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On the Number of Minimum Neighbourhood Sets in Paths and Cycles

Sumathi. M.P.[1] **and N.D. Soner**[2]

[1]Department of Studies in Mathematics, University of Mysore, Mysore 570006, India.
E-mail: sumathideepak@gmail.com
[2]Department of Studies in Mathematics, University of Mysore, Mysore 570006, India.
E-mail: ndsoner@yahoo.co.in

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Abstract

This paper is concerned with total number of minimum neighbourhood sets in paths and cycles. We establish the relation between minimum number of neighbourhood sets in paths and cycles and the blocks of partial balanced incomplete block design with m-association scheme.

Keywords: *Minimum neighbourhood sets, neighbourhood number and PBIBD with m-association scheme.*

1 Introduction

By a graph we mean a finite undirected graph without loops or multiple lines. For a graph G, let $V(G)$ and $E(G)$ respectively denote the point set and the line set of graph G.
For $v \in V$, the closed neighbourhood of V is $N[v] = \{u \in v : uv \in E(G)\} \bigcup \{v\}$. A subset S of $V(G)$ is a neighbourhood set of G if $G = \bigcup_{v \in S} \langle N(v) \rangle$. Where $\langle N(v) \rangle$ is the subgraph of G induced by $N[v]$. The neighbourhood number $\eta(G)$ of G is the minimum cardinality of a neighbourhood set of G. This number was introduced by Sampathkumar and Neeralagi in the year 1985.
We now determine the total number of minimum neighbourhood sets in paths

and cycles. We denote $\eta_M(G)$ as the total number of minimum neighbourhood sets (mn-sets) in a graph G.
We also consider the set of all minimum neighbourhood sets in cycles as the blocks of some partial balanced incomplete block designs with m-association scheme. Any undefined terms and notation, reader may refer to F.Harary[5] and for more details about neighbourhood set see [3,4]. We refer the reader to see [1,2] for more details about PBIBD and dominating set.

2 Minimum Neighbourhood Sets in Paths and Cycles

We denote $\eta_M(G)$ as the total number of minimum neighbourhood sets in a graph G. Here we find $\eta_M(P_n)$ by characterizing the path Pn into two classes $n = 2k$ and $2k+1$ for any integer $k \geq 0$. Similarly we characterize the C_n into two classes $n = 2k$ and $2k+1$, for any integer $k \geq 2$.

2.1 Basic Results

Observation 2.1. *In P_n with the labellings $v_1, v_2, v_3, ...v_{n-2}, v_{n-1}, v_n$, one of the following holds:*
(i) $v_1, v_{n-1} \in S$
(ii) $v_2, v_n \in S$
(iii) $v_2, v_{n-1} \in S$,
where S is any arbitrary minimum neighbourhood sets in P_n. From here onwards, we write mn-sets for minimum neighbourhood sets.

The following theorem is straightforward from the observation 1.

Theorem 2.2. $\eta_M(P_{2k+1}) = 1 fork \geq 0$ *We prove the above theorem straightforward from the observation* 1.

Lemma 2.3. *There is only one minimum neighbourhood set containing v_1 and v_{2k+1} in P_{2k+2} with the labellings $v_1, v_2, ...v_{2k}, v_{2k+1}, v_{2k+2}$.*

Proof. Clearly $S = \{v_1, v_3, ...v_{2k-1}, v_{2k+1}\}$ is an mn-set in P_{2k+2} containing v_1 and v_{2k+1} which proves the existence.
To prove the uniqueness, consider $S - \{v_1, v_{2k+1}\}$ which is a neighbourhood set in P_{2k-2} with the labellings $\{v_3, v_4, ...v_{2k-2}, v_{2k-1}\}$. The set $S - \{v_1, v_{2k+1}\}$ is not only a neighbourhood set, but also minimum.
Since $|S - \{v_1, v_{2k+1}\}| = (k+1) - 2, = \eta(P_{2k-1})$
By Theorem 1, $S - \{v_1, v_{2k+1}\}$ is unique mn-set in P_{2k-1}, therefore S is unique in P_{2k+2}.

Lemma 2.4. *There is only one mn-set containing v_2 and v_{2k+2} in P_{2k+2} with the labellings $v_1, v_2, ...v_{2k+2}$.*

Proof. The proof is similar that of Lemma 2, by counting mn-sets from the right of the labellings of P_{2k+2} as was done from the right of the labellings of P_{2k+2} as was done from the left in Lemma 2.

Lemma 2.5. *There are exactly k mn-sets containing v_2 and v_{2k+1} in P_{2k+2} with labellings $v_1, v_2, ...v_{2k}, v_{2k+1}, v_{2k+2}$.*

Proof. For $k = 1$, the set $\{v_2, v_3\}$ is an mn-set in $P_{2(1)+2} = P_4$ and $k = 2$, the sets are $\{v_2, v_4, v_5\}$ and $\{v_2, v_3, v_5\}$ are mn-sets in $P_{2(2)+2} = P_6$ as stated in lemma. Thus the result is true for $k = 1$ and $k = 2$.
Assume the result is true for $k - 1$, that there are exactly $k - 1$ mn-sets in $P_{2(k-1)+2} = P_{2k}$ containing v_2 and v_{2k-1}.
Let $S_1, S_2, ...S_{k-1}$ be mn-sets containing v_2 and v_{2k-1}. Now consider P_{2k+2} with the labellings $v_1, v_2, ...v_{2k}, v_{2k+1}, v_{2k+2}$. Clearly $S_i \bigcup \{v_{2k+1}\}$ is an mn-set containing v_2 and v_{2k-1} in P_{2k+2}.
Since $|S_i \bigcup \{v_{2k+1}\}| = k + 2$
$= \eta(P_{2k+2})$ for $i = 1, 2, ...(k - 1)$.

Lemma 2.6. *There are exactly $(k - 1)$ mn-sets containing v_2 and v_{2k-1} in P_{2k} with labellings $v_1, v_2, ...v_{2k-1}, v_{2k}$.*

Proof. We prove the lemma by induction on K. For $k = 2$ the set $\{v_2, v_3\}$ is an mn-set in $P_{2(2)} = P_4$. For $k = 3$, the set $\{v_2, v_4, v_5\}$ and $\{v_2, v_3, v_5\}$ are mn-sets in $P_{2(2)} = P_6$.
Thus the result is true for $k = 2$ and $k = 3$. Assume that the result is true for $k - 2$, that is there are exactly $k - 2$ mn-sets in $P_{2(k-2)} = P_{2k-4}$ containing v_2 and v_{2k-3} in P_{2k-4}. Let $A_1, A_2, ...A_{k-2}$ be mn-sets containing v_2 and v_{2k-3} in P_{2k-4} with the labellings $v_1, v_2, ...v_{2k-1}, v_{2k}$.
clearly $A \bigcup \{v_{2k-1}\}$ is an mn-set containing v_2 and v_{2k-1} in P_{2k}. Since
$|A_i \bigcup \{v_{2k-1}\}| = k$
$= \eta(P_{2k})$, for $i = 1, 2, 3, ...(k - 2)$.
Hence the result is true for $(k - 1)$.

Lemma 2.7. *There is no mn-set containing v_1 and v_{2k+2} in P_{2k+2}.*

Proof. On the contrary, assume D be a mn-set containing v_1 and v_{2k+2} in P_{2k+2}. Then $D = \{v_1, v_{2k+2}\}$ is a neighbourhood of P_{2k} with the labellings $v_3, v_4, v_5, ...v_{2k}$. Since v_2 and v_{2k+1} are neighbourhoods of v_1 and v_{2k+2} in P_{2k+2}. Thus,
$K = \eta(P_{2k}) \leq |D - \{v_1, v_{2k+2}\}|$
$= k + 1 - 2$
$= k - 1$ a contradiction.

Lemma 2.8. *There is no mn-set containing v_1 and v_{2k+1} in P_{2k+1}.*

Proof. On contrary, assume that S be a mn-set containing v_1 and v_{2k+1} in P_{2k+1}. Then, $S = \{v_1, v_{2k+1}\}$ is a neighbourhood set in P_{2k-1} with labellings $v_2, v_3, ...v_{2k}$, since v_2 and v_{2k} are neighbourhoods of v_1 and v_{2k+1} of mn-sets S in P_{2k+1}.
Thus,
$K = \eta(P_{2k+1}) \leq |S - \{v_1, v_{2k+1}\}|$
$= k - 2$, a contradiction.

2.2 Main Results

Lemma 2.9. *There is no mn-set containing v_1 and v_{2k} in P_{2k}.*

Proof. The proof is similar to the above Lemma 5.

Theorem 2.10. *For any integer $k \geq 0$,*
$\eta(P_{2k+4}) = \eta_M(P_{2k+2}) + 1$

Proof. Let us label the vertices of P_{2k+4} as $v_1, v_2, ...v_{2k+3, v_{2k+4}}$. Let S be the mn-set containing v_1 and v_{2k+1} in P_{2k+4} which is the only one mn-set follows from Lemma 2.
Therefore $S_1 = S \bigcup \{v_{2k+3}\}$
$= \eta(P_{2k+2}) + 1$
$= (k + 2) + 1$
$= k + 3$
$= \eta P_{2k+4}$.
Let S' be the mn-set containing v_2 and v_{2k+1} in P_{2k+2}, which is the only one mn-set. Similarly follows, from the Lemma 3, therefore
$S_2 = S' \bigcup v_{2k+4}$ is the mn-set for P_{2k+4}, since
$|S_2| = |S' \bigcup \{v_{2k+4}\}|$
$= |S'| + 1$
$= \eta(P_{2k+2}) + 1$
$= k + 3$
$= \eta(P_{2k+4})$
Again by the above Lemma let $S_1, S_2, ...S_k$ be the mn-set containing v_1 and v_{2k+1} in P_{2k+4}.
For each set S_i containing v_1 and v_{2k+1} in P_{2k+4}, there is an mn-set $S_i' = S_1 \bigcup \{v_{2k+3}\}$ in P_{2k+4}. Hence by the Lemma$(2, 3, 4)$ and Observation 1 the number of mn-sets are
$\eta_M(P_{2k+2}) = k + 2$ thus
$\eta_M(P_{2k+4}) = k + 3$
$= (k + 2) + 1$
$= \eta_M(P_{2k+2}) + 1.$

Theorem 2.11. *$\eta_M(C_{2k}) = 2$ for $k \geq 1$.*

Example 2.12. $k = 4$,$\eta_M(C_8) = 2$

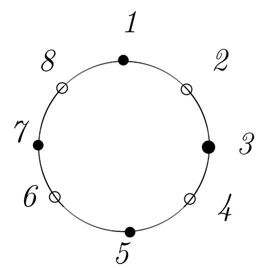

Figure 1

The minimum neighbourhood sets are $\{1,3,5,7\}$ and $\{2,4,6,8\}$.

Lemma 2.13. *There is one and only one minimum neighbourhood set containing two adjacent vertices in C_{2k+3}.*

Proof. Let us label the vertices of C_{2k+3} as $v_1, v_2, ...v_{2k-1}, v_{2k}, v_{2k+1}, v_{2k+2}, v_{2k+3}$. Clearly the set $\eta = v_1, v_2, v_4....v_{2k-1}, v_{2k}, v_{2k+2}$ is an mn-set in C_{2k+3} containing two adjacent vertices for example v_1 and v_2 which proves the existence. Also D is unique. Since $D - \{v_1, v_2\}$ is mn-set in P_{2k-2} with the labeling $v_3, v_4, v_5, ...v_{2k}$. But by previous theorem, $D - \{v_1, v_2\}$ is uniqueness. P_{2k-2} and hence D is unique in C_{2k+3}.

3 Minimum Neighbourhood Sets and PBIBD, Association Scheme Related To Cycle Graphs

The relation between graphs and PBIBD have been studied by many authors. For example H.B.Walikar [2], has been studied the relation between the minimum dominating sets and PBIBD, Anwar Alwardi [1] has been studied the relation between the dominating sets and PBIBD with association scheme for special case of graphs called strongly regular graphs. This motivated us to study the general case for cycles and for the other graphs still open for research. In our paper we denote the parameters of PBIBD in the following order $(v, b, r, k, \lambda_1, \lambda_i)$.

Definition 3.1. *Given v objects a relation satisfying the following conditions is said to be an association scheme with m classes:*

(i) any two objects are either first associates, or second associates,..., or m^{th} associates, the relation of association being symmetric.

(ii) each object α has n_i i^{th} associates, the number n_i being independent of α.

(iii) if two objects α and β are i^{th} associates, then the number of objects which are j^{th} associates of α and k^{th} associates of β is p^i_{jk} and is independent of the pair of i^{th} associates α and β. Also p^i_{jk}=p^i_{kj}.

If we have association scheme for the v objects we can define a PBIBD as the following definition.

Definition 3.2. *The PBIB design is arrangement of v objects into b sets (called blocks) of size k where $k < v$ such that*

(i) every object is contained in exactly r blocks.

(ii) each block contains k distinct objects.

(iii) Any two objects which are i^{th} associates occur together in exactly λ_i blocks.

Proposition 3.3. *From the neighbourhood sets of C_5 we get the PBIBD with parameters (5,5,3,3,1,2) with two association scheme.*

Proof: The cycle C_5 with the labelling $1, 2, 3, 4, 5$ as in Figure 2 whose mn-sets are (1,3,5), (1,2,4), (2,3,5), (1,3,4) and (2,4,5) and these are the blocks of the PBIBD with parameters (5,5,3,3,1,2) with two association scheme defined as following:
Any two vertices say α and β, α is first associates to β if α and β appear together once in the minimum neighbourhood sets, and α is second associates to β if α and β appear together twice in the minimum neighbourhood sets we give the table of association scheme as following:

Elements	First Associates	Second Associates
1	2,5	3,4
2	1,3	4,5
3	2,4	1,5
4	3,5	1,2
5	1,4	2,3

and $P_1 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ and $P_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

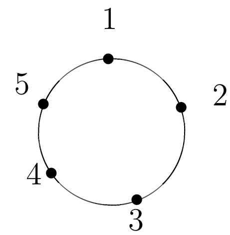

Figure 2

Proposition 3.4. *From the neighbourhood sets of C_7 we get the PBIBD with parameters (7,7,4,4,1,2,3) with three association scheme.*

Proof: The cycle C_7 with the labelling $1, 2, 3, 4, 5, 6, 7$ as in Figure 3 whose mn-sets are (1,3,5,6), (1,2,4,6), (2,3,5,7), (1,3,4,6), (2,4,5,7), (2,4,6,7) and (1,3,5,7) and these are the blocks of the PBIBD with parameters (7,7,4,4,1,2,3) with three association scheme defined as following:
Any two vertices say α and β, α is first associates to β if α and β appear together once in the minimum neighbourhood sets, and α is second associates to β if α and β appear together twice in the minimum neighbourhood sets and α is third associates to β if α and β appear together thrice in the minimum neighbourhood sets we give the table of association scheme as following:

Elements	First Associates	Second Associates	Third Associates
1	2,7	4,5	3,6
2	1,3	5,6	4,7
3	2,4	6,7	1,5
4	3,5	1,7	2,6
5	4,6	1,2	3,7
6	5,7	2,3	1,4
7	1,6	3,4	2,5

and $P_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $P_2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ and $P_3 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

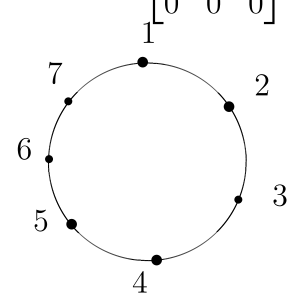

Figure 3

Proposition 3.5. *From the neighbourhood sets of C_9 we get the PBIBD with parameters (9,9,5,5,1,2,3,4) with four association scheme.*

Proof: The cycle C_9 with the labelling $1, 2, 3, 4, 5, 6, 7, 8, 9$ as in Figure 4 whose mn-sets are (1,3,5,6,8), (1,2,4,6,8), (2,3,5,7,9), (1,3,4,6,8), (2,4,5,7,9), (2,4,6,7,9), (1,3,5,7,9), (1,3,5,7,8), and (2,4,6,8,9) and these are the blocks of the PBIBD with parameters (9,9,5,5,1,2,3,4) with four association scheme defined as following:
Any two vertices say α and β, α is first associates to β if α and β appear together once in the minimum neighbourhood sets, and α is second associates to β if α and β appear together twice in the minimum neighbourhood sets and α is third associates to β if α and β appear together thrice in the minimum neighbourhood sets and α is fourth associates to β if α and β appear together four times in the minimum neighbourhood sets we give the table of association scheme as following:

Elements	First Associates	Second Associates	Third Associates	Fourth Associates
1	2,9	4,7	5,6	3,8
2	1,3	5,8	6,7	4,9
3	2,4	6,9	7,8	1,5
4	3,5	1,7	8,9	2,6
5	4,6	2,8	1,9	3,7
6	5,7	3,9	1,2	4,8
7	6,8	1,4	2,3	5,9
8	7,9	2,5	3,4	1,6
9	1,8	3,6	4,5	2,7

and

$$P_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}, P_2 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, P_3 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\text{and } P_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

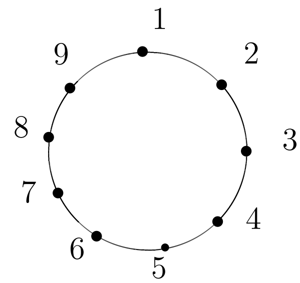

Figure 4

Proposition 3.6. *From the neighbourhood sets of C_{11} we get the PBIBD with parameters (11,11,6,6,1,2,3,4,5) with five association scheme.*

Proof: The cycle C_{11} with the labelling $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$ as in Figure 5 whose mn-sets are (1,3,5,6,8,10), (1,2,4,6,8,10), (2,3,5,7,9,11), (1,3,4,6,8,10), (2,4,5,7,9,11), (2,4,6,7,9,11), (1,3,5,7,9,10), (1,3,5,7,8,10), (2,4,6,8,9,11), (1,3,5,7,9,11) and (2,4,6,8,10,11) and these are the blocks of the PBIBD with parameters (11,11,6,6,1,2,3,4,5) with five association scheme defined as following:
Any two vertices say α and β, α is first associates to β if α and β appear together once in the minimum neighbourhood sets, α is second associates to β if α and β appear together twice in the minimum neighbourhood sets, α is third associates to β if α and β appear together thrice in the minimum neighbourhood sets, α is fourth associates to β if α and β appear together four times in the minimum neighbourhood sets and α is fifth associates to β if α and β appear together five times in the minimum neighbourhood sets, we give the table of association scheme as following:

Elements	1^{st} Associates	2^{nd} Associates	3^{rd}Associates	4^{th} Associates	5^{th} Associates
1	2,11	4,9	6,7	5,8	3,10
2	1,3	5,10	7,8	6,9	4,1
3	2,4	6,11	8,9	7,10	1,5
4	3,5	1,7	9,10	8,11	2,6
5	4,6	2,8	10,11	1,9	3,7
6	5,7	3,9	1,11	2,10	4,8
7	6,8	4,10	1,2	3,11	5,9
8	7,9	5,11	2,3	1,4	6,10
9	8,10	1,6	3,4	2,5	7,11
10	9,11	2,7	4,5	3,6	1,8
11	1,10	3,8	5,6	4,7	2,9

and

$$P_1 = \begin{bmatrix} 0&0&0&0&1\\ 0&0&0&1&1\\ 0&0&1&1&0\\ 0&1&1&0&0\\ 1&1&0&0&0 \end{bmatrix},\ P_2 = \begin{bmatrix} 0&0&0&1&1\\ 0&0&1&0&0\\ 0&1&0&0&1\\ 1&0&0&1&0\\ 1&0&1&0&0 \end{bmatrix},\ P_3 = \begin{bmatrix} 0&0&1&1&0\\ 0&1&0&0&1\\ 1&0&0&0&0\\ 1&0&0&0&1\\ 0&1&0&1&0 \end{bmatrix}$$

$$P_4 = \begin{bmatrix} 0&1&1&0&0\\ 1&0&0&1&0\\ 1&0&0&0&1\\ 0&1&0&0&0\\ 0&0&1&0&1 \end{bmatrix} \text{ and } P_5 = \begin{bmatrix} 1&0&0&0&1\\ 1&0&1&0&0\\ 0&1&0&1&0\\ 0&0&1&0&1\\ 0&0&0&1&0 \end{bmatrix}$$

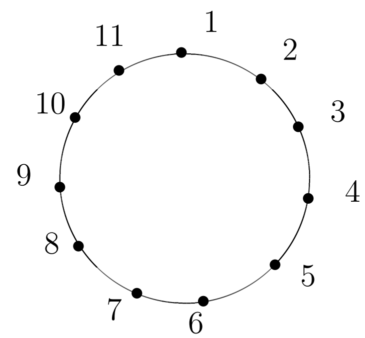

Figure 5

From the previous examples we can deduce that for any cycle graph C_{2k+1}, where $k \geq 2$, we can define PBIBD from the minimum neighbourhood set with $2k+1$ points and also $2k+1$ blocks because as we have proved previously. Also it is clear that the size of any block is the neighbouhood number of C_{2k+1} and association scheme with $\lfloor (2k+1)/2 \rfloor$.

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