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The Common Neighborhood Graph and Its Energy

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Abstract. Let G be a simple graph with vertex set $\{v_1, v_2, \ldots, v_n\}$. The common neighborhood graph (congraph) of G, denoted by $con(G)$, is the graph with vertex set $\{v_1, v_2, \ldots, v_n\}$, in which two vertices are adjacent if and only they have at least one common neighbor in the graph G. The basic properties of $con(G)$ and of its energy are established.

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1. Introduction

In this paper we are concerned with simple graphs, that is, graphs without multiple, weighted or directed edges, and without self–loops. Let G be such a graph with vertex set $\mathbf{V} = \mathbf{V}(G) = \{v_1, v_2, \ldots, v_n\}$. Thus, the number of vertices G is n.

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The adjacency matrix of the graph G is the symmetric square matrix $\mathbf{A} = \mathbf{A}(G) = ||a_{ij}||$ of order n whose (i,j)-entry is defined as [6]

$$a_{ij} = \begin{cases} 1 & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent} \\ \\ 0 & \text{otherwise .} \end{cases} \tag{1.1}$$

The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $\mathbf{A}$ are the (ordinary) eigenvalues of the graph G and form the (ordinary) spectrum of G [6].

A much studied spectrum–based invariant of graphs is the *energy*, defined as

$$E(G) = \sum_{i=1}^{n} |\lambda_i| \ . \tag{1.2}$$

Details on the theory of graph energy can be found in the reviews [9, 12] and elsewhere [1, 2].

For $i \neq j$, the *common neighborhood* of the the vertices v_i and v_j, denoted by $\Gamma(v_i, v_j)$, is the set of vertices, different from v_i and v_j, that are adjacent to both v_i and v_j . In a recent paper [3], a new graph matrix $\mathbf{CN} = \mathbf{CN}(G) = ||\gamma_{ij}||$ was considered, named *common–neighborhood matrix* whose (i,j)-entry was defined as

$$\gamma_{ij} = \begin{cases} |\Gamma\{v_i, v_j\}| & \text{if } \ i \neq j \\ \\ 0 & \text{otherwise .} \end{cases} \tag{1.3}$$

Recall that the diagonal elements of $\mathbf{CN}$ are all equal to zero. The off-diagonal elements assume integer values between 0 and $n-2$. Only in some exceptional cases is $\mathbf{CN}$ related to the adjacency matrix [3]; for example, $\mathbf{CN}(K_n) = (n-2)\,\mathbf{A}(K_n)$.

Bearing in mind Eqs. (1.1) and (1.3), as a sort of compromise we now introduce a symmetric square matrix $||a'_{ij}||$ of order n, whose (i,j)-entry is defined as

$$a'_{ij} = \begin{cases} 1 & \text{if } |\Gamma\{v_i, v_j\}| \geq 1 \text{ and } i \neq j \\ \\ 0 & \text{otherwise .} \end{cases} \tag{1.4}$$

Evidently, this matrix can be viewed as the adjacency matrix of some graph. We call it the *common neighborhood graph* or, shorter, the *congraph* of the graph G, and denote it by $con(G)$.

In the following section we establish properties of the congraphs, and in the next section properties of their energy.

At this point it should be noted that in two earlier works [5, 4] the so-called *derived graph* $G^\dagger$ of the graph G was considered. The derived graph $G^\dagger$ has the same vertex set as the parent graph G, and two vertices of $G^\dagger$ are adjacent if and only if their distance in G is equal to two.

It is immediately seen that $G^\dagger \cong con(G)$ if and only if the parent graph G does not contain triangles. Thus, in particular, $G^\dagger \cong con(G)$ holds whenever G is bipartite.

2. Properties of common neighborhood graphs

Denote by $G_1 \cup G_2$ the graph consisting of (disconnected) components G_1 and G_2. Denote by $\overline{G}$ the complement of the graph G. As usual, P_n , C_n , and K_n , are the n-vertex path, cycle, and complete graph. In addition, $K_{a,b}$ is the complete bipartite graph on $a+b$ vertices. Recall that $K_{1,n-1}$ is called the star and often denoted by S_n . The following simple relations can easily be verified.

Example 2.1.

$$\begin{aligned} con(K_n) &\cong K_n && (2.1)\\ con(\overline{K_n}) &\cong \overline{K_n} && (2.2)\\ con(P_n) &\cong P_{\lfloor n/2 \rfloor} \cup P_{\lceil n/2 \rceil} && (2.3)\\ con(K_{a,b}) &\cong K_a \cup K_b && (2.4) \end{aligned}$$

and

$$con(C_n) \cong \begin{cases} C_n & \text{if } n \text{ is odd and } n \geq 3 \\ P_2 \cup P_2 & \text{if } n = 4 \\ C_{n/2} \cup C_{n/2} & \text{if } n \text{ is even and } n \geq 6 \ . \end{cases} \tag{2.5}$$

As a special case of Eq. (2.4) *we have* $con(S_n) \cong K_{n-1} \cup K_1$.

Since, evidently,

$$con(G_1 \cup G_2) \cong con(G_1) \cup con(G_2) \tag{2.6}$$

it is seen that the congraph of a disconnected graph is necessarily disconnected. We, however, have a somewhat stronger claim:

Theorem 2.2. *The common neighborhood graph $con(G)$ is connected if and only if the parent graph G is connected and non-bipartite.*

Proof. In view of Eq. (2.6), we only need to consider the case when the parent graph G is connected.

Case 1: $\underline{G \text{ is connected bipartite.}}$ Assume that the vertex set of G is partitioned as $\mathbf{V}(G) = \mathbf{V}_1 \cup \mathbf{V}_2$, $\mathbf{V}_1 \cap \mathbf{V}_2 = \emptyset$, so that no two adjacent vertices belong to either $\mathbf{V}_1$ or $\mathbf{V}_2$.

Let $x, y \in \mathbf{V}_1$. Since G is connected, there exists a path in G, connecting x and y. Let $(x, v_1, v_2, \ldots, v_p, y)$ be such a path. Since G is bipartite, p must be odd. Therefore in $con(G)$ the vertex x is adjacent to v_2 (because v_1 is their common neighbor), v_2 is adjacent to v_4 (because v_3 is their common neighbor), $\ldots$, v_{p-1} is adjacent to y (because v_p is their common neighbor). Thus $(x, v_2, v_4, \ldots, v_{p-1}, y)$ is a path in $con(G)$, connecting the vertices x and y. Therefore x and y belong to the same component of $con(G)$.

In an analogous manner, if $x, y \in \mathbf{V}_2$, then these two vertices belong to the same component of $con(G)$.

Let now $x \in \mathbf{V}_1$ and $y \in \mathbf{V}_2$. Then these two vertices cannot be adjacent in $con(G)$. Namely, if x and y were adjacent in $con(G)$, then there would exist a vertex z adjacent to both x and y in G. Then z could not belong to either $\mathbf{V}_1$ or $\mathbf{V}_2$, which is impossible.

Therefore, no pair of vertices x, y such that $x \in \mathbf{V}_1$ and $y \in \mathbf{V}_2$ is adjacent in $con(G)$. Consequently, the vertices from $\mathbf{V}_1$ belong to one, and those from $\mathbf{V}_2$ to another component of $con(G)$.

Case 2: G is connected non-bipartite. Then G possesses an odd cycle, and by Eq. (2.5) this cycle is contained also in $con(G)$. Let y and y' be two adjacent vertices of the odd cycle of G, and let x be any other vertex of G. Since G is connected, there exists a path $(x, v_1, v_2, \ldots, v_p, y)$ in G, connecting x and y. This time p may be either odd or even. If p is odd, than by the same reasoning as above we conclude that there is a path in $con(G)$, connecting x and y. If p is even, then in an analogous manner there is a path in $con(G)$, connecting x and y'. Thus all vertices of $con(G)$ belong to the same component, i. e., $con(G)$ is connected. $\square$

Corollary 2.3. *If G is a connected bipartite graph, then $con(G)$ has exactly two components.*

Theorem 2.4. *If G is connected, then $con(G)$ is bipartite if and only if $G \cong C_{4k}$, $k \geq 1$ or $G \cong P_n$.*

Proof. That the congraphs of C_{4k} and P_n are bipartite is seen from Eqs. (2.3) and (2.5). If G is the cycle whose size is not divisible by 4, then by (2.5) its congraph is non-bipartite. Any other connected graph possesses a vertex x whose degree is three or greater. This vertex x implies the existence of $\binom{3}{2}$ or more pairs of vertices in G having x as a common neighbor, i. e., $\binom{3}{2}$ or more mutually adjacent vertices in $con(G)$. Consequently, $con(G)$ possesses triangles and is thus not bipartite. $\square$

Corollary 2.5. *$con(G)$ cannot be a connected bipartite graph. In particular, $con(G)$ cannot be a tree.*

Corollary 2.6. *If G is connected, and $con(G)$ is a forest, then $con(G) \cong P_{\lfloor n/2 \rfloor} \cup P_{\lceil n/2 \rceil}$ i. e., either $G \cong C_4$ or $G \cong P_n$.*

For $v_i \in \mathbf{V}(G)$ by d_i we denote the degree (= number of first neighbors) of v_i. Then $d_1, d_2, \ldots, d_n$ is said to be the degree sequence of the graph G. For details on degree sequences see [7, 17] and the references cited therein.

Theorem 2.7. *If G has degree sequence $d_1, d_2, \ldots, d_n$, and m is the number of edges of $con(G)$, then*

$$m \leq \sum_{i=1}^{n} \binom{d_i}{2} \tag{2.7}$$

and equality holds if and only if G is quadrangle-free.

Proof. Let $v_i \in \mathbf{V}(G)$ and let d_i be the degree (= number of first neighbors) of v_i. Then v_i is a common neighbor of exactly $\binom{d_i}{2}$ pairs of vertices. The upper bound follows.

Equality in (2.7) will be violated if and only if in G there exists a pair of vertices, say x and y, having more than one common neighbor. Let z' and z'' be two common neighbors of x and y. Then x, z', y, z'' form a quadrangle. Thus, if G possesses at least one quadrangle, then the inequality (2.7) is strict. $\square$

For the considerations that follow it is important to note that a congraph possesses much less structural information than the parent graph. In particular, there exist numerous pairs and larger families of graph, whose congraphs are isomorphic. We point out here a few such examples.

Example 2.8. *(a) Let no component of the graph G has more than two vertices, i. e., $G \cong \alpha K_2 \cup \beta K_1$, for any non-negative integers α and β such that $2\alpha + \beta = n$. Then $con(G) \cong \overline{K_n}$, cf. Eq. (2.2).*
(b) By Eqs. (2.5) and (2.6) we have for any $k \geq 1$, $con(C_{4k+2}) \cong con(C_{2k+1} \cup C_{2k+1}) \cong C_{2k+1} \cup C_{2k+1}$.
(c) $con(K_{a+b}) \cong con(\overline{K_{a,b}}) \cong con(K_a \cup K_b) \cong K_a \cup K_b$, cf. Eq. (2.4).
(d) A strongly regular graph with parameters (n, k, s, t) is a k-regular graph with n vertices, such that any two adjacent vertices have s common neighbors, and any two non-adjacent vertices have t common neighbors. The congraph of any strongly regular graph with $s > 0$ is the complete graph K_n.

With regard to Example 2.8 (d) it is interesting to note the following:

Lemma 2.9. *If G is a strongly regular graph with parameters (n, k, s, t) and if $s = 0$, then $con(G) = \overline{G}$.*

Proof. If $s = 0$ then it must be $t > 0$ since otherwise the graph G would be edgeless. Because $s = 0$, any two vertices adjacent in G are not adjacent in $con(G)$. Because $t > 0$, any two vertices not adjacent in G are adjacent in $con(G)$. $\square$

Corollary 2.10. *If G is a strongly regular graph with parameters $(n, k, 0, t)$, then $con(G)$ is a strongly regular graph with parameters $(n, n-k-1, n-2k+t-2, n-2k)$.*

3. Energy of common neighborhood graphs

In this section we are concerned with the energy of congraphs. This energy is calculated by means of Eq. (1.2), with the only difference that instead of the eigenvalues of the graph G we use the eigenvalues of $con(G)$. By this, and by taking into account the properties of congraphs established in the preceding section, the numerous results known for graph energy [9, 12] can be straightforwardly applied to the energy of congraphs.

First we note that the energy of a congraph may be greater than, smaller than, or equal to the energy of the parent graph. This is illustrated by the following simple examples.

Example 3.1.

$$E(P_4) = 2\sqrt{5} \quad ; \quad E(con(P_4)) = E(P_2 \cup P_2) = 2+2=4$$

$$E(K_{1,3}) = 2\sqrt{3} \quad ; \quad E(con(K_{1,3})) = E(K_3 \cup K_1) = 4+0=4$$

$$E(C_6) = 8 \quad ; \quad E(con(C_6)) = E(C_3 \cup C_3) = 4+4=8$$

A graph G on n vertices is said to be *hypoenergetic* [13, 11, 10] if $E(G) < n$.

Claim 3.2. *There exist hypoenergetic congraphs of connected graphs. In particular, $con(G)$ is hypoenergetic if $G \cong P_1, P_2, P_3, P_5, P_6$. We deem that this list may be complete.*

Claim 3.3. *There exist congraphs of connected graphs with property $E(con(G)) = n$. Such are the congraphs of $C_4, C_8, P_4, K_{1,3}$. We deem that this list may be complete.*

The energy of the complete graph K_n is equal to $2(n-1)$. Therefore, by Eq. (2.1) the energy of $con(K_n)$ is also equal to $2(n-1)$. An n-vertex graph G is said to be *hyperenergetic* [8, 16] if $E(G) > E(K_n)$. Details on hyperenergetic graphs can be found in the review [10].

Finding hyperenergetic congraphs is not an easy task. This, for instance, is seen from Example 2.8 (d), according to which no strongly regular graph with parameters (n, k, s, t) , $s > 0$ is hyperenergetic. Recall that just these strongly regular graphs have the greatest possible energy among all n-vertex graphs [15, 14, 19].

We, nevertheless, established the following:

Claim 3.4. *There exist hyperenergetic congraphs.*

In fact, we established a result much stronger than Claim 3.4:

Theorem 3.5. *The congraphs of all strongly regular graphs with parameters $(n, k, 0, t)$, except of C_5 , are hyperenergetic.*

Proof. By direct calculation we first check that $con(C_5) \cong C_5$ is not hyperenergetic.

Let G be any strongly regular graph with parameters $(n, k, 0, t)$. Thus G is triangle–free. Let the eigenvalues of G be k, ρ, and σ, such that σ is the negative eigenvalue. Let their multiplicities be 1, f, and g, respectively. Then, in view of Corollary 2.10, the eigenvalues of $con(G)$ are $n-k-1$, ρ', and σ' with multiplicities 1, f', and g', respectively, where

$$\begin{aligned} \rho' &= -(\sigma+1) \\ \sigma' &= -(\rho+1) \\ f' &= g \\ g' &= f \ . \end{aligned}$$

From the spectral theory of strongly regular graphs [6, 18] it is known that

$$\rho = \frac{1}{2}\Big[-t+\sqrt{t^2-4(k-t)}\Big] \quad ; \quad \sigma = \frac{1}{2}\Big[-t-\sqrt{t^2-4(k-t)}\Big] \ .$$

The energy of $con(G)$ is given by

$$E(con(G)) = (n-k-1) + g\rho' + f|\sigma'|$$

and since $(n-k-1) + g\rho' + f\sigma' = 0$, we have $f|\sigma'| = n-k-1+g\rho'$ which implies

$$E(con(G)) = 2f|\sigma'| \ . \tag{3.1}$$

The congraph of G will be hyperenergetic if $E(con(G)) > 2(n-1) = 2(f+g)$. Hence from Eq. (3.1) we get

$$f(|\sigma'|-1) > g \tag{3.2}$$

Since $\sigma' = -(\rho+1)$, and $|-(\rho+1)| = \rho+1$, we can write the condition (3.2) as

$$f\rho > g \ . \tag{3.3}$$

Two cases need to be distinguished: either $f = g$ or $f \neq g$. In the former case G is a conference graph [18]. The only triangle–free conference graph is C_5 which is not hyperenergetic. If $f \neq g$, then there are exactly six strongly regular graphs without triangles [18], and these all satisfy the inequality (3.3). $\square$

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