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# The Edge $C_4$ Signed Graph of a Signed Graph

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**Abstract.** A *signed graph (marked graph)* is an ordered pair $S = (G, \sigma)$ $(S = (G, \mu))$, where $G = (V, E)$ is a graph called the *underlying graph* of $S$ and $\sigma : E \to \{+, -\}$ $(\mu : V \to \{+, -\})$ is a function. The *edge $C_4$ graph* $E_4(G)$ of a graph $G = (V, E)$ has all edges of $G$ as it vertices, two vertices in $E_4(G)$ are adjacent if their corresponding edges in $G$ are either incident or are opposite edges of some $C_4$. Analogously, one can define the *the edge $C_4$ signed graph* of a signed graph $S = (G, \sigma)$ as a signed graph $E_4(S) = (E_4(G), \sigma')$, where $E_4(G)$ is the underlying graph of $E_4(S)$, where for any edge $e_1e_2$ in $E_4(S)$, $\sigma'(e_1e_2) = \sigma(e_1)\sigma(e_2)$. It is shown that for any signed graph $S$, its edge $C_4$ signed graph $E_4(S)$ is balanced. We then give structural characterization of edge $C_4$ signed graphs. Two signed graphs $S_1$ and $S_2$ are *switching equivalent* written $S_1 \sim S_2$, whenever there exists a marking $\mu$ of $S_1$ such that the signed graph $\mathcal{S}_\mu(S_1)$ obtained by changing the sign of every edge of $S_1$ to its opposite whenever its end vertices are of opposite signs, is isomorphic to $S_2$. Further, we obtain a structural characterization of signed graphs that are switching equivalent to their edge $C_4$ signed graphs.

**Keywords:** Signed graphs; Balance; Switching; Edge $C_4$ signed graph; Negation of a signed graph.

## 1. Introduction

For graph theoretical terminologies and notations in this paper we follow the book [4]; the non-standard will be given in this paper as and when required. We treat only finite simple graphs without self loops and isolates.

A *signed graph* is an ordered pair $S = (G, \sigma)$, where $G = (V, E)$ is a graph called *underlying graph of* $S$ and $\sigma : E \to \{+, -\}$ is a function. A signed graph $S = (G, \sigma)$ is *balanced* if every cycle in $S$ has an even number of negative edges (See [2]). Equivalently a signed graph is balanced if product of signs of the edges on every cycle of $S$ is positive. A *marking* of $S$ is a function $\mu : V(G) \to \{+, -\}$; A signed graph $S$ together with a marking $\mu$ is denoted by $S_\mu$. In a signed graph $S = (G, \sigma)$, for any $A \subseteq E(G)$ the *sign* $\sigma(A)$ is the product of the signs on the edges of $A$.

The following characterization of balanced signed graphs is well known.

**Proposition 1.1.** [9] *A signed graph $S = (G, \sigma)$ is balanced if and only if there exist a marking $\mu$ of its vertices such that each edge $uv$ in $S$ satisfies $\sigma(uv) = \mu(u)\mu(v)$.*

The idea of switching a signed graph was introduced by Abelson and Rosenberg [1] in connection with structural analysis of marking $\mu$ of a signed graph $S$. Switching $S$ with respect to a marking $\mu$ is the operation of changing the sign of every edge of $S$ to its opposite whenever its end vertices are of opposite signs. The signed graph obtained in this way is denoted by $\mathcal{S}_\mu(S)$ and is called *$\mu$-switched signed graph* or just *switched signed graph*. Two signed graphs $S_1 = (G, \sigma)$ and $S_2 = (G', \sigma')$ are said to be *isomorphic*, written as $S_1 \cong S_2$ if there exists a graph isomorphism $f : G \to G'$ (that is a bijection $f : V(G) \to V(G')$ such that if $uv$ is an edge in $G$ then $f(u)f(v)$ is an edge in $G'$) such that for any edge $e \in G$, $\sigma(e) = \sigma'(f(e))$. Further a signed graph $S_1 = (G, \sigma)$ *switches* to a signed graph $S_2 = (G', \sigma')$ (or that $S_1$ and $S_2$ are *switching equivalent*) written $S_1 \sim S_2$, whenever there exists a marking $\mu$ of $S_1$ such that $\mathcal{S}_\mu(S_1) \cong S_2$. Note that $S_1 \sim S_2$ implies that $G \cong G'$, since the definition of switching does not involve change of adjacencies in the underlying graphs of the respective signed graphs. In [11], the authors extended the above notion 'switching' to signed directed graphs (see also, E. Sampathkumar et al. [10]).

Two signed graphs $S_1 = (G, \sigma)$ and $S_2 = (G', \sigma')$ are said to be *weakly isomorphic* (see [12]) or *cycle isomorphic* (see [13]) if there exists an isomorphism $\phi : G \to G'$ such that the sign of every cycle $Z$ in $S_1$ equals to the sign of $\phi(Z)$ in $S_2$. The following result is well known (See [13]):

**Proposition 1.2.** [13] *Two signed graphs $S_1$ and $S_2$ with the same underlying graph are switching equivalent if and only if they are cycle isomorphic.*

## 2. Edge $C_4$ Signed Graph of a Signed Graph

The *edge $C_4$ graph* $E_4(G)$ of a graph $G$ is defined in [8] as follows:The edge

$C_4$ graph of a graph $G = (V, E)$ is a graph $E_4(G) = (V', E')$, with vertex set $V' = E(G)$ such that two vertices $e$ and $f$ are adjacent if and only if the corresponding edges in $G$ are either incident or opposite edges of some $C_4$. So two vertices are adjacent vertices in $E_4(G)$ if the union of the corresponding edges in $G$ induces any one of the graphs $P_3, C_3, C_4, K_4 - \{e\}, K_4$.

Clearly the edge $C_4$ graph coincides with the line graph for any acyclic graph. But they differ in many properties. In [7], the authors proved that the edge $C_4$ graph $E_4(G)$ has no forbidden subgraphs.

In this paper, we extend the notion of $E_4(G)$ to realm of signed graphs: Given a signed graph $S = (G, \sigma)$ its *edge $C_4$ signed graph* $E_4(S) = (E_4(G), \sigma')$ is that signed graph whose underlying graph is $E_4(G)$, the edge $C_4$ graph of $G$, where for any edge $e_1e_2$ in $E_4(S)$, $\sigma'(e_1e_2) = \sigma(e_1)\sigma(e_2)$.

Hence, we shall call a given signed graph an edge $C_4$ signed graph if there exists a signed graph $S'$ such that $S \cong E_4(S')$. In the following subsection, we shall present a characterization of edge $C_4$ signed graphs.

The following result indicates the limitations of the notion of edge $C_4$ signed graphs as introduced above, since the entire class of unbalanced signed graphs is forbidden to be edge $C_4$ signed graphs.

**Proposition 2.1.** *For any signed graph $S = (G, \sigma)$, its edge $C_4$ signed graph $E_4(S)$ is balanced.*

*Proof.* Let $\sigma^{'}$ denote the signing of $E_4(S)$ and let the signing $\sigma$ of $S$ be treated as a marking of the vertices of $E_4(S)$. Then by definition of $E_4(S)$ we see that $\sigma'(e_1e_2) = \sigma(e_1)\sigma(e_2)$, for every edge $e_1e_2$ of $E_4(S)$ and hence, by Proposition 1.1, the result follows. ■

For any positive integer $k$, the $k^{th}$ iterated edge $C_4$ signed graph, $E_4^k(S)$ of $S$ is defined as follows:

$$E_4^0(S) = S,\ E_4^k(S) = E_4(E_4^{k-1}(S)).$$

**Corollary 2.2.** *For any signed graph $S = (G, \sigma)$ and any positive integer $k$, $E_4^k(S)$ is balanced.*

Recall the edge $C_4$ graph coincides with the line graph for any acyclic graph. As a case, for a connected graph $G$, $E_4(G) = G$ if and only if $G = C_n, n \neq 4$ (see [8]).

We now characterize signed graphs that are switching equivalent to their edge $C_4$ signed graphs.

**Proposition 2.3.** *For any signed graph $S = (G, \sigma)$, $S \sim E_4(S)$ if and only if $G \cong C_n$, where $n \neq 4$ and $S$ is balanced.*

*Proof.* Suppose $S \sim E_4(S)$. This implies, $G \cong E_4(G)$ and hence by the above observation we see that the graph $G$ must be isomorphic to $C_n$, where $n \neq 4$. Now, if $S$ is signed graph on $C_n$, Proposition 2.1 implies that $E_4(S)$ is balanced and hence if $S$ is unbalanced its $E_4(S)$ being balanced cannot be switching equivalent to $S$ in accordance with Proposition 1.2. Therefore, $S$ must be balanced.

Conversely, suppose that $S$ is balanced signed graph on $C_n$, where $n \neq 4$. Then, since $E_4(S)$ is balanced as per Proposition 2.1 and since $G \cong C_n$, where $n \neq 4$, the result follows from Proposition 1.2 again. ∎

The notion of *negation* $\eta(S)$ of a given signed graph $S$ defined in [5] as follows: $\eta(S)$ has the same underlying graph as that of $S$ with the sign of each edge opposite to that given to it in $S$. However, this definition does not say anything about what to do with nonadjacent pairs of vertices in $S$ while applying the unary operator $\eta(.)$ of taking the negation of $S$.

Proposition 2.3 provides easy solutions to two other signed graph switching equivalence relations, which are given in the following results.

**Corollary 2.4.** *For any signed graph $S = (G, \sigma)$, $\eta(S) \sim E_4(S)$ if and only if $G \cong C_n$, where $n \geq 5$ and $S$ is unbalanced.*

**Corollary 2.5.** *For any signed graph $S = (G, \sigma)$, $E_4(\eta(S)) \sim E_4(S)$.*

For a signed graph $S = (G, \sigma)$, the $E_4(S)$ is balanced (Proposition 2.1). We now examine, the conditions under which negation $\eta(S)$ of $E_4(S)$ is balanced.

**Proposition 2.6.** *Let $S = (G, \sigma)$ be a signed graph. If $E_4(G)$ is bipartite then $\eta(E_4(S))$ is balanced.*

*Proof.* Since, by Proposition 2.1, $E_4(S)$ is balanced, if each cycle $C$ in $E_4(S)$ contains even number of negative edges. Also, since $E_4(G)$ is bipartite, all cycles have even length; thus, the number of positive edges on any cycle $C$ in $E_4(S)$ is also even. Hence $\eta(E_4(S))$ is balanced. ∎

In [7], the authors proved that for a connected complete multipartite graph $G$, $E_4(G)$ is complete.

The following result follows from the above observation and Proposition 2.1.

**Proposition 2.7.** *For a connected signed graph $S = (G, \sigma)$, $E_4(S)$ is complete balanced signed graph if and only if $G$ is complete multipartite graph.*

**Proposition 2.8.** [7] *For a connected graph $G$, $E_4(G)$ is bipartite if and only if $G$ is either a path or $C_{2n}$, $n \geq 3$.*

From the above Proposition we have the following result:

**Proposition 2.9.** *For a connected signed graph* $S = (G, \sigma)$, $E_4(S)$ *is bipartite balanced signed graph if and only if* $G$ *is isomorphic to either path or* $C_{2n}$, $n \geq 3$.

*Proof.* Suppose $E_4(S)$ is bipartite balanced signed graph. This implies $E_4(G)$ is bipartite and hence by Proposition 2.8 we see that the graph $G$ must be isomorphic to either a path or $C_{2n}$, $n \geq 3$.

Conversely, suppose that $S$ is a signed graph on path or $C_{2n}$, $n \geq 3$. Then, since $E_4(S)$ is balanced as per Proposition 2.1 and since $E_4(G)$ is bipartite by Proposition 2.8. ■

## 3. Characterization of Edge $C_4$ Signed Graphs

The following result characterize signed graphs which are edge $C_4$ signed graphs.

**Proposition 3.1.** *A signed graph* $S = (G, \sigma)$ *is an edge* $C_4$ *signed graph if and only if* $S$ *is balanced signed graph and its underlying graph* $G$ *is an edge* $C_4$ *graph.*

*Proof.* Suppose that $S$ is balanced and $G$ is an edge $C_4$ graph. Then there exists a graph $H$ such that $E_4(H) \cong G$. Since $S$ is balanced, by Proposition 1.1, there exists a marking $\mu$ of $G$ such that each edge $uv$ in $S$ satisfies $\sigma(uv) = \mu(u)\mu(v)$. Now consider the signed graph $S' = (H, \sigma')$, where for any edge $e$ in $H$, $\sigma'(e)$ is the marking of the corresponding vertex in $G$. Then clearly, $E_4(S') \cong S$. Hence $S$ is an edge $C_4$ signed graph.

Conversely, suppose that $S = (G, \sigma)$ is an edge $C_4$ signed graph. Then there exists a signed graph $S' = (H, \sigma')$ such that $E_4(S') \cong S$. Hence $G$ is the edge $C_4$ graph of $H$ and by Proposition 2.1, $S$ is balanced. ■

The reader may refer the papers [5, 6] for some more related topics in concerning the minimum number of vertices and edges in a graph with a given vertex conecticity, edge conectivity and mimimm degree. We investigate these notions of $E_4(S)$ in a separate paper.

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