaped for some $x_0\in E$ if for each $x\in E$ the segment $\{\alpha x_0+(1-\alpha)x;0\leq\alpha\leq1\}$ is contained in E. A convex set is star shaped with respect to any of its points.

[2] Note that $\mathfrak{F}[\phi]=\mathfrak{F}[\phi+C]$ for any constant C. Hence the first component in solutions to (1.3) is only determined up to additive constants.

As with the time-homogeneous case, there exists a unique solution $(\mathbb{P}^{v,x}_T)_{x\in E}$ on $(\Omega, \mathcal{F}_T)$ to the generalized martingale problem for $\mathcal{L}^{v,T-\cdot}$. Both $(\mathbb{P}^{\phi,x})_{x\in E}$ and $(\mathbb{P}^{v,x}_T)_{x\in E}$ satisfy the strong Markov property. The martingale problem for $\mathcal{L}^{\phi}$ (resp. $\mathcal{L}^{v,T-\cdot}$) is *well-posed* if the coordinate process does not explode $\mathbb{P}^{\phi,x}$ a.s. (resp. before T, $\mathbb{P}^{v,x}_T$ a.s) for any $x \in E$.

In preparation for the convergence results, let us first establish existence and uniqueness of classical solutions to (1.2) and (1.3). For (1.3), as in [27, 25, 20, 26], the following assumption on the *Lyapunov function* helps to construct its solution.

Assumption 2.2. There exists a non-negative $\phi_0 \in C^3(E)$ such that

$$\lim_{n\uparrow\infty} \sup_{x\in E\setminus E_n} \mathfrak{F}[\phi_0](x) = -\infty. \tag{2.3}$$

Given the Lyapunov function ϕ_0, the following proposition is a collection of results in [27, 25, 26, 20], whose proofs will be discussed briefly in Section 4.

Proposition 2.3. *Let Assumption 2.1 and 2.2 hold. There exists a unique $\hat{\lambda} \in \mathbb{R}$ such that the following statements hold:*

i) There exists a unique (up to an additive constant) $\hat{v} \in C^2(E)$ solving (1.3) with $\hat{\lambda}$ such that $\left(\mathbb{P}^{\hat{v},x}\right)_{x\in E}$ is ergodic with an invariant density $\hat{m}$;
ii) $\sup_{x\in E}(\hat{v} - \phi_0)(x) < \infty$;
iii) $e^{-\underline{\kappa}(\hat{v}-\phi)} \in \mathbb{L}^1(E, \hat{m})$, for any $\phi \in C^2(E)$ with $\lim_{n\uparrow\infty} \sup_{x\in E\setminus E_n} \mathfrak{F}[\phi](x) = -\infty$.

The following assumption enables construction of both super and sub-solutions to (1.2), which in turn establishes existence of solutions to (1.2).

Assumption 2.4. For ϕ_0 as in Assumption 2.2, the martingale problem for $\mathcal{L}^{\phi_0}$ is well-posed.

Proposition 2.5. *Let Assumptions 2.1, 2.2, and 2.4 hold. For any v_0 satisfying*

$$\sup_{x\in E}(v_0 - \phi_0)(x) < \infty, \tag{2.4}$$

there exists at least one solution $v \in C^{1,2}((0,\infty)\times E) \cap C([0,\infty)\times E)$ solving (1.2) such that

$$\sup_{(t,x)\in[0,T]\times E} (v(t,x) - \phi_0(x)) < \infty, \qquad \text{for each } T \geq 0. \tag{2.5}$$

The uniqueness of classical solutions to (1.2) in the class of functions satisfying (2.5) follows from the following comparison result, which requires a strengthening of Assumption 2.2.

Assumption 2.6. For the ϕ_0 as in Assumption 2.2,

$$\lim_{n\uparrow\infty} \inf_{x\in E\setminus E_n} \phi_0(x) = \infty \quad \text{and} \quad \exists\ \delta > 1 \text{ such that } \lim_{n\uparrow\infty} \sup_{x\in E\setminus E_n} \mathfrak{F}[\delta\phi_0](x) = -\infty. \tag{2.6}$$

Proposition 2.7. *Let Assumptions 2.1, 2.2, 2.4 and 2.6 hold. Let $v_0, \tilde{v}_0$ satisfy (2.4) and denote by $v, \tilde{v}$ the respective solutions to (1.2) from Proposition 2.5. Then $v_0 \leq \tilde{v}_0$ on E implies*

$$v \leq \tilde{v}, \qquad \text{on } [0,\infty)\times E.$$

2.2. **Convergence.** To study the large time behavior of $v(t,\cdot)$, we restrict the initial condition v_0 in (1.2) from the larger class of functions satisfying (2.4) to the class of functions satisfying

$$\sup_{x\in E}\left(|v_0|-\phi_0\right)(x)<\infty. \tag{2.7}$$

Note that $v_0\equiv 0$ satisfies the above bound since $\phi_0\geq 0$. For v_0 satisfying (2.7), let v be the unique classical solution to (1.2) from Proposition 2.5.

We define the difference between v and $\hat\lambda\cdot+\hat v$, where $(\hat v,\hat\lambda)$ comes from Proposition 2.3, as

$$h(t,x):=v(t,x)-\hat\lambda t-\hat v(x),\qquad (t,x)\in[0,\infty)\times E. \tag{2.8}$$

Hence $h\in C^{1,2}((0,\infty)\times E)\cap C([0,\infty)\times E)$ and a direct calculation using (1.2) and (1.3) yields

$$\partial_t h=\mathcal{L}^{\hat v}h+\frac{1}{2}\nabla h'\overline{A}\nabla h,\quad \text{ on } (0,\infty)\times E,\qquad h(0,x)=(v_0-\hat v)(x). \tag{2.9}$$

Using (2.9) and Assumption 2.1, it follows (cf. equation (4.7), Lemma 4.3, and Remark 4.2 below) that the functions $\{h(t,\cdot)\}_{t\geq 1}$ are bounded from below by an $\hat m$ integrable function. To obtain a corresponding upper bound, crucial for proving convergence, the following assumption is made.

Assumption 2.8. There exists $\psi_0\in C^3(E)$ such that

$$\lim_{n\uparrow\infty}\inf_{x\in E\setminus E_n}\mathfrak{F}[\psi_0](x)=\infty,\qquad \lim_{n\uparrow\infty}\inf_{x\in E\setminus E_n}(\phi_0-\psi_0)(x)=\infty; \tag{2.10}$$

$$\inf_{x\in E}(\psi_0+K\phi_0)>-\infty,\qquad \sup_{x\in E}(\mathfrak{F}[\delta\phi_0]+\alpha(\delta\phi_0(x)-\psi_0))<\infty, \tag{2.11}$$

for some $\alpha,K>0$ and δ from (2.6).

As shown in [26, Lemma 4.5], (2.10) provides a lower bound for $\hat v$ in that

$$\inf_E(\hat v-\psi_0)>-\infty. \tag{2.12}$$

Furthermore, (2.11) provides an upper bound on $h(t,\cdot)$ for $t\geq 0$ (cf. Lemma 4.5 below), which is key for establishing convergence of h. With all the assumptions in place, we now state first convergence result.

Theorem 2.9. *Let Assumptions 2.1, 2.2, 2.4, 2.6, and 2.8 hold. Then, for v_0 satisfying* (2.7) *and any $t\geq 0$, as functions of $x\in E$,*

i) $\lim_{T\to\infty}\mathbb{E}^{\mathbb{P}^{\hat v,x}}\left[\int_0^t(\nabla h)'\overline{A}\nabla h(T-s,X_s)\,ds\right]=0$ *in $C(E)$;*

ii) $\lim_{T\to\infty}\mathbb{E}^{\mathbb{P}^{\hat v,x}}\left[\sup_{0\leq s\leq t}|h(T,x)-h(T-s,X_s)|\right]=0$ *in $C(E)$.*

Remark 2.10. As mentioned in the introduction, convergence in Theorem 2.9 can be understood in the context of BSDEs. As generalizations of the Feynman-Kac formula, solutions to BSDEs provide stochastic representations to solutions of semi-linear PDEs (cf. [33]). Given $T>0$, a solution v to (1.2) and a solution $(\hat\lambda,\hat v)$ to (1.3) define (Y^T,Z^T) and $(\hat Y,\hat Z)$ by

$$\begin{aligned}(Y_t^T,Z_t^T)\;&:=\;(v(T-t,X_t),a'\nabla v(T-t,X_t)),\qquad t\leq T,\\ (\hat Y_t,\hat Z_t)\;&:=\;(\hat v(X_t),a'\nabla\hat v(X_t)),\qquad t\geq 0,\end{aligned} \tag{2.13}$$

where $a=\sqrt{A}$. Then, (Y^T,Z^T) solves the *quadratic* BSDE:

$$Y_t=v_0(X_T)+\int_t^T\left(V(X_u)-\frac{1}{2}(Z_u^T)'M(X_u)Z_u^T\right)du-\int_t^T(Z_u^T)'dW_u^T,\quad t\leq T. \tag{2.14}$$

Here, W^T is a $\mathbb{P}^{v,x}_T$−Brownian motion and $M(x) := a^{-1}\overline{A}a^{-1}(x)$ [3]. In a similar manner, $(\hat{Y}, \hat{Z})$ solves the *ergodic* BSDE:

$$\hat{Y}_t = \hat{Y}_s + \int_t^s \left(V(X_u) - \frac{1}{2}\hat{Z}_u' M(X_u)\hat{Z}_u - \hat{\lambda}\right) du - \int_t^s \hat{Z}_u' d\hat{W}_u, \quad \text{for any } t \leq s, \tag{2.15}$$

where $\hat{W}$ is a $\mathbb{P}^{\hat{v},x}$−Brownian motion. This type of ergodic BSDE has been introduced in [16] and studied in [37], [10]. Now set $\mathcal{Y}^T := Y^T - \hat{Y} - \hat{\lambda}(T - \cdot)$ and $\mathcal{Z}^T := Z^T - \hat{Z}$. A direct calculation using (2.8) and (2.13) shows

$$\int_0^t \|\mathcal{Z}_s^T\|^2 ds = \int_0^t \nabla h' A \nabla h(T-s, X_s) ds; \qquad \mathcal{Y}_t^T - \mathcal{Y}_0^T = h(T-t, X_t) - h(T, x).$$

Thus, Theorem 2.9 and Assumption 2.1 *ii*) imply

$$\lim_{T\to\infty} \mathbb{E}^{\mathbb{P}^{\hat{v},x}} \left[\int_0^t \|\mathcal{Z}_s^T\|^2 ds\right] = 0 \quad \text{and} \quad \lim_{T\to\infty} \mathbb{E}^{\mathbb{P}^{\hat{v},x}} \left[\sup_{0\leq s\leq t} \left|\mathcal{Y}_s^T - \mathcal{Y}_0^T\right|\right] = 0, \quad \text{for any } t > 0.$$

In addition to the convergence in Theorem 2.9, the function $h(t, \cdot)$ and its gradient also converge pointwise as $t \to \infty$. Such result has been proved in [26] when $E = \mathbb{R}^d$ and $A = I_d$.

Theorem 2.11. *Let Assumptions 2.1, 2.2, 2.4, 2.6, and 2.8 hold. Then, for v_0 satisfying* (2.7)*,*

i) $\lim_{t\to\infty} h(t, \cdot) = C$ *in* $C(E)$ *for some constant* C*;*
ii) $\lim_{t\to\infty} \nabla h(t, \cdot) = 0$ *in* $C(E)$*.*

In the next section, Theorems 2.9 and 2.11 are applied to domains $\mathbb{R}^d$ and $\mathbb{S}^d_{++}$ respectively. There, easy-to-verify growth conditions on coefficients are given so that ϕ_0 and ψ_0 satisfying all requirements are constructed, thus implying the conclusions in Theorems 2.9 and 2.11.

3. Convergence results when the state space is $\mathbb{R}^d$ or $\mathbb{S}^d_{++}$

3.1. **The $\mathbb{R}^d$ case.** This case has been studied in [26] when $A(x) = I_d$. Here, we present an extension when A is locally elliptic. Other than the regularity assumptions at the beginning of Section 2, and Assumption 2.1, the coefficients in $\mathfrak{F}$ satisfy the following growth conditions:

Assumption 3.1.

i) A is bounded and B has at most linear growth. In particular, there exists an $\alpha_1 > 0$ such that $x'A(x)x \leq \alpha_1(1 + |x|^2)$, for $x \in \mathbb{R}^d$.
ii) There exist $\beta_1 \in \mathbb{R}$ and $C_1 > 0$ such that

$$B(x)'x \leq -\beta_1|x|^2 + C_1, \quad x \in \mathbb{R}^d.$$

iii) There exist $\gamma_1, \gamma_2 \in \mathbb{R}$ and $C_2 > 0$ such that

$$-\gamma_2|x|^2 - C_2 \leq V(x) \leq -\gamma_1|x|^2 + C_2, \quad x \in \mathbb{R}^d.$$

iv) $\max\{\beta_1, \gamma_1\} > 0$. Additionally
   a) When $\beta_1 \leq 0$ and $\gamma_1 > 0$ there exist $\alpha_2, C_3 > 0$ such that

$$x'A(x)x \geq \alpha_2|x|^2 - C_3, \quad x \in \mathbb{R}^d;$$

   b) When $\beta_1 > 0$ and $\gamma_1 < 0$, for the α_1 of part *i*),

$$\beta_1^2 + 2\gamma_1\overline{\kappa}\alpha_1 > 0.$$

[3] Note that Assumption 2.1 *ii*) implies $\underline{\kappa} I_d \leq M \leq \overline{\kappa} I_d$, hence the generator of (2.14) has quadratic growth in Z.

However, when $\beta_1 > 0$ and $\gamma_1 \geq 0$, no additional conditions are needed.

Remark 3.2. To understand Assumption 3.1 *iv*), consider a $\mathbb{R}^d$-valued diffusion X with dynamics

$$dX_t = B(X_t)dt + a(X_t)dW_t, \quad X_0 = x \in \mathbb{R}^d, \tag{3.1}$$

where W is a d dimensional Brownian motion and $a = \sqrt{A}$. By Assumption 3.1 *i*) and the regularity assumptions on A and B, (3.1) admits a global strong solution $(X_t)_{t\geq 0}$. If $\beta_1 > 0$ then X is mean-reverting. On the other hand, if $\gamma_1 > 0$, V decays to $-\infty$ on the boundary. Thus, part *iv*) requires either mean reversion or a decaying potential. If both happen, then no additional parameter restrictions are necessary. However, if mean reversion fails we require uniform ellipticity for $A(x)$ in the direction of x. If $\gamma_1 < 0$ then a delicate relationship between the growth and degeneracy of A, mean reversion of B and the growth of V is needed to ensure convergence results.

Under these growth assumptions on model coefficients, it follows that with $\phi_0(x) = (c/2)|x|^2$ and $\psi_0(x) = -(\tilde{c}/2)|x|^2$ for some $c, \tilde{c} > 0$, Assumptions 2.2, 2.4, 2.6, and 2.8 hold; see Section 5.1 below. In this case, the main convergence result reads:

Theorem 3.3. *Suppose that Assumptions 2.1 and 3.1 are satisfied. Then, for any v_0 satisfying* (2.7)*, the statements of Theorems 2.9 and 2.11 hold.*

3.2. **The $\mathbb{S}^d_{++}$ case.** Though $\mathbb{S}^d_{++}$ cannot be set as E directly, it can be identified with an open set $E \subset \mathbb{R}^{d(d+1)/2}$ which is filled up by subregions E_n satisfying the given assumptions. This identification, discussed in detail in Appendix A, allows one to freely go back and forth between E and $\mathbb{S}^d_{++}$ and hence results are presented in this section using matrix, rather than vector, notation.

To define $\mathfrak{F}$ in (1.1) using matrix notation, note that $\mathfrak{F}$ takes the form

$$\mathfrak{F} = \mathcal{L} + \frac{1}{2}\sum_{i,j=1}^{d} \bar{A}_{ij} D_i D_j + V, \tag{3.2}$$

where the linear operator $\mathcal{L}$ is given in (2.1) with $\phi \equiv 0$ and is the generator associated to (3.1).

To define $\mathcal{L}$ in the matrix setting, we follow the notation used in [30, Section 3]. Let $B : \mathbb{S}^d_{++} \to \mathbb{S}^d$ be locally Lipschitz and $F, G : \mathbb{S}^d_{++} \to \mathbb{M}^d$ be such that $G' \otimes F(x)$ [4] is locally Lipschitz. Consider

$$dX_t = B(X_t)dt + F(X_t)dW_t G(X_t) + G(X_t)' dW_t' F(X_t)'; \qquad X_0 = x \in \mathbb{S}^d_{++}, \tag{3.3}$$

where $W = (W^{ij})_{1\leq i,j\leq d}$ is a $\mathbb{M}^d$-valued Brownian motion. Defining the functions $a^{ij} : \mathbb{S}^d_{++} \to \mathbb{M}^d, i, j = 1, ..., d$ by

$$a^{ij}_{kl} := F^{ik}G^{lj} + F^{jk}G^{li}, \quad k, l = 1, \cdots, d, \tag{3.4}$$

the system in (3.3) takes the form

$$dX^{ij}_t = B_{ij}(X_t)dt + \mathrm{Tr}(a^{ij}(X_t)dW_t'), \quad i, j = 1, \ldots d. \tag{3.5}$$

Thus $\mathcal{L}$ is set as the generator associated to X:

$$\mathcal{L} = \frac{1}{2}\sum_{i,j,k,l=1}^{d} \mathrm{Tr}\left(a^{ij}(a^{kl})'\right) D^2_{(ij),(kl)} + \sum_{i,j=1}^{d} B_{ij} D_{(ij)}, \tag{3.6}$$

[4]Here $\otimes$ is the Kronecker product between two matrices whose definition is recalled at the end of Section 1.

where $D_{(ij)} = \partial_{x^{ij}}$ and $D^2_{(ij),(kl)} = \partial^2_{x^{ij}x^{kl}}$. Now, let $\bar{A}_{(ij),(kl)}, i,j,k,l = 1,...,d$ be functions on $\mathbb{S}^d_{++}$ which are symmetric (in an analogous manner to $\bar{A}_{ij} = \bar{A}_{ji}$ in the $\mathbb{R}^d$ case):

$$\overline{A}_{(ij),(kl)} = \overline{A}_{(ji),(kl)} = \overline{A}_{(ij),(lk)} = \overline{A}_{(kl),(ij)}, \quad \text{for } i,j,k,l = 1,\cdots,d. \tag{3.7}$$

Given such an $\bar{A}$ and $V : \mathbb{S}^d_{++} \to \mathbb{R}$ the operator $\mathfrak{F}$ is defined by

$$\mathfrak{F} := \mathcal{L} + \frac{1}{2}\sum_{i,j,k,l=1}^{d} D_{(ij)}\overline{A}_{(ij),(kl)}D_{(kl)} + V. \tag{3.8}$$

As in Section 2, we assume that $\text{Tr}(a^{ij}(a^{kl})') \in C^{2,\gamma}(\mathbb{S}^d_{++},\mathbb{R})$, $\overline{A}_{(ij),(kl)} \in C^{2,\gamma}(\mathbb{S}^d_{++},\mathbb{R})$, $B \in C^{1,\gamma}(\mathbb{S}^d_{++},\mathbb{S}^d)$, and $V \in C^{1,\gamma}(\mathbb{S}^d_{++},\mathbb{R})$, for some $\gamma \in (0,1]$ and any $i,j,k,l = 1,\cdots,d$. The analogue of (1.2) and (1.3) are:

$$\partial_t v = \mathfrak{F}[v], \quad (t,x) \in (0,\infty)\times\mathbb{S}^d_{++}, \quad v(0,x) = v_0(x); \tag{3.9}$$

$$\lambda = \mathfrak{F}[v], \quad x \in \mathbb{S}^d_{++}. \tag{3.10}$$

The notion of classical solutions to the above equations is defined in the same manner as in Section 2. Appendix A below shows that equations (3.9) and (3.10) can be treated as special cases of (1.2) and (1.3). Hence existence and uniqueness of classical solutions to (3.9) and (3.10) follow from Propositions 2.3, 2.5, and 2.7, provided the requisite assumptions are met.

We now specify Assumption 2.1 to the matrix setting. In particular, the first item below implies that $\mathfrak{F}$ in (3.8) is locally elliptic; cf. Lemma 5.1 below. Before stating the assumptions, define

$$f(x) := FF'(x) \quad \text{and} \quad g(x) := G'G(x), \qquad x \in \mathbb{S}^d_{++}. \tag{3.11}$$

Calculation shows that $\text{Tr}\left(a^{ij}(a^{kl})'\right) = f^{ik}g^{jl}+f^{il}g^{jk}+f^{jk}g^{il}+f^{jl}g^{ik}$. To keep the notation compact, the assumption giving bounds on $\bar{A}$ below uses the matrices a^{ij} while all other assumptions use the functions f and g.

Assumption 3.4. The functions f, g, and $\overline{A}$ satisfy

i) For any $n \in \mathbb{N}$, $x \in E_n \subset \mathbb{S}^d_{++}$, and $\xi \in \mathbb{R}^d$, $\xi' f(x)\xi \geq c_n|\xi|^2$ and $\xi' g(x)\xi \geq c_n|\xi|^2$, for some constant $c_n > 0$;
ii) There exist $\overline{\kappa} \geq \underline{\kappa} > 0$ such that, for any $x \in \mathbb{S}^d_{++}$ and $\theta \in \mathbb{S}^d$,

$$\underline{\kappa}\sum_{i,j,k,l=1}^{d}\theta_{ij}\text{Tr}(a^{ij}(a^{kl})')(x)\theta_{kl} \leq \sum_{i,j,k,l=1}^{d}\theta_{ij}\overline{A}_{(ij),(kl)}(x)\theta_{kl} \leq \overline{\kappa}\sum_{i,j,k,l=1}^{d}\theta_{ij}\text{Tr}(a^{ij}(a^{kl})')(x)\theta_{kl}.$$

As in the $\mathbb{R}^d$ case, growth assumptions on the coefficients are needed to construct the Lyapunov function. However, unlike $\mathbb{R}^d$, there are two types of boundaries to $\mathbb{S}^d_{++}$: $\{\|x\| = \infty\}$ and $\{\det(x) = 0\}$. Therefore separate growth assumptions are needed as x approaches each boundary. Let us first present growth assumptions when $\|x\|$ is large. Here, the assumptions are similar to those in Assumption 3.1 : cf. Remark 3.2 for a qualitative explanation of the restriction in part *iv*).

Assumption 3.5. There exists $n_0 > 0$ such that for $\|x\| \geq n_0$ the following conditions hold:

i) B has at most linear growth and there exist $\alpha_1 > 0$ such that $\text{Tr}(f(x))\text{Tr}(g(x)) \leq \alpha_1\|x\|$.
ii) There exist $\beta_1 \in \mathbb{R}$ and $C_1 > 0$ such that

$$\text{Tr}(B(x)'x) \leq -\beta_1\|x\|^2 + C_1.$$

iii) There exist constants $\gamma_1, \gamma_2 \in \mathbb{R}$ and $C_2 > 0$ such that

$$-\gamma_2\|x\| - C_2 \leq V(x) \leq -\gamma_1\|x\| + C_2.$$

Furthermore, $V(x)$ is uniformly bounded from above for $\|x\| \leq n_0$.

iv) $\max\{\beta_1, \gamma_1\} > 0$. Additionally

a) When $\beta_1 \leq 0$ and $\gamma_1 > 0$, there exists $\alpha_3, C_3 > 0$ such that

$$\mathrm{Tr}(f(x)xg(x)x) \geq \alpha_3\|x\|^3 - C_3.$$

b) When $\beta_1 > 0$ and $\gamma_1 < 0$, for α_1 of part *i*)

$$\beta_1^2 + 16\overline{\kappa}\alpha_1\gamma_1 > 0.$$

However, when $\beta_1 > 0$ and $\gamma_1 \geq 0$, no additional conditions are needed.

For small $\det(x)$, different growth assumptions are needed. To precisely state them, for $\delta \in \mathbb{R}$ and $x \in \mathbb{S}^d_{++}$ define

$$H_\delta(x) := \mathrm{Tr}(B(x)x^{-1}) - (1+\delta)\,\mathrm{Tr}(f(x)x^{-1}g(x)x^{-1}) - \mathrm{Tr}(f(x)x^{-1})\,\mathrm{Tr}(g(x)x^{-1}). \tag{3.12}$$

The function H_0 controls the explosion of solutions to (3.5). Indeed, as shown in [30, Theorem 3.4], (3.5) admits a global strong solution when $H_0(x)$ is uniformly bounded from below on $\mathbb{S}^d_{++}$.

Assumption 3.6. There exits $\epsilon, c_0, c_1 > 0$ such that

i) $\inf_{x\in\mathbb{S}^d_{++}} H_\epsilon(x) > -\infty$.

ii) $\liminf_{\det(x)\downarrow 0} \left(H_\epsilon(x) + c_0\log(\det(x))\right) > -\infty$.

iii) $\lim_{\det(x)\downarrow 0} \left(H_0(x) + c_1 V(x)\right) = \infty$.

Remark 3.7. Lemma 5.1 below shows that H_δ is decreasing in δ and hence part *i*) of Assumption 3.6 implies $\inf_{x\in\mathbb{S}^d_{++}} H_0(x) > -\infty$ so that [30, Theorem 3.4] yields the existence of global strong solution $(X_t)_{t\in\mathbb{R}_+}$ to (3.5). Part *ii*) implies that ϕ_0 can be chosen (up to additive and multiplicative constants) as $-\log(\det(x))$ when $\det(x)$ is small. Since part *ii*) implies $\lim_{\det(x)\downarrow 0} H_0(x) = \infty$, part *iii*) allows for the potential to decay to $-\infty$ as $\det(x) \downarrow 0$ but at a rate slower than the rate at which H_0 goes to ∞.

Example 3.8. The primary example for (3.5) is when X follows a Wishart process:

$$dX_t = (LL' + KX_t + X_tK')\,dt + \sqrt{X_t}dW_t\Lambda' + \Lambda dW_t'\sqrt{X_t}, \tag{3.13}$$

where $K, L, \Lambda \in \mathbb{M}^d$ with Λ invertible. Here, f and g from (3.11) specify to $f(x) = x$ and $g(x) = \Lambda\Lambda'$. Thus, part *i*) of Assumption 3.4 as well as parts *i*), *ii*) of Assumption 3.5 readily follow. H_δ from (3.12) takes the form $H_\delta(x) = \mathrm{Tr}((LL' - (d+1+\delta)\Lambda\Lambda')x^{-1}) + 2\mathrm{Tr}(K)$. Then $LL' \geq (d+1)\Lambda\Lambda'$ ensures that H_0 is uniformly bounded from below on $\mathbb{S}^d_{++}$, and hence (3.13) admits a unique global strong solution. However, the slightly stronger assumption: $LL' > (d+1)\Lambda\Lambda'$, is needed to satisfy Assumption 3.6. Indeed, for $LL' > (d+1)\Lambda\Lambda'$, part *i*) of Assumption 3.6 is evident, and part *ii*) holds because, as $\det(x) \downarrow 0$, $\mathrm{Tr}(Cx^{-1}) + \log(\det(x)) \to \infty$ for any $C \in \mathbb{S}^d_{++}$. Lastly, any potential V which is bounded from below by $-\mathrm{Tr}((LL' - (d+\delta+1)\Lambda\Lambda')x^{-1})$, for some $\delta > 0$ and small $\det(x)$ satisfies part *iii*).

Let n_0 be from Assumption 3.5 and let $\overline{c}, \underline{c}, C > 0$ be constants. Under Assumptions 3.5 and 3.6 a candidate Lyapunov function ϕ_0 is given by

$$\phi_0(x) := -\underline{c}\log(\det(x)) + \overline{c}\|x\|\eta(\|x\|) + C, \tag{3.14}$$

where the cutoff function $\eta \in C^\infty(0,\infty)$ is such that $0 \le \eta \le 1$, $\eta(x) = 1$ when $x > n_0 + 2$, and $\eta(x) = 0$ for $x < n_0 + 1$. Furthermore, for $\underline{k}, \overline{k} > 0$, ψ_0 is chosen as

$$\psi_0(x) := \underline{k}\log(\det(x)) - \overline{k}\|x\|\eta(\|x\|), \quad x \in \mathbb{S}^d_{++}.$$

Section 5.2 proves that, under Assumptions 3.4 - 3.6, there exist $\underline{c}$, $\overline{c}$, C, $\underline{k}$, and $\overline{k}$ such that Assumptions 2.2, 2.4, 2.6, and 2.8 are satisfied. Then the main convergence result in the $\mathbb{S}^d_{++}$ case readily follows:

Theorem 3.9. *Suppose that Assumptions 3.4, 3.5, and 3.6 are satisfied. Then, for any v_0 satisfying* (2.7)*, the statements of Theorems 2.9 and 2.11 hold.*

4. Proofs in Section 2

4.1. **Proofs in Section 2.1.** Let us first briefly discuss proofs for Propositions 2.3 and 2.5. Proposition 2.3 i) essentially follows from [20, Theorems 13, 18], with only the following minor modifications. First, in [20] it is assumed that $\sup_{x\in E} V(x) < \infty$ and that $\bar{A}(x)$ takes a particular form. However, $\sup_{x\in E} V(x) < \infty$ is not actually necessary in the presence of Assumption 2.2 and the only essential fact used regarding $\bar{A}$ (labeled $\hat{A}$ therein) is that Assumption 2.1 holds: see equation (91) therein. To see this, when repeating the proof of Theorem 13 on page 272 of [20] note that since $\sup_{x\in E}\mathfrak{F}[\phi_0](x) < \infty$, it follows that $\mathfrak{F}[\phi_0] - \lambda < 0$ on E for sufficient large λ. Then, since the generalized principal eigenfunction for the operator L^c therein with $c = \underline{\kappa}$ is finite (as can be seen by repeating the argument on page 272), it follows again that for λ large enough there exist strictly positive solutions g of $L^c g = \lambda g$, at which point setting $f = (1/c)\log(g)$ and using Assumption 2.1 it follows that $\mathfrak{F}[f] - \lambda > 0$ on E. From here the result follows exactly as in [20, Theorem 13]. The proof of [20, Theorem 18] follows with only notational modifications.

To prove Proposition 2.3 *ii*), calculation shows that under Assumption 2.1, for any two $\phi, \psi \in C^2(E)$, the function $w := e^{-\underline{\kappa}(\psi-\phi)}$ satisfies

$$\mathcal{L}^\psi w \le \underline{\kappa} w(\mathfrak{F}[\phi] - \mathfrak{F}[\psi]) \quad \text{on } E, \tag{4.1}$$

where $\mathcal{L}^\psi$ is defined in (2.1). This is exactly [25, Lemma 4.2 (b)][5] with $\epsilon = 0$ in (A5) therein. Then repeating remaining arguments in [25, Section 4], the statement follows from [25, Theorem 2.2 $(i) \Rightarrow (iv)$].

Define the stopping times $\{\tau_n\}_{n\in\mathbb{N}}$ as the first exit time of X from E_n:

$$\tau_n := \inf\{t \ge 0 : X_t \notin E_n\}. \tag{4.2}$$

Proposition 2.3 *iii*) essentially follows from [26, Proposition 2.4]. To connect to the proof therein, note that (4.1) with $\psi = \hat{v}$ and $x \le \max\{x+1, 0\}$ combined yield:

$$\mathcal{L}^{\hat{v}} w \le \underline{\kappa} w(\mathfrak{F}[\phi] - \hat{\lambda}) \le \underline{\kappa} w\left(\max\{\mathfrak{F}[\phi] - \hat{\lambda} + 1, 0\}\right).$$

Since $\mathfrak{F}[\phi](x) \to -\infty$ as $x \to \partial E$ there is a constant M so that $\mathcal{L}^{\hat{v}} w \le M$ on E. Thus, by first stopping at τ_n and then using Fatou's lemma, there is a constant $C = C(x)$ such that $\mathbb{E}^{\mathbb{P}^{\hat{v},x}}\left[e^{-\underline{\kappa}(\hat{v}-\phi)(X_t)}\right] \le C + Mt$. The result now follows by repeating the argument in [26, Proposition 2.4] starting right after equation (2.4) therein.

Proposition 2.5 is proved by first constructing super- and sub-solutions ψ_1 and ψ_2 to (1.2) and then repeating the arguments in [26, Theorems 3.8, 3.9]. Even though the equation is uniformly

[5] Note a negative sign needs to added to ϕ and ψ in [25] to fit our context.

parabolic in [26], the solution v is constructed, using the given super- and sub-solutions, via a sequence of localized problems, each of which is uniformly parabolic, cf. [26, Equation (3.6)]. Here, the sequence of localized problems can be considered on $(E_n)_{n\in\mathbb{N}}$, where A is uniformly elliptic in each E_n due to Assumption 2.1 i). To construct the super- and sub-solutions ψ_1 and ψ_2, for $\zeta > 0$, define

$$\psi(t,x;\zeta) := \phi_0(x) + \frac{1}{\zeta}\log\left(\mathbb{E}^{\mathbb{P}^{\phi_0,x}}\left[\exp\left(\zeta(v_0-\phi_0)(X_t) + \zeta\int_0^t \mathfrak{F}[\phi_0](X_s)ds\right)\right]\right).$$

In view of Assumptions 2.2, 2.4 and equation (2.4) it follows that $\psi(t,x;\zeta)$ is well-defined and finite for $(t,x)\in[0,\infty)\times E$. With $\psi_1 = \psi(\cdot;\overline{\kappa})$ and $\psi_2 = \psi(\cdot;\underline{\kappa})$, Hölder's inequality implies $\psi_2 \le \psi_1$. Moreover, one can check that ψ_1 and ψ_2 are super- and sub-solutions of (1.2) respectively. This fact follows from the extension of the classical Feynman-Kac formula to the current, locally elliptic, setup; see [23, 19]. Thus, Proposition 2.5 holds.

Now we prove Proposition 2.7 which does not follow from [26, Theorem 3.6]. Let us first prepare a prerequisite result.

Lemma 4.1. *Let Assumptions 2.1, 2.2, and 2.4 and 2.6 hold. Let v be a classical solution to* (1.2) *in Proposition 2.5 with initial condition v_0 satisfying* (2.4). *Then, for any $T>0$, the martingale problem for $\mathcal{L}^{v,T-\cdot}$ on E is well-posed. Hence the coordinate process does not hit the boundary of E before T, $\mathbb{P}^{v,x}_T$ a.s., for any $x\in E$.*

Proof. Set $\tilde{v}(t,x) = v(t,x) - \delta\phi_0(x)$ for $(t,x)\in[0,T]\times E$, where δ is from (2.6). It follows from (2.5) and (2.6) that

$$\lim_{n\uparrow\infty}\sup_{(t,x)\in[0,T]\times E\setminus E_n}\tilde{v}(t,x) = \lim_{n\uparrow\infty}\sup_{(t,x)\in[0,T]\times E\setminus E_n}\left(v(t,x)-\phi_0(x)-(\delta-1)\phi_0(x)\right) = -\infty. \tag{4.3}$$

A direct calculation shows (note: $\partial_t\tilde{v} = \partial_t v$)

$$\mathcal{L}^{v,T-\cdot}\tilde{v} = \partial_t\tilde{v} - \mathfrak{F}[\delta\phi_0] + \frac{1}{2}\left(\nabla v - \delta\nabla\phi_0\right)'\bar{A}\left(\nabla v - \delta\nabla\phi_0\right) \ge \partial_t\tilde{v} - \mathfrak{F}[\delta\phi_0].$$

Since (2.6) assumes $\lim_{n\uparrow\infty}\sup_{x\in E\setminus E_n}\mathfrak{F}[\delta\phi_0] = -\infty$, there exists a constant C such that

$$-\partial_t\tilde{v} + \mathcal{L}^{v,T-\cdot}\tilde{v} \ge C, \quad \text{on } (0,T]\times E. \tag{4.4}$$

The well-posedness of the martingale problem for $\mathcal{L}^{v,T-\cdot}$ on E now follows from [38, Theorem 10.2.1], by defining $\phi_T(t,x) := -\tilde{v}(T-t,x)+K$ for some K so that $\phi_T(t,x)\ge 1, (t,x)\in[0,T]\times E$. Such a K exists in view of (4.3). Note also that the coefficients a_n, b_n in [38, Theorem 10.2.1] can easily be constructed in the present setup, cf. [38, p.250], and λ there can be chosen as any positive constant larger than $-C$.

□

Proof of Proposition 2.7. For the given $\tilde{v}_0 \ge v_0$ and associated solutions $\tilde{v}, v$ in Proposition 2.5, fix a $T>0$ and set $w(t,x) = \tilde{v}(T-t,x) - v(T-t,x)$, for $t\le T$ and $x\in E$. Since $\tilde{v}, v$ solve the differential expression in (1.2) it follows that

$$\partial_t w + \mathcal{L}^{v,T-\cdot}w = -(1/2)\nabla w'\bar{A}\nabla w.$$

Then under $\mathbb{P}^{v,x}_T$, which is the solution to the martingale problem for $\mathcal{L}^{v,T-\cdot}$ in Lemma 4.1, we have

$$\underline{\kappa}\left(w(T,X_T) - w(0,x)\right) \le \underline{\kappa}\int_0^T \nabla w' a(s,X_s)dW^v_s - \frac{1}{2}\underline{\kappa}^2\int_0^T \nabla w' A\nabla w(s,X_s)dx,$$

where W^v is a $\mathbb{P}^{v,x}_T$-Brownian Motion and the inequality follows from $\overline{A} \geq \underline{\kappa} A$. Exponentiating both sides of the previous inequality and taking $\mathbb{P}^{v,x}_T$-expectations, we obtain

$$e^{-\underline{\kappa} w(0,x)}\mathbb{E}^{\mathbb{P}^{v,x}_T}\left[e^{\underline{\kappa} w(T,X_T)}\right] \leq \mathbb{E}^{\mathbb{P}^{v,x}_T}\left[\mathcal{E}\left(\underline{\kappa}\int_0^\cdot \nabla w' a(s,X_s)dW^v_s\right)_T\right] \leq 1.$$

Plugging in for $w = \tilde{v} - v$ and using $\tilde{v}_0 \geq v_0$ gives

$$1 \geq e^{-\underline{\kappa}(\tilde{v}(T,x)-v(T,x))}\mathbb{E}^{\mathbb{P}^{v,x}_T}\left[e^{\underline{\kappa}(\tilde{v}_0-v_0)(X_T)}\right] \geq e^{-\underline{\kappa}(\tilde{v}(T,x)-v(T,x))},$$

which confirms the assertion since $\underline{\kappa} > 0$. □

4.2. **Proofs in Section 2.2.** Theorems 2.9 and 2.11 are proved in this section. For $\hat{v}$ in Proposition 2.3 and $x \in E$, to simplify notation, we denote

$$\hat{\mathbb{P}}^x := \mathbb{P}^{\hat{v},x} \quad \text{and} \quad \hat{\mathbb{E}}^x := \mathbb{E}^{\mathbb{P}^{\hat{v},x}}.$$

Throughout this section C is a universal constant which may be different in different places and the assumptions of Theorem 2.9 are enforced. In particular, v_0 is chosen to satisfy (2.7). The following facts regarding ergodic diffusions are used repeatedly throughout the sequel:

Remark 4.2 (Ergodic results). Recall from Proposition 2.3 *i*) yields that X is ergodic under $(\hat{\mathbb{P}}^x)_{x\in E}$ with invariant density $\hat{m}$. Given a continuous non-negative function f such that $f \in \mathbb{L}^1(E, \hat{m})$, [34] and [35, Corollary 5.2] prove

i) $\hat{\mathbb{E}}^x[f(X_t)] < \infty$ for any $x \in E$ and $t > 0$;
ii) $\sup_{t\geq\delta}\sup_{x\in E_n}\hat{\mathbb{E}}^x[f(X_t)] < \infty$ for any $\delta > 0$ and integer n;
iii) $\lim_{t\to\infty}\hat{\mathbb{E}}^x[f(X_t)] = \int_E f(x)\hat{m}(x)\,dx$ in $C(E)$.

To prove Theorems 2.9 and 2.11, we first prepare several results.

Lemma 4.3. *For ϕ_0 in Assumption 2.2 and $\hat{m}$ in Proposition 2.3 i), $\phi_0 \in \mathbb{L}^1(E,\hat{m})$.*

Proof. Set $\tilde{\phi}_0 := \delta\phi_0$. From Proposition 2.3 *ii*), $\underline{\kappa}(\delta-1)\phi_0 = \underline{\kappa}(\tilde{\phi}_0 - \phi_0) \leq \underline{\kappa}(\tilde{\phi}_0 - \hat{v}) + \underline{\kappa}C = -\underline{\kappa}(\hat{v} - \tilde{\phi}_0) + \underline{\kappa}C$ for some $C > 0$. Then $e^{\underline{\kappa}(\delta-1)\phi_0} \in \mathbb{L}^1(E,\hat{m})$ follows Proposition 2.3 *iii*) and Assumption 2.6. Since ϕ_0 is non-negative, then the statement is confirmed. □

Corollary 4.4. *Let $x \in E$, $0 \leq t \leq T$ and $\{\tau_n\}_{n\in\mathbb{N}}$ be as in* (4.2)*. Then the family of random variables*

$$\{h(T - t\wedge\tau_n, X_{t\wedge\tau_n});\, n \in \mathbb{N}\},$$

is $\hat{\mathbb{P}}^x$-uniformly integrable.

Proof. Applying Ito's formula to $h(t-\cdot, X_\cdot)$ and utilizing both (2.9) and $\overline{A} \geq \underline{\kappa} A$, we obtain, for any stopping time τ for which $\tau \leq t$,

$$\underline{\kappa} h(t-\tau, X_\tau) \leq \underline{\kappa} h(t,x) - \frac{\underline{\kappa}^2}{2}\int_0^\tau \nabla h' A \nabla h(t-u,X_u)\,du + \int_0^\tau \underline{\kappa}\nabla h' a(t-u,X_u)\,d\hat{W}_u,$$

where $\hat{W}$ is a $\hat{\mathbb{P}}^x$−Brownian motion. Exponentiating both sides of the previous inequality and taking expectations gives

$$\hat{\mathbb{E}}^x\left[e^{\underline{\kappa} h(t-\tau,X_\tau)}\right] \leq e^{\underline{\kappa} h(t,x)}\hat{\mathbb{E}}^x\left[\mathcal{E}\left(\underline{\kappa}\int_0^\cdot \nabla h' a(t-u,X_u)d\hat{W}_u\right)_\tau\right] \leq e^{\underline{\kappa} h(t,x)}, \tag{4.5}$$

and thus, at $\tau = s$ for the fixed time $s \le t$:

$$\frac{1}{\underline{\kappa}} \log \hat{\mathbb{E}}^x \left[e^{\underline{\kappa} h(t-s, X_s)} \right] \le h(t,x). \tag{4.6}$$

Proposition 2.3 *ii)* and (2.7) imply both $\hat{v} \le \phi_0 + C$ and $v_0 \ge -\phi_0 - C$. Thus, (4.6) with $s = t$ and Jensen's inequality combined imply

$$h(t,x) \ge \frac{1}{\underline{\kappa}} \log \hat{\mathbb{E}}^x \left[e^{\underline{\kappa}(v_0 - \hat{v})(X_t)} \right] \ge \hat{\mathbb{E}}^x \left[(v_0 - \hat{v})(X_t) \right] \ge -2\hat{\mathbb{E}}^x \left[\phi_0(X_t) \right] - C, \tag{4.7}$$

for some constant C. Therefore, with $h_- := \max\{-h, 0\}$, the Markov property and $\phi_0 \ge 0$ combined yield

$$h_-(T - t\wedge\tau_n, X_{t\wedge\tau_n}) \le C + 2\hat{\mathbb{E}}^{X_{t\wedge\tau_n}} \left[\phi_0(X_{T - t\wedge\tau_n}) \right] = C + 2\hat{\mathbb{E}}^x [\phi_0(X_T) \,|\, \mathcal{F}_{t\wedge\tau_n}],$$

By Lemma 4.3 and Remark 4.2 *i)*, we have $\hat{\mathbb{E}}^x[\phi_0(X_T)] < \infty$. Thus, the random variables $\{h_-(T - t\wedge\tau_n, X_{t\wedge\tau_n}); n = 1, 2, ...\}$ are uniformly integrable under $\hat{\mathbb{P}}^x$.

As for the positive part, set $h_+ := \max\{h, 0\}$. Since for any constant $k > 0$, $e^{kh_+} \le 1 + e^{kh}$, (4.5) implies there is a $C = C(T,x) > 0$ so that

$$\hat{\mathbb{E}}^x \left[e^{\underline{\kappa} h(T - t\wedge\tau_n, X_{t\wedge\tau_n})_+} \right] \le C, \quad \text{for any } n.$$

The uniform integrability of $\{h(T - t\wedge\tau_n, X_{t\wedge\tau_n}); n \in \mathbb{N}\}$ now follows, finishing the proof. ☐

The next result identifies an upper bound on $h(t,\cdot)$, uniformly in $t \ge 0$. The statement and proof are similar to [26, Lemma 4.7].

Lemma 4.5. *Let $\mathcal{J}(x) := J(1 + \phi_0(x) + \hat{v}_-(x))$, for $x \in E$. Here $\hat{v}_- := \max\{-\hat{v}, 0\}$. Then $\mathcal{J} \in C(E,\mathbb{R}) \cap \mathbb{L}^1(E,\hat{m})$ and there exists a sufficiently large constant J such that*

$$\sup_{t\ge0} h(t,x) \le \mathcal{J}(x), \quad x \in E. \tag{4.8}$$

Proof. Due to (2.12) and the first inequality in (2.11), $\hat{v}_- \le C - \psi_0 \le C + K\phi_0$, hence $\mathcal{J} \in \mathbb{L}^1(E,\hat{m})$ follows from Lemma 4.3. Moreover it is clear that $\mathcal{J} \in C(E,\mathbb{R})$.

Let us prove (4.8). Since v_0 satisfies (2.7), $v_0 \le \phi_0 + C$ for some constant C. Thus, by the comparison principle in Proposition 2.7 it suffices to prove (4.8) when $v_0 = \phi_0 + C$. Additionally, since $\mathfrak{F}[v + C] = \mathfrak{F}[v]$ for any constant C, one can set $v_0 = \phi_0$ without loss of generality. Thus, let v be the solution of (1.2) with initial condition ϕ_0 and let $h(t,x) = v(t,x) - \hat{\lambda}t - \hat{v}(x)$.

Set $w(t,x) := \delta\phi_0(x) + \hat{\lambda}t - v(t,x)$. We first derive upper and lower bounds for w. On the one hand, note that $w(t,x) = \delta\phi_0(x) - \hat{v}(x) + (\hat{v}(x) + \hat{\lambda}t - v(t,x))$ and that $\hat{v}(x) + \hat{\lambda}t$ satisfies (1.2) with the initial condition $\hat{v}$. Proposition 2.3 *ii)* and $\phi_0 = v_0$ give $\hat{v}(x) \le v_0(x) + C$ and hence a second application of Proposition 2.7 yields $\hat{v}(x) + \hat{\lambda}t \le v(t,x) + C$ on $[0,T] \times E$. Thus

$$w(t,x) \le \delta\phi_0(x) - \hat{v}(x) + C, \quad \text{on } [0,T] \times E. \tag{4.9}$$

On the other hand, (2.5) implies the existence of constant C_T, which may depend on T, such that $v(t,x) \le \phi_0(x) + C_T$ on $[0,T] \times E$. Then

$$w(t,x) \ge (\delta - 1)\phi_0(x) + \hat{\lambda}t - C_T \ge \tilde{C}_T, \quad \text{on } [0,T] \times E, \tag{4.10}$$

for some constant $\tilde{C}_T$, where the second inequality follows from $\delta > 1$ and $\phi_0 \ge 0$.

A direct calculation shows $\mathcal{L}^{v,T-\cdot}w = \mathfrak{F}[\delta\phi_0] + w_t - \hat{\lambda} - (1/2)\nabla w'\bar{A}\nabla w$, which implies $-w_t + \mathcal{L}^{v,T-\cdot}w \leq \mathfrak{F}[\delta\phi_0] - \hat{\lambda}$. For the given α in (2.11), applying Ito's formula to $e^{\alpha\cdot}w(T-\cdot, X_\cdot)$ and utilizing the previous inequality, we obtain for each n (recall τ_n from (4.2)):

$$\mathbb{E}^{\mathbb{P}_T^{v,x}}\left[e^{\alpha(T\wedge\tau_n)}w(T-T\wedge\tau_n, X_{T\wedge\tau_n})\right] \leq w(T,x) + \mathbb{E}^{\mathbb{P}_T^{v,x}}\left[\int_0^{T\wedge\tau_n} e^{\alpha s}(\mathfrak{F}[\delta\phi_0] - \hat{\lambda} + \alpha w)(T-s, X_s)ds\right],$$

Since w is bounded from below (cf. (4.10)), applying Fatou's lemma on the left-hand-side yields

$$e^{\alpha T}\mathbb{E}^{\mathbb{P}_T^{v,x}}\left[w(0, X_T)\right] \leq w(T,x) + \liminf_{n\uparrow\infty}\left(\mathbb{E}^{\mathbb{P}_T^{v,x}}\left[\int_0^{T\wedge\tau_n} e^{\alpha s}(\mathfrak{F}[\delta\phi_0] - \hat{\lambda} + \alpha w)(T-s, X_s)ds\right]\right).$$

On the right-hand-side, (2.11) implies $M := \sup_{x\in E}(\mathfrak{F}[\delta\phi_0] + \alpha(\delta\phi_0(x) - \psi_0)) < \infty$. Therefore (4.9) and (2.12) combined yield

$$\mathfrak{F}[\delta\phi_0] - \hat{\lambda} + \alpha w \leq \mathfrak{F}[\delta\phi_0] - \hat{\lambda} + \alpha(\delta\phi_0 - \hat{v} + C) \leq \mathfrak{F}[\delta\phi_0] - \hat{\lambda} + \alpha(\delta\phi_0 - \psi_0) + C \leq M - \hat{\lambda} + C.$$

Set $\hat{M} = \max\{M - \hat{\lambda} + C, 0\}/\alpha$. Combining the previous two inequalities and using $w(0,x) = \delta\phi_0(x) - v_0(x) = (\delta-1)\phi_0(x)$, we obtain

$$(\delta-1)e^{\alpha T}\mathbb{E}^{\mathbb{P}_T^{v,x}}\left[\phi_0(X_T)\right] \leq w(T,x) + \hat{M}(e^{\alpha T} - 1) \leq e^{\alpha T}\left(\delta\phi_0(x) + \hat{v}_-(x) + C + \hat{M}\right),$$

where the second inequality follows from (4.9). Thus, by taking $C > 0$ sufficiently large,

$$\mathbb{E}^{\mathbb{P}_T^{v,x}}\left[\phi_0(X_T)\right] \leq C(1 + \phi_0(x) + \hat{v}_-(x)), \quad \text{for all } x \in E \text{ and } T \geq 0. \tag{4.11}$$

Calculation shows that h satisfies $h_t = \mathcal{L}^{v,T-\cdot}h - (1/2)\nabla h'\bar{A}\nabla h \leq \mathcal{L}^{v,T-\cdot}h - (1/2)\underline{\kappa}\nabla h' A\nabla h$, since $\bar{A} \geq \underline{\kappa}A$. Applying Ito's formula to $\underline{\kappa}h(T-\cdot, X_\cdot)$ yields

$$\begin{aligned}\underline{\kappa}\left(\mathbb{E}^{\mathbb{P}_T^{v,x}}[h(0,X_T)] - h(T,x)\right) &\geq \mathbb{E}^{\mathbb{P}_T^{v,x}}\left[\int_0^T \frac{\underline{\kappa}^2}{2}(\nabla h)'A\nabla h(T-s,X_s)\,ds + \underline{\kappa}\int_0^T(\nabla h)'adW_s^T\right]\\ &\geq -\log\mathbb{E}^{\mathbb{P}_T^{v,x}}\left[\mathcal{E}\left(-\underline{\kappa}\int(\nabla h)'adW_s^T\right)_T\right] \geq 0,\end{aligned}$$

where W^T is a $\mathbb{P}_T^{v,x}-$Brownian motion and the second inequality follows from Jensen's inequality. Thus, since $h(0,x) = \phi_0(x) - \hat{v}(x)$, for any $T \geq 0$ and $x \in E$,

$$h(T,x) \leq \mathbb{E}^{\mathbb{P}_T^{v,x}}\left[\phi_0(X_T) - \hat{v}(X_T)\right] \leq C + (K+1)\mathbb{E}^{\mathbb{P}_T^{v,x}}\left[\phi_0(X_T)\right] \leq C + (K+1)C(1 + \phi_0(x) + \hat{v}_-(x)),$$

where the second inequality uses the first inequality in (2.11) and (2.12), the third inequality uses (4.11). Hence (4.8) now holds by taking J large enough, finishing the proof. ☐

For $0 \leq t \leq T$ and $x \in E$ set

$$f^{t,T}(x) := \frac{1}{2}\hat{\mathbb{E}}^x\left[\int_0^t(\nabla h)'\overline{A}\nabla h(T-s,X_s)\,ds\right]. \tag{4.12}$$

The next result gives a weak form of the convergence in Theorem 2.9 *i*).

Proposition 4.6. *For all $t \geq 0$,*

$$\lim_{T\to\infty}\int_E f^{t,T}(x)\hat{m}(x)\,dx = 0. \tag{4.13}$$

Proof. Corollary 4.4 and Ito's formula applied to $h(T-\cdot, X_\cdot)$ imply that

$$f^{t,T}(x) = h(T,x) - \hat{\mathbb{E}}^x\left[h(T-t, X_t)\right]. \tag{4.14}$$

Let $\hat{p}(t,x,y)$ denote the transition density of X under $\hat{\mathbb{P}}^x$. Recall from [36, pp. 179] that

$$\hat{m}(y) = \int_E \hat{p}(t,x,y)\hat{m}(x)dx, \quad \text{for any } t > 0 \text{ and } y \in E. \tag{4.15}$$

Thus

$$\begin{aligned} \int_E f^{t,T}(x)\hat{m}(x)dx &= \int_E h(T,x)\hat{m}(x)dx - \iint_E \hat{p}(t,x,y)h(T-t,y)\hat{m}(x)dydx \\ &= \int_E h(T,x)\hat{m}(x)dx - \int_E h(T-t,y)\hat{m}(y)dy. \end{aligned} \tag{4.16}$$

Set $l(T) := \int_E h(T,x)\hat{m}(x)dx$. It then follows from (4.16) and $f^{t,T}(x) \geq 0$ that $l(T) \geq l(T-t)$ for all $0 \leq t \leq T$. Therefore $l(T)$ is increasing in T and hence $\lim_{T\to\infty} l(T)$ exists. Furthermore, by (4.8) we know that $\sup_{T\geq 0} l(T) \leq \int_E \mathcal{J}(x)\hat{m}(x)dx < \infty$, hence $\lim_{T\to\infty} l(T) = l < \infty$. Sending $T \to \infty$ on both sides of (4.16), we have

$$\lim_{T\to\infty} \int_E f^{t,T}(x)\hat{m}(x)dx = \lim_{T\to\infty} \left(l(T) - l(T-t)\right) = l - l = 0.$$

□

In order to remove the integral with respect to the invariant density in (4.13), we need the following result.

Lemma 4.7. *For any fixed $t > 0$ and $n \in \mathbb{N}$, the family of functions on E given by $\left\{f^{t,T}(\cdot); T \geq t\right\}$ is uniformly bounded and equicontinuous on E_n.*

Proof. Define $k^{t,T}(s,x) := \hat{\mathbb{E}}^x\left[h(T-t, X_s)\right]$, for $s \leq t \leq T$ and $x \in E$, so that (4.14) becomes $f^{t,T}(x) = h(T,x) - k^{t,T}(t,x)$. We will prove, for any $E_n \subset E_m$ and $t > 0$,

a) $\{k^{t,T}(s,\cdot); T \geq t, t \geq s \geq t/2\}$ is uniformly bounded on E_m.
b) $\{h(T,\cdot); T \geq t/2\}$ is uniformly bounded on E_m.
c) both $\{k^{t,T}(t,\cdot); T \geq t\}$ and $\{h(T,\cdot); T \geq t\}$ are equicontinuous in E_n.

Let us first handle $k^{t,T}$. We have from (4.7) and (4.8) that

$$-C - 2\hat{\mathbb{E}}^x\left[\phi_0(X_{T-t+s})\right] = -C - 2\hat{\mathbb{E}}^x\left[\hat{\mathbb{E}}^{X_s}[\phi_0(X_{T-t})]\right] \leq k^{t,T}(s,x) \leq \hat{\mathbb{E}}^x\left[\mathcal{J}(X_s)\right],$$

for $T - t \geq 0$ and $t \geq s \geq t/2$. Since $\phi_0, \mathcal{J} \in \mathbb{L}^1(E,\hat{m})$ from Lemmas 4.3 and 4.5, it then follows from Remark 4.2 *ii)* that both $\sup_{s\geq t/2} \hat{\mathbb{E}}^x[\mathcal{J}(X_s)]$ and $\sup_{T\geq t, s\geq t/2} \hat{\mathbb{E}}^x[\phi_0(X_{T-t+s})]$ are bounded in E_m. Therefore, assertion a) is verified. Similarly, (4.7) and (4.8) imply that

$$-C - 2\hat{\mathbb{E}}^x\left[\phi_0(X_T)\right] \leq h(T,x) \leq \mathcal{J}(x), \quad \text{for} \quad T \geq t/2.$$

Then again, assertion b) follows from $\phi_0 \in \mathbb{L}^1(E,\hat{m})$ and Remark 4.2 *ii)*.

To prove $\{k^{t,T}(t,\cdot); T \geq t\}$ is equicontinuous in E_n, one can show that $k^{t,T} \in C^{1,2}((0,t)\times E_n) \cap C([0,t]\times\overline{E_n})$ and satisfies

$$\partial_s k = \mathcal{L}^{\hat{v}} k \quad \text{in } (0,t)\times E_n.$$

This result essentially follows from [23], and its proof is carried out in [21, Lemma A.3]. It then follows from the interior Schauder estimates (cf. [15, Theorem 2.15]) that, for any $E_n \subset E_m$ with $n < m$, $\max_{\overline{E_n}} |\nabla k^{t,T}(t,\cdot)|$ is bounded from above by a constant which only depend on the dimension of the problem, $\max_{[t/2,t]\times\overline{E_m}}|k^{t,T}|$, maximum and minimum of eigenvalues of A in $\overline{E_m}$, the distance from the boundary of E_n to the boundary of E_m, and finally t. In particular, the uniform bounds in a) implies that this upper bound on $\max_{\overline{E_n}}|\nabla k^{t,T}(t,\cdot)|$ is independent of T. Therefore $\{k^{t,T}(t,\cdot); T\geq t\}$ is equicontinuous in E_n.

Now, h satisfies (2.9) for all $T>0$ and $x\in E_m$. Moreover, we have seen from b) that $\{h(T,\cdot); T\geq t/2\}$ is uniformly bounded in E_m. It then follows from [29, Theorem V.3.1] that, for any $E_n\subset E_m$ with $n<m$ and $T\geq t$, $\max_{\overline{E_n}}|\nabla h(T,\cdot)|$ is bounded by a constant which only depends on the dimension of the problem, uniform bounds for h in b), the minimum and maximum eigenvalue of A in $\overline{E_m}$, distance from boundary of E_n to boundary of E_m, and finally t. Therefore, $\{h(T,\cdot); T\geq t\}$ is equicontinuous in E_n as well. □

Remark 4.8. For later development, we record from the previous proof that $\{h(T,\cdot); T\geq t\}$ is uniformly bounded and equicontinuous on E_n for any n.

With these preparations we are able to prove Theorems 2.9 and 2.11.

Proof of Theorem 2.9. Suppose that the convergence in i) does not hold, then there exist $\epsilon>0$, E_n, and a sequence $(T_i)_i$ such that $\sup_{E_n} f^{t,T_i}(x)\geq\epsilon$ for all i. Owing to Lemma 4.7, the Arzela-Ascoli theorem implies, taking a subsequence if necessary, f^{t,T_i} converge to some continuous function $\hat{f}$ uniformly in E_n. Note that $\sup_{E_n}|f^{t,T_i}-\hat{f}|+\sup_{E_n}\hat{f}\geq\sup_{E_n}f^{t,T_i}$. Sending $T_i\to\infty$, the uniform convergence and the choice of f^{t,T_i} implies $\sup_{E_n}\hat{f}(x)\geq\epsilon$. Since $\hat{f}$ is continuous, there exists a subdomain of $D\subset E_n$ such that $\hat{f}\geq\epsilon/2$ on D. However, this contradicts with Proposition 4.6 when the bounded convergence theorem is applied to the family of functions $(f^{t,T_i}\mathbb{I}_D)_{i\in\mathbb{N}}$.

To prove the statement ii), utilizing (2.9) and applying Ito's formula to $h(T-\cdot, X_\cdot)$, we obtain

$$\sup_{0\leq u\leq t}|h(T,x)-h(T-u,X_u)|\leq\frac{1}{2}\int_0^t(\nabla h)'\overline{A}\nabla h(T-s,X_s)\,ds+\sup_{0\leq u\leq t}\left|\int_0^u(\nabla h)'\,a\,d\hat{W}_s\right|.$$

Taking the $\hat{\mathbb{P}}^x$-expectation on both sides and using Burkholder-Davis-Gundy inequality, we obtain

$$\begin{aligned}&\sup_{E_n}\widehat{\mathbb{E}}^x\left[\sup_{0\leq u\leq t}|h(T,x)-h(T-u,X_u)|\right]\\&\leq\frac{1}{2}\sup_{E_n}\widehat{\mathbb{E}}^x\left[\int_0^t(\nabla h)'\overline{A}\nabla h(T-s,X_s)\,ds\right]+c\left(\sup_{E_n}\widehat{\mathbb{E}}^x\left[\int_0^t(\nabla h)'A\nabla h(T-s,X_s)\,ds\right]\right)^{\frac{1}{2}}\\&\to 0,\quad\text{ as } T\to\infty,\ \text{for any } E_n,\end{aligned}$$

where the last convergence follows from i) and $A\leq\overline{A}/\underline{\kappa}$. □

Proof of Theorem 2.11. The proof of Theorem 2.11 follows a similar argument as those presented in the proofs of [26, Proposition 4.3 and Theorems 1.3, 1.4, 4.1], and hence only connections to those proofs are given. Regarding [26, Proposition 4.3], for constants $S,T>0$, using (4.5) at $t=S+T$ and $\tau=S$ it follows that

$$\hat{\mathbb{E}}^x\left[e^{\underline{\kappa}h(T,X_S)}\right]\leq e^{\underline{\kappa}h(T+S,x)}.$$

Furthermore, (2.4) implies, for any fixed $T>0$ that $h(T,x)\leq\phi_0(x)-\hat{v}(x)+C_T$ for some $C_T>0$. This, combined with Proposition 2.3 iii) (with $\phi=\phi_0$) imply that $e^{\underline{\kappa}h(T,\cdot)}\in\mathbb{L}^1(E,\hat{m})$ for any fixed $T>0$ and hence Remark 4.2 iii) implies

$$\lim_{S\uparrow\infty}\hat{\mathbb{E}}^x\left[e^{\underline{\kappa}h(T,X_S)}\right]=\int_E e^{\underline{\kappa}h(T,y)}\hat{m}(y)dy.$$

Thus, the conclusions of [26, Proposition 4.3] follow by repeating their proof, noting that the role of $-k_1 w(t,x)$ therein is now played by $\underline{\kappa} h(t,x)$ here. Now, $i)$ in Theorem 2.11 follows by repeating the argument of [26, Theorem 4.1] and using Remark 4.8.

As for part $ii)$ in Theorem 2.11, we essentially repeat the steps within the proof of [26, Theorem 1.4]. Namely, using interior estimates for quasi-linear parabolic equations in [29, Theorem V.3.1] and Remark 4.8 it follows that there are constants $C_n > 0$ and $\gamma \in (0,1)$ such that for $x, y \in E_n$ and $s, \tilde{s} > t$: $|\nabla h(s,y) - \nabla h(\tilde{s}, x)| \leq C_n |s - \tilde{s}|^\gamma$. Now, define

$$f(n,T) := \int_{E_n} (\nabla h)' \overline{A} \nabla h(T,y) \hat{m}(y)\, dy. \tag{4.17}$$

It thus follows that $f(n,T)$ is uniformly continuous in (t,∞). Next we claim that $\lim_{T\uparrow\infty} f(n,T) = 0$ for any n. Indeed, recall from Proposition 4.6 that

$$0 = \lim_{T\to\infty} \int_E \hat{\mathbb{E}}^x \left[\int_0^t (\nabla h)' \overline{A} \nabla h(T-s, X_s)\, ds \right] \hat{m}(x)\, dx = 0, \quad \text{for any } t > 0.$$

Applying Fubini's theorem and (4.15) to the previous convergence yields

$$\lim_{T\to\infty} \int_0^t f(n, T-s) ds = 0. \tag{4.18}$$

Therefore, as shown in the proof of [26, Theorem 1.4], that $f(n,T) \to 0$ follows by the uniform continuity of $f(n,\cdot)$. The remaining steps of the proof are identical to those in [26, Theorem 1.4]. $\square$

5. Proofs from Section 3

5.1. **Proof of Theorem 3.3.** Theorem 2.11 has been proved in [26] when $E = \mathbb{R}^d$ and $A = I_d$. When A and $\overline{A}$ are local elliptic satisfying Assumption 2.1, the same calculation as in [26, Proposition 5.1] shows, when Assumption 3.1 holds, there exist $\epsilon_0, C > 0$ and $0 < \underline{c} < \overline{c}$ such that $\phi_0 = (c/2)|x|^2$ for any $c \in (\underline{c}, \overline{c})$ satisfies

$$\mathfrak{F}[\phi_0](x) = \frac{1}{2} c \operatorname{Tr}(A) + \frac{1}{2} c^2 x' \bar{A} x + c x' B + V \leq C + \left(\frac{\overline{\kappa}}{2} \alpha_1 c^2 - \beta_1 c - \gamma_1 \right) |x|^2 \leq C - \epsilon_0 |x|^2, \quad x \in \mathbb{R}^d.$$

Indeed, for $\gamma_1 > 0$ one can take $0 \leq \underline{c} < \overline{c}$ for $\overline{c}$ sufficiently small, while for $\gamma_1 < 0, \beta_1 > 0$ one can use part $iv-b)$ of Assumption 3.1 to find $0 < \underline{c} < \overline{c}$. Therefore, Assumption 2.2 is satisfied and Assumption 2.6 holds when $\delta > 1$ satisfies $c\delta < \overline{c}$. On the other hand, Assumption 3.1 i) implies that A is bounded and $B + \overline{A}\nabla\phi_0$ has at most linear growth. Thus the coordinate process does not explode $\mathbb{P}^{\phi_0,x}$-a.s. for any $x \in \mathbb{R}^d$, implying that Assumption 2.4 holds. As for Assumption 2.8, take $\psi_0(x) = -(\tilde{c}/2)|x|^2$ for $\tilde{c} > 0$, the second convergence in (2.10) and the first inequality in (2.11) clearly hold. For the second inequality in (2.11),

$$\mathfrak{F}[\delta\phi_0] + \alpha(\delta\phi_0 - \psi_0) \leq C - (\epsilon_0 - (\alpha/2)(\delta c + \tilde{c}))|x|^2,$$

which is bounded from above by C when α is sufficiently small. Finally, it remains to find $\tilde{c}$ such that the first convergence in (2.10) is verified. To this end,

$$\mathfrak{F}[\psi_0](x) = -\frac{1}{2}\tilde{c}\operatorname{Tr}(A) + \frac{1}{2}\tilde{c}^2 x' \overline{A} x - \tilde{c} x' B + V \geq C + \frac{\underline{\kappa}}{2}\tilde{c}^2 x' A x + (\tilde{c}\beta_1 - \gamma_2)|x|^2,$$

where the inequality is a result of $\overline{A} \geq \underline{\kappa} A$ and Assumption 3.1 i)-iii). When $\beta_1 > 0$, choose $\tilde{c}$ sufficiently large such that $\tilde{c}\beta_1 > \gamma_2$. When $\gamma_1 > 0$ and $\beta_1 \leq 0$, Assumption 3.1 iv-a) yields $(\underline{\kappa}/2)\tilde{c}^2 x' A x \geq (\underline{\kappa}/2)\tilde{c}^2(\alpha_2|x|^2 - C_3)$. Thus choose $\tilde{c}$ sufficiently large such that $(\underline{\kappa}/2)\tilde{c}^2\alpha_2 + \tilde{c}\beta_1 - \gamma_2 >$

0. In conclusion, all assumptions of Theorem 2.9 are satisfied, hence statements of Theorems 2.9 and 2.11 follow.

5.2. **Proof of Theorem 3.9.**

5.2.1. *Preliminaries.* The assumptions of Theorems 2.9 and 2.11 are now verified via Assumptions 3.4 – 3.6, which are enforced throughout. To ease notation, the argument x is suppressed when writing any function $f(x)$; for example, $\mathrm{Tr}(f(x)xg(x)x)$ will be written as $\mathrm{Tr}(fxgx)$. The following basic identities and inequalities are used repeatedly. The first one concerns derivatives of the functions $\log(\det(x))$ and $\|x\|$ respectively, and holds for $i,j,k,l = 1,...,d$:

$$
\begin{aligned}
&D_{(ij)} \log(\det(x)) = x^{-1}_{ij}, \qquad D^2_{(ij),(kl)} \log(\det(x)) = -(x^{-1})_{il}(x^{-1})_{jk},\\
&D_{(ij)}\|x\| = \frac{x_{ij}}{\|x\|}, \qquad D^2_{(ij),(kl)}\|x\| = \frac{\delta_{(ij),(kl)}}{\|x\|} - \frac{x_{ij}x_{kl}}{\|x\|^3},
\end{aligned} \tag{5.1}
$$

where $\delta_{(ij),(kl)} = 1$ if $i = k, j = l$ and 0 otherwise. Next, we give an identity, which follows from the discussion below (3.11):

$$
\sum_{i,j,k,l=1}^{d} \theta_{ij} \mathrm{Tr}\left(a^{ij}(a^{kl})'\right)\psi_{kl} = 4\mathrm{Tr}(f\psi g\theta); \qquad \theta,\psi \in \mathbb{S}^d_{++}, \tag{5.2}
$$

Now, (5.1), along with the definitions of $\mathcal{L}$ and H_δ from (3.6) and (3.12) respectively, give

$$
\mathcal{L}(\log(\det(x))) = H_0; \qquad \mathcal{L}(\|x\|) = \frac{1}{\|x\|}\left(\mathrm{Tr}(f'g) + \mathrm{Tr}(f)\mathrm{Tr}(g) - \frac{2}{\|x\|^2}\mathrm{Tr}(fxgx) + \mathrm{Tr}(B'x)\right). \tag{5.3}
$$

On the other hand, for $\theta,\psi,\eta \in \mathbb{S}^d_{++}$:

$$
\mathrm{Tr}(\theta\psi) \le \mathrm{Tr}(\theta)\mathrm{Tr}(\psi); \qquad \mathrm{Tr}(\theta\psi\eta\psi) \le \mathrm{Tr}(\theta)\mathrm{Tr}(\eta)\|\psi\|^2. \tag{5.4}
$$

Note that the first inequality in (5.4) also holds for $\theta \in \mathbb{S}^d_{++}$ and $\psi \in \mathbb{M}^d$ with $\psi + \psi' \in \mathbb{S}^d_{++}$. This is because $\mathrm{Tr}(\theta\psi) = (1/2)\mathrm{Tr}(\theta(\psi+\psi')) \le (1/2)\mathrm{Tr}(\theta)\mathrm{Tr}(\psi+\psi') = \mathrm{Tr}(\theta)\mathrm{Tr}(\psi)$. Lastly for any constants $a, b > 0$,

$$
\lim_{\theta\to\partial\mathbb{S}^d_{++}} -a\log(\det(\theta)) + b\|\theta\| = \infty. \tag{5.5}
$$

This convergence is clear when $\det(\theta) \downarrow 0$. When $\|\theta\| \uparrow \infty$, since $\det(\theta) = \prod_{i=1}^d \lambda_i$ and $\|\theta\| = \sqrt{\sum_{i=1}^d \lambda_i^2}$, where $(\lambda_i)_{i=1...d} > 0$ are the eigenvalues of θ, counting multiplicity, then (5.5) follows from Jensen and Hölder's inequalities.

5.2.2. *Proofs.* Let us first show $\mathfrak{F}$ in (3.8) is locally elliptic.

Lemma 5.1. *Let Assumption 3.4 i) hold. Then for each $E_n \subset \mathbb{S}^d_{++}$, there exists $c_n > 0$ such that*

$$
\sum_{i,j,k,l=1}^{d} \theta_{ij}\, Tr(a^{ij}(a^{kl})')(x)\theta_{kl} = 4\, Tr(f\theta g\theta)(x) \ge c_n\|\theta\|^2, \quad \textit{for any } x \in E_n, \theta \in \mathbb{S}^d.
$$

Proof. Applying (5.2) for $\theta = \psi \in \mathbb{S}^d_{++}$ gives $\sum_{i,j,k,l=1}^d \theta_{ij}\mathrm{Tr}\left(a^{ij}(a^{kl})'\right)\theta_{kl} = 4\,\mathrm{Tr}(f\theta g\theta)$. Now, $\mathrm{Tr}(f\theta g\theta) = (\mathrm{vec}\theta)'(f\otimes g)(\mathrm{vec}\theta)$, cf. [24, Chapter 4, Problem 25], where $\mathrm{vec}(\theta) \in \mathbb{R}^{d^2}$ is obtained by stacking columns of θ on top of one another. From [24, Corollary 4.2.13] it follows that $f \otimes g$ is positive definite if both f and g are positive definite. Hence Assumption 3.4 $i)$ ensures the existence of $c_n > 0$ such that $(\mathrm{vec}\theta)'(f\otimes g)(\mathrm{vec}\theta) \ge c_n|\mathrm{vec}\theta|^2 = c_n\|\theta\|^2$ on E_n, proving the result. ∎

Let us now study the Lyapunov function ϕ_0. Recall ϕ_0 and the cutoff function η from (3.14) and, for given $\overline{c}, \underline{c} > 0$ set $\phi_0^{(1)}(x) := -\underline{c}\log(\det(x))$ and $\phi_0^{(2)}(x) = \overline{c}\|x\|\eta(\|x\|)$ so that $\phi_0 = \phi_0^{(1)} + \phi_0^{(2)} + C$. We first derive an upper bound for $\mathfrak{F}[\phi_0]$.

Lemma 5.2. *There exists a constant C, depending on $\overline{c}$ but not on $\underline{c}$, such that*

$$\mathfrak{F}[\phi_0](x) \le -\underline{c}H_{4\overline{\kappa}\underline{c}}(x) - \left(\gamma_1 + \beta_1\overline{c} - 4\overline{\kappa}\alpha_1\overline{c}^2\right)\|x\|\,\mathbb{I}_{\{\|x\|>n_0+2\}} + C, \quad \text{for } x \in \mathbb{S}^d_{++}. \tag{5.6}$$

Proof. By the definition of $\mathfrak{F}$ and Assumption 3.4 *ii*):

(5.7)

$$\begin{aligned}\mathfrak{F}[\phi_0] &\le \mathcal{L}\phi_0^{(1)} + \mathcal{L}\phi_0^{(2)} + \frac{\overline{\kappa}}{2}\sum_{i,j,k,l=1}^{d}(D_{ij}\phi_0^{(1)} + D_{(ij)}\phi_0^{(2)})\mathrm{Tr}\left(a^{ij}(a^{kl})'\right)(D_{(kl)}\phi_0^{(1)} + D_{(kl)}\phi_0^{(2)}) + V,\\
&\le \mathcal{L}\phi_0^{(1)} + \overline{\kappa}\sum_{i,j,k,l=1}^{d} D_{(ij)}\phi_0^{(1)}\mathrm{Tr}\left(a^{ij}(a^{kl})'\right)D_{(kl)}\phi_0^{(1)}\\
&\quad + \mathcal{L}\phi_0^{(2)} + \overline{\kappa}\sum_{i,j,k,l=1}^{d} D_{(ij)}\phi_0^{(2)}\mathrm{Tr}\left(a^{ij}(a^{kl})'\right)D_{(kl)}\phi_0^{(2)} + V.\end{aligned}$$

In what follows, each term on the right-hand-side will be estimated. First, (5.2), (5.3), and the definition of H_δ in (3.12) yield:

$$\mathcal{L}\phi_0^{(1)} + \overline{\kappa}\sum_{i,j,k,l=1}^{d} D_{(ij)}\phi_0^{(1)}\mathrm{Tr}\left(a^{ij}(a^{kl})'\right)D_{(kl)}\phi_0^{(1)} = -\underline{c}H_{4\overline{\kappa}\underline{c}}. \tag{5.8}$$

As for $\phi_0^{(2)}$, when $\|x\| > n_0 + 2$, $\phi_0^{(2)}(x) = \|x\|$ and hence by (5.2) and (5.3):

$$\begin{aligned}&\mathcal{L}\phi_0^{(2)} + \overline{\kappa}\sum_{i,j,k,l=1}^{d} D_{(ij)}\phi_0^{(2)}\mathrm{Tr}\left(a^{ij}(a^{kl})'\right)D_{(kl)}\phi_0^{(2)}\\
&\quad = \frac{\overline{c}}{\|x\|}\left(\mathrm{Tr}(f'g) + \mathrm{Tr}(f)\mathrm{Tr}(g) + \left(\frac{4\overline{c}\overline{\kappa}}{\|x\|} - \frac{2}{\|x\|^2}\right)\mathrm{Tr}(fxgx) + \mathrm{Tr}(B'x)\right).\end{aligned} \tag{5.9}$$

Assumption 3.5 is now used to refine the right-hand-side. Since $\mathrm{Tr}(f'g) \le \mathrm{Tr}(f)\mathrm{Tr}(g)$ for $f, g \in \mathbb{S}^d_{++}$ (cf. (5.4)), Assumption 3.5 *i*) yields

$$\frac{\overline{c}}{\|x\|}\left(\mathrm{Tr}(f'g) + \mathrm{Tr}(f)\mathrm{Tr}(g)\right) \le 2\alpha_1\overline{c}, \quad \|x\| > n_0 + 2.$$

Lemma 5.1 implies $\mathrm{Tr}(fxgx) \ge 0$. This, and $\mathrm{Tr}(fxgx) \le \mathrm{Tr}(f)\mathrm{Tr}(g)\|x\|^2$ (cf. (5.4) again) gives, in light of Assumption 3.5 *i*), that

$$\frac{\overline{c}}{\|x\|}\left(\frac{4\overline{c}\overline{\kappa}}{\|x\|} - \frac{2}{\|x\|^2}\right)\mathrm{Tr}(fxgx) \le 4\overline{c}^2\overline{\kappa}\alpha_1\|x\|, \quad \|x\| > n_0 + 2.$$

Lastly, Assumption 3.5 *ii*) gives

$$\frac{\overline{c}}{\|x\|}\mathrm{Tr}(B'x) \le -\beta_1\overline{c}\|x\| + \frac{C_1\overline{c}}{n_0 + 2}, \quad \|x\| > n_0 + 2.$$

Putting previous three estimates back to (5.9) yields

$$\mathcal{L}\phi_0^{(2)} + \overline{\kappa}\sum_{i,j,k,l=1}^{d} D_{(ij)}\phi_0^{(2)}\mathrm{Tr}\left(a^{ij}(a^{kl})'\right)D_{(kl)}\phi_0^{(2)} \le C - \left(\beta_1\overline{c} - 4\overline{\kappa}\alpha_1\overline{c}^2\right)\|x\|, \tag{5.10}$$

when $\|x\| > n_0 + 2$. Here C is a constant which depends linearly on $\overline{c}$.

On the other hand, when $\|x\| \leq n_0 + 2$, since $\phi_0^{(2)}$ and its derivatives are bounded for bounded $\|x\|$, one can show the left-hand-side of (5.9) is bounded from above by a constant. Combining previous estimates on different parts of $\mathbb{S}^d_{++}$ yields

$$\mathcal{L}\phi_0^{(2)} + \overline{\kappa} \sum_{i,j,k,l=1}^{d} D_{(ij)}\phi_0^{(2)} \mathrm{Tr}\left(a^{ij}(a^{kl})'\right) D_{(kl)}\phi_0^{(2)} \leq C - \left(\beta_1 \overline{c} - 4\overline{\kappa}\alpha_1 \overline{c}^2\right) \|x\| \, \mathbb{I}_{\{\|x\|>n_0+2\}}. \tag{5.11}$$

Now putting (5.8) and (5.11) back into (5.7), and utilizing the upper bound of V in Assumption 3.5 *iii*), we confirm (5.6). □

The upper bound in (5.6) is then used to identify the Lyapunov function and verify its properties.

Lemma 5.3. *For the ϵ of Assumption 3.6, there exist $C > 0$ and $0 < \overline{c}_l < \overline{c}_h$ such that for any $0 < \underline{c} < \epsilon/(4\overline{\kappa})$ and $\overline{c}_l < \overline{c} < \overline{c}_h$ the function ϕ_0 in (3.14) is nonnegative on $\mathbb{S}^d_{++}$ and satisfies $\lim_{n\uparrow\infty} \sup_{x\in\mathbb{S}^d_{++}\setminus E_n} \mathfrak{F}[\phi_0](x) = -\infty$. Therefore, Assumption 2.2 holds.*

Proof. Since $\mathrm{Tr}(fxgx) > 0$ for $x \in \mathbb{S}^d_{++}$, H_δ is decreasing in δ. Hence for $\underline{c} < \epsilon/(4\overline{\kappa})$, (5.6) gives

$$\mathfrak{F}[\phi_0] \leq -\underline{c}H_\epsilon(x) - \left(\gamma_1 + \beta_1\overline{c} - 4\overline{\kappa}\alpha_1\overline{c}^2\right) \|x\| \, \mathbb{I}_{\{\|x\|>n_0+2\}} + C.$$

Assume for now that there exist $\epsilon_0 > 0$ and $0 < \overline{c}_l < \overline{c}_h$ such that

$$\gamma_1 + \beta_1\overline{c} - 4\overline{\kappa}\alpha_1\overline{c}^2 \geq \epsilon_0, \quad \text{for any } \overline{c} \in (\overline{c}_l, \overline{c}_h). \tag{5.12}$$

For such $\underline{c}$ and $\overline{c}$, the previous two inequalities combined imply

$$\mathfrak{F}[\phi_0] \leq -\underline{c}H_\epsilon(x) - \epsilon_0\|x\| \, \mathbb{I}_{\{\|x\|>n_0+2\}} + C. \tag{5.13}$$

By Assumption 3.6 *i*), $\mathfrak{F}[\phi_0] \leq C - \epsilon_0\|x\| \, \mathbb{I}_{\{\|x\|>n_0+2\}} \to -\infty$, as $\|x\| \uparrow \infty$. Moreover, Assumption 3.6 *ii*) implies $\lim_{\det(x)\downarrow 0} H_\epsilon(x) = \infty$ and thus $\mathfrak{F}[\phi_0] \leq C - \underline{c}H_\epsilon(x) \to -\infty$ as $\det(x) \downarrow 0$. Combining these two cases, the assertion $\lim_{n\uparrow\infty} \sup_{x\in\mathbb{S}^d_{++}\setminus E_n} \mathfrak{F}[\phi_0](x) = -\infty$ is confirmed.

To show (5.12), we use Assumption 3.5 *iv*). When $\gamma_1 > 0$ one can take $\epsilon_0 = \gamma_1/3$ and $\overline{c}_l = \overline{c}_h/2$ for some small enough $\overline{c}_h > 0$. When $\gamma_1 \leq 0$ and $\beta_1 > 0$, $\beta_1^2 + 16\overline{\kappa}\alpha_1\gamma_1 > 0$ holds due to Assumption 3.5 $iv - b$). Then there exists some sufficiently small ϵ_0 such that $\beta_1^2 - 16\overline{\kappa}\alpha_1(-\gamma_1 + \epsilon_0) > 0$. Hence one can take any $\overline{c}_l < \overline{c}_h$ satisfying $\overline{c}^- < \overline{c}_l < \overline{c}_h < \overline{c}^+$, where $\overline{c}^\pm > 0$ are two roots of $-4\overline{\kappa}\alpha_1\overline{c}^2 + \beta_1\overline{c} + \gamma_1 - \epsilon_0 = 0$.

Finally, it follows from (5.5) that ϕ_0 can be made nonnegative by adding a sufficiently large constant C to $\phi_0^{(1)} + \phi_0^{(2)}$. □

Corollary 5.4. *The following statements hold.*

i) When $\underline{c} < \epsilon/(8\underline{\kappa})$, the martingale problem for $\mathcal{L}^{\phi_0}$ is well-posed on $\mathbb{S}^d_{++}$. Hence, Assumption 2.4 is satisfied.

ii) There exists $\delta > 1$ such that $\lim_{n\uparrow\infty} \sup_{x\in\mathbb{S}^d_{++}\setminus E_n} \mathfrak{F}[\delta\phi_0](x) = -\infty$. Hence Assumption 2.6 is satisfied.

Proof. Part *ii*) follows from (3.14) and from Lemma 5.3 by taking $\delta > 1$ such that $\underline{c}\delta < \epsilon/(4\underline{\kappa})$ and $\overline{c}_l < \overline{c}\delta < \overline{c}_h$.

To prove part *i*), note that $\mathcal{L}^{\phi_0}\phi_0 = \mathcal{L}\phi_0 + \sum_{i,j,k,l=1}^d D_{(ij)}\phi_0 \bar{A}_{(ij),(kl)} D_{(kl)}\phi_0$, an upper bound for which is obtained by following (5.7), replacing $\underline{\kappa}$ by $2\underline{\kappa}$ and disregarding V. Then the same estimates leading to (5.8) and (5.11) yield

$$\mathcal{L}^{\phi_0}(\phi_0)(x) \leq -\underline{c}H_{8\overline{\kappa}\underline{c}}(x) - (\beta_1\overline{c} - 8\overline{\kappa}\alpha_1\overline{c}^2)\|x\|\,\mathbb{I}_{\{\|x\|>n_0+2\}} + C. \tag{5.14}$$

From (5.5), $\phi_0(x) \uparrow \infty$ as either $\det(x) \downarrow 0$ or $\|x\| \uparrow \infty$. If we can find $\lambda > 0$ such that $(\mathcal{L}^{\phi_0}(\phi_0) - \lambda\phi_0)(x) \leq 0, x \in \mathbb{S}^d_{++}/E_n$, for some n, then the martingale problem for $\mathcal{L}^{\phi_0}$ is well-posed; cf. [36, Theorem 6.7.1]. To find such a λ, (5.14) implies

$$\begin{aligned}\mathcal{L}^{\phi_0}\phi_0 - \lambda\phi_0 &\leq -\underline{c}H_{8\overline{\kappa}\underline{c}}(x) + \lambda\underline{c}\log(\det(x)) - (\beta_1\overline{c} - 8\overline{\kappa}\alpha_1\overline{c}^2 + \lambda\overline{c})\|x\|\,\mathbb{I}_{\{\|x\|>n_0+2\}} + C,\\ &\leq \lambda\underline{c}\log(\det(x)) - (\beta_1\overline{c} - 8\overline{\kappa}\alpha_1\overline{c}^2 + \lambda\overline{c})\|x\|\,\mathbb{I}_{\{\|x\|>n_0+2\}} + C,\end{aligned}$$

where the second inequality follows from $H_{8\overline{\kappa}\underline{c}} \geq H_\epsilon$, for $8\overline{\kappa}\underline{c} < \epsilon$, which is bounded from below on $\mathbb{S}^d_{++}$ by Assumption 3.6 *i*). For large enough λ, $\beta_1\overline{c} - 8\overline{\kappa}\alpha_1\overline{c}^2 + \lambda\overline{c} > 0$. Then, using (5.5), we conclude that $\mathcal{L}^{\phi_0}\phi_0 \leq \lambda\phi_0$ outside a sufficiently large E_n. □

Let us now switch our attention to ψ_0 in Assumption 2.8.

Lemma 5.5. *For $\underline{k}, \overline{k} > 0$ set*

$$\psi_0(x) := \underline{k}\log(\det(x)) - \overline{k}\|x\|\eta(\|x\|), \quad x \in \mathbb{S}^d_{++}.$$

Recall the constant c_1 from Assumption 3.6. Then, there exists a $\overline{k}^h > 0$ such that for all $\overline{k} > \overline{k}^h$ and $\underline{k} > c_1^{-1}$, (2.10) is satisfied.

Proof. $\lim_{x\to\partial\mathbb{S}^d_{++}} \psi_0(x) = -\infty$ holds by (5.5). Since $\phi_0 \geq 0$, this yields $\lim_{x\to\partial\mathbb{S}^d_{++}}(\psi_0 - \phi_0)(x) = -\infty$. Hence it suffices to find $\underline{k}, \overline{k} > 0$ such that $\lim_{x\to\partial\mathbb{S}^d_{++}} \mathfrak{F}[\psi_0](x) = \infty$.

Set $\psi_0^{(1)}(x) := \underline{k}\log(\det(x))$ and $\psi_0^{(2)}(x) := -\overline{k}\|x\|\eta(\|x\|)$. By Assumption 3.4 *ii*):

$$\begin{aligned}\mathfrak{F}[\psi_0] \geq& \mathcal{L}\psi_0^{(1)} + \mathcal{L}\psi_0^{(2)} + V\\ &+ \frac{\underline{\kappa}}{2}\sum_{i,j,k,l=1}^d \left(D_{(ij)}\psi_0^{(1)} + D_{(ij)}\psi_0^{(2)}\right)\mathrm{Tr}(a^{ij}(a^{kl})')\left(D_{(kl)}\psi_0^1 + D_{(kl)}\psi_0^{(2)}\right).\end{aligned} \tag{5.15}$$

From (5.3), for $\|x\| \geq n_0 + 2$,

$$\mathcal{L}\psi_0^{(2)} = -\frac{\overline{k}}{\|x\|}\left(\mathrm{Tr}(f'g) + \mathrm{Tr}(f)\mathrm{Tr}(g) - \frac{2}{\|x\|^2}\mathrm{Tr}(fxgx) + \mathrm{Tr}(B'x)\right). \tag{5.16}$$

For the right-hand-side, $\mathrm{Tr}(f'g) \leq \mathrm{Tr}(f)\mathrm{Tr}(g)$ for $f, g \in \mathbb{S}^d_{++}$ and Assumption 3.5 *i*) imply that $\mathrm{Tr}(f'g) + \mathrm{Tr}(f)\mathrm{Tr}(g) \leq 2\alpha_1\|x\|$. Combining the previous inequality with $\mathrm{Tr}(fxgx) > 0$ and Assumption 3.5 *ii*), we obtain

$$\mathcal{L}\psi_0^{(2)} \geq C + \overline{k}\beta_1\|x\| \quad \text{for } \|x\| > n_0 + 2.$$

On the other hand, when $\|x\| \leq n_0 + 2$, similar to the discussion before (5.11), one can show $\mathcal{L}\psi_0^{(2)} \geq C$. Therefore, the previous two estimates combined yield

$$\mathcal{L}\psi_0^{(2)} \geq C + \overline{k}\beta_1\|x\|\,\mathbb{I}_{\{\|x\|>n_0+2\}} \quad \text{for } x \in \mathbb{S}^d_{++}. \tag{5.17}$$

Bypassing V for the moment, the quadratic term on the right hand side of (5.15) is estimated. We only consider $\{x : \|x\| > n_0 + 2\}$ since the quadratic term is nonnegative and we are looking for a lower bound. Here, $\psi_0^{(2)}(x) = -\overline{k}\|x\|$ and hence (5.1) and (5.2) give

$$\begin{aligned}
&\sum_{i,j,k,l=1}^{d} \left(D_{(ij)}\psi_0^{(1)} + D_{(ij)}\psi_0^{(2)}\right) \mathrm{Tr}(a^{ij}(a^{kl})') \left(D_{(kl)}\psi_0^{(1)} + D_{(kl)}\psi_0^{(2)}\right) \\
&= 4\underline{k}^2 \mathrm{Tr}(fx^{-1}gx^{-1}) - \frac{8\underline{k}\overline{k}}{\|x\|}\mathrm{Tr}(fx^{-1}gx) + \frac{4\overline{k}^2}{\|x\|^2}\mathrm{Tr}(fxgx) \\
&\geq -8\underline{k}\overline{k}\alpha_1 + 4\frac{\overline{k}^2}{\|x\|^2}\mathrm{Tr}(fxgx),
\end{aligned} \tag{5.18}$$

where the inequality holds due to $\mathrm{Tr}(fx^{-1}gx^{-1}) \geq 0$, $\mathrm{Tr}(fx^{-1}gx) \leq \mathrm{Tr}(f)\mathrm{Tr}(x^{-1}gx) = \mathrm{Tr}(f)\mathrm{Tr}(g)$ (cf. the discussion after (5.4)), and Assumption 3.5 *i*). Using $\mathcal{L}\psi_0^{(1)} = \underline{k}H_0$ from (5.3) and putting (5.17), (5.18) back to (5.15) and utilizing Assumption 3.5 *iii*), we obtain

$$\mathfrak{F}[\psi_0] \geq \underline{k}H_0 + V\mathbb{I}_{\|x\|\leq n_0+2} + \left[2\frac{\overline{k}^2\underline{\kappa}}{\|x\|^2}\mathrm{Tr}(fxgx) + (\overline{k}\beta_1 - \gamma_2)\|x\|\right] \mathbb{I}_{\{\|x\|>n_0+2\}} + C.$$

Consider when $\|x\|$ is large. When $\beta_1 > 0$, $\mathrm{Tr}(fxgx) > 0$ and the uniform lower bound for $H_0(x)$ on $\mathbb{S}^d_{++}$ in Assumption 3.6 *i*) imply $\lim_{\|x\|\uparrow\infty} \mathfrak{F}[\phi_0](x) = \infty$ for $\overline{k} > \gamma_2/\beta_1$. On the other hand, when $\beta_1 \leq 0$, Assumption 3.5 *iv* $-$ *a*) gives $2\overline{k}^2\underline{\kappa}\mathrm{Tr}(fxgx)/(\|x\|^2) + (\overline{k}\beta_1 - \gamma_2)\|x\| \geq C + (2\overline{k}^2\underline{\kappa}\alpha_2 + \overline{k}\beta_1 - \gamma_2)\|x\|$. Then taking $\overline{k}$ sufficiently large gives $\lim_{\|x\|\uparrow\infty} \mathfrak{F}[\phi_0](x) = \infty$.

Consider now when $\|x\| \leq n_0 + 2$ but $\det(x) \downarrow 0$. Note $\underline{k}H_0 + V = (H_0 + c_1 V)/c_1 + (\underline{k} - c_1^{-1})H_0$. It then follows from Assumption 3.6 *i*) and *iii*) that $\lim_{\det(x)\downarrow 0} \underline{k}H_0(x) + V(x) = \infty$ when $\underline{k} > c_1^{-1}$, hence $\lim_{\det(x)\downarrow 0} \mathfrak{F}[\phi_0](x) = \infty$. Therefore, the first convergence in (2.10) is confirmed. □

Finally, it remains to verify (2.11).

Lemma 5.6. *For the δ from Corollary 5.4 ii), there exists $\alpha > 0$ such that* (2.11) *holds.*

Proof. Using Lemma 5.5 and the construction of ψ_0, ϕ_0, for any $K > 0$

$$\psi_0(x) + K\phi_0(x) = C - (K\underline{c} - \underline{k})\log(\det(x)) + (K\overline{c} - \overline{k})\|x\|\eta(\|x\|).$$

That the first inequality in (2.11) for large enough K now follows from (5.5). As for the second inequality in (2.11), the same estimate as in (5.13) yields the existence of $\epsilon_0 > 0$ such that

$$\mathfrak{F}[\delta\phi_0](x) \leq -\delta\underline{c}H_\epsilon(x) - \epsilon_0\|x\|1_{\|x\|>n_0+2} + C.$$

Then choose $\alpha > 0$ such that $\alpha(\delta\overline{c} + \overline{k}) < \epsilon_0$ and $\alpha(1 + \underline{k}/(\delta\underline{c})) < c_0$. It follows from the previous inequality and Lemma 5.5 that

$$\begin{aligned}
\mathfrak{F}[\delta\phi_0] + \alpha(\delta\phi_0 - \psi_0) &\leq -\delta\underline{c}H_\epsilon(x) - \alpha(\delta\underline{c} + \underline{k})\log\det(x) - (\epsilon_0 - \alpha(\delta\overline{c} + \overline{k}))\|x\|1_{\|x\|>n_0+2} + C \\
&\leq -\delta\underline{c}\left[H_\epsilon(x) + c_0\log\det(x)\right] + C,
\end{aligned}$$

which is bounded from above when $\det(x)$ is small, due to Assumption 3.6 *ii*). If $\det(x)$ is bounded away from zero, both $H_\epsilon(x)$ and $\log\det(x)$ are bounded from below. Combining the previous two cases, we confirm the second inequality in (2.11). □

Appendix A. Going between $\mathbb{S}^d_{++}$ and E

This appendix shows how to consider (3.9) and (3.10) as special cases of (1.2) and (1.3), respectively. Set $\tilde{d} = d(d+1)/2$ and let $I : \{1, 2, \dots, \tilde{d}\} \mapsto \{(i,j) \,:\, i = 1, \dots, d; j = i, \dots, d\}$ be a bijection such that $I(p) = (p,p)$ for $p = 1, ..., d$. If $I(p) = (i,j)$, we write $I'(p) = (j,i)$. Define $\ell : \mathbb{S}^d \to \mathbb{R}^{\tilde{d}}$ via $\ell(x)_p := x_{I(p)}$, for $p = 1, \dots, \tilde{d}, x \in \mathbb{S}^d$. Thus, ℓ maps upper triangle entries of x to entries in the vector $\ell(x)$. Denote by ℓ^{-1} the inverse of ℓ.

Set $E = \ell(\mathbb{S}^d_{++})$. It can be shown that E is an open, convex subset of $\mathbb{R}^{\tilde{d}}$ which can be filled up by open, bounded sets $(E_n)_{n\in\mathbb{N}}$ with smooth boundaries. Such E_n is created by smoothing out the boundary of the set $\{y \in E \,:\, \det(\ell^{-1}(y)) > 1/n, |y| < n\}$.

Given X following (3.5), one can then verify that $Y := \ell(X)$ satisfies

$$dY_t = \hat{B}(Y_t)\,dt + \hat{a}(Y_t)\,d\,\mathrm{vec}(W_t),$$

where, for $y \in E$

$$\hat{B}_p(y) := B_{I(p)}(\ell^{-1}(y)), \quad p = 1, \dots, \tilde{d},$$
$$\hat{a}_{pq}(y) := a^{I(p)}_{J(q)}(\ell^{-1}(y)), \quad p = 1, \dots, \tilde{d}, q = 1, \dots, d^2.$$

Here, $J : \left\{1, ..., d^2\right\} \mapsto \{(i,j); i,j = 1, ..., d\}$ is given by $J(1) = (1,1), \dots, J(d) = (d,1), J(d+1) = (1,2), \dots, J(2d) = (d,2), \dots, J(d,d) = d^2$.

Define $\hat{A} := \hat{a}\hat{a}'$ and $\hat{\overline{A}}_{pq}(y) := \overline{A}_{I(p),I(q)}(\ell^{-1}(y))$ for $p,q = 1, \cdots \tilde{d}$ and $y \in \mathbb{R}^{\tilde{d}}$. Then Assumption 3.4 for A and $\bar{A}$ is equivalent to Assumption 2.1 for $\hat{A}$ and $\hat{\bar{A}}$. Indeed, for any $\xi \in \mathbb{R}^{\tilde{d}}$, denote $\theta = \ell^{-1}(\xi)$. When $y = \ell(x)$,

$$\begin{aligned}
4\sum_{p,q=1}^{\tilde{d}} \xi_p (\hat{a}\hat{a}')_{pq}(y)\xi_q &= 4\sum_{p,q=1}^{\tilde{d}} \xi_p \mathrm{Tr}\left(a^{I(p)}(a^{I(q)})'\right)(x)\xi_q \\
&= 4\sum_{i,k=1}^{d} \theta_{ii}\mathrm{Tr}(a^{ii}(a^{kk})')(x)\theta_{kk} + 4\sum_{i=1}^{d}\sum_{l=1}^{d}\sum_{k=1}^{l-1} \theta_{ii}\mathrm{Tr}(a^{ii}(a^{kl})')(x)\theta_{kl} \\
&\quad + 4\sum_{j=1}^{d}\sum_{i=1}^{j-1}\sum_{k=1}^{d} \theta_{ij}\mathrm{Tr}(a^{ij}(a^{kk})')(x)\theta_{kk} + 4\sum_{j=1}^{d}\sum_{i=1}^{j-1}\sum_{l=1}^{d}\sum_{k=1}^{l-1} \theta_{ij}\mathrm{Tr}(a^{ij}(a^{kl})')(x)\theta_{kl} \\
&= \sum_{i,k=1}^{d} (2\theta_{ii})\mathrm{Tr}(a^{ii}(a^{kk})')(x)(2\theta_{kk}) + \sum_{i=1}^{d}\sum_{\substack{l,k=1\\ l\neq k}}^{d} (2\theta_{ii})\mathrm{Tr}(a^{ii}(a^{kl})')(x)\theta_{kl} \\
&\quad + \sum_{\substack{i,j=1\\ i\neq j}}^{d}\sum_{k=1}^{d} \theta_{ij}\mathrm{Tr}(a^{ij}(a^{kk})')(x)(2\theta_{kk}) + \sum_{\substack{i,j=1 \;\; k,l=1\\ i\neq j\;,\; k\neq l}}^{d} \theta_{ij}\mathrm{Tr}(a^{ij}(a^{kl})')(x)\theta_{kl} \\
&= \sum_{i,j,k,l=1}^{d} ({}^D\theta)_{ij}\mathrm{Tr}\left(a^{ij}(a^{kl})'\right)(x)({}^D\theta)_{kl},
\end{aligned}$$

where the third identity follows from $a^{ij} = a^{ji}$, and ${}^D\theta \in \mathbb{S}^d$ is obtained by doubling all diagonal entries of θ. Note that $\|\theta\|^2 \leq \|{}^D\theta\|^2 \leq 2\|\theta\|^2$. Therefore Assumption 3.4 *i)* for A is equivalent to Assumption 2.1 *i)* for $\hat{A}$. The equivalence between Assumption 3.4 *ii)* and Assumption 2.1 *ii)* can be proved similarly.

Now let us connect operators $\mathfrak{F}$ in (1.1) and (3.8). Let g be a smooth function on $\mathbb{S}^d_{++}$ and define $\tilde{g} : E \to \mathbb{R}$ by $\tilde{g}(y) := g(x)$ where $x = \ell^{-1}(y)$. Calculations show that $\partial_p \tilde{g}(y) = D_{I(p)} g(\ell^{-1}(y))$ when $I(p)$ is diagonal, or $(D_{I(p)} + D_{I'(p)}) g(\ell^{-1}(y))$ when $I(p)$ is off-diagonal. It then follows that

$$\begin{aligned}\sum_{p=1}^{\tilde{d}} \hat{B}_p(y)\partial_p \tilde{g}(y) &= \sum_{i=1}^{d} B_{ii}(x) D_{(ii)} g(x) + \sum_{j=1}^{d}\sum_{i=1}^{j-1} B_{ij}(x) D_{(ij)} g(x) + \sum_{j=1}^{d}\sum_{i=1}^{j-1} B_{ij}(x) D_{(ji)} g(x)\\ &= \sum_{i,j=1}^{d} B_{ij}(x) D_{(ij)} g(x),\end{aligned}$$

where the second identity above follows from $B_{ij} = B_{ji}$. A similar (but longer) calculation using $a^{ij} = a^{ji}$ and $\bar{A}_{(ij),(kl)} = \bar{A}_{(ji),(kl)} = \bar{A}_{(ij),(lk)}$ (cf. (3.7)) shows

$$\begin{aligned}\sum_{p,q=1}^{\tilde{d}} \left(\hat{a}\hat{a}'\right)_{pq}(y)\partial^2_{pq}\tilde{g}(y) &= \sum_{i,j,k,l=1}^{d} \mathrm{Tr}(a^{ij}(a^{kl})')(x) D^2_{(ij),(kl)} g(x),\\ \sum_{p,q=1}^{\tilde{d}} \partial_p \tilde{g}(y)\hat{\bar{A}}_{pq}(y)\partial_q \tilde{g}(y) &= \sum_{i,j,k,l=1}^{d} D_{(ij)} g(x)\bar{A}_{(ij),(kl)}(x) D_{(kl)} g(x).\end{aligned}$$

Write $\hat{V}(y) = V(x)$ where $x = \ell^{-1}(y)$. The previous three identities combined yield $\mathfrak{F}[g](x) = \mathfrak{F}[\tilde{g}](\ell(x))$. Therefore, (3.9) and (3.10) can be considered as special cases of (1.2) and (1.3), respectively.

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