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Note on Path Signed Graphs

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Abstract

Data in the social sciences can often modeled using *signed graphs*, graphs where every edge has a sign + or −, or *marked graphs*, graphs where every vertex has a sign + or −. The *path graph* $P_k(G)$ of a graph G is obtained by representing the paths P_k in G by vertices whenever the corresponding paths P_k in G from a path P_{k+1} or a cycle C_k. In this note, we introduce a natural extension of the notion of path graphs to the realm of signed graphs. It is shown that for any signed graph S, $P_k(S)$ is balanced. The concept of a line signed graph is generalized to that of a path signed graphs. Further, in this note we discuss the structural characterization of path signed graphs. Also, we characterize signed graphs which are switching equivalent to their path signed graphs $P_3(S)$ ($P_4(S)$).

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1 Introduction

For standard terminology and notion in graph theory we refer the reader to West [11]; the non-standard will be given in this paper as and when required. We treat only finite simple graphs without self loops and isolates.

A *signed graph* is an ordered pair $S = (G, \sigma)$, where $G = (V, E)$ is a graph called *underlying graph of* S and $\sigma : E \to \{+, -\}$ is a function. A signed graph $S = (G, \sigma)$ is *balanced* if every cycle in S has an even number of negative edges (See [4]). Equivalently, a signed graph is balanced if product of signs of the edges on every cycle of S is positive.

A *marking* of S is a function $\mu : V(G) \to \{+, -\}$; A signed graph S together with a marking μ is denoted by S_μ.

The following characterization of balanced signed graphs is well known.

Proposition 1. (**E. Sampathkumar** [8]) *A signed graph $S = (G, \sigma)$ is balanced if, and only if, there exist a marking μ of its vertices such that each edge uv in S satisfies $\sigma(uv) = \mu(u)\mu(v)$.*

The idea of switching a signed graph was introduced by Abelson and Rosenberg [1] in connection with structural analysis of marking μ of a signed graph S. Switching S with respect to a marking μ is the operation of changing the sign of every edge of S to its opposite whenever its end vertices are of opposite signs. The signed graph obtained in this way is denoted by $\mathcal{S}_\mu(S)$ and is called *μ-switched signed graph* or just *switched signed graph*. Two signed graphs $S_1 = (G, \sigma)$ and $S_2 = (G', \sigma')$ are said to be *isomorphic*, written as $S_1 \cong S_2$ if there exists a graph isomorphism $f : G \to G'$ (that is a bijection $f : V(G) \to V(G')$ such that if uv is an edge in G then $f(u)f(v)$ is an edge in G') such that for any edge $e \in G$, $\sigma(e) = \sigma'(f(e))$. Further, a signed graph $S_1 = (G, \sigma)$ *switches* to a signed graph $S_2 = (G', \sigma')$ (or that S_1 and S_2 are *switching equivalent*) written $S_1 \sim S_2$, whenever there exists a marking μ of S_1 such that $\mathcal{S}_\mu(S_1) \cong S_2$. Note that $S_1 \sim S_2$ implies that $G \cong G'$, since the definition of switching does not involve change of adjacencies in the underlying graphs of the respective signed graphs.

Two signed graphs $S_1 = (G, \sigma)$ and $S_2 = (G', \sigma')$ are said to be *cycle isomorphic* (see [12]) if there exists an isomorphism $\phi : G \to G'$ such that the sign of every cycle Z in S_1 equals to the sign of $\phi(Z)$ in S_2. The following result will be useful in our further investigation (See [12]):

Proposition 2. (**T. Zaslavsky** [12]) *Two signed graphs S_1 and S_2 with the same underlying graph are switching equivalent if, and only if, they are cycle isomorphic.*

2 Path Signed Graphs

Broersma and Hoede [2] generalized the concept of line graphs to that of path graphs. Let P_k and C_k denote a path and a cycle with k vertices, respectively.

Denote $\Pi_k(G)$ the set of all paths of G on k vertices ($k \geq 1$). The *path graph* $P_k(G)$ of a graph G has vertex set $\Pi_k(G)$ and edges joining pairs of vertices that represent two paths P_k, the union of which forms either a path P_{k+1} or a cycle C_k in G. A graph is called a *P_k-graph*, if it is isomorphic to $P_k(H)$ for some graph H. If $k = 2$, then the P_2-graph is exactly the line graph. The way of describing a line graph stresses the adjacency concept, whereas the way of describing a path graph stresses concept of the path generation by consecutive paths.

For P_3-graphs, Broersma and Hoede [2] gave a solution to the characterization problem, which contained flaw. Later, Li and Lin [6] presented corrected form of the characterization of P_3-graphs. For $k \geq 4$, the problems becomes more difficult. Although the determination and characterization problems for P_k-graphs for $k \geq 4$ have not been completely solved.

We extend the notion of $P_k(G)$ to the realm of signed graphs. In a signed graph $S = (G, \sigma)$, for any $A \subseteq E(G)$ the sign $\sigma(A)$ is the product of the signs on the edges of A. The *path signed graph* $P_k(S) = (P_k(G), \sigma')$ of a signed graph $S = (G, \sigma)$ is a signed graph whose underlying graph is $P_k(G)$ called *path graph* and sign of any edge $e = P_k P'_k$ in $P_k(S)$ is $\sigma'(P_k P'_k) = \sigma(P_k)\sigma(P'_k)$. Further, a signed graph $S = (G, \sigma)$ is called *path signed graph*, if $S \cong P_k(S')$, for some signed graph S'. We now gives a straightforward, yet interesting, property of path signed graphs.

Proposition 3. *For any signed graph $S = (G, \sigma)$, its path signed graph $P_k(S)$ is balanced.*

Proof. Since sign of any edge $\sigma'(e = P_k P'_k)$ in $P_k(S)$ is $\sigma(P_k)\sigma(P'_k)$, where σ is the marking of $P_k(S)$, by Proposition 1, $P_k(S)$ is balanced. □

Remark: For any two signed graphs S and S' with same underlying graph, their path signed graphs are switching equivalent.

In [3], the author defined line signed graph of a signed graph $S = (G, \sigma)$ as follows:
The *line signed graph* of a signed graph $S = (G, \sigma)$ is a signed graph $L(S) = (L(G), \sigma')$, where for any edge ee' in $L(S)$, $\sigma'(ee') = \sigma(e)\sigma(e')$ (see also, E. Sampathkumar et al. [9]). Hence, we shall call a given signed graph S a *line signed graph* if it is isomorphic to the line signed graph $L(S')$ of some signed graph S'. By the definition of path signed graphs, we observe that $P_2(S) = L(S)$.

Corollary 4. *For any signed graph $S = (G, \sigma)$, its $P_2(S)$ (=$L(S)$) is balanced.*

In [10], the authors obtain structural characterization of line signed graphs as follows:

Proposition 5. (**E. Sampathkumar et al. [10]**)
A signed graph $S = (G, \sigma)$ is a line signed graph (or P_2-signed graph) if, and only if, S is balanced and G is a line graph (or P_2-graph).

Proof. Suppose that S is balanced and G is a line graph. Then there exists a graph H such that $L(H) \cong G$. Since S is balanced, by Proposition 1, there exists a marking μ of G such that each edge uv in S satisfies $\sigma(uv) = \mu(u)\mu(v)$. Now consider the signed graph $S' = (H, \sigma')$, where for any edge e in H, $\sigma'(e)$ is the marking of the corresponding vertex in G. Then clearly, $L(S') \cong S$. Hence S is a line signed graph.

Conversely, suppose that $S = (G, \sigma)$ is a line signed graph. Then there exists a signed graph $S' = (H, \sigma')$ such that $L(S') \cong S$. Hence G is the line graph of H and by Corollary 4, S is balanced. □

We strongly believe that the above Proposition can be generalized to path signed graphs $P_k(S)$ for $k \geq 3$. Hence, we pose it as a problem:

Problem 6. *If $S = (G, \sigma)$ is a balanced signed graph and its underlying graph G is a path graph, then S is a path signed graph.*

We now characterize those signed graphs that are switching equivalent to their P_3 (P_4)-signed graphs. In the case of graphs the following results is due to Broersma and Hoede [2] and Li and Zhao [7] respectively.

Proposition 7. (**Broersma and Hoede [2]**)
A connected graph G is isomorphic to its path graph $P_3(G)$ if, and only if, G is a cycle.

Proposition 8. (**Li and Zhao [7]**)
A connected graph G is isomorphic to its path graph $P_4(G)$ if, and only if, G is a cycle of length at least 4.

Proposition 9. *For any connected signed graph $S = (G, \sigma)$ satisfies*

(i) $S \sim P_3(S)$ if, and only if, S is a balanced signed graph on a cycle.
(ii) $S \sim P_4(S)$ if, and only if, S is a balanced signed graph on a cycle of length at least 4.

Proof. (i) Suppose that $S \sim P_3(S)$. This implies, $G \cong P_3(G)$ and hence by Proposition 7, we see that the graph G must be a cycle. Now, if S is any signed graph on a cycle, Proposition 3 implies that $P_3(S)$ is balanced and hence if S is unbalanced, $P_3(S)$ being balanced cannot be switching equivalent to S in accordance with Proposition 2. Therefore, S must be balanced.

Conversely, suppose that S is a balanced signed graph on a cycle. Then, since $P_3(S)$ is balanced as per Proposition 3 and since $P_3(G) \cong G$, the result follows from Proposition 2.
Similarly, we can prove (ii) using Proposition 8. □

The notion of *negation* $\eta(S)$ of a given signed graph S defined in [5] as follows: $\eta(S)$ has the same underlying graph as that of S with the sign of each edge opposite to that given to it in S. However, this definition does not say anything about what to do with nonadjacent pairs of vertices in S while applying the unary operator $\eta(.)$ of taking the negation of S.

For a signed graph $S = (G, \sigma)$, the $P_k(S)$ is balanced (Proposition 3). We now examine, the condition under which negation of $P_k(S)$ (i.e., $\eta(P_k(S))$) is balanced.

Proposition 10. *Let $S = (G, \sigma)$ be a signed graph. If $P_k(G)$ is bipartite then $\eta(P_k(S))$ is balanced.*

Proof. Since, by Proposition 3, $P_k(S)$ is balanced, then every cycle in $P_k(S)$ contains even number of negative edges. Also, since $P_k(G)$ is bipartite, all cycles have even length; thus, the number of positive edges on any cycle C in $P_k(S)$ are also even. This implies that the same thing is true in negation of $P_k(S)$. Hence $\eta(P_k(S))$ is balanced. □

Proposition 8 provides easy solutions to three other signed graph switching equivalence relations, which are given in the following results.

Corollary 11. *For any signed graph $S = (G, \sigma)$,*

(i) $\eta(S) \sim P_3(S)$ if, and only if, S is an unbalanced signed graph on any odd cycle.

(ii) $\eta(S) \sim P_4(S)$ if, and only if, S is an unbalanced signed graph on any odd cycle of length at least 5.

Corollary 12. *For any signed graph $S = (G, \sigma)$ and for any integer $k \geq 1$, $P_k(\eta(S)) \sim P_k(S)$.*

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References

[1] R. P. Abelson and M. J. Rosenberg, Symoblic psychologic : A model of attitudinal cognition, *Behav. Sci.*, 3 (1958), 1-13.

[2] H. J. Broersma and C. Hoede, Path graphs, *J. Graph Theory*, 13 (1989), 427-444.

[3] M. K. Gill, **Contributions to some topics in graph theory and its applications**, Ph.D. thesis, The Indian Institute of Technology, Bombay, 1983.

[4] F. Harary, On the notion of balance of a sigraph, *Michigan Math. J.*, 2(1953), 143-146.

[5] F. Harary, Structural duality, *Behav. Sci.*, 2(4) (1957), 255-265.

[6] H. Li and Y. Lin, On the characterization of path graphs, *J. Graph Theory*, 17 (1993), 463-466.

[7] X. Li and B. Zhao, Isomorphisms of P_4-graphs, *Australas. J. Combin.*, 15 (1997), 135-143.

[8] E. Sampathkumar, Point signed and line signed graphs, *Nat. Acad. Sci. Letters*, 7(3) (1984), 91-93.

[9] E. Sampathkumar, P. Siva Kota Reddy, and M. S. Subramanya, The Line n-sigraph of a symmetric n-sigraph, *Southeast Asian Bull. Math.*, to appear.

[10] E. Sampathkumar, M. S. Subramanya and P. Siva Kota Reddy, Characterization of Line Sidigraphs, *Southeast Asian Bull. Math.*, to appear.

[11] D. B. West, **Introduction to Graph Theory**, Prentice-Hall of India Pvt. Ltd., 1996.

[12] T. Zaslavsky, Signed Graphs, *Discrete Appl. Math.*, 4(1)(1982), 47-74.