PERGAMON

Computers & Operations Research 29 (2002) 403–422

computers & operations research

www.elsevier.com/locate/dsw

A Bayesian approach to a dynamic inventory model under an unknown demand distribution

K. Rajashree Kamath, T.P.M. Pakkala*

Department of Statistics, Mangalore University, Mangalagangothri 574 199, India

Received 1 March 1999; received in revised form 1 November 1999

Abstract

In this paper, the Bayesian approach to demand estimation is outlined for the cases of stationary as well as non-stationary demand. The optimal policy is derived for an inventory model that allows stock disposal, and is shown to be the solution of a dynamic programming backward recursion. Then, a method is given to search for the optimal order level around the myopic order level. Finally, a numerical study is performed to make a profit comparison between the Bayesian and non-Bayesian approaches, when the demand follows a stationary lognormal distribution. A profit comparison is also made between the stationary and non-stationary Bayesian approaches to observe whether the Bayesian approach incorporates non-stationarity in the demand. And, it is observed whether stock disposal reduces the losses due to ignoring non-stationarity in the demand.

Scope and purpose

In the context of inventory models, one of the crucial factors to determine an optimal inventory policy, is the accurate forecasting or estimation of the demand for items in the inventory. The assumption of a constant demand is seriously questioned in recent times, since in reality the demand is generally uncertain and may even vary with time. For instance, the demand for new products, spare parts, or style goods, is likely to fluctuate widely, the average demand is quite likely to be low, and may exhibit a trend. In such situations, the Bayesian approach is a very useful tool for demand estimation, which is applicable even when past observations are scarce. In this paper, we use this approach to estimate the demand for an item, and obtain the expressions for finding the optimal inventory policies. We give a simpler method to find the optimal inventory policy, since the procedure to obtain the optimal inventory policy in the Bayesian framework, is quite tedious especially for long planning horizons, and in cases where the future demand becomes unpredictable. To widen the application of the method, we have given a general procedure which is not restricted to any particular probability distribution for the demand. We compare the Bayesian approach with the corresponding non-Bayesian approach, in terms of the optimum expected profits, when the demand

* Corresponding author. Tel.: +91-824-742796; fax: +91-824-742367.
E-mail address: tpm_pakkala@yahoo.com (T.P.M. Pakkala).

PII: S0305-0548(00)00075-7

follows a lognormal distribution. We also investigate how well the Bayesian approach incorporates non-stationarity in the demand.

Keywords: Inventory; Periodic review; Bayesian; Non-stationary demand; Dynamic programming

1. Introduction

The problem of demand estimation, is an important aspect in the analysis of probabilistic inventory systems. It is generally assumed that the demand distribution has known parameters and is static throughout the planning horizon. In practice, the parameters have to be fixed subjectively, or statistically estimated using past demand information. But it is almost impossible to specify exactly the true values of the parameters, especially in the absence of abundant demand information, as in the case of demand for new products. Moreover, sometimes, due to many reasons, demand may exhibit a trend. And the optimal solutions are very sensitive to the changes in the demand rate. To incorporate this or otherwise, it is more appropriate to assume randomness in the parameters as well. For this purpose, a prior distribution is considered for the unknown parameters of the demand distribution, based on past experience or intuition. This distribution can be updated as and when fresh demand occurs. One of the best systematic methods for incorporating current demand information and updating the demand distribution, is known to be the Bayesian approach.

The Bayesian approach can be applied to inventory systems with either a finite or an infinite planning horizon. Items like computers and related products or even motor vehicles, are being continuously updated and new versions are introduced in the market. Inventory of such items generally have finite planning horizons with fluctuating demands, and the Bayesian set-up could be appropriate. Brown and Rogers [1] and Eppen and Iyer [2] have considered specific finite horizon problems under the Bayesian framework. The Bayesian approach can also be applied to infinite horizon problems in the initial stages, until the demand for the product stabilizes or enough data accumulates for using other estimation procedures. In this age of information technology, obtaining data from time to time for updating the information about uncertain quantities like demand, deterioration, or supply, is not a problem. Hence, with the easy availability of such information, the Bayesian approach is expected to give better results.

The Bayesian approach to inventory modeling, has been investigated earlier for specific cases. The optimal policy has been characterized when the demand distribution belongs to a particular case of the one-parameter exponential or range family of distributions, by Scarf [3] and Karlin [4]. The dynamic programming procedure for finding the optimal solutions for an inventory model with an unknown Poisson demand, has been outlined by Brown and Rogers [1] and Zacks [5]. For particular demand distributions, the procedure to find the optimal policy can be slightly simplified as shown by Azoury [6]. In the same situation, it can be approximated by a myopic policy without significant losses, as shown by Lovejoy [7]. Bounds for the optimal order level in the lost sales inventory problem, have been given by Morton and Pentico [8]. The Bayesian approach has been compared with the non-Bayesian approach in specific situations by Azoury and Miller [9], Kaplan [10] and Hill [11]. Bayesian inventory models with time-dependent demand have been investigated by Popovic [12] and Eppen and Iyer [2].

Our aim is to investigate the utility of the Bayesian approach without restricting the demand distribution to any particular family of distributions. Also, previous numerical studies were restricted to a planning horizon of at the most two periods, in which case the optimal policy can be directly obtained by backward recursion. However, the problem becomes quite difficult to solve for longer planning horizons. Hence, we propose a simpler method to obtain the optimal policy, which can be easily implemented even for long planning horizons. We compare the policy so obtained with the corresponding non-Bayesian policy, for various cost and demand parameters. We also consider simple non-stationary demand structures and investigate whether the Bayesian approach implicitly accounts for such non-stationarity.

The rest of the paper is organized as follows. In Section 2, the Bayesian approach is outlined for any given demand distribution, and is extended to the case where there is non-stationarity in the mean demand. In Section 3, the relevant inventory model is described, and a method to find the optimal policy is proposed. The algorithm to find the optimal policy, is outlined in Section 4. Section 5 illustrates the Bayesian approach for a lognormal demand distribution. In Section 6, the numerical study performed to compare the Bayesian and non-Bayesian approaches, as well as the stationary and non-stationary Bayesian approaches, is described. A general summary and conclusions are given in Section 7.

2. The Bayesian approach

The demand for an item is generally random, and its distribution is not known completely. The reason may be that the item is newly introduced in the market, or its demand is changing with time as in the fashion industry. In such situations, it is sensible to subjectively assign a particular form for the demand distribution, and update it as fresh information is obtained. The Bayesian method of updating the demand distribution as and when fresh data becomes available, continuously improves the probability distribution, so that it may adequately represent the demand at any given point of time.

2.1. Stationary demand

Consider a periodic review inventory system, where the demands in successive periods, are independent and have demand densities $f(.|\theta)$ with unknown parameter θ. Let $\pi(\theta)$ be the prior density function of θ. Then, given the past demand observations $d_1, d_2, \ldots, d_n$, or equivalently the sufficient statistic $S_n = T(d_1, d_2, \ldots, d_n)$, the parameter θ is updated in terms of its posterior distribution, which is given by

$$g(\theta|S_n) = \frac{\prod_{i=1}^{n} f(d_i|\theta) f_\theta(\theta)}{\int \prod_{i=1}^{n} f(d_i|\theta) f_\theta(\theta)\, \mathrm{d}\theta}. \tag{2.1}$$

It is noted that θ and hence S_n, may be vector valued. Given S_n, the distribution of the demand in the $(n+1)$th period, is given by

$$f_{n+1}(d|S_n) = \int_0^\infty f_{n+1}(d|\theta) g(\theta|S_n)\, \mathrm{d}\theta, \tag{2.2}$$

where $g(\theta|S_n)$ can be found using (2.1).

Generally, for a given demand distribution, there are corresponding prior distributions called as conjugate prior distributions which can be assumed for the demand parameters. When conjugate priors are used, the posterior distribution also belongs to the same family as the prior, a property which is intuitively appealing. DeGroot [13] gives a good discussion for many standard distributions. Although assuming a particular distribution for the demand parameters may appear restrictive, the uncertainty about the prior distributions can be expressed by considering their variances to be large or by considering improper priors.

2.2. Non-stationary demand

In the above discussion, it was assumed that the demand although uncertain, is stationary throughout the planning horizon. But, this is not always the case, that is, the demand may be changing with time. For example, the demand for fashion goods generally decreases with time. Based on the customers' assessment of the utility of the product, the demand may increase or decrease from period to period. It is interesting to know how the Bayesian approach accounts for such a non-stationarity. For this purpose, we model non-stationarity in the demand through its mean. In particular, a random trend component is included in the mean demand. This model could be additive or multiplicative. Similar models have been considered by Popovic [12] and Reyman [14]. Such a form of the mean demand can also be used when the demand rate continuously varies with time, as shown below.

Suppose the mean demand in the ith period, denoted by M_i, has the following form:

$$M_i = M + m_i \Delta, \tag{2.3}$$

where the stationary component M, as well as the trend component Δ, are both random, and m_i is a known constant. Also, M is a function of the parameter θ, of the demand distribution. Alternatively, for some distributions like the lognormal, the multiplicative model could be more appropriate. That is, the mean demand M_i is given by

$$M_i = M\delta^{m_i}, \tag{2.4}$$

where δ represents the trend component. The Bayesian approach can be applied in such cases, by considering a prior distribution for M, or equivalently for θ, which can be updated using (2.1). Then, the demand distribution can be updated as in the stationary case, using (2.2).

Model (2.3) ((2.4)) includes the particular case $m_i = (i-1)$. This case can be interpreted as: the difference (ratio) between successive mean demands, is independent of the time point apart from being random. In addition to this, if it is known that on an average, the mean demand itself is stationary throughout the planning horizon, then, (2.3) ((2.4)) can be applied with $E(\Delta) = 0$ ($E(\delta) = 1$), where 'E' denotes expectation with respect to the underlying random variable. Model (2.3) or (2.4) can also be used to describe the situation studied by Popovic [12], as shown below.

Suppose the demand rate function (the number of units demanded per unit time), $\Lambda(t)$, varies continuously over time. As given by Popovic [12], the mean demand in the ith period is given by

$$M_i = \int_{i-1}^{i} \Lambda(t)\,dt. \tag{2.5}$$

For instance, the demand for a new model of a computer, may linearly or exponentially decrease over time, respectively, as follows:

$$\Lambda(t) = M - \Delta t$$

or

$$\Lambda(t) = \Delta \exp(-t).$$

Then, from (2.5), M_i is in the form of (2.3) where

$$m_i = ((i-1)^2 - i^2)/2$$

or

$$m_i = \exp(-(i-1)) - \exp(-i),$$

respectively.

Another form of the demand rate function, suggested by Popovic [12], is as follows:

$$\Lambda(t) = (c+1)\Delta t^c, \tag{2.6}$$

where Δ is a function of the mean demand, and c is a known constant representing the degree of the demand rate function. This implies, if $c = 0$, (2.6) gives a constant demand rate, and if $c = 1$, it gives a linear demand rate, and so on. In this case, (2.5) simplifies to

$$\begin{aligned} M_i &= \Delta \int_{i-1}^{i} (c+1) t^c \, \mathrm{d}t \\ &= m_i \Delta, \end{aligned} \tag{2.7}$$

where $m_i = (i^{(c+1)} - (i-1)^{(c+1)})$.
This structure of the mean demand is a particular case of (2.3), with $E(M) = 0$, $V(M) \to 0$ and m_i as given above.

3. Model development

In this section, the inventory model relevant to the Bayesian approach is described. Most of the earlier work related to periodic review models concentrated on models where stock disposal was not an option. However, in the case of items that could perish, deteriorate or become obsolete with time, for example fashion goods, it is beneficial to dispose off excess stock in each review period, for a reduced price (cf. Eppen and Iyer [2], Zacks [5], and Lovejoy [7]). The model is not restricted to the stock disposal option. On the contrary, it incorporates the model which does not allow such an option as a particular case, as shown in Section 3.2.

3.1. Models with the stock disposal option

A periodic review probabilistic inventory system is considered with a finite planning horizon consisting of N review periods. Review periods are of equal length. Suppose C_1, C_2 and C_3 are the holding, shortage and ordering costs per unit and r denotes the selling price per unit. Let there be an option to dispose off some stock if necessary, at the beginning of every period at a price of C_s per unit. C_s may include the disposal cost. Since the stock purchased in a period will not be disposed off in the same period, we have $C_s \leqslant C_3$. The objective function is the discounted profit function, where the future profit is discounted by a factor α. If y denotes the order level, and z denotes the disposal level in any given period, the inventory system operates in the following sequence:

(1) If the inventory level, x, in the beginning of the period, is less than y, an order is placed to bring the inventory level up to y units. The order is immediately received and a proportional purchase cost is incurred.
(2) If $x > z$, $(x - z)$ units are disposed and z is the inventory level after disposal.
(3) Then, the demands begin to occur. An amount equal to the selling price is received for each unit of demand that is immediately met. Back-ordered demand if any, will be met at the beginning of the next period when the order of that period arrives. This implies that only the discounted selling price is received for the back-ordered demand. This approach to back-ordered demand is given by Taha [15]. The holding or shortage costs are incurred for the balance inventory level.

It is noted that the reorder level and the order level, y, are one and the same. This is because there is only a proportional ordering cost C_3. Morton and Pentico [8] have the following to say about this case " … the ordering frequency is often set by the company to produce roughly an economical size order each period or to coordinate orders for multiple items from the same supplier. Thus, in many cases, setup costs may be reasonably ignored". If there is a fixed cost of ordering, also called the setup cost, then there will be distinct order and reorder levels. However, this problem is analytically different from the case of zero setup cost. Further, the solution procedure is also expected to be quite different.

The discounted expected profit function over the planning horizon is written as

$$\sum_{i=1}^{N} \alpha^{(i-1)}\{C_s(x_i - z_i)^+ - C_3(y_i - x_i)^+ + L_i(a_i)\} + \alpha^N\{\sigma_s x_{N+1}^+ - \sigma_p x_{N+1}^-\}, \tag{3.1}$$

where

$$a^+ = \begin{cases} a & \text{if } a \geqslant 0, \\ 0 & \text{if } a < 0 \end{cases}$$

and for the ith period, x_i denotes the inventory before ordering, y_i denotes the desired inventory level to order up to, z_i denotes the desired inventory level to dispose down to, and a_i denotes the inventory after ordering or disposal, that is,

$$a_i = \begin{cases} \max(x_i, y_i) & \text{if } x_i < z_i, \\ \min(x_i, z_i) & \text{if } x_i > y_i. \end{cases}$$

σ_p and σ_s are proportional penalty and salvage costs, respectively, and

$$L_i(y) = \int_0^\infty \{rt\delta(y-t) - C_1(y-t)^+ + (\alpha r - C_2)(t-y)^+ + ry\delta(t-y)\} f_i(t)\,\mathrm{d}t$$

gives the sum of expected net one-period profit, for a demand distribution with the density $f_i(.)$, where

$$\delta(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x \leqslant 0 \end{cases}$$

is the Kronecker delta function.

The first term in the integrand of $L_i(y)$, corresponds to the net profit when stock is in excess of the demand, and the rest of the terms correspond to the net profit when there is a shortage of stock at the end of the period. $L_i(y)$ can be simplified to

$$L_i(y) = (C_1 + r)E(D_i) - C_1 y - \{(C_1 + C_2 + r(1-\alpha)\}\int_y^\infty (t-y) f_i(t)\,\mathrm{d}t. \tag{3.2}$$

Now, let $P_n(x)$ denote the maximum expected profit function from the nth period to the end of the planning horizon, for a given initial inventory level x, when an optimal policy is followed. Then, the dynamic programming formulation of the expected profit function can be derived from (3.1) as

$$P_n(x) = \begin{cases} \max\limits_{y > x} \{-C_3(y-x) + H_n(y)\} & \text{if } x < y, \\ H_n(x) & \text{if } y \leqslant x \leqslant z, \\ \max\limits_{z > x} \{C_s(y-x) + H_n(z)\} & \text{if } x > z, \end{cases} \tag{3.3}$$

where

$$H_n(x) = L_n(x) + \alpha \int_0^\infty P_{n+1}(x-t) f_n(t)\,\mathrm{d}t \tag{3.4}$$

for $n = 1, 2, \ldots, N$ and

$$P_{N+1}(x) = \begin{cases} \sigma_p x & \text{if } x < 0, \\ \sigma_s x & \text{if } x \geqslant 0. \end{cases} \tag{3.5}$$

Here, $\sigma_p \geqslant C_3$, as demand in excess of the inventory level at the end of the planning horizon, has to be met through a special order. Also, since any excess inventory left at the end of the planning horizon, has to be salvaged for a price which is less than the order cost, we have $\sigma_s \leqslant C_3$.

The value of y (z) that maximizes $P_n(x)$ is called the optimal ordering (disposal) level, denoted by $s^*_{o,n}$ ($s^*_{d,n}$). These optimal levels are determined by a backward recursion on (3.3). In the following discussion, the subscript n is dropped unless necessary, to avoid complexities in notation.
Observing (3.3), it is obvious that the optimal policy will be:
If $x < s^*_o$, order up to s^*_o.
If $x > s^*_d$, dispose down to s^*_d.
Otherwise, x itself is optimal in the situation.

The following inequality holds in most inventory systems:

$$C_2 + (r - C_3)(1-\alpha) > 0, \tag{3.6}$$

where the L.H.S. is the derivative of the profit function at zero. If (3.6) is not satisfied, then the optimal policy will be never to order or dispose. Eq. (3.6) is satisfied if, for instance, the selling price is more than the ordering cost. Mathematically, (3.6) ensures that the optimal order level is positive. And since $C_s \leqslant C_3$, the optimal disposal level is positive as well.

The following theorem gives the properties of $P_n(x)$, and the optimal ordering and disposal levels. The proof is given in the appendix.

Theorem 3.1. *The optimal order level, s_o^*, and, the optimal disposal level, s_d^*, are positive and finite, and $P_n(x)$ is a concave function of x.*

In general, the optimal policy has to be found by solving recursively Eqs. (A.2) and (A.3) mentioned in the appendix, starting from the last period N. This is quite tedious especially for planning horizons spanning more than two periods. The following theorem is helpful in deriving a simple method to find the optimal policy. It states that the optimal solutions are bounded by the myopic ordering and disposal levels. The proof is given in the appendix. Here, the myopic ordering (disposal) policy assumes that ordering (disposal) is made in the next period. The myopic order (disposal) level, s_o^c (s_d^c), is shown to be an upper bound of the actual optimal order (disposal) level, s_o^* (s_d^*).

Theorem 3.2. (a) *The optimal order-up-to level in the nth period, s_o^*, is less than or equal to the myopic order level s_o^c, where s_o^c is the solution of*

$$F_n(y) = \frac{C_2 + (r - C_3)(1 - \alpha)}{(C_1 + C_2) + r(1 - \alpha)}. \tag{3.7}$$

(b) *The optimal dispose-down-to level in the nth period, s_d^*, is more than or equal to the myopic disposal level s_d^c, where s_d^c is the solution of*

$$F_n(y) = \frac{C_2 + (r - C_s)(1 - \alpha)}{(C_1 + C_2) + r(1 - \alpha)}. \tag{3.8}$$

Theorem 3.2 highlights the fact that if the myopic policy is used, there is a chance of either over-stocking or over-disposal which will result in reduced profits. Also, the theorem gives bounds on the optimal stock levels. This result is used while suggesting a method to find the optimal policy, as outlined in Section 4. Before that, an inventory model that does not allow the option to dispose stock, is derived in Section 3.2 below, as a particular case of the model just discussed.

3.2. Models without the stock disposal option

The option to dispose stock is generally allowed in inventory models. However, there are many situations where stock disposal is not considered. Since disposal can be seen as a measure to correct for excess inventory level, we would like to observe whether it can reduce losses due to incorrect modeling of the demand. Analytically, the model without the stock disposal option can be derived as a particular case of the model discussed in Section 3.1 above.

The profit function for an inventory model without the stock disposal option can be derived from (3.1) with $z_i \to \infty$. The maximum expected profit function from the nth period to the end of the planning horizon, for a given initial inventory level x, is obtained from (3.3) with $z \to \infty$. That is, the dynamic programming formulation of the expected profit function can be derived from (3.3) as

$$P_n(x) = \begin{cases} \max\limits_{y>x} \{ -C_3(y-x) + L_n(y) + \alpha\int_0^\infty P_{n+1}(y-t) f_n(t)\,dt \} & \text{if } x < y, \\ L_n(x) + \alpha\int_0^\infty P_{n+1}(x-t) f_n(t)\,dt & \text{if } x \geqslant z \end{cases} \tag{3.9}$$

for $n = 1, 2, \ldots, N$. The profit function in the $(N+1)$th period is the same as (3.5). Theorems similar to Theorems 3.1 and 3.2, can be proved in this situation as well. These are stated without proof, below.

Theorem 3.3. (a) *The optimum order level in the nth period*, s_o^*, *is positive and finite and*, (b) $P_n(x)$ *is a concave function of* x.

Theorem 3.4. *The optimal order level in the nth period*, *is less than or equal to the myopic order level* s_o^c, *and* s_o^c *is the solution of*

$$F_n(y) = \frac{C_2 + (r - C_3)(1-\alpha)}{(C_1 + C_2) + r(1-\alpha)}.$$

Using Theorems 3.2 and 3.4, a method has been derived to search for the optimal policy within a bounded region. The details are given in the next section.

4. Algorithm to find the optimal policy

The general dynamic programming procedure used to find the optimal stock levels is quite tedious especially for long planning horizons. Hence, a simpler method is proposed in this section, to search for the optimal policy. The basic idea behind this method is that the optimal ordering level is the ideal inventory level. Hence, in any review period, the probability of placing an order should be quite high. Translated in terms of the demand, this gives an upper bound on the optimal ordering level. In the following discussion, the idea behind the method is explained and then, an algorithm is given to obtain the optimal policy.

It has been shown by Karlin [4] that if the demand densities are stochastically ordered, then $s_{o,n}^* \leqslant s_{o,n+1}^*$ for every n, when the planning horizon is infinite. In such a case, the myopic ordering policy will be optimal. But the condition of stochastic ordering is only a sufficient condition. The necessary and sufficient condition regardless of whether the planning horizon is finite or infinite, is that an order up to the optimal ordering level, is placed in every period, that is, for every n and a given demand D_n,

$$s_{o,n}^* - D_n \leqslant s_{o,n+1}^*. \tag{4.1}$$

Now, by analyzing the form of the optimal policy, the optimal ordering level can be viewed as the ideal inventory level. Hence, the probability of attaining this level should be quite high if an optimal policy is being followed. That is, (4.1) should be highly probable. In other words, the probability of not placing an order in the $(n+1)$th period, is close to zero, that is

$$P(D_n \leqslant s^*_{o,n} - \hat{s}^c_{o,(n+1)}) \leqslant \rho, \tag{4.2}$$

where ρ is a small specified value that could vary with n, and $\hat{s}^c_{o,(n+1)}$ is the estimated myopic order level in the $(n+1)$th period obtained by estimating the demand in the nth period by its mean. Eq. (4.2) can be written as

$$s^*_{o,n} \leqslant \hat{s}^c_{o,(n+1)} + F_n^{-1}(\rho), \tag{4.3}$$

where $F_n^{-1}(\rho)$ is the inverse distribution function of D_n at ρ.

Eq. (4.3) gives an approximate upper bound on the optimal ordering level. Since the disposal of stock is only a secondary option, the decision to dispose is taken based on the myopic disposal level itself. The algorithm given below outlines the proposed method.

(1) For each of the first $(N-1)$ periods, compute the value of s^c_o as given by (3.7), for the demand distribution which has been updated based on the demand observations of the previous periods.
(2) If s^c_o satisfies (4.3), then $s^*_o = s^c_o$. Otherwise, using the Lagrange multiplier method, find the value of s^c_o which satisfies (4.3).
(3) For the Nth period, compute $s^*_{o,N}$ as the solution of

$$F_N(y) = \frac{C_2 - C_3 + r(1-\alpha) + \alpha\sigma_p}{(C_1 + C_2) + r(1-\alpha) + \alpha(\sigma_p - \sigma_s)}. \tag{4.4}$$

(4) Compute the myopic disposal level as the solution of (3.8).
(5) For the Nth period, compute $s^*_{d,N}$ as the solution of

$$F_N(y) = \frac{C_2 - C_s + r(1-\alpha) + \alpha\sigma_p}{(C_1 + C_2) + r(1-\alpha) + \alpha(\sigma_p - \sigma_s)}. \tag{4.5}$$

It is noted that when $\sigma_p = C_3$ and $\sigma_s = C_3$, we get $s^*_{o,N} = s^c_{o,N}$ and $s^*_{d,N} = s^c_{d,N}$. Also, when $C_s = C_3$, the optimal policy is characterized by a single value, since s^*_o and s^*_d will be solutions of the same equation, that is, we get, $s^*_o = s^*_d$.

The above procedure can be easily adopted to find the approximate policy using MATLAB which gives the quantiles, that is, the values of the inverse distribution function, for any distribution. But, if the quantiles are not readily available, one can appeal to the generalized lambda distribution (GLD) approximation (cf. Ramberg et al. [16]), provided the distribution is unimodal, and its first four moments exist. The family of GLDs is characterized by the quantile function, so that for a given value p, of the distribution function $F(.)$, the corresponding quantile x, is given in terms of the parameters $\lambda_1, \lambda_2, \lambda_3$, and λ_4, as

$$x = F^{-1}(p) = \lambda_1 + \{p^{\lambda_3} - (1-p)^{\lambda_4}\}/\lambda_2. \tag{4.6}$$

Ramberg et al. [16] have tabulated the parameter values for various values of the coefficients of skewness and kurtosis (α_3 and α_4) for any standardized distribution. Hence, the required quantiles can be obtained in terms of the λ-values using (4.6).

Using the algorithm given above, the optimal inventory policy can be easily obtained, when the demand distribution is updated using the Bayesian approach. This algorithm was used while performing a numerical study to compare the optimum expected profit by using the Bayesian approach, with that for the corresponding non-Bayesian approach. Similarly, the stationary and non-stationary Bayesian approaches are also compared. The results of the numerical study are given in Section 6.

5. Lognormal demand

The results of the previous sections, are applied to the case where the demand in any review period, has a lognormal distribution with known scale parameter and unknown location parameter. The lognormal distribution is considered for the following reasons:

(i) The lognormal random variable is positive valued. This is suitable in the case of low-demand items which show wide variation. Examples of such demand are the demand for new products, spare parts, or style goods. If the normal distribution which has been widely used as a demand distribution in the Bayesian framework, was used, it would invariably assign a positive probability to negative values.
(ii) Previous research has considered a particular case of the one-parameter exponential and range family of distributions, which includes only the normal and the Poisson distributions. The Poisson distribution is not suitable when the demand is continuous. The normal distribution received criticism because it allows negative values.
(iii) The lognormal distribution is well-suited for economic variables such as demand, and can closely approximate a normal distribution as mentioned by Johnson and Kotz [17].
(iv) To describe inventory demand, it has been established that lognormal distribution is one of the suitable probability distributions. This is based on extensive numerical work on inventory demand done by Brown [18].

Consider the case of demand in the ith period, having a lognormal distribution with a known parameter γ and unknown parameter θ_i. Suppose the mean demand is of form (2.4). Then, θ_i can be written in a form analogous to (2.3), by substituting

$$M_i = \exp(\theta_i + (2\gamma)^{-1}),$$

and $M = \exp(\theta + (2\gamma)^{-1})$ in (2.4).

This gives

$$\theta_i = \theta + m_i \ln(\delta). \tag{5.1}$$

For a given γ, let θ have a normal distribution with known mean μ and known precision (inverse of the variance) $\tau\gamma$. Also, let the prior distribution of δ be lognormal with known parameters v and $\eta\gamma$. Then, given the sufficient statistic $\mathbf{S}_n = (S_{1n}, S_{2n})$ of the past n demand observations, the distribution of D_{n+1} is seen to be a lognormal distribution with parameters as given below. The notation '$\sum$' mentioned hereafter indicates summation taken over $i = 1$ to n.

$$\theta_{n+1} = (S_{1n} + v\eta)(m_{n+1}(n+\tau) - \sum m_i) + \eta_n^{-1}(S_{2n} + \tau\mu)(\sum m_i^2 + \eta - m_{n+1}\sum m_i) \tag{5.2}$$

and

$$\gamma_{n+1} = \gamma\eta_n/\eta_{n+1}, \tag{5.3}$$

where

$$\eta_n = (n+\tau)(\sum m_i^2 + \eta) - (\sum m_i)^2,$$

and

$$\mathbf{S}_n = (\sum m_i \ln(d_i), \sum \ln(d_i)).$$

Model 1: Time-dependent demand-rate: Suppose the demand rate function is non-stationary and given by (2.6). Then, the mean demand in the ith period M_i is given by (2.7) where Δ is of the form $\exp(\theta + (2\gamma)^{-1})$. In terms of θ_i, this reduces to

$$\theta_i = \theta + m_i, \tag{5.4}$$

where $m_i = \ln(i^{(c+1)} - (i-1)^{(c+1)})$. We call this as Model 1.
Eq. (5.4) is a particular case of (5.1) with $\ln(\delta) \to 1$ in distribution. That is, the marginal distribution of D_{n+1} is the limit of the distribution given by (5.2) and (5.3), when $v = 1$ and $\eta \to \infty$. This results in a lognormal distribution with parameters

$$\theta_{n+1} = m_{n+1} + (S_{2n} + \tau\mu - \sum m_i)/(n+\tau) \tag{5.5}$$

and

$$\gamma_{n+1} = \gamma(n+\tau)/(n+1+\tau). \tag{5.6}$$

The marginal distribution of the demand in the $(n+1)$th period, when the mean demand is stationary, is obtained through Model 1 by substituting $m_i = 0$ for every i, in (5.5) and (5.6). This gives the distribution of D_{n+1} to be lognormal with parameters

$$\theta_{n+1} = (S_{2n} + \tau\mu)/(n+\tau) \tag{5.7}$$

and

$$\gamma_{n+1} = \gamma(n+\tau)/(n+1+\tau). \tag{5.8}$$

Model 2: Linear trend for the mean demand: In (5.1), let $m_i = (i-1)$ and $E\{\ln(\delta)\} = v = 0$. In other words, on an average, the mean demand is stationary throughout the planning horizon. That is, θ_i is of the form

$$\theta_i = \theta + (i-1)\Delta, \tag{5.9}$$

where $\Delta = \ln(\delta)$ follows a normal distribution with mean $v = 0$, and precision $\eta\gamma$. Let this model be named Model 2. Then, the distribution of D_{n+1} is found to be lognormal with parameters

$$\theta_{n+1} = nS_{1n}(\tau + (n+1)/2) + \eta_n^{-1}(S_{2n} + \tau\mu)(\eta - n(n^2-1)/6) \tag{5.10}$$

and

$$\gamma_{n+1} = \gamma\eta_n/\eta_{n+1}, \tag{5.11}$$

where

$$\eta_n = (n+\tau)(\sum(i-1)^2 + \eta) - (\sum(i-1))^2, \tag{5.12}$$

and

$$\mathbf{S}_n = (\sum(i-1)\ln(d_i), \sum\ln(d_i)).$$

The optimal policies and expected profits are obtained for the distributions derived above, for various values of the parameters and given costs. The results of the numerical study are presented and interpreted in Section 6.

6. Numerical study

A numerical study was made to observe the advantage over the non-Bayesian approach, as well as the effect of ignoring demand non-stationarity. Specifically, the case of lognormal demand was considered for both the models given in Sections 3 and 4. The results are given below for each of these cases. Before doing the comparison, a simulation was performed for a three-period inventory model, to test whether there is any deviation from the actual optimum profit, using the methods suggested in Sections 3 and 4. The dynamic programming recursion was used to evaluate the actual optimal policies, for various values of the parameters for the same cost parameters as given below. It was found that a maximum loss of 0.00008% was incurred. (The values have not been reproduced here as there are practically no differences except in extreme cases.) This loss is expected to be lesser for longer horizons, as the Bayesian policy ultimately converges to the maximum likelihood policy. Hence, it was found in this example, that the optimal policies obtained by the methods suggested in Sections 3 and 4, are very good approximations of the dynamic programming solutions.

6.1. Profit comparison of Bayesian and non-Bayesian models

Consider the case where demand has a lognormal distribution with parameters θ and γ, where γ is known. In practice, this situation corresponds to the case of a known coefficient of variation. Given that the expected prior mean demand $e_m = 100$ units, the simulation was carried out for various values of the coefficient of variation in the demand c_d, and the coefficient of variation in the expected demand c_m. The parameters of the prior distributions were obtained, using the relations between the parameters of a lognormal distribution, as follows:

$$\gamma = 1/\ln(1+c_d^2),$$

$$\tau = 1/\{\gamma\ln(1+c_m^2)\},$$

and

$$\mu = \ln(e_m) - (1+1/\tau)/2\gamma.$$

Table 1
The percentage loss in profit if randomness in the mean demand, is ignored[a]

cm \ cd	0.05		0.10		0.15		0.20		0.25		0.30	
	(1)	(2)	(1)	(2)	(1)	(2)	(1)	(2)	(1)	(2)	(1)	(2)
0.05	0.66	0.63	0.66	0.65	0.93	0.93	1.30	1.32	1.72	1.73	2.19	2.19
0.10	2.17	2.16	1.43	1.47	1.30	1.26	1.49	1.46	1.78	1.80	2.21	2.22
0.15	4.56	4.75	2.65	2.96	2.14	2.29	2.09	2.11	2.19	2.25	2.48	2.44
0.20	7.48	7.33	5.07	4.93	3.99	3.43	3.09	3.24	2.91	3.04	3.07	3.13
0.25	9.40	10.44	7.03	7.55	5.49	5.82	4.46	5.01	4.05	4.28	3.95	4.08
0.30	12.42	13.09	10.18	10.53	8.12	7.73	6.19	6.28	5.79	5.86	5.15	5.41

[a](1) relates to the model which does not allow stock disposal, (2) relates to the model which allows stock disposal.

Then, the parameters of the updated demand distribution were computed using (5.7) and (5.8). The optimal policies were computed using the algorithms given in Sections 3 and 4. The percentage decrease in profit if the demand is assumed to follow a lognormal distribution with mean 100 and known coefficient of variation c_d, was computed for the standard model as well as the disposal model. The cost parameters were taken as follows:

$$C_3 = 1, \quad C_s = 0.9, \quad r = 2.5, \quad C_1 = 0.1, \quad C_2 = 1.5, \quad \sigma_s = 0.1 \quad \text{and} \quad \sigma_p = 2.5.$$

The percentage loss in profit is listed in Table 1. In each column, (1) relates to the standard model, and (2) relates to the disposal model.

It is observed that, the results are very similar in both the cases, that is the standard model as well as the model with stock disposal. The percentage decrease in profit increases with increasing variation in the mean demand. This is expected since ignoring the variation in the mean demand, affects the optimal solutions more and more as the variation increases. For a given value of c_m, the loss is a convex function of c_d, that is, it first decreases and then increases with c_d. The value of c_d for which the loss is minimum, increases with c_m. These tendencies can be attributed to the relative behaviour of the parameter values involved in the two cases being compared, as c_d increases.

6.2. Profit comparison of stationary and non-stationary Bayesian models

To study how well the Bayesian approach incorporates non-stationary demand, the non-stationary demand structures given in Section 5, namely Models 1 and 2, are each compared with the corresponding stationary demand structure. The various cost values are the same as in Section 6.1. The details of the two comparisons are given below.

Model 1: The demand distribution is lognormal with parameters γ and θ_i which is as outlined in (5.4). The expected mean demand of the first period is $e_m = 100$ units per period. The values of the parameters γ, τ and μ, were computed as described in Section 6.1. The parameters of the updated demand distribution were computed using (5.5) and (5.6) for the non-stationary mean demand. The

Table 2
The percentage loss in profit if non-stationarity in the mean demand as given by Model 1 with $c = 1$, is ignored[a]

cd \ cm	0.05		0.10		0.15		0.20		0.25		0.30	
	(1)	(2)	(1)	(2)	(1)	(2)	(1)	(2)	(1)	(2)	(1)	(2)
0.05	65.38	55.48	48.57	21.89	37.00	40.81	29.81	60.53	25.16	76.91	21.93	91.07
0.10	68.99	82.72	60.49	52.74	50.21	30.18	40.55	25.98	33.32	35.07	27.87	47.65
0.15	68.31	90.49	63.01	71.58	56.05	50.48	48.19	35.07	40.36	29.07	34.32	30.72
0.20	67.78	93.93	63.51	80.94	58.27	64.51	51.99	48.71	45.33	38.06	39.04	32.87
0.25	67.67	96.14	63.78	87.12	59.07	73.58	53.76	59.75	48.01	48.18	42.45	40.44
0.30	67.15	97.88	63.71	90.65	59.53	79.39	54.83	67.60	49.64	56.27	44.69	47.82

[a](1) and (2) as defined in the footnote of Table 1.

Table 3
The percentage loss in profit if non-stationarity in the mean demand as given by Model 1 with $c = 2$, is ignored[a]

cd \ cm	0.05		0.10		0.15		0.20		0.25		0.30	
	(1)	(2)	(1)	(2)	(1)	(2)	(1)	(2)	(1)	(2)	(1)	(2)
0.05	102.04	111.26	90.33	61.41	76.31	31.41	65.50	41.08	57.98	58.19	52.39	73.51
0.10	103.76	129.83	99.43	111.09	93.17	84.13	85.12	59.01	76.91	42.90	68.70	38.69
0.15	103.38	134.19	100.89	125.14	97.14	110.78	92.36	92.51	86.12	73.88	80.52	59.10
0.20	102.50	136.50	100.71	132.04	98.38	123.02	95.30	111.03	90.99	96.80	86.09	83.05
0.25	102.00	138.06	100.44	135.31	98.23	129.51	95.97	121.91	92.51	111.24	89.44	99.49
0.30	101.47	139.74	99.90	138.03	98.12	134.26	96.36	128.31	94.19	121.07	91.34	112.58

[a](1) and (2) as defined in the footnote of Table 1.

stationary mean demand was considered to be

$$\bar{M} = \frac{1}{N}\int_0^N (c+1)Mt^c\,\mathrm{d}t = MN^c.$$

The updated demand distribution for this case would be of form (5.5) and (5.6), with m_i replaced by $\ln(N^c)$ for all i. The percentage loss in profit by using the stationary Bayesian approach was computed for the same set of values as in Section 6.1. The values obtained are listed in Table 2 for the case where $c = 1$, and Table 3 for the case where $c = 2$.

As seen from Table 2, there is a significant percentage loss in profit if non-stationarity as given by (5.4), is ignored. For the standard model, the percentage loss increases with c_m, until it becomes almost a constant. Further, for given values of c_m, the loss decreases with c_d. This decrease is because Model 1 will progressively be a less-accurate representation of the mean demand as c_d increases, and hence the trend becomes similar to the stationary model.

Table 4
The percentage loss in profit if non-stationarity in the mean demand as given by Model 2, is ignored[a]

cd / cm	0.05		0.10		0.15		0.20		0.25		0.30	
	(1)	(2)	(1)	(2)	(1)	(2)	(1)	(2)	(1)	(2)	(1)	(2)
0.05	10.84	10.89	11.18	11.09	10.93	10.97	10.85	10.63	10.49	10.25	9.94	9.46
0.10	29.28	16.54	30.16	17.98	28.96	19.75	29.83	21.00	29.66	22.00	30.40	22.81
0.15	69.39	16.71	69.27	18.79	58.42	21.74	55.52	23.85	52.93	26.64	51.97	28.86
0.20	96.19	16.67	98.07	19.40	87.30	22.66	81.76	25.66	77.00	28.49	72.13	31.63
0.25	110.12	16.61	115.50	18.95	116.92	22.23	113.59	24.90	112.45	29.19	99.90	32.61
0.30	118.63	17.67	134.95	19.51	140.40	22.77	143.69	25.80	143.79	29.61	146.36	33.47

[a](1) and (2) as defined in the footnote of Table 1.

For the model with stock disposal, the percentage loss is convex in c_m, as well as c_d. This may be explained as follows. If there had been no option to dispose off excess stock, the behaviour would have been monotonic as mentioned above. This indicates that the disposal of stock reduces the loss when c_m is small, but increases it when c_d is large. It may be observed that the stationary model tends to overestimate the mean demand through most of the planning horizon, and hence overstocks the item. When c_m is small, the overstocking is clearly identified by the stationary model and brought under control by disposal of the excess stock. However, overstocking is not discernible when c_m or c_d is large, and the stationary model behaves as it would for the non-disposal model.

Comparing the values in Tables 2 and 3, it is clear that the percentage loss in profit is higher when the demand rate is a quadratic function of time, than when it is linear. The loss otherwise, behaves in a similar pattern with respect to c_d and c_m, whether the demand rate is linear or quadratic.

Model 2: The effect of ignoring the component δ, that is using a stationary model instead of the non-stationary model (5.9), is observed here. The cost and parameter values were taken as in the case of Model 1. Since, the component δ is more important in this case, the percentage loss in profit by ignoring δ, was computed for various values of the coefficient of variation of δ, denoted by c_δ. The parameter η was computed as follows:

$$\eta = 1/\{\gamma \ln(1 + c_\delta^2)\}.$$

The updated demand distributions for the non-stationary case were computed using (5.10) and (5.11). The coefficient of variation of the mean demand c_m is fixed at a fairly high value of 0.2. The percentage loss is listed in Table 4.

When stock disposal is not allowed, it has been observed that by ignoring δ, a significant loss is incurred and it increases with c_δ. Thus, an increase in c_δ has the same effect as an increase in c_m, and the behaviour is expected. For any given c_δ, the percentage loss may slightly decrease but is almost constant over c_d.

When there is an option to dispose stock, for any given c_d, the percentage loss in profit increases with c_δ until it becomes almost a constant. For any given c_δ, the percentage loss increases with c_d. The increase is because the stationary model may not call for disposal when c_d is large. However, the losses are not as high as in the model without this option. That is, stock disposal seems to control the loss in profit to some extent, although it is still significant.

7. Summary and conclusion

In this paper, the Bayesian approach to demand estimation has been investigated. An algorithm has been given to obtain the optimal policy, which is particularly useful for long planning horizons. Using this method to find optimal policies, profit comparisons have been made to observe the loss incurred by ignoring prior information about the demand parameters, or by ignoring non-stationarity in the average demand. Although the conclusions are made for a lognormal distribution, it is expected that the similar results could be obtained when the demand has some other probability distribution.

The method suggested to find the optimal policy, was found to be a very good approximation of the dynamic programming solutions, for the example considered. And, it is expected to perform well in longer planning horizons also, since the Bayesian policy ultimately converges to the maximum likelihood policy. By applying the suggested method, it has been observed that ignoring prior information could prove very costly, regardless of whether stock disposal is allowed or not. The loss increases as the variation in the mean demand increases. Hence, the more uncertain one is about the mean demand, the more important it is to consider the Bayesian approach. A significant outcome of this study has been that the option to dispose stock could scale down the consequences of ignoring some kinds of demand non-stationarity like Model 2. Also, it can control the losses due to ignoring non-stationarity of the form of Model 1, provided the mean demand does not vary too much. And, the exact form of the demand rate function is crucial especially when the variation in the demand is high, because the loss in profit decreases as this variation increases.

Acknowledgements

The first author gratefully acknowledges the financial support of The Council of Scientific and Industrial Research, India, in undertaking this research work. We also thank the referees for their useful suggestions which have contributed to improving the content of the paper.

Appendix

Proof of Theorem 3.1. The proof is by induction on n, starting with the Nth period. For any n, (3.3) can be written as

$$P_n(x) = \begin{cases} -C_3(s_o^* - x) + H_n(s_o^*) & \text{if } x < s_o^*, \\ H_n(x) & \text{if } s_o^* \leqslant x \leqslant s_d^*, \\ C_s(x - s_d^*) + H_n(s_d^*) & \text{if } x > s_d^*. \end{cases} \tag{A.1}$$

It is observed from (A.1) that s_o^* is the solution of $(\mathrm{d}/\mathrm{d}y)H_n(y) = C_3$, that is,

$$C_2 - C_3 + r(1-\alpha) - \{C_1 + C_2 + r(1-\alpha)\}F_n(y) + \alpha\int_0^\infty \frac{\partial}{\partial y}P_{n+1}(y-t)f_n(t)\,\mathrm{d}t = 0. \quad \text{(A.2)}$$

Similarly, s_d^* is the solution of $(\mathrm{d}/\mathrm{d}y)H_n(y) = C_s$, that is,

$$C_2 - C_s + r(1-\alpha) - \{C_1 + C_2 + r(1-\alpha)\}F_n(y) + \alpha\int_0^\infty \frac{\partial}{\partial y}P_{n+1}(y-t)f_n(t)\,\mathrm{d}t = 0. \quad \text{(A.3)}$$

Here, the derivative of $P_n(.)$ is obtained by differentiating (A.1), and gives

$$\frac{\mathrm{d}}{\mathrm{d}x}P_n(x) = \begin{cases} C_3 & \text{if } x < s_o^*, \\ \dfrac{\mathrm{d}}{\mathrm{d}x}H_n(x) & \text{if } s_o^* \leqslant x \leqslant s_d^*, \\ C_s & \text{if } x > s_d^*. \end{cases} \quad \text{(A.4)}$$

For $n = N$, the first and second derivatives of $H_n(y)$ are given by

$$\frac{\mathrm{d}}{\mathrm{d}y}H_N(y) = C_2 + r(1-\alpha) + \alpha\sigma_p - \{C_1 + C_2 + r(1-\alpha) + \alpha(\sigma_p - \sigma_s)\}F_n(y) \quad \text{(A.5)}$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2}H_N(y) = -\{C_1 + C_2 + r(1-\alpha) + \alpha(\sigma_p - \sigma_s)\}f_n(y), \quad \text{(A.6)}$$

respectively. As $\sigma_s \leqslant C_3 \leqslant \sigma_p$, the second derivative of $H_N(y)$ is non-positive, that is, $H_N(y)$ is concave. Also, as $\sigma_p \geqslant C_3$, (3.6), (A.5) and (A.6) together imply that the derivative of $H_N(y)$ is decreasing and positive when $y = 0$. Hence, the optimal stock level in the Nth period is positive, and since $H_N(y) \to -\infty$ as $y \to \infty$, s_o^* and s_d^* are finite.

As $(\mathrm{d}/\mathrm{d}x)H_N(s_o^*) = C_3$, and $(\mathrm{d}/\mathrm{d}x)H_N(s_d^*) = C_s$, it is clear that $(\mathrm{d}/\mathrm{d}x)P_N(x)$ is continuous and non-increasing. And, although $(\mathrm{d}^2/\mathrm{d}x^2)P_N(x)$ may not exist at s_o^* and s_d^*, bounded left- and right-hand derivatives exist. So, from (A.4) and (A.6), we get

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}P_N(x) \leqslant 0$$

except possibly at s_o^* and s_d^*, which is sufficient to prove that $P_N(x)$ is concave. Hence, the theorem holds for $n = N$. Suppose, it holds for some $n+1$. We prove that it holds for n. That is, we have in the $(n+1)$th period,

s_o^* and s_d^* are finite and $P_{n+1}(x)$ is a concave function of x.

To prove that $P_n(x)$ is concave, consider

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2}H_n(y) = -\{C_1 + C_2 + r(1-\alpha)\}f_n(y) + \alpha\int_0^\infty \frac{\partial^2}{\partial y^2}P_{n+1}(y-t)f_n(t)\,\mathrm{d}t.$$

The integral is non-positive because of the induction hypothesis. Hence,

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2}H_n(y) \leqslant 0,$$

so that $H_n(y)$ is concave. The rest of the proof is identical to that for $n = N$. Thus, the theorem is proved for any n. □

Proof of Theorem 3.2. As seen in the proof of Theorem 3.1, s_o^* and s_d^* are the solutions of (A.2) and (A.3), respectively. By Theorem 3.1, $(\mathrm{d}/\mathrm{d}x)P_N(x)$ is a non-increasing function of x so that from (A.4), for $x > s_o^*$, we have $(\mathrm{d}/\mathrm{d}x)H_n(x) \leqslant C_3$. As $C_s \leqslant C_3$, by replacing $(\partial/\partial y)P_{n+1}(y-t)$ in (A.2) by C_3, we get a larger quantity than the L.H.S. of (A.2). That is, the solution of

$$C_2 + (r - C_3)(1-\alpha) - \{C_1 + C_2 + r(1-\alpha)\}F_n(y) = 0 \tag{A.7}$$

which is s_o^c, is more than or equal to the optimal order level, s_o^*. This proves (a). A similar argument proves (b). □

References

[1] Brown Jr GF, Rogers WF. A Bayesian approach to demand estimation and inventory provisioning. Naval Research Logistics Quarterly 1973;20:607–24.
[2] Eppen GD, Iyer AV. Improved fashion buying with Bayesian updates. Operations Research 1997;45:805–19.
[3] Scarf H. Bayes solutions of the statistical inventory problem. Annals of Mathematical Statistics 1959;30:490–508.
[4] Karlin S. Dynamic inventory policy with varying stochastic demands. Management Science 1960;6:231–58.
[5] Zacks S. Bayes sequential design of stock levels. Naval Research Logistics Quarterly 1969;16:143–55.
[6] Azoury KS. Bayes solution to dynamic inventory models under unknown demand distribution. Management Science 1985;31:1150–60.
[7] Lovejoy WS. Myopic policies for some inventory models with uncertain demand distributions. Management Science 1990;36:724–38.
[8] Morton TE, Pentico DW. The finite horizon nonstationary stochastic inventory problem: near-myopic bounds, heuristics, testing. Management Science 1995;41:334–43.
[9] Azoury KS, Miller BL. A comparison of the optimal ordering levels of Bayesian and non-Bayesian inventory models. Management Science 1984;30:993–1003.
[10] Kaplan AJ. Bayesian approach to inventory control of new parts. IIE Transactions 1988;20:151–6.
[11] Hill RM. Applying Bayesian methodology with a uniform prior to the single period inventory model. European Journal of Operational Research 1997;98:555–62.
[12] Popovic JB. Decision making on stock levels in cases of uncertain demand rate. European Journal of Operational Research 1987;32:276–90.
[13] DeGroot MH. Optimal statistical decisions. New York: McGraw-Hill, 1970.
[14] Reyman G. State reduction in a dependent demand inventory model given by a time series. European Journal of Operational Research 1989;41:174–80.
[15] Taha HA. Operations research: an introduction, 5th ed. New York: McMillan, 1992.
[16] Ramberg JS, Dudewicz EJ, Tadikamalla PR, Mykytka EF. A probability distribution and its uses in fitting data. Technometrics 1979;21:201–14.
[17] Johnson NL, Kotz S. Continuous univariate distributions – 1. New York: Wiley, 1970.
[18] Brown RG. Statistical forecasting for inventory control. New York: McGraw-Hill, 1959.

Rajashree Kamath K is a Senior Research Fellow in the Department of Statistics, Mangalore University. She was awarded the research fellowship by the Council of Scientific and Industrial Research, India in 1995. She holds a M.Sc. degree in Statistics from Mangalore University. Her research interests include estimation problems in Inventory systems.

T.P.M. Pakkala is a Reader in Statistics at Mangalore University. He holds a M.Sc. degree in Statistics from the University of Mysore, and a Ph.D. from Mangalore University. He has worked for the University of New Brunswick, Canada. He has publications in journals like *European Journal of Operational Research*, *Journal of the Operational Research Society*, *International Journal of Reliability*, *Quality and Safety Engineering*, *Opsearch*, *International Journal of Quality Management*, and *Information Systems and Operational Research*.