

THEORY OF THE NEAREST SQUARE CONTINUED FRACTION*

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4. Special Critical Fractions

4.1. IN §2 of our previous communication† we have called the surds (i) $\frac{1}{2} + \frac{\sqrt{R}}{Q} - \frac{\sqrt{4R - Q^2}}{2Q}$ ($|Q| < 2\sqrt{R}$), and (ii) $\frac{1}{2} + \frac{\sqrt{R}}{Q}$ ($|Q| > 2\sqrt{R}$), critical fractions, since they decide the nature of the representations to be assigned to $\frac{P + \sqrt{R}}{Q}$ in a B.c.f. development. Ambiguities arise when

(iii) $\frac{P + \sqrt{R}}{Q} - \frac{1}{2} - \frac{\sqrt{R}}{Q} + \frac{\sqrt{4R - Q^2}}{2Q} = \frac{P}{Q} - \frac{1}{2} + \frac{\sqrt{4R - Q^2}}{2Q} = \text{an integer}$, ($|Q| < 2\sqrt{R}$) which implies $4R - Q^2 = t^2$, $\frac{2P + t}{Q}$ an odd integer, Q, t both even integers, and R is the sum of two squares; and

(iv) $\frac{P + \sqrt{R}}{Q} - \frac{1}{2} - \frac{\sqrt{R}}{Q} = \frac{P}{Q} - \frac{1}{2} = \text{an integer}$ ($|Q| > 2\sqrt{R}$); but these cases have been circumvented by appropriate conventions.

If $\frac{P + \sqrt{R}}{Q}$ be a special surd with $\frac{P_1 + \sqrt{R}}{Q_1}$ as its successor, and $R = Q_1^2 + \frac{1}{4}Q^2 > Q^2 + \frac{1}{4}Q_1^2$, then it is easily seen that the fractional part of $\frac{P + \sqrt{R}}{Q}$ in its positive representation is equal to the corresponding critical fraction which takes the special form $\frac{1}{2} + \frac{\sqrt{R} - Q_1}{Q}$, where $Q_1 > |Q|$.

Definition.—A proper fraction of the form $\frac{q - p + \sqrt{R}}{2q}$ (R a non-square positive integer) is called a special critical fraction when $R = p^2 + q^2$, and $p > 2q > 0$.

* This is a continuation of the memoir published in the *Journal of the Mysore University*, Vol. I, Part II, pp. 21–32.

† See *ibid.*, Vol. I, Part II, p. 26.

4.2. THEOREM VIII. If $\frac{P_{v-1} + \sqrt{R}}{Q_{v-1}}$ is a special surd with successors

$\frac{P_v + \sqrt{R}}{Q_v}$, $\frac{P_{v+1} + \sqrt{R}}{Q_{v+1}}$, $\frac{P_{v+2} + \sqrt{R}}{Q_{v+2}}$ in a B. c.f. development, then $\frac{P_{v+1} + \sqrt{R}}{Q_v}$

is a successor of $\frac{P_{v+2} + \sqrt{R}}{Q_{v+1}}$ in all cases except when $R = Q_v^2 + \frac{1}{4} Q_{v-1}^2$.

Let
$$\frac{P_{v+1} + \sqrt{R}}{Q_{v+1}} = b_{v+1} + \frac{\epsilon_{v+2} Q_{v+2}}{P_{v+2} + \sqrt{R}}.$$

Then,
$$\frac{P_{v+1} - \sqrt{R}}{Q_{v+1}} = b_{v+1} + \frac{\epsilon_{v+2} Q_{v+2}}{P_{v+2} - \sqrt{R}}$$

i.e.,
$$\frac{-\epsilon_{v+1} Q_v}{P_{v+1} + \sqrt{R}} = b_{v+1} - \frac{P_{v+2} + \sqrt{R}}{Q_{v+1}}$$

Hence
$$\frac{P_{v+2} + \sqrt{R}}{Q_{v+1}} = b_{v+1} + \frac{\epsilon_{v+1} Q_v}{P_{v+1} + \sqrt{R}} \quad (1)$$

where $Q_v^2 + \frac{1}{4} Q_{v+1}^2 \leq R$, $Q_{v+1}^2 + \frac{1}{4} Q_v^2 \leq R$.

Therefore $\frac{P_{v+1} + \sqrt{R}}{Q_v}$ will be a Bhaskara successor of $\frac{P_{v+2} + \sqrt{R}}{Q_{v+1}}$ in all cases except when

$$Q_v^2 + \frac{1}{4} Q_{v+1}^2 = R \text{ and } \epsilon_{v+1} = 1,$$

which will violate our convention in the ambiguous case.

Similarly,
$$\frac{P_{v+1} + \sqrt{R}}{Q_v} = b_v + \frac{\epsilon_v Q_{v-1}}{P_v + \sqrt{R}} \quad (2)$$

where $Q_{v-1}^2 + \frac{1}{4} Q_v^2 \leq R$, $Q_v^2 + \frac{1}{4} Q_{v-1}^2 \leq R$.

If $\epsilon_{v+1} = 1$, and $Q_v^2 + \frac{1}{4} Q_{v+1}^2 = R$, we have

$$P_{v+1}^2 = R - Q_v Q_{v+1} = (Q_v - \frac{1}{2} Q_{v+1})^2;$$

since $Q_v^2 + \frac{1}{4} Q_{v+1}^2 = R \geq Q_{v+1}^2 + \frac{1}{4} Q_v^2$,

$$Q_v > Q_{v+1}, \text{ and } P_{v+1} = Q_v - \frac{1}{2} Q_{v+1} \quad (3)$$

Hence,
$$\begin{aligned} \frac{P_{v+1} + \sqrt{R}}{Q_v} &= \frac{\sqrt{R} + Q_v - \frac{1}{2} Q_{v+1}}{Q_v} \\ &= 2 + \frac{\sqrt{R} - Q_v - \frac{1}{2} Q_{v+1}}{Q_v} \\ &= 2 - \frac{Q_{v+1}}{\sqrt{R} + Q_v + \frac{1}{2} Q_{v+1}} \end{aligned} \quad (4)$$

Comparing (2) and (4), we have $\epsilon_v = -1$, $Q_{v+1} = Q_{v-1}$, $P_v - Q_v = \frac{1}{2} Q_{v+1}$, and therefore $\frac{1}{4} Q_{v-1}^2 + Q_v^2 = R$.

Conversely, if $Q_v^2 + \frac{1}{4} Q_{v-1}^2 = R$, we see that $Q_v > |Q_{v-1}|$, $P_v = Q_v - \frac{1}{2} \epsilon_v Q_{v-1}$, where $\epsilon_v Q_{v-1}$ is negative, and

$$\frac{P_v + \sqrt{R}}{Q_v} = 2 + \frac{\sqrt{R} + \frac{1}{2} |Q_{v-1}| - Q_v}{Q_v} = 2 + \frac{|Q_{v-1}|}{\sqrt{R} - \frac{1}{2} |Q_{v-1}| + Q_v},$$

so that $\epsilon_{v+1} = 1$, $Q_{v+1} = |Q_{v-1}|$, and $Q_v^2 + \frac{1}{4} Q_{v+1}^2 = R$.

Hence, $\frac{P_{v+1} + \sqrt{R}}{Q_v}$ will fail to be a successor of $\frac{P_{v+2} + \sqrt{R}}{Q_{v+1}}$, when and only when $Q_v^2 + \frac{1}{4} Q_{v-1}^2 = R$.

4.3. THEOREM IX: *Two different semi-reduced surds cannot have the same Bhaskara successor unless they are conjugates of $-g$ and $1-g$, g being any special critical fraction.*

If possible, let two different semi-reduced surds

$\xi_v = \frac{P_v + \sqrt{R}}{Q_v}$, $\xi'_v = \frac{P'_v + \sqrt{R}}{Q'_v}$ have the same successor ξ_{v+1} (where $P'_v > P_v$), while the predecessor of ξ_v is ξ_{v-1} .

$$\text{Then } \frac{P_v + \sqrt{R}}{Q_v} \pm \frac{P'_v + \sqrt{R}}{Q'_v} = \text{an integer.} \quad (1)$$

Hence $Q_v = \pm Q'_v$ the irrational part being equated to zero.

But Q_v, Q'_v are both positive, the surds being semi-reduced; hence $Q_v = Q'_v$ and the sign to be chosen in (1) is negative, so that

$$P_v - P'_v \equiv 0 \pmod{Q_v} \quad (2)$$

Arguing as in the proof of Theorem VII* replacing therein P_{v+1} by P'_v but omitting the consideration $P'_v < R^{\frac{1}{2}}$, $P_v > R^{\frac{1}{2}}$, which has obviously no application in the present context, we get $P'_v - P_v \leq Q_v$. (3)

From (2) and (3), $P'_v = P_v$, or $P'_v - P_v = Q_v$, and in the latter case, $P_v^2 + P_v^2 = 2R$, from which we derive $P_v = |Q_{v-1}| - \frac{1}{2} Q_v$,

$P'_v = |Q_{v-1}| + \frac{1}{2} Q_v$, $R = Q_{v-1}^2 + \frac{1}{4} Q_v^2$, since we may put $P_v^2 = R - |Q_{v-1}| Q_v$.

Thus the two surds which have the same successor are of the form

$$\xi_v = \frac{|Q_{v-1}| - \frac{1}{2} Q_v + \sqrt{R}}{Q_v}; \quad \xi'_v = \frac{|Q_{v-1}| + \frac{1}{2} Q_v + \sqrt{R}}{Q_v} = 1 + \xi_v;$$

where $R = Q_{v-1}^2 + \frac{1}{4} Q_v^2$, Q_v is even and less than $|Q_{v-1}|$.

* See *Journal of the Mysore University*, Vol I, Part II, page 31.

Obviously, $\frac{\frac{1}{2}Q_v - |Q_{v-1}| + \sqrt{R}}{Q_v}$ is a special critical fraction, g say, ξ_v is the conjugate of $-g$ and ξ'_v is the conjugate of $(1-g)$. This proves the proposition.

4.4. THEOREM X: *If g be a special critical fraction, then g^{-1} has no Bhaskara predecessor, $(1-g)^{-1}$ is semi-reduced, and the Bhaskara successors of g^{-1} and $(1-g)^{-1}$ are respectively the conjugates of $1-g$ and $-g$; the conjugate of $1-g$ has no semi-reduced predecessor, while the conjugate of $-g$ has a unique semi-reduced predecessor.*

Let $g = \frac{q-p + \sqrt{p^2+q^2}}{2q}$, ($p > 2q > 0$). Then a predecessor of g^{-1} or $(1-g)^{-1}$ will be of the form $a \pm g$, where a is an integer.

Put $\frac{P + \sqrt{R}}{Q} = a + g = a + \frac{p}{p-q + \sqrt{R}} = a + 1 - (1-g) = a + 1 - \frac{p}{p+q + \sqrt{R}}$, where $R = p^2 + q^2$.

Then $Q = 2q < p < R^{\frac{1}{2}}$; $p^2 + \frac{1}{4}Q^2 = R > Q^2 + \frac{1}{4}p^2$, so that $\frac{P + \sqrt{R}}{Q}$ is a special surd.

Hence g^{-1} has no predecessor of the form $a + g$, while $(1-g)^{-1}$, has one of the form $a + 1 - (1-g)$.

Similarly, it can be shown that g^{-1} has no predecessor of the form $a - g$, while $(1-g)^{-1}$ has a predecessor of the form $a - 1 + (1-g)$.

$$\begin{aligned} \text{Now } g^{-1} &= \frac{p-q + \sqrt{R}}{p} = 1 + \frac{p}{q + \sqrt{R}} = 2 - \frac{2q}{p+q + \sqrt{R}} \\ &= 2 - \frac{1}{\text{conjugate of } (1-g)}; \end{aligned}$$

$$\begin{aligned} \text{and } (1-g)^{-1} &= \frac{p+q + \sqrt{R}}{p} = 3 - \frac{3p-4q}{2p-q + \sqrt{R}} \\ &= 2 + \frac{2q}{p-q + \sqrt{R}} = 2 + \frac{1}{\text{conjugate of } (-g)}. \end{aligned}$$

Since $2q < p < 3p - 4q$, the Bhaskara successors of g^{-1} and $(1-g)^{-1}$ are respectively the conjugates of $(1-g)$ and $-g$.

Any predecessor of the conjugate of $(1-g)$ must be of the form $a \pm \frac{p+q - \sqrt{R}}{p}$,

where a is an integer. For a semi-reduced predecessor, $a + \frac{p+q - \sqrt{R}}{p}$ is inadmissible and a must be an integer such that $p(a-1) - q > 0$, and $(pa - p - q)^2$ is nearest to R ; all these conditions are satisfied only

when $a = 2$, for it can be easily verified that $p - q < \sqrt{R}$, $pa - p - q > \sqrt{R}$ when $a > 2$, and $R - (p - q)^2 < (2p - q)^2 - R$, when $p > 2q$. Thus the only possible semi-reduced predecessor of the conjugate of $1 - g$ is g^{-1} . But since g^{-1} has no Bhaskara predecessor, it cannot be semi-reduced.

Similarly, the possible semi-reduced predecessors of the conjugate of $-g$ must be of the form $\frac{pa - p + q + \sqrt{R}}{p}$, where a is an integer such that $pa - p - q > 0$, and $(pa - p - q)^2$ is nearest to R . Obviously $a = 2$, since when $a \geq 2$, $pa - p + q > \sqrt{R}$, and when $a = 1$, $q < \sqrt{R}$, while $(p + q)^2 - R < R - q^2$. Thus the possible semi-reduced predecessor is $(1 - g)^{-1}$, which is certainly semi-reduced with a 'special' surd as its predecessor.

Hence the proposition is proved.

Cor. 1. Two different reduced surds cannot have the same successor.

Cor. 2. Neither the conjugate of $-g$ nor that of $(1 - g)$ can be the successor of a standard surd of the form $\frac{\sqrt{R}}{Q}$.

5. Pure Recurring Bhaskara Continued Fractions

5.1 Definition.—A pure recurring B.c.f. is one in which the complete quotients recur from the first.

We have already seen that the complete quotients in a B.c.f. development are ultimately reduced surds. Hence a pure recurring B.c.f. is equal to a reduced surd.

The converse of this will now be proved.

5.2. THEOREM XI: *The Bhaskara development of a reduced surd is a pure recurring half-regular continued fraction.*

Let $\xi_0 = \frac{P_0 + \sqrt{R}}{Q_0}$ be a reduced surd and if possible, let its B.c.f. development be the periodic h.r.c.f.

$$b_0 + \frac{\epsilon_1}{b_1} + \dots + \frac{\epsilon_{k-1}}{b_{k-1}} + \frac{\epsilon_k}{b_k} + \dots + \frac{\epsilon_{k+n-1}}{b_{k+n-1}}$$

where $\xi_{k+v} = \xi_{k+v+tn}$ ($v = 0, 1, \dots, n-1$), t a positive integer, and $b_{k+v} = b_{k+v+tn}$.

Since ξ_0 is reduced, ξ_{k-1} and ξ_{k+n-1} are also reduced; but their respective successors ξ_k and ξ_{k+n} are equal.

By Theorem X, Cor. (1), therefore, $\xi_{k-1} = \xi_{k+n-1}$.

If $\epsilon_{k-1} \neq \epsilon_{k+n-1}$, then $\xi_{k-2} \neq \xi_{k+n-2}$ which will contradict Theorem X, Cor. (1), so that $\epsilon_{k-1} = \epsilon_{k+n-1}$, i.e., the recurrence begins one step earlier. This process can be evidently continued backwards until ξ_0 is reached. The first complete quotient therefore recurs and the h.r.c.f. is a pure recurring one, of the form $b_0 + \frac{\epsilon_1}{b_1} + \dots + \frac{\epsilon_k}{b_0}$.

5.3. THEOREM, XII: *The B.c.f. development of the standard surd $\frac{\sqrt{R}}{Q}$ (> 1) has only one term in the acyclic part.*

Proof: Let $\xi_0 = \frac{\sqrt{R}}{Q} = b_0 + \frac{\epsilon_1}{\xi_1}$ (a B.R.), where $\xi_1 = \frac{P_1 + \sqrt{R}}{Q_1}$.

Then $P_1 = b_0 Q, \epsilon_1 Q Q_1 = R - P_1^2$;

$\frac{\sqrt{R}}{Q}$ being in the standard form, we may write $R = QQ'$, where Q, Q' are positive integers having no common factor; hence $\epsilon_1 Q_1 = Q' - b_0^2 Q$.

By Theorem I, since $Q < \sqrt{R}, P_1, Q_1$ are positive and $|Q_1 - \frac{1}{2} \epsilon_1 Q| \leq P_1$. (1)

When $Q < \frac{1}{2} Q_1$, and $\epsilon_1 = 1, \frac{1}{2} Q_1 - Q < Q_1 - \frac{1}{2} Q \leq P_1$ by (1).

We shall now prove that $|Q - \frac{1}{2} \epsilon_1 Q_1| \leq P_1$, which is equivalent to

$$Q - \frac{1}{2} Q' + \frac{1}{2} b_0^2 Q < b_0 Q,$$

i.e., $Q(1 - b_0 + \frac{1}{2} b_0^2) < \frac{1}{2} Q'$,

i.e., $(b_0 - 1)^2 < \frac{Q'}{Q} - 1$, when $Q > \frac{1}{2} Q_1$ (2)

If $\epsilon_1 = 1, b_0 Q < \sqrt{R} = \sqrt{QQ'},$ i.e., $b_0^2 < \frac{Q'}{Q}$, so that

$$(b_0 - 1)^2 \leq b_0^2 - 1 < \frac{Q'}{Q} - 1. \tag{3}$$

If $\epsilon_1 = -1$, we have from (1), $Q_1 + \frac{1}{2} Q \leq P_1$,

i.e., $b_0^2 Q - Q' + \frac{1}{2} Q \leq b_0 Q,$

i.e., $b_0^2 - b_0 + \frac{1}{2} \leq Q' / Q.$

When $\epsilon_1 = -1, b_0 \geq 1$, and so $(b_0 - 1)^2 + 1 < b_0^2 - b_0 + \frac{1}{2}$;

hence $(b_0 - 1)^2 < \frac{Q'}{Q} - 1. \tag{4}$

Thus, in all cases, $|Q - \frac{1}{2} \epsilon_1 Q_1| < P_1 \tag{5}$

In case (i), since $p/q = \frac{n^2 - n + 2}{n - \frac{1}{2}} = \frac{n^2 - n + \frac{1}{2}}{n - \frac{1}{2}}$ is impossible, simultaneous equality has to be excluded.

The upper limits (U) and the lower limits (L) of p/q corresponding to the four cases are as follows:

Case	U		L	
(i)	$\frac{n^2 - n + 2}{n - \frac{1}{2}}$,	$\frac{n^2 - n + \frac{1}{2}^*}{n - \frac{1}{2}}$	$\frac{n^2 - 3n + 3}{n - \frac{3}{2}}$,	$n - 1^*$
(ii)	n ,	$\frac{n^2 - n + 2^*}{n - \frac{1}{2}}$	$\frac{n^2 - 3n + 3}{n - \frac{3}{2}}$,	$\frac{n^2 - n + \frac{1}{2}^*}{n - \frac{1}{2}}$
(iii)	$\frac{n^2 + n + 1}{n + \frac{1}{2}}$,	$\frac{n^2 - n + \frac{1}{2}^*}{n - \frac{1}{2}}$	$n - 1$,	$\frac{n^2 - n + 2^*}{n - \frac{1}{2}}$
(iv)	$\frac{n^2 + n + 1}{n + \frac{1}{2}}$,	n^*	$\frac{n^2 - n + 2^*}{n - \frac{1}{2}}$,	$\frac{n^2 - n + \frac{1}{2}}{n - \frac{1}{2}}$

the starred expressions signifying lesser upper limits (U) and greater lower limits (L) when $n > 4$.

Case (iii) is impossible since the lesser upper limit is obviously less than the greater lower limit; since $p > 2q$, $n = 2$ is impossible in all the four cases; when $n = 3, 4$, the limits for p/q are respectively (2 and 13/5) and (3, 25/7) in case (i) and (13/5 and 3) and (25/7, 4) in case (ii), while case (iv) is inapplicable.

For integral $n > 4$, the first two and the last case are applicable, in order, for the values of p/q in the intervals I_1, I_2, I_3 corresponding respectively to $(n - 1, \frac{n^2 - n + \frac{1}{2}}{n - \frac{1}{2}})$, $(\frac{n^2 - n + \frac{1}{2}}{n - \frac{1}{2}}, \frac{n^2 - n + 2}{n - \frac{1}{2}})$, $(\frac{n^2 - n + 2}{n - \frac{1}{2}}, n)$, closed on the right and open to the left.

Thus, for every value of p/q greater than 2, we can always fix up a unique value of n also greater than 2, since p/q is bound to lie in one and only one of the rational intervals (closed on the right and open on the left), (2, 13/5), (13/5, 3), (3, 25/7), (25/7, 4), (4, 41/9), (41/9, 44/9), (44/9, 5) and so on, which cover the entire set of rational numbers greater than 2. This proves our theorem.

5.5. Before discussing further the properties of the recurring B.c.f., we require certain lemmas on the behaviour of unit partial quotients in simple continued fractions.

Lemma (1): If $\xi = \frac{P + \sqrt{R}}{Q}$ develops as a pure recurring simple continued fraction with a set of successive unit partial quotients preceded and

followed by other partial quotients, then the denominators of the complete quotients corresponding to the unit partial quotients other than the first and the last of the set are less than \sqrt{R} .

$$\text{Let } \xi = \frac{P + \sqrt{R}}{Q} = \frac{1}{a_0} + \frac{1}{a_1 + \frac{1}{a_2 + \dots + \frac{1}{a_r + \frac{1}{1 + \frac{1}{1 + \dots + \frac{1}{1 + \frac{1}{a_{r+n+1} + \dots + \frac{1}{a_p}}}}}}}}}$$

$$= (a_0, a_1, a_2, \dots, a_r, 1_{[n]}, a_{r+n+1}, \dots, a_p), \text{ and}$$

$$\xi_{r+v} = \frac{P_{r+v} + \sqrt{R}}{Q_{r+v}} = (1_{[n-v+1]}, a_{n+r+1}, \dots, a_p, a_0, a_1, \dots, a_r, 1_{[v-1]})$$

$$= f, \text{ (say)} \tag{1}$$

By* Galois's theorem of inverse periods,

$$\frac{Q_{r+v}}{P_{r+v} + \sqrt{R}} = (1_{[v-1]}, a_r, \dots, a_0, a_p, \dots, a_{n+r+1}, 1_{[n-v+1]}).$$

Hence,
$$\frac{P_{r+v} + \sqrt{R}}{Q_{r+v}} = (0, 1_{[v-1]}, \dots, 1_{[n-v+1]}) = f', \text{ (say)}. \tag{2}$$

Adding (1) and (2), $\frac{2\sqrt{R}}{Q_{r+v}} = f + f'$, so that

$$Q_{r+v} \leq \sqrt{R}, \text{ according as } f \cong (2 - f') = f'' \text{ (say).}$$

But,
$$2 - f' = (2 - (0, 1_{[v-1]}, \dots, 1_{[n-v+1]}))$$

$$= (1, 1, 0, 1_{[v-2]}, \dots, \dots)$$

$$= (1, 2, 1_{[v-3]}, \dots, \dots)$$

If $n - 1 \geq v \geq 3$, the second complete quotient of f is less than the corresponding complete quotient of f'' and therefore $f > f''$, implying $Q_{r+v} < \sqrt{R}$. If $v = 2$ and $n \geq 3$, we have $f'' = 2 - (0, 1, a_r, \dots) = (1, 1 + a_r, \dots) < f$ and again $Q_{r+v} < \sqrt{R}$.

Thus for all values of v greater than 1 and less than n (≥ 3), $Q_{r+v} < \sqrt{R}$.

The lemma is therefore proved.

Cor. (1).—If $n > 2$, and $a_r > 2$, then $Q'_{r+1} > \sqrt{R}$; if $n > 2$, and $a_{r+n+1} > 2$ then $Q'_{r+n} > \sqrt{R}$.

Cor. (2).—If $n = 2$, and $1 + a_r < a_{r+n+1}$, then $Q'_{r+1} < \sqrt{R}$ and $Q'_{r+n} > \sqrt{R}$; the inequalities are reversed when $1 + a_{r+n+1} < a_r$, while both the Q 's are less than \sqrt{R} if $a_r = a_{r+n+1}$.

* Vide pp. 82-85, *Die Lehre von den Kettenbrüchen*, by O. Perron, 1929.

Lemma (2): In the simple continued fraction development of a surd of the form $\frac{q + \sqrt{p^2 + q^2}}{p}$, p, q , being integers such that $p > 2q > 0$, there cannot occur a complete quotient of the same form more than once in the recurring period; when such a complete quotient does occur, the recurring period is symmetric, with an even number of terms, which include a central set of an even number of unit partial quotients.

Let $\xi_0 = \frac{P_0 + \sqrt{R}}{Q_0} = \frac{q + \sqrt{p^2 + q^2}}{p}$, with $\xi_v \left(= \frac{P_v + \sqrt{R}}{Q_v} \right)$ as the v -th successor of ξ_0 . Let $\bar{\xi}_0$ be the conjugate of ξ_0 . Then, $\xi_0 \bar{\xi}_0 = -1$; $1 < \xi_0 < \frac{1 + \sqrt{5}}{2} = (1, 1, \dots) = (1)$ or (1_∞) and $-1 < \bar{\xi}_0 < 0$. (1)

By a well-known theorem of Galois, the simple continued fraction for ξ_0 has a pure recurring period (a_0, a_1, \dots, a_n) , say.

From (1), $a_0 = 1$ and if a_m is the first partial quotient greater than 1, m must be odd; for, if m be even, we have successively $(a_m, \dots) > (1_\infty)$; $(1, a_m, \dots) < (1_\infty)$; $(1_{[2]}, a_m, \dots) > (1_\infty)$; $\dots, (1_{[m]}, a_m, \dots) > (1_\infty)$, which contradicts (1).

Hence $\xi_0 = (1_{[m]}, a_m, \dots, a_n)$. (2)

Again, $\xi_0 = -1/\bar{\xi}_0 = (a_n, \dots, 1_{[m]})$. (3)

Comparing (2) and (3), we have $a_n = a_{n-1} = \dots = a_{n-m+1} = 1$, $a_m = a_{n-m}$; i.e., the period is a symmetric one, beginning and ending with an odd number of unit partial quotients.

The comparison of the complete quotients in (2) and (3) gives

$$\frac{P_v + \sqrt{R}}{Q_v} = \frac{P_{n+1-v} + \sqrt{R}}{Q_{n-v}}, \quad (v \leq n), \quad \text{i.e.,} \quad P_v = P_{n-v+1}, \quad Q_v = Q_{n-v}$$

$$\begin{aligned} \text{If } Q_v = Q_{v-1}, \text{ then } \xi_v &= \frac{P_v + \sqrt{R}}{Q_v} = \frac{P_{n+1-v} + \sqrt{R}}{Q_{v-1}} \\ &= \frac{P_{n+1-v} + \sqrt{R}}{Q_{n+1-v}} = \xi_{n+1-v} \end{aligned}$$

and so $v = n+1-v$, i.e., $v = \frac{n+1}{2}$, which implies that n should be odd.

Thus, only when n is odd, $\frac{Q_{n+1}}{2} = \frac{Q_{n-1}}{2}$ and these are the only consecutive Q 's which can be equal to each other. (4)

If a complete quotient, say, ξ_v , should be of the same form as ξ_0 , its simple continued fraction development should have the same properties. Writing $Q_0, Q_1 \dots Q_n$ round a circle at the vertices of a regular polygon of $n+1$ sides, we find that they arrange themselves symmetrically about a diameter, such that the Q 's symmetrically placed about this diameter are also equal, since $Q_v = Q_{n-v}$.

The symmetry of the Q 's corresponding to ξ_v imply that $Q_v = Q_{v-1}$ just as $Q_0 = Q_n$. From (4) we see that this can happen only once and so, there cannot be more than one ξ_v of the same form as ξ and it occurs when n is odd and $v = \frac{n+1}{2}$. In this case we realise the same symmetry of Q 's starting from $Q_{\frac{n+1}{2}}$, going round the circle and ending with $Q_{\frac{n-1}{2}}$ as in the first set $(Q_0, Q_1, \dots Q_n)$.

This proves the existence of ξ_v of the same form as ξ_0 , only when $Q_{\frac{n \pm r}{2}} = 1$, where $r = 1, 3, 5 \dots (2k-1)$, and k, n are both odd.

Hence, if ξ_0 should have a remote successor of the same form as itself in the recurring period of its simple continued fraction development, then the recurring period must consist of an even number of symmetrically disposed partial quotients including an initial, a central and a final set of unit partial quotients. In order that the recurring cycle may not lose its character as a primitive period, it is necessary that the first half of the cycle is not itself symmetrical.

Example.— $\frac{27 + \sqrt{27^2 + 82^2}}{82} = (1, 2, 1_{[6]}, 2, 1)$ has a remote successor within the recurring period of the same form $\frac{37 + \sqrt{37^2 + 78^2}}{78}$.

Lemma (3): If the standard surd of the form $\frac{\sqrt{R}}{Q_0}$ have in its simple continued fraction development a complete quotient of the form $\frac{q + \sqrt{R}}{p}$, where $R = p^2 + q^2, p > 2q > 0$, then the symmetric portion of the recurring period of partial quotients will include a central even number, of the form $4n-2$, of unit partial quotients; and there cannot occur any other complete quotient of a similar form within the recurring period, which must consist of an odd number of terms.

Conversely, if any simple continued fraction development of the standard surd $\frac{\sqrt{R}}{Q_0}$ has in its recurring period an odd number of partial quotients with

a central even number $(4n-2)$ of unit partial quotients, in the symmetric part, then $R = p^2 + q^2$, $p > 2q > 0$ and the complete quotient $\frac{q + \sqrt{R}}{p}$ occurs just once in the recurring period.

$$\text{Let } \frac{\sqrt{R}}{Q_0} = (a_0, \underset{x}{a_1}, a_2, \dots, a_{k-1}, \underset{x}{2a_0}) \quad (1)$$

From Lemma (2), a complete quotient, say ξ_v , of the form in question in (1) cannot have either a_1 or $2a_0$ (obviously $\neq 1$) as its first partial quotient so that we may write $\xi_v = (\underset{x}{a_v}, \dots, \underset{x}{a_{v-1}})$, where $a_v \neq a_1$ or $2a_0$. From the equality of the first and last Q's in ξ_v , we must have $Q_v = Q_{v-1}$ in (1), which implies, by a well-known theorem of Muir,* that k is odd and $v = \frac{k+1}{2}$; and in this case, it is easily seen that $\xi_v = \frac{P_v + \sqrt{R}}{Q_v}$, and $R = P_v^2 + Q_v^2$.

Further, there cannot be another complete quotient of the same form in the recurring period, since it is possible only when the number of terms in the recurring period is even.

We infer therefore that $\xi_{\frac{k+1}{2}} = (\underset{\frac{x}{2}}{a_{\frac{k+1}{2}}}, \dots, a_{k-1}, 2a_0, a_1, \dots, \underset{\frac{x}{2}}{a_{\frac{k+1}{2}}})$, where an odd number of unit partial quotients must begin with $\underset{\frac{x}{2}}{a_{\frac{k+1}{2}}}$ and also an equal odd number of such partial quotients end with $\underset{\frac{x}{2}}{a_{\frac{k+1}{2}}}$.

Thus $\frac{\sqrt{R}}{Q_0}$ must contain in its period an even number, of the form $4n-2$, of unit partial quotients in the centre of the symmetric portion, as, for example, $\sqrt{58} = (7, \underset{x}{1_{[6]}}, \underset{x}{14})$; $\sqrt{97} = (9, 1, 5, \underset{x}{1_{[6]}}, \underset{x}{5}, 1, 18)$.

In this case, $\xi_{\frac{k+1}{2}}$ is of the form $\frac{q + \sqrt{p^2 + q^2}}{p} < (1_\infty)$, as the continued fraction begins with an odd number of unit partial quotients.

Hence, $\frac{q + \sqrt{p^2 + q^2}}{p} < \frac{1 + \sqrt{5}}{2}$, $\frac{-q + \sqrt{p^2 + q^2}}{p} > \frac{-1 + \sqrt{5}}{2}$, so that, subtracting the second from the first, $2q/p < 1$, and obviously p and q are positive in a recurring period.

This completes our proof.

* Vide p. 91, Perron, *loc. cit.*

5.5 1. We will now point out an application of the last two lemmas to the most rapidly convergent continued fractions. Tietze* has shown that such continued fractions are characterised by the property that the complete quotients are, after a certain point, always greater than $\frac{1+\sqrt{5}}{2}$. The B.c.f.'s are therefore of this class. We have proved elsewhere† that the only transformations (apart from the P-transformation) which convert a simple continued fraction into one of the most rapidly convergent h.r.c.f.'s are the annihilatory transformations which we have called the C_1 , C_2 , and C_1C_2 types. The effect of an annihilatory transformation applied to a unit partial quotient is obviously to increase the following complete quotient by 1, without affecting the preceding complete quotient.

From these considerations, we see that a complete quotient of the form $\frac{q+p+\sqrt{p^2+q^2}}{p}$ will occur in any most rapidly convergent h.r.c.f. development (not involving a P-transformation) of \sqrt{R}/Q_0 (> 1 , and in the standard form), when and only when either $\frac{q+\sqrt{p^2+q^2}}{p}$ or $\frac{q+p+\sqrt{p^2+q^2}}{p}$ occurs in the simple continued fraction development. But $\frac{q+p+\sqrt{p^2+q^2}}{p}$ is not a reduced surd in Perron's sense‡ and therefore cannot occur in the recurring period of the simple continued fraction, while $\frac{q+\sqrt{p^2+q^2}}{p}$ will occur just once in the recurring period under the conditions of Lemma (3).

Hence, every most rapidly convergent h.r.c.f. development of \sqrt{R}/Q_0 (not involving a P-transformation) will contain in its period $\frac{q+p+\sqrt{p^2+q^2}}{p}$ as a complete quotient just once when the unit partial quotient corresponding to $\frac{q+\sqrt{p^2+q^2}}{p}$ in the simple continued fraction is not annihilated.

If $\sqrt{R}/Q_0 = (a_0, a_1, a_2, \dots, a_p, 1_{[4l+2]}, a_p, \dots, a_1, 2a_0)$, where ξ_{p+2l+2} is the only complete quotient of the form $\frac{q+\sqrt{p^2+q^2}}{p}$, the result of applying the C_1 -transformation gives the complete quotient $1 + \xi_{p+2l+2}$, while the C_2 -transformation will annihilate the unit partial quotient corresponding to

* Tietze, H., *Monatshfte für Mathematik und Physik*, 1913, 24.

† Ayyangar, A. A. K., *Maths. Student*, 1938, 6.

‡ *Vidz* p. 79, Perron, *loc. cit.*

ξ_{p+2t+2} and so there will be no complete quotient of the form in question. To preserve the complete quotient, we may also apply the eclectic transformation C_1C_2 , provided that C_1 process is continued at least until it annihilates the $(2t+1)$ -th central unit partial quotient. Hence we may state that it is possible to have a complete quotient of the form in question in the B.c.f. development as well as in the continued fraction to the nearest integer, but not in the singular continued fraction (all of which do not involve the P-transformation*).

5.6. We are now in a position to resume our original thread of discussion and study the nature of the recurring period of the B.c.f. development of $\frac{\sqrt{R}}{Q_0}$. We at once recognize three possible types:

Type I.—This occurs when the recurring cycle does not contain any complete quotient of the form $(1-g)^{-1}$, i.e., $\frac{q+p+\sqrt{p^2+q^2}}{p}$, g being a special critical fraction pertaining to R . Evidently, this type must occur when R cannot be expressed as the sum of two squares, or when \sqrt{R}/Q_0 does not satisfy the conditions of Lemma (3). We will presently show that the characteristic property of this type is that it simulates the simple continued fraction period in its symmetries and also in the property of the last partial quotient. e.g., $\sqrt{46} = 7 - \frac{1}{x} - \frac{1}{2} + \frac{1}{2} + \frac{1}{6} + \frac{1}{2} + \frac{1}{2} - \frac{1}{x} - \frac{1}{x}$.

Type II.—This occurs when the recurring cycle contains a complete quotient of the form $\frac{q+p+\sqrt{p^2+q^2}}{p}$. We call this 'almost' symmetrical, as the symmetries are slightly disturbed, as for example, $\sqrt{58} = 8 - \frac{1}{x} - \frac{1}{2} - \frac{1}{2} - \frac{1}{x}$.

Type III.—This is an extreme case of Type II, with only two terms in the recurring period, e.g., $\sqrt{n^2+n+\frac{1}{2}} = n+1 - \frac{1}{2} + \frac{1}{2n+1}$.

$$5.6.1. \quad \text{Let} \quad \xi_0 = b_0 + \frac{\epsilon_1}{b_1 + \dots + \frac{\epsilon_{k-1}}{b_{k-1} + \frac{\epsilon_k}{b_0}}}, \quad \xi_v = \frac{P_v + \sqrt{R}}{Q_v},$$

$$\bar{\xi}_v = \frac{P_v - \sqrt{R}}{Q_v}, \quad \text{and} \quad \zeta_v = -\frac{\epsilon_{k-v}}{\bar{\xi}_{k-v}}, \quad (v = 0, 1, \dots, k-1),$$

where ξ_v is the v -th successor of ξ_0 , $\xi_k = \xi_0$, $\bar{\xi}_k = \bar{\xi}_0$.

* *Vide Maths. Student*, 6, 63; and *Journal of the Mysore University*, Vol. I, Part II, Note (2), Th. II.

Then, as in the simple continued fraction it is easily seen that

$$\zeta_0 = b_{k-1} + \frac{\epsilon_{k-1}}{b_{k-2} + \frac{\epsilon_{k-2}}{b_{k-3} + \dots + \frac{\epsilon_1}{b_0 + \frac{\epsilon_k}{x}}}};$$

$$\zeta_{v-1} = b_{k-v} + \frac{\epsilon_{k-v}}{\zeta_v} = \frac{P_{k-v+1} + \sqrt{R}}{Q_{k-v}}. \quad (1)$$

By Theorem VIII, ζ_v is the Bhaskara successor of ζ_{v-1} in all cases except when $Q_{k-v-1}^2 + \frac{1}{4}Q_{k-v-2}^2 = R$, which implies that $\epsilon_{k-v-1} = -1$, $\epsilon_{k-v} = 1$ and ξ_{k-v-1} is of the form $(1-g)^{-1}$, g being a special critical fraction.

When no successor (immediate or remote) of $\sqrt{R}/Q_0 = \sqrt{D}$ (say)

$$= b_0 + \frac{\epsilon_1}{b_1 + \frac{\epsilon_2}{b_2 + \dots + \frac{\epsilon_{k-1}}{b_{k-1} + \frac{\epsilon_k}{b_1 + \frac{\epsilon_2}{x}}}}} \quad (2)$$

is of the form in question, we may write

$$\frac{\epsilon_1}{\sqrt{D} - b_0} = b_1 + \frac{\epsilon_2}{b_2 + \dots + \frac{\epsilon_{k-1}}{b_{k-1} + \frac{\epsilon_k}{b_1}}}$$

and by (1), $\epsilon_1 \epsilon_k (\sqrt{D} + b_0) = b_{k-1} + \frac{\epsilon_{k-1}}{b_{k-2} + \dots + \frac{\epsilon_2}{b_1 + \frac{\epsilon_k}{x}}}. \quad (3)$

Since the r.h.s. is positive $\epsilon_1 \epsilon_k = 1$.

Comparing (2) and (3) which are both B.c.f.'s we get $b_{k-1} = 2b_0$ and the symmetries, which may be characterised thus :

$$b_{v-1} = b_{k-v} \quad (v = 2, 3, \dots, k-1);$$

$$Q_{v-1} = Q_{k-v} \quad (v = 2, 3, \dots, k-1);$$

$$\epsilon_v = \epsilon_{k-v} \quad (v = 1, 2, \dots, k-1);$$

$$P_v = P_{k-v} \quad (v = 1, 2, \dots, k-1).$$

When k is even, or the number of terms in the recurring period is odd, two consecutive b 's and two consecutive Q 's are equal, viz., $\frac{b_{k-2}}{2} = b_k, \frac{Q_{k-2}}{2} = Q_k$.

When k is odd or the number of terms in the recurring period is even, we have two consecutive ϵ 's and P 's equal, viz., $\frac{\epsilon_{k-1}}{2} = \frac{\epsilon_{k+1}}{2}; \frac{P_{k-1}}{2} = \frac{P_{k+1}}{2}$.

Conversely, if two consecutive Q 's are equal in the recurring cycle, say, $Q_v = Q_{v-1}$, then $\xi_{k-v} = \frac{P_{k-v} + \sqrt{R}}{Q_{k-v}} = \frac{P_v + \sqrt{R}}{Q_{v-1}} = \xi_v$, so that $v = k/2$ and

k is even. Similarly for two consecutive P 's, $v = \frac{k+1}{2}$ and k is odd.

THEOREM XIV: If $\sqrt{R}/Q_0 (> 1)$ develops as a Type I. B.c.f., R is a non-square positive integer divisible by Q_0 , and the number of terms in the recurring cycle is odd, then R is either a sum of two squares or a composite number or is equal to 3.

Let k be the number of terms in the recurring cycle.

Then, $P_{v+1}^2 + \epsilon_{v+1} Q_{v+1} Q_v = R$, and $Q_v = Q_{v+1}$, when $v = \frac{k-1}{2}$.

If $\epsilon_{v+1} = +1$, R is evidently the sum of two squares.

If $\epsilon_{v+1} = -1$, and R is a prime, $P_{v+1} + Q_{v+1} = R$, and $P_{v+1} - Q_{v+1} = 1$, so that $Q_{v+1} = \frac{R-1}{2} < \sqrt{R}$, and therefore R is either 3 or 5. In both these cases, it is easily verified that $k = 1$.

When R is neither 3, nor a sum of two squares, $\epsilon_{v+1} = -1$, and R is composite.

Cor.—When R is a prime ($\neq 3$) and is not the sum of two squares, k is even.

5.62. If in the B.c.f. development of \sqrt{R}/Q_0 given in (2) of § 5.61, ξ_{k-2} happens to be of the form $(1-g)^{-1}$, then ξ_{k-1} is the conjugate of $-g$ (vide Theorem X), and being the predecessor of ξ_k is also of the form $\sqrt{D} + \mu$, where μ is an integer; i.e., $p-q$ is divisible by $2q$ ($p > 2q > 0$). Hence, we may put $p = (2n+1)q$, $R = p^2 + q^2 = q^2(4n^2 + 4n + 2)$,

$\xi_{k-1} = n + \frac{\sqrt{R}}{2q}$, so that \sqrt{D} is of the form $\sqrt{4n^2 + 4n + 2} / 2$.

The B.c.f. development of $\frac{\sqrt{4n^2 + 4n + 2}}{2}$ is $n + 1 - \frac{1}{2 + \frac{1}{2n+1}}$.

This is what we have called Type III.

5.63. As we have already seen, the recurring period in Type II will contain one and only one complete quotient of the form $\frac{p+q+\sqrt{p^2+q^2}}{p}$, and therefore, the recurring cycle will be merely a cyclic permutation of that of this complete quotient.

By Theorem XIII, $(1-g)^{-1} = \frac{p+q+\sqrt{p^2+q^2}}{p}$, ($p > 2q > 0$) develops as a pure recurring B.c.f. We will now proceed to study its nature.

Let $\xi'_0 = (1-g)^{-1} = \frac{p+q+\sqrt{R}}{p}$; $\xi'_v = \frac{P'_v + \sqrt{R}}{Q'_v}$; $\bar{\xi}'_0 = \frac{p+q-\sqrt{R}}{p}$

By Theorem X, $\xi'_0 = 2 + 1/(\text{conjugate of } -g) = 2 + \frac{2q}{p-q+\sqrt{R}}$
 $= 2 + \frac{1}{\frac{1}{b'_1 + \frac{\epsilon'_2}{b'_2 + \frac{\epsilon'_3}{b'_3 + \dots + \frac{\epsilon'_{k'-1}}{b'_{k'-1} + \frac{\epsilon'_{k'}}{2}}}}}$ ($= \xi'_{k'}$) (1)

As in § 5.6 1, we write

$$\epsilon'_{k'} (-1 + \text{conjugate of } -g) = -\frac{\epsilon'_{k'}}{\xi'_0} = b'_{k'-1} + \frac{\epsilon'_{k'-1}}{b'_{k'-2} + \dots + \frac{\epsilon'_2}{b'_1 + 2 + \frac{\epsilon'_{k'}}{x}}}$$

which will be a B.c.f. development for the first $(k' - 2)$ terms, since ξ'_v is not of the form $(1 - g)^{-1}$ for $v = 1, 2, 3, \dots (k' - 1)$.

Hence,

$$\text{conjugate of } -g = -1 - \epsilon'_{k'} b'_{k'-1} - \frac{\epsilon'_{k'} \epsilon'_{k'-1}}{b'_{k'-2}} + \frac{\epsilon'_{k'-2}}{b'_{k'-3} + \dots + \frac{\epsilon'_3}{b'_2 + \dots}}$$
 (2)

$$\text{From (1) the conjugate of } -g = b'_1 + \frac{\epsilon'_2}{b'_2 + \dots + \frac{\epsilon'_{k'}}{2} + \dots}$$
 (3)

Comparing the first $(k' - 1)$ complete quotients and the first $(k' - 2)$ terms of (2) and (3), which correspond to the B.c.f. developments of the same number we obtain the following properties of (1):

(i) $-\epsilon'_{k'} b'_{k'-1} = b'_1 + 1$, a positive integer, so that $\epsilon'_{k'} = -1$, and $b'_{k'-1} = b'_1 + 1$.

(ii) The symmetries $b'_v = b'_{k'-v}$ ($v = 2, 3, \dots k - 2$);

$$Q'_v = Q'_{k'-v} \quad (v = 1, 2, \dots k - 1);$$

$$\epsilon'_v = \epsilon'_{k'-v+1} \quad (v = 2, 3, \dots k - 1);$$

$$P'_v = P'_{k'-v+1} \quad (v = 2, 3, \dots k - 1).$$

(iii) $P'_1 = p - q$, $Q'_1 = 2q$, $P'_{k'} = P'_0 = p + q$, $P'_{k'-1} = q(2n - 1) - p$, $Q'_{k'-1} = 2q$, where $n = b'_{k'-1}$ = the integer just greater than p/q when p is not divisible by q , and $n = p/q$ otherwise.*

As in § 5.6 1, we can prove that two consecutive Q's will be equal only when k' is odd, and that two consecutive P's will be equal only when, k' is even. For example, if $P'_v = P'_{v+1}$,

$$\text{then } \xi'_v = \frac{P'_v + \sqrt{R}}{Q'_v} = \frac{P'_{v+1} + \sqrt{R}}{Q'_v} = \frac{P'_{k'-v} + \sqrt{R}}{Q'_{k'-v}} = \xi'_{k'-v}$$

so that $v = k' - v$, or $v = k'/2$, i.e., k' is even.

5.6 4. Reverting to the B.c.f. development of \sqrt{D} ($= \sqrt{R}/Q_0 = \xi_0$) and following the notation of § 5.6 1, we notice that, if ξ_{k-v-1} ($v > 1$), is the only

*Vide Theorem XIII.

complete quotient of the form $(1-g)^{-1}$ in the period of \sqrt{D} , then $b_{k-1} = 2b_0$, $b_{k-2} = b_1$, $\epsilon_1 = \epsilon_{k-1}$, $P_1 = P_{k-1}$, $Q_1 = Q_{k-2}$ (1)

As observed already, the recurring periods of \sqrt{D} and ξ_{k-v-1} of the form $(1-g)^{-1}$

viz.,
$$\left(\frac{\epsilon_1}{b_1 + b_2 + \dots + b_{k-1}} \right), \tag{a}$$

and
$$\left(\frac{1}{b'_1 + b'_2 + \dots + b'_{k'-2} + b'_{k'-1} - 2} \right) \tag{\beta}$$

are cyclic permutations of each other.

Now $\frac{\epsilon_{k-1}}{b_{k-1}}$ cannot occur as the first partial fraction in (β) , for it will lead to Type III with b_{k-1} as an odd integer, while b_{k-1} is equal to $2b_0$, which is a contradiction. Again if $\frac{\epsilon_{k-1}}{b_{k-1}}$ is the last partial fraction of (β) it will contradict Theorem X, Cor. (2).

Hence $\frac{\epsilon_{k-1}}{b_{k-1}}$ will occur somewhere in the middle of the period (β) , coinciding with $\frac{\epsilon'_v}{b'_v}$, say; then by (1) $P'_{v+1} = P'_v$, indicating $v = k'/2$, and k' is even ($= k-1$). The period of Type II, viz., (a) is of the form

$$\sqrt{D} = b_0 + \frac{\epsilon_1}{b_1 + \dots} + \frac{\epsilon_{\frac{k-3}{2}}}{b_{\frac{k-3}{2}} - 2} + \frac{1}{b_{\frac{k+1}{2}}} + \frac{1}{b_{\frac{k+3}{2}}} + \dots + \frac{\epsilon_{k-1}}{2b_0}$$

having an even number of recurring terms and possessing the same symmetries as Type I with the following exceptions:

$b_{\frac{k-1}{2}} = 2$, $\epsilon_{\frac{k-1}{2}} = -1$, $\epsilon_{\frac{k+1}{2}} = 1$, $b_{\frac{k-3}{2}} = b_{\frac{k+1}{2}} + 1$, $P_{\frac{k-1}{2}} \neq P_{\frac{k+1}{2}}$; which justify our characterisation of this type as 'almost' symmetric.

It may be useful to telescope the results of this section applicable to the case of \sqrt{R} where R is a non-square positive integer, in the form of a theorem.

THEOREM XV: *The period of the B.c.f. development of \sqrt{R} is either a completely symmetrical type simulating the corresponding simple continued fraction, or an almost symmetrical type consisting of an even number of partial quotients, say, $2v$ with a central set of three unsymmetrical terms of the form*

$$\frac{\epsilon_{v-1}}{b_{v-1} - 2} + \frac{1}{b_{v-1} - 1}$$

Cor.—In the almost symmetrical type of $2v$ terms, Q_v is always greater than 4.

For $\frac{P_v + \sqrt{R}}{Q_v}$ is of the form $\frac{p+q + \sqrt{p^2+q^2}}{p}$, so that $Q_v = p > 2q$.

If $q = 1$, $\sqrt{R} = \sqrt{p^2+1} = p + \frac{1}{2p}$, which is not of Type II. Hence $q \geq 2$ and $Q_v > 4$. In fact, when $Q_v = 5$, $q = 2$, $\sqrt{29} = 5 + \frac{1}{3} - \frac{1}{2} + \frac{1}{2} + \frac{1}{10}$ (Type II).

We give below a table of B.c.f.'s equal to the square-roots of non-square integers less than 100.

R	B.c.f.	R	B.c.f.
2	$1 + \frac{1}{2}$	23	$5 - \frac{1}{3} - \frac{1}{10}$
3	$2 - \frac{1}{4}$	24	$5 - \frac{1}{10}$
5	$2 + \frac{1}{2}$	26	$5 + \frac{1}{10}$
6	$2 + \frac{1}{2} + \frac{1}{4}$	27	$5 + \frac{1}{2} + \frac{1}{10}$
7	$3 - \frac{1}{3} - \frac{1}{6}$	28	$5 + \frac{1}{3} + \frac{1}{2} + \frac{1}{3} + \frac{1}{10}$
8	$3 - \frac{1}{6}$	29	$5 + \frac{1}{3} - \frac{1}{2} + \frac{1}{2} + \frac{1}{10}$
10	$3 + \frac{1}{6}$	30	$5 + \frac{1}{2} + \frac{1}{10}$
11	$3 + \frac{1}{3} + \frac{1}{6}$	31	$6 - \frac{1}{2} + \frac{1}{3} + \frac{1}{8} + \frac{1}{3} + \frac{1}{2} - \frac{1}{12}$
12	$3 + \frac{1}{2} + \frac{1}{6}$	32	$6 - \frac{1}{3} - \frac{1}{12}$
13	$4 - \frac{1}{2} + \frac{1}{2} - \frac{1}{8}$	33	$6 - \frac{1}{4} - \frac{1}{12}$
14	$4 - \frac{1}{2} - \frac{1}{8}$	34	$6 - \frac{1}{6} - \frac{1}{12}$
15	$4 - \frac{1}{8}$	35	$6 - \frac{1}{12}$
17	$4 + \frac{1}{8}$	37	$6 + \frac{1}{12}$
18	$4 + \frac{1}{4} + \frac{1}{8}$	38	$6 + \frac{1}{6} + \frac{1}{12}$
19	$4 + \frac{1}{3} - \frac{1}{6} - \frac{1}{3} + \frac{1}{8}$	39	$6 + \frac{1}{4} + \frac{1}{12}$
20	$4 + \frac{1}{2} + \frac{1}{8}$	40	$6 + \frac{1}{3} + \frac{1}{12}$
21	$5 - \frac{1}{2} + \frac{1}{2} + \frac{1}{2} - \frac{1}{10}$	41	$6 + \frac{1}{2} + \frac{1}{2} + \frac{1}{12}$
22	$5 - \frac{1}{3} + \frac{1}{4} + \frac{1}{3} - \frac{1}{10}$	42	$6 + \frac{1}{2} + \frac{1}{12}$

R	B.c.f.	R	B.c.f.
43	$7 - \frac{1}{2} + \frac{1}{4} - \frac{1}{7} - \frac{1}{4} + \frac{1}{2} - \frac{1}{14}$	72	$8 + \frac{1}{2} + \frac{1}{16}$
44	$7 - \frac{1}{3} - \frac{1}{4} - \frac{1}{3} - \frac{1}{14}$	73	$9 - \frac{1}{2} + \frac{1}{5} + \frac{1}{5} + \frac{1}{2} - \frac{1}{18}$
45	$7 - \frac{1}{3} + \frac{1}{2} + \frac{1}{3} - \frac{1}{14}$	74	$9 - \frac{1}{2} + \frac{1}{2} - \frac{1}{18}$
46	$7 - \frac{1}{5} - \frac{1}{2} + \frac{1}{2} + \frac{1}{6} + \frac{1}{2} + \frac{1}{2} - \frac{1}{5} - \frac{1}{14}$	75	$9 - \frac{1}{3} - \frac{1}{18}$
47	$7 - \frac{1}{7} - \frac{1}{14}$	76	$9 - \frac{1}{3} + \frac{1}{2} - \frac{1}{6} + \frac{1}{4} + \frac{1}{6} - \frac{1}{2} + \frac{1}{3} - \frac{1}{18}$
48	$7 - \frac{1}{14}$	77	$9 - \frac{1}{4} + \frac{1}{2} + \frac{1}{4} - \frac{1}{18}$
50	$7 + \frac{1}{14}$	78	$9 - \frac{1}{6} - \frac{1}{18}$
51	$7 + \frac{1}{7} + \frac{1}{14}$	79	$9 - \frac{1}{2} - \frac{1}{18}$
52	$7 + \frac{1}{5} - \frac{1}{4} - \frac{1}{5} + \frac{1}{14}$	80	$9 - \frac{1}{18}$
53	$7 + \frac{1}{4} - \frac{1}{2} + \frac{1}{3} + \frac{1}{14}$	82	$9 + \frac{1}{18}$
54	$7 + \frac{1}{3} - \frac{1}{8} - \frac{1}{3} + \frac{1}{14}$	83	$9 + \frac{1}{6} + \frac{1}{18}$
55	$7 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{14}$	84	$9 + \frac{1}{6} + \frac{1}{18}$
56	$7 + \frac{1}{2} + \frac{1}{14}$	85	$9 + \frac{1}{5} - \frac{1}{2} + \frac{1}{4} + \frac{1}{18}$
57	$8 - \frac{1}{2} + \frac{1}{4} + \frac{1}{2} - \frac{1}{16}$	86	$9 + \frac{1}{4} - \frac{1}{3} - \frac{1}{10} - \frac{1}{3} - \frac{1}{4} - \frac{1}{18}$
58	$8 - \frac{1}{3} - \frac{1}{2} + \frac{1}{2} - \frac{1}{16}$	87	$9 + \frac{1}{3} + \frac{1}{18}$
59	$8 - \frac{1}{3} + \frac{1}{7} + \frac{1}{3} - \frac{1}{16}$	88	$9 + \frac{1}{3} - \frac{1}{3} - \frac{1}{3} + \frac{1}{18}$
60	$8 - \frac{1}{4} - \frac{1}{16}$	89	$9 + \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{2} + \frac{1}{18}$
61	$8 - \frac{1}{5} + \frac{1}{4} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{16}$	90	$9 + \frac{1}{2} + \frac{1}{18}$
62	$8 - \frac{1}{8} - \frac{1}{16}$	91	$10 - \frac{1}{2} + \frac{1}{6} - \frac{1}{6} + \frac{1}{2} - \frac{1}{16}$
63	$8 - \frac{1}{16}$	92	$10 - \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{2} + \frac{1}{2} - \frac{1}{16}$
65	$8 + \frac{1}{16}$	93	$10 - \frac{1}{3} - \frac{1}{5} + \frac{1}{6} + \frac{1}{5} - \frac{1}{3} - \frac{1}{16}$
66	$8 + \frac{1}{8} + \frac{1}{16}$	94	$10 - \frac{1}{3} + \frac{1}{3} + \frac{1}{2} - \frac{1}{7} - \frac{1}{16} - \frac{1}{7} - \frac{1}{2} + \frac{1}{3} + \frac{1}{3} - \frac{1}{16}$
67	$8 + \frac{1}{5} + \frac{1}{2} + \frac{1}{2} - \frac{1}{5} - \frac{1}{2} + \frac{1}{2} + \frac{1}{5} + \frac{1}{16}$	95	$10 - \frac{1}{4} - \frac{1}{16}$
68	$8 + \frac{1}{4} + \frac{1}{16}$	96	$10 - \frac{1}{5} - \frac{1}{16}$
69	$8 + \frac{1}{3} + \frac{1}{4} - \frac{1}{6} - \frac{1}{4} + \frac{1}{3} + \frac{1}{16}$	97	$10 - \frac{1}{7} - \frac{1}{3} - \frac{1}{2} + \frac{1}{2} - \frac{1}{7} - \frac{1}{16}$
70	$8 + \frac{1}{3} - \frac{1}{4} - \frac{1}{3} + \frac{1}{16}$	98	$10 - \frac{1}{16} - \frac{1}{16}$
71	$8 + \frac{1}{2} + \frac{1}{3} - \frac{1}{5} - \frac{1}{3} + \frac{1}{2} + \frac{1}{16}$	99	$10 - \frac{1}{16}$

The basic elements of the theory are now fairly complete, and it should be obvious that the B.c.f. has a complicated individuality of its own, that claims recognition and cannot easily be brushed aside by such remarks as "Bhaskara's method is the same as that rediscovered by Lagrange". We have only constructed "an arch, wherethro' gleam untravelled and partly travelled regions", such as the character of the acyclic part, the transformations that convert the simple continued fraction into the continued fraction to the nearest square, and the associated quadratic forms. These difficult problems need further investigation.