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The Line n-Sigraph of a Symmetric n-Sigraph*

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Abstract. An n-tuple $(a_1, a_2, ..., a_n)$ is symmetric, if $a_k = a_{n-k+1}, 1 \leq k \leq n$. Let $H_n = \{(a_1, a_2, ..., a_n) : a_k \in \{+, -\}, a_k = a_{n-k+1}, 1 \leq k \leq n\}$ be the set of all symmetric n-tuples. A symmetric n-sigraph (symmetric n-marked graph) is an ordered pair $S_n = (G, \sigma)$ $(S_n = (G, \mu))$, where $G = (V, E)$ is a graph called the underlying graph of S_n and $\sigma : E \to H_n$ $(\mu : V \to H_n)$ is a function. Analogous to the concept of the line sigraph of a sigraph, the line symmetric n-sigraph of a symmetric n-sigraph is defined. We then give a structural characterization of line symmetric n-sigraphs, extending the well known characterization of line graphs due to Beineke. Analogous to the concept of switching in sigraphs we introduce the notion of switching in symmetric n-sigraphs and obtain some relationships between jump symmetric n-sigraphs and line

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symmetric n-sigraphs.

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1. Introduction

For graph theoretical terminologies and notations in this paper we follow the book [7].

Let $n \geq 1$ be an integer. An n-tuple $(a_1, a_2, ..., a_n)$ is symmetric, if $a_k = a_{n-k+1}, 1 \leq k \leq n$. Let $H_n = \{(a_1, a_2, ..., a_n) : a_k \in \{+, -\}, a_k = a_{n-k+1}, 1 \leq k \leq n\}$ be the set of all symmetric n-tuples. Note that H_n is a group under coordinate wise multiplication, and the order of H_n is 2^m, where $m = \lceil \frac{n}{2} \rceil$.

A symmetric n-sigraph (symmetric n-marked graph) is an ordered pair $S_n = (G, \sigma)$ $(S_n = (G, \mu))$, where $G = (V, E)$ is a graph called the underlying graph of S_n and $\sigma : E \rightarrow H_n$ $(\mu : V \rightarrow H_n)$ is a function.

A sigraph (marked graph) is an ordered pair $S = (G, \sigma)$ $(S = (G, \mu))$, where $G = (V, E)$ is a graph called the underlying graph of S and $\sigma : E \rightarrow \{+, -\}$ $(\mu : V \rightarrow \{+, -\})$ is a function. Thus a symmetric 1-sigraph (symmetric 1-marked graph) is a sigraph (marked graph). Sigraphs (Marked graphs) are well studied in literature (See for example [8, 12, 13, 17, 18, 3, 11, 12, 13]).

The line graph $L(G)$ of graph G has the edges of G as the vertices and two vertices of $L(G)$ are adjacent if the corresponding edges of G are adjacent.

The jump graph $J(G)$ of a graph $G = (V, E)$ is $\overline{L(G)}$, the complement of the line graph $L(G)$ of G (See [5, 7]).

Behzad and Chartrand [4] introduced the notion of line sigraph $L(S)$ of a given sigraph S as follows: Given a signed graph $S = (G, \sigma)$ its line sigraph $L(S) = (L(G), \sigma')$ is that signed graph whose underlying graph is $L(G)$, the line graph of G, where for any edge e_ie_j in $L(S)$, $\sigma'(e_ie_j)$ is negative if and only if both e_i and e_j are adjacent negative edges in S. Another notion of line sigraph introduced in [6], is as follows: The line sigraph of a sigraph $S = (G, \sigma)$ is a sigraph $L(S) = (L(G), \sigma')$, where for any edge ee' in $L(S)$, $\sigma'(ee') = \sigma(e)\sigma(e')$ (See also [15]).

In this paper by an n-tuple/n-sigraph/n-marked graph we always mean a symmetric n-tuple/symmetric n-sigraph/symmetric n-marked graph.

Analogous to the concept of line sigraph defined in [6] we define line n-sigraph as follows: A line n-sigraph $L(S_n)$ of an n-sigraph $S_n = (G, \sigma)$ is an n-sigraph $L(S_n) = (L(G), \sigma')$ where for any edge ee' in $L(G)$, $\sigma'(ee') = \sigma(e)\sigma(e')$.

Hence, we shall call a given n-sigraph S_n a line n-sigraph if it is isomorphic to the line n-sigraph $L(S'_n)$ of some n-sigraph S'_n.

In the following section, we shall present an extension of well known characterization of a line graph given in most of the standard text-books on graph theory (e.g., see [7]), originally due to [2].

The jump n-sigraph $J(S_n)$ of an n-sigraph $S_n = (G, \sigma)$ is defined as follows (See [14]): $J(S_n) = (J(G), \sigma')$, where for any edge ee' in $J(G)$, $\sigma'(ee') = \sigma(e)\sigma(e')$.

Further, an n-tuple $(a_1, a_2, ..., a_n)$ is the identity n-tuple, if $a_k = +$, for $1 \leq k \leq n$, otherwise it is a non-identity n-tuple. In an n-sigraph $S_n = (G, \sigma)$ an edge labeled with the identity n-tuple is called an identity edge, otherwise it is a non-identity edge.

In an n-sigraph $S_n = (G, \sigma)$, for any $A \subseteq E(G)$ the n-tuple $\sigma(A)$ is the product of the n-tuples on the edges of A.

2. Balance in an n-Sigraph

In [14], we define two notions of balance in n-sigraph $S_n = (G, \sigma)$ as follows:

Definition 1. *Let $S_n = (G, \sigma)$ be an n-sigraph. Then,*

(i) *S_n is identity balanced (or i-balanced), if product of n-tuples on each cycle of G is the identity n-tuple, and*

(ii) *S_n is balanced, if every cycle in G contains an even number of non-identity edges.*

Note 2. An i-balanced n-sigraph need not be balanced and vice versa.

Proposition 3. [14] *An n-sigraph $S_n = (G, \sigma)$ is i-balanced if and only if it is possible to assign n-tuples to its vertices such that the n-tuple of each edge uv is equal to the product of the n-tuples of u and v.*

In [14], we define switching and cycle isomorphism of an n-sigraph $S_n = (G, \sigma)$ as follows:

Let $S_n = (G, \sigma)$ and $S'_n = (G', \sigma')$, be two n-sigraphs. Then S_n and S'_n are said to be isomorphic, written as $S_n \cong S'_n$, if there exists an isomorphism $\phi : G \to G'$ such that if uv is an edge in S_n with label $(a_1, a_2, ..., a_n)$ then $\phi(u)\phi(v)$ is an edge in S'_n with label $(a_1, a_2, ..., a_n)$.

Given an n-marking μ of an n-sigraph $S_n = (G, \sigma)$, switching S_n with respect to μ is the operation of changing the n-tuple of every edge uv of S_n by $\mu(u)\sigma(uv)\mu(v)$. The n-sigraph obtained in this way is denoted by $\mathcal{S}_\mu(S_n)$ and is called the μ-switched n-sigraph or just switched n-sigraph.

Further, an n-sigraph S_n switches to n-sigraph S'_n (or they are witching equivalent to each other), written as $S_n \sim S'_n$, whenever there exists an n-marking of S_n such that $\mathcal{S}_\mu(S_n) \cong S'_n$.

Two n-sigraphs $S_n = (G, \sigma)$ and $S'_n = (G', \sigma')$ are said to be cycle isomorphic, if there exists an isomorphism $\phi : G \to G'$ such that the n-tuple $\sigma(C)$ of every cycle C in S_n equals to the n-tuple $\sigma(\phi(C))$ in S'_n. We make use of the following known result (see [14]).

Proposition 4. [14] *Given a graph G, any two n-sigraphs with G as underlying graphs are switching equivalent if and only if they are cycle isomorphic.*

In [14], it has been shown that the jump n-sigraph of an n-sigraphs $S_n = (G, \sigma)$ is i-balanced. This is also true for line n-sigraph of an n-sigraph.

Proposition 5. *For any n-sigraph $S_n = (G, \sigma)$, its line n-sigraph $L(S_n) = (L(G), \sigma')$ is i-balanced.*

Proof. We first note that the labeling σ of S_n can be treated as an n-marking of vertices of $L(S_n)$. Then by definition of $L(S_n)$ we see that $\sigma'(ee') = \sigma(e)\sigma(e')$, for every edge ee' of $L(S_n)$ and hence, by proposition 1, the result follows. ■

Remark 6. In [1], M. Acharya has proved the above result in the special case when $n = 1$. The proof given here is different from that given in [1].

The following result characterizes n-sigraphs which are line n-sigraphs.

Proposition 7. *An n-sigraph $S_n = (G, \sigma)$ is a line n-sigraph if and only if S_n is i-balanced n-sigraph and its underlying graph G is a line graph.*

Proof. Suppose that S_n is i-balanced and G is a line graph. Then there exists a graph H such that $L(H) \cong G$. Since S_n is i-balanced, by Proposition 3, there exists an n-marking μ of G such that each edge uv in S_n satisfies $\sigma(uv) = \mu(u)\mu(v)$. Now consider the n-sigraph $S'_n = (H, \sigma')$, where for any edge e in H, $\sigma'(e)$ is the n-marking of the corresponding vertex in G. Then clearly, $L(S'_n) \cong S_n$. Hence S_n is a line n-sigraph.

Conversely, suppose that $S_n = (G, \sigma)$ is a line n-sigraph. Then there exists an n-sigraph $S'_n = (H, \sigma')$ such that $L(S'_n) \cong S_n$. Hence G is the line graph of H and by Proposition 5, S_n is i-balanced. ■

3. Operations on an n-tuple $(a_1, a_2, ..., a_n)$

For any $a \in \{+, -\}$, let $\overline{a} \in \{+, -\} \setminus \{a\}$. In an n-tuple $(a_1, a_2, ..., a_n)$, the elements $a_{\lceil \frac{n}{2} \rceil}$ and $a_{\lceil \frac{n+1}{2} \rceil}$ are called middle elements. Note that an n-tuple has two middle elements if n is even and exactly one if n is odd. We now define various operations on an n-tuple $(a_1, a_2, ..., a_n)$ as follows:

(i) f-complement, $(a_1, a_2, ..., a_n)^f = (\overline{a}_1, \overline{a}_2, ..., \overline{a}_n)$.
(ii) m-complement $(a_1, a_2, ..., a_n)^m = (b_1, b_2, ..., b_n)$, where

$$b_k = \begin{cases} \overline{a}_k, & \text{if} \quad a_k \text{ is a middle element,} \\ a_k, & \text{otherwise.} \end{cases}$$

(iii) e-complement $(a_1, a_2, ..., a_n)^e = (b_1, b_2, ..., b_n)$, where,

$$b_k = \begin{cases} \overline{a}_k, & \text{if} \quad a_k \text{ is not a middle element,} \\ a_k, & \text{otherwise.} \end{cases}$$

Let $t \in \{f, e, m\}$. Then t-complement S_n^t of an n-sigraph $S_n = (G, \sigma)$ is obtained from S_n by replacing each n-tuple on the edges of S_n by its t-complement.

For an n-sigraph $S_n = (G, \sigma)$, the $L(S_n)$ is i-balanced (Proposition 5) and $J(S_n)$ is also i-balanced (see [14]). We now examine, the conditions under which t-complements of $L(S_n)$ and $J(S_n)$ are i-balanced, where $t \in \{f, e, m\}$.

Proposition 8. *Let $S_n = (G, \sigma)$ be an n-sigraph. Then, for any $t \in \{f, e, m\}$,*

(i) *If $L(G)$ is bipartite then $(L(S_n))^t$ is i-balanced.*

(ii) *If $J(G)$ is bipartite then $(J(S_n))^t$ is i-balanced.*

Proof. (i) Since, by Proposition 5, $L(S_n)$ is i-balanced, for each k, $1 \le k \le n$, the number of n-tuples on any cycle C in $L(S_n)$ whose k^{th} co-ordinate are $-$ is even. Also, since $L(G)$ is bipartite, all cycles have even length; thus, for each k, $1 \le k \le n$, the number of n-tuples on any cycle C in $L(S_n)$ whose k^{th} co-ordinate are $+$ is also even. This implies that the same thing is true in any t-complement, where $t \in \{f, e, m\}$. Hence $(L(S_n))^t$ is i-balanced.

Similarly (ii) follows. ■

4. Complement of an n-Sigraph $S_n = (G, \sigma)$

Let $S_n = (G, \sigma)$ be an n-sigraph. Consider the n-marking μ on vertices of S_n defined as follows: for each vertex $v \in V$, $\mu(v)$ is the n-tuple which is the product of the n-tuples on the edges incident with v. *Complement* of S_n is an n-sigraph $\overline{S_n} = (\overline{G}, \sigma^c)$, where for any edge $e = uv \in \overline{G}$, $\sigma^c(uv) = \mu(u)\mu(v)$. Clearly, $\overline{S_n}$ as defined here is an i-balanced n-sigraph due to Proposition 3.

The following result is easy to verify using Proposition 4.

Proposition 9. *For any n-sigraph $S_n = (G, \sigma)$, $\overline{L(S_n)} \sim J(S_n)$.*

5. Switching Equivalence of Jump n-Sigraphs and Line n-sigraphs

Towards searching for an ideal notion of the complement of a given n-sigraph, one is naturally led to look for the analogue of the graph equation, $J(G) \cong L(G)$ for the case of jump n-sigraphs. Since $J(G) = \overline{L(G)}$, the solutions to the above equation would be graphs whose line graphs are self-complementary; these graphs have been determined already.

Proposition 10. [16] *The graph equation $J(G) \cong L(G)$ has only six solutions; namely $K_2, P_5, P_3 \circ K_1, K_{3,3} - e, K_{3,3}$.*

Proposition 11. *An n-sigraph $S_n = (G, \sigma)$ satisfies $L(S_n) \sim J(S_n)$ if and only if G is isomorphic to any of the graphs $K_2, P_5, P_3 \circ K_1, K_{3,3} - e, K_{3,3}$.*

The reader may refer the papers [9, 10] for some more related topics in concerning the minimum number of vertices and edges in a graph with a given vertex conecticity, edge conectivity and mimimm degree. Further, we can easily extend the same to n-sigraphs.

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