

ON NEAR EQUITABLE DOMINATION IN GRAPHS

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ABSTRACT

Let D be a dominating set of a graph G . Then D is called a near equitable dominating set of G if for every vertex $v \in V - D$, there exists a vertex $u \in D$ such that u is adjacent to v and $|od_D(u) - od_{V-D}(v)| \leq 1$. The minimum cardinality of such a near equitable dominating set is called the near equitable domination number of G and is denoted by $\gamma_{ne}(G)$. In this paper, we introduce the concept of near equitable domination in graphs. The minimal near equitable dominating sets are established. The relation between $\gamma_{ne}(G)$, $\gamma_e(G)$ and $\gamma(G)$ are obtained, bounds for $\gamma_{ne}(G)$ are found. Near equitable domatic partition in a graph G is also studied.

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1 INTRODUCTION

By a graph $G = (V, E)$ we mean a finite, undirected graph with neither loops nor multiple edges. The order and size of G are denoted by n and m , respectively. For graph theoretic terminology we refer to Chartrand and Lesnaik [2].

Let $G = (V, E)$ be a graph and let $v \in V$. The open neighborhood and the closed neighborhood of v are denoted by $N(v) = \{u \in V : uv \in E\}$ and $N[v] = N(v) \cup \{v\}$, respectively. If $D \subseteq V$ then $N(D) = \cup_{v \in D} N(v)$ and $N[D] = N(D) \cup D$.

A subset D of V is called a dominating set if $N[D] = V$. The minimum (maximum) cardinality of a minimal dominating set of G is called the domination number (upper domination number) of G and is denoted by $\gamma(G)$ ($\Gamma(G)$). A subset D of V is an independent set if no two vertices in D are adjacent. The minimum cardinality of an independent dominating set of G is called the independent domination number of G and is denoted by $i(G)$.

A dominating set D is called a perfect dominating set if $|N(v) \cap D| = 1$ for each $v \in V - D$. The perfect domination number $\gamma_p(G)$ is the minimum cardinality of a perfect dominating set of G . An excellent treatment of the fundamentals of domination is given in the book by Haynes et al. [4]. A survey of several advanced topics in domination is given in the book edited by Haynes et al. [5]. Various types of domination have been defined and studied by several authors and more than 75 models of domination are listed in the appendix of Haynes et al. [4].

A double star is the tree obtained from two disjoint stars $K_{1,n}$ and $K_{1,m}$ by connecting their centers.

For any graph G , an induced subgraph H isomorphic to $K_{1,3}$ is called a claw of G , and the only vertex of degree 3 of H is the center of the claw. A graph G is claw free if it does not contain a claw.

A subset D of V is called an equitable dominating set if for every $v \in V - D$, there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|d(u) - d(v)| \leq 1$. The minimum cardinality of such an equitable dominating set is denoted by $\gamma_e(G)$ and is called the equitable domination number of G . A vertex $u \in V$ is said to be degree equitable with a vertex $v \in V$ if $|d(u) - d(v)| \leq 1$. An equitable dominating set D is said to be a minimal equitable dominating set if no proper subset of D is an equitable dominating set.

Equitable domination has interesting application in the context of social networks. In a network, nodes with nearly equal capacity may interact with each other in a better way. In the society persons with nearly equal status, tend to be friendly.

In this paper, we introduce the concept of near equitable domination in graphs. The minimal near equitable

dominating sets are established. The relation between $\gamma_{ne}(G)$, $\gamma_e(G)$ and $\gamma(G)$ are obtained, bounds for $\gamma_{ne}(G)$ are found. Near equitable domatic partition in a graph G is also studied.

We need the following definition and theorem.

Theorem 1.1[4]. If G is a claw-free graph, then $\gamma(G) = i(G)$.

2 Near Equitable Domination Number of Graphs

Definition 2.1 Let $G = (V, E)$ be a graph, $D \subseteq V(G)$ and u be any vertex in D . The out degree of u with respect to D denoted by $od_D(u)$, is defined as $od_D(u) = |N(u) \cap (V - D)|$.

Definition 2.2 Let D be a dominating set of a graph G . Then D is called a near equitable dominating set of G if for every $v \in V - D$, there exists a vertex $u \in D$ such that u is adjacent to v and $|od_D(u) - od_{V-D}(v)| \leq 1$. The minimum cardinality of such a near equitable dominating set is called the near equitable domination number of G and is denoted by $\gamma_{ne}(G)$.

It is obvious that any near equitable dominating set in a graph G is also a dominating set, and thus we obtain the obvious bound $\gamma(G) \leq \gamma_{ne}(G)$. Furthermore, the difference $\gamma_{ne}(G) - \gamma(G)$ can be arbitrarily large in a graph G . It can be easily checked that $\gamma(K_{1,n}) = 1$, while $\gamma_{ne}(K_{1,n}) = n - 1$.

Observation 2.3 For any connected graph G of order n , $\gamma_{ne}(G) = \gamma_e(G) = \gamma(G) = 1$ if and only if $n \leq 3$.

Observation 2.4 For $G \cong nK_2 \cup mK_1$, $m \geq 1$, $\gamma_{ne}(G) = \gamma_e(G) = \gamma(G) = n + m$.

Definition 2.5 Let G be a graph and let D be a near equitable dominating set of G . The near equitable neighbourhood of $u \in D$, denoted by $N_D^{ne}(u)$ is defined as $N_D^{ne}(u) = \{v \in V - D : v \in N(u), |od_D(u) - od_{V-D}(v)| \leq 1\}$.

Definition 2.6 Let G be a graph and let D be a near equitable dominating set of G . The maximum and minimum near equitable degree of D are denoted by Δ_D^{ne} and δ_D^{ne} , respectively. That is $\Delta_D^{ne} = \max_{u \in D} |N_D^{ne}(u)|$ and $\delta_D^{ne} = \min_{u \in D} |N_D^{ne}(u)|$.

For example, let $G \cong mK_2$, $m \geq 1$, if D is a near equitable dominating set of G , then $\Delta(G) = \Delta_D^{ne} = \delta(G) = \delta_D^{ne} = 1$.

From the definition 2.6, we have the following propositions.

Proposition 2.7 If D is a near equitable dominating set of graph G , then, $\Delta_D^{ne} \leq \Delta(G)$.

Proposition 2.8 Let G be a graph containing isolated vertices. If D is a near equitable dominating set of G , then $\delta(G) = \delta_D^{ne}$.

Proposition 2.9 If D is a near equitable dominating set of a tree T , then $\delta(T) \leq \delta_D^{ne}$.

We now proceed to compute $\gamma_{ne}(G)$ for some standard graphs. It can be easily verified that

1. For a path P_n , $\gamma_{ne}(P_n) = \gamma_e(P_n) = \gamma(P_n) = \lceil \frac{n}{3} \rceil$.
2. For a cycle C_n , $\gamma_{ne}(C_n) = \gamma_e(C_n) = \gamma(C_n) = \lceil \frac{n}{3} \rceil$.
3. For a complete graph K_n , $\gamma_{ne}(K_n) = \lfloor \frac{n}{2} \rfloor$.
4. For a double star $S_{n,m}$,

$$\gamma_{ne}(S_{n,m}) = \begin{cases} 2, & \text{if } n, m \leq 2; \\ n + m - 2, & \text{if } n, m \geq 2 \text{ and } n \text{ or } m \geq 3. \end{cases}$$

Theorem 2.10 For the complete bipartite graph $G \cong K_{n,m}$ with $1 < m \leq n$, we have

$$\gamma_{ne}(K_{n,m}) = \begin{cases} m - 1, & \text{if } n = m \text{ and } m \geq 3; \\ m, & \text{if } n - m = 1 \text{ or } n, m \leq 2; \\ n - 1, & \text{if } n - m \geq 2. \end{cases}$$

Proof. Let $V_1 = \{u_1, u_2, \dots, u_n\}$ and $V_2 = \{v_1, v_2, \dots, v_m\}$ be the bipartition of $K_{n,m}$. We consider the following cases.

Case 1: $n = m \geq 3$. We consider the following subcases.

Subcase 1.1 : $n = m = 3$. Let $D = \{u_1, v_1\}$ be a minimum dominating set of $K_{n,m}$. Then, $|od_D(u_1) - od_{V_2-D}(v_1)| \leq 1$, for all $v_i \in V_2 - D$ and $|od_D(v_1) - od_{V_1-D}(u_i)| \leq 1$, for all $u_i \in V_1 - D$. Hence, D is a near equitable dominating set of $K_{n,m}$. Therefore, $\gamma_{ne}(K_{n,m}) \leq \gamma(K_{n,m})$. But we have, $\gamma(K_{n,m}) \leq \gamma_{ne}(K_{n,m})$. Hence, $\gamma(K_{n,m}) = \gamma_{ne}(K_{n,m})$. Thus, D is a minimum near equitable dominating set.

Subcase 1.2 : $n = m \geq 4$. We have the following subsubcases.

Subsubcase 1.2.1 : n and m are odd.

Consider a dominating set $D = \{u_1, u_2, \dots, u_{\lfloor \frac{n}{2} \rfloor}, v_1, v_2, \dots, v_{\lfloor \frac{m}{2} \rfloor}\}$ such that $|D| = m - 1$. Then $od_D(u_i) = \lceil \frac{m}{2} \rceil$, $od_D(v_i) = \lceil \frac{n}{2} \rceil$, $od_{V_1-D}(u_j) = \lfloor \frac{m}{2} \rfloor$ and $od_{V_2-D}(v_j) = \lfloor \frac{n}{2} \rfloor$. Since $n = m$, we have, $|od_D(u_i) - od_{V_2-D}(v_j)| \leq 1$, for all $v_j \in V_2 - D$ and $|od_D(v_i) - od_{V_1-D}(u_j)| \leq 1$, for all $u_j \in V_1 - D$. Therefore, D is a near equitable dominating set.

Now, if $D_1 = \{u_1, u_2, \dots, u_s, v_1, v_2, \dots, v_{n-s-2}\}$, $s < \lfloor \frac{n}{2} \rfloor$ is a near equitable dominating set of $K_{n,m}$, then $|od_{D_1}(u_i) - od_{V_2-D_1}(v_j)| = 2$, and $|od_{D_1}(v_i) - od_{V_1-D_1}(u_j)| = 2$, a contradiction. Therefore, D is a minimum near equitable dominating set.

Subsubcase 1.2.2 : n and m are even.

Consider a dominating set $D = \{u_1, u_2, \dots, u_{\frac{n}{2}}, v_1, v_2, \dots, v_{\frac{m-1}{2}}\}$ such that $|D| = m - 1$. Then $od_D(u_i) = \frac{m}{2} + 1$, $od_D(v_i) = \frac{n}{2}$, $od_{V_1-D}(u_j) = \frac{m}{2} - 1$ and $od_{V_2-D}(v_j) = \frac{n}{2}$. Since $n = m$, we have, $|od_D(u_i) - od_{V_2-D}(v_j)| \leq 1$, for all $v_j \in V_2 - D$ and $|od_D(v_i) - od_{V_1-D}(u_j)| \leq 1$, for all $u_j \in V_1 - D$. Therefore, D is a near equitable dominating set. Now,

if $D_1 = \{u_1, u_2, \dots, u_s, v_1, v_2, \dots, v_{n-s-2}\}$, $s < \lfloor \frac{n}{2} \rfloor$ is a near equitable dominating set of $K_{n,m}$, then $|od_{D_1}(u_i) - od_{V_2-D_1}(v_j)| = 2$, and $|od_{D_1}(v_i) - od_{V_1-D_1}(u_j)| = 2$, a contradiction. Therefore, D is a minimum near equitable dominating set.

Case 2 : $n \neq m$. We consider the following subcases.

Subcase 2.1 : $n - m = 1$.

Consider a dominating set $D = \{u_1, u_2, \dots, u_{\lfloor \frac{n}{2} \rfloor}, v_1, v_2, \dots, v_{\lfloor \frac{m}{2} \rfloor}\}$ such that $|D| = m$. Since $n = m + 1$, we have, $|od_D(u_i) - od_{V_2-D}(v_j)| \leq 1$, for all $v_j \in V_2 - D$ and $|od_D(v_i) - od_{V_1-D}(u_j)| \leq 1$, for all $u_j \in V_1 - D$. Therefore, D is a near equitable dominating set. Now, if n is odd and m even, we have, $|D| = m$. Consider a near equitable dominating set $D_1 = \{u_1, u_2, \dots, u_s, v_1, v_2, \dots, v_{n-s-2}\}$, $s < \lfloor \frac{n}{2} \rfloor$. Then, $|od_{D_1}(v_i) - od_{V_1-D_1}(u_j)| = 2$. Similarly, if m is odd and n even, $|od_{D_1}(v_i) - od_{V_1-D_1}(u_j)| = 2$, a contradiction. Therefore, D is a minimum near equitable dominating set.

Subcase 2.2 : $n - m \geq 2$.

Consider a dominating set $D = \{u_1, u_2, \dots, u_{n-m-1}, v_1, v_2, \dots, v_m\}$, $|D| = n - 1$. Then, $|od_D(u_i) - od_{V_2-D}(v_j)| = 0$, for all $v_j \in V_2 - D$ and $|od_D(v_i) - od_{V_1-D}(u_j)| = 1$, for all $u_j \in V_1 - D$. Therefore, D is a near equitable dominating set.

Now, if $D_1 = D - \{u_{n-m-1}\}$ or $D - \{v_m\}$ is a near equitable dominating set, then $D_1 = \{u_1, u_2, \dots, u_{n-m-2}, v_1, v_2, \dots, v_m\}$ or $D_1 = \{u_1, u_2, \dots, u_{n-m-1}, v_1, v_2, \dots, v_{m-1}\}$. Therefore, $|od_{D_1}(v_i) - od_{V_1-D_1}(u_j)| = 2$, a contradiction. Thus, D is a minimum near equitable dominating set.

Theorem 2.11 For the wheel $W_{1,n}$, $n \geq 5$,

$$\gamma_{ne}(W_{1,n}) = \lceil \frac{n}{3} \rceil + 1$$

Proof. Let $V(W_{1,n}) = \{u, v_1, v_2, \dots, v_n\}$, where u is the center vertex and $v_i, 1 \leq i \leq n$ is on the cycle. We know that $\gamma_{ne}(C_n) = \lceil \frac{n}{3} \rceil$. Consider a minimum near equitable dominating set $D = \{v_1, v_2, \dots, v_{\lceil \frac{n}{3} \rceil}\}$ of C_n , then $od_D(v_i) = 3$, for

all $i, 1 \leq i < n, od_{V-D}(v_j) = 1, j \neq i, 1 \leq j < n$ and $od_{V-D}(u) = \lceil \frac{n}{3} \rceil$. Therefore, $|od_D(v_i) - od_D(v_j)| = 2$. and

$|od_{V-D}(u) - od_D(v_i)| \geq 2$. Hence, D is not near equitable dominating set. So, $u \in D$. Thus, $\gamma_{ne}(W_{1,n}) = \lceil \frac{n}{3} \rceil + 1$.

Theorem 2.12 Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs such that $|V_1| = n$ and $|V_2| = m, n \leq m, m - n \leq 1$. Then $\gamma_{ne}(G_1 + G_2) = n$.

Proof. Let $G = G_1 + G_2$. For any $u \in V_1$ and $v \in V_2, u$ and v are adjacent. Since $m - n \leq 1$, it follows that $|od_{V_1}(u) - od_{V_2}(v)| \leq 1$ in G . Since $n \leq m, V_1$ is a minimum near equitable dominating set of G . Thus, $\gamma_{ne}(G) = n$.

Theorem 2.13 Let G be a graph with any minimum perfect dominating set D having the following property, for every $u \in D, od_D(u) \leq 2$. Then $\gamma_{ne}(G) \leq \gamma_p(G)$.

Proof. Let G be a graph and a subset D of G be a minimum perfect dominating set, then for every $v \in V - D, v$ is dominated by exactly one vertex of D . Since for every $u \in D, od_D(u) \leq 2$, we have $|od_D(u) - od_{V-D}(v)| \leq 1$. Hence

D is a near equitable dominating set. Since $\gamma_{ne}(G) \leq |D|$. Thus $\gamma_{ne}(G) \leq \gamma_p(G)$.

3 Graphs with $\gamma_{ne}(G) = \gamma(G)$ or $\gamma_{ne}(G) = \gamma_e(G)$.

In this section, we present several families of graphs for which $\gamma_{ne}(G)$ and $\gamma_e(G)$ or $\gamma_{ne}(G)$ and $\gamma(G)$ are equal. Now, we define the near equitable pendant vertex as follows.

Definition 3.1. Let $G = (V, E)$ be a graph and let D be a near equitable dominating set of G . Then $u \in D$ is a near equitable pendant vertex if $od_D(u) = 1$. A set D is called a near equitable pendant dominating set if every vertex in D is an equitable pendant vertex.

Proposition 3.2 Let G be a graph and D be a near equitable dominating set of G such that $u \in D$. If u support a pendant vertex v and $od_D(u) \geq 3$. Then $v \in D$.

Theorem 3.3 Let $G = (V, E)$ be a graph and let D be a near equitable pendant set. Then $V - D$ is minimum near equitable dominating set of G .

Proof. Suppose that D is near equitable pendant set, then for every $u \in D, od_D(u) = 1$ and $od_{V-D}(v) \leq 2$ for every $v \in V - D$. Therefore for any $u \in D$ there exists $v \in V - D$ such that v is adjacent to u and $|od_D(u) - od_{V-D}(v)| \leq 1$, that $V - D$ is near equitable dominating set. Since for every $u \in D, od_D(u) = 1$ and $od_{V-D}(v) \leq 2$ for every $v \in V - D$, then $|V - D| \leq |D|$. So, $V - D$ is a minimum near equitable dominating set of G .

Theorem 3.4 Let T be a wounded spider obtained from the star $K_{1,n-1}, n \geq 5$ by subdividing m edges exactly once. Then

$$\gamma_{ne}(T) = \begin{cases} n, & \text{if } m = n - 1; \\ n - 1, & \text{if } m = n - 2; \\ n - 2, & \text{if } m \leq n - 3. \end{cases}$$

Proof. Let $K_{1,n-1}$ be a star with central vertex u . Then $V(K_{1,n-1}) = \{u, u_1, u_2, \dots, u_{n-1}\}$ with $deg(u) = n - 1$. Let v_i be the vertex subdividing the edge uu_i . Then we consider the following cases.

Case 1: $m = n - 1. D = \{u, v_1, v_2, \dots, v_{m-1}, v_{n-1}\}$ is a near equitable dominating set and hence $\gamma_{ne}(T) \leq |D| = n$. But $\gamma(T) = n - 1$ and $\gamma(G) \leq \gamma_{ne}(G)$, it follows that $n - 1 \leq \gamma_{ne}(T) \leq n$. Now, if $\gamma_{ne}(T) = n - 1$, then consider a near equitable dominating set, $D = \{u, v_1, v_2, \dots, v_r, u_1, u_2, \dots, u_s\}$ such that $r + s + 1 = n - 1$. We consider the following subcases.

Subcase 1.1 : $r = 0$ (or $s = 0$), then $s = n - 2$ (or $r = n - 2$), so that there exists a vertex u_i which is not dominated by any vertex of D , a contradiction.

Subcase 1.2 : $u \notin D$, then $D = \{v_1, v_2, \dots, v_r, u_1, u_2, \dots, u_s\}$ such that $r + s = n - 1$. Since $n \geq 5$, $od_{V-D}(u) \geq 4$ and for any $v \in D$, $od_D(v) \leq 2$, so that $|od_D(v) - od_{V-D}(u)| \geq 2$, a contradiction. So, $|D| = n$. Hence $\gamma_{ne}(T) = n$.

Case 2: $m = n - 2$. Since $n \geq 5$, then $D = \{u, v_1, v_2, \dots, v_{m-1}, v_{n-2}\}$ is a near equitable dominating set and hence $\gamma_{ne}(T) \leq |D| = n - 1$. Since $\gamma(T) = n - 1$, we have $\gamma_{ne} \geq n - 1$ and hence $\gamma_{ne}(T) = n - 1$.

Case 3: $m \leq n - 3$. Since $n \geq 5$, then $D = \{u, v_1, v_2, \dots, v_{m-1}, v_{n-3}\}$ is a near equitable dominating set and hence $\gamma_{ne}(T) \leq |D| = n - 2$. Since $\gamma(T) = n - 2$, we have $\gamma_{ne}(T) \geq n - 2$ and hence $\gamma_{ne}(T) = n - 2$.

Corollary 3.5 Let T be a wounded spider obtained from the star $K_{1,n-1}$, $n \geq 5$ by subdividing m edges exactly once. Then $\gamma_{ne}(T) = \gamma_e(T) = n$ if and only if $m = n - 1$

Proof. Proof follows from Theorem 3.4.

Theorem 3.6 Let G be a connected claw free graph and D be a minimum dominating set of G . If $deg(u) \leq 2$, for any $u \in D$, then $\gamma_{ne}(G) = \gamma(G)$.

Proof. Let D be a maximal independent set of minimum cardinality. It follows from Theorem 1.1 that $\gamma(G) = |D|$. Now since $od_D(u) \leq 2$, for any $u \in D$, and G is claw free, then every vertex u in D has at most two neighbors in $V - D$ and every vertex v in $V - D$ has either one or two neighbors in D . Therefore for every $v \in V - D$, $|od_D(u) - od_{V-D}(v)| \leq 1$. Hence D is near equitable dominating set of G . Since $\gamma_{ne}(G) \leq |D| = \gamma(G)$ and $\gamma(G) \leq \gamma_{ne}(G)$, $\gamma_{ne}(G) = \gamma(G)$.

Remark 3.7 Let $G \cong mK_2$, $m \geq 1$. Then $\gamma_{ne}(G) = \gamma_e(G) = \gamma(G) = m$.

Theorem 3.8 Let D be a minimum dominating set of a graph G . If D is a perfect dominating set such that for any $u \in D$, $od_D(u) \leq 2$, then $\gamma_{ne}(G) = \gamma(G)$.

Proof. Suppose D is a perfect dominating set a graph G . Then every vertex of $V - D$ has one neighbor in D . Since for any $u \in D$, $od_D(u) \leq 2$, it follows that for every $v \in V - D$, we have $|od_D(u) - od_{V-D}(v)| \leq 1$. Hence D is a near dominating set of G . Since $\gamma_{ne}(G) \leq |D| = \gamma(G)$ and $\gamma(G) \leq \gamma_{ne}(G)$, it follows that $\gamma_{ne}(G) = \gamma(G)$.

Theorem 3.9 Let T be a tree in which every non-pendant vertex is either a support or adjacent to a support and every non-pendant vertex which is non support is adjacent to at most three support and every support is adjacent to at most one non-support and one pendant vertex. Then $\gamma_{ne}(T) = \gamma(T)$.

Proof. Let D denote set of all supports of T . Clearly, D is a γ -set. Since by hypothesis, the out degree of any support vertex is at most two and the out degree of any non support vertex is at least one and at most three, then D is a γ_{ne} -set. So $\gamma_{ne}(G) \leq \gamma(G)$. But, $\gamma(G) \leq \gamma_{ne}(G)$. Hence $\gamma_{ne}(T) = \gamma(T)$.

Theorem 3.10 For any positive integer m , there exists a graph G such that $\gamma_{ne}(G) - \lfloor \frac{n}{\Delta+1} \rfloor = m$, where $\lfloor x \rfloor$ denotes the greatest integer not exceeding x .

Proof. For $m = 1$, take $G = K_{2,2}$, $\gamma_{ne}(G) - \lfloor \frac{n}{\Delta+1} \rfloor = 2 - 1 = 1$.

For $m = 2$, take $G = K_{1,4}$, $\gamma_{ne}(G) - \lfloor \frac{n}{\Delta+1} \rfloor = 3 - 1 = 2$.

For $m \geq 3$, take $G = S_{r,s}$, where $r + s = m + 3$, $s \geq r + 3$ and $r \geq 2$

$$\gamma_{ne}(G) = r + s - 2 = m + 1,$$

$$\lfloor \frac{n}{\Delta+1} \rfloor = \lfloor \frac{r+s+2}{s+2} \rfloor = 1,$$

$$\gamma_{ne}(G) - \lfloor \frac{n}{\Delta+1} \rfloor = r + s - 3 = m.$$

4 Minimal Near Equitable Dominating Sets

Definition 4.1 A near equitable dominating set D is said to be a minimal near equitable dominating set if no proper subset of D is near equitable dominating set.

Theorem 4.2 Let D be a near equitable dominating set of a graph G . Then for any $v \in D$, D is minimal near equitable dominating set of G if and only if one of the following axioms holds.

(i) D is minimal dominating set.

(ii) There exists a vertex $y \in V - D$ such that for each $x \in N(y)$ in D , $od_{V-D}(y) \leq od_D(x)$, and for any $z \in N(y)$, $od_D(x) < od_D(z)$, the set U_v is nonempty, where $U_v = \{x \in N(y) : od_D(x) - od_{V-D}(y) = 0, \text{ and } v \in N(x) \cap N(y) \text{ or } od_D(x) - od_{V-D}(y) = 1, \text{ and } v \in N(x) \text{ or } v \in N(y)\}$.

Proof. Suppose that D is a minimal near equitable dominating set of G . Then for any $v \in D$, $D - \{v\}$ is not near equitable dominating set. Since D is a near equitable dominating set, D is a dominating set. If D is a minimal dominating set, then we are done. If not, then for any $v \in D$, let $U_v = \{x \in N(y) : od_D(x) - od_{V-D}(y) = 0, \text{ and } v \in N(x) \cap N(y) \text{ or } od_D(x) - od_{V-D}(y) = 1, \text{ and } v \in N(x) \text{ or } v \in N(y)\}$. Since D is a minimal near equitable dominating set, it follows that there exists $y \in V - (D - \{v\})$ such that for any $x \in D - \{v\}$, $|od_{D-\{v\}}(x) - od_{V-(D-\{v\})}(y)| > 1$. If v is not adjacent to both x and y , then $|od_{D-\{v\}}(x) - od_{V-(D-\{v\})}(y)| = |od_D(x) - od_{V-D}(y)| \leq 1$, a contradiction. If v is adjacent to x , then using triangle inequality, we obtain,

$$1 < |od_{D-\{v\}}(x) - od_{V-(D-\{v\})}(y)| = |od_D(x) + 1 - od_{V-D}(y)| \leq |od_D(x) - od_{V-D}(y)| + 1$$

So, $|od_D(x) - od_{V-D}(y)| > 0$.

But, $|od_D(x) - od_{V-D}(y)| \leq 1$.

Hence, $|od_D(x) - od_{V-D}(y)| = 1$.

If $od_{V-D}(y) > od_D(x)$, then $|od_{D-\{v\}}(x) - od_{V-(D-\{v\})}(y)| \leq 1$, a contradiction. Therefore, $od_{V-D}(y) \leq od_D(x)$, and for any $z \in N(y)$, $od_D(x) < od_D(z)$. Hence, $od_D(x) - od_{V-D}(y) = 1$. Similarly, if v is adjacent to y , we have $od_D(x) - od_{V-D}(y) = 1$. Now, if v is adjacent to both x and y , then using triangle inequality, we obtain,

$$1 < |od_{D-\{v\}}(x) - od_{V-(D-\{v\})}(y)| = |od_D(x) + 1 - od_{V-D}(y) - 1| \leq |od_D(x) - od_{V-D}(y)| + 2$$

So, $|od_D(x) - od_{V-D}(y)| > -1$.

But, $|od_D(x) - od_{V-D}(y)| \leq 1$.

So, $|od_D(x) - od_{V-D}(y)| = 0$ or 1 .

Since $od_{V-D}(y) \leq od_D(x)$, it follows that, $od_D(x) - od_{V-D}(y) = 0$ or 1 . Hence U_v is not empty.

Conversely, let D be a near equitable dominating set and suppose that D is a minimal dominating set. Suppose to the contrary D is not minimal near equitable dominating set. Then there exists $v \in D$ such that $D - \{v\}$ is a near equitable dominating set. So, D is not minimal dominating set, a contradiction. Next, suppose that D is a near equitable dominating set and (ii) holds. Then for every $v \in D$, U_v is not empty. So, for every $v \in D$, there exist two adjacent vertices $x \in D$ and $y \in V - D$ such that $od_D(x) - od_{(V-D)}(y) = 0$ and $v \in N(x) \cap N(y)$, or $od_D(x) - od_{(V-D)}(y) = 1$ and $v \in N(x)$ or $v \in N(y)$.

Suppose to the contrary D is not minimal near equitable dominating set. Then there exists $v \in D$ such that $D - \{v\}$ is a near equitable dominating set. So,

$$|od_{D-\{v\}}(x) - od_{V-(D-\{v\})}(y)| \leq 1$$

If v is adjacent to x , then using triangle inequality, we obtain

$$1 \geq |od_{D-\{v\}}(x) - od_{V-(D-\{v\})}(y)| = |od_D(x) + 1 - od_{V-D}(y)| \leq |od_D(x) - od_{(V-D)}(y)| + 1 = 2$$

Similarly, if v is adjacent to y , then using triangle inequality, we obtain

$$1 \geq |od_{D-\{v\}}(x) - od_{V-(D-\{v\})}(y)| = |od_D(x) - od_{V-D}(y) - 1| \leq |od_D(x) - od_{(V-D)}(y)| + 1 = 2$$

New, if v is adjacent to both x and y , then using triangle inequality, we obtain

$$\begin{aligned} 1 &\geq |od_{D-\{v\}}(x) - od_{V-(D-\{v\})}(y)| = |od_D(x) + 1 - od_{V-D}(y) - 1| \\ &\leq |od_D(x) - od_{V-D}(y)| + 2 = 3 \end{aligned}$$

Therefore, if $|od_{D-\{v\}}(x) - od_{V-(D-\{v\})}(y)| = 2$ or 3 , then we have a contradiction to the fact that

$D - \{v\}$ is a near equitable dominating set. If $|od_{D-\{v\}}(x) - od_{V-(D-\{v\})}(y)| \leq 1$, then

$|od_{D-\{v\}}(x) - od_{V-(D-\{v\})}(y)| = |od_D(x) - od_{V-D}(y)|$, and v is not adjacent to x or y , a contradiction.

5 Bounds

In this section, we present sharp bounds for $\gamma_{ne}(G)$.

Theorem 5.1 Let G be a connected graph of order n , $n \geq 3$. Then $\gamma_{ne}(G) \leq n - 2$.

Proof. It is enough to show that for any minimum connected near equitable dominating set D of G , $|V - D| \geq 2$. Since G is a connected graph, it follows that $\delta(G) \geq 1$. Suppose $v \in V - D$ and is adjacent to $u \in D$. Since $od_{V-D}(v) \geq 1$, then $od_D(u) \geq 2$.

The bound is sharp for $K_{1,n}$, $n \geq 2$.

Theorem 5.2 Let G be a graph of order n and D be a dominating set of G . If $V - D$ is near equitable pendant dominating set. Then $\gamma_{ne}(G) \leq \frac{n}{2}$.

Proof. Let D be a dominating set of G . Suppose $V - D$ is a near equitable pendant dominating set, then by Theorem 3.3, D is a minimum near equitable dominating set. Therefore,

$$\begin{aligned} \gamma_{ne}(G) &\leq |V - D| \\ &= n - |D| \\ &= n - \gamma_{ne}(G) \\ &= \frac{n}{2} \end{aligned}$$

Proposition 5.3 Let T be a tree of order n with r support vertices such that for any support vertex u , $deg(u) \leq 2$, then $\gamma_{ne}(T) \leq n - r$.

Proof. Let D be the set of all support vertices. Then $|D| = r$. Since every vertex of D has at most two neighbors in $V - D$ one of them pendant vertex, $V - D$ is near equitable dominating set of T . Hence $\gamma_{ne}(T) \leq |V - D| = n - |D| = n - r$.

6 Near Equitable Domatic Partition.

The maximum order of a partition of the vertex set V of a graph G into dominating sets is called the domatic number of G and is denoted by $d(G)$. For a survey of results on domatic number and their variants we refer to Zelinka [8]. In this section we present a few basic results on the near equitable domatic number of a graph.

Definition 6.1. A near equitable domatic partition of G is a partition $\{V_1, V_2, \dots, V_k\}$ of $V(G)$ in which each V_i is a near equitable dominating set of G . The maximum order of a near equitable domatic partition of G is called the near equitable domatic number of G and is denoted by $d_{ne}(G)$.

We now proceed to compute $d_{ne}(G)$ for some standard graphs. It can be easily verified that

1. For any complete graph K_n , $n \geq 4$, $d_{ne}(K_n) = 2$.
2. For any cycle C_n , $n \geq 4$, $d_{ne}(C_n) = 2$.
3. For any path P_n , $d_{ne}(P_n) = 2$.
4. For any star $K_{1,n}$, $n \geq 3$, $d_{ne}(K_{1,n}) = 1$.
5. If $W_{1,n}$ denotes the wheel on n vertices, then $d_{ne}(W_{1,n}) = 1$.
6. For the complete bipartite graph $G \cong K_{n,m}$ with $m \leq n$, we have

$$d_{ne}(K_{n,m}) = \begin{cases} 2, & \text{if } n - m \leq 2; \\ 1, & \text{if } n - m \geq 3, n, m \geq 2. \end{cases}$$

Theorem 6.2 For any graph G , $d_{ne}(G) \leq d(G)$.

Proof. Let $G = (V, E)$ be a graph. Since it is clear that any partition of V into near equitable dominating set is also partition of V into dominating set, then $d_{ne}(G) \leq d(G)$.

Theorem 6.3 For any graph G , $d_{ne}(G) \leq \delta(G) + 1$.

Proof. Let D be any near equitable dominating set. Then for any $v \in V(G)$, $D \cap N[v] \neq \emptyset$. Let $v \in V(G)$ such that $deg(v) = \delta(G)$ and $N[v] = \{v, u_1, u_2, \dots, u_\delta\}$. If $d_{ne}(G) > \delta(G) + 1$, then there exist at least $(\delta(G) + 2)$ sets of the near equitable domatic partition of G , each containing at least one element of $N[v]$. so that $deg(v) \geq \delta(G) + 1$, a contradiction. Hence $d_{ne}(G) \leq \delta(G) + 1$.

Theorem 6.4 For any graph G of order n , $d_{ne}(G) \leq \frac{n}{\gamma_{ne}(G)}$.

Proof. Suppose that $d_{ne}(G) = t$, for some positive integer t . Let $P = \{D_1, D_2, \dots, D_t\}$ be the near equitable domatic partition of G . Obviously, $|V(G)| = \sum_{i=1}^t |D_i|$ and from definition of near equitable domination number $\gamma_{ne}(G)$, we have $\gamma_{ne}(G) \leq |D_i|$, $i = 1, 2, \dots, t$. Hence $n = \sum_{i=1}^t |D_i| \geq t\gamma_{ne}(G)$. Thus $d_{ne}(G) \leq \frac{n}{\gamma_{ne}(G)}$.

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