DEMONSTRATIO MATHEMATICA
Vol. XLV No 3 2012

L. Dileep and S. Latha

ON P-VALENT FUNCTIONS OF COMPLEX ORDER

Abstract. The purpose of the present paper is to derive characterization theorem and radius of starlikeness for certain class of p- valent analytic functions. In connection with these results some more properties are discussed.

1. Introduction

Let $\mathcal{A}_p$ denote the class of analytic functions defined in the unit disc $\mathcal{U} = \{z : |z| < 1\}$ of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad (p \in \mathbb{N} := \{1, 2, 3, \cdots\}). \tag{1.1}$$

For $-1 \leq B < A \leq 1$, let $\mathcal{P}(A, B)$ [8] denote the class of functions which are of the form

$$p_1(z) = \frac{1 + A\omega(z)}{1 + B\omega(z)}, \tag{1.2}$$

where ω is bounded analytic functions satisfying the conditions $\omega(0) = 0$ and $|\omega(z)| < 1$.

Let $\mathcal{P}(A, B, p, \alpha)$ denote the class of functions $p(z) = p + \sum_{n=1}^{\infty} a_n z^n$ which are analytic in $\mathcal{U}$ and

$$p(z) = (p - \alpha)p_1(z) + \alpha \quad p_1(z) \in \mathcal{P}(A, B). \tag{1.3}$$

Using (1.2) and (1.3), one can show that $p(z) \in \mathcal{P}(A, B, p, \alpha)$ if and only if

$$p(z) = \frac{p + \gamma\omega(z)}{1 + B\omega(z)}, \quad \omega(z) \in \mathcal{U}, \tag{1.4}$$

where $\gamma = (p - \alpha)A + \alpha B$.

2000 *Mathematics Subject Classification*: 30C45.
Key words and phrases: p- valent functions, Janowski class.

For some real λ, $|\lambda| < \frac{\pi}{2}$, b a non-zero complex number, we designate $\mathcal{S}_p^\lambda(A,B,b)$ as the class of functions $f(z) \in \mathcal{A}_p$ such that

$$p + d\left(\frac{zf'(z)}{f(z)} - p\right) = p(z), \quad \text{for } p(z) \in \mathcal{P}(A,B,p,\alpha), \tag{1.5}$$

$$\text{where } d = \frac{e^{i\lambda}}{b\cos\lambda}.$$

This class generalizes various classes studied earlier by Aouf [2], Janowski [8], Golzina [5], Ganesan [4], Silverman [10] and Polatoglu et al. [11] respectively. In particular, $\mathcal{S}(A,B,p,\alpha)$, $\mathcal{S}^*(A,B), \mathcal{S}_\alpha(p)$, $\mathcal{S}_p(A,B)$, $\mathcal{S}(a,b)$ and $\mathcal{S}^\lambda(A,B,b)$ are all contained in the class $\mathcal{S}_p^\lambda(A,B,b)$.

2. Some preliminaries

Lemma 2.1. [2] *Let $p(z) \in \mathcal{P}(A,B,p,\alpha)$. Then, for $|z| \leq r$, we have*

$$\left|p(z) - \frac{p - [pB + (A-B)(p-\alpha)]Br^2}{1 - B^2r^2}\right| \leq \frac{(A-B)(p-\alpha)r}{1-B^2r^2}, \quad z \in \mathcal{U}. \tag{2.1}$$

Lemma 2.2. [2] *$f(z) \in \mathcal{S}(A,B,p,\alpha)$ if and only if*

$$f(z) = z^p\left[\frac{f_1(z)}{z}\right]^{p-\alpha}, \quad f_1(z) \in \mathcal{S}^*(A,B), \quad z \in \mathcal{U}. \tag{2.2}$$

Lemma 2.3. [9] *If $f(z) \in \mathcal{S}^*(A,B)$ $(-1 \leq B < A \leq 1)$, then*

$$\left|\arg\frac{f(z)}{z}\right| \leq \frac{2(A-B)}{(1-B)}\arcsin r, \ |z| = r < 1. \tag{2.3}$$

3. Main results

In this section, a necessary and sufficient condition, radius of starlikeness for functions belonging to class $\mathcal{S}_p^\lambda(A,B,b)$ are determined.

Theorem 3.1. *A function $f(z) = z^p + a_{p+1}z^{p+1} + a_{p+2}z^{p+2} + \ldots.$, belongs to the class $\mathcal{S}_p^\lambda(A,B,b)$ if and only if*

$$e^{i\lambda}\left(\frac{zf'(z)}{f(z)} - p\right) \prec \frac{(\gamma - pB)b\cos\lambda z}{1+Bz}, \quad z \in \mathcal{U}.$$

Proof. Let

$$e^{i\lambda}\left(\frac{zf'(z)}{f(z)} - p\right) \prec \frac{(\gamma - pB)b\cos\lambda z}{1+Bz}.$$

Using subordination principle, it follows that

$$p+\frac{e^{i\lambda}}{b\cos\lambda}\left(\frac{zf'(z)}{f(z)}-p\right)=p+\frac{(\gamma-pB)\omega(z)}{1+B\omega(z)}=\frac{p+\gamma\omega(z)}{1+B\omega(z)}.$$

This implies $f(z)\in\mathcal{S}_p^{\lambda}(A,B,b)$.

Conversely if $f(z)\in\mathcal{S}_p^{\lambda}(A,B,b)$, then for some $p(z)\in\mathcal{P}(A,B,p,\alpha)$ and $z\in\mathcal{U}$,

$$p+\frac{e^{i\lambda}}{b\cos\lambda}\left(\frac{zf'(z)}{f(z)}-p\right)=p(z).$$

Hence, we have

$$p+\frac{e^{i\lambda}}{b\cos\lambda}\left(\frac{zf'(z)}{f(z)}-p\right)=\frac{p+\gamma\omega(z)}{1+B\omega(z)}.$$

This simplifies into

$$\frac{e^{i\lambda}}{b\cos\lambda}\left(\frac{zf'(z)}{f(z)}-p\right)=\frac{(\gamma-pB)\omega(z)}{1+B\omega(z)}.$$

By subordination principle, it follows that

$$e^{i\lambda}\left(\frac{zf'(z)}{f(z)}-p\right)\prec\frac{(\gamma-pB)b\cos\lambda z}{1+Bz}. \blacksquare$$

For parametric values $p=1$ and $\alpha=0$, we get the Theorem 1 in [11] which reads as follows:

Corollary 3.2. $f(z)=z+a_2z^2+a_3z^3+\cdots$, *belongs to* $\mathcal{S}^{\lambda}(A,B,b)$ *if and only if*

$$e^{i\lambda}\left(\frac{zf'(z)}{f(z)}-1\right)\prec\begin{cases}\dfrac{(A-B)b\cos\lambda z}{1+Bz}, & B\neq 0\\ Ab\cos\lambda z, & B=0.\end{cases}$$

Theorem 3.3. *If* $f(z)\in\mathcal{S}_p^{\lambda}(A,B,b)$, *then*

$$\left|1-\left(\frac{f(z)}{z^p}\right)^{\frac{Bd}{(\gamma-pB)}}\right|<1,\qquad z\in\mathcal{U}. \tag{3.1}$$

Proof. We define the function $\omega(z)$ by

$$\frac{f(z)}{z^p}=(1+B\omega(z))^{\frac{(\gamma-pB)}{Bd}}$$

where the exponent is so chosen that $(1+B\omega(z))^{\frac{(\gamma-pB)}{Bd}}$ has the value 1 at the origin. Then, $\omega(z)$ is analytic in $\mathcal{U}$ and $\omega(0)=0$. Logarithmic differentiation, yields

$$e^{i\lambda}\frac{zf'(z)}{f(z)}-e^{i\lambda}p=\frac{(\gamma-pB)b\cos\lambda\omega'(z)}{1+B\omega(z)}.$$

By Theorem 3.1, it follows that $|\omega(z)| < 1$ for all $z \in \mathcal{U}$ with $|\omega(z_1)| = 1$ such that $|\omega(z)|$ attains its maximum value on the circle $|z| = z_1 < 1$ at the point z_1.
Using Jack's Lemma in this equality, since $|\omega(z_1)| = 1$ and $k \geq 1$, we obtain

$$e^{i\lambda} z_1 \frac{f'(z_1)}{f(z_1)} - e^{i\lambda} p = \frac{(\gamma - pB) b \cos\lambda \omega(z_1)}{1 + B\omega(z_1)} = F(\omega(z_1)) \notin F(\mathcal{U}).$$

But this contradicts Theorem 3.1. Now using (3.1), we obtain

$$\left| 1 - \left(\frac{f(z)}{z^p} \right)^{\frac{Bd}{(\gamma - pB)}} \right| = |B\omega(z)| < |B|,$$

which completes the proof. ■

For parametric values $p = 1$ and $\alpha = 0$, we get the Theorem 2 in [11] which reads as follows:

Corollary 3.4. *If $f(z) \in \mathcal{S}_p(A, B, b)$, then*

$$\left| 1 - \left(\frac{f(z)}{z} \right)^{\frac{Bd}{(A - B)}} \right| < 1, \qquad z \in \mathcal{U}.$$

Theorem 3.5. *If $f(z) = z^p + a_{p+2} z^{p+2} + a_{p+3} z^{p+3} + \cdots$, belongs to $\mathcal{S}_p^{\lambda}(A, B, b)$, then*

$$G(r, -A, -B, |b|) \leq |f(z)| \leq G(r, A, B, |b|), \quad z \in \mathcal{U}$$

where,

$$G(r, A, B, |b|) = \begin{cases} r^p (1 + Br)^{\frac{(\gamma - pB)\cos\lambda(|b| + \Re\{b\}\cos\lambda)}{2B}}, & B \neq 0 \\ r^p e^{A(p-\alpha)|b|\cos\lambda r}, & B = 0\,. \end{cases}$$

This bound is sharp, being attained by the extremal function

$$f_*(z) = \begin{cases} z^p (1 + Bz)^{\frac{(\gamma - pB)}{Bd}}, & B \neq 0 \\ z^p e^{\frac{A(p-\alpha)}{d}}, & B = 0\,. \end{cases} \tag{3.2}$$

The radius of starlikeness of the class $\mathcal{S}_p^{\lambda}(A, B, b)$ is

$$r_s = \frac{(\gamma - pB)|b|\cos\lambda - \sqrt{(\gamma - pB)^2 |b|^2 \cos^2\lambda + 4B^2 + 4B(\gamma - pB)\Re\{b\}\cos^2\lambda}}{2[-B^2 - B(\gamma - pB)\Re\{b\}\cos^2\lambda]}.$$

This radius is sharp, being attained by the extremal function

$$f_*(z) = z(1 + Bz)^{\frac{(\gamma - pB)e^{-i\lambda} b \cos\lambda}{B}}.$$

Proof. Let $f(z) \in \mathcal{S}_p^\lambda(A, B, b)$ and $B = 0$. Therefore, from (2.1) we have

$$\left| p + d\left(\frac{zf'(z)}{f(z)} - p\right) - \frac{p - [pB + (\gamma - pB)]Br^2}{1 - B^2r^2} \right| \leq \frac{(\gamma - pB)|b|\cos\lambda r}{1 - B^2r^2}.$$

Hence we get

$$\left| \frac{zf'(z)}{f(z)} - \frac{p(1 - B^2r^2) - B(\gamma - pB)b\cos^2\lambda r^2}{1 - B^2r^2} \right| \leq \frac{(\gamma - pB)|b|\cos\lambda r}{1 - B^2r^2}. \tag{3.3}$$

The set of the values of $\dfrac{zf'(z)}{f(z)}$ in the closed disc with center

$$C(r) = \left(\frac{p(1 - B^2r^2) - B(\gamma - pB)b\cos^2\lambda r^2}{1 - B^2r^2}, \frac{p(1 - B^2r^2) + B(\gamma - pB)b\cos^2\lambda r^2}{1 - B^2r^2}\right) \tag{3.4}$$

and radius

$$\rho(r) = \frac{(\gamma - pB)|b|\cos\lambda r}{1 - B^2r^2}.$$

The inequality (3.3) can be written in the form,

$$M_1(r) \leq \Re\left\{\frac{zf'(z)}{f(z)}\right\} \leq M_2(r), \tag{3.5}$$

where

$$M_1(r) = \frac{p - (\gamma - pB)|b|\cos\lambda r + p(B^2 + B(\gamma - pB)\Re\{b\}\cos^2\lambda)r^2}{1 - B^2r^2},$$

$$M_2(r) = \frac{p + (\gamma - pB)|b|\cos\lambda r - p(B^2 + B(\gamma - pB)\Re\{b\}\cos^2\lambda)r^2}{1 - B^2r^2}.$$

On the other hand

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} = r\frac{\partial}{\partial r}\log|f(z)|. \tag{3.6}$$

By considering (3.5) and (3.6) we can write

$$M_1(r) \leq r\frac{\partial}{\partial r}\log|f(z)| \leq M_2(r),$$

which yields the desired result on integration. If we take $B = 0$ in the inequality (3.3) we obtain the complete result.

From (3.3), we have

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} \geq \frac{p - (\gamma - pB)|b|\cos\lambda r - (B^2 + B(\gamma - pB)\Re\{b\}\cos^2\lambda)r^2}{1 - B^2r^2}.$$

For $r < r_s$ the right hand side of the preceding inequality is positive, which implies

$$r_s = \frac{(\gamma - pB)|b|\cos\lambda - \sqrt{(\gamma - pB)^2|b|^2\cos^2\lambda + 4B^2 + 4B(\gamma - pB)\Re\{b\}\cos^2\lambda}}{2[-B^2 - B(\gamma - pB)\Re\{b\}\cos^2\lambda]}.$$

We also note that the inequality (3.3) becomes an equality for the function

$$f_*(z) = z(1 + Bz)^{\frac{(\gamma - pB)be^{-i\lambda}\cos\lambda}{B}}.$$

Hence the proof is complete. ■

Theorem 3.6. *If* $f(z) \in \mathcal{S}_p^\lambda(A, B, b)$, *then for* $|z| = r < 1$,

$$\left|\arg\frac{f(z)}{z^p}\right| \leq \frac{2(\gamma - pB)}{(1 - B)d}, \quad z \in \mathcal{U}.$$

Proof. From (2.2) $\dfrac{f(z)}{z^p} = \left(\dfrac{f_1(z)}{z}\right)^{\left(\frac{p-\alpha}{d}\right)}$ where $f_1(z) \in \mathcal{S}^*(A, B)$. From (2.3), the desired inequality follows. ■

Acknowledgement. This work was supported by UGC Major Research Fund F. No. 38-268/2009(SR).

References

[1] F. M. AL-Oboudi, M. M. Haidan, *Spirallike functions of complex order*, J. Nat. Geom. 19 (2000), 53–72.

[2] M. K. Aouf, *On class of p-valent starlike functions of order α*, Internat. J. Math. Math. Sci. 10(4) (1987), 733–744.

[3] M. K. Aouf, F. M. Al-Oboudi, M. M. Haidan, *On some results for λ- spirallike and λ-Robertson functions of complex order*, Publ. Instit. Math. Belgrade 77 (2005), 93–98.

[4] M. S. Ganesan, *A study in the theory of univalent function and multivalent functions*, Ph. D Thesis.

[5] E. G. Golzina, *On the coefficients of a class of functions regular in a disk and having an integral representation in it*, J. of Soviet Math. 6(2) (1974), 606–617.

[6] A. W. Goodman, *Univalent Functions*, Vol. I & II, Mariner Publishing Comp. Inc., Tampa, Florida, 1983.

[7] I. S. Jack, *Functions starlike and convex of order α*, J. London Math. Soc. 3(2) (1971), 469–474.

[8] W. Janowski, *Some extremal problems for certain families of analytic functions*, I. Ann. Polon. Math. 28 (1973), 297–326.

[9] B. Pinchuk, *On starlike and convex functions of order α*, Duke Math. J. 35 (1968), 721–734.

[10] H. Silverman, *Subclasses of starlike functions*, Rev. Roumaine Math. Pures Appl. 23 (1978), 1093–1099.

[11] Y. Polatoglu, A. Sen, *Some results on subclasses of Janowski λ-spirallike functions of complex order*, Gen. Math. 15(2–3) (2007), 88–97.

L. Dileep
DEPARTMENT OF MATHEMATICS
VIDYAVARDHAKA COLLEGE OF ENGINEERING
Gokolum 3rd stage
MYSORE, INDIA
E-mail: dileepl84@gmail.com

S. Latha
DEPARTMENT OF MATHEMATICS
YUVARAJA'S COLLEGE
UNIVERSITY OF MYSORE
MYSORE - 570 005, INDIA
E-mail: drlatha@gmail.com

Received May 16, 2010; revised version January 11, 2011.