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# Entangling capabilities of symmetric two-qubit gates

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**Abstract.** Our work addresses the problem of generating maximally entangled two spin-1/2 (qubit) symmetric states using NMR, NQR, Lipkin–Meshkov–Glick Hamiltonians. Time evolution of such Hamiltonians provides various logic gates which can be used for quantum processing tasks. Pairs of spin-1/2s have modelled a wide range of problems in physics. Here, we are interested in two spin-1/2 symmetric states which belong to a subspace spanned by the angular momentum basis $\{|j = 1, \mu\rangle; \mu = +1, 0, -1\}$. Our technique relies on the decomposition of a Hamiltonian in terms of $SU(3)$ basis matrices. In this context, we define a set of linearly independent, traceless, Hermitian operators which provides an alternate set of $SU(n)$ generators. These matrices are constructed out of angular momentum operators $\mathbf{J}_x, \mathbf{J}_y, \mathbf{J}_z$. We construct and study the properties of perfect entanglers acting on a symmetric subspace, i.e., spin-1 operators that can generate maximally entangled states from some suitably chosen initial separable states in terms of their entangling power.

**Keywords.** Quantum entanglement; $SU(3)$ generators; entangling power.

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## 1. Introduction

In the last few years, there has been considerable increase in experimental activity aiming to create entangled quantum states in a wide range of physical phenomena like neutral atoms in an optical lattice, exchange of photons in cavity quantum electrodynamics, generating and manipulating $N$-particle entangled states in ion traps, NMR, NQR [1] etc. In practice, these states are created by some physical operations involving the interaction between several systems. Thus, analysing these operations with regard to the possibility of creating maximally entangled states from an initial unentangled one is very important. However, for arbitrary $N$-particle spin-$\frac{1}{2}$ ensembles (two-level systems), these operations are exponentially difficult to compute because a general state of the ensemble resides in the Hilbert space $C_2^{\otimes N}$ and the dimension of the density matrix scales as $2^N \times 2^N$. Computational investigation of entanglement of such ensembles is therefore impractical for all but the smallest values of $N$. Fortunately, many experimentally relevant states possess symmetry under particle exchange and this property allows us to significantly reduce the computational complexity. Pairs of spin-$\frac{1}{2}$s have modelled a wide range of problems in physics. Considering two spin-$\frac{1}{2}$s (two qubits) in the symmetric subspace – the set of those $N$-particle pure states that remain unchanged by permutations of individual particles, one can then produce any entangled symmetric state by the time evolution of properly chosen Hamiltonian, eg., NMR, NQR, quantum optics, Lipkin Hamiltonian [2] which is widely used in nuclear physics etc. As pointed out in [3], this necessarily does not lead to the most efficient way of creating a particular state. Knowing which states are prohibitively expensive to produce is an important experimental question. An interesting, but difficult, way to characterize this is by quantifying the resources needed to create that state, given a certain set of generators. In this context, we define a new set of $SU(3)$ basis matrices constructed out of angular momentum operators $\mathbf{J}_x, \mathbf{J}_y, \mathbf{J}_z$. These matrices constitute a $3 \times 3$ linearly independent, experimentally realizable [4] Cartesian tensor operators which can also provide different symmetric logic gates for quantum processing tasks. As these two qubit symmetric gates are capable of producing entanglement of quantum states, quantifying their entangling capability is very important. Makhlin [5] has analysed nonlocal properties of general two-qubit gates and also studied some basic properties of perfect entanglers which are defined as the unitary operators that can generate maximally entangled states from some suitably chosen separable states. Zanardi *et al* [6] have explored the entangling power of quantum evolutions in terms of mean linear entropy produced when unitary operator acts on a given distribution of pure product states. Kraus and Cirac [7] and Rezakhani [8] have given the tools to find the best separable two-qubit input orthonormal product states such that some given unitary transformation can create maximally entangled quantum states. The entangling capability of a unitary quantum gate can be quantified by its entangling power $e_p(U)$ [6]. Balakrishnan *et al* [9] have derived $e_p(U)$ in terms of the local invariant $G_1$. In this paper, we show that five of the eight, two-qubit symmetric quantum gates expressed in terms of our newly defined basis set belong to the class of perfect entanglers which can generate maximally entangled states from some suitably chosen product states. Further, we show that these gates belong to a family of special perfect entanglers under certain conditions. This is a very relevant problem from both theoretical as well as experimental points of views.

### 1.1 *Symmetric states*

Our interest here is on two-qubit states, which are symmetric under particle interchange. Symmetric states offer elegant mathematical analysis as the dimension of the Hilbert space reduces drastically from $2^N$ to $(N + 1)$, when $N$ qubits respect exchange symmetry. Such a Hilbert space is considered to be spanned by the eigenstates $\{|j, \mu\rangle; -j \leq \mu \leq +j\}$ of angular momentum operators $J^2$ and $J_z$, where $j = N/2$. The corresponding density matrix gets transformed to a $3 \times 3$ block form in the symmetric subspace characterized by the maximal value of total angular momentum $j_{\max} = 1$. The symmetric subspace provides a convenient, computationally accessible class of spin states relevant to many experimental situations such as spin squeezing. Completely symmetric systems are experimentally interesting, largely because it is often easier to nonselectively address an entire ensemble of particles rather than individually address each member as in the case of atomic assembly. Permutationally, symmetric states are useful in a variety of quantum information processing tasks and a class of these states has recently been implemented experimentally [10,11].

In §2, we define an alternate representation of $SU(n)$ generators using the well-known spherical tensor operators. Explicit form of these basis matrices in three-dimensional representation is given in §2.1. Section 3 deals with two-qubit symmetric gates and their entangling power in terms of local invariant $G_1$. In §3.1, we identify the conditions under which the perfect entangler can be classified as special perfect entangler. Further, as a realistic example the entangling property of Lipkin–Meshkov–Glick Hamiltonian is studied in the spin-1 subspace.

## 2. Alternate representation of $SU(n)$ generators

It is well known that any density operator for a spin $j$ system is given by [12]

$$\rho(\vec{J}) = \frac{1}{(2j+1)} \sum_{k=0}^{2j} \sum_{q=-k}^{+k} t_q^k \, \tau_q^{k^\dagger}(\vec{J}), \tag{1}$$

where $\tau_q^k$'s (with $\tau_0^0 = I$, the identity operator) are irreducible spherical tensor operators of rank $k$ in the $2j+1$-dimension spin space with projection $q$ along the axis of quantization in the real three-dimensional space. The $\tau_q^k$'s satisfy the orthogonality relation

$$\mathrm{Tr}(\tau_q^{k^\dagger} \tau_{q'}^{k'}) = (2j+1)\, \delta_{kk'} \delta_{qq'} \,. \tag{2}$$

Here the normalization has been chosen so as to be in agreement with Madison convention [13]. The spherical tensor parameters $t_q^k$ which characterize the given density operator $\rho$ are given by $t_q^k = \mathrm{Tr}(\rho \tau_q^k)$. As $\rho$ is Hermitian and $\tau_q^{k^\dagger} = (-1)^q \tau_{-q}^k$, $t_q^k$'s satisfy the condition $t_q^{k^*} = (-1)^q \, t_{-q}^k$. The spherical tensor parameters $t_q^k$'s have simple transformation properties under coordinate rotation [14] in the three-dimensional space, i.e.,

$$(t_q^k)^R = \sum_{q'=-k}^{+k} D_{q'q}^k(\alpha\beta\gamma) t_{q'}^k, \tag{3}$$

where $D_{q'q}^k(\alpha\beta\gamma)$ denotes Wigner-D matrix parametrized by Euler angles $(\alpha\beta\gamma)$.

Following the well-known Weyl construction [14] for $\tau_q^k$ in terms of angular momentum operators $\mathbf{J}_x$, $\mathbf{J}_y$ and $\mathbf{J}_z$, we have

$$\tau_q^k(\vec{J}) = \mathcal{N}_{kj} (\vec{J} \cdot \vec{\nabla})^k \, r^k \, Y_q^k(\hat{r}) \,, \tag{4}$$

where

$$\mathcal{N}_{kj} = \frac{2^k}{k!} \sqrt{\frac{4\pi (2j-k)!(2j+1)}{(2j+k+1)!}}, \tag{5}$$

are the normalization factors and $Y_q^k(\hat{r})$ are the spherical harmonics. Under rotations $\tau_q^k$'s transform according to Wigner-D matrices, i.e.,

$$(\tau_q^k(\vec{J}))^R = \sum_{k=0}^{2j} \sum_{q=-k}^{+k} D_{q'q}^k(\alpha\beta\gamma) \tau_{q'}^k(\vec{J}) \,. \tag{6}$$

We now define a set of linearly independent, traceless (except $(T^0)^0_0$), orthonormal Hermitian basis matrices $(T^\alpha)^k_q$, where $\alpha = +, -, 0$, $k = 1 \ldots 2j$, and $q = 1$ to $+k$ as follows:

$$(T^+)^k_q = \frac{\tau^k_q + (\tau^k_q)^\dagger}{\sqrt{2(2j+1)}}, \tag{7}$$

$$(T^-)^k_q = \frac{i(\tau^k_q - (\tau^k_q)^\dagger)}{\sqrt{2(2j+1)}} \tag{8}$$

and

$$(T^0)^k_0 = \frac{\tau^k_0}{\sqrt{2j+1}}. \tag{9}$$

Observe that these matrices satisfy the relation $\mathrm{Tr}((T^\alpha)^k_q (T^\beta)^{k'}_{q'}) = \delta_{\alpha\beta}\delta_{kk'}\delta_{qq'}$. In our new representation the most general density matrix can be written as

$$\rho = (r^0)^0_0 (T^0)^0_0 + \sum_{k=1\ldots 2j} (r^0)^k_0 (T^0)^k_0 + \sum_{\alpha=+,-} \sum_{k=1\ldots 2j} \sum_{q=1\ldots k} (r^\alpha)^k_q (T^\alpha)^k_q. \tag{10}$$

Apart from $(T^0)^0_0$ which is proportional to identity matrix, there are $2j$ diagonal matrices namely $(T^0)^k_0$, $k = 1 \ldots 2j$ and the rest are off-diagonal.

### 2.1 *SU(3) basis set*

In the particular case of two-qubit symmetric subspace, our set of basis matrices (the above matrices are equivalent to the set of matrices with different normalization defined by R J Morris [15]) can be obtained from eqs (7)–(9) as (we have used a different notation for the set for the sake of simplicity)

$$M_0 = \sqrt{\frac{2}{3}}\tau^0_0, \quad M_1 = \frac{\tau^1_1 + \tau^{1\dagger}_1}{\sqrt{3}}, \quad M_2 = \frac{i(\tau^1_1 - \tau^{1\dagger}_1)}{\sqrt{3}}, \tag{11}$$

$$M_3 = \sqrt{\frac{2}{3}}\tau^1_0, \quad M_4 = \frac{i(\tau^2_2 - \tau^{2\dagger}_2)}{\sqrt{3}}, \quad M_5 = \frac{i(\tau^2_1 - \tau^{2\dagger}_1)}{\sqrt{3}}, \tag{12}$$

$$M_6 = \frac{\tau^2_1 + \tau^{2\dagger}_1}{\sqrt{3}}, \quad M_7 = \frac{\tau^2_2 + \tau^{2\dagger}_2}{\sqrt{3}}, \quad M_8 = \sqrt{\frac{2}{3}}\tau^2_0. \tag{13}$$

These operators are explicitly represented in $|1m\rangle$ basis where $m = 1, 0, -1$ as follows:

$$M_0 = \sqrt{\frac{2}{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix},$$

$$M_2 = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$M_4 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad M_5 = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

$$M_6 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad M_7 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad M_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The above matrices are normalized, i.e., $\mathrm{Tr}(M_k M_{k'}) = 2\,\delta_{kk'}$ and $M_1, ..., M_7$ have eigenvalues 1, 0, −1. In this representation the most general spin-1 Hamiltonian can be written as

$$\mathcal{H}(t) = \frac{1}{2} \sum_{k=0}^{8} h_k(t) M_k. \tag{14}$$

Here $M_k$'s in terms of angular momentum operators $\mathbf{J}_x, \mathbf{J}_y, \mathbf{J}_z$ are given by

$$M_1 = -(J_x), \quad M_2 = (J_y), \quad M_3 = (J_z),$$
$$M_4 = -(J_x J_y + J_y J_x), \quad M_5 = (J_y J_z + J_z J_y),$$
$$M_6 = -(J_x J_z + J_z J_x), \quad M_7 = (J_x^2 - J_y^2), \quad M_8 = (3J_z^2 - 2).$$

Note that the expansion coefficients $h_k = \mathrm{Tr}(\mathcal{H} M_k)$ are real and hence they constitute an experimentally measurable set of parameters.

## 3. Two-qubit symmetric gates

Hamiltonian evolution provides the hardware for quantum gates, i.e., the time evolution of the operators $M_k$'s provides various symmetric logic gates for quantum computation. The closed form expression for $\mathrm{e}^{iM_k\theta}$ is given by $B_k = \mathrm{e}^{iM_k\theta} = I + (\cos\theta - 1)M_k^2 + i \sin\theta M_k$. Here $k = 1 \ldots 7$, $\theta = \zeta_k t$, where $\zeta_k$ is the coupling constant corresponding to each $M_k$ and $I$ is the $3 \times 3$ identity matrix. Following are the explicit forms of the gates $B_k$'s in the angular momentum basis $|11\rangle, |10\rangle, |1{-}1\rangle$:

$$B_1 = \begin{pmatrix} \cos^2\theta/2 & -i\sin\theta/\sqrt{2} & -\sin^2\theta/2 \\ -i\sin\theta/\sqrt{2} & \cos\theta & -i\sin\theta/\sqrt{2} \\ -\sin^2\theta/2 & -i\sin\theta/\sqrt{2} & \cos^2\theta/2 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} \cos^2\theta/2 & \sin\theta/\sqrt{2} & \sin^2\theta/2 \\ -\sin\theta/\sqrt{2} & \cos\theta & \sin\theta/\sqrt{2} \\ \sin^2\theta/2 & -\sin\theta/\sqrt{2} & \cos^2\theta/2 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} \cos\theta + i\sin\theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos\theta - i\sin\theta \end{pmatrix}, \quad B_4 = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix},$$

$$B_5 = \begin{pmatrix} \cos^2\theta/2 & \sin\theta/\sqrt{2} & -\sin^2\theta/2 \\ -\sin\theta/\sqrt{2} & \cos\theta & -\sin\theta/\sqrt{2} \\ -\sin^2\theta/2 & \sin\theta/\sqrt{2} & \cos^2\theta/2 \end{pmatrix},$$

$$B_6 = \begin{pmatrix} \cos^2\theta/2 & -i\sin\theta/\sqrt{2} & \sin^2\theta/2 \\ -i\sin\theta/\sqrt{2} & \cos\theta & i\sin\theta/\sqrt{2} \\ \sin^2\theta/2 & i\sin\theta/\sqrt{2} & \cos^2\theta/2 \end{pmatrix},$$

$$B_7 = \begin{pmatrix} \cos\theta & 0 & i\sin\theta \\ 0 & 1 & 0 \\ i\sin\theta & 0 & \cos\theta \end{pmatrix}, \quad B_8 = \begin{pmatrix} e^{i\theta/\sqrt{3}} & 0 & 0 \\ 0 & e^{-2i\theta/\sqrt{3}} & 0 \\ 0 & 0 & e^{i\theta/\sqrt{3}} \end{pmatrix}.$$

A useful property of a two-qubit symmetric gate is its ability to produce a maximally entangled state from an unentangled one. It is well known that perfect entanglers are those unitary operators that can generate maximally entangled states from some suitably chosen separable states. The entangling properties of quantum operators have already been discussed in [6,9,16]. Here, we calculate the entangling power of two-qubit symmetric gates following the simplified expression given by Balakrishnan *et al* [9] according to which the gate $B$ is a perfect entangler if its entangling power, $e_p(B) = \frac{2}{9}(1 - |G_1|)$ has the range $\frac{1}{6} \le e_p \le \frac{2}{9}$.

The local invariant $G_1$ ([5], table II) in terms of symmetric, unitary matrix $m$ is given by $G_1 = \mathrm{tr}^2 m/16\det[B]$. Here $m = \mathcal{B}_{\mathrm{Bell}}^T \mathcal{B}_{\mathrm{Bell}}$ where the gates in the Bell basis are given by $\mathcal{B}_{\mathrm{Bell}} = U\mathcal{B}U^\dagger$ and $\mathcal{B} = \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}$. $U$ is the transformation matrix

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -\sqrt{2}i & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} \\ -i & 0 & i & 0 \end{pmatrix},$$

connecting the angular momentum basis $\{|11\rangle, |10\rangle, |1\,-1\rangle, |00\rangle\}$ to the Bell basis

$$\left\{ \frac{|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle}{\sqrt{2}}, \frac{i(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)}{\sqrt{2}}, \frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}, \frac{i(|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle)}{\sqrt{2}} \right\}.$$

The relation $e_p(B) = \frac{2}{9}(1 - |G_1|)$ implies that gates having the same $|G_1|$ must necessarily possess the same entangling power $e_p$.

It is obvious that $B_1$, $B_2$, $B_3$ do not produce entanglement as they represent rotations. Note that $|G_1| = 1$ and $e_p = 0$ for the above gates. Interestingly, for the gates $B_4$, $B_5$, $B_6$ and $B_7$, $|G_1| = \cos^4\theta$. Observe that since $0 \le G_1 \le 1$, it is clear that $0 \le e_p(B)_k \le \frac{2}{9}$ ($k = 4 \ldots 7$). All these above-mentioned gates entangle for all values of $\theta$ except when $\theta = 0, \pi, 2\pi, 3\pi$, etc. But they are perfect entanglers for $(2n+1)(\pi/4) \le \theta \le (2n+3)(\pi/4)$ where $n = 0, 2, 4, 6, \ldots$. Similarly, the gate $B_8$ has maximum entangling power, i.e., $e_p = 2/9$ when $\theta = \sqrt{3}(\pi/2)$.

As an example, consider the direct product state $|\psi_{12}\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$, of two spinors in the qubit basis. Let

$$|\psi_{12}\rangle = \begin{pmatrix} \cos(\alpha_1/2) \\ \sin(\alpha_1/2)e^{i\phi_1} \end{pmatrix} \otimes \begin{pmatrix} \cos(\alpha_2/2) \\ \sin(\alpha_2/2)e^{i\phi_2} \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\alpha_1/2)\cos(\alpha_2/2) \\ \cos(\alpha_1/2)\sin(\alpha_2/2)e^{i\phi_2} \\ \sin(\alpha_1/2)\cos(\alpha_2/2)e^{i\phi_1} \\ \sin(\alpha_1/2)\sin(\alpha_2/2)e^{i(\phi_1+\phi_2)} \end{pmatrix},$$

where $0 \leq \alpha_{1,2} \leq \pi$, $0 \leq \phi_{1,2} < 2\pi$. Note that a separable state in the symmetric, angular momentum basis has the form

$$|\psi_{12}\rangle_{\text{sym}} = \begin{pmatrix} \cos^2(\alpha/2) \\ \sqrt{2}\sin(\alpha/2)\cos(\alpha/2)\mathrm{e}^{i\phi} \\ \sin^2(\alpha/2)\mathrm{e}^{2i\phi} \end{pmatrix},$$

where $\alpha_1 = \alpha_2 = \alpha$ and $\phi_1 = \phi_2 = \phi$. It is a well-known fact that for a pure state of two qubits $|\psi\rangle = a|\uparrow\uparrow\rangle + b|\uparrow\downarrow\rangle + c|\downarrow\uparrow\rangle + d|\downarrow\downarrow\rangle$, the expression for concurrence is $C(\psi) = 2|ad - bc|$ [17]. For a maximally entangled quantum state concurrence $C = 1$. It can be observed that under the action of the gates $B_4$, $B_7$ and $B_8$ (with $e_p$ being maximum i.e., $2/9$), $|\psi_{12}\rangle_{\text{sym}}$ becomes maximally entangled state when $\alpha = \pi/2$, i.e.,

$$B_4|\psi_{12}\rangle_{\text{sym}} \xrightarrow{\alpha=\pi/2} -\frac{1}{2}\mathrm{e}^{2i\phi}|\uparrow\uparrow\rangle + \frac{1}{2}\mathrm{e}^{i\phi}|\uparrow\downarrow\rangle + \frac{1}{2}\mathrm{e}^{i\phi}|\downarrow\uparrow\rangle + \frac{1}{2}|\downarrow\downarrow\rangle,$$

$$B_7|\psi_{12}\rangle_{\text{sym}} \xrightarrow{\alpha=\pi/2} \frac{i}{2}\mathrm{e}^{2i\phi}|\uparrow\uparrow\rangle + \frac{1}{2}\mathrm{e}^{i\phi}|\uparrow\downarrow\rangle + \frac{1}{2}\mathrm{e}^{i\phi}|\downarrow\uparrow\rangle + \frac{i}{2}|\downarrow\downarrow\rangle,$$

$$B_8|\psi_{12}\rangle_{\text{sym}} \xrightarrow{\alpha=\pi/2} -\frac{i}{2}|\uparrow\uparrow\rangle + \frac{1}{2}\mathrm{e}^{i\phi}|\uparrow\downarrow\rangle + \frac{1}{2}\mathrm{e}^{i\phi}|\downarrow\uparrow\rangle + \frac{i}{2}\mathrm{e}^{2i\phi}|\downarrow\downarrow\rangle.$$

Similarly, the gates $B_5$, $B_6$ acting on the symmetric separable state, transform it into maximally entangled one when $\alpha = 0, \pi$. For e.g.,

$$B_5|\psi_{12}\rangle_{\text{sym}} \xrightarrow{\alpha=0} \frac{1}{2}|\uparrow\uparrow\rangle - \frac{1}{2}|\uparrow\downarrow\rangle - \frac{1}{2}|\downarrow\uparrow\rangle - \frac{1}{2}|\downarrow\downarrow\rangle,$$

$$B_6|\psi_{12}\rangle_{\text{sym}} \xrightarrow{\alpha=0} \frac{1}{2}|\uparrow\uparrow\rangle - \frac{i}{2}|\uparrow\downarrow\rangle - \frac{i}{2}|\downarrow\uparrow\rangle + \frac{1}{2}|\downarrow\downarrow\rangle.$$

Note that concurrence $C = 1$ in all these cases. It can be noted that the operators $B_8$ and $B_4$ produce spin squeezing resulting from a single axis twisting and two-axis countertwisting, respectively [18]. Also, possibilities of physical realization of these spin squeezing operators are given in [4].

### 3.1 *Special perfect entanglers*

Rezakhani [8] has analysed the perfect entanglers and found that some of them have the unique property of maximally entangling a complete set of orthonormal product vectors. Such operators belong to a well-known family of special perfect entanglers. A study of using such special perfect entanglers as the building blocks of the most efficient universal gate simulation is also given in [8]. Let us now study the conditions under which the perfect entanglers $B_4, \ldots B_8$ can be classified as special perfect entanglers. $B_4, \ldots B_8$ in the qubit basis $|\uparrow\uparrow\rangle$, $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$ are given by

$$B_4 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad B_5 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & 1 & 1 & 1 \end{pmatrix},$$

$$B_6 = \frac{1}{2}\begin{pmatrix} 1 & -i & -i & 1 \\ -i & 1 & -1 & i \\ -i & -1 & 1 & i \\ 1 & i & i & 1 \end{pmatrix}, \quad B_7 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad B_8 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & i \end{pmatrix}.$$

Following Rezakhani [8], the most general separable basis (upto general phase factors for each vector) can be written as

$$|\psi_1\rangle = (a\,|\uparrow\rangle + b\,|\downarrow\rangle) \otimes (c\,|\uparrow\rangle + d\,|\downarrow\rangle),$$

$$|\psi_2\rangle = (-b^*\,|\uparrow\rangle + a^*\,|\downarrow\rangle) \otimes (c\,|\uparrow\rangle + d\,|\downarrow\rangle),$$

$$|\psi_3\rangle = (e\,|\uparrow\rangle + f\,|\downarrow\rangle) \otimes (-d^*\,|\uparrow\rangle + c^*\,|\downarrow\rangle),$$

$$|\psi_4\rangle = (-f^*\,|\uparrow\rangle + e^*\,|\downarrow\rangle) \otimes (-d^*\,|\uparrow\rangle + c^*\,|\downarrow\rangle).$$

Here $|a|^2 + |b|^2 = |c|^2 + |d|^2 = |e|^2 + |f|^2 = 1$.

When the gates $B_4$, $B_7$ and $B_8$ as perfect entanglers act on the state – say $|\psi_1\rangle$, we obtain

$$[B_{4,7,8}]|\psi_1\rangle = -bd\ |\uparrow\uparrow\rangle + ad\ |\uparrow\downarrow\rangle + bc\ |\downarrow\uparrow\rangle + ac\ |\downarrow\downarrow\rangle.$$

This state is maximally entangled if its concurrence, $C = 4|abcd| = 1$. Thus, these two-qubit symmetric gates transform the orthonormal states $|\psi_1\rangle$, $|\psi_2\rangle$, $|\psi_3\rangle$ and $|\psi_4\rangle$ into maximally entangled ones if $|abcd| = |cdef| = \frac{1}{4}$. Similarly, for the gates $B_5$ and $B_6$, condition for finding a full set of orthonormal product states is $|(a^2 + b^2)(c^2 + d^2)| = |(e^2 + f^2)(c^2 + d^2)| = 1$.

*Example*: Let us consider the example of Lipkin–Meshkov–Glick interaction Hamiltonian [2,4] which is widely used in nuclear physics.

$$\mathcal{H}_L = \mathcal{G}_1(J_+^2 + J_-^2) + \mathcal{G}_2(J_+J_- + J_-J_+). \tag{15}$$

Here $\mathcal{G}_1$ and $\mathcal{G}_2$ are the coupling constants. In terms of our operators $M_k$'s,

$$\mathcal{H}_L = \mathcal{G}_1' M_7 + \mathcal{G}_2'(\sqrt{8}M_0 - M_8), \tag{16}$$

where $\mathcal{G}_1' = 2\mathcal{G}_1$ and $\mathcal{G}_2' = (2/\sqrt{3})\mathcal{G}_2$. Since $[M_7, M_8] = 0$, we have

$$\mathrm{e}^{iH_Lt} = B_L = \begin{pmatrix} \mathrm{e}^{\sqrt{3}i\beta}\cos\xi & 0 & i\mathrm{e}^{\sqrt{3}i\beta}\sin\xi \\ 0 & \mathrm{e}^{2\sqrt{3}i\beta} & 0 \\ i\mathrm{e}^{\sqrt{3}i\beta}\cos\xi & 0 & \mathrm{e}^{\sqrt{3}i\beta}\cos\xi \end{pmatrix},$$

in spin-1 subspace. Here $\xi = \mathcal{G}_1' t$ and $\beta = \mathcal{G}_2' t$ and $e_p = \frac{2}{9}$ for $2\mathcal{G}_2 t = \frac{\pi}{2} + 2\mathcal{G}_1 t$. Under the action of this gate (with $e_p = \frac{2}{9}$), the separable state $|\uparrow\uparrow\rangle(|\downarrow\downarrow\rangle)$ becomes entangled for all values of $t$ except when $t = n\pi/4\mathcal{G}_1$; $n = 0, 1, 2\ldots$ and maximally entangled when $4\mathcal{G}_1 t = (2n+1)(\pi/2)$. For e.g.,

$$B_L|\psi_{12}\rangle_{\rm sym} \xrightarrow{\alpha=0} \cos(2\mathcal{G}_1 t)\ |\uparrow\uparrow\rangle + i\sin(2\mathcal{G}_1 t)\ |\downarrow\downarrow\rangle.$$

## 4. Conclusion

In conclusion, we have constructed a traceless, Hermitian and linearly independent set of basis matrices which provides an alternate representation of $SU(n)$ generators. As these basis matrices are constructed out of various powers of angular momentum operators $\mathbf{J}_x$, $\mathbf{J}_y$, $\mathbf{J}_z$, their physical interpretation is easier compared to the Gellmann matrices in higher dimensions. We have considered unitary evolutions of two spin-$1/2$ states in angular momentum subspace ($j = 1$) and constructed physically realizable logic gates using $(2j+1)$-dimensional representation of the above set of basis matrices. Entangling properties of these gates have been studied in terms of their entangling power $e_p$. $e_p$ is found to be maximum (2/9) for $B_4$, ..., $B_8$ under certain conditions which is the signature for special perfect entanglers. These logic gates are obtained by the exponentiation of the quadratic form of angular momentum operators $\mathbf{J}_x, \mathbf{J}_y, \mathbf{J}_z$. As an example, we have taken the well-known Lipkin–Meshkov–Glick Hamiltonian and studied its entangling properties in spin-1 subspace. Further, we have shown precisely at what time the initial separable state becomes maximally entangled under the action of perfect entanglers which consists of one-axis twisting and two-axis twisting Hamiltonians that produce spin squeezing.

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