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Coefficient Inequalities and Convolution Conditions

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Abstract. Some interesting sufficient conditions involving coefficient inequalities for functions belonging to the classes $\mathcal{S}^*(A,B)$, $\mathcal{K}(A,B)$, $\mathcal{S}^*_\lambda(A,B)$ and $\mathcal{K}_\lambda(A,B)$ are derived. Several known convolution conditions, coefficient inequalities are special cases and consequences of these coefficient inequalities.

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1. Introduction

Let $\mathcal{N}$ denote the class of normalized analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

defined in the unit disc $\mathcal{U} = \{z : |z| < 1\}$. For any two real numbers A and B, $-1 \leq B < A \leq 1$, the class $\mathcal{P}(A,B)$ [1] consists of functions of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$$

which are analytic in the unit disc $\mathcal{U}$ such that

$$p(z) = \frac{1 + A\omega(z)}{1 + B\omega(z)}, \ z \in \mathcal{U}$$

where $\omega(z)$ is analytic in $\mathcal{U}$ satisfying the conditions $\omega(0) = 0$, $|\omega(z)| < 1$, $z \in \mathcal{U}$. The classes $\mathcal{S}^*(A,B)$, $\mathcal{K}(A,B)$, $\mathcal{S}^*_\lambda(A,B)$ and $\mathcal{K}_\lambda(A,B)$ are defined as follows:

$$\mathcal{S}^*(A,B) := \left\{ f(z) \in \mathcal{N} : \frac{zf'(z)}{f(z)} \in \mathcal{P}(A,B) \right\}$$

$$\mathcal{K}(A,B) := \left\{ f(z) \in \mathcal{N} : \; 1 + \frac{zf''(z)}{f'(z)} \in \mathcal{P}(A,B) \right\}$$

$$\mathcal{S}^*_\lambda(A,B) := \left\{ f(z) \in \mathcal{N} : \; \frac{e^{i\lambda}\left(\frac{zf'}{f}\right) - i\sin\lambda}{\cos\lambda} \in \mathcal{P}(A,B) \right\}$$

$$\mathcal{K}_\lambda(A,B) := \left\{ f(z) \in \mathcal{N} : \; \frac{e^{i\lambda}\left(\frac{zf'}{f}\right)' - i\sin\lambda}{\cos\lambda} \in \mathcal{P}(A,B) \right\}$$

For $A = 1$ and $B = -1$ we get the well known classes $\mathcal{S}^*$, $\mathcal{K}$, $\mathcal{S}^*_\lambda$ and $\mathcal{K}_\lambda$.

2. Preliminary Lemmas

In this section we prove the following necessary and sufficient conditions.

Lemma 2.1. *A function* $p(z) \in \mathcal{P}(A,B)$ *if and only if*

$$p(z) \neq \frac{1 + A\zeta}{1 + B\zeta} \quad (z \in \mathcal{U}, \; \zeta \in \mathbb{C}, \; |\zeta| = 1).$$

Proof. The proof is quite obvious. For, consider the linear transformation

$$\omega = \frac{1 + Az}{1 + Bz}$$

which maps the unit circle $\delta\mathcal{U}$ onto the imaginary axis $\Re\{\omega\} = 0$. Indeed, for all ζ such that $|\zeta| = 1$ ($\zeta \in \mathbb{C}$), set

$$\omega = \frac{1 + A\zeta}{1 + B\zeta} \quad (\zeta \in \mathbb{C}, \; |\zeta| = 1)$$

so that

$$|\zeta| = \left| \frac{\omega - 1}{A - \omega B} \right| = 1$$

which shows that $\Re\{\omega\} = \Re\left\{\dfrac{1 + Az}{1 + Bz}\right\} = 0 \quad (\zeta \in \mathbb{C}, \; |\zeta| = 1)$. Since $p(0) = 1$, for $p(z) \in \mathcal{P}(A,B)$ we know that

$$p(z) \neq \frac{1 + A\zeta}{1 + B\zeta} \quad (z \in \mathcal{U}, \; \zeta \in \mathbb{C}, \; |\zeta| = 1)$$

which completes the proof. □

Lemma 2.2. *A function* $f(z) \in \mathcal{N}$ *is in the class* $\mathcal{S}^*(A,B)$ *if and only if*

$$1 + \sum_{n=2}^{\infty} A_n z^{n-1} \neq 0 \tag{2.1}$$

where

$$A_n = \frac{(n-1) + (nB - A)\zeta}{\zeta(B - A)} \, a_n.$$

Proof. A function $f(z) \in \mathcal{N}$ is in the class $\mathcal{S}^*(A,B)$ if and only if

$$\frac{zf'(z)}{f(z)} \neq \frac{1+A\zeta}{1+B\zeta}$$

That is,

$$(1+B\zeta)\,(zf'(z)) - (1+A\zeta)f(z) \neq 0$$

which implies

$$(B-A)\zeta z + \sum_{n=2}^{\infty} [n(1+B\zeta) - (1+A\zeta)]\, a_n z^n \neq 0.$$

This simplifies into

$$(B-A)\zeta z \left[1 + \sum_{n=2}^{\infty} \frac{n(1+B\zeta)-(1+A\zeta)}{\zeta(B-A)}\, a_n z^{n-1}\right] \neq 0. \tag{2.2}$$

Dividing both sides of (2.2) by $(B-A)\zeta z \ (z \neq 0)$ we obtain

$$1 + \sum_{n=2}^{\infty} \frac{(n-1)+(nB-A)\zeta}{\zeta(B-A)}\, a_n z^{n-1} \neq 0 \quad (z \in \mathcal{U},\ \zeta \in \mathbb{C},\ |\zeta| = 1)$$

which completes the proof. □

Remark 2.3. *It follows from the normalization conditions $a_0 = 0$ and $a_1 = 1$ that*

$$A_0 = -\frac{(1+A\zeta)}{B-A}\, a_0 = 0, \quad A_1 = \frac{(1+B\zeta)-(1+A\zeta)}{(B-A)\zeta}\, a_1 = 1.$$

Remark 2.4. *The assertion (2.1) of Lemma 2.2 is equivalent to*

$$\frac{1}{z}\left(f(z) * \frac{\zeta z(B-A)+(1+A\zeta)z^2}{(1-z)^2}\right) \neq 0 \ \ (|z| < R,\ |\zeta| = 1)$$

which was given earlier by Ganesan [2].

Remark 2.5. *Further more, for $A = 1$ and $B = -1$ as a special case we get convolution conditions characterizing starlike functions as in* [3] *with a suitable modification.*

3. Main Results

In this section, we determine certain conditions for functions in the class $\mathcal{N}$.

Theorem 3.1. *If $f(z) \in \mathcal{N}$ satisfies the following condition*

$$\sum_{n=2}^{\infty}\left[\left|\sum_{k=1}^{n}\left(\sum_{j=1}^{k}(-1)^{k-j}(j-1)\binom{\beta}{k-j}a_j\right)\binom{\gamma}{n-k}\right| + \right.$$

$$\left.\left|\sum_{k=1}^{n}\left(\sum_{j=1}^{k}(-1)^{k-j}(jB-A)\binom{\beta}{k-j}a_j\right)\binom{\gamma}{n-k}\right|\right] \leq B-A \tag{3.1}$$

with $\beta,\ \gamma \in \mathbb{R}$ and $-1 \leq B < A \leq 1$ then $f(z) \in \mathcal{S}^*(A,B)$.

Proof. We note that $(1-z)^\beta \neq 0$ and $(1+z)^\gamma \neq 0$ for β, $\gamma \in \mathbb{R}$ and $z \in \mathcal{U}$. If the inequality

$$\left(1+\sum_{n=2}^{\infty} A_n z^{n-1}\right)(1-z)^\beta (1+z)^\gamma \neq 0 \ (z \in \mathcal{U} \ \beta, \gamma \in \mathbb{R}) \tag{3.2}$$

holds true, then we have,

$$1+\sum_{n=2}^{\infty} A_n z^{n-1} \neq 0,$$

which is the relation (2.1) of Lemma 2.2. Equation (2.1) is equivalent to

$$\left(1+\sum_{n=2}^{\infty} A_n z^{n-1}\right)\left(\sum_{n=0}^{\infty}(-1)^n b_n z^n\right)\left(\sum_{n=0}^{\infty} c_n z^n\right) \neq 0 \tag{3.3}$$

where, $b_n = \binom{\beta}{n}$ and $c_n = \binom{\gamma}{n}$.

Considering the Cauchy product of the first two factors, expression (3.3) can be rewritten as

$$\left(1+\sum_{n=2}^{\infty} B_n z^{n-1}\right)\left(\sum_{n=0}^{\infty} c_n z^n\right) \neq 0 \tag{3.4}$$

where

$$B_n = \sum_{j=1}^{n}(-1)^{n-j} A_j b_{n-j}.$$

Further, by applying the Cauchy product again in (3.4) we find that

$$1+\sum_{n=2}^{\infty}\left(\sum_{k=1}^{n} B_k c_{n-k}\right) z^{n-1} \neq 0 \quad (z \in \mathcal{U}).$$

Equivalently, we have

$$1+\sum_{n=2}^{\infty}\left[\left(\sum_{k=1}^{n}(-1)^{k-j} A_j b_{k-j}\right) c_{n-k}\right] z^{n-1} \neq 0 \quad (z \in \mathcal{U}).$$

If $f(z) \in \mathcal{N}$ satisfies the following inequality

$$\sum_{n=2}^{\infty}\left|\sum_{k=1}^{n}\left(\sum_{j=1}^{k}(-1)^{k-j} A_j b_{k-j}\right) c_{n-k} z^{n-1}\right| \neq 0 \quad (z \in \mathcal{U}).$$

That is if

$$\frac{1}{\zeta(B-A)}\sum_{n=2}^{\infty}\left|\sum_{k=1}^{n}\left(\sum_{j=1}^{k}(-1)^{k-j}\left[(j-1)+(jB-A)\zeta\right] a_j b_{k-j}\right) c_{n-k}\right|$$

$$\leq \frac{1}{(B-A)}\sum_{n=2}^{\infty}\left(\left|\sum_{k=1}^{n}\left[\sum_{j=1}^{k}(-1)^{k-j}(j-1) a_j b_{k-j}\right] c_{n-k}\right| + \right.$$

$$\left|\sum_{k=1}^{n}\left[\sum_{j=1}^{k}(-1)^{k-j}(jB-A)a_j b_{k-j}\right]c_{n-k}\right|\Bigg) \leq 1$$

for $-1 \leq B < A \leq 1,\ \zeta \in \mathbb{C},\ |\zeta| = 1$ then, $f(z) \in \mathcal{S}^*(A,B)$ which establishes the result. □

Corollary 3.2. *For $A = 1-2\alpha$ and $B = -1$ we get coefficient conditions for functions in the class $\mathcal{S}^*(\alpha)$ [2] with suitable modifications.*

Moreover, for $\beta = \gamma = 0$, $A = 1$, $B = -1$ in Theorem 3.1 we obtain the following result.

Corollary 3.3. *If $f(z) \in \mathcal{N}$ satisfies the following coefficient inequality*

$$\sum_{n=2}^{\infty}(n-\alpha)\,|a_n| \leq 1-\alpha\ (0 \leq \alpha < 1) \tag{3.5}$$

then $f(z) \in S^*(\alpha)$.

In particular, by putting $\alpha = 0$ in (3.5) we get the following well - known coefficient condition for the class of starlike functions in $\mathcal{U}$.

Corollary 3.4. *If $f(z) \in \mathcal{N}$ satisfies*

$$\sum_{n=2}^{\infty} n\,|a_n| \leq 1 \tag{3.6}$$

then $f(z) \in \mathcal{S}^*$.

The following theorem gives the coefficient condition for functions $f(z)$ to be in the class $\mathcal{K}(A,B)$.

Theorem 3.5. *If $f(z) \in \mathcal{N}$ satisfies the condition*

$$\sum_{n=2}^{\infty}\left[\left|\sum_{k=1}^{n}\left(\sum_{j=1}^{k}(-1)^{k-j}j(j-1)\binom{\beta}{k-j}a_j\right)\binom{\gamma}{n-k}\right| + \left|\sum_{k=1}^{n}\left(\sum_{j=1}^{k}(-1)^{k-j}j(jB-A)\binom{\beta}{k-j}a_j\right)\binom{\gamma}{n-k}\right|\right] \leq B-A \tag{3.7}$$

for $-1 \leq B < A \leq 1,\ \beta, \gamma \in \mathbb{R}$, then $f(z) \in \mathcal{K}(A,B)$.

Proof. Since $f(z)$ is in the class $\mathcal{K}(A,B)$ if and only if $zf'(z)$ belongs to the class $\mathcal{S}^*(A,B)$, replacing a_j in the statement of the Theorem 3.1 by ja_j we get the required result. □

Corollary 3.6. *For $A = 1-2\alpha$ and $B = -1$ we get coefficient conditions for functions in the class $\mathcal{K}(\alpha)$ [2] with suitable modifications.*

Moreover, for $\beta = \gamma = 0$, $A = 1$, $B = -1$ in Theorem 3.5 we obtain the following result.

Corollary 3.7. *If $f(z) \in \mathcal{N}$ satisfies the following coefficient inequality*

$$\sum_{n=2}^{\infty} n(n-\alpha)\,|a_n| \le 1-\alpha \ (0 \le \alpha < 1) \tag{3.8}$$

then $f(z) \in \mathcal{K}(\alpha)$.

In particular, by putting $\alpha = 0$ in (3.5) we get the following well - known coefficient condition for the class of convex functions in $\mathcal{U}$.

Corollary 3.8. *If $f(z) \in \mathcal{N}$ satisfies*

$$\sum_{n=2}^{\infty} n^2\,|a_n| \le 1 \tag{3.9}$$

then $f(z) \in \mathcal{K}$.

4. Coefficient conditions for functions in the classes $\mathcal{S}^*_\lambda(A,B)$ and $\mathcal{K}_\lambda(A,B)$.

In this section, we obtain coefficient conditions for functions belonging to the classes $\mathcal{S}^*_\lambda(A,B)$ and $\mathcal{K}_\lambda(A,B)$.

Lemma 4.1. *A function $f(z) \in \mathcal{N}$ is in the class $\mathcal{S}^*_\lambda(A,B)$ if and only if*

$$1 + \sum_{n=2}^{\infty} d_n z^{n-1} \ne 0 \tag{4.1}$$

where

$$d_n = \frac{(n-1) + (nB-\gamma)\zeta}{\zeta(B-A)} a_n \text{ and } \gamma = (A\cos\lambda + iB\sin\lambda)e^{-i\lambda}.$$

Proof. A function $f(z) \in \mathcal{N}$ is in the class $\mathcal{S}^*_\lambda(A,B)$ if and only if

$$e^{i\lambda}\frac{\frac{zf'}{f} - i\sin\lambda}{\cos\lambda} \ne \frac{1+A\zeta}{1+B\zeta}.$$

This simplifies into

$$(1+B\zeta)(zf'(z)) - (1+\gamma\zeta)f(z) \ne 0$$

where $\gamma = (A\cos\lambda + iB\sin\lambda)e^{-i\lambda}$. The rest of the proof follows as in Lemma 2.2. ∎

Theorem 4.2. *If $f(z) \in \mathcal{N}$ satisfies the following condition:*

$$\sum_{n=2}^{\infty}\left[\left|\sum_{k=1}^{n}\left(\sum_{j=1}^{k}(-1)^{k-j}(jB-\gamma)\begin{pmatrix}\beta\\ k-j\end{pmatrix}a_j\right)\begin{pmatrix}\gamma\\ n-k\end{pmatrix}\right|+\right.$$
$$\left.+\left|\sum_{k=1}^{n}\left(\sum_{j=1}^{k}(-1)^{k-j}(j-1)\begin{pmatrix}\beta\\ k-j\end{pmatrix}a_j\right)\begin{pmatrix}\gamma\\ n-k\end{pmatrix}\right|\right]\leq B-A$$

for $-1 \leq B < A \leq 1$ then, $f(z) \in \mathcal{S}^*_\lambda(A,B)$ where $\gamma = (A\cos\lambda + iB\sin\lambda)e^{-i\lambda}$.

Proof. Applying the same methods as in Theorem 3.1 we get the result. □

Theorem 4.3. *If $f(z) \in \mathcal{N}$ satisfies the following condition:*

$$\sum_{n=2}^{\infty}\left[\left|\sum_{k=1}^{n}\left(\sum_{j=1}^{k}(-1)^{k-j}j(jB-\gamma)\begin{pmatrix}\beta\\ k-j\end{pmatrix}a_j\right)\begin{pmatrix}\gamma\\ n-k\end{pmatrix}\right|+\right.$$
$$\left.+\left|\sum_{k=1}^{n}\left(\sum_{j=1}^{k}(-1)^{k-j}j(j-1)\begin{pmatrix}\beta\\ k-j\end{pmatrix}a_j\right)\begin{pmatrix}\gamma\\ n-k\end{pmatrix}\right|\right]\leq B-A$$

for $-1 \leq B < A \leq 1$ then, $f(z) \in \mathcal{K}_\lambda(A,B)$ where $\gamma = (A\cos\lambda + iB\sin\lambda)e^{-i\lambda}$.

Proof. Since $f(z)$ is in the class $\mathcal{K}_\lambda(A,B)$ if and only if $zf'(z)$ belongs to the class $\mathcal{S}^*_\lambda(A,B)$, replacing a_j in the statement of the Theorem 4.2 by ja_j we get the required result. □

Corollary 4.4. *For parametric values $A = 1-2\alpha$ and $B = -1$ we get coefficient conditions for functions in the class $\mathcal{S}^*_\lambda(\alpha)$ and $\mathcal{K}_\lambda(\alpha)$ [2] with suitable modifications.*

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