



# Diversification and Financial Stability

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**Abstract**

This paper contributes to a growing literature on the pitfalls of diversification by shedding light on a new mechanism under which, full risk diversification can be sub-optimal. In particular, banks must choose the optimal level of diversification in a market where returns display a bimodal distribution. This feature results from the combination of two opposite economic trends that are weighted by the probability of being either in a bad or in a good state of the world. Banks have also interlocked balance sheets, with interbank claims marked-to-market according to the individual default probability of the obligor. Default is determined by extending the Black and Cox (1976) first-passage-time approach to a network context. We find that, even in the absence of transaction costs, the optimal level of risk diversification is interior. Moreover, in the presence of market externalities, individual incentives favor a banking system that is over-diversified with respect to the level of socially desirable diversification.

JEL classification: G20, G28

Keywords: Naive Diversification, Leverage, Default Probability

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# Diversification and Financial Stability

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## Abstract

This paper contributes to a growing literature on the pitfalls of diversification by shedding light on a new mechanism under which, full risk diversification can be sub-optimal. In particular, banks must choose the optimal level of diversification in a market where returns display a bimodal distribution. This feature results from the combination of two opposite economic trends that are weighted by the probability of being either in a bad or in a good state of the world. Banks have also interlocked balance sheets, with interbank claims marked-to-market according to the individual default probability of the obligor. Default is determined by extending the Black and Cox (1976) first-passage-time approach to a network context. We find that, even in the absence of transaction costs, the optimal level of risk diversification is interior. Moreover, in the presence of market externalities, individual incentives favor a banking system that is over-diversified with respect to the level of socially desirable diversification.

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## 1. Introduction

The folk wisdom of “not putting all of your eggs in one basket” has been a dominant paradigm in the financial community in recent decades. Pioneered by the works of Markowitz (1952), Tobin (1958) and Samuelson (1967), analytic tools have been developed to quantify the benefits derived from increased risk diversification. However, recent theoretical studies have begun to challenge this view by investigating the conditions under which diversification may have undesired effects (see, e.g., Battiston et al., 2012b; Ibragimov et al., 2011; Wagner, 2011; Stiglitz, 2010; Brock et al., 2009; Wagner, 2009; Goldstein and Pauzner, 2004). These works have found various types of mechanisms leading to the result that full diversification may not be optimal. For instance, Battiston et al. (2012b) assume an amplification mechanism in the dynamics of the financial robustness of banks; Wagner (2011) assumes non-constant asset liquidation costs; Stiglitz (2010) assumes that the default of one actor implies the default of all counterparties; and Wagner (2009) assumes that in the presence of a systemic default there is an additional cost of recovery for each bank.

The present paper contributes to the aforementioned literature by shedding light on a new mechanism under which, full risk diversification can be sub-optimal. In particular, we assume an arbitrage-free market where price returns are normally distributed and uncorrelated. However, the market may follow positive or negative trends that are ex-ante unpredictable and persist over a certain period of time. This incomplete information framework leads to a problem of portfolio diversification under uncertainty. In fact, portfolio returns display a bimodal distribution resulting from the combination of two opposite trends weighted by the probability of being either in a bad or in a good state of the world.

We find that even in the absence of transaction costs, optimal diversification can be *interior*. This result holds both at the *individual* and at the *social* welfare level. Moreover, we find that individual incentives favor a financial system that is over-diversified with respect to what is socially efficient.

More in detail, we consider a banking system composed of *leveraged* and risk-averse financial institutions (hereafter, “banks”) that invest in two asset classes. The first class consists of debts issued by other banks in the network (hereafter, “interbank claims”). The value of these securities depends, in turn, on the leverage of the issuers. The second class represents risky assets that are external to the financial network and may include, e.g., mortgages

on real estate, loans to firms and households and other real economy-related activities (hereafter, “external assets”). The underlying economic cycle is the primary source of external asset price fluctuations, but it is unknown *ex ante* to the banks and, with a certain probability  $p$ , it may be positive (hereafter, “uptrend”) or negative (hereafter, “downtrend”).

As a first result, we find that if the future economic trend is unknown, the expected utilities (both at the banking and social system levels) are inverse U-shaped functions of the diversification level. Thus, optimal risk diversification is interior and its level depends on the probability  $p$  of the trend and on the magnitude of the expected profit and loss. The intuition behind this results is as follows. Diversification of idiosyncratic risks lowers the volatility of a bank’s portfolio and increases the likelihood of the portfolio to follow the economic trend underlying the price movements. Therefore, risk diversification is beneficial in the presence of a positive economic trend because it reduces the *downside risk*, and it is detrimental in the presence of a negative economic trend because it reduces the *upside potential*. As a result, there exists an optimal intermediate level of risk diversification that depends on the probability of the stochastic trend to be positive or negative.

As a second result, we find that the incentives of individual banks favor a banking system that is over-diversified with respect to the level of diversification that is socially desirable. Hence, tension arises between individual incentives and system efficiency. The fact that the trade-off is more pronounced at the social welfare level stems from the assumption of additional recovery costs faced by the social planner (hereafter, “regulator”). In other words, we assume an asymmetry in the expected losses between individual banks and the system. Although, the losses of individual banks are bounded from above in the presence of limited liability, externalities associated with the failure of interconnected institutions amplify the expected losses at a system level. It follows that the optimal diversification level from the social system perspective is always smaller than the optimal level for individual banks.

### 1.1. Related work

One of the novelties of our work is the fact that the result about interior optimal diversification holds even in the absence of asymmetric information, behavioral biases or transaction costs and taxes. Moreover, we do not need to impose *ad hoc* asset price distributions as in the literature on diversification

pitfalls in portfolios with fat-tailed distributions (to name a few, Zhou, 2010; Ibragimov et al., 2011; Mainik and Embrechts, 2012).

In our model, because external assets carry idiosyncratic risks, banks have an incentive to diversify across them. In this respect, similar to Evans and Archer (1968); Statman (1987); Elton and Gruber (1977); Johnson and Shannon (1974); Bird and Tippett (1986), we measure how the benefit of diversification vary as the number of external assets in an equally weighted portfolio is increased. This benchmark is the so-called  $1/n$  or *naive* rule. However, because banks are debt financed, we depart from the methods of those previous studies by modeling risk not in terms of a portfolio's standard deviation but in terms of the *default probability*. Indeed, in that literature the relationship between default probability and portfolio size has not been investigated in depth.

In order to investigate the notion of default probability in a system context, we build on the framework in which banks are connected in a network of liabilities as in the stream of works pioneered by Eisenberg and Noe (2001). However, that literature considers only the liquidation value of corporate debts at the time of default. In particular, in the works based on the notion of “clearing payment vector” (e.g., Cifuentes et al., 2005; Elsinger et al., 2006), the value of interbank claims depends on the solvency of the counterparties at the maturity of the contracts and it is determined as the fixed point of a so-called “fictitious sequential default” algorithm. Starting from a given *exogenous* shock on one or more banks, one can measure *ex-post* the impact of the shock in the system and investigate, for instance, which structure are more resilient to systemic risk (Battiston et al., 2012a; Roukny et al., 2013).

Our object instead here is to derive the default probability of individual banks, in a system context, that can be computed by regulators and market players *ex-ante*, i.e. before the shocks are realized and before the maturity of the claims. A related question was addressed in (Shin, 2008) where one assumes that asset values are random variables that move altogether according to a same scaling factor. The expected value of the assets is plugged into the Eisenberg-Noe fixed point algorithm yielding an estimate of the values of the liabilities before the observation of the shocks. However, the latter approach does not apply if assets are independent random variables and, more importantly, it does not address the issue of how the default probability of the various banks are related.

Strictly speaking, default means that the bank is not able to meet its obligations at the time of their maturity. Therefore, in principle, it does

not matter whether, any time before the maturity of the liabilities, the total asset value of a firm falls beneath the book value of its debts as long as it can recover by the time of the maturity. In practice, however, it does matter a lot. This is the case, for instance, if the bank has also some short-term liabilities and short-term creditors decide to run on the bank. Indeed there is a whole literature that building on Black and Cox (1976) investigates the notion of *time to default* in various settings. Such notion extends the framework of Merton (1974) by allowing defaults to occur at any *random* time before the maturity of the debt, as soon as the firm assets value falls to some prescribed lower threshold. In this paper, we combine the Eisenberg-Noe approach with the Black-Cox approach, by modeling the evolution over time of banks assets as stochastic processes where, at the same time, the value of interbank claims is a function of the financial fragility of the counterparties as reflected by the credit-liability network.

Although from a mathematical point of view, the framework requires to deploy the machinery of continuous stochastic processes, this work offers a valuable way to compute the default probability in system context under mild assumptions. The default probability can be written in analytical form in simple cases and it can be computed numerically in more complicated cases. An underlying assumption in the model is to consider the credit spread of counterparties as an increasing function of their leverage, i.e. the higher the leverage the higher the credit spread. As a benchmark, in this paper we assume that such a dependence is linear.

In general, the framework developed here allows to investigate how the probability of defaults depends on certain characteristics of the network such as the number of interbank contracts and the number of external assets. In this paper, we focus on the diversification level across external assets and we look at the limit in which analytical results can be obtained. The assumption we make is that the interbank market is relatively tightly knit and banks are sufficiently homogeneous in balance sheet composition and investment strategies. Indeed, it has been argued that the financial sector has undergone increasing levels of homogeneity, Haldane (2009). Moreover, empirical evidence shows that bank networks feature a core-periphery structure with a core of big and densely connected banks and a periphery of smaller banks. Thus, our hypothesis of homogeneity applies to the banks in such a core (see, e.g., Elsinger et al., 2006; Iori et al., 2006; Battiston et al., 2012c; Fricke and Lux, 2012).

The paper is organized as follows. In Section 2, we introduce the model.



Section 3 adopts a marginal benefit analysis by formalizing the single bank utility maximization problem with respect to the number of external assets in the portfolio. In Section 4, we compare private incentives of risk diversification with social welfare effects. Section 5 concludes the paper and considers some policy implications.

## 2. Model

Let time be indexed by  $t \in [0, \infty]$  in a system of  $N$  risk-averse leveraged banks with mean-variance utility function. To ensure simplicity in notation, we omit the time subscripts whenever there is no confusion. For the bank  $i \in \{1, \dots, N\}$ , the balance sheet identity conceives the equilibrium between the asset and liability sides as follows:

$$a_i = l_i + e_i, \quad \forall t \geq 0 \quad (1)$$

where  $\mathbf{a} := (a_1, \dots, a_N)^\top$  is the column vector of bank assets at market value.  $\mathbf{l} := (l_1, \dots, l_N)^\top$  is the column vector of bank debts at book face value. There is an homogeneous class of debt with maturity  $T$  and zero coupons, i.e., defaultable zero-coupon bonds.  $\mathbf{e} := (e_1, \dots, e_N)^\top$  is the column vector of equity values. Notably, the market for investment opportunities is complete and composed of two asset classes that are perfectly divisible and traded continuously: (i)  $N$  interbank claims, and (ii)  $M$  external assets related to the real side of the economy. There are no transaction costs or taxes. However, there are borrowing and short-selling restrictions. Each bank selects a portfolio composed of  $n \leq N - 1$  interbank claims and  $m \leq M$  external assets. Then, the asset side in Eq. (1) can be decomposed as<sup>1</sup>:

$$a_i := \sum_j z_{ij} \nu_j + \sum_k w_{ik} \hat{l}_k. \quad (2)$$

$\mathbf{Z} := [z_{ij}]_{N \times M}$  is the  $N \times M$  weighting matrix of external investments in which each entry  $z_{ij} \geq 0$  is the number of units of external asset  $j$  at price  $\nu_j$  hold by bank  $i$ .  $\mathbf{W} := [w_{ik}]_{N \times N}$  is the  $N \times N$  adjacency matrix in which each

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<sup>1</sup>Notice that, for the sake of simplicity, we omit the lower and upper bounds of the summations. It remains understood that, in the summation for external assets, the index ranges from 1 to  $M$ , and that, in the summation for banks, the index ranges from 1 to  $N$  (with the condition that  $w_{ii} = 0$  for all  $i \in \{1, \dots, N\}$ ).

entry  $w_{ik} \geq 0$  is the number of units of bank  $i$ ' interbank claim on bank  $k$ . Interbank claims are marked-to-market. In line with the practice, we assume that bonds are priced according to the discounted value of future payoffs at maturity:

$$\hat{l}_i = \frac{l_i}{(1 + r_i)^{T-t}} \quad (3)$$

where  $r_i$  is the rate of return on  $(T-t)$ -years maturity obligations.  $c_i = r_i - r_f$  is the credit spread (premium), over the risk free rate  $r_f$ , paid by the bank to the bond holders. In a stylized form, each bank's balance-sheet is as follows:

**Bank- $i$  balance-sheet**

Assets	Liabilities
$\sum_j z_{ij} \nu_j$	$l_i$
$\sum_k w_{ik} \hat{l}_k$	$e_i$

*2.1. Leverage and Default Event*

Our approach to define the default event builds on Black and Cox (1976), who extends Merton (1974) by allowing for a premature default when the asset value of the firm falls beneath the book value of its debt. From a technical point of view, what matters is the debt-to-asset ratio:

$$\phi_i := \frac{l_i}{a_i}, \quad (4)$$

with natural bound  $[\varepsilon, 1]$  where  $\begin{cases} 1 & \text{default boundary} \\ \varepsilon \rightarrow 0^+ & \text{safe boundary} \end{cases}$ .

**Definition of Default Event:** The probability of the default event is the probability that  $\phi_i$ , initially at an arbitrary level  $\phi_i(0) \in (\varepsilon, 1)$ , exits for the first time through the *default boundary* 1, after time  $t > 0$ . More precisely, we use the concept of first exit time,  $\tau$ , through a particular end of the interval  $(\varepsilon, 1)$ . Namely,

$$\tau := \inf \{t \geq 0 \mid \mathbf{1}_{\phi_i(t) \leq \varepsilon} + \mathbf{1}_{\phi_i(t) \geq 1} \geq 1\}. \quad (5)$$

If the default event is defined as  $(\text{default}_i) := \{\phi_i(\tau) \geq 1\}$ , then the default probability is the probability of this event:

$$\mathbb{P}(\text{default}_i) = \mathbb{P}(\phi_i(\tau) \geq 1).$$

With a slight abuse of notation we rewrite  $\mathbb{P}(\text{default}_i)$  as:

$$\mathbb{P}(\text{default}_i) = \mathbb{P}(\phi_i \geq 1). \quad (6)$$

**Leverage in System Context:** Combining together Eq. (2)-(3)-(4), we obtain:

$$\phi_i = l_i / \left( \sum_j z_{ij} \nu_j + \sum_k w_{ik} \left( \frac{l_k}{(1+r_k)^{(T-t)}} \right) \right). \quad (7)$$

Theoretical and empirical evidence have shown that there are multiple control variables affecting the credit spread, such as the firm's leverage, the volatility of the underlying assets or the liquidity risk (see, e.g., Collin-Dufresne et al., 2001). Since we are interested to study Eq. (6) in a system context, in order to better isolate the explanatory power of leverage, we assume that the credit spread depends only on leverage in a linear fashion:

$$c_i = \beta \phi_i. \quad (8)$$

In reality, the relation between credit spread and leverage can be more complicated. However, as it will be more clear in the following, we establish a useful benchmark for a number of exercises.

The parameter  $\beta (> 0)$  is the factor loading on  $i$ 's leverage  $\phi_i$  and can be understood as the responsiveness of the rate of return to the leverage. Then, by replacing Eq. (8) into Eq. (3) we have:

$$\hat{l}_i = \frac{l_i}{(1+r_f + \beta \phi_i)} \quad (9)$$

where, w.l.g.  $T - t = 1$ . This means that banks issue 1-year maturity obligations that are continuously rolled over. Notice that, by Eq. (6) and Eq. (9), even in the case of a high default probability, bank debts are still priced at a positive market value. Namely, for  $\phi_i \rightarrow 1$ ,  $\hat{l}_i > 0$ . This means that, creditors are assumed to partially recover their credits in case of default. The recovery rate can be implicitly determined as shown in Appendix A.

Now, by using Eq. (9) we can rewrite Eq. (7) as:

$$\phi_i = l_i / \left( \sum_j z_{ij} \nu_j + \sum_k w_{ik} \left( \frac{l_k}{1+r_f + \beta \phi_k} \right) \right). \quad (10)$$

Eq. (10) highlights a *non-linear* dependence of  $\phi_i$  from the leverage  $\phi_{k=1,\dots,n}$  of the other banks to whom  $i$  is exposed via the matrix  $\mathbf{W}$ .<sup>2</sup>

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<sup>2</sup>The obligation of each bank  $i$  can be considered as an  $n$ -order derivative, the price of which is derived from the risk-free rate  $r_f$  and from the leverage  $\phi_i$  of  $i$ . The latter, in turn, depends on the leverage  $\phi_{k=1,\dots,n}$  of the other banks obligors of  $i$ .

Recent works based on the “clearing payment vector” mechanism (see, e.g., Eisenberg and Noe, 2001; Cifuentes et al., 2005) provide a “fictitious sequential default” algorithm to determine the liquidation equilibrium value of interbank claims at their maturity. In reality, however, defaults may happen before the maturity of the debts. In this respect, Eq. (9) together with Eq. (10) captures, even before the maturity of the debts, the market value of interbank claims in the building up of the distress spreading from one bank to another. In matrix notation, this value depends on the solution of a second order polynomial equation in  $\Phi$ :

$$\Phi \mathbf{H} \beta \Phi + \Phi \mathbf{H} \mathbf{R} - \mathbf{W}^{-1} \beta \Phi + \Phi = \mathbf{W}^{-1} \mathbf{R} \quad (11)$$

where  $\mathbf{H} := \mathbf{ZV}(\mathbf{WL})^{-1}$  and  $\Phi := \text{diag}(\phi_1, \phi_2, \dots, \phi_N)$ ;  $\mathbf{L} := \text{diag}(l_1, l_2, \dots, l_N)$ ;  $\mathbf{V} := \text{diag}(\nu_1, \nu_2, \dots, \nu_M)$ ;  $\mathbf{R} := \text{diag}(R, R, \dots, R)$  with  $R = 1 + r_f$ ;  $\mathbf{W} := [w_{ik}]_{N \times N}$ ;  $\mathbf{Z} := [z_{ij}]_{N \times M}$ . See Appendix A.

Along this line of reasoning, one can notice that the default probability of a given bank depends on the likelihood of its leverage to hit the default boundary. This, in turn, depends on the joint probability of the other banks’ leverages, to whom this banks is connected, of hitting the default boundary. In order to account for these network effects, in the next section we will provide an explicit form of default probability in system context. This *forward looking* measure of systemic risk estimates at any point in time the likelihood of a system collapse and combines into a single figure asset values, business risk, and leverage.

### 3. Benefits of Diversification in External Assets

Similar to Evans and Archer (1968); Statman (1987); Elton and Gruber (1977); Johnson and Shannon (1974); Bird and Tippett (1986), in this section we measure the advantage of diversification by determining the rate at which risk reduction benefits are realized as the number  $m$  ( $\leq M$ ) of external assets in an *equally weighted* portfolio is increased. In contrast with those studies, rather than minimizing the variance of the banks’ assets, we maximize their expected utility with respect to  $m$ . The methodology is explained in the following subsections. In 3.1 we formalize the equally weighted portfolio of external assets. 3.2 defines the systemic default event. In 3.3 we formalize the bank utility function and in 3.4 we maximize the utility function with respect to the control variable  $m$ .

### 3.1. Equally Weighted Portfolio of External Assets

From Eq. (2), let bank  $i$ 's portfolio of external assets be defined as:

$$s_i := \sum_j z_{ij} \nu_j. \quad (12)$$

To study the benefits of diversification, in isolation, we need to consider the  $1/m$  (equally weighted) portfolio allocation. This means that portfolio allocation is respect to the number. This allocation is adopted as a metric to measure the rate at which risk-reduction benefits are realized as the number of assets held in the portfolio is increased. Therefore, external assets are assumed to be equally weighted in banks' portfolios. Formally, for every external asset  $j \in \{1, \dots, M\}$  and each bank  $i \in \{1, \dots, N\}$ , the fraction of portfolio  $s_i$  invested by bank  $i$  in the external asset  $j$  is:

$$\frac{1}{m} = \frac{z_{ij} \nu_j}{s_i}. \quad (13)$$

The minimum conditions that allow us to apply the  $1/m$  rule, as a benchmark, without violating the mean-variance dominance criterion, is to assume the external assets to be indistinguishable, i.e., they have the same drift, the same variance and they are uncorrelated.<sup>3</sup> Thus, the price of external assets is properly characterised by following time-homogenous diffusion process:<sup>4</sup>

$$\frac{d\nu_j}{\nu_j(t)} = \mu dt + \sigma d\tilde{B}_j(t), \quad j = 1, \dots, M \quad (14)$$

Using the expression in Eq. (13), we arrive after some transformations at the following dynamics for the the portfolio in Eq. (12):

$$\frac{ds_i}{s_i(t)} = \mu dt + \frac{\sigma}{\sqrt{m}} dB_i(t). \quad (15)$$

$B_i = \frac{1}{m} \sum_j \tilde{B}_j$  is an equally weighted linear combination of Brownian shocks s.t.  $dB_j \sim N(0, dt)$ .

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<sup>3</sup>Notice that under those conditions, the  $1/m$  portfolio allocation is Pareto optimal. See e.g., Rothschild and Stiglitz (1971); Samuelson (1967); Windcliff and Boyle (2004).

<sup>4</sup>Where  $\tilde{B}_j(t)$  is a standard Brownian motion defined on a complete filtered probability space  $(\Omega; \mathcal{F}; \{\mathcal{F}_t\}; \mathbb{P})$ , with  $\mathcal{F}_t = \sigma_y\{\tilde{B}(s) : s \leq t\}$ ,  $\mu$  is the instantaneous risk-adjusted expected growth rate,  $\sigma > 0$  is the volatility of the growth rate and  $\mathbb{E}(d\tilde{B}_j, d\tilde{B}_y) := \rho_{jy} = 0$ .

**Properties of the Portfolio:** There exist two states of the world,  $\theta = \{0, 1\}$ . This captures a situation in which the economy is either in a boom ( $\theta = 1$ ) or in a bust ( $\theta = 0$ ) state and is reminiscent of a stylized economic cycle. The probability that the world is in state  $\theta$  is denoted as  $\mathbb{P}(\{\theta\})$  with  $\mathbb{P}(\{0\}) = p$  and  $\mathbb{P}(\{1\}) = 1 - p$ . According to the state of the world, the market of external assets is assumed to follow a given constant stochastic trend under a certain probability space  $(\Omega_\mu, \mathcal{A}, \mathbb{P})$ . The sample space  $\Omega_\mu = \{\mu^-, \mu^+\}$  is the set of the outcomes. We use the convention:

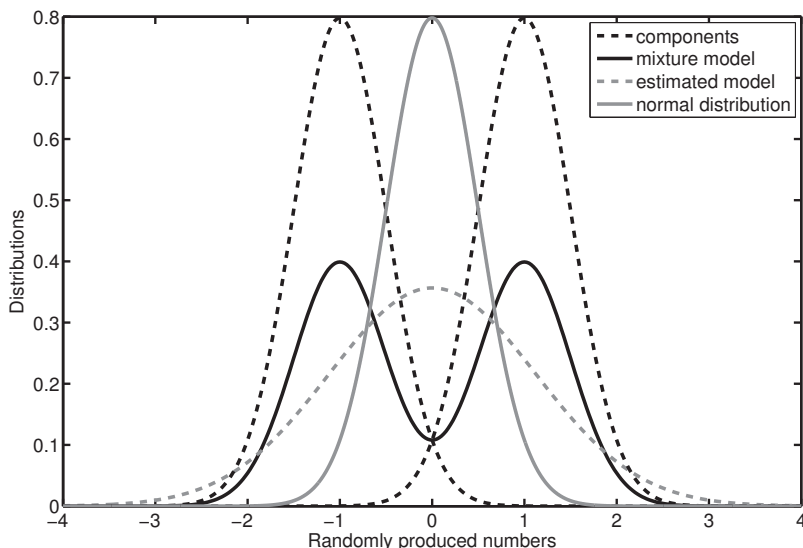
$$\begin{cases} \mu < 0 := \mu^- & \text{if } \theta = 0, \\ \mu \geq 0 := \mu^+ & \text{if } \theta = 1 \end{cases}$$

with  $|\mu^+| = |\mu^-|$ . The  $\sigma$ -algebra  $\mathcal{A}$  is the power set of all the subsets of the sample space,  $\mathcal{A} = 2^{\Omega_\mu} = 2^2 = \{\{\mu^-\}, \{\mu^+\}, \{\mu^-, \mu^-\}, \{\}\}$ .  $\mathbb{P}$  is the probability measure,  $\mathbb{P} : \mathcal{A} \rightarrow [0, 1]$  with  $\mathbb{P}(\{\}) = 0$ ,  $\mathbb{P}(\{\mu^-\}) = p$ ,  $\mathbb{P}(\{\mu^+\}) = 1 - p$  and  $\mathbb{P}(\{\mu^-, \mu^+\}) = 1$ . That is,  $p$  and  $(1 - p)$  are the probabilities of having a downtrend and an uptrend, respectively. To conclude, portfolio returns display a mixture distribution expressed by the convex combination of two normal distributions weighted by  $p$  and  $1 - p$ . Namely,

$$\frac{ds_i}{s_i} \sim pN\left(\mu^-, \frac{\sigma}{\sqrt{m}}\right) + (1 - p)N\left(\mu^+, \frac{\sigma}{\sqrt{m}}\right) \quad (16)$$

$$\text{with } \begin{cases} \mathbb{E}\left[\frac{ds_i}{s_i}\right] := \hat{\mu} = p\mu^+ + (1 - p)\mu^- \\ \mathbb{E}\left[\left(\frac{ds_i}{s_i} - \hat{\mu}\right)^2\right] := \hat{\sigma}^2 = p\left[(\mu^+ - \hat{\mu})^2 + \frac{\sigma^2}{m}\right] + (1 - p)\left[(\mu^- - \hat{\mu})^2 + \frac{\sigma^2}{m}\right]. \end{cases}$$

**Figure 1:** Distribution of portfolio returns (mixture model).



Comparison between a portfolio with returns following a bimodal distribution (black color) and a portfolio with normally distributed returns (gray colour). Parameters:  $\mu = 0$ ,  $\mu^- = -1$ ,  $\mu^+ = 1$ ,  $\sigma = 1$ ,  $p = 0.5$ .

Figure 1 illustrates this result by comparing two probability density functions (pdf). The first (gray color) curve represents the pdf of a portfolio with returns  $X$  normally distributed, i.e.,  $X \sim N(0, 1)$ . The second (black color) curve represents the pdf of a portfolio with returns distributed as  $Y1 \sim N(0.5, 1)$  with probability  $p = 0.5$  and as  $Y2 \sim N(-0.5, 1)$  with probability  $1 - p = 0.5$ . Notice that in a world where the market displays a normal distribution, the middle part of the distribution range (the “belly”) is the most likely outcome. In contrast, in a bimodal world, the belly is the least likely outcome. Moreover, the tails of the bimodal distribution are higher than those of the normal distribution. This indicates the higher probability of severe left and right side events. The distance between the two peaks depends on the difference between the means of the two normal distributions,  $|(\mu^+) - (\mu^-)|$ . The (possible) asymmetry between the two peaks and the skewness of the bimodal distribution depends on the difference between the

probability of having an uptrend or a downtrend,  $|(1 - p) - p|$ . As a result, portfolio diversification choice is subject to much more uncertainty in a bimodal world than in a normal one. The next sections show how the statistical properties of the bimodal distribution impact on the risk diversification effects.

### 3.2. Systemic Default Probability

**Interbank Network structure and Contagion.** We leave aside issues related to endogenous interbank network formation, optimal interbank network structures and network efficiency. See Leitner (2005), Gale and Kariv (2007), Castiglionesi and Navarro (2007) and the survey by Allen and Babus (2009) for discussion of these topics.

In our analysis, we assume the presence of a *tightly knit* network of homogeneous banks holding balance sheets and portfolios that look alike. Indeed, it has been argued that the financial sector has undergone increasing levels of homogeneity, Haldane (2009). Moreover, empirical evidence shows that bank networks feature a core-periphery structure with a dense core of fully connected banks and a periphery of small banks. Thus, our hypothesis of homogeneity is realistic for the banks in the core (see, e.g., Elsinger et al., 2006; Iori et al., 2006; Battiston et al., 2012c).

Despite, one might distinguish between two channels of contagion by which shocks may propagate: direct asset price contagion (via overlapping portfolios) and indirect asset price contagion (via interbank claims), strict interdependencies make it difficult to characterize the propagation of contagion in the system. The potential spread of contagion is high and immediate. Since, the size and structure of interbank linkages are hold constant, all the banks are likely to be hit as defaults propagate through the system.<sup>5</sup>

**Relation between Individual and Systemic Default.** Under the property of homogeneity, banks are assumed to adopt the same capital structure:

- the portfolio of external assets is similar across banks, i.e.,  $s_i = s$  for all  $i \in \{1, \dots, N\}$ ;
- the book value of promised payments at maturity is equal for every bank, i.e.,  $l_i = l$  for all  $i \in \{1, \dots, N\}$ .

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<sup>5</sup>Arguably, it is appropriate to assume that the network remains static, especially in downtrend periods. See Gai and Kapadia (2007).



Under those conditions, leverage ratios may differ across banks and over time, but they will remain close to the mean leverage over all banks. Namely, the leverage of every bank “converges” in distribution to the *market leverage*:

$$\phi_i \stackrel{d}{\simeq} \frac{1}{N} \sum_j^N \phi_j := \phi ,$$

for all  $i \in \{1, \dots, N\}$ . The above assumption, together with the results from the previous section allow us to rewrite Eq. (10) as:

$$\phi = l / \left( s + \frac{l}{1 + r_f + \beta\phi} \right) . \quad (17)$$

which is a quadratic expression in  $\phi$ :

$$\phi^2 \beta s + \phi (sR + l(1 - \beta)) - lR = 0 , \quad R = 1 + r_f . \quad (18)$$

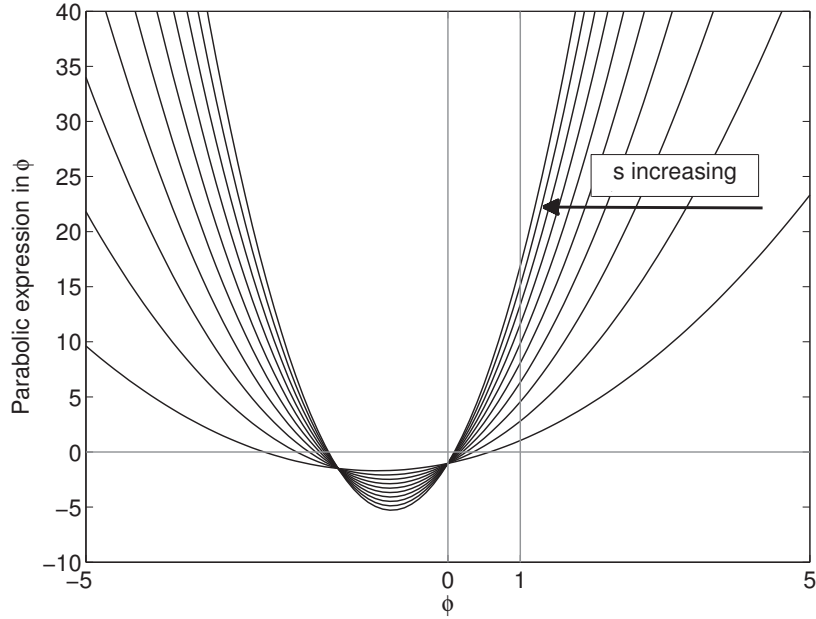
Heuristically, we can proof that Eq. (18) has always one positive and one negative root for any values of the parameters ( $\beta > 0$ ,  $R \geq 1$ ,  $s > 0$ ,  $l > 0$ ) in their range of variation:

$$\begin{cases} \phi_{pos} &= \frac{1}{2\beta s} \left[ l(\beta - 1) - Rs + (4\beta lRs + (l(1 - \beta) + Rs)^2)^{1/2} \right] , \\ \phi_{neg} &= \frac{1}{2\beta s} \left[ l(\beta - 1) - Rs - (4\beta lRs + (l(1 - \beta) + Rs)^2)^{1/2} \right] . \end{cases}$$

Since, by definition  $\phi$  can only be positive, we exclude the negative solution. Therefore, one can always find a unique positive solution to Eq. (18):

$$\phi := \phi_{pos} . \quad (19)$$

**Figure 2:** Parabolic expression of the market leverage.



The market leverage is the solution of a parabolic expression that implies a non linear relationship between banks' leverage. Parameters:  $l = 0.9$ ,  $s \in \{1, 2, \dots, 10\}$ ,  $r_f = 0.03$ ,  $\beta = 0.5$ .

The parabola in Eq. (18) is depicted in Figure 2. For different values of  $s$  the roots are those points crossing zero in the interval  $[\varepsilon, 1]$ , which is the natural bound of the leverage, see Eq. (4).

Under the setting described above, we assign a systemic meaning to the concept of default probability:

$$\mathbb{P}(\text{default}_i) = \mathbb{P}(\phi_i \geq 1) \simeq \mathbb{P}(\text{default}) = \mathbb{P}(\phi \geq 1), \quad (20)$$

for all  $i \in \{1, \dots, N\}$ .

**Remark 1.** Let  $\mathbb{P}(\phi_i \in C)$  and  $\mathbb{P}(\phi \in C)$  represent the default probability of a single bank and the system, respectively. Since, under the mean-field approximation,  $\phi_i \stackrel{d}{\simeq} \phi$ , for the simplified version of the Continuous Mapping

Theorem,  $\mathbb{P}(\phi_i \in C) \simeq \mathbb{P}(\phi \in C)$  for all continuity sets  $C \subseteq \Omega_\phi := [\varepsilon, 1]$  and all  $i \in \{1, \dots, N\}$ .

$\mathbb{P}(\text{default})$  depends on the distribution of  $\phi$  that, in turn, depends on the distribution of  $s$ . Hence, the systemic default probability can be also defined w.r.t.  $s$ :

$$\mathbb{P}(\text{default}) = \mathbb{P}(\phi \geq 1) \equiv \mathbb{P}(s \leq s^-) \quad (21)$$

$$\text{with } \begin{cases} s^+ = \frac{l(\beta\varepsilon - \varepsilon + 1)}{(\beta\varepsilon^2 + \varepsilon)} & \text{safe boundary} \\ s^- = \frac{l(r_f + \beta)}{R + \beta} & \text{default boundary} . \end{cases}$$

See Appendix A. Given the distribution properties of the portfolio in Eq. (16) and the default conditions in Eq.(21),  $\mathbb{P}(\text{default})$  can be expressed as:

$$\mathbb{P}(\text{default}) = \left( \int_{s_0}^{s^+} ds \psi(x) \right) / \left( \int_{s^-}^{s^+} dx \psi(x) \right), \text{ where } \psi(x) = \exp \left( \int_0^x -\frac{2\hat{\mu}}{\hat{\sigma}^2} ds \right) . \quad (22)$$

Eq. (22) has the following closed form solution:

$$\mathbb{P}(\text{default}) = \left( \exp \left[ -\frac{2\hat{\mu}s_0}{\hat{\sigma}^2} \right] - \exp \left[ -\frac{2\hat{\mu}s^+}{\hat{\sigma}^2} \right] \right) / \left( \exp \left[ -\frac{2\hat{\mu}s^-}{\hat{\sigma}^2} \right] - \exp \left[ -\frac{2\hat{\mu}s^+}{\hat{\sigma}^2} \right] \right) .$$

See Appendix A. Eq. (22) is the probability that  $s$ , initially at an arbitrary level  $s(0) := s_0 \in (s^-, s^+)$ , exits through  $s^-$  before  $s^+$  after time  $t > 0$ . This can be related to the problem of first exit time through a particular end of the interval  $(s^-, s^+)$ , see Gardiner (1985). Now, we define the systemic default probability, *conditional* to a given trend followed by the external assets, as:

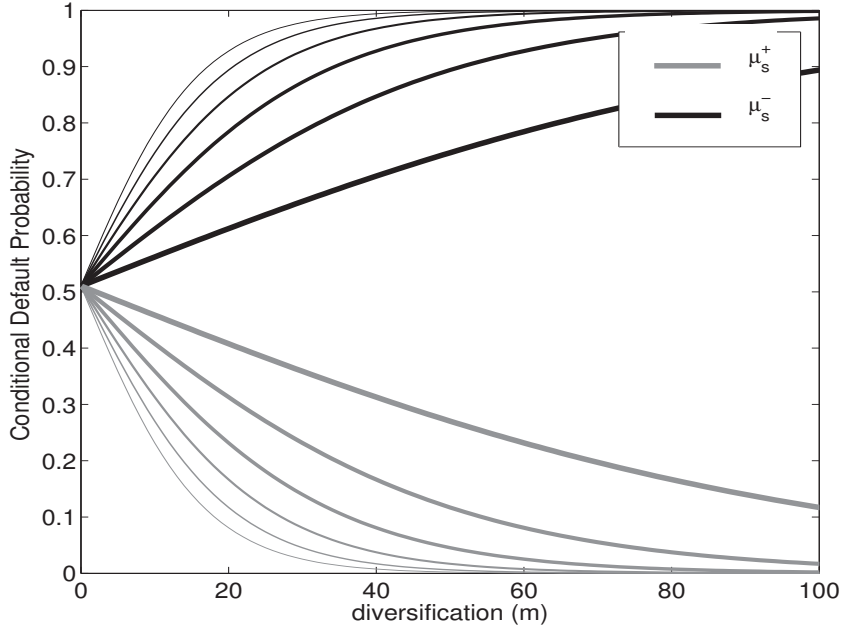
$$\begin{cases} q := \mathbb{P}(\text{default} \mid \mu^-) & \text{def. prob. in the case of a downtrend ,} \\ g := \mathbb{P}(\text{default} \mid \mu^+) & \text{def. prob. in the case of an uptrend} \end{cases}$$

with the following closed form solutions:

$$\begin{cases} q = \left( \exp \left[ -\frac{(2\mu^-)s_0}{\sigma^2/m} \right] - \exp \left[ -\frac{(2\mu^-)s^+}{\sigma^2/m} \right] \right) / \left( \exp \left[ -\frac{(2\mu^-)s^-}{\sigma^2/m} \right] - \exp \left[ -\frac{(2\mu^-)s^+}{\sigma^2/m} \right] \right) , \\ g = \left( \exp \left[ -\frac{(2\mu^+)s_0}{\sigma^2/m} \right] - \exp \left[ -\frac{(2\mu^+)s^+}{\sigma^2/m} \right] \right) / \left( \exp \left[ -\frac{(2\mu^+)s^-}{\sigma^2/m} \right] - \exp \left[ -\frac{(2\mu^+)s^+}{\sigma^2/m} \right] \right) . \end{cases}$$

See Appendix A.

**Figure 3:** Conditional Default Prob. for different levels of risk diversification.



Conditional def. prob.  $q$  given a downtrend (black lines). Conditional def. prob.  $g$  given an uptrend (gray lines).  $q$  increases with diversification. Instead,  $g$  decreases with diversification. The elasticity of  $q$  and  $g$  w.r.t.  $m$  depends on the magnitude of the trend. Parameters:  $|\mu^+| = |\mu^-| \in \{0.005, 0.01, 0.02, 0.025, 0.03\}$ ,  $l = 0.9$ ,  $s_0 = (s^+ - s^-)/2$ ,  $\sigma = 0.5$ ,  $r_f = 0.01$ ,  $\varepsilon = 0.1$ ,  $\beta = 0.2$ .

***Diversification effects on Default Probability.*** An asymptotic analysis of  $q$  and  $g$  reveals that, in an idealized world without transaction costs and infinite population size of external assets (i.e.,  $M \rightarrow \infty$ ), at increasing levels of risk diversification (i.e.,  $m \rightarrow M$ ), the default probability exhibits a bifurcated behavior. Precisely,  $g$  ( $q$ ) decreases (increases) with diversification.<sup>6</sup> See Figure 3. We conclude with the following general proposition:

<sup>6</sup>In both cases, trends are assumed to be persistent (i.e., approximately constant during a given period  $\Delta t$ ).

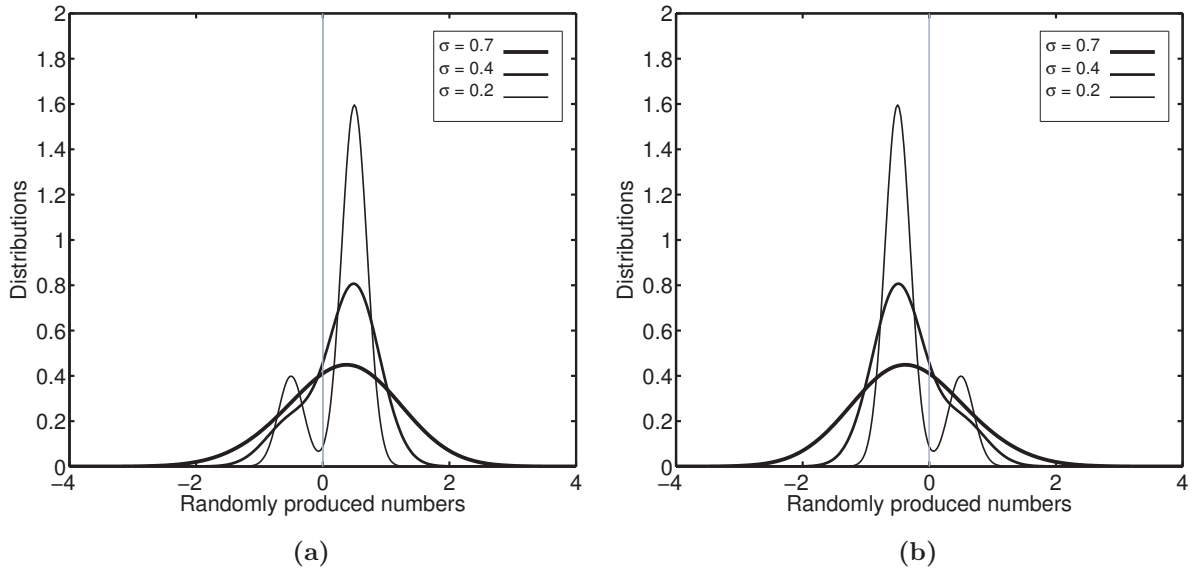
**Proposition 1.** *Consider a debt-financed portfolio subject to a fix default threshold. Then, the mitigation of idiosyncratic risks via portfolio diversification is desirable when asset prices uptrend and undesirable when asset prices downtrend. The degree of (un)desirability, which is measured in terms of default probability, increases with the level of diversification.*

See Appendix A. The intuition behind the polarization of the probability to “survive” and the probability to “fail” is beguilingly simple, but its implications are profound. In brief, the diversification of idiosyncratic risks reduces the volatility of the portfolio. The lower volatility increases the likelihood of the portfolio to follow an underlying economic trend. Therefore:

- in uptrend periods, diversification is beneficial because it reduces the downside risk and highlights the positive trend; thus, the default probability decreases;
- in downtrend periods, diversification is detrimental because it reduces the upside potential and highlights the negative trend; therefore, the default probability increases.

Figure 4 explains this intuition by showing how the pdf of portfolio returns is influenced by the downtrend probability and by the level of diversification. As one may observe, for increasing diversification, viz., lower volatility, the pdf changes shape by moving from the thick black curve ( $\sigma = 0.7$ ), to the medium black curve ( $\sigma = 0.4$ ), and finally to the thin black curve ( $\sigma = 0.2$ ). In Figure 4 a) the probability of a positive trend is greater than the probability of a negative trend,  $p = 0.2$ . Therefore, diversification is desirable because it reduces the volatility and, by doing so, it shifts to the right the density of the distribution of portfolio returns. Instead, In Figure 4 b) the probability of a negative trend is greater than the probability of a positive trend,  $p = 0.8$ . In this case, diversification is undesirable because, by reducing the volatility, it shifts to the left the density of the distribution.

**Figure 4:** Distribution of portfolio returns for different levels of volatility and different prob. of downtrend.



Comparison between the probability density functions of three portfolios with returns following a bimodal distribution with the same expected value but decreasing volatility. Thick black curve ( $\sigma = 0.7$ ), medium black curve ( $\sigma = 0.4$ ), thin black curve ( $\sigma = 0.2$ ). **(a)** Each curve represents the pdf of a portfolio with returns distributed as  $Y1 \sim N(-0.5, \sigma)$  with probability  $p = 0.2$  and as  $Y2 \sim N(0.5, \sigma)$  with probability  $1 - p = 0.8$ . **(b)** Each curve represents the pdf of a portfolio with returns distributed as  $Y1 \sim N(0.5, \sigma)$  with probability  $p = 0.8$  and as  $Y2 \sim N(-0.5, \sigma)$  with probability  $1 - p = 0.2$ .

### 3.3. Bank Utility Function

In this section we formalize the bank utility maximization problem with respect to the number  $m$  of external assets held in the equally weighted portfolio described in Section 3.1.

The bank's payoff from investing in external assets is a random variable  $\Pi_m$  that depends on the number  $m$  of external assets in portfolio and on their values. It takes the value  $\pi$  in the set  $\Omega_\pi = [\pi^-, \dots, \pi^+]$ , where:

$$\begin{cases} \pi^- := s^- - s_0 & \text{max attainable profit,} \\ \pi^+ := s^+ - s_0 & \text{max attainable loss.} \end{cases}$$

More specifically, given  $m$  mutually exclusive choices (i.e., the bank portfolio can be composed of 1, 2, ... external assets) and their corresponding random return  $\Pi_1, \Pi_2, \dots$ , with distribution function  $F_1(\pi), F_2(\pi), \dots$ , preferences that satisfy the von Neumann-Morgenstern axioms imply the existence of a measurable, continuous utility function  $U(\pi)$  such that  $\Pi_1$  is preferred to  $\Pi_2$  if and only if  $\mathbb{E}U(\Pi_1) > \mathbb{E}U(\Pi_2)$ . We assume that banks are mean-variance (MV) decision makers, such that the utility function  $\mathbb{E}U(\Pi_m)$  may be written as a smooth function  $V(\mathbb{E}(\Pi_m), \sigma^2(\Pi_m))$ <sup>7</sup> of the mean  $\mathbb{E}(\Pi_m)$  and the variance  $\sigma^2(\Pi_m)$  of  $\Pi_m$  or

$$V(\mathbb{E}(\Pi_m), \sigma^2(\Pi_m)) := \mathbb{E}U(\Pi_m) = \mathbb{E}(\Pi_m) - (\lambda\sigma^2(\Pi_m))/2$$

such that  $\Pi_1$  is preferred to  $\Pi_2$  if and only if <sup>8</sup>

$$V(\mathbb{E}(\Pi_1), \sigma^2(\Pi_1)) > V(\mathbb{E}(\Pi_2), \sigma^2(\Pi_2)).$$

Then, the maximization problem is as follows:

$$\begin{aligned} \max_m \mathbb{E}U(\Pi_m) &= \mathbb{E}(\Pi_m) - \frac{\lambda\sigma^2(\Pi_m)}{2} & (23) \\ \text{s.t.} & \begin{cases} 1 \leq m \leq M \\ l > 0 \\ s^- < s_0 < s^+ \\ p\mu_s^- + (1-p)\mu_s^+ > 0 \end{cases} \\ \text{with} & \begin{cases} \mathbb{E}(\Pi_m) = p[q\pi^- + (1-q)\pi^+] + (1-p)[g\pi^- + (1-g)\pi^+] \\ \sigma^2(\Pi_m) = p \left[ q(\pi^- - \mathbb{E}(\Pi_m))^2 + (1-q)(\pi^+ - \mathbb{E}(\Pi_m))^2 \right] \\ \quad + (1-p) \left[ g(\pi^- - \mathbb{E}(\Pi_m))^2 + (1-g)(\pi^+ - \mathbb{E}(\Pi_m))^2 \right] \end{cases} \end{aligned}$$

Notice that Eq.(23) is a *static* non-linear optimization problem w.r.t.  $m$ , with inequality constraints. The first constraint means that  $m$  can take only positive values between 1 and  $M$ . The second constraint requires banks to be

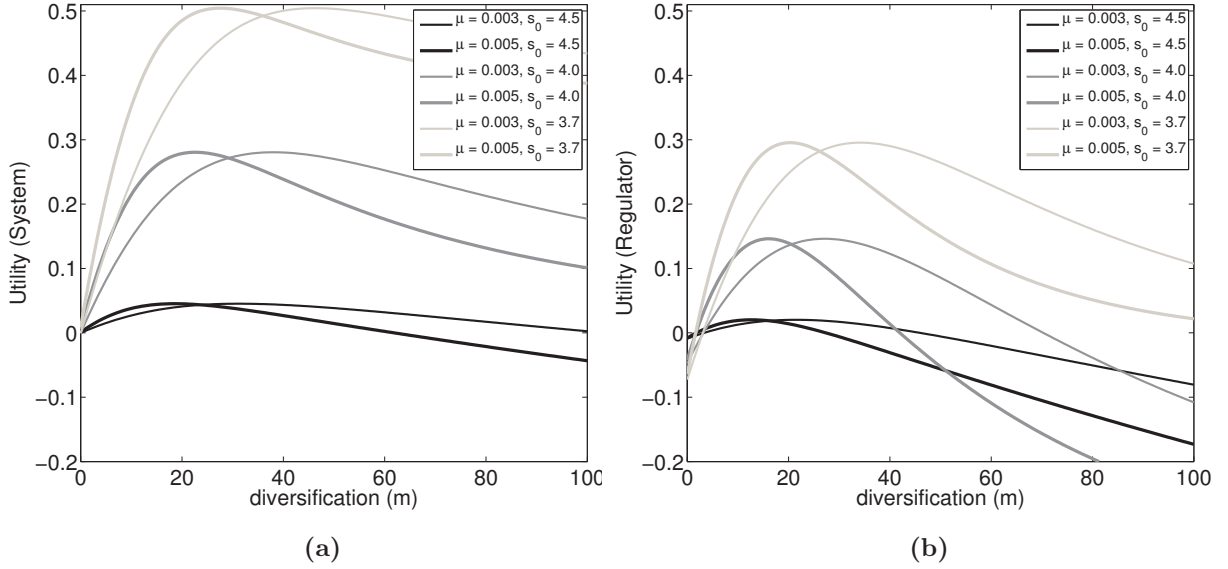
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<sup>7</sup>To describe  $V$  as smooth, it simply means that  $V$  is a twice differentiable function of the parameters  $\mathbb{E}(\Pi_m)$  and  $\sigma^2(\Pi_m)$ .

<sup>8</sup>Only the first two moments are relevant for the decision maker; thus, the expected utility can be written as a function in terms of the expected return (increasing) and the variance (decreasing) only, with  $\partial V(\mathbb{E}(\Pi_m), \sigma^2(\Pi_m)) / \partial \mathbb{E}(\Pi_m) > 0$  and  $\partial V(\mathbb{E}(\Pi_m), \sigma^2(\Pi_m)) / \partial \sigma^2(\Pi_m) < 0$ .

debt-financed. The third constraint requires that banks are not yet in default when implementing their asset allocation, i.e., the initial portfolio value must lie between the lower default boundary and the upper safe boundary. The last constraint represents the “economic growth” condition. That is, the expected economic trend of the real economy-related assets has to be positive. Since by definition,  $|\mu_s^-| = |\mu_s^+|$ , this condition is equivalent to impose an upper bound to the downtrend probability, namely  $p \in \Omega_p := [0, \frac{1}{2})$ . Then, given the above constraints, at time  $t$  banks randomly select (and fix) the number  $m$  of external assets to hold in their portfolio in order to minimize their default probability. This, in turn, maximizes their expected utility.

**Figure 5:** Expected MV Utility for different levels of risk diversification.



Fixed downtrend probability,  $p = 0.4$ . The figure compares the expected MV utility of the banking system vs. the expected MV utility of the regulator. The curves represent different initial asset values,  $s_0$ , and different drifts,  $\mu$ . **(a)** Exp. MV Utility of the banking system  $\mathbb{E}U(\Pi_m)$ . **(b)** Exp. MV Utility of the regulator  $\mathbb{E}U_r(\Pi_m)$ . Parameters:  $\sigma^2 = 0.5$ ,  $r_f = 0.001$ ,  $\varepsilon = 0.1$ ,  $\lambda = 0.1$ ,  $\beta = 0.2$ ,  $l = 0.5$ ,  $k = 2$ ,  $m \in \{1, \dots, 100\}$ ,  $s_0 \in \{3.7, 4, 4.5\}$ ,  $|\mu^+| = |\mu^-| \in \{0.003, 0.005\}$ .



### 3.4. Solution of the Bank Max Problem

The analysis of Eq. (23) leads to the following proposition:

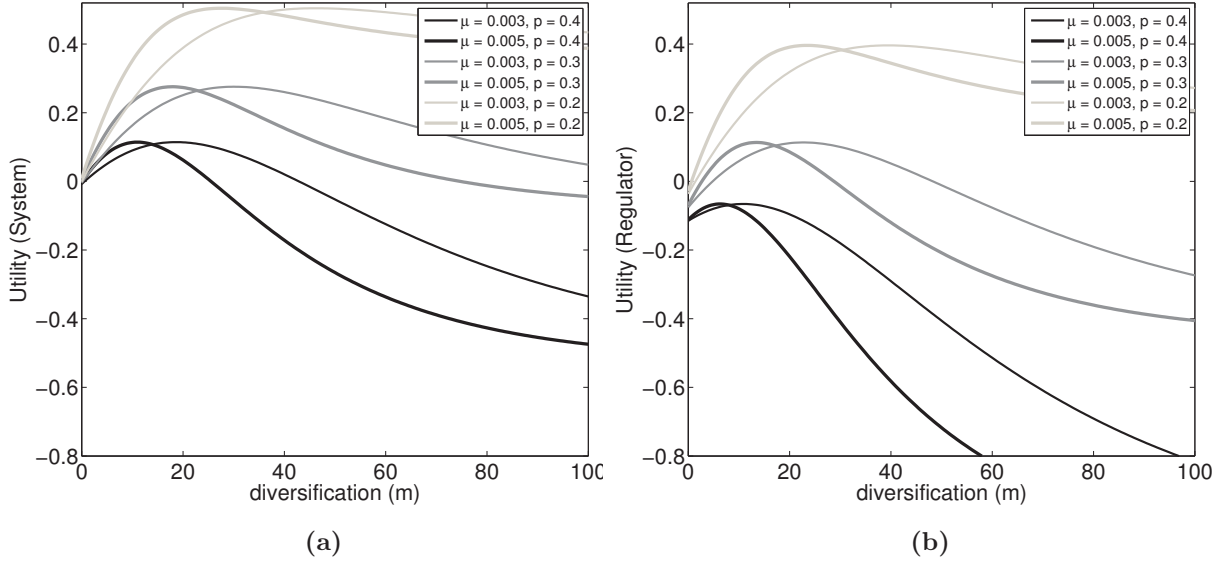
**Proposition 2.** *Given the probability interval  $\Omega_p := [0, \frac{1}{2})$ , there exists a subinterval  $\Omega_{p^*} \subset \Omega_p$  s.t., to each  $p^* \in \Omega_{p^*}$  corresponds an optimal level of diversification  $m^*$  in the open ball  $B(\frac{1+M}{2}, r) = \{m^* \in \mathbb{R} \mid d(m^*, \frac{1+M}{2}) < r\}$  with center  $\frac{1+M}{2}$  and radius  $r \in [0, \alpha]$  where  $\alpha = f(q, g)$ . Then,  $\mathbb{E}U(\Pi_m) \leq \mathbb{E}U(\Pi_{m^*})$ , for all  $m \notin B(\frac{1+M}{2}, r)$ .*

See Appendix A. Proposition 2 states that optimal diversification may be an interior solution. Namely, when banks maximize their MV utility they may choose an intermediate level of diversification, viz.,  $m^* \in (1, M)$ .  $m^*$  is the unique optimal solution and its level depends on the market size and on the likelihood of incurring in a negative or positive trend. In the words of Haldane (2009), we show that diversification is a *double-edged* strategy. Values of  $m \gtrsim m^*$  are second-best choices. Precisely, by increasing  $m$  to approach  $m^*$  from below, banks increase their utility. However, by increasing  $m$  beyond  $m^*$ , banks decrease their utility. In summary, the MV utility exhibits inverse U-shaped non-monotonic behavior with respect to  $m$ . These results hold under the market structure described in the previous sections. Briefly, banks are fully rational agents with incomplete information about the future state of the world. There are no transaction costs, negative externalities or market asymmetries. Market returns exhibit a bimodal distribution.

For a fixed probability  $p$ , Figure 5 a) shows how the utility changes for different levels of diversification  $m$ , different magnitudes of the trend and different initial asset values,  $s_0$ . Instead, for a fixed initial asset value  $s_0$ , Figure 6 a) shows how the utility changes for different levels of diversification  $m$ , different magnitudes of the trend and different probability of downtrend,  $p$ . Notice that,  $m$  enters into the maximization problem via Eq. (15). In particular, the portfolio volatility decreases with  $m$ . Therefore, an outward movement along the  $x$ -axis in Figures 5–6, i.e., increasing  $m$ , is equivalent to a market condition where the volatility of the assets is low,  $\sigma = \sigma_{low}$ . Conversely, an inward movement along the  $x$ -axis in Figures 5–6, i.e., decreasing  $m$ , is equivalent to a market condition where the volatility of the assets is high,  $\sigma = \sigma_{high}$ . To conclude, one might observe that if banks are already in  $m^*$ , an abrupt increases in the volatility of the assets (equivalent

to an inward movement from  $m^*$ ) decreases the utility of the banks.<sup>9</sup> In an optimization problem similar to our own, but with endogenous equilibrium asset pricing, Danielsson and Zigrand (2008) show that an increase in the volatility of both assets and portfolios can be generated by imposing strict risk-sensitive constraints of the VaR type.

**Figure 6:** Expected MV Utility for different levels of risk diversification.



Fixed initial asset value,  $s_0 = 3.69$ . The figure compares the expected MV utility of the banking system vs. the expected MV utility of the regulator. The curves represent different probabilities of downtrend,  $p$  and different drifts,  $\mu$ . **(a)** Exp. MV Utility of the banking system  $\mathbb{E}U(\Pi_m)$ . **(b)** Exp. MV Utility of the regulator  $\mathbb{E}U_r(\Pi_m)$ . Parameters:  $\sigma^2 = 0.5$ ,  $r_f = 0.001$ ,  $\varepsilon = 0.1$ ,  $\lambda = 0.1$ ,  $\beta = 0.2$ ,  $l = 0.5$ ,  $k = 2$ ,  $m \in \{1, \dots, 100\}$ ,  $p \in \{0.2, 0.3, 0.4\}$ ,  $|\mu^+| = |\mu^-| \in \{0.003, 0.005\}$ .

<sup>9</sup>This result could uprise as a possible explanation for the empirical findings that solvency condition across financial institutions, in the recent US 2007-08 crises, has been driven by an increase in the volatility of the firm's assets, Atkeson et al. (2013).

#### 4. Private Incentives vs. Social Welfare

Consider a sophisticated bank that, differently from the other banks, internalizes in its maximization problem the *social costs* due to multiple bank failures. These are negative externalities that limited-liability banks commonly do not account for. The definition of social losses is rather flexible since it depends on the characteristics of the financial system under analysis. In our model, we remain generic regarding the structure of these social costs.

In this section, we formulate the utility maximization problem of this sophisticated bank that we call “regulator” and compare it to the utility maximization problem of individual banks in Eq. (23). Let  $K$  be the number of simultaneously crashing banks. Then, it is reasonable to assume the followings:

**Assumption 1.** *The total loss to be accounted for by the regulator in down-trend periods is a monotonically increasing function  $f(k, \pi^-) := k\pi^-$  of: (i) the expected number  $k$  of bank crashes given a collapse of at least one bank  $\mathbb{E}(K|K \geq 1) = k$ , (ii) the magnitude of the loss  $\pi^-$ .*

##### 4.1. Regulator Utility Function

Therefore, the regulator’s utility maximization problem is as follows,

$$\begin{aligned} \max_m \mathbb{E}U_r(\Pi_m) = \mathbb{E}_r(\Pi_m) - \frac{\lambda\sigma_r^2(\Pi_m)}{2} \quad (24) \\ \text{s.t.:} \begin{cases} 1 \leq m \leq M \\ l > 0 \\ s^- < s_0 < s^+ \\ p\mu_s^- + (1-p)\mu_s^+ > 0 \\ k > 1 \end{cases} \end{aligned}$$

$$\text{with} \begin{cases} \mathbb{E}_r(\Pi_m) = p[qk\pi^- + (1-q)\pi^+] + (1-p)[g\pi^- + (1-g)\pi^+] \\ \sigma_r^2(\Pi_m) = p\left[q(k\pi^- - \mathbb{E}(\Pi_m))^2 + (1-q)(\pi^+ - \mathbb{E}(\Pi_m))^2\right] \\ \quad + (1-p)\left[g(\pi^- - \mathbb{E}(\Pi_m))^2 + (1-g)(\pi^+ - \mathbb{E}(\Pi_m))^2\right]. \end{cases}$$

Notice that, with respect to the optimization in Eq. (23), the regulator is subject to the additional constraint  $k > 1$  that amplifies both the expected loss and its variance. In summary, we compare the general solution of the maximization problem in Eq. (23) with that in Eq. (24).

#### 4.2. Results

The analysis shows that the optimal level of diversification for the regulator,  $m^r$ , is left-shifted with respect to the one desirable from the financial system point of view,  $m^*$ . Therefore,

**Proposition 3.** *The incentives of individual banks favor a banking system that is over-diversified in external assets w.r.t. to the level of diversification that is socially desirable:*

$$m^* \geq m^r . \quad (25)$$

See Appendix A. This result can be also illustrated by comparing Figure 5 a) with Figure 5 b) and Figure 6 a) with Figure 6 b).

To conclude, our model suggests that over-diversification occurs because banks do not internalize in their utility maximization problem the fact that their failure may also drag down other banks causing generalized losses to the whole system. As a final result, as soon as the probability of downtrend is moderately high, diversification turns to be a negative strategy that decreases the utility of the system.

### 5. Concluding Remarks

This paper provides a new modeling framework to measure the benefits that are associated with holding a diversified portfolio of assets in a system context of banks with interlocked balance sheets. In particular, we use the Black-Cox first-passage-time approach to measure the default probability of individual banks in a network context *a' la* Eisenberg and Noe. Indeed, we model the evolution over time of banks assets as stochastic processes where, at the same time, interbank assets value are a function of the financial fragility of the counterparties.

A first contribution of our dynamic and stochastic portfolio approach lies in measuring the benefits of risk diversification not in terms of portfolio variance but in terms of default probability and expected utility. The advantage of the stochastic approach is to provide an ex-ante estimation of default probability as opposed to what is usually done in the works building on the Eisenberg-Noe approach.

A second contribution is to compare the optimal diversification at individual and system levels. In contrast with previous studies, we find that even in the absence of transaction costs, the optimal level of diversification

is interior both for the individuals and the system. The mechanism behind this result functions as follows. The mitigation of idiosyncratic risks reduces the volatility of the portfolio of assets in the balance sheet. A lower level of volatility reduces the likelihood of the portfolio return to deviate from the underlying economic trend. Therefore, if the assets in portfolio are trending upward, then increasing diversification is a good strategy that reduces the default probability and increases the expected utility. Conversely, if the assets are in a downtrend, then increasing diversification is a poor strategy that increases the default probability and reduces the expected utility.

Notice that, to better isolate the effectiveness of diversification in mitigating idiosyncratic risks to which banks are exposed via external assets holding, in our setting the interbank diversification is fixed and homogeneous. However, one could introduce some heterogeneity in the balance sheet structure and in the portfolio holdings of the banks and answer the question whether external diversification and interbank diversification are substitutes or complements.

Overall, an important point stemming from our analysis lies in the recognition that the objective of a regulator is not to target a specific diversification level of risk but rather to manage the trade-off between the social losses from defaults (because of excessive risk spreading in economic downturn) and the social costs of avoiding defaults (because of excessive risk diversification in economic booms).

## Appendix A. Proofs

*Proof. Implicit Recovery Rate* The recovery rate is the proportion of face value that is recovered through bankruptcy procedures in the event of a default. Therefore, the general formula for the discounted recovery rate  $\hat{\delta}$  is

$$\begin{aligned}\hat{\delta} &= \frac{l_i}{(1+r_f+\beta\phi_i)} \left(\frac{1}{l_i}\right) \\ &= \frac{1}{(1+r_f+\beta\phi_i)}.\end{aligned}\tag{A.1}$$

That is

$$\hat{\delta} = \begin{cases} \frac{1}{1+r_f} & \text{in case of no default} \\ \frac{1}{1+r_f+\beta} & \text{in case of default} \end{cases}$$

The actual recovery rate at the maturity of the debt is  $\delta = \hat{\delta}(1 + r_f)$ . That is

$$\delta = \begin{cases} 1 & \text{in case of no default} \\ \frac{1+r_f}{1+r_f+\beta} & \text{in case of default} \end{cases}$$

□

*Proof. Quadratic leverage* The leverage at banking system level can be derived from Eq. (10) rewritten as:

$$l_i = \phi_i \times \left( \sum_j z_{ij} s_j + \sum_k w_{ik} l_k / (1 + r_f + \beta \phi_k) \right). \quad (\text{A.2})$$

In vector notation, Eq.(A.2) is equivalent to

$$\mathbf{L} = \Phi \times [\mathbf{ZV} + (\mathbf{R} + \beta\Phi)^{-1}\mathbf{WL}]$$

that we explicit for  $\Phi$ :

$$\begin{aligned} \mathbf{L} &= \Phi\mathbf{ZV} + \Phi(\mathbf{R} + \beta\Phi)^{-1}\mathbf{WL} \\ \mathbf{L}(\mathbf{WL})^{-1} &= \Phi\mathbf{ZV}(\mathbf{WL})^{-1} + \Phi(\mathbf{R} + \beta\Phi)^{-1}\mathbf{WL}(\mathbf{WL})^{-1} \\ \mathbf{W}^{-1} &= \Phi\mathbf{ZV}(\mathbf{WL})^{-1} + \Phi(\mathbf{R} + \beta\Phi)^{-1} \\ \mathbf{W}^{-1}(\mathbf{R} + \beta\Phi) &= \Phi\mathbf{ZV}(\mathbf{WL})^{-1}(\mathbf{R} + \beta\Phi) + \Phi(\mathbf{R} + \beta\Phi)^{-1}(\mathbf{R} + \beta\Phi) \\ \mathbf{W}^{-1}(\mathbf{R} + \beta\Phi) &= \Phi\mathbf{ZV}(\mathbf{WL})^{-1}(\mathbf{R} + \beta\Phi) + \Phi \\ \mathbf{W}^{-1}\mathbf{R} + \mathbf{W}^{-1}\beta\Phi &= \Phi\mathbf{ZV}\mathbf{L}^{-1}\mathbf{W}^{-1}\mathbf{R} + \Phi\mathbf{ZV}\mathbf{L}^{-1}\mathbf{W}^{-1}\beta\Phi + \Phi \\ \Phi\mathbf{H}\beta\Phi + \Phi\mathbf{H}\mathbf{R} - \mathbf{W}^{-1}\beta\Phi + \Phi &= \mathbf{W}^{-1}\mathbf{R} \end{aligned}$$

where  $\mathbf{H} := \mathbf{ZV}(\mathbf{WL})^{-1}$ . □

*Proof. Systemic Default Probability* In compact form, Eq. (19) reads as  $\phi = f(s, l, r_f, \beta)$ . The dynamics of  $\phi$  depends directly on the dynamics of  $s$  in Eq. (15) because both  $r_f$  and  $l$  are real const. and  $\beta$  is a coefficient. Therefore, to derive the systemic default probability  $\mathbb{P}(\phi \geq 1)$  in a close form, one could find, via Ito's Lemma, the dynamics of  $\phi$  from the dynamics of  $s$  and observe whether  $\phi(0) \in (\varepsilon, 1)$  exits after time  $t \geq 0$  through the *upper default boundary fixed at one*. However,  $f$  is highly non linear in  $s$ , see Eq. (19). Thus, it is convenient to derive the systemic default probability directly from the dynamics of  $s$  by mapping the sample space of  $\phi$  into the sample space of  $s$ . Since the partial derivative of  $f$  w.r.t.  $s$ , i.e,  $\frac{\partial f}{\partial s} = -l / (2s\sqrt{lRs})$ , is negative for all  $s$  and for any value of  $r_f, l$  in

their range of variation, the Inverse Function Theorem implies that  $f$  is invertible on  $\mathbb{R}^+$ :

$$f^{-1}(\phi) = \frac{l(2\phi + R)}{\phi(\phi + R)}, \quad \text{s.t.} \quad (f^{-1})'(\phi) = \frac{1}{f'(s)}.$$

Observe that, by definition, the value of external assets cannot be negative,  $s \in \mathbb{R}^+$ . The partial derivative  $f'(s)$  w.r.t.  $s$  is negative for all  $s$  and for any value of  $(l, r_f, \beta)$  in their range of variation:

$$f'(s) = \frac{-l \left( (\beta - 1)^2 l + (1 + \beta) R s + (-1 + \beta) \sqrt{(-1 + \beta)^2 l^2 + 2(1 + \beta) p R s + R^2 s^2} \right)}{2\beta s^2 \sqrt{4\beta l R s (l - \beta l + R s)^2}} < 0$$

Then, by the Inverse Function Theorem  $f$  is invertible on  $\mathbb{R}^+$

$$f^{-1}(\phi) = \frac{l(\phi + \beta\phi + R)}{\phi(\beta\phi + R)}, \quad \text{s.t.} \quad (f^{-1})'(\phi) = \frac{1}{f'(y)}.$$

Moreover, one can show that the inverse  $f^{-1}$  is continuous. Given the above result, we obtain the following mapping between the values of  $\phi$  and  $s$ :

$$\begin{cases} f(s) = 1 & \text{iff } f^{-1}(\phi) = \frac{l(r_f + \beta)}{(R + \beta)} := s^- \\ f(s) = \varepsilon & \text{iff } f^{-1}(\phi) = \frac{l(\beta\varepsilon - \varepsilon + 1)}{(\beta\varepsilon^2 + \varepsilon)} := s^+ . \end{cases}$$

Hence, the systemic default probability can be defined also w.r.t.  $s$ :

$$\mathbb{P}(\text{default}) = \mathbb{P}(\phi \geq 1) \equiv \mathbb{P}(s \leq s^-).$$

This is the probability that  $s$ , initially at an arbitrary level  $s(0) := s_0 \in (s^-, s^+)$ , exits through the *lower default boundary*  $s^-$  after time  $t \geq 0$ . From Gardiner (1985),  $\mathbb{P}(s \leq s^-)$  has the following explicit form:

$$\mathbb{P}(s \leq s^-) = \left( \int_{s_0}^{s^+} ds \psi(x) \right) / \left( \int_{s^-}^{s^+} dx \psi(x) \right). \quad (\text{A.3})$$

with  $\psi(x) = \exp\left(\int_0^x -\frac{2\hat{\mu}}{\hat{\sigma}^2} ds\right)$ . Eq. (A.3) has the following closed form solution:

$$\mathbb{P}(s \leq s^-) = \left( \exp\left[-\frac{2\hat{\mu}s_0}{\hat{\sigma}^2}\right] - \exp\left[-\frac{2\hat{\mu}s^+}{\hat{\sigma}^2}\right] \right) / \left( \exp\left[-\frac{2\hat{\mu}s^-}{\hat{\sigma}^2}\right] - \exp\left[-\frac{2\hat{\mu}s^+}{\hat{\sigma}^2}\right] \right), \quad (\text{A.4})$$

with  $\hat{\mu} = p\mu^+ + (1-p)\mu^-$  and  $\hat{\sigma}^2 = p\left[(\mu^+ - \hat{\mu})^2 + \frac{\sigma^2}{m}\right] + (1-p)\left[(\mu^- - \hat{\mu})^2 + \frac{\sigma^2}{m}\right]$ . During a downtrend, Eq. (A.3) yields the conditional default probability given a downtrend

$$q := \mathbb{P}(\text{default} \mid \mu^-) = \mathbb{P}(s \leq s^- \mid \mu^-)$$

with the following closed form solution:

$$q = \left( \exp \left[ -\frac{(2\mu^-)s_0}{\sigma^2/m} \right] - \exp \left[ -\frac{(2\mu^-)s^+}{\sigma^2/m} \right] \right) / \left( \exp \left[ -\frac{(2\mu^-)s^-}{\sigma^2/m} \right] - \exp \left[ -\frac{(2\mu^-)s^+}{\sigma^2/m} \right] \right). \quad (\text{A.5})$$

During an uptrend, Eq. (A.3) yields the conditional default probability given an uptrend:

$$g := \mathbb{P}(\text{default} \mid \mu^+) = \mathbb{P}(s \leq s^- \mid \mu^+)$$

with the following closed form solution:

$$g = \left( \exp \left[ -\frac{(2\mu^+)s_0}{\sigma^2/m} \right] - \exp \left[ -\frac{(2\mu^+)s^+}{\sigma^2/m} \right] \right) / \left( \exp \left[ -\frac{(2\mu^+)s^-}{\sigma^2/m} \right] - \exp \left[ -\frac{(2\mu^+)s^+}{\sigma^2/m} \right] \right). \quad (\text{A.6})$$

□

*Proof. Bifurcation of the conditional default probability*

We provide an asymptotic analysis that explains the results presented in Proposition 1. Let rewrite Eq. (A.5) and Eq. (A.6) as

$$\begin{aligned} q &= \frac{M_{(-)}^{s_0} - M_{(-)}^{s^+}}{M_{(-)}^{s^-} - M_{(-)}^{s^+}} \\ &= M_{(-)}^{(s_0-s^+)} \left[ M_{(-)}^{(s^- - s^+)} - 1 \right] - \frac{1}{M_{(-)}^{(s^- - s^+)} - 1} \end{aligned} \quad (\text{A.7a})$$

$$\begin{aligned} g &= \frac{M_{(+)}^{s_0} - M_{(+)}^{s^+}}{M_{(+)}^{s^-} - M_{(+)}^{s^+}} \\ &= M_{(+)}^{(s_0-s^+)} \left[ M_{(+)}^{(s^- - s^+)} - 1 \right] - \frac{1}{M_{(+)}^{(s^- - s^+)} - 1} \end{aligned} \quad (\text{A.7b})$$

with  $M_{(-)} = \exp \left[ \frac{-(2\mu^-)m}{\sigma^2} \right]$  and  $M_{(+)} = \exp \left[ \frac{-(2\mu^+)m}{\sigma^2} \right]$ . Then, the following result is straightforward:

$$\begin{cases} \lim_{m \rightarrow +\infty} q = 0 - (-1) = 1 \\ \lim_{m \rightarrow +\infty} g = 0 - 0 = 0 \end{cases}$$

To conclude, for any arbitrary small number  $\epsilon \in (0, 1)$

$$\exists \bar{m} > 1 \mid (q - g) > 1 - \epsilon \quad \forall m > \bar{m}$$

□



*Proof.* Existence Proof of the Optimal Intermediate Diversification Level Proposition 1 shows that in a (arbitrage free) complete market of assets with a stochastic trend, even in the absence of transaction costs banks maximize their MV utility by selecting an intermediate level of diversification  $m^*$ . To prove it, let rewrite Eq. (A.5) and Eq. (A.6) as

$$q = \frac{\exp[-(2\mu^-)s_0m] - \exp[-(\mu^-)m]}{1 - \exp[-(\mu^-)m]} \quad (\text{A.8a})$$

$$g = \frac{\exp[-(2\mu^+)s_0m] - \exp[-(\mu^+)m]}{1 - \exp[-(\mu^+)m]} \quad (\text{A.8b})$$

where, we discard the trivial case  $\mu = 0$ , and w.l.g. we set  $\sigma^2 = 1$ . The partial derivatives of (A.8a) and (A.8b) w.r.t.  $m$  are positive and negative, respectively:

$$\frac{\partial q}{\partial m} = \frac{(\mu^-) \exp[m(1-2s_0)(\mu^-)] \left( \exp[(2\mu^-)s_0m] - 2s_0 \left( \exp[(\mu^-)m] - 1 \right) - 1 \right)}{\left( \exp[(\mu^-)m] - 1 \right)^2} > 0, \quad (\text{A.9a})$$

$$\frac{\partial g}{\partial m} = \frac{(\mu^+) \exp[m(1-2s_0)(\mu^+)] \left( \exp[(2\mu^+)s_0m] - 2s_0 \left( \exp[(\mu^+)m] - 1 \right) - 1 \right)}{\left( \exp[(\mu^+)m] - 1 \right)^2} < 0. \quad (\text{A.9b})$$

Now, we decompose the MV utility in Eq. (23) as follows:

$$\mathbb{E}U(\Pi_m)_{\mu^-} = p \left[ (q\pi^- + (1-q)\pi^+) - \frac{\lambda}{2} \left( q(\Delta_-^-)^2 + (1-q)(\Delta_-^+)^2 \right) \right], \quad (\text{A.10a})$$

$$\mathbb{E}U(\Pi_m)_{\mu^+} = (1-p) \left[ (g\pi^- + (1-g)\pi^+) - \frac{\lambda}{2} \left( g(\Delta_+^-)^2 + (1-g)(\Delta_+^+)^2 \right) \right] \quad (\text{A.10b})$$

$$\text{where } \begin{cases} \pi^+ := s^+ - s_0, \\ \pi^- := s^- - s_0, \end{cases}$$

$$\text{and } \begin{cases} \Delta_-^- := \pi^- - p[q\pi^- + (1-q)\pi^+], \\ \Delta_-^+ := \pi^+ - p[q\pi^- + (1-q)\pi^+], \\ \Delta_+^- := \pi^- - (1-p)[g\pi^- + (1-g)\pi^+], \\ \Delta_+^+ := \pi^+ - (1-p)[g\pi^- + (1-g)\pi^+]. \end{cases}$$

Let observe that Eq. (A.10a) is decreasing in  $m$ , while Eq. (A.10b) is increasing in  $m$ :

$$\frac{\partial \mathbb{E}U(\Pi_m)_{\mu^-}}{\partial m} = p \left( \frac{\partial q}{\partial m} \right) (\pi^- - \pi^+) < 0 \quad (\text{A.11a})$$

$$\frac{\partial \mathbb{E}U(\Pi_m)_{\mu^+}}{\partial m} = (1-p) \left( \frac{\partial g}{\partial m} \right) (\pi^- - \pi^+) > 0 \quad (\text{A.11b})$$

Eq. (23) can be interpreted as a linear combination of Eq. (A.10a) and Eq. (A.10b) that are weighted by  $p$  and  $(1-p)$ , respectively. Then, when the partial derivative in Eq. (A.11a) is equal to the partial derivative in Eq. (A.11b),  $\mathbb{E}U(\Pi_m)$  is maximized w.r.t.  $m$ . The condition to be verified is to find the probability  $p^*$  that makes the two equations to be equivalent:

$$\text{FOC: } p \left( \frac{\partial q}{\partial m} \right) (\pi^- - \pi^+) = (1-p) \left( \frac{\partial g}{\partial m} \right) (\pi^- - \pi^+).$$

The condition is satisfied for all

$$p^* = 1 / \left( 1 + \frac{\partial q}{\partial g} \right) \in \Omega_{p^*} \subset \Omega_P := [0, 1] \quad (\text{A.12})$$

with  $q = q(m^*)$ ,  $g = g(m^*)$ . In a more general form, Eq. (A.12) can be written as  $p^* = f[g(m^*); q(m^*)]$ . Since  $g$ ,  $q$  and  $f$  are all one-to-one, for the Inversion Function Theorem are invertible functions. Hence, for a fixed value of  $p^*$ , must exists an  $m^*$  such that:

$$m^* = [(g^{-1}; q^{-1}) \circ f^{-1}](p^*) \Rightarrow \exists \quad \mathbb{E}U(\Pi_{m^*}) \geq \mathbb{E}U(\Pi_m) \quad \forall m \geq m^*.$$

The economic growth condition implies  $p^* < \frac{1}{2}$ . From (A.12), this is equivalent to write:

$$\begin{aligned} \frac{1}{2} &> \frac{\frac{\partial g}{\partial m}}{\frac{\partial q}{\partial m} + \frac{\partial g}{\partial m}} \\ \frac{\partial g}{\partial m} &< \frac{1}{2} \left[ \frac{\partial q}{\partial m} + \frac{\partial g}{\partial m} \right] \\ \frac{\partial g}{\partial m} + \frac{\partial g}{\partial m} &< \frac{\partial q}{\partial m} + \frac{\partial g}{\partial m} \\ \frac{\partial g}{\partial m} &< \frac{\partial q}{\partial m} \end{aligned}$$

which is always true because from Eq. (A.9a)–(A.9b)  $\frac{\partial g}{\partial m} < 0$  and  $\frac{\partial q}{\partial m} > 0$ . To conclude, to each  $p^* \in \Omega_{p^*} \subset \Omega_P := [0, \frac{1}{2})$  corresponds an optimal level of diversification  $m^*$  in the open ball  $B\left(\frac{1+M}{2}, r\right) = \left\{ m^* \in \mathbb{R} \mid d\left(m^*, \frac{1+M}{2}\right) < r \right\}$  with

center  $\frac{1+M}{2}$  and radius  $r \in [0, \alpha]$  where  $\alpha = f(q, g)$ . Then,  $\mathbb{E}U(\Pi_m) \leq \mathbb{E}U(\Pi_{m^*})$ , for all  $m \notin B\left(\frac{1+M}{2}, r\right)$ .  $\square$

*Proof. Banks are Over-Diversified:  $m^* \geq m^r$ .* Following the same line of reasoning used in the previous proof, we decompose  $\mathbb{E}U_r(\Pi_m)$  as follows:

$$\mathbb{E}U_r(\Pi_m)_{\mu^-} = p \left[ (qk\pi^- + (1-q)\pi^+) - \frac{\lambda}{2} \left( q(k\Delta_-^-)^2 + (1-q)(\Delta_-^+)^2 \right) \right] \quad (\text{A.13a})$$

$$\mathbb{E}U_r(\Pi_m)_{\mu^+} = (1-p) \left[ (g\pi^- + (1-g)\pi^+) - \frac{\lambda}{2} \left( g(\Delta_+^-)^2 + (1-g)(\Delta_+^+)^2 \right) \right]. \quad (\text{A.13b})$$

Eq. (A.13b) is not affected by Assumption 1 in Section 4 and remains equivalent to (A.10b). Hence, the partial derivative w.r.t.  $m$  of Eq. (A.13b) is equal to Eq (A.11b):

$$\frac{\partial \mathbb{E}U_r(\Pi_m)_{\mu^+}}{\partial m} \equiv \frac{\partial \mathbb{E}U(\Pi_m)_{\mu^+}}{\partial m} = (1-p) \left( \frac{\partial g}{\partial m} \right) (\pi^- - \pi^+).$$

However, because of the factor  $k$ , the partial derivative w.r.t.  $m$  of Eq. (A.13a) is steeper than Eq (A.11a). It is easy to see that for any  $k > 1$ ,

$$\frac{\partial \mathbb{E}U_r(\Pi_m)_{\mu^-}}{\partial m} = p \left( \frac{\partial q}{\partial m} \right) (k\pi^- - \pi^+) < \frac{\partial \mathbb{E}U(\Pi_m)_{\mu^-}}{\partial m} = p \left( \frac{\partial q}{\partial m} \right) (\pi^- - \pi^+) < 0$$

The condition to be verified is to find the probability  $p^r$  that makes the two equations to be equivalent:

$$\text{FOC: } (1-p) \left( \frac{\partial g}{\partial m} \right) (\pi^- - \pi^+) = p \left( \frac{\partial q}{\partial m} \right) (k\pi^- - \pi^+).$$

Discarding the trivial solution  $\mu_s = 0$ , the condition is satisfied for all

$$p^r = 1 / \left( 1 + \left( \frac{\partial q}{\partial g} \right) v \right) \in \Omega_{P^r} \subset \Omega_P := [0, 1] \quad (\text{A.14})$$

with  $q = q(m^r)$ ,  $g = g(m^r)$  and  $v = \frac{(k\pi^- - \pi^+)}{(\pi^- - \pi^+)}$ .

In a more general form, (A.14) can be written as

$$p^r = f(v[g(m^*); q(m^*)]).$$

Since  $g$ ,  $q$ ,  $v$  and  $f$  are all one-to-one and hence invertible functions, for a fixed value of  $p^r$ , must exists an  $m^r$  such that

$$m^r = [(g^{-1}; q^{-1}) \circ v^{-1} \circ f^{-1}] (p^r).$$

Eq.(A.14) is a decreasing function w.r.t.  $v$  which is a constant function bigger than one because  $k > 1$ . Then,  $p^r < p^*$ . This implies that the diversification level  $m^r$  is lower than the level  $m^*$ :

$$m^r = [(g^{-1}; q^{-1}) \circ v^{-1} \circ f^{-1}] (p^r) < m^* = [(g^{-1}; q^{-1}) \circ f^{-1}] (p^*)$$

□

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