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THE LAW OF THE ITERATED LOGARITHM FOR A MARKOV PROCESS

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1. Introduction. The purpose of this paper is to prove the law of the iterated logarithm for a sequence $\{f(x_n)\}$, where f is a real-valued function defined on the state space of a discrete Markov Process $\{x_n\}$ satisfying Doeblin's hypothesis [3].

Most of the known results concerning the law of the iterated logarithm are obtained for a sequence of independent random variables and the proofs mainly depend on (i) the rate of convergence in the central limit theorem and (ii) certain inequalities due to Kolmogorov/Lévy [5]. In Section 3 we obtain the rate of convergence of $n^{-\frac{1}{2}}\sum_{j=1}^{n} f(x_j)$ to the normal distribution. In Section 5 we obtain the rate of convergence of the maximum of the partial sums of the random variables $f(x_j)$ to the positive normal distribution and use this rate in the place of Kolmogorov/Lévy inequalities.

2. Preliminary assumptions and lemmas. Let X be a space of points ξ and let $\mathscr{F}_X$ be a Borel field of subsets of X. Let $\{x_n\}$ be a Markov process with state space X and stationary transition probabilities:

$$P(\xi, A) = P(x_{n+1} \in A \mid x_n = \xi). \tag{2.1}$$

That is, $\{x_n\}$ is a sequence of measurable functions from some probability space $(\Omega, \mathscr{B}, P)$ to X such that (2.1) holds where the transition function $p(\xi, A)$ is a measurable function of ξ for fixed $A \in \mathscr{F}_X$ and is a probability measure on $\mathscr{F}_X$ for fixed ξ. The initial distribution π is defined by $\pi(A) = P\{x_1 \in A\}$ and the n-step transition probabilities are given by $P^{(n)}(\xi, A) = P\{x_{n+1} \in A \mid x_1 = \xi\}$. Throughout the following discussion Doeblin's condition will be assumed. In fact, we shall assume the hypothesis (D_0):

(a) Doeblin's condition is satisfied.

(b) There is only a single ergodic set and this contains no cyclically moving subsets.

It is known that if (D_0) holds then there exist positive constants γ and ρ, $\rho < 1$, and a unique stationary absolute distribution p such that $|p^{(n)}(\xi, A) - p(A)| < \gamma\rho^n$ for all $\xi \in X$ and $A \in \mathscr{F}_X$ and $n \geqq 1$. Throughout the following discussion we shall make the assumption:

$$\pi = p. \tag{2.2}$$

Let $C_1, C_2, \cdots$ be absolute constants. We shall now obtain two lemmas which will be used in the later analysis. Let ${}_r\mathscr{F}_m$ denote the σ-field generated by the random variables (rv's) $x_r, \cdots, x_m$.

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LEMMA 2.1. *If f is measurable with respect to ${}_1\mathscr{F}_m$ and g is a bounded function measurable with respect to ${}_{k+m}\mathscr{F}_\infty$, $|g| \leqq M$, then $|E\{g|f\} - E\{g\}| \leqq 2M\gamma\rho^k$.*

PROOF. Since $E\{g|f\} = E\{E(g|x_1, \cdots, x_m)|f\}$

$$|E\{g|f\} - E\{g\}| \leqq E\{|E\{g|x_1, \cdots, x_m\} - E\{g\}||f\}.$$

The result follows from Lemma 7.2 page 224 of [3].

COROLLARY 2.1. *If $A \in {}_1\mathscr{F}_m$ and $B \in {}_{k+m}\mathscr{F}_\infty$ then $|P(B|A) - P(B)| \leqq 2\gamma\rho^k$.*

LEMMA 2.2. *If $A \in {}_1\mathscr{F}_m$ and g is a function measurable with respect to ${}_{k+m}\mathscr{F}_\infty$ and if x is a possible value of the* rv *g then $|P(A|g = x) - P(A)| \leqq 2\gamma\rho^k$.*

PROOF. Define for each integer m the events

$$H_m(x) = \{[x2^m]2^{-m} \leqq g < ([x2^m]+1)2^{-m}\}$$

where $[a]$ is the largest integer less than or equal to a. Notice that $P(H_m(x)) > 0$ for all m. It is known (page 335 of [5], that $P(A|g = x) = \lim_{m\to\infty} P(A|H_m(x))$. Then we have by Lemma 7.1 page 222 [3]

$$\begin{aligned} |P(A|g = x) - P(A)| &= \lim_{m\to\infty} |P(A|H_m(x)) - P(A)| \\ &= \lim_{m\to\infty} |E(\chi_A \chi_{H_m}) - E(\chi_A)E(\chi_{H_m})| E^{-1}(\chi_{H_m}) \\ &\leqq \lim_{m\to\infty} 2\gamma^{1/s}\rho^{k/s}E^{1/r}(\chi_A)E^{1/s}(\chi_{H_m})E^{-1}(\chi_{H_m}) \end{aligned} \tag{2.3}$$

for $r, s > 1$, $(1/r)+(1/s) = 1$.

Take $s = 1+(1/m)E(\chi_{H_m})$. Then $s(m, x) \to 1$ and $E^{1/s}(\chi_{H_m})E^{-1}(\chi_{H_m}) \to 1$ as $m \to \infty$. We therefore have from (2.3)

$$|P(A|g = x) - P(A)| \leqq 2\gamma\rho^k.$$

3. Convergence of partial sums. Let f be a real-valued function measurable with respect to $\mathscr{F}_X$ such that $E\{f(x_1)\} = 0$ and $E\{f^2(x_1)\} = \sigma^2$. In view of (2.2) we have for every k, $E\{f^2(x_k)\} = \sigma^2$. Without loss of generality σ may be taken to be 1 which we do. Then

$$\lim_{n\to\infty} E\{(n^{-\frac{1}{2}}\textstyle\sum_{j=1}^n f(x_j))^2\} = \sigma_1^2$$

exists. If $\sigma_1^2 > 0$ and if

$$E\{|f(x_1)|^{2+\delta}\} < \infty \tag{3.1}$$

for some $\delta > 0$ then it has been proved (Theorem 7.5 page 228 [3]) that

$$\lim_{n\to\infty} P(S_n \leqq x\sigma_1 n^{\frac{1}{2}}) = (2\pi)^{-\frac{1}{2}}\textstyle\int_{-\infty}^x e^{-(\frac{1}{2})t^2}dt = \Phi(x) \tag{3.2}$$

where $S_n = \sum_{j=1}^n f(x_j)$. Throughout this paper we shall assume that $\sigma_1^2 > 0$ and that (3.1) holds for some $\delta \leqq 1$.

The purpose of this section is to obtain an estimate of the difference between the distribution of $(S_n)/\sigma_1 n^{\frac{1}{2}}$ and the standard normal distribution.

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Let $\alpha_n = [n^{\frac{3}{4}}]$ and $\beta_n = [n^{\frac{1}{4}}]$. Then $\mu_n = [n(\alpha_n+\beta_n)^{-1}] \sim \beta_n$. For notational convenience we shall ignore the suffix n and write $\alpha_n = \alpha$, $\beta_n = \beta$ and $\mu_n = \mu$. Define

$$y_m = \sum_{j=(m-1)(\alpha+\beta)+1}^{(m-1)(\alpha+\beta)+\alpha} f(x_j) \qquad m = 1, 2, \cdots, \mu$$

$$(3.3) \qquad y_\mu' = \sum_{j=(m-1)(\alpha+\beta)+\alpha+1}^{m(\alpha+\beta)} f(x_j) \qquad m = 1, 2, \cdots, \mu$$

$$y'_{m+1} = \sum_{j=\mu(\alpha+\beta)+1}^{n} f(x_j).$$

Write

$$(3.4) \qquad T_r = \sum_{m=1}^{r} y_m \quad \text{and} \quad V_n = \sum_{m=1}^{\mu+1} y_m'.$$

Under the assumption (2.2) y_m's are identically distributed. Let $F(x)$ denote the distribution function of y_1.

THEOREM 3.1. *There exists N_0 such that for $n \geqq N_0$*

$$\sup_x |P(S_n \leqq x\sigma_1 n^{\frac{1}{2}}) - \Phi(x)| \leqq C_4 \max\{n^{-\delta/8}, n^{-1/12}\}.$$

PROOF. Let $\eta = \eta(n)$ be an arbitrary positive number.

$$P(S_n \leqq x\sigma_1 n^{\frac{1}{2}}) = P(T_\mu + V_n \leqq x\sigma_1 n^{\frac{1}{2}}, |V_n| \leqq \eta\sigma_1 n^{\frac{1}{2}})$$
$$+ P(T_\mu + V_n \leqq x\sigma_1 n^{\frac{1}{2}}, |V_n| > \eta\sigma_1 n^{\frac{1}{2}})$$
$$(3.5) \qquad \leqq P(T_\mu \leqq (x+\eta)\sigma_1 n^{\frac{1}{2}}) + P(|V_n| > \eta\sigma_1 n^{\frac{1}{2}}).$$

Also

$$(3.6) \qquad P(S_n \leqq x\sigma_1 n^{\frac{1}{2}}) \geqq P(T_\mu \leqq (x-\eta)\sigma_1 n^{\frac{1}{2}}) - P(|V_n| > \eta\sigma_1 n^{\frac{1}{2}}).$$

Now consider

$$P(T_\mu \leqq u) = \int_{-\infty}^{\infty} P(T_\mu - y_1 \leqq u - x_1 \mid y_1 = x_1)\, dF(x_1)$$
$$= \int_{-\infty}^{\infty} P(T_\mu - y_1 \leqq u - x_1)\, dF(x_1) + \theta_1(n), \quad \text{say}.$$

By Corollary 2.1, $|\theta_1(n)| \leqq 2\gamma\rho^\beta$. Also $P(T_\mu - y_1 \leqq u - x_1) = \int_{-\infty}^{\infty} P(T_\mu - y_1 - y_2 \leqq u - x_1 - x_2)\, dF(x_2) + \theta_2^*(n)$ so that

$$P(T_\mu \leqq u) = \int_{-\infty}^{\infty} \{\int_{-\infty}^{\infty} P(T_\mu - y_1 - y_2 \leqq u - x_1 - x_2)\, dF(x_2)\}\, dF(x_1) + \theta_1(n) + \theta_2(n)$$

where $|\theta_2(n)| \leqq \int_{-\infty}^{\infty} |\theta_2^*(n)|\, dF(x_1) \leqq 2\gamma\rho^\beta$.

Proceeding as above we get

$$(3.7) \qquad P(T_\mu \leqq u) = P(Z_1 + \cdots + Z_\mu \leqq u) + \sum_{j=1}^{\mu-1} \theta_j(n)$$

where $Z_1, \cdots, Z_\mu$ are independent random variables each distributed like y_1 and $|\theta_j(n)| \leqq 2\gamma\rho^\beta$, $1 \leqq j \leqq \mu - 1$. Also $E(Z_1 \alpha^{-\frac{1}{2}} \sigma_1^{-1})^2 \to 1$ as $n \to \infty$. It therefore follows that

$$\lim_{n\to\infty} P(Z_1 + \cdots + Z_\mu \leqq x\sigma_1 n^{\frac{1}{2}}) = \Phi(x).$$

In fact using Esseen's estimate [4] we get

$$(3.8) \qquad \sup_x |P(Z_1 + \cdots + Z_\mu \leqq x\sigma_1 n^{\frac{1}{2}}) - \Phi(x)| \leqq C_2 \mu^{-\delta/2} = C_2 n^{-\delta/8}$$

where C_2 does not depend on n.

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From relations (3.5) to (3.8) we have

$$(3.9)\qquad \begin{aligned}&\Phi(x-\eta)-C_2 n^{-\delta/8}+\textstyle\sum_{j=1}^{\mu-1}\theta_j(n)-P(|V_n|>\eta\sigma_1 n^{\frac{1}{2}})\\ &\leqq P(S_n\leqq x\sigma_1 n^{\frac{1}{2}})\\ &\leqq \Phi(x+\eta)+C_2 n^{-\delta/8}+\textstyle\sum_{j=1}^{\mu-1}\theta_j(n)+P(|V_n|>\eta\sigma_1 n^{\frac{1}{2}}).\end{aligned}$$

But $|\Phi(x)-\Phi(x\pm\eta)|\leqq\eta$. Following the discussion to prove (7.16), page 229 [3] we obtain $E(V_n{}^2)=O(n^{\frac{3}{4}})$. Applying Tchebyshev's inequality we get

$$P(|V_n|>\eta\sigma_1 n^{\frac{1}{2}})<C_3\eta^{-2}n^{-\frac{1}{4}}$$

where C_3 depends on σ_1 only.

We have then from (3.9)

$$\sup_x|P(S_n\leqq x\sigma_1 n^{\frac{1}{2}})-\Phi(x)|\leqq\eta+C_2 n^{-\delta/8}+2\gamma\mu\rho^\beta+C_3\eta^{-2}n^{-\frac{1}{4}}.$$

Taking $\eta=\max\{n^{-\delta/8},n^{-1/12}\}$ we get for n large, say, $\geqq N_0$

$$\sup_x|P(S_n\leqq x\sigma_1 n^{\frac{1}{2}})-\Phi(x)|\leqq C_4\eta.$$

4. An approximation theorem for a multidimensional distribution. Set

$\varepsilon_1=1/(3+\delta)$, $\varepsilon_2=\varepsilon_1\delta/4$, $\eta_1(n)=n^{-\varepsilon_2/2}(\log n)^{(1+\delta/2)\varepsilon_1}$ and $k(n)=[n^{\varepsilon_2}(\log n)^{\varepsilon_1}]$.

Define $\mu_i=[i\mu/k]$, $i=1,2,\cdots,k$.

In this section we approximate the distribution function of $(T_{\mu_1},\cdots,T_{\mu_k})$ with an appropriate k-dimensional normal distribution function. We follow the method of Chung [2].

Consider independent rv's $\xi_1,\cdots,\xi_k$ where ξ_j is distributed like $T_{\mu_j}-T_{\mu_{j-1}}$, $1\leqq j\leqq k$. $(T_{\mu_0}=0)$.

Denote $\zeta_i=\sum_{j=1}^i\xi_j$. Then

$$\begin{aligned}&P(T_{\mu_1}\leqq x_1,\cdots,T_{\mu_k}\leqq x_k)\\ &\quad=\textstyle\int_{-\infty}^{\infty}P(T_{\mu_1}\leqq x_1,\cdots,T_{\mu_{k-1}}\leqq\min(x_{k-1},x_k-u_k)\mid T_{\mu_k}-T_{\mu_{k-1}}=u_k)\,dP(\xi_k\leqq u_k)\\ &\quad=\textstyle\int_{-\infty}^{\infty}P(T_{\mu_1}\leqq x_1,\cdots,T_{\mu_{k-1}}\leqq\min(x_{k-1},x_k-u_k))\,dP(\xi_k\leqq u_k)+\Delta_1(n)\end{aligned}$$

where $|\Delta_1(n)|\leqq 2\gamma\rho^\beta$ by Lemma 2.2.

Also

$$\begin{aligned}&P(T_{\mu_1}\leqq x_1,\cdots,T_{\mu_{k-1}}\leqq\min(x_{k-1},x_k-u_k))\\ &\qquad=\textstyle\int_{-\infty}^{\infty}P(T_{\mu_1}\leqq x_1,\cdots,T_{\mu_{k-2}}\\ &\qquad\leqq\min(x_{k-2},x_{k-1}-u_{k-1},x_k-u_k-u_{k-1})\mid T_{\mu_{k-1}}-T_{\mu_{k-2}}=u_{k-1})\\ &\qquad\qquad\cdot dP(\xi_{k-1}\leqq u_{k-1})\end{aligned}$$

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so that

$$P(T_{\mu_1} \leqq x_1, \cdots, T_{\mu_k} \leqq x_k)$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(T_{\mu_1} \leqq x_1, \cdots, T_{\mu_{k-2}} \leqq \min(x_{k-2}, x_{k-1} - u_{k-1}, x_k - u_k - u_{k-1}))$$
$$\cdot dP(\xi_{k-1} \leqq u_{k-1})\, dP(\xi_k \leqq u_k) + \Delta_1(n) + \Delta_2(n)$$

where $|\Delta_2(n)| \leqq 2\gamma\rho^{\beta}$.

Proceeding as above we arrive at

$$P(T_{\mu_1} \leqq x_1, \cdots, T_{\mu_k} \leqq x_k)$$
$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} P(\xi_1 \leqq \min(x_1, x_2 - u_2, \cdots, x_k - \textstyle\sum_{j=2}^{k} u_j)) \tag{4.1}$$
$$\cdot dP(\xi_2 \leqq u_2) \cdots dP(\xi_k \leqq u_k) + \textstyle\sum_{j=1}^{k-1} \Delta_j(n)$$
$$= P(\zeta_1 \leqq x_1, \cdots, \zeta_k \leqq x_k) + \textstyle\sum_{j=1}^{k-1} \Delta_j(n).$$

Denote $F_j(x_1, \cdots, x_j) = P(\zeta_1 \leqq x_1, \cdots, \zeta_j \leqq x_j)$, $1 \leqq j \leqq k$.

Let Φ_j be the j dimensional normal distribution function with the same first and second order moments as F_j. Let Φ_j^* be the one dimensional normal distribution function with mean zero and variance $= E(\xi_j^2)$. Denote $F_j(x_1, \cdots, x_j) - \Phi_j(x_1, \cdots, x_j) = R_j(x_1, \cdots, x_j)$ and $P(\xi_j \leqq u) - \Phi_j^*(u) = R_j^*(u)$.

In view of (3.7) and (3.8) there exists a constant C_5 such that

$$\sup |R_1| \leqq C_5 k^{\delta/2} n^{-\delta/8} \quad \text{and} \quad \sup |R_j^*| \leqq C_5 k^{\delta/2} n^{-\delta/8} \qquad 1 \leqq j \leqq k \tag{4.2}$$

for n large.

Consider

$$F_{j+1}(x_1, \cdots, x_{j+1})$$
$$= \int_{-\infty}^{\infty} F_j(x_1, \cdots, x_{j-1}, \min(x_j, x_{j+1} - u))\, dP(\xi_{j+1} \leqq u)$$
$$= \int_{-\infty}^{\infty} \{\Phi_j(x_1, \cdots, x_{j-1}, \min(x_j, x_{j+1} - u))$$
$$+ R_j(x_1, \cdots, x_{j-1}, \min(x_j, x_{j+1} - u))\}\, dP(\xi_{j+1} \leqq u)$$
$$= \int_{-\infty}^{\infty} \Phi_j(x_1, \cdots, x_{j-1}, \min(x_j, x_{j+1} - u))\, d\Phi_{j+1}^*(u)$$
$$+ \int_{-\infty}^{\infty} \Phi_j\, dR_{j+1}^* + \int_{-\infty}^{\infty} R_j\, dP(\xi_{j+1} \leqq u).$$

That is

$$R_{j+1}(x_1, \cdots, x_{j+1}) = \int_{-\infty}^{\infty} \Phi_j\, dR_{j+1}^* + \int_{-\infty}^{\infty} R_j\, dP(\xi_{j+1} \leqq u). \tag{4.3}$$

Now

$$\left|\int_{-\infty}^{\infty} R_j\, dP(\xi_{j+1} \leqq u)\right| \leqq \sup |R_j|.$$

$$\int_{-\infty}^{\infty} \Phi_j(x_1, \cdots, x_{j-1}, \min(x_j, x_{j+1} - u))\, dR_{j+1}^*(u)$$
$$= \int_{-\infty}^{x_{j+1} - x_j} \Phi_j(x_1, \cdots, x_j)\, dR_{j+1}^*(u)$$
$$+ \int_{x_{j+1} - x_j}^{\infty} \Phi_j(x_1, \cdots, x_{j-1}, x_{j+1} - u)\, dR_{j+1}^*(u)$$
$$= \Phi_j(x_1, \cdots, x_j) R_{j+1}^*(x_{j+1} - x_j) - \Phi_j(x_1, \cdots, x_j) R_{j+1}^*(x_{j+1} - x_j)$$
$$- \int_{x_{j+1} - x_j}^{\infty} R_{j+1}^*(u)\, d\Phi_j(x_1, \cdots, x_{j+1} - u)$$

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on integration by parts. Hence $|\int_{-\infty}^{\infty} \Phi_j \, dR_{j+1}^*| \leqq \sup |R_{j+1}^*|$. From (4.3) we therefore have

$$\sup |R_{j+1}| \leqq \sup |R_j| + \sup |R_{j+1}^*|.$$

Using the relations at (4.2) we get by induction $\sup |R_k| \leqq C_5 k^{1+\delta/2} n^{-\delta/8}$ for n large. From (4.1) and the above result we have

$$|P(T_{\mu_1} \leqq x_1, \cdots, T_{\mu_k} \leqq x_k) - \Phi_k(x_1, \cdots, x_k)| \leqq C_5 k^{1+\delta/2} n^{-\delta/8} + 2\gamma k \rho^{\beta}$$
$$\leqq C_6 k^{1+\delta/2} n^{-\delta/8}.$$

We thus proved

LEMMA 4.1. *There exist constants C_6 and N_1 such that for all $n \geqq N_1$*

$$\sup_{x_i, 1 \leqq i \leqq k} |F_k(x_1, \cdots, x_k) - \Phi_k(x_1, \cdots, x_k)| \leqq C_6 \eta_1,$$

where η_1 is defined at the beginning of this section.

5. Rate of convergence of $\max_{1 \leqq r \leqq n} S_r$. Set $S_n^* = \max_{1 \leqq r \leqq n} S_r$ and $S_n^{**} = \max_{1 \leqq j \leqq \mu} S_{(\alpha+\beta)j}$. The limit distribution of S_n^* has been obtained by Billingsley [1].

We shall write $\alpha + \beta = \alpha_1$. Observe that

$$P(S_n^* \leqq x\sigma_1 n^{\frac{1}{2}}) \leqq P(S_n^{**} \leqq x\sigma_1 n^{\frac{1}{2}}). \tag{5.1}$$

Let for each r, $\alpha_1(j(r)-1) < r \leqq \alpha_1 j(r)$. Define $D_r = \{S_{r-1}^* \leqq x\sigma_1 n^{\frac{1}{2}}, S_r > x\sigma_1 n^{\frac{1}{2}}\}$ so that

$$\sum_{r=1}^n P(D_r) = P(S_n^* > x\sigma_1 n^{\frac{1}{2}}). \tag{5.2}$$

Write $D_r = D_r^{(1)} \cup D_r^{(2)}$ where $D_r^{(1)} = \{D_r \cap \{|S_{\alpha_1 j(r)} - S_r| \leqq \eta_1 \sigma_1 n^{\frac{1}{2}}\}\}$ and $D_r^{(2)} = \{D_r \cap \{|S_{\alpha_1 j(r)} - S_r| > \eta_1 \sigma_1 n^{\frac{1}{2}}\}\}$.

$$\sum_{r=1}^n P(D_r^{(1)}) \leqq P(S_n^{**} > (x - \eta_1)\sigma_1 n^{\frac{1}{2}}). \tag{5.3}$$

In order to analyze $P(D_r^{(2)})$ we set $\delta_n = [n^{3\delta/(8+4\delta)}]$. Then if $\alpha_1 j(r) - r > \delta_n$

$$P(D_r^{(2)}) \leqq P(D_r \cap \{|S_{\alpha_1 j(r)} - S_{r+\delta_n}| > (\tfrac{1}{2})\eta_1 \sigma_1 n^{\frac{1}{2}}\})$$
$$+ P(|S_{r+\delta_n} - S_r| > (\tfrac{1}{2})\eta_1 \sigma_1 n^{\frac{1}{2}})$$
$$\leqq P(D_r)\{P(|S_{\alpha_1 j(r)} - S_{r+\delta_n}| > (\tfrac{1}{2})\eta_1 \sigma_1 n^{\frac{1}{2}}) + 2\gamma\rho^{\delta_n}\}$$
$$+ C_7 \delta_n^{(1+\delta/2)} \eta_1^{-(2+\delta)} n^{-(1+\delta/2)}$$

by Corollary 2.1 and Tchebyshev's inequality. Therefore

$$P(D_r^{(2)}) \leqq P(D_r)\{C_8 \alpha_1^{(1+\delta/2)} \eta_1^{-(2+\delta)} n^{-(1+\delta/2)} + 2\gamma\rho^{\delta_n}\}$$
$$+ C_7 \delta_n^{(1+\delta/2)} \eta_1^{-(2+\delta)} n^{-(1+\delta/2)}.$$

$$\sum_{r=1}^n P(D_r^{(2)}) \leqq C_9 \eta_1^{-(2+\delta)} n^{-(2+\delta)/8} + 2\gamma\rho^{\delta_n} + C_7 \delta_n^{(1+\delta/2)} \eta_1^{-(2+\delta)} n^{-\delta/2}.$$

If $\alpha_1 j(r) - r < \delta_n$ also this inequality holds.

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We have from (5.2), (5.3) and the above inequality

$$P(S_n^* > x\sigma_1 n^{\frac{1}{2}}) \leqq P(S_n^{**} > (x-\eta_1)\sigma_1 n^{\frac{1}{2}}) + C_9 \eta_1^{-(2+\delta)} n^{-(2+\delta)/8} + 2\gamma\rho^{\delta_n} + C_7 \delta_n^{(1+\delta/2)} \eta_1^{-(2+\delta)} n^{-\delta/2}.$$

This together with (5.1) gives

LEMMA 5.1. *For η_1 as defined in Section 4 there exist constants C_{10} and N_2 such that for $n \geqq N_2$*

$$P(S_n^{**} \leqq (x-\eta_1)\sigma_1 n^{\frac{1}{2}}) - C_{10}\eta_1 \leqq P(S_n^* \leqq x\sigma_1 n^{\frac{1}{2}}) \leqq P(S_n^{**} \leqq x\sigma_1 n^{\frac{1}{2}}).$$

Denote $U_j = y_1' + \cdots + y_j'$ and the event $\{|U_j| \leqq \eta_1 \sigma_1 n^{\frac{1}{2}}\} = M_j$. Then

$$\begin{aligned} P(S_n^{**} \leqq x\sigma_1 n^{\frac{1}{2}}) &= P(\{S_n^{**} \leqq x\sigma_1 n^{\frac{1}{2}}\} \cap \{\textstyle\bigcap_{j=1}^{\mu} M_j\}) \\ &\quad + P(\{S_n^{**} \leqq x\sigma_1 n^{\frac{1}{2}}\} \cap \{\textstyle\bigcap_{j=1}^{\mu} M_j\}') \\ &\leqq P(\max_{1 \leqq r \leqq \mu} T_r \leqq (x+\eta_1)\sigma_1 n^{\frac{1}{2}}) + P(\textstyle\bigcup_{j=1}^{\mu} M_j') \qquad (5.4) \\ &\leqq P(\max_{1 \leqq r \leqq \mu} T_r \leqq (x+\eta_1)\sigma_1 n^{\frac{1}{2}}) + C_{11}\eta_1^{-2} n^{-\frac{1}{2}}. \end{aligned}$$

Similarly

$$P(S_n^{**} \leqq x\sigma_1 n^{\frac{1}{2}}) \geqq P(\max_{1 \leqq r \leqq \mu} T_r \leqq (x-\eta_1)\sigma_1 n^{\frac{1}{2}}) - C_{11}\eta_1^{-2} n^{-\frac{1}{2}}. \qquad (5.5)$$

From (5.4), (5.5) and Lemma 5.1 we have the following

LEMMA 5.2. *For η_1 as defined in Section 4 there exist constants C_{12} and N_3 such that for $n \geqq N_3$*

$$\begin{aligned} P(\max_{1 \leqq r \leqq \mu} T_r &\leqq (x-2\eta_1)\sigma_1 n^{\frac{1}{2}}) - C_{12}\eta_1 \\ &\leqq P(S_n^* \leqq x\sigma_1 n^{\frac{1}{2}}) \\ &\leqq P(\max_{1 \leqq r \leqq \mu} T_r \leqq (x+\eta_1)\sigma_1 n^{\frac{1}{2}}) + C_{12}\eta_1. \end{aligned}$$

Set $T_\mu^* = \max_{1 \leqq r \leqq \mu} T_r$ and $T_\mu^{**} = \max_{1 \leqq i \leqq k} T_{\mu_i}$ where μ_i's are as defined in Section 4.

LEMMA 5.3. *We can find constants C_{16} and N_4 such that for $n \geqq N_4$*

$$P(T_\mu^{**} \leqq (x-\eta_1)\sigma_1 n^{\frac{1}{2}}) - C_{16}\eta_1 \leqq P(T_\mu^* \leqq x\sigma_1 n^{\frac{1}{2}}) \leqq P(T_\mu^{**} \leqq x\sigma_1 n^{\frac{1}{2}}).$$

PROOF. It is easily seen that

$$P(T_\mu^* \leqq x\sigma_1 n^{\frac{1}{2}}) \leqq P(T_\mu^{**} \leqq x\sigma_1 n^{\frac{1}{2}}). \qquad (5.6)$$

Define the events

$$E_r = \{T_{r-1}^* \leqq x\sigma_1 n^{\frac{1}{2}},\ T_r > x\sigma_1 n^{\frac{1}{2}}\}.$$

Then

$$\textstyle\sum_{r=1}^{\mu} P(E_r) = P(T_\mu^* > x\sigma_1 n^{\frac{1}{2}}). \qquad (5.7)$$

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Suppose $\mu_{j(r)} < r \leqq \mu_{j(r)+1}$. Then for any positive number $\eta_1 = \eta_1(n)$

(5.8) $$E_r = \{E_r \cap \{|T_{\mu_{j(r)+1}} - T_r| > \eta_1 \sigma_1 n^{\frac{1}{2}}\}\} \cup \{E_r \cap \{|T_{\mu_{j(r)+1}} - T_r| \leqq \eta_1 \sigma_1 n^{\frac{1}{2}}\}\}$$
$$= E_r^{(1)} \cup E_r^{(2)}, \quad \text{say.}$$

(5.9) $$\sum_{r=1}^{\mu} P(E_r^{(2)}) \leqq P(T_\mu^{**} > (x-\eta_1)\sigma_1 n^{\frac{1}{2}}).$$

By Corollary 2.1

(5.10) $$P(E_r^{(1)}) \leqq P(E_r)\{P(|T_{\mu_{j(r)+1}} - T_r| > \eta_1 \sigma_1 n^{\frac{1}{2}}) + 2\gamma\rho^\beta\}.$$

It is easily shown as in Theorem 3.1 that

(5.11) $$P(|T_{\mu_{j(r)+1}} - T_r| > \eta_1 \sigma_1 n^{\frac{1}{2}}) \leqq P(|Z_1 + \cdots + Z_m| > \eta_1 \sigma_1 n^{\frac{1}{2}}) + 4m\gamma\rho^\beta,$$

where $m = \mu_{j(r)+1} - r$.

Let $B = B(n) = [n^{\frac{1}{4}}]$. If $m \leqq B$ then by Tchebyshev's inequality and Lemma 7.4, page 225, [3] we have

(5.12) $$P(|Z_1 + \cdots + Z_m| > \eta_1 \sigma_1 n^{\frac{1}{2}}) \leqq C_{13}\eta_1^{-(2+\delta)} n^{-(2+\delta)/8} B.$$

If $m > B$ using Lemma 7.4 and the Esseen's estimate we get

(5.13) $$|P(|Z_1 + \cdots + Z_m| > \eta_1 \sigma_1 n^{\frac{1}{2}}) - 2^{\frac{1}{2}}\pi^{-\frac{1}{2}} \int_v^\infty \exp(-(\tfrac{1}{2})t^2)\,dt| \leqq C_{14} B^{-\delta/2}$$

where C_{14} depends only on σ_1; and $v = \eta_1 n^{\frac{1}{2}} m^{-\frac{1}{2}}$. Since $m \leqq \mu/k$, $v > \eta_1 k^{\frac{1}{2}}$. Now

(5.14) $$2^{\frac{1}{2}}\pi^{-\frac{1}{2}} \int_v^\infty \exp(-(\tfrac{1}{2})t^2)\,dt \leqq C_{15} v^{-1} e^{-(\frac{1}{2})v^2} \leqq C_{15}\eta_1^{-1} k^{-\frac{1}{2}} e^{-(\frac{1}{2})\eta_1^2 k}$$
$$= C_{15}\eta_1^{-1} k^{-\frac{1}{2}} n^{-\frac{1}{2}}.$$

F rom the relations (5.7) to (5.14) we get

$$P(T_\mu^* > x\sigma_1 n^{\frac{1}{2}}) \leqq P(T_\mu^{**} > (x-\eta_1)\sigma_1 n^{\frac{1}{2}}) + C_{16}\eta_1.$$

This together with (5.6) gives the result.

From Lemmas 4.1, 5.2 and 5.3 we have with some constant C_{17}

(5.15) $$\Phi_k((x-3\eta_1)\sigma_1 n^{\frac{1}{2}}, \cdots, (x-3\eta_1)\sigma_1 n^{\frac{1}{2}}) - C_{17}\eta_1 \leqq P(S_n^* \leqq x\sigma_1 n^{\frac{1}{2}})$$
$$\leqq \Phi_k((x+\eta_1)\sigma_1 n^{\frac{1}{2}}, \cdots, (x+\eta_1)\sigma_1 n^{\frac{1}{2}}) + C_{17}\eta_1$$

for $n \geqq \max(N_1, N_3 \text{ and } N_4)$.

If $\{x_n\}$ is a sequence of independent Bernoulli variables defined by $P(x_n = \pm 1) = \frac{1}{2}$ and $f(x) = x$ then it is well known that

(5.16) $$|P(S_n^* \leqq x\sigma_1 n^{\frac{1}{2}}) - I^*(x)| \leqq C_{18} n^{-\frac{1}{2}}$$ where

(5.17) $$I^*(x) = 2^{\frac{1}{2}}\pi^{-\frac{1}{2}} \int_0^x \exp(-(\tfrac{1}{2})t^2)\,dt.$$

We have $\sigma_1 = 1$ in this case. Applying the inequality (5.15) to the Bernoulli variables and using (5.16) we have

(5.18) $$\Phi_k((x-3\eta_1)\sigma_1 n^{\frac{1}{2}}, \cdots, (x-3\eta_1)\sigma_1 n^{\frac{1}{2}}) - C_{18}\eta_1$$
$$\leqq I^*(x) \leqq \Phi_k((x+\eta_1)\sigma_1 n^{\frac{1}{2}}, \cdots, (x+\eta_1)\sigma_1 n^{\frac{1}{2}}) + C_{18}\eta_1.$$

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Replacing x by $x+4\eta$ and $x-4\eta$ and using the fact that $|I^*(x)-I^*(x\pm 4\eta)| \leqq C_{19}\eta_1$ we get from (5.15) and (5.18) the following

THEOREM 5.1. *There exist constants C_{20} and N_5 such that for $n \geqq N_5$*

$$\sup_x |P(S_n{}^* \leqq x\sigma_1 n^{\frac{1}{2}}) - I^*(x)| \leqq C_{20}(\log n)^{\varepsilon_1(1+\delta/2)} n^{-\varepsilon_2/2}$$

where $\varepsilon_1 = 1/(3+\delta)$, $\varepsilon_2 = \varepsilon_1 \delta/4$ *and $I^*(x)$ is defined at* (5.17).

6. The law of the iterated logarithm.

THEOREM 6.1.

$$P\{\limsup \{(S_n)/(2\sigma_1{}^2 n \log\log n)^{\frac{1}{2}}\} = 1\} = 1.$$

PROOF. Write $\chi(n) = (2\sigma_1{}^2 n \log\log n)^{\frac{1}{2}}$.

From Theorem 3.1 we get for every b

$$|P(S_n \leqq b\chi(n)) - \Phi(b(2\log\log n)^{\frac{1}{2}})| \leqq C_{21} \max(n^{-\delta/8}, n^{-1/12}).$$

Using the asymptotic relation for $1-\Phi(x)$ we get from the above inequality

$$(\log n)^{-(1+\theta)b^2} < P(S_n > b\chi(n)) < (\log n)^{-b^2} \tag{6.1}$$

for any positive constants θ and b.

Corresponding to every $\tau < 1$ and integer k we can find an n_k such that $n_k \to \infty$ as $k \to \infty$ and $n_{k-1} < \tau^k \leqq n_k$, $k = 1, 2, \cdots$. We assume that $n_0 = 0$. Then

$$n_k \sim \tau^k \quad \text{and} \quad n_k - n_{k-1} \sim n_k(\tau-1)/\tau. \tag{6.2}$$

We have from Theorem 5.1 for any $\xi > 0$

$$P(S_{n_k}^* > (1+\xi)\chi(n)) \leqq 1 - I^*((1+\xi)(2\log\log n_k)^{\frac{1}{2}}) + C_{20}(\log n_k)^{\varepsilon_1(1+\delta/2)} n_k{}^{-\varepsilon_2/2}.$$

For k large, say, $\geqq K$, the right-hand side

$$\leqq C_{22}(2\log\log n_k)^{-\frac{1}{2}}(\log n_k)^{-(1+\xi)^2} + C_{20}(\log n_k)^{\varepsilon_1(1+\delta/2)} n_k{}^{-\varepsilon_2/2}$$
$$\leqq C_{23}\, k^{-(1+\xi)^2} + C_{24}\, k^{\varepsilon_1(1+\delta/2)} \tau^{-k\varepsilon_2/2}$$

so that

$$\textstyle\sum_{k=K}^{\infty} P(S_{n_k}^* > (1+\xi)\chi(n_k)) < \infty. \tag{6.3}$$

Let ε be an arbitrary positive number. Consider

$$P(S_n > (1+\varepsilon)\chi(n)\,\text{i.o.}) \leqq P\{\max_{n_{k-1} \leqq n \leqq n_k} S_n > (1+\varepsilon)\chi(n_{k-1})\,\text{i.o.}\}$$
$$\leqq P\{\max_{1 \leqq n \leqq n_k} S_n > (1+\varepsilon)\chi(n_{k-1})\,\text{i.o.}\}$$

By (6.2) $\{\chi(n_k)\}/\{\chi(n_{k-1})\} \leqq (2\tau-1)^{\frac{1}{2}}$ for large k. Let τ be chosen such that $(1+\varepsilon)(2\tau-1)^{-\frac{1}{2}} > 1+\xi$. Then

$$P(S_n > (1+\varepsilon)\chi(n)\,\text{i.o.}) \leqq P(S_{n_k}^* > (1+\xi)\chi(n_k)\,\text{i.o.}). \tag{6.4}$$

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By the Borel–Cantelli lemma we get from (6.3) and (6.4)

$$P(S_n > (1+\varepsilon)\chi(n)\,\text{i.o.}) = 0 \tag{6.5}$$

for any $\varepsilon > 0$.

Proof of the theorem will be complete if we show that $P(S_n > (1-\varepsilon)\chi(n)\,\text{i.o.}) = 1$ for any $\varepsilon > 0$.

Let us denote $\psi(n_k) = [2\sigma_1{}^2(n_k - n_{k-1})\log\log(n_k - n_{k-1})]^{\frac{1}{2}}$. Set $m_k = [n_{k-1} + \tau^{2\log k}]$. Consider for any positive $\xi < 1$

$$\begin{aligned} P(W_k) &= P(S_{n_k} - S_{m_k} > (1-\xi)\psi(n_k)) \\ &\geqq P(\{S_{n_k} > (1-(\tfrac{1}{2})\xi)\psi(n_k)\} \cap \{S_{m_k} > (\tfrac{1}{2})\xi\psi(n_k)\}) \\ &\geqq P(S_{n_k} > (1-(\tfrac{1}{2})\xi)\psi(n_k)) - P(S_{m_k} > (\tfrac{1}{2})\xi\psi(n_k)). \end{aligned} \tag{6.6}$$

Now $\{\psi(n_k)\}/\{\chi(m_k)\} \to (\tau-1)^{\frac{1}{2}}$ and $\{\psi(n_k)\}/\{\chi(n_k)\} \to ((\tau-1)/\tau)^{\frac{1}{2}} < 1$. Using (6.1) we then have from (6.6) for any positive constant θ

$$\begin{aligned} P(W_k) &\geqq (\log n_k)^{-(1+\theta)(1-(\frac{1}{2})\xi)^2} - (\log n_{k-1})^{-\xi^2(\tau-1)/5} \\ &\geqq C_{25}\{k^{-(1+\theta)(1-(\frac{1}{2})\xi)^2} - k^{-\xi^2(\tau-1)/5}\} \\ &\geqq (\tfrac{1}{2})C_{25}\, k^{-(1+\theta)(1-(\frac{1}{2})\xi)^2} \end{aligned}$$

for sufficiently large k and τ. The constant C_{25} in the above inequality is independent of k. If we choose θ sufficiently small so that $(1+\theta)(1-(\frac{1}{2})\xi)^2 < 1$ we obtain

$$\textstyle\sum_{k=K}^{\infty} P(W_k) = \infty. \tag{6.7}$$

By Corollary 2.1

$$|P(W_k \mid W_{k-1}, \cdots, W_1) - P(W_k)| \leqq 2\gamma\rho^{\tau^{2\log k}}.$$

Since $\sum_{k=1}^{\infty} \rho^{\tau^{2\log k}}$ converges, we get from (6.7) $\sum_{k=K}^{\infty} P(W_k \mid W_{k-1}, \cdots, W_1) = \infty$. Then by Corollary 2 page 324 [3] we have

$$P(W_k\ \text{i.o.}) = 1 \tag{6.8.}$$

for any positive $\xi < 1$. Now as $k \to \infty$

$$(1-\xi)\psi(n_k) - 2\chi(m_k) \sim \{(1-\xi)(\tau-1)^{\frac{1}{2}}\tau^{-\frac{1}{2}} - 2\tau^{-\frac{1}{2}}\}\chi(n_k).$$

If ε is an arbitrary fixed positive constant, we can choose positive numbers ξ and τ so that $(1-\xi)(\tau-1)^{\frac{1}{2}}\tau^{-\frac{1}{2}} - 2\tau^{-\frac{1}{2}} > 1-\varepsilon$. Then

$$\begin{aligned} &P(S_{n_k} > (1-\varepsilon)\chi(n_k)\,\text{i.o.}) \\ &\qquad \geqq P(S_{n_k} > (1-\xi)\psi(n_k) - 2\chi(m_k)\,\text{i.o.}) \\ &\qquad \geqq P(S_{n_k} - S_{m_k} > (1-\xi)\psi(n_k)\,\text{i.o.}) \end{aligned}$$

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because from (6.5) $|S_n| \leqq 2\chi(n)$ for $n \geqq N_5(\omega)$ and all $\omega \in \Omega$ except for a set of probability measure zero. It now follows from (6.8) that

$$P(S_{n_k} > (1-\varepsilon)\chi(n_k)\,\text{i.o.}) = 1. \tag{6.9}$$

The assertion is an immediate consequence of (6.9).

NOTE. By standard arguments we relax the assumption (2.2) that the initial distribution is the stationary absolute probability distribution.

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