Statistical inference for 2-type doubly symmetric critical irreducible continuous state and continuous time branching processes with immigration

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Abstract

We study asymptotic behavior of conditional least squares estimators for 2-type doubly symmetric critical irreducible continuous state and continuous time branching processes with immigration based on discrete time (low frequency) observations.

1 Introduction

Asymptotic behavior of conditional least squares (CLS) estimators for critical continuous state and continuous time branching processes with immigration (CBI processes) is available only for single-type processes. Huang et al. [11] considered a single-type CBI process which can be represented as a pathwise unique strong solution of the stochastic differential equation (SDE)

$$\begin{aligned} X_t = X_0 + \int_0^t (\beta + \widetilde{B} X_s)\, \mathrm{d}s + \int_0^t \sqrt{2cX_s^+}\, \mathrm{d}W_s \\ + \int_0^t \int_0^\infty \int_0^\infty z \mathbb{1}_{\{u \leqslant X_{s-}\}}\, \widetilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t \int_0^\infty z\, M(\mathrm{d}s, \mathrm{d}z) \end{aligned} \tag{1.1}$$

for $t \in [0, \infty)$, where $\beta, c \in [0, \infty)$, $\widetilde{B} \in \mathbb{R}$, and $(W_t)_{t \geqslant 0}$ is a standard Wiener process, N and M are independent Poisson random measures on $(0, \infty)^3$ and on $(0, \infty)^2$ with

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intensity measures $\mathrm{d}s\,\mu(\mathrm{d}z)\,\mathrm{d}u$ and $\mathrm{d}s\,\nu(\mathrm{d}z)$, respectively, $\widetilde{N}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) := N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) - \mathrm{d}s\,\mu(\mathrm{d}z)\,\mathrm{d}u$, the measures μ and ν satisfy some moment conditions, and $(W_t)_{t\geqslant 0}$, N and M are independent. The model is called subcritical, critical or supercritical if $\widetilde{B} < 0$, $\widetilde{B} = 0$ or $\widetilde{B} > 0$, see Huang et al. [11, page 1105] or Definition 2.8. Based on discrete time (low frequency) observations $(X_k)_{k\in\{0,1,\ldots,n\}}$, $n \in \{1,2,\ldots\}$, Huang et al. [11] derived weighted CLS estimator of $(\beta, \widetilde{B})$. Under some regularity assumptions, they showed that the estimator of $(\beta, \widetilde{B})$ is asymptotically normal in the subcritical case, the estimator of $\widetilde{B}$ has a non-normal limit in the critical case, and the estimator of $\widetilde{B}$ is asymptotically normal with a random scaling in the supercritical case.

Overbeck and Rydén [22] considered CLS and weighted CLS estimators for the well-known Cox–Ingersoll–Ross model, which is, in fact, a single-type diffusion CBI process (without jump part), i.e., when $\mu = 0$ and $\nu = 0$ in (1.1). Based on discrete time observations $(X_k)_{k\in\{0,1,\ldots,n\}}$, $n \in \{1,2,\ldots\}$, they derived CLS estimator of $(\beta, \widetilde{B}, c)$ and proved its asymptotic normality in the subcritical case. Note that Li and Ma [21] started to investigate the asymptotic behaviour of the CLS and weighted CLS estimators of the parameters $(\beta, \widetilde{B})$ in the subcritical case for a Cox–Ingersoll–Ross model driven by a stable noise, which is again a special single-type CBI process (with jump part).

In this paper we consider a 2-type CBI process which can be represented as a pathwise unique strong solution of the SDE

$$\begin{aligned}\boldsymbol{X}_t = \boldsymbol{X}_0 &+ \int_0^t (\boldsymbol{\beta} + \widetilde{\boldsymbol{B}}\boldsymbol{X}_s)\,\mathrm{d}s + \sum_{i=1}^2 \int_0^t \sqrt{2c_i X_{s,i}^+}\,\mathrm{d}W_{s,i}\,\boldsymbol{e}_i \\ &+ \sum_{j=1}^2 \int_0^t \int_{\mathcal{U}_2} \int_0^\infty \boldsymbol{z}\mathbb{1}_{\{u\leqslant X_{s-,j}\}}\,\widetilde{N}_j(\mathrm{d}s,\mathrm{d}\boldsymbol{z},\mathrm{d}u) + \int_0^t \int_{\mathcal{U}_2} \boldsymbol{z}\,M(\mathrm{d}s,\mathrm{d}\boldsymbol{z})\end{aligned} \tag{1.2}$$

for $t \in [0,\infty)$. Here $X_{t,i}$, $i \in \{1,2\}$, denotes the coordinates of $\boldsymbol{X}_t$, $\boldsymbol{\beta} \in [0,\infty)^2$, $\widetilde{\boldsymbol{B}} \in \mathbb{R}^{2\times 2}$ has non-negative off-diagonal entries, $c_1, c_2 \in [0,\infty)$, $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d$ denotes the natural basis in $\mathbb{R}^d$, $\mathcal{U}_2 := [0,\infty)^2 \setminus \{(0,0)\}$, $(W_{t,1})_{t\geqslant 0}$ and $(W_{t,2})_{t\geqslant 0}$ are independent standard Wiener processes, N_j, $j \in \{1,2\}$, and M are independent Poisson random measures on $(0,\infty)\times\mathcal{U}_2\times(0,\infty)$ and on $(0,\infty)\times\mathcal{U}_2$ with intensity measures $\mathrm{d}s\,\mu_j(\mathrm{d}\boldsymbol{z})\,\mathrm{d}u$, $j \in \{1,2\}$, and $\mathrm{d}s\,\nu(\mathrm{d}\boldsymbol{z})$, respectively, $\widetilde{N}_j(\mathrm{d}s,\mathrm{d}\boldsymbol{z},\mathrm{d}u) := N_j(\mathrm{d}s,\mathrm{d}\boldsymbol{z},\mathrm{d}u) - \mathrm{d}s\,\mu_j(\mathrm{d}\boldsymbol{z})\,\mathrm{d}u$, $j \in \{1,2\}$. We suppose that the measures μ_j, $j \in \{1,2\}$, and ν satisfy some moment conditions, and $(W_{t,1})_{t\geqslant 0}$, $(W_{t,2})_{t\geqslant 0}$, N_1, N_2 and M are independent. We will suppose that the process $(\boldsymbol{X}_t)_{t\geqslant 0}$ is doubly symmetric in the sense that

$$\widetilde{\boldsymbol{B}} = \begin{bmatrix} \gamma & \kappa \\ \kappa & \gamma \end{bmatrix},$$

where $\gamma \in \mathbb{R}$ and $\kappa \in [0,\infty)$. Note that the parameters γ and κ might be interpreted as the transformation rates of one type to the same type and one type to the other type, respectively, compare with Xu [25]; that's why the model can be called doubly symmetric.

The model will be called subcritical, critical or supercritical if $s < 0$, $s = 0$ or $s > 0$, respectively, where $s := \gamma + \kappa$ denotes the criticality parameter, see Definition 2.8.

For the simplicity, we suppose $\boldsymbol{X}_0 = (0,0)^\top$. We suppose that $\boldsymbol{\beta}, c_1, c_2, \mu_1, \mu_2$ and ν are known, and we derive the CLS estimators of the parameters s, γ and κ based on a discrete time (low frequency) observations $(\boldsymbol{X}_k)_{k\in\{1,\ldots,n\}}$, $n \in \{1,2,\ldots\}$. In the irreducible and critical case, i.e, when $\kappa > 0$ and $s = \gamma + \kappa = 0$, under some moment conditions, we describe the asymptotic behavior of these CLS estimators as $n \to \infty$, provided that $\boldsymbol{\beta} \neq (0,0)^\top$ or $\nu \neq 0$, see Theorem 3.1. We point out that the limit distributions are non-normal in general. In the present paper we do not investigate the asymptotic behavior of CLS estimators of s, γ and κ in the subcritical and supercritical cases, it could be the topic of separate papers.

Xu [25] considered a 2-type diffusion CBI process (without jump part), i.e., when $\mu_j = 0$, $j \in \{1,2\}$, and $\nu = 0$ in (1.2). Based on discrete time (low frequency) observations $(\boldsymbol{X}_k)_{k\in\{1,\ldots,n\}}$, $n \in \{1,2,\ldots\}$, Xu [25] derived CLS estimators and weighted CLS estimators of $(\boldsymbol{\beta}, \widetilde{\boldsymbol{B}}, c_1, c_2)$. Provided that $\boldsymbol{\beta} \in (0,\infty)^2$, the diagonal entries of $\widetilde{\boldsymbol{B}}$ are negative, the off-diagonal entries of $\widetilde{\boldsymbol{B}}$ are positive, the determinant of $\widetilde{\boldsymbol{B}}$ is positive and $c_i > 0$, $i \in \{1,2\}$ (which yields that the process $\boldsymbol{X}$ is irreducible and subcritical, see Xu [25, Theorem 2.2] and Definitions 2.7 and 2.8), it was shown that these CLS estimators are asymptotically normal, see Theorem 4.6 in Xu [25].

Finally, we give an overview of the paper. In Section 2, for completeness and better readability, from Barczy et al. [5] and [7], we recall some notions and statements for multi-type CBI processes such as the form of their infinitesimal generator, Laplace transform, a formula for their first moment, the definition of subcritical, critical and supercritical irreducible CBI processes, see Definitions 2.7 and 2.8. We recall a result due to Barczy and Pap [7, Theorem 4.1] stating that, under some fourth order moment assumptions, a sequence of scaled random step functions $(n^{-1}\boldsymbol{X}_{\lfloor nt\rfloor})_{t\geqslant 0}$, $n \geqslant 1$, formed from a critical, irreducible multi-type CBI process $\boldsymbol{X}$ converges weakly towards a squared Bessel process supported by a ray determined by the Perron vector of a matrix related to the branching mechanism of $\boldsymbol{X}$.

In Section 3, first we derive formulas of CLS estimators of the transformed parameters $\mathrm{e}^{\gamma+\kappa}$ and $\mathrm{e}^{\gamma-\kappa}$, and then of the parameters γ and κ. The reason for this parameter transformation is to reduce the minimization in the CLS method to a linear problem. Then we formulate our main result about the asymptotic behavior of CLS estimators of s, γ and κ in the irreducible and critical case, see Theorem 3.1. These results will be derived from the corresponding statements for the transformed parameters $\mathrm{e}^{\gamma+\kappa}$ and $\mathrm{e}^{\gamma-\kappa}$, see Theorem 3.5.

In Section 4, we give a decomposition of the process $\boldsymbol{X}$ and of the CLS estimators of the transformed parameters $\mathrm{e}^{\gamma+\kappa}$ and $\mathrm{e}^{\gamma-\kappa}$ as well, related to the left eigenvectors of $\widetilde{\boldsymbol{B}}$ belonging to the eigenvalues $\gamma + \kappa$ and $\gamma - \kappa$, see formulas (4.5) and (4.6). By the help of these decompositions, Theorem 3.5 will follow from Theorems 4.1, 4.2 and 4.3.

Sections 5, 6 and 7 are devoted to the proofs of Theorems 4.1, 4.2 and 4.3, respectively. The proofs are heavily based on a careful analysis of the asymptotic behavior of some martingale differences related to the process $\boldsymbol{X}$ and the decompositions given in Section 4, and delicate moment estimations for the process $\boldsymbol{X}$ and some auxiliary processes.

In Appendix A we recall a representation of multi-type CBI processes as pathwise unique strong solutions of certain SDEs with jumps based on Barczy et al. [5]. In Appendix B we recall some results about the asymptotic behaviour of moments of irreducible and critical multi-type CBI processes based on Barczy, Li and Pap [6], and then, presenting new results as well, the asymptotic behaviour of the moments of some auxiliary processes is also investigated. Appendix C is devoted to study of the existence of the CLSE of the transformed parameters $\mathrm{e}^{\gamma+\kappa}$ and $\mathrm{e}^{\gamma-\kappa}$. In Appendix D, we present a version of the continuous mapping theorem. In Appendix E, we recall a useful result about convergence of random step processes towards a diffusion process due to Ispány and Pap [15, Corollary 2.2].

In some cases the proofs are omitted or condensed, however in these cases we always refer to our ArXiv preprint Barczy et al. [8] for a detailed discussion.

2 Multi-type CBI processes

Let $\mathbb{Z}_+$, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{R}_{++}$ denote the set of non-negative integers, positive integers, real numbers, non-negative real numbers and positive real numbers, respectively. For $x, y \in \mathbb{R}$, we will use the notations $x \wedge y := \min\{x, y\}$ and $x^+ := \max\{0, x\}$. By $\|\boldsymbol{x}\|$ and $\|\boldsymbol{A}\|$, we denote the Euclidean norm of a vector $\boldsymbol{x} \in \mathbb{R}^d$ and the induced matrix norm of a matrix $\boldsymbol{A} \in \mathbb{R}^{d\times d}$, respectively. The natural basis in $\mathbb{R}^d$ will be denoted by $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d$. The null vector and the null matrix will be denoted by $\boldsymbol{0}$. By $C^2_{\mathrm{c}}(\mathbb{R}^d_+, \mathbb{R})$ we denote the set of twice continuously differentiable real-valued functions on $\mathbb{R}^d_+$ with compact support. Convergence in distribution and in probability will be denoted by $\stackrel{\mathcal{D}}{\longrightarrow}$ and $\stackrel{\mathbb{P}}{\longrightarrow}$, respectively. Almost sure equality will be denoted by $\stackrel{\mathrm{a.s.}}{=}$.

2.1 Definition. *A matrix $\boldsymbol{A} = (a_{i,j})_{i,j\in\{1,\ldots,d\}} \in \mathbb{R}^{d\times d}$ is called essentially non-negative if $a_{i,j} \in \mathbb{R}_+$ whenever $i, j \in \{1, \ldots, d\}$ with $i \neq j$, that is, if $\boldsymbol{A}$ has non-negative off-diagonal entries. The set of essentially non-negative $d \times d$ matrices will be denoted by $\mathbb{R}^{d\times d}_{(+)}$.*

2.2 Definition. *A tuple $(d, \boldsymbol{c}, \boldsymbol{\beta}, \boldsymbol{B}, \nu, \boldsymbol{\mu})$ is called a set of admissible parameters if*

(i) $d \in \mathbb{N}$,

(ii) $\boldsymbol{c} = (c_i)_{i\in\{1,\ldots,d\}} \in \mathbb{R}^d_+$,

(iii) $\boldsymbol{\beta} = (\beta_i)_{i\in\{1,\ldots,d\}} \in \mathbb{R}^d_+$,

(iv) $\boldsymbol{B} = (b_{i,j})_{i,j\in\{1,\ldots,d\}} \in \mathbb{R}^{d\times d}_{(+)}$,

(v) ν *is a Borel measure on* $\mathcal{U}_d := \mathbb{R}^d_+ \setminus \{\boldsymbol{0}\}$ *satisfying* $\int_{\mathcal{U}_d} (1 \wedge \|\boldsymbol{z}\|)\, \nu(\mathrm{d}\boldsymbol{z}) < \infty$,

(vi) $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_d)$, *where, for each* $i \in \{1, \ldots, d\}$, μ_i *is a Borel measure on* $\mathcal{U}_d$ *satisfying*

$$\int_{\mathcal{U}_d} \Bigg[(1 \wedge z_i)^2 + \sum_{j \in \{1,\ldots,d\} \setminus \{i\}} (1 \wedge z_j) \Bigg] \mu_i(\mathrm{d}\boldsymbol{z}) < \infty.$$

2.3 Remark. Our Definition 2.2 of the set of admissible parameters is a special case of Definition 2.6 in Duffie et al. [9], which is suitable for all affine processes, see Barczy et al. [5, Remark 2.3]. □

2.4 Theorem. *Let* $(d, \boldsymbol{c}, \boldsymbol{\beta}, \boldsymbol{B}, \nu, \boldsymbol{\mu})$ *be a set of admissible parameters. Then there exists a unique transition semigroup* $(P_t)_{t \in \mathbb{R}_+}$ *acting on the Banach space (endowed with the supremum norm) of real-valued bounded Borel-measurable functions on the state space* $\mathbb{R}_+^d$ *such that its infinitesimal generator is*

$$(2.1)\qquad \begin{aligned} (\mathcal{A}f)(\boldsymbol{x}) = &\sum_{i=1}^d c_i x_i f_{i,i}''(\boldsymbol{x}) + \langle \boldsymbol{\beta} + \boldsymbol{B}\boldsymbol{x}, \boldsymbol{f}'(\boldsymbol{x}) \rangle + \int_{\mathcal{U}_d} \big(f(\boldsymbol{x} + \boldsymbol{z}) - f(\boldsymbol{x}) \big) \, \nu(\mathrm{d}\boldsymbol{z}) \\ &+ \sum_{i=1}^d x_i \int_{\mathcal{U}_d} \big(f(\boldsymbol{x} + \boldsymbol{z}) - f(\boldsymbol{x}) - f_i'(\boldsymbol{x})(1 \wedge z_i) \big) \, \mu_i(\mathrm{d}\boldsymbol{z}) \end{aligned}$$

for $f \in C_{\mathrm{c}}^2(\mathbb{R}_+^d, \mathbb{R})$ *and* $\boldsymbol{x} \in \mathbb{R}_+^d$, *where* f_i' *and* $f_{i,i}''$, $i \in \{1, \ldots, d\}$, *denote the first and second order partial derivatives of* f *with respect to its* i*-th variable, respectively, and* $\boldsymbol{f}'(\boldsymbol{x}) := (f_1'(\boldsymbol{x}), \ldots, f_d'(\boldsymbol{x}))^\top$. *Moreover, the Laplace transform of the transition semigroup* $(P_t)_{t \in \mathbb{R}_+}$ *has a representation*

$$\int_{\mathbb{R}_+^d} \mathrm{e}^{-\langle \boldsymbol{\lambda}, \boldsymbol{y} \rangle} P_t(\boldsymbol{x}, \mathrm{d}\boldsymbol{y}) = \mathrm{e}^{-\langle \boldsymbol{x}, \boldsymbol{v}(t, \boldsymbol{\lambda}) \rangle - \int_0^t \psi(\boldsymbol{v}(s, \boldsymbol{\lambda})) \, \mathrm{d}s}, \qquad \boldsymbol{x} \in \mathbb{R}_+^d, \quad \boldsymbol{\lambda} \in \mathbb{R}_+^d, \quad t \in \mathbb{R}_+,$$

where, for any $\boldsymbol{\lambda} \in \mathbb{R}_+^d$, *the continuously differentiable function* $\mathbb{R}_+ \ni t \mapsto \boldsymbol{v}(t, \boldsymbol{\lambda}) = (v_1(t, \boldsymbol{\lambda}), \ldots, v_d(t, \boldsymbol{\lambda}))^\top \in \mathbb{R}_+^d$ *is the unique locally bounded solution to the system of differential equations*

$$(2.2)\qquad \partial_t v_i(t, \boldsymbol{\lambda}) = -\varphi_i(\boldsymbol{v}(t, \boldsymbol{\lambda})), \qquad v_i(0, \boldsymbol{\lambda}) = \lambda_i, \qquad i \in \{1, \ldots, d\},$$

with

$$\varphi_i(\boldsymbol{\lambda}) := c_i \lambda_i^2 - \langle \boldsymbol{B}\boldsymbol{e}_i, \boldsymbol{\lambda} \rangle + \int_{\mathcal{U}_d} \big(\mathrm{e}^{-\langle \boldsymbol{\lambda}, \boldsymbol{z} \rangle} - 1 + \lambda_i (1 \wedge z_i) \big) \, \mu_i(\mathrm{d}\boldsymbol{z})$$

for $\boldsymbol{\lambda} \in \mathbb{R}_+^d$ *and* $i \in \{1, \ldots, d\}$, *and*

$$\psi(\boldsymbol{\lambda}) := \langle \boldsymbol{\beta}, \boldsymbol{\lambda} \rangle + \int_{\mathcal{U}_d} \big(1 - \mathrm{e}^{-\langle \boldsymbol{\lambda}, \boldsymbol{z} \rangle} \big) \, \nu(\mathrm{d}\boldsymbol{z}), \qquad \boldsymbol{\lambda} \in \mathbb{R}_+^d.$$

2.5 Remark. This theorem is a special case of Theorem 2.7 of Duffie et al. [9] with $m = d$, $n = 0$ and zero killing rate. The unique existence of a locally bounded solution to the system of differential equations (2.2) is proved by Li [20, page 45]. □

2.6 Definition. *A Markov process with state space* $\mathbb{R}_+^d$ *and with transition semigroup* $(P_t)_{t\in\mathbb{R}_+}$ *given in Theorem 2.4 is called a multi-type CBI process with parameters* $(d, \boldsymbol{c}, \boldsymbol{\beta}, \boldsymbol{B}, \nu, \boldsymbol{\mu})$. *The function* $\mathbb{R}_+^d \ni \boldsymbol{\lambda} \mapsto (\varphi_1(\boldsymbol{\lambda}), \ldots, \varphi_d(\boldsymbol{\lambda}))^\top \in \mathbb{R}^d$ *is called its branching mechanism, and the function* $\mathbb{R}_+^d \ni \boldsymbol{\lambda} \mapsto \psi(\boldsymbol{\lambda}) \in \mathbb{R}_+$ *is called its immigration mechanism.*

Note that the branching mechanism depends only on the parameters $\boldsymbol{c}$, $\boldsymbol{B}$ and $\boldsymbol{\mu}$, while the immigration mechanism depends only on the parameters $\boldsymbol{\beta}$ and ν.

Let $(\boldsymbol{X}_t)_{t\in\mathbb{R}_+}$ be a multi-type CBI process with parameters $(d, \boldsymbol{c}, \boldsymbol{\beta}, \boldsymbol{B}, \nu, \boldsymbol{\mu})$ such that the moment conditions

$$\int_{\mathcal{U}_d} \|\boldsymbol{z}\|^q \mathbb{1}_{\{\|\boldsymbol{z}\|\geqslant 1\}} \, \nu(\mathrm{d}\boldsymbol{z}) < \infty, \qquad \int_{\mathcal{U}_d} \|\boldsymbol{z}\|^q \mathbb{1}_{\{\|\boldsymbol{z}\|\geqslant 1\}} \, \mu_i(\mathrm{d}\boldsymbol{z}) < \infty, \quad i \in \{1, \ldots, d\} \tag{2.3}$$

hold with $q = 1$. Then, by formula (3.4) in Barczy et al. [5],

$$\mathbb{E}(\boldsymbol{X}_t \,|\, \boldsymbol{X}_0 = \boldsymbol{x}) = \mathrm{e}^{t\widetilde{\boldsymbol{B}}}\boldsymbol{x} + \int_0^t \mathrm{e}^{u\widetilde{\boldsymbol{B}}}\widetilde{\boldsymbol{\beta}} \, \mathrm{d}u, \qquad \boldsymbol{x} \in \mathbb{R}_+^d, \quad t \in \mathbb{R}_+, \tag{2.4}$$

where

$$\widetilde{\boldsymbol{B}} := (\widetilde{b}_{i,j})_{i,j\in\{1,\ldots,d\}}, \qquad \widetilde{b}_{i,j} := b_{i,j} + \int_{\mathcal{U}_d} (z_i - \delta_{i,j})^+ \, \mu_j(\mathrm{d}\boldsymbol{z}), \tag{2.5}$$

$$\widetilde{\boldsymbol{\beta}} := \boldsymbol{\beta} + \int_{\mathcal{U}_d} \boldsymbol{z} \, \nu(\mathrm{d}\boldsymbol{z}), \tag{2.6}$$

with $\delta_{i,j} := 1$ if $i = j$, and $\delta_{i,j} := 0$ if $i \neq j$. Note that $\widetilde{\boldsymbol{B}} \in \mathbb{R}_{(+)}^{d\times d}$ and $\widetilde{\boldsymbol{\beta}} \in \mathbb{R}_+^d$, since

$$\int_{\mathcal{U}_d} \|\boldsymbol{z}\| \, \nu(\mathrm{d}\boldsymbol{z}) < \infty, \qquad \int_{\mathcal{U}_d} (z_i - \delta_{i,j})^+ \, \mu_j(\mathrm{d}\boldsymbol{z}) < \infty, \quad i, j \in \{1, \ldots, d\}, \tag{2.7}$$

see Barczy et al. [5, Section 2]. One can give probabilistic interpretations of the modified parameters $\widetilde{\boldsymbol{B}}$ and $\widetilde{\boldsymbol{\beta}}$, namely, $\mathrm{e}^{\widetilde{\boldsymbol{B}}}\boldsymbol{e}_j = \mathbb{E}(\boldsymbol{Y}_1 \,|\, \boldsymbol{Y}_0 = \boldsymbol{e}_j)$, $j \in \{1, \ldots, d\}$, and $\widetilde{\boldsymbol{\beta}} = \mathbb{E}(\boldsymbol{Z}_1 \,|\, \boldsymbol{Z}_0 = \boldsymbol{0})$, where $(\boldsymbol{Y}_t)_{t\in\mathbb{R}_+}$ and $(\boldsymbol{Z}_t)_{t\in\mathbb{R}_+}$ are multi-type CBI processes with parameters $(d, \boldsymbol{c}, \boldsymbol{0}, \boldsymbol{B}, 0, \boldsymbol{\mu})$ and $(d, \boldsymbol{0}, \boldsymbol{\beta}, \boldsymbol{0}, \nu, \boldsymbol{0})$, respectively, see formula (2.4). The processes $(\boldsymbol{Y}_t)_{t\in\mathbb{R}_+}$ and $(\boldsymbol{Z}_t)_{t\in\mathbb{R}_+}$ can be considered as pure branching (without immigration) and pure immigration (without branching) processes, respectively. Consequently, $\mathrm{e}^{\widetilde{\boldsymbol{B}}}$ and $\widetilde{\boldsymbol{\beta}}$ may be called the branching mean matrix and the immigration mean vector, respectively.

Next we recall a classification of multi-type CBI processes. For a matrix $\boldsymbol{A} \in \mathbb{R}^{d\times d}$, $\sigma(\boldsymbol{A})$ will denote the spectrum of $\boldsymbol{A}$, that is, the set of the eigenvalues of $\boldsymbol{A}$. Then $r(\boldsymbol{A}) := \max_{\lambda\in\sigma(\boldsymbol{A})} |\lambda|$ is the spectral radius of $\boldsymbol{A}$. Moreover, we will use the notation

$$s(\boldsymbol{A}) := \max_{\lambda\in\sigma(\boldsymbol{A})} \mathrm{Re}(\lambda).$$

A matrix $\boldsymbol{A} \in \mathbb{R}^{d\times d}$ is called reducible if there exist a permutation matrix $\boldsymbol{P} \in \mathbb{R}^{d\times d}$ and an integer r with $1 \leqslant r \leqslant d - 1$ such that

$$\boldsymbol{P}^\top \boldsymbol{A} \boldsymbol{P} = \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{A}_2 \\ \boldsymbol{0} & \boldsymbol{A}_3 \end{bmatrix},$$

where $\boldsymbol{A}_1 \in \mathbb{R}^{r\times r}$, $\boldsymbol{A}_3 \in \mathbb{R}^{(d-r)\times(d-r)}$, $\boldsymbol{A}_2 \in \mathbb{R}^{r\times(d-r)}$, and $\boldsymbol{0} \in \mathbb{R}^{(d-r)\times r}$ is a null matrix. A matrix $\boldsymbol{A} \in \mathbb{R}^{d\times d}$ is called irreducible if it is not reducible, see, e.g., Horn and Johnson [10, Definitions 6.2.21 and 6.2.22]. We do emphasize that no 1-by-1 matrix is reducible.

2.7 Definition. *Let $(\boldsymbol{X}_t)_{t\in\mathbb{R}_+}$ be a multi-type CBI process with parameters $(d, \boldsymbol{c}, \boldsymbol{\beta}, \boldsymbol{B}, \nu, \boldsymbol{\mu})$ such that the moment conditions* (2.3) *hold with $q = 1$. Then $(\boldsymbol{X}_t)_{t\in\mathbb{R}_+}$ is called irreducible if $\widetilde{\boldsymbol{B}}$ is irreducible.*

2.8 Definition. *Let $(\boldsymbol{X}_t)_{t\in\mathbb{R}_+}$ be a multi-type CBI process with parameters $(d, \boldsymbol{c}, \boldsymbol{\beta}, \boldsymbol{B}, \nu, \boldsymbol{\mu})$ such that $\mathbb{E}(\|\boldsymbol{X}_0\|) < \infty$ and the moment conditions* (2.3) *hold with $q = 1$. Suppose that $(\boldsymbol{X}_t)_{t\in\mathbb{R}_+}$ is irreducible. Then $(\boldsymbol{X}_t)_{t\in\mathbb{R}_+}$ is called*

$$\begin{cases} \textit{subcritical} & \textit{if } s(\widetilde{\boldsymbol{B}}) < 0, \\ \textit{critical} & \textit{if } s(\widetilde{\boldsymbol{B}}) = 0, \\ \textit{supercritical} & \textit{if } s(\widetilde{\boldsymbol{B}}) > 0. \end{cases}$$

For motivations of Definitions 2.7 and 2.8, see Barczy et al. [7, Section 3].

Next we will recall a convergence result for irreducible and critical multi-type CBI processes.

A function $f : \mathbb{R}_+ \to \mathbb{R}^d$ is called *càdlàg* if it is right continuous with left limits. Let $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ and $\mathbb{C}(\mathbb{R}_+, \mathbb{R}^d)$ denote the space of all $\mathbb{R}^d$-valued càdlàg and continuous functions on $\mathbb{R}_+$, respectively. Let $\mathcal{D}_\infty(\mathbb{R}_+, \mathbb{R}^d)$ denote the Borel σ-field in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ for the metric characterized by Jacod and Shiryaev [16, VI.1.15] (with this metric $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ is a complete and separable metric space). For $\mathbb{R}^d$-valued stochastic processes $(\boldsymbol{\mathcal{Y}}_t)_{t\in\mathbb{R}_+}$ and $(\boldsymbol{\mathcal{Y}}_t^{(n)})_{t\in\mathbb{R}_+}$, $n \in \mathbb{N}$, with càdlàg paths we write $\boldsymbol{\mathcal{Y}}^{(n)} \xrightarrow{\mathcal{D}} \boldsymbol{\mathcal{Y}}$ as $n \to \infty$ if the distribution of $\boldsymbol{\mathcal{Y}}^{(n)}$ on the space $(\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d), \mathcal{D}_\infty(\mathbb{R}_+, \mathbb{R}^d))$ converges weakly to the distribution of $\boldsymbol{\mathcal{Y}}$ on the space $(\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d), \mathcal{D}_\infty(\mathbb{R}_+, \mathbb{R}^d))$ as $n \to \infty$. Concerning the notation $\xrightarrow{\mathcal{D}}$ we note that if ξ and ξ_n, $n \in \mathbb{N}$, are random elements with values in a metric space (E, ρ), then we also denote by $\xi_n \xrightarrow{\mathcal{D}} \xi$ the weak convergence of the distributions of ξ_n on the space $(E, \mathcal{B}(E))$ towards the distribution of ξ on the space $(E, \mathcal{B}(E))$ as $n \to \infty$, where $\mathcal{B}(E)$ denotes the Borel σ-algebra on E induced by the given metric ρ.

The proof of the following convergence theorem can be found in Barczy and Pap [7, Theorem 4.1 and Lemma A.3].

2.9 Theorem. *Let $(\boldsymbol{X}_t)_{t\in\mathbb{R}_+}$ be a multi-type CBI process with parameters $(d, \boldsymbol{c}, \boldsymbol{\beta}, \boldsymbol{B}, \nu, \boldsymbol{\mu})$ such that $\mathbb{E}(\|\boldsymbol{X}_0\|^4) < \infty$ and the moment conditions* (2.3) *hold with $q = 4$. Suppose that $(\boldsymbol{X}_t)_{t\in\mathbb{R}_+}$ is irreducible and critical. Then*

$$(2.8) \qquad (\boldsymbol{\mathcal{X}}_t^{(n)})_{t\in\mathbb{R}_+} := (n^{-1}\boldsymbol{X}_{\lfloor nt\rfloor})_{t\in\mathbb{R}_+} \xrightarrow{\mathcal{D}} (\boldsymbol{\mathcal{X}}_t)_{t\in\mathbb{R}_+} := (\mathcal{Y}_t \boldsymbol{u}_{\mathrm{right}})_{t\in\mathbb{R}_+} \qquad \textit{as } n \to \infty$$

in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$, where $\boldsymbol{u}_{\mathrm{right}} \in \mathbb{R}^d_{++}$ is the right Perron vector of $\mathrm{e}^{\widetilde{\boldsymbol{B}}}$ (corresponding to the eigenvalue 1), $(\mathcal{Y}_t)_{t\in\mathbb{R}_+}$ is the pathwise unique strong solution of the SDE

$$(2.9) \qquad \mathrm{d}\mathcal{Y}_t = \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}}\rangle \,\mathrm{d}t + \sqrt{\langle \overline{\boldsymbol{C}} \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{u}_{\mathrm{left}}\rangle \mathcal{Y}_t^+}\,\mathrm{d}\mathcal{W}_t, \qquad t \in \mathbb{R}_+, \qquad \mathcal{Y}_0 = 0,$$

where $\boldsymbol{u}_{\mathrm{left}} \in \mathbb{R}_{++}^{d}$ *is the left Perron vector of* $\mathrm{e}^{\widetilde{\boldsymbol{B}}}$ *(corresponding to the eigenvalue 1),* $(\mathcal{W}_t)_{t\in\mathbb{R}_+}$ *is a standard Brownian motion and*

$$\overline{\boldsymbol{C}} := \sum_{k=1}^{d} \langle \boldsymbol{e}_k, \boldsymbol{u}_{\mathrm{right}} \rangle \, \boldsymbol{C}_k \in \mathbb{R}_{+}^{d\times d} \tag{2.10}$$

with

$$\boldsymbol{C}_k := 2c_k \boldsymbol{e}_k \boldsymbol{e}_k^\top + \int_{\mathcal{U}_d} \boldsymbol{z}\boldsymbol{z}^\top \mu_k(\mathrm{d}\boldsymbol{z}) \in \mathbb{R}_{+}^{d\times d}, \qquad k \in \{1, \ldots, d\}. \tag{2.11}$$

The moment conditions (2.3) with $q = 4$ in Theorem 2.9 are used only for checking the conditional Lindeberg condition, namely, condition (ii) of Theorem E.1. For a more detailed discussion, see Barczy and Pap [7, Remark 4.2]. Note also that Theorem 2.9 is in accordance with Theorem 3.1 in Ispány and Pap [15].

2.10 Remark. The SDE (2.9) has a pathwise unique strong solution $(\mathcal{Y}_t^{(y)})_{t\in\mathbb{R}_+}$ for all initial values $\mathcal{Y}_0^{(y)} = y \in \mathbb{R}$, and if the initial value y is nonnegative, then $\mathcal{Y}_t^{(y)}$ is nonnegative for all $t \in \mathbb{R}_+$ with probability one, since $\langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle \in \mathbb{R}_+$, see, e.g., Ikeda and Watanabe [12, Chapter IV, Example 8.2]. □

2.11 Remark. Note that for the definition of $\boldsymbol{C}_k$, $k \in \{1, \ldots, d\}$ and $\overline{\boldsymbol{C}}$, the moment conditions (2.3) are needed only with $q = 2$. Moreover, $\langle \overline{\boldsymbol{C}} \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{u}_{\mathrm{left}} \rangle = 0$ if and only if $\boldsymbol{c} = \boldsymbol{0}$ and $\boldsymbol{\mu} = \boldsymbol{0}$, when the pathwise unique strong solution of (2.9) is the deterministic function $\mathcal{Y}_t = \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle \, t$, $t \in \mathbb{R}_+$. Indeed,

$$\langle \overline{\boldsymbol{C}} \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{u}_{\mathrm{left}} \rangle = \sum_{k=1}^{d} \langle \boldsymbol{e}_k, \boldsymbol{u}_{\mathrm{right}} \rangle \biggl(2c_k \langle \boldsymbol{e}_k, \boldsymbol{u}_{\mathrm{left}} \rangle^2 + \int_{\mathcal{U}_d} \langle \boldsymbol{z}, \boldsymbol{u}_{\mathrm{left}} \rangle^2 \, \mu_k(\mathrm{d}\boldsymbol{z}) \biggr).$$

Further, $\overline{\boldsymbol{C}}$ in (2.9) can be replaced by

$$\widetilde{\boldsymbol{C}} := \sum_{i=1}^{d} \langle \boldsymbol{e}_i, \boldsymbol{u}_{\mathrm{right}} \rangle \boldsymbol{V}_i = \operatorname{Var}(\boldsymbol{Y}_1 \,|\, \boldsymbol{Y}_0 = \boldsymbol{u}_{\mathrm{right}}), \tag{2.12}$$

where $(\boldsymbol{Y}_t)_{t\in\mathbb{R}_+}$ is a multi-type CBI process with parameters $(d, \boldsymbol{c}, \boldsymbol{0}, \boldsymbol{B}, 0, \boldsymbol{\mu})$ such that the moment conditions (2.3) hold with $q = 2$, see Proposition B.3. Indeed, by the spectral mapping theorem, $\boldsymbol{u}_{\mathrm{left}}$ is a left eigenvector of $\mathrm{e}^{s\widetilde{\boldsymbol{B}}}$, $s \in \mathbb{R}_+$, belonging to the eigenvalue 1, hence $\langle \widetilde{\boldsymbol{C}} \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{u}_{\mathrm{left}} \rangle = \langle \overline{\boldsymbol{C}} \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{u}_{\mathrm{left}} \rangle$. In fact, $(\boldsymbol{Y}_t)_{t\in\mathbb{R}_+}$ is a multi-type CBI process without immigration such that its branching mechanism is the same as that of $(\boldsymbol{X}_t)_{t\in\mathbb{R}_+}$. Note that for each $i \in \{1, \ldots, d\}$, $\boldsymbol{V}_i = \sum_{j=1}^{d} (\boldsymbol{e}_j^\top \boldsymbol{e}_i) \boldsymbol{V}_j = \operatorname{Var}(\boldsymbol{Y}_1 \,|\, \boldsymbol{Y}_0 = \boldsymbol{e}_i)$. Clearly, $\overline{\boldsymbol{C}}$ and $\widetilde{\boldsymbol{C}}$ depend only on the branching mechanism. □

3 Main results

Let $(\boldsymbol{X}_t)_{t\in\mathbb{R}_+}$ be a 2-type CBI process with parameters $(2, \boldsymbol{c}, \boldsymbol{\beta}, \boldsymbol{B}, \nu, \boldsymbol{\mu})$ such that the moment conditions (2.3) hold with $q = 1$. We call the process $(\boldsymbol{X}_t)_{t\in\mathbb{R}_+}$ doubly symmetric if $\widetilde{b}_{1,1} = \widetilde{b}_{2,2} =: \gamma \in \mathbb{R}$ and $\widetilde{b}_{1,2} = \widetilde{b}_{2,1} =: \kappa \in \mathbb{R}_+$, where $\widetilde{\boldsymbol{B}} = (\widetilde{b}_{i,j})_{i,j\in\{1,2\}}$ is defined in (2.5), that is, if $\widetilde{\boldsymbol{B}}$ takes the form

$$\widetilde{\boldsymbol{B}} = \begin{bmatrix} \gamma & \kappa \\ \kappa & \gamma \end{bmatrix} \tag{3.1}$$

with some $\gamma \in \mathbb{R}$ and $\kappa \in \mathbb{R}_+$. For the sake of simplicity, we suppose $\boldsymbol{X}_0 = \boldsymbol{0}$. In the sequel we also assume that $\boldsymbol{\beta} \neq \boldsymbol{0}$ or $\nu \neq 0$ (i.e., the immigration mechanism is non-zero), equivalently, $\widetilde{\boldsymbol{\beta}} \neq \boldsymbol{0}$ (where $\widetilde{\boldsymbol{\beta}}$ is defined in (2.6)), otherwise $\boldsymbol{X}_t = \boldsymbol{0}$ for all $t \in \mathbb{R}_+$, following from (2.4). Clearly $\widetilde{\boldsymbol{B}}$ is irreducible if and only if $\kappa \in \mathbb{R}_{++}$, since $\boldsymbol{P}^\top \widetilde{\boldsymbol{B}} \boldsymbol{P} = \widetilde{\boldsymbol{B}}$ for both permutation matrices $\boldsymbol{P} \in \mathbb{R}^{2\times 2}$. Hence $(\boldsymbol{X}_t)_{t\in\mathbb{R}_+}$ is irreducible if and only if $\kappa \in \mathbb{R}_{++}$, see Definition 2.7. The eigenvalues of $\widetilde{\boldsymbol{B}}$ are $\gamma - \kappa$ and $\gamma + \kappa$, thus $s := s(\widetilde{\boldsymbol{B}}) = \gamma + \kappa$, which is called *criticality parameter*, and $(\boldsymbol{X}_t)_{t\in\mathbb{R}_+}$ is critical if and only if $s = 0$, see Definition 2.8.

For $k \in \mathbb{Z}_+$, let $\mathcal{F}_k := \sigma(\boldsymbol{X}_0, \boldsymbol{X}_1, \ldots, \boldsymbol{X}_k)$. Since $(\boldsymbol{X}_k)_{k\in\mathbb{Z}_+}$ is a time-homogeneous Markov process, by (2.4),

$$\mathbb{E}(\boldsymbol{X}_k \mid \mathcal{F}_{k-1}) = \mathbb{E}(\boldsymbol{X}_k \mid \boldsymbol{X}_{k-1}) = \mathrm{e}^{\widetilde{\boldsymbol{B}}} \boldsymbol{X}_{k-1} + \overline{\boldsymbol{\beta}}, \qquad k \in \mathbb{N}, \tag{3.2}$$

where

$$\overline{\boldsymbol{\beta}} := \int_0^1 \mathrm{e}^{s\widetilde{\boldsymbol{B}}} \widetilde{\boldsymbol{\beta}} \, \mathrm{d}s \in \mathbb{R}_+^d. \tag{3.3}$$

Note that $\overline{\boldsymbol{\beta}} = \mathbb{E}(\boldsymbol{X}_1 \mid \boldsymbol{X}_0 = \boldsymbol{0})$, see (2.4). Note also that $\overline{\boldsymbol{\beta}}$ depends both on the branching and immigration mechanisms, although $\widetilde{\boldsymbol{\beta}}$ depends only on the immigration mechanism. Let us introduce the sequence

$$\boldsymbol{M}_k := \boldsymbol{X}_k - \mathbb{E}(\boldsymbol{X}_k \mid \mathcal{F}_{k-1}) = \boldsymbol{X}_k - \mathrm{e}^{\widetilde{\boldsymbol{B}}} \boldsymbol{X}_{k-1} - \overline{\boldsymbol{\beta}}, \qquad k \in \mathbb{N}, \tag{3.4}$$

of martingale differences with respect to the filtration $(\mathcal{F}_k)_{k\in\mathbb{Z}_+}$. By (3.4), the process $(\boldsymbol{X}_k)_{k\in\mathbb{Z}_+}$ satisfies the recursion

$$\boldsymbol{X}_k = \mathrm{e}^{\widetilde{\boldsymbol{B}}} \boldsymbol{X}_{k-1} + \overline{\boldsymbol{\beta}} + \boldsymbol{M}_k, \qquad k \in \mathbb{N}. \tag{3.5}$$

By the so-called Putzer's spectral formula, see, e.g., Putzer [23], we have

$$\mathrm{e}^{t\widetilde{\boldsymbol{B}}} = \frac{\mathrm{e}^{(\gamma+\kappa)t}}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{\mathrm{e}^{(\gamma-\kappa)t}}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \mathrm{e}^{\gamma t} \begin{bmatrix} \cosh(\kappa t) & \sinh(\kappa t) \\ \sinh(\kappa t) & \cosh(\kappa t) \end{bmatrix}, \qquad t \in \mathbb{R}_+.$$

Consequently,

$$\mathrm{e}^{\widetilde{\boldsymbol{B}}} = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix} \qquad \text{with} \qquad \alpha := \mathrm{e}^{\gamma} \cosh(\kappa), \qquad \beta := \mathrm{e}^{\gamma} \sinh(\kappa).$$

Considering the eigenvalues $\varrho := \alpha + \beta$ and $\delta := \alpha - \beta$ of $\mathrm{e}^{\widetilde{\boldsymbol{B}}}$, we have $\alpha = (\varrho + \delta)/2$ and $\beta = (\varrho - \delta)/2$, thus the recursion (3.5) can be written in the form

$$\boldsymbol{X}_k = \frac{1}{2} \begin{bmatrix} \varrho + \delta & \varrho - \delta \\ \varrho - \delta & \varrho + \delta \end{bmatrix} \boldsymbol{X}_{k-1} + \boldsymbol{M}_k + \overline{\boldsymbol{\beta}}, \qquad k \in \mathbb{N}.$$

For each $n \in \mathbb{N}$, a CLS estimator $(\widehat{\varrho}_n, \widehat{\delta}_n)$ of (ϱ, δ) based on a sample $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ can be obtained by minimizing the sum of squares

$$\sum_{k=1}^{n} \left\| \boldsymbol{X}_k - \frac{1}{2} \begin{bmatrix} \varrho + \delta & \varrho - \delta \\ \varrho - \delta & \varrho + \delta \end{bmatrix} \boldsymbol{X}_{k-1} - \overline{\boldsymbol{\beta}} \right\|^2$$

with respect to (ϱ, δ) over $\mathbb{R}^2$, and it has the form

$$(3.6) \qquad \widehat{\varrho}_n := \frac{\sum_{k=1}^{n} \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{X}_k - \overline{\boldsymbol{\beta}} \rangle \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{X}_{k-1} \rangle}{\sum_{k=1}^{n} \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{X}_{k-1} \rangle^2}, \qquad \widehat{\delta}_n := \frac{\sum_{k=1}^{n} \langle \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{X}_k - \overline{\boldsymbol{\beta}} \rangle \langle \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{X}_{k-1} \rangle}{\sum_{k=1}^{n} \langle \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{X}_{k-1} \rangle^2}$$

on the set $H_n \cap \widetilde{H}_n$, where

$$\boldsymbol{u}_{\mathrm{left}} := \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in \mathbb{R}^2_{++}, \qquad \boldsymbol{v}_{\mathrm{left}} := \begin{bmatrix} 1 \\ -1 \end{bmatrix} \in \mathbb{R}^2,$$

$$H_n := \left\{ \omega \in \Omega : \sum_{k=1}^{n} \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{X}_{k-1}(\omega) \rangle^2 > 0 \right\}, \qquad \widetilde{H}_n := \left\{ \omega \in \Omega : \sum_{k=1}^{n} \langle \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{X}_{k-1}(\omega) \rangle^2 > 0 \right\},$$

see Ispány et al. [13, Lemma A.1]. Here $\boldsymbol{u}_{\mathrm{left}}$ and $\boldsymbol{v}_{\mathrm{left}}$ are left eigenvectors of $\widetilde{\boldsymbol{B}}$ belonging to the eigenvalues $\gamma + \kappa$ and $\gamma - \kappa$, respectively, hence they are left eigenvectors of $\mathrm{e}^{\widetilde{\boldsymbol{B}}}$ belonging to the eigenvalues $\varrho = \mathrm{e}^{\gamma+\kappa}$ and $\delta = \mathrm{e}^{\gamma-\kappa}$, respectively. In a natural way, one can extend the CLS estimators $\widehat{\varrho}_n$ and $\widehat{\delta}_n$ to the set H_n and $\widetilde{H}_n$, respectively. By Lemma C.3, $\mathbb{P}(H_n) \to 1$ and $\mathbb{P}(\widetilde{H}_n) \to 1$ as $n \to \infty$ under appropriate assumptions.

Let us introduce the function $h : \mathbb{R}^2 \to \mathbb{R}^2_{++}$ by

$$h(\gamma, \kappa) := (\mathrm{e}^{\gamma+\kappa}, \mathrm{e}^{\gamma-\kappa}) = (\varrho, \delta), \qquad (\gamma, \kappa) \in \mathbb{R}^2.$$

Note that h is bijective having inverse

$$h^{-1}(\varrho, \delta) = \left(\frac{1}{2} \log(\varrho\delta), \frac{1}{2} \log\left(\frac{\varrho}{\delta}\right) \right) = (\gamma, \kappa), \qquad (\varrho, \delta) \in \mathbb{R}^2_{++}.$$

Theorem 3.5 will imply that the CLSE $(\widehat{\varrho}_n, \widehat{\delta}_n)$ of (ϱ, δ) is weakly consistent (in the critical case), hence $(\widehat{\varrho}_n, \widehat{\delta}_n)$ falls into the set $\mathbb{R}^2_{++}$ for sufficiently large $n \in \mathbb{N}$ with probability converging to one. Hence one can introduce a natural estimator of (γ, κ) by applying the inverse of h to the CLSE of (ϱ, δ), that is,

$$(\widehat{\gamma}_n, \widehat{\kappa}_n) := \left(\frac{1}{2} \log(\widehat{\varrho}_n \widehat{\delta}_n), \frac{1}{2} \log\left(\frac{\widehat{\varrho}_n}{\widehat{\delta}_n}\right) \right), \qquad n \in \mathbb{N},$$

on the set $\{\omega \in \Omega : (\widehat{\varrho}_n(\omega), \widehat{\delta}_n(\omega)) \in \mathbb{R}^2_{++}\}$. We also obtain

$$(3.7) \qquad \big(\widehat{\gamma}_n, \widehat{\kappa}_n\big) = \arg\min_{(\gamma,\kappa)\in\mathbb{R}^2} \sum_{k=1}^n \left\| \boldsymbol{X}_k - \mathrm{e}^\gamma \begin{bmatrix} \cosh(\kappa) & \sinh(\kappa) \\ \sinh(\kappa) & \cosh(\kappa) \end{bmatrix} \boldsymbol{X}_{k-1} - \overline{\boldsymbol{\beta}} \right\|^2$$

for sufficiently large $n \in \mathbb{N}$ with probability converging to one, hence $\big(\widehat{\gamma}_n, \widehat{\kappa}_n\big)$ is the CLSE of (γ, κ) for sufficiently large $n \in \mathbb{N}$ with probability converging to one. In a similar way,

$$\widehat{s}_n := \log \widehat{\varrho}_n, \qquad n \in \mathbb{N},$$

is the CLSE of the criticality parameter $s = \gamma + \kappa$ on the set $\{\omega \in \Omega : \widehat{\varrho}_n(\omega) \in \mathbb{R}_{++}\}$ with probability converging to one. We would like to stress the point that the estimators $\big(\widehat{\gamma}_n, \widehat{\kappa}_n\big)$ and $\widehat{s}_n$ exist only for sufficiently large $n \in \mathbb{N}$ with probability converging to 1. However, as all our results are asymptotic, this will not cause a problem.

3.1 Theorem. *Let $(\boldsymbol{X}_t)_{t\in\mathbb{R}_+}$ be a 2-type CBI process with parameters $(2, \boldsymbol{c}, \boldsymbol{\beta}, \boldsymbol{B}, \nu, \boldsymbol{\mu})$ such that $\boldsymbol{X}_0 = \boldsymbol{0}$, the moment conditions (2.3) hold with $q = 8$, $\boldsymbol{\beta} \neq \boldsymbol{0}$ or $\nu \neq 0$, and (3.1) holds with some $\gamma \in \mathbb{R}$ and $\kappa \in \mathbb{R}_{++}$ such that $s = \gamma + \kappa = 0$ (hence it is irreducible and critical). Then the probability of the existence of the estimator $\widehat{s}_n$ converges to 1 as $n \to \infty$ and*

$$(3.8) \qquad n\widehat{s}_n \xrightarrow{\mathcal{D}} \frac{\int_0^1 \mathcal{Y}_t \, \mathrm{d}(\mathcal{Y}_t - (\widetilde{\beta}_1 + \widetilde{\beta}_2)t)}{\int_0^1 \mathcal{Y}_t^2 \, \mathrm{d}t} \qquad \text{as } n \to \infty,$$

where $(\mathcal{Y}_t)_{t\in\mathbb{R}_+}$ is the pathwise unique strong solution of the SDE (2.9).

If $\boldsymbol{c} = \boldsymbol{0}$ and $\boldsymbol{\mu} = \boldsymbol{0}$, then

$$(3.9) \qquad n^{3/2}\widehat{s}_n \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \frac{3}{(\widetilde{\beta}_1 + \widetilde{\beta}_2)^2} \int_{\mathcal{U}_2} (z_1 + z_2)^2 \, \nu(\mathrm{d}\boldsymbol{z})\right) \qquad \text{as } n \to \infty.$$

If $\|\boldsymbol{c}\|^2 + \sum_{i=1}^2 \int_{\mathcal{U}_2} (z_1 - z_2)^2 \, \mu_i(\mathrm{d}\boldsymbol{z}) > 0$, then the probability of the existence of the estimators $\widehat{\gamma}_n$ and $\widehat{\kappa}_n$ converges to 1 as $n \to \infty$ and

$$(3.10) \qquad \begin{bmatrix} n^{1/2}(\widehat{\gamma}_n - \gamma) \\ n^{1/2}(\widehat{\kappa}_n - \kappa) \end{bmatrix} \xrightarrow{\mathcal{D}} \frac{1}{2} \sqrt{\mathrm{e}^{2(\kappa-\gamma)} - 1} \, \frac{\int_0^1 \mathcal{Y}_t \, \mathrm{d}\widetilde{\mathcal{W}}_t}{\int_0^1 \mathcal{Y}_t \, \mathrm{d}t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \text{as } n \to \infty,$$

where $(\widetilde{\mathcal{W}}_t)_{t\in\mathbb{R}_+}$ is a standard Wiener process, independent from $(\mathcal{W}_t)_{t\in\mathbb{R}_+}$.

If $\|\boldsymbol{c}\|^2 + \sum_{i=1}^2 \int_{\mathcal{U}_2} (z_1 - z_2)^2 \, \mu_i(\mathrm{d}\boldsymbol{z}) = 0$ and $(\widetilde{\beta}_1 - \widetilde{\beta}_2)^2 + \int_{\mathcal{U}_2} (z_1 - z_2)^2 \, \nu(\mathrm{d}\boldsymbol{z}) > 0$, then the probability of the existence of the estimators $\widehat{\gamma}_n$ and $\widehat{\kappa}_n$ converges to 1 as $n \to \infty$, and

$$(3.11) \qquad \begin{bmatrix} n^{1/2}(\widehat{\gamma}_n - \gamma) \\ n^{1/2}(\widehat{\kappa}_n - \kappa) \end{bmatrix} \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \frac{\mathrm{e}^{2(\kappa-\gamma)} - 1}{8(\kappa - \gamma)M} \int_{\mathcal{U}_2} (z_1 - z_2)^2 \, \nu(\mathrm{d}\boldsymbol{z})\right) \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \text{as } n \to \infty,$$

where

$$(3.12) \qquad M := \frac{1}{2(\kappa - \gamma)} \int_{\mathcal{U}_2} (z_1 - z_2)^2 \, \nu(\mathrm{d}\boldsymbol{z}) + \left(\frac{\widetilde{\beta}_1 - \widetilde{\beta}_2}{\kappa - \gamma}\right)^2.$$

Under the assumptions of Theorem 3.1, we have the following remarks.

3.2 Remark. If $(\widetilde{\beta}_1 - \widetilde{\beta}_2)^2 + \int_{\mathcal{U}_2} (z_1 - z_2)^2 \, \nu(\mathrm{d}\boldsymbol{z}) > 0$, then $M > 0$. $\square$

3.3 Remark. If $\|\boldsymbol{c}\|^2 + \sum_{i=1}^2 \int_{\mathcal{U}_2} (z_1 - z_2)^2 \, \mu_i(\mathrm{d}\boldsymbol{z}) = 0$ and $(\widetilde{\beta}_1 - \widetilde{\beta}_2)^2 + \int_{\mathcal{U}_2} (z_1 - z_2)^2 \, \nu(\mathrm{d}\boldsymbol{z}) = 0$, then, by Lemma C.2, $X_{k,1} \stackrel{\text{a.s.}}{=} X_{k,2}$ for all $k \in \mathbb{N}$, hence there is no unique CLS estimator for δ, thus $(\widehat{\gamma}_n, \widehat{\kappa}_n)$, $n \in \mathbb{N}$, are not defined. $\square$

3.4 Remark. For each $n \in \mathbb{N}$, consider the random step process

$$\boldsymbol{\mathcal{X}}_t^{(n)} := n^{-1} \boldsymbol{X}_{\lfloor nt \rfloor}, \qquad t \in \mathbb{R}_+.$$

Theorem 2.9 implies convergence

$$\boldsymbol{\mathcal{X}}^{(n)} \stackrel{\mathcal{D}}{\longrightarrow} \boldsymbol{\mathcal{X}} := \mathcal{Y} \boldsymbol{u}_{\mathrm{right}} \qquad \text{as} \quad n \to \infty, \tag{3.13}$$

where the process $(\mathcal{Y}_t)_{t \in \mathbb{R}_+}$ is the pathwise unique strong solution of the SDE (2.9) with initial value $\mathcal{Y}_0 = 0$, and

$$\boldsymbol{u}_{\mathrm{right}} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Note that convergence (3.13) holds even if $\langle \overline{\boldsymbol{C}} \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{u}_{\mathrm{left}} \rangle = 0$, which is equivalent to $\boldsymbol{c} = \boldsymbol{0}$ and $\boldsymbol{\mu} = \boldsymbol{0}$ (see Remark 2.11), when the pathwise unique strong solution of (2.9) is the deterministic function $\mathcal{Y}_t = \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle \, t$, $t \in \mathbb{R}_+$, further, by (3.8), $n \widehat{s}_n \stackrel{\mathcal{D}}{\longrightarrow} 0$, and hence $n \widehat{s}_n \stackrel{\mathbb{P}}{\longrightarrow} 0$ as $n \to \infty$. $\square$

Theorem 3.1 will follow from the following statement.

3.5 Theorem. *Under the assumptions of Theorem 3.1, the probability of the existence of a unique CLS estimator $\widehat{\varrho}_n$ converges to 1 as $n \to \infty$ and*

$$n(\widehat{\varrho}_n - 1) \stackrel{\mathcal{D}}{\longrightarrow} \frac{\int_0^1 \mathcal{Y}_t \, \mathrm{d}(\mathcal{Y}_t - (\widetilde{\beta}_1 + \widetilde{\beta}_2) t)}{\int_0^1 \mathcal{Y}_t^2 \, \mathrm{d}t} \qquad \text{as} \quad n \to \infty. \tag{3.14}$$

If $\boldsymbol{c} = \boldsymbol{0}$ and $\boldsymbol{\mu} = \boldsymbol{0}$, then

$$n^{3/2}(\widehat{\varrho}_n - 1) \stackrel{\mathcal{D}}{\longrightarrow} \mathcal{N}\left(0, \frac{3}{(\widetilde{\beta}_1 + \widetilde{\beta}_2)^2} \int_{\mathcal{U}_2} (z_1 + z_2)^2 \, \nu(\mathrm{d}\boldsymbol{z})\right) \qquad \text{as} \quad n \to \infty. \tag{3.15}$$

If $\|\boldsymbol{c}\|^2 + \sum_{i=1}^2 \int_{\mathcal{U}_2} (z_1 - z_2)^2 \, \mu_i(\mathrm{d}\boldsymbol{z}) > 0$, then the probability of the existence of a unique CLS estimator $\widehat{\delta}_n$ converges to 1 as $n \to \infty$ and

$$n^{1/2}(\widehat{\delta}_n - \delta) \stackrel{\mathcal{D}}{\longrightarrow} \sqrt{1 - \delta^2} \, \frac{\int_0^1 \mathcal{Y}_t \, \mathrm{d}\widetilde{\mathcal{W}}_t}{\int_0^1 \mathcal{Y}_t \, \mathrm{d}t} \qquad \text{as} \quad n \to \infty, \tag{3.16}$$

where $(\widetilde{\mathcal{W}}_t)_{t\in\mathbb{R}_+}$ *is a standard Wiener process, independent from* $(\mathcal{W}_t)_{t\in\mathbb{R}_+}$*.*

If $\|\boldsymbol{c}\|^2 + \sum_{i=1}^2 \int_{\mathcal{U}_2}(z_1 - z_2)^2\,\mu_i(\mathrm{d}\boldsymbol{z}) = 0$ *and* $(\widetilde{\beta}_1 - \widetilde{\beta}_2)^2 + \int_{\mathcal{U}_2}(z_1 - z_2)^2\,\nu(\mathrm{d}\boldsymbol{z}) > 0$, *then the probability of the existence of a unique CLS estimator* $\widehat{\delta}_n$ *converges to 1 as* $n \to \infty$, *and*

$$n^{1/2}(\widehat{\delta}_n - \delta) \stackrel{\mathcal{D}}{\longrightarrow} \mathcal{N}\left(0, \frac{1-\delta^2}{2M\log(\delta^{-1})}\int_{\mathcal{U}_2}(z_1 - z_2)^2\,\nu(\mathrm{d}\boldsymbol{z})\right) \qquad \text{as } n \to \infty \tag{3.17}$$

with

$$M = \frac{1}{2\log(\delta^{-1})}\int_{\mathcal{U}_2}(z_1 - z_2)^2\,\nu(\mathrm{d}\boldsymbol{z}) + \left(\frac{\widetilde{\beta}_1 - \widetilde{\beta}_2}{\log(\delta^{-1})}\right)^2.$$

Proof of Theorem 3.1. We can use the so-called delta method (see, e.g., Theorem 11.2.14 in Lehmann and Romano [19]). Indeed, $\widehat{s}_n = g(\widehat{\varrho}_n) - g(1)$ on the set $\{\omega \in \Omega : \widehat{\varrho}_n(\omega) \in \mathbb{R}_{++}\}$ with the function $g(x) := \log(x)$, $x \in \mathbb{R}_{++}$, where $g'(1) = 1$, hence (3.14) and (3.15) imply (3.8) and (3.9), respectively.

In a similar way, (3.16) and (3.17) imply

$$n^{1/2}(\widehat{\Delta}_n - \Delta) \stackrel{\mathcal{D}}{\longrightarrow} \sqrt{\mathrm{e}^{-2\Delta} - 1}\,\frac{\int_0^1 \mathcal{Y}_t\,\mathrm{d}\widetilde{\mathcal{W}}_t}{\int_0^1 \mathcal{Y}_t\,\mathrm{d}t} \qquad \text{as } n \to \infty \tag{3.18}$$

and

$$n^{1/2}(\widehat{\Delta}_n - \Delta) \stackrel{\mathcal{D}}{\longrightarrow} \mathcal{N}\left(0, \frac{\mathrm{e}^{-2\Delta} - 1}{2(-\Delta)M}\int_{\mathcal{U}_2}(z_1 - z_2)^2\,\nu(\mathrm{d}\boldsymbol{z})\right) \qquad \text{as } n \to \infty, \tag{3.19}$$

respectively, for $\widehat{\Delta}_n := \log(\widehat{\delta}_n)$ on the set $\{\omega \in \Omega : \widehat{\delta}_n(\omega) \in \mathbb{R}_{++}\}$ for $n \in \mathbb{N}$, and $\Delta := \log(\delta) = \gamma - \kappa$, since $g(\delta) = \log(\delta) = \Delta$ and $g'(\delta) = 1/\delta = \mathrm{e}^{-\Delta}$. We have $\widehat{\gamma}_n = (\widehat{s}_n + \widehat{\Delta}_n)/2$, $\widehat{\kappa}_n = (\widehat{s}_n - \widehat{\Delta}_n)/2$, $\gamma = (s + \Delta)/2$ and $\kappa = (s - \Delta)/2$, thus

$$\begin{bmatrix}\widehat{\gamma}_n - \gamma\\ \widehat{\kappa}_n - \kappa\end{bmatrix} = \frac{\widehat{s}_n - s}{2}\begin{bmatrix}1\\1\end{bmatrix} + \frac{\widehat{\Delta}_n - \Delta}{2}\begin{bmatrix}1\\-1\end{bmatrix}$$

on the set $\{\omega \in \Omega : (\widehat{\varrho}_n(\omega), \widehat{\delta}_n(\omega)) \in \mathbb{R}^2_{++}\}$. We have $s = 0$ by criticality, hence (3.8) or (3.9) yields $n^{1/2}(\widehat{s}_n - s) = n^{1/2}\widehat{s}_n \stackrel{\mathbb{P}}{\longrightarrow} 0$ as $n \to \infty$, and hence, by Slutsky's lemma, (3.18) and (3.19) imply (3.10) and (3.11), respectively. □

4 Decomposition of the process

Let us introduce the sequence

$$U_k := \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{X}_k\rangle = X_{k,1} + X_{k,2}, \qquad k \in \mathbb{Z}_+,$$

where $\boldsymbol{X}_k =: (X_{k,1}, X_{k,2})^\top$. One can observe that $U_k \geqslant 0$ for all $k \in \mathbb{Z}_+$, and, by (3.5),

$$U_k = U_{k-1} + \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}}\rangle + \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{M}_k\rangle, \qquad k \in \mathbb{N}, \tag{4.1}$$

since $\langle \boldsymbol{u}_{\mathrm{left}}, \mathrm{e}^{\widetilde{\boldsymbol{B}}} \boldsymbol{X}_{k-1}\rangle = \boldsymbol{u}_{\mathrm{left}}^\top \mathrm{e}^{\widetilde{\boldsymbol{B}}} \boldsymbol{X}_{k-1} = \boldsymbol{u}_{\mathrm{left}}^\top \boldsymbol{X}_{k-1} = U_{k-1}$ and $\langle \boldsymbol{u}_{\mathrm{left}}, \overline{\boldsymbol{\beta}}\rangle = \int_0^1 \langle \boldsymbol{u}_{\mathrm{left}}, \mathrm{e}^{s\widetilde{\boldsymbol{B}}} \widetilde{\boldsymbol{\beta}}\rangle \,\mathrm{d}s = \int_0^1 \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}}\rangle \,\mathrm{d}s = \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}}\rangle$, because $\boldsymbol{u}_{\mathrm{left}}$ is a left eigenvector of $\mathrm{e}^{s\widetilde{\boldsymbol{B}}}$, $s \in \mathbb{R}_+$, belonging to the eigenvalue $\varrho = 1$. Hence $(U_k)_{k\in\mathbb{Z}_+}$ is a nonnegative unstable AR(1) process with positive drift $\langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}}\rangle$ and with heteroscedastic innovation $(\langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{M}_k\rangle)_{k\in\mathbb{N}}$. Note that the solution of the recursion (4.1) is

$$U_k = \sum_{j=1}^{k} \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{M}_j + \widetilde{\boldsymbol{\beta}}\rangle, \qquad k \in \mathbb{N}. \tag{4.2}$$

Moreover, let

$$V_k := \langle \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{X}_k\rangle = X_{k,1} - X_{k,2}, \qquad k \in \mathbb{Z}_+.$$

By (3.5), we have

$$V_k = \delta V_{k-1} + \widetilde{\delta}\langle \boldsymbol{v}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}}\rangle + \langle \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{M}_k\rangle, \qquad k \in \mathbb{N}, \tag{4.3}$$

where

$$\widetilde{\delta} := \frac{1-\delta}{\log(\delta^{-1})},$$

since $\langle \boldsymbol{v}_{\mathrm{left}}, \mathrm{e}^{\widetilde{\boldsymbol{B}}} \boldsymbol{X}_{k-1}\rangle = \boldsymbol{v}_{\mathrm{left}}^\top \mathrm{e}^{\widetilde{\boldsymbol{B}}} \boldsymbol{X}_{k-1} = \delta \boldsymbol{v}_{\mathrm{left}}^\top \boldsymbol{X}_{k-1} = \delta V_{k-1}$ and $\langle \boldsymbol{v}_{\mathrm{left}}, \overline{\boldsymbol{\beta}}\rangle = \int_0^1 \langle \boldsymbol{v}_{\mathrm{left}}, \mathrm{e}^{s\widetilde{\boldsymbol{B}}} \widetilde{\boldsymbol{\beta}}\rangle \,\mathrm{d}s = \int_0^1 \delta^s \langle \boldsymbol{v}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}}\rangle \,\mathrm{d}s = \frac{1-\delta}{\log(\delta^{-1})} \langle \boldsymbol{v}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}}\rangle$, because $\boldsymbol{v}_{\mathrm{left}}$ is a left eigenvector of $\mathrm{e}^{s\widetilde{\boldsymbol{B}}}$, $s \in \mathbb{R}_+$, belonging to the eigenvalue δ^s. Thus $(V_k)_{k\in\mathbb{Z}_+}$ is a stable AR(1) process with drift $\widetilde{\delta}\langle \boldsymbol{v}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}}\rangle$ and with heteroscedastic innovation $(\langle \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{M}_k\rangle)_{k\in\mathbb{N}}$, since $\gamma + \kappa = 0$, $\gamma \in \mathbb{R}$ and $\kappa \in \mathbb{R}_{++}$ yield $\delta = \mathrm{e}^{\gamma-\kappa} = \mathrm{e}^{-2\kappa} \in (0,1)$. Note that the solution of the recursion (4.3) is

$$V_k = \sum_{j=1}^{k} \delta^{k-j} \big\langle \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{M}_j + \widetilde{\delta}\,\widetilde{\boldsymbol{\beta}}\big\rangle, \qquad k \in \mathbb{N}. \tag{4.4}$$

Observe that

$$X_{k,1} = (U_k + V_k)/2, \qquad X_{k,2} = (U_k - V_k)/2, \qquad k \in \mathbb{Z}_+. \tag{4.5}$$

By (3.6), for each $n \in \mathbb{N}$, we have

$$\widehat{\varrho}_n - 1 = \frac{\sum_{k=1}^n \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{M}_k\rangle U_{k-1}}{\sum_{k=1}^n U_{k-1}^2}, \qquad \widehat{\delta}_n - \delta = \frac{\sum_{k=1}^n \langle \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{M}_k\rangle V_{k-1}}{\sum_{k=1}^n V_{k-1}^2}, \tag{4.6}$$

on the sets H_n and $\widetilde{H}_n$, respectively.

Theorem 3.5 will follow from the following statements by the continuous mapping theorem and by Slutsky's lemma.

4.1 Theorem. *Under the assumptions of Theorem 3.1, we have*

$$\sum_{k=1}^{n} \begin{bmatrix} n^{-3} U_{k-1}^2 \\ n^{-2} V_{k-1}^2 \\ n^{-2} \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{M}_k\rangle U_{k-1} \\ n^{-3/2} \langle \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{M}_k\rangle V_{k-1} \end{bmatrix} \xrightarrow{\mathcal{D}} \begin{bmatrix} \int_0^1 \mathcal{Y}_t^2 \,\mathrm{d}t \\ (1-\delta^2)^{-1} \langle \widetilde{\boldsymbol{C}} \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}}\rangle \int_0^1 \mathcal{Y}_t \,\mathrm{d}t \\ \int_0^1 \mathcal{Y}_t \,\mathrm{d}(\mathcal{Y}_t - \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}}\rangle\, t) \\ (1-\delta^2)^{-1/2} \langle \widetilde{\boldsymbol{C}} \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}}\rangle \int_0^1 \mathcal{Y}_t \,\mathrm{d}\widetilde{\mathcal{W}}_t \end{bmatrix} \qquad \text{as } n \to \infty.$$

In case of $\langle \widetilde{\boldsymbol{C}} \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}} \rangle = 0$ the second and fourth coordinates of the limit vector is 0 in Theorem 4.1, thus other scaling factors should be chosen for these coordinates, described in the following theorem.

4.2 Theorem. *Suppose that the assumptions of Theorem 3.1 hold. If* $\langle \widetilde{\boldsymbol{C}} \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}} \rangle = 0$, *then*

$$n^{-1} \sum_{k=1}^{n} V_{k-1}^2 \xrightarrow{\mathbb{P}} \frac{\langle \boldsymbol{V}_0 \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}} \rangle}{1-\delta^2} + \left(\frac{\widetilde{\delta} \langle \boldsymbol{v}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle}{1-\delta} \right)^2 = M \qquad \text{as } n \to \infty,$$

$$\sum_{k=1}^{n} \begin{bmatrix} n^{-3} U_{k-1}^2 \\ n^{-2} \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{M}_k \rangle U_{k-1} \\ n^{-1/2} \langle \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{M}_k \rangle V_{k-1} \end{bmatrix} \xrightarrow{\mathcal{D}} \begin{bmatrix} \int_0^1 \mathcal{Y}_t^2 \, \mathrm{d}t \\ \int_0^1 \mathcal{Y}_t \, \mathrm{d}(\mathcal{Y}_t - \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle \, t) \\ \langle \boldsymbol{V}_0 \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}} \rangle^{1/2} M^{1/2} \, \widetilde{\mathcal{W}}_1 \end{bmatrix} \qquad \text{as } n \to \infty,$$

where $\boldsymbol{V}_0$ *is defined in Proposition B.3.*

In case of $\langle \widetilde{\boldsymbol{C}} \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{u}_{\mathrm{left}} \rangle = 0$ the second coordinate of the limit vector of the second convergence is 0 in Theorem 4.2, since $(\mathcal{Y}_t)_{t \in \mathbb{R}_+}$ is the deterministic function $\mathcal{Y}_t = \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle t$, $t \in \mathbb{R}_+$ (see Remark 2.11), hence another scaling factor should be chosen for this coordinate, as given in the following theorem.

4.3 Theorem. *Suppose that the assumptions of Theorem 3.1 hold. If* $\langle \widetilde{\boldsymbol{C}} \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{u}_{\mathrm{left}} \rangle = 0$, *then*

$$n^{-3} \sum_{k=1}^{n} U_{k-1}^2 \xrightarrow{\mathbb{P}} \frac{\langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle^2}{3} \qquad \text{as } n \to \infty,$$

$$n^{-1} \sum_{k=1}^{n} V_{k-1}^2 \xrightarrow{\mathbb{P}} \frac{\langle \boldsymbol{V}_0 \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}} \rangle}{1-\delta^2} + \left(\frac{\widetilde{\delta} \langle \boldsymbol{v}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle}{1-\delta} \right)^2 = M \qquad \text{as } n \to \infty,$$

$$\sum_{k=1}^{n} \begin{bmatrix} n^{-3/2} \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{M}_k \rangle U_{k-1} \\ n^{-1/2} \langle \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{M}_k \rangle V_{k-1} \end{bmatrix} \xrightarrow{\mathcal{D}} \mathcal{N}_2(\boldsymbol{0}, \boldsymbol{\Sigma}) \qquad \text{as } n \to \infty$$

with

$$\boldsymbol{\Sigma} := \begin{bmatrix} \frac{1}{3} \langle \boldsymbol{V}_0 \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{u}_{\mathrm{left}} \rangle \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle^2 & \frac{\widetilde{\delta}}{2(1-\delta)} \langle \boldsymbol{V}_0 \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{u}_{\mathrm{left}} \rangle \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle \langle \boldsymbol{v}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle \\ \frac{\widetilde{\delta}}{2(1-\delta)} \langle \boldsymbol{V}_0 \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}} \rangle \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle \langle \boldsymbol{v}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle & \langle \boldsymbol{V}_0 \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}} \rangle M \end{bmatrix}.$$

Proof of Theorem 3.5. First note that $\langle \widetilde{\boldsymbol{C}} \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}} \rangle = 0$ if and only if $\|\boldsymbol{c}\|^2 + \sum_{i=1}^{2} \int_{\mathcal{U}_2} (z_1 - z_2)^2 \, \mu_i(\mathrm{d}\boldsymbol{z}) = 0$. Indeed, by the spectral mapping theorem, $\boldsymbol{v}_{\mathrm{left}}$ is a left eigenvector of $\mathrm{e}^{s\widetilde{\boldsymbol{B}}}$, $s \in \mathbb{R}_+$, belonging to the eigenvalue δ^s and $\boldsymbol{u}_{\mathrm{right}}$ is a right eigenvector of $\mathrm{e}^{s\widetilde{\boldsymbol{B}}}$, $s \in \mathbb{R}_+$, belonging to the eigenvalue 1, hence $\langle \widetilde{\boldsymbol{C}} \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}} \rangle = \frac{1-\delta^2}{2\log(\delta^{-1})} \langle \overline{\boldsymbol{C}} \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}} \rangle$. Thus $\langle \widetilde{\boldsymbol{C}} \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}} \rangle = 0$ if and only if $\langle \overline{\boldsymbol{C}} \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}} \rangle = 0$. Recalling

$$\langle \overline{\boldsymbol{C}} \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}} \rangle = \sum_{k=1}^{2} \langle \boldsymbol{e}_k, \boldsymbol{u}_{\mathrm{right}} \rangle \langle \boldsymbol{C}_k \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}} \rangle,$$

one can observe that $\langle \overline{\boldsymbol{C}} \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}} \rangle = 0$ if and only if $\langle \boldsymbol{C}_k \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}} \rangle = 2c_k + \int_{\mathcal{U}_2} \langle \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{z} \rangle^2 \, \mu_k(\mathrm{d}\boldsymbol{z}) = 0$ for each $k \in \{1, 2\}$, which is equivalent to $\boldsymbol{c} = \boldsymbol{0}$ and $\int_{\mathcal{U}_2} (z_1 - z_2)^2 \, \mu_k(\mathrm{d}\boldsymbol{z}) = 0$ for each $k \in \{1, 2\}$.

Further note that $\langle \widetilde{\boldsymbol{C}} \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{u}_{\mathrm{left}} \rangle = 0$ if and only if $\boldsymbol{c} = \boldsymbol{0}$ and $\boldsymbol{\mu} = \boldsymbol{0}$. Indeed, by Remark 2.11, we have $\langle \widetilde{\boldsymbol{C}} \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{u}_{\mathrm{left}} \rangle = \langle \overline{\boldsymbol{C}} \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{u}_{\mathrm{left}} \rangle$, and $\langle \overline{\boldsymbol{C}} \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{u}_{\mathrm{left}} \rangle = 0$ if and only if $\boldsymbol{c} = \boldsymbol{0}$ and $\boldsymbol{\mu} = \boldsymbol{0}$. Hence, $\langle \widetilde{\boldsymbol{C}} \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{u}_{\mathrm{left}} \rangle = 0$ or $\langle \overline{\boldsymbol{C}} \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{u}_{\mathrm{left}} \rangle = 0$ implies $\langle \widetilde{\boldsymbol{C}} \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}} \rangle = 0$ and $\langle \overline{\boldsymbol{C}} \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}} \rangle = 0$ as well.

The statements about the existence of unique CLS estimators $\widehat{\varrho}_n$ and $\widehat{\delta}_n$ under the given conditions follow from Lemma C.3.

In order to derive the statements, we can use the continuous mapping theorem and Slutsky's lemma. Theorem 4.1 and (4.6) imply (3.14) and (3.16). Indeed, since $\widetilde{\boldsymbol{\beta}} \neq \boldsymbol{0}$, by the SDE (2.9), we have $\mathbb{P}\big(\mathcal{Y}_t = 0, \, t \in [0, 1]\big) = 0$, which implies $\mathbb{P}\big(\int_0^1 \mathcal{Y}_t^2 \, \mathrm{d}t > 0\big) = 1$. By Remark 2.10, $\mathbb{P}(\mathcal{Y}_t \geqslant 0, \, t \in \mathbb{R}_+) = 1$, and hence $\mathbb{P}(\int_0^1 \mathcal{Y}_t \, \mathrm{d}t > 0) = 1$. Moreover, as we have already proved, the assumption $\|\boldsymbol{c}\|^2 + \sum_{i=1}^2 \int_{\mathcal{U}_2} (z_1 - z_2)^2 \, \mu_i(\mathrm{d}\boldsymbol{z}) > 0$ implies $\langle \widetilde{\boldsymbol{C}} \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}} \rangle > 0$. Theorem 4.3 and (4.6) imply (3.15), since $\widetilde{\beta}_1 + \widetilde{\beta}_2 = \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle \neq 0$, and the assumption $\langle \widetilde{\boldsymbol{C}} \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{u}_{\mathrm{left}} \rangle = 0$ yields $\boldsymbol{c} = \boldsymbol{0}$ and $\boldsymbol{\mu} = \boldsymbol{0}$, consequently $\boldsymbol{C}_\ell = \boldsymbol{0}$, $\ell \in \{1, 2\}$, and hence $\langle \boldsymbol{V}_0 \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{u}_{\mathrm{left}} \rangle = \int_{\mathcal{U}_2} (z_1 + z_2)^2 \, \nu(\mathrm{d}\boldsymbol{z})$. Theorem 4.2 and (4.6) imply (3.17), since $\delta \in (0, 1)$, $M \neq 0$ and $\langle \boldsymbol{V}_0 \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}} \rangle = \frac{1-\delta^2}{2 \log(\delta^{-1})} \int_{\mathcal{U}_2} (z_1 - z_2)^2 \, \nu(\mathrm{d}\boldsymbol{z})$. □

5 Proof of Theorem 4.1

Consider the sequence of stochastic processes

$$\boldsymbol{\mathcal{Z}}_t^{(n)} := \begin{bmatrix} \boldsymbol{\mathcal{M}}_t^{(n)} \\ \boldsymbol{\mathcal{N}}_t^{(n)} \\ \boldsymbol{\mathcal{P}}_t^{(n)} \end{bmatrix} := \sum_{k=1}^{\lfloor nt \rfloor} \boldsymbol{Z}_k^{(n)},$$

with

$$\boldsymbol{Z}_k^{(n)} := \begin{bmatrix} n^{-1} \boldsymbol{M}_k \\ n^{-2} \boldsymbol{M}_k U_{k-1} \\ n^{-3/2} \boldsymbol{M}_k V_{k-1} \end{bmatrix} = \begin{bmatrix} n^{-1} \\ n^{-2} U_{k-1} \\ n^{-3/2} V_{k-1} \end{bmatrix} \otimes \boldsymbol{M}_k$$

for $t \in \mathbb{R}_+$ and $k, n \in \mathbb{N}$, where $\otimes$ denotes Kronecker product of matrices. Theorem 4.1 follows from Lemma C.1 and the following theorem (this will be explained after Theorem 5.1).

5.1 Theorem. *Under the assumptions of Theorem 3.1, we have*

$$\boldsymbol{\mathcal{Z}}^{(n)} \stackrel{\mathcal{D}}{\longrightarrow} \boldsymbol{\mathcal{Z}}, \qquad \text{as } n \to \infty, \tag{5.1}$$

where the process $(\boldsymbol{\mathcal{Z}}_t)_{t \in \mathbb{R}_+}$ with values in $(\mathbb{R}^2)^3$ is the pathwise unique strong solution of the

SDE

$$\mathrm{d}\boldsymbol{\mathcal{Z}}_t = \gamma(t, \boldsymbol{\mathcal{Z}}_t) \begin{bmatrix} \mathrm{d}\boldsymbol{\mathcal{W}}_t \\ \mathrm{d}\widetilde{\boldsymbol{\mathcal{W}}}_t \end{bmatrix}, \qquad t \in \mathbb{R}_+, \tag{5.2}$$

with initial value $\boldsymbol{\mathcal{Z}}_0 = \mathbf{0}$, *where* $(\boldsymbol{\mathcal{W}}_t)_{t\in\mathbb{R}_+}$ *and* $(\widetilde{\boldsymbol{\mathcal{W}}}_t)_{t\in\mathbb{R}_+}$ *are independent 2-dimensional standard Wiener processes, and* $\gamma : \mathbb{R}_+ \times (\mathbb{R}^2)^3 \to (\mathbb{R}^{2\times 2})^{3\times 2}$ *is defined by*

$$\gamma(t, \boldsymbol{x}) := \begin{bmatrix} (\langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{x}_1 + t\widetilde{\boldsymbol{\beta}}\rangle^+)^{1/2}\, \widetilde{\boldsymbol{C}}^{1/2} & \mathbf{0} \\ (\langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{x}_1 + t\widetilde{\boldsymbol{\beta}}\rangle^+)^{3/2}\, \widetilde{\boldsymbol{C}}^{1/2} & \mathbf{0} \\ \mathbf{0} & \left(\frac{\langle \widetilde{\boldsymbol{C}}\boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}}\rangle}{1-\delta^2}\right)^{1/2} \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{x}_1 + t\widetilde{\boldsymbol{\beta}}\rangle\, \widetilde{\boldsymbol{C}}^{1/2} \end{bmatrix}$$

for $t \in \mathbb{R}_+$ *and* $\boldsymbol{x} = (\boldsymbol{x}_1^\top, \boldsymbol{x}_2^\top, \boldsymbol{x}_3^\top)^\top \in (\mathbb{R}^2)^3$.

(Note that the statement of Theorem 5.1 holds even if $\langle \widetilde{\boldsymbol{C}}\boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}}\rangle = 0$, when the last 2-dimensional coordinate process of the pathwise unique strong solution $(\boldsymbol{\mathcal{Z}}_t)_{t\in\mathbb{R}_+}$ is $\mathbf{0}$.)

The SDE (5.2) has the form

$$\mathrm{d}\boldsymbol{\mathcal{Z}}_t =: \begin{bmatrix} \mathrm{d}\boldsymbol{\mathcal{M}}_t \\ \mathrm{d}\boldsymbol{\mathcal{N}}_t \\ \mathrm{d}\boldsymbol{\mathcal{P}}_t \end{bmatrix} = \begin{bmatrix} (\langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{\mathcal{M}}_t + t\widetilde{\boldsymbol{\beta}}\rangle^+)^{1/2}\, \widetilde{\boldsymbol{C}}^{1/2}\, \mathrm{d}\boldsymbol{\mathcal{W}}_t \\ (\langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{\mathcal{M}}_t + t\widetilde{\boldsymbol{\beta}}\rangle^+)^{3/2}\, \widetilde{\boldsymbol{C}}^{1/2}\, \mathrm{d}\boldsymbol{\mathcal{W}}_t \\ \left(\frac{\langle \widetilde{\boldsymbol{C}}\boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}}\rangle}{1-\delta^2}\right)^{1/2} \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{\mathcal{M}}_t + t\widetilde{\boldsymbol{\beta}}\rangle\, \widetilde{\boldsymbol{C}}^{1/2}\, \mathrm{d}\widetilde{\boldsymbol{\mathcal{W}}}_t \end{bmatrix}, \qquad t \in \mathbb{R}_+. \tag{5.3}$$

One can prove that the first 2-dimensional equation of the SDE (5.3) has a pathwise unique strong solution $(\boldsymbol{\mathcal{M}}_t^{(\boldsymbol{y}_0)})_{t\in\mathbb{R}_+}$ with arbitrary initial value $\boldsymbol{\mathcal{M}}_0^{(\boldsymbol{y}_0)} = \boldsymbol{y}_0 \in \mathbb{R}^2$. Indeed, it is equivalent to the existence of a pathwise unique strong solution of the SDE

$$\begin{cases} \mathrm{d}\mathcal{S}_t = \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}}\rangle\, \mathrm{d}t + (\mathcal{S}_t^+)^{1/2}\, \boldsymbol{u}_{\mathrm{left}}^\top \widetilde{\boldsymbol{C}}^{1/2}\, \mathrm{d}\boldsymbol{\mathcal{W}}_t, \\ \mathrm{d}\boldsymbol{\mathcal{Q}}_t = -\boldsymbol{\Pi}\widetilde{\boldsymbol{\beta}}\, \mathrm{d}t + (\mathcal{S}_t^+)^{1/2}\, \big(\boldsymbol{I}_2 - \boldsymbol{\Pi}\big)\widetilde{\boldsymbol{C}}^{1/2}\, \mathrm{d}\boldsymbol{\mathcal{W}}_t, \end{cases} \qquad t \in \mathbb{R}_+, \tag{5.4}$$

with initial value $\big(\mathcal{S}_0^{(\boldsymbol{y}_0)},\, \boldsymbol{\mathcal{Q}}_0^{(\boldsymbol{y}_0)}\big) = \big(\langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{y}_0\rangle,\, (\boldsymbol{I}_2 - \boldsymbol{\Pi})\boldsymbol{y}_0\big) \in \mathbb{R} \times \mathbb{R}^2$, where $\boldsymbol{I}_2$ denotes the 2-dimensional unit matrix and $\boldsymbol{\Pi} := \boldsymbol{u}_{\mathrm{right}}\boldsymbol{u}_{\mathrm{left}}^\top$, since we have the correspondences

$$\mathcal{S}_t^{(\boldsymbol{y}_0)} = \boldsymbol{u}_{\mathrm{left}}^\top(\boldsymbol{\mathcal{M}}_t^{(\boldsymbol{y}_0)} + t\widetilde{\boldsymbol{\beta}}), \qquad \boldsymbol{\mathcal{Q}}_t^{(\boldsymbol{y}_0)} = \boldsymbol{\mathcal{M}}_t^{(\boldsymbol{y}_0)} - \mathcal{S}_t^{(\boldsymbol{y}_0)}\boldsymbol{u}_{\mathrm{right}}$$

$$\boldsymbol{\mathcal{M}}_t^{(\boldsymbol{y}_0)} = \boldsymbol{\mathcal{Q}}_t^{(\boldsymbol{y}_0)} + \mathcal{S}_t^{(\boldsymbol{y}_0)}\boldsymbol{u}_{\mathrm{right}},$$

see the proof of Ispány and Pap [15, Theorem 3.1]. By Remark 2.10, $\mathcal{S}_t^+$ may be replaced by $\mathcal{S}_t$ for all $t \in \mathbb{R}_+$ in the first equation of (5.4) provided that $\langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{y}_0\rangle \in \mathbb{R}_+$, hence $\langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{\mathcal{M}}_t + t\widetilde{\boldsymbol{\beta}}\rangle^+$ may be replaced by $\langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{\mathcal{M}}_t + t\widetilde{\boldsymbol{\beta}}\rangle$ for all $t \in \mathbb{R}_+$ in (5.3). Thus the SDE (5.2) has a pathwise unique strong solution with initial value $\boldsymbol{\mathcal{Z}}_0 = \mathbf{0}$, and we have

$$\boldsymbol{\mathcal{Z}}_t = \begin{bmatrix} \boldsymbol{\mathcal{M}}_t \\ \boldsymbol{\mathcal{N}}_t \\ \boldsymbol{\mathcal{P}}_t \end{bmatrix} = \begin{bmatrix} \int_0^t \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{\mathcal{M}}_s + s\widetilde{\boldsymbol{\beta}}\rangle^{1/2}\, \widetilde{\boldsymbol{C}}^{1/2}\, \mathrm{d}\boldsymbol{\mathcal{W}}_s \\ \int_0^t \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{\mathcal{M}}_s + s\widetilde{\boldsymbol{\beta}}\rangle\, \mathrm{d}\boldsymbol{\mathcal{M}}_s \\ \left(\frac{\langle \widetilde{\boldsymbol{C}}\boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}}\rangle}{1-\delta^2}\right)^{1/2} \int_0^t \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{\mathcal{M}}_s + s\widetilde{\boldsymbol{\beta}}\rangle\, \widetilde{\boldsymbol{C}}^{1/2}\, \mathrm{d}\widetilde{\boldsymbol{\mathcal{W}}}_s \end{bmatrix}, \qquad t \in \mathbb{R}_+.$$

By the method of the proof of $\mathcal{X}^{(n)} \stackrel{\mathcal{D}}{\longrightarrow} \mathcal{X}$ in Theorem 3.1 in Barczy et al. [1], applying Lemma D.2, one can easily derive

$$\begin{bmatrix} \boldsymbol{\mathcal{X}}^{(n)} \\ \boldsymbol{\mathcal{Z}}^{(n)} \end{bmatrix} \stackrel{\mathcal{D}}{\longrightarrow} \begin{bmatrix} \widetilde{\boldsymbol{\mathcal{X}}} \\ \boldsymbol{\mathcal{Z}} \end{bmatrix}, \qquad \text{as } n \to \infty, \tag{5.5}$$

where

$$\boldsymbol{\mathcal{X}}_t^{(n)} = n^{-1} \boldsymbol{X}_{\lfloor nt \rfloor}, \qquad \widetilde{\boldsymbol{\mathcal{X}}}_t := \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{\mathcal{M}}_t + t\widetilde{\boldsymbol{\beta}} \rangle \boldsymbol{u}_{\mathrm{right}}, \qquad t \in \mathbb{R}_+, \qquad n \in \mathbb{N}.$$

Now, with the process

$$\widetilde{\mathcal{Y}}_t := \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\mathcal{X}}}_t \rangle = \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{\mathcal{M}}_t + t\widetilde{\boldsymbol{\beta}} \rangle, \qquad t \in \mathbb{R}_+, \tag{5.6}$$

we have

$$\widetilde{\boldsymbol{\mathcal{X}}}_t = \widetilde{\mathcal{Y}}_t \boldsymbol{u}_{\mathrm{right}}, \qquad t \in \mathbb{R}_+,$$

since $\langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{u}_{\mathrm{right}} \rangle = 1$. By Itô's formula and the first 2-dimensional equation of the SDE (5.3) we obtain

$$\mathrm{d}\widetilde{\mathcal{Y}}_t = \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle \, \mathrm{d}t + (\widetilde{\mathcal{Y}}_t^+)^{1/2} \boldsymbol{u}_{\mathrm{left}}^\top \widetilde{\boldsymbol{C}}^{1/2} \mathrm{d}\boldsymbol{\mathcal{W}}_t, \qquad t \in \mathbb{R}_+.$$

If $\langle \widetilde{\boldsymbol{C}} \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{u}_{\mathrm{left}} \rangle = \| \boldsymbol{u}_{\mathrm{left}}^\top \widetilde{\boldsymbol{C}}^{1/2} \|^2 = 0$ then $\boldsymbol{u}_{\mathrm{left}}^\top \widetilde{\boldsymbol{C}}^{1/2} = \boldsymbol{0}$, hence $\mathrm{d}\widetilde{\mathcal{Y}}_t = \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle \mathrm{d}t$, $t \in \mathbb{R}_+$, implying that the process $(\widetilde{\mathcal{Y}}_t)_{t \in \mathbb{R}_+}$ satisfies the SDE (2.9). If $\langle \widetilde{\boldsymbol{C}} \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{u}_{\mathrm{left}} \rangle \neq 0$ then the process

$$\widetilde{\widetilde{\mathcal{W}}}_t := \frac{\langle \widetilde{\boldsymbol{C}}^{1/2} \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{\mathcal{W}}_t \rangle}{\langle \widetilde{\boldsymbol{C}} \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{u}_{\mathrm{left}} \rangle^{1/2}}, \qquad t \in \mathbb{R}_+,$$

is a (one-dimensional) standard Wiener process, hence the process $(\widetilde{\mathcal{Y}}_t)_{t \in \mathbb{R}_+}$ satisfies the SDE (2.9). Consequently, $\widetilde{\mathcal{Y}} = \mathcal{Y}$ (due to pathwise uniqueness), and hence $\widetilde{\boldsymbol{\mathcal{X}}} = \boldsymbol{\mathcal{X}}$. Next, by Lemma D.3, convergence (5.5) with $U_{k-1} = \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{X}_{k-1} \rangle$ and Lemma C.1 imply

$$\sum_{k=1}^n \begin{bmatrix} n^{-3} U_{k-1}^2 \\ n^{-2} V_{k-1}^2 \\ n^{-2} \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{M}_k \rangle U_{k-1} \\ n^{-3/2} \langle \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{M}_k \rangle V_{k-1} \end{bmatrix} \stackrel{\mathcal{D}}{\longrightarrow} \begin{bmatrix} \int_0^1 \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{\mathcal{X}}_t \rangle^2 \, \mathrm{d}t \\ \frac{\langle \widetilde{\boldsymbol{C}} \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}} \rangle}{1-\delta^2} \int_0^1 \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{\mathcal{X}}_t \rangle \, \mathrm{d}t \\ \int_0^1 \mathcal{Y}_t \, \mathrm{d} \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{\mathcal{M}}_t \rangle \\ \left(\frac{\langle \widetilde{\boldsymbol{C}} \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}} \rangle}{1-\delta^2} \right)^{1/2} \int_0^1 \mathcal{Y}_t \, \mathrm{d} \langle \boldsymbol{v}_{\mathrm{left}}, \widetilde{\boldsymbol{C}}^{1/2} \widetilde{\boldsymbol{\mathcal{W}}}_t \rangle \end{bmatrix},$$

as $n \to \infty$. This limiting random vector can be written in the form as given in Theorem 4.1, since $\langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{\mathcal{X}}_t \rangle = \mathcal{Y}_t$, $\langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{\mathcal{M}}_t \rangle = \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{\mathcal{X}}_t \rangle - \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle \, t = \mathcal{Y}_t - \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle \, t$ (using (5.6)), and $\langle \boldsymbol{v}_{\mathrm{left}}, \widetilde{\boldsymbol{C}}^{1/2} \widetilde{\boldsymbol{\mathcal{W}}}_t \rangle = \langle \widetilde{\boldsymbol{C}} \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}} \rangle^{1/2} \, \widetilde{\mathcal{W}}_t$ for all $t \in \mathbb{R}_+$ with a (one-dimensional) standard Wiener process $(\widetilde{\mathcal{W}}_t)_{t \in \mathbb{R}_+}$.

Proof of Theorem 5.1. In order to show convergence $\boldsymbol{\mathcal{Z}}^{(n)} \stackrel{\mathcal{D}}{\longrightarrow} \boldsymbol{\mathcal{Z}}$, we apply Theorem E.1 with the special choices $\boldsymbol{\mathcal{U}} := \boldsymbol{\mathcal{Z}}$, $\boldsymbol{U}_k^{(n)} := \boldsymbol{Z}_k^{(n)}$, $n, k \in \mathbb{N}$, $(\mathcal{F}_k^{(n)})_{k \in \mathbb{Z}_+} := (\mathcal{F}_k)_{k \in \mathbb{Z}_+}$ and the function γ which is defined in Theorem 5.1. Note that the discussion after Theorem 5.1 shows that the SDE (5.2) admits a pathwise unique strong solution $(\boldsymbol{\mathcal{Z}}_t^{\boldsymbol{z}})_{t\in\mathbb{R}_+}$ for all initial values $\boldsymbol{\mathcal{Z}}_0^{\boldsymbol{z}} = \boldsymbol{z} \in (\mathbb{R}^2)^3$. Applying Cauchy–Schwarz inequality and Corollary B.5, one can check that $\mathbb{E}(\|\boldsymbol{U}_k^{(n)}\|^2) < \infty$ for all $n, k \in \mathbb{N}$.

Now we show that conditions (i) and (ii) of Theorem E.1 hold. The conditional variance $\operatorname{Var}\big(\boldsymbol{Z}_k^{(n)} \,|\, \mathcal{F}_{k-1}\big)$ has the form

$$\begin{bmatrix} n^{-2} & n^{-3}U_{k-1} & n^{-5/2}V_{k-1} \\ n^{-3}U_{k-1} & n^{-4}U_{k-1}^2 & n^{-7/2}U_{k-1}V_{k-1} \\ n^{-5/2}V_{k-1} & n^{-7/2}U_{k-1}V_{k-1} & n^{-3}V_{k-1}^2 \end{bmatrix} \otimes \boldsymbol{V}_{\boldsymbol{M}_k}$$

for $n \in \mathbb{N}$, $k \in \{1, \ldots, n\}$, with $\boldsymbol{V}_{\boldsymbol{M}_k} := \operatorname{Var}(\boldsymbol{M}_k \,|\, \mathcal{F}_{k-1})$, and $\gamma(s, \boldsymbol{\mathcal{Z}}_s^{(n)})\gamma(s, \boldsymbol{\mathcal{Z}}_s^{(n)})^\top$ has the form

$$\begin{bmatrix} \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{\mathcal{M}}_s^{(n)} + s\widetilde{\boldsymbol{\beta}}\rangle & \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{\mathcal{M}}_s^{(n)} + s\widetilde{\boldsymbol{\beta}}\rangle^2 & \boldsymbol{0} \\ \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{\mathcal{M}}_s^{(n)} + s\widetilde{\boldsymbol{\beta}}\rangle^2 & \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{\mathcal{M}}_s^{(n)} + s\widetilde{\boldsymbol{\beta}}\rangle^3 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \frac{\langle \widetilde{\boldsymbol{C}}\boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}}\rangle}{1-\delta^2}\langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{\mathcal{M}}_s^{(n)} + s\widetilde{\boldsymbol{\beta}}\rangle^2 \end{bmatrix} \otimes \widetilde{\boldsymbol{C}}$$

for $s \in \mathbb{R}_+$, where we used that $\langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{\mathcal{M}}_s^{(n)} + s\widetilde{\boldsymbol{\beta}}\rangle^+ = \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{\mathcal{M}}_s^{(n)} + s\widetilde{\boldsymbol{\beta}}\rangle$, $s \in \mathbb{R}_+$, $n \in \mathbb{N}$. Indeed, by (3.4), we get

$$\langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{\mathcal{M}}_s^{(n)} + s\widetilde{\boldsymbol{\beta}}\rangle = \frac{1}{n}U_{\lfloor ns\rfloor} + \frac{ns - \lfloor ns\rfloor}{n}\langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}}\rangle \in \mathbb{R}_+ \tag{5.7}$$

for $s \in \mathbb{R}_+$, $n \in \mathbb{N}$, since $\boldsymbol{u}_{\mathrm{left}}^\top \mathrm{e}^{\widetilde{\boldsymbol{B}}} = \boldsymbol{u}_{\mathrm{left}}^\top$ implies $\langle \boldsymbol{u}_{\mathrm{left}}, \mathrm{e}^{\widetilde{\boldsymbol{B}}}\boldsymbol{X}_{k-1}\rangle = \boldsymbol{u}_{\mathrm{left}}^\top \mathrm{e}^{\widetilde{\boldsymbol{B}}}\boldsymbol{X}_{k-1} = \boldsymbol{u}_{\mathrm{left}}^\top \boldsymbol{X}_{k-1} = \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{X}_{k-1}\rangle$.

In order to check condition (i) of Theorem E.1, we need to prove that for each $T > 0$, as $n \to \infty$,

$$\sup_{t\in[0,T]} \left\| \frac{1}{n^2}\sum_{k=1}^{\lfloor nt\rfloor} \boldsymbol{V}_{\boldsymbol{M}_k} - \int_0^t \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{\mathcal{M}}_s^{(n)} + s\widetilde{\boldsymbol{\beta}}\rangle\, \widetilde{\boldsymbol{C}}\,\mathrm{d}s \right\| \stackrel{\mathbb{P}}{\longrightarrow} 0, \tag{5.8}$$

$$\sup_{t\in[0,T]} \left\| \frac{1}{n^3}\sum_{k=1}^{\lfloor nt\rfloor} U_{k-1}\boldsymbol{V}_{\boldsymbol{M}_k} - \int_0^t \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{\mathcal{M}}_s^{(n)} + s\widetilde{\boldsymbol{\beta}}\rangle^2\, \widetilde{\boldsymbol{C}}\,\mathrm{d}s \right\| \stackrel{\mathbb{P}}{\longrightarrow} 0, \tag{5.9}$$

$$\sup_{t\in[0,T]} \left\| \frac{1}{n^4}\sum_{k=1}^{\lfloor nt\rfloor} U_{k-1}^2\boldsymbol{V}_{\boldsymbol{M}_k} - \int_0^t \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{\mathcal{M}}_s^{(n)} + s\widetilde{\boldsymbol{\beta}}\rangle^3\, \widetilde{\boldsymbol{C}}\,\mathrm{d}s \right\| \stackrel{\mathbb{P}}{\longrightarrow} 0, \tag{5.10}$$

$$\sup_{t\in[0,T]} \left\| \frac{1}{n^3}\sum_{k=1}^{\lfloor nt\rfloor} V_{k-1}^2\boldsymbol{V}_{\boldsymbol{M}_k} - \frac{\langle \widetilde{\boldsymbol{C}}\boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}}\rangle}{1-\delta^2}\int_0^t \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{\mathcal{M}}_s^{(n)} + s\widetilde{\boldsymbol{\beta}}\rangle^2\, \widetilde{\boldsymbol{C}}\,\mathrm{d}s \right\| \stackrel{\mathbb{P}}{\longrightarrow} 0, \tag{5.11}$$

$$\sup_{t\in[0,T]} \left\| \frac{1}{n^{5/2}}\sum_{k=1}^{\lfloor nt\rfloor} V_{k-1}\boldsymbol{V}_{\boldsymbol{M}_k} \right\| \stackrel{\mathbb{P}}{\longrightarrow} 0, \tag{5.12}$$

$$\sup_{t\in[0,T]} \left\| \frac{1}{n^{7/2}} \sum_{k=1}^{\lfloor nt\rfloor} U_{k-1} V_{k-1} \boldsymbol{V}_{\boldsymbol{M}_k} \right\| \xrightarrow{\mathbb{P}} 0. \tag{5.13}$$

First we show (5.8). By (5.7), $\int_0^t \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{\mathcal{M}}_s^{(n)} + s\widetilde{\boldsymbol{\beta}} \rangle \, \mathrm{d}s$ has the form

$$\frac{1}{n^2} \sum_{k=1}^{\lfloor nt\rfloor -1} U_k + \frac{nt - \lfloor nt\rfloor}{n^2} U_{\lfloor nt\rfloor} + \frac{\lfloor nt\rfloor + (nt - \lfloor nt\rfloor)^2}{2n^2} \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle.$$

Using Proposition B.3, formula (4.5) and $\widetilde{\boldsymbol{C}} = (\boldsymbol{V}_1 + \boldsymbol{V}_2)/2$, we obtain

$$\begin{aligned} \boldsymbol{V}_{\boldsymbol{M}_k} &= \operatorname{Var}(\boldsymbol{M}_k \mid \mathcal{F}_{k-1}) = \frac{1}{2} U_{k-1} (\boldsymbol{V}_1 + \boldsymbol{V}_2) + \frac{1}{2} V_{k-1} (\boldsymbol{V}_1 - \boldsymbol{V}_2) + \boldsymbol{V}_0 \\ &= U_{k-1} \widetilde{\boldsymbol{C}} + \frac{1}{2} V_{k-1} (\boldsymbol{V}_1 - \boldsymbol{V}_2) + \boldsymbol{V}_0. \end{aligned} \tag{5.14}$$

Thus, in order to show (5.8), it suffices to prove

$$n^{-2} \sum_{k=1}^{\lfloor nT\rfloor} |V_k| \xrightarrow{\mathbb{P}} 0, \qquad n^{-2} \sup_{t\in[0,T]} U_{\lfloor nt\rfloor} \xrightarrow{\mathbb{P}} 0, \tag{5.15}$$

$$n^{-2} \sup_{t\in[0,T]} \left[\lfloor nt\rfloor + (nt - \lfloor nt\rfloor)^2 \right] \to 0, \tag{5.16}$$

as $n \to \infty$. Using (B.3) with $(\ell, i, j) = (2, 0, 1)$ and (B.4) with $(\ell, i, j) = (2, 1, 0)$, we have (5.15). Clearly, (5.16) follows from $|nt - \lfloor nt\rfloor| \leqslant 1$, $n \in \mathbb{N}$, $t \in \mathbb{R}_+$, thus we conclude (5.8).

The proofs of (5.9) and (5.10) can be carried out similarly, for a detailed discussion, see Barczy et al. [8].

Next we turn to prove (5.11). First we show that

$$n^{-3} \sup_{t\in[0,T]} \left\| \sum_{k=1}^{\lfloor nt\rfloor} V_{k-1}^2 \boldsymbol{V}_{\boldsymbol{M}_k} - \frac{\langle \widetilde{\boldsymbol{C}} \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}} \rangle}{1 - \delta^2} \sum_{k=1}^{\lfloor nt\rfloor} U_{k-1}^2 \widetilde{\boldsymbol{C}} \right\| \xrightarrow{\mathbb{P}} 0, \tag{5.17}$$

as $n \to \infty$ for all $T > 0$. By (5.14),

$$\sum_{k=1}^{\lfloor nt\rfloor} V_{k-1}^2 \boldsymbol{V}_{\boldsymbol{M}_k} = \sum_{k=1}^{\lfloor nt\rfloor} U_{k-1} V_{k-1}^2 \widetilde{\boldsymbol{C}} + \frac{1}{2} \sum_{k=1}^{\lfloor nt\rfloor} V_{k-1}^3 (\boldsymbol{V}_1 - \boldsymbol{V}_2) + \sum_{k=1}^{\lfloor nt\rfloor} V_{k-1}^2 \boldsymbol{V}_0.$$

Using (B.3) with $(\ell, i, j) = (6, 0, 3)$ and $(\ell, i, j) = (4, 0, 2)$, we have

$$n^{-3} \sum_{k=1}^{\lfloor nT\rfloor} |V_k|^3 \xrightarrow{\mathbb{P}} 0, \qquad n^{-3} \sum_{k=1}^{\lfloor nT\rfloor} V_k^2 \xrightarrow{\mathbb{P}} 0, \qquad \text{as } n \to \infty,$$

hence (5.17) will follow from

$$n^{-3} \sup_{t\in[0,T]} \left| \sum_{k=1}^{\lfloor nt\rfloor} U_{k-1} V_{k-1}^2 - \frac{\langle \widetilde{\boldsymbol{C}} \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}} \rangle}{1 - \delta^2} \sum_{k=1}^{\lfloor nt\rfloor} U_{k-1}^2 \right| \xrightarrow{\mathbb{P}} 0, \tag{5.18}$$

as $n \to \infty$ for all $T > 0$. By the method of the proof of Lemma C.1 (see also Ispány et al. [13, page 16 of arXiv version]), applying Proposition B.4 with $q = 3$, we obtain a decomposition of $\sum_{k=1}^{\lfloor nt\rfloor} U_{k-1} V_{k-1}^2$, namely,

$$\sum_{k=1}^{\lfloor nt\rfloor} U_{k-1}V_{k-1}^2 = \frac{1}{1-\delta^2}\sum_{k=2}^{\lfloor nt\rfloor}\big[U_{k-1}V_{k-1}^2 - \mathbb{E}(U_{k-1}V_{k-1}^2 \,|\, \mathcal{F}_{k-2})\big]$$

$$+ \frac{\langle \widetilde{\boldsymbol{C}}\boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}}\rangle}{1-\delta^2}\sum_{k=2}^{\lfloor nt\rfloor} U_{k-2}^2 - \frac{\delta^2}{1-\delta^2} U_{\lfloor nt\rfloor -1} V_{\lfloor nt\rfloor -1}^2 + \mathrm{O}(n)$$

$$+ \text{lin. comb. of } \sum_{k=2}^{\lfloor nt\rfloor} U_{k-2}V_{k-2}, \ \sum_{k=2}^{\lfloor nt\rfloor} V_{k-2}^2, \ \sum_{k=2}^{\lfloor nt\rfloor} U_{k-2} \ \text{and} \ \sum_{k=2}^{\lfloor nt\rfloor} V_{k-2}.$$

Note that Proposition B.4 with $q = 3$ is needed above in order to express products $\mathbb{E}(M_{k-1,i_1}M_{k-1,i_2}M_{k-1,i_2} \,|\, \mathcal{F}_{k-2})$, $i_1, i_2, i_3 \in \{1, 2\}$, as a first order polynomial of $\boldsymbol{X}_{k-2}$, and hence, by (4.5), as a linear combination of U_{k-2}, V_{k-2} and 1. Using (B.5) with $(\ell, i, j) = (8, 1, 2)$ we have

$$n^{-3}\sup_{t\in[0,T]}\left|\sum_{k=2}^{\lfloor nt\rfloor}\big[U_{k-1}V_{k-1}^2 - \mathbb{E}(U_{k-1}V_{k-1}^2 \,|\, \mathcal{F}_{k-2})\big]\right| \stackrel{\mathbb{P}}{\longrightarrow} 0, \qquad \text{as } n \to \infty.$$

In order to show (5.18), it suffices to prove

$$n^{-3}\sum_{k=1}^{\lfloor nT\rfloor}|U_kV_k| \stackrel{\mathbb{P}}{\longrightarrow} 0, \qquad n^{-3}\sum_{k=1}^{\lfloor nT\rfloor} V_k^2 \stackrel{\mathbb{P}}{\longrightarrow} 0, \tag{5.19}$$

$$n^{-3}\sum_{k=1}^{\lfloor nT\rfloor} U_k \stackrel{\mathbb{P}}{\longrightarrow} 0, \qquad n^{-3}\sum_{k=1}^{\lfloor nT\rfloor} |V_k| \stackrel{\mathbb{P}}{\longrightarrow} 0, \tag{5.20}$$

$$n^{-3}\sup_{t\in[0,T]} U_{\lfloor nt\rfloor}V_{\lfloor nt\rfloor}^2 \stackrel{\mathbb{P}}{\longrightarrow} 0, \qquad n^{-3/2}\sup_{t\in[0,T]} U_{\lfloor nt\rfloor} \stackrel{\mathbb{P}}{\longrightarrow} 0, \tag{5.21}$$

as $n \to \infty$. Using (B.3) with $(\ell, i, j) = (2, 1, 1)$, $(\ell, i, j) = (4, 0, 2)$, $(\ell, i, j) = (2, 1, 0)$ and $(\ell, i, j) = (2, 0, 1)$, we have (5.19) and (5.20). By (B.4) with $(\ell, i, j) = (4, 1, 2)$ and $(\ell, i, j) = (3, 1, 0)$, we have (5.21). Thus we conclude (5.17). By (5.14) and (B.3) with $(\ell, i, j) = (2, 1, 1)$ and $(\ell, i, j) = (2, 1, 0)$, we get

$$n^{-3}\sup_{t\in[0,T]}\left\|\sum_{k=1}^{\lfloor nt\rfloor} U_{k-1}\boldsymbol{V}_{\boldsymbol{M}_k} - \sum_{k=1}^{\lfloor nt\rfloor} U_{k-1}^2\widetilde{\boldsymbol{C}}\right\| \stackrel{\mathbb{P}}{\longrightarrow} 0, \tag{5.22}$$

as $n \to \infty$ for all $T > 0$. As a last step, using (5.9), we obtain (5.11). Convergences (5.12) and (5.13) can be proved similarly (see also the same considerations in Ispány et al. [13, pages 17-20 of arXiv version]).

Finally, we check condition (ii) of Theorem E.1, that is, the conditional Lindeberg condition

$$\sum_{k=1}^{\lfloor nT\rfloor} \mathbb{E}\left(\|\boldsymbol{Z}_k^{(n)}\|^2 \mathbb{1}_{\{\|\boldsymbol{Z}_k^{(n)}\|>\theta\}} \,\big|\, \mathcal{F}_{k-1}\right) \stackrel{\mathbb{P}}{\longrightarrow} 0, \qquad \text{as} \quad n \to \infty \tag{5.23}$$

for all $\theta > 0$ and $T > 0$. We have $\mathbb{E}\left(\|\boldsymbol{Z}_k^{(n)}\|^2 \mathbb{1}_{\{\|\boldsymbol{Z}_k^{(n)}\|>\theta\}} \,\big|\, \mathcal{F}_{k-1}\right) \leqslant \theta^{-2}\,\mathbb{E}\left(\|\boldsymbol{Z}_k^{(n)}\|^4 \,\big|\, \mathcal{F}_{k-1}\right)$ and

$$\|\boldsymbol{Z}_k^{(n)}\|^4 \leqslant 3\left(n^{-4} + n^{-8}U_{k-1}^4 + n^{-6}V_{k-1}^4\right)\|\boldsymbol{M}_k\|^4.$$

Hence, for all $\theta > 0$ and $T > 0$, we have

$$\sum_{k=1}^{\lfloor nT\rfloor} \mathbb{E}\left(\|\boldsymbol{Z}_k^{(n)}\|^2 \mathbb{1}_{\{\|\boldsymbol{Z}_k^{(n)}\|>\theta\}}\right) \to 0, \qquad \text{as} \quad n \to \infty,$$

since $\mathbb{E}(\|\boldsymbol{M}_k\|^4) = \mathrm{O}(k^2)$, $\mathbb{E}(\|\boldsymbol{M}_k\|^4 U_{k-1}^4) \leqslant \sqrt{\mathbb{E}(\|\boldsymbol{M}_k\|^8)\,\mathbb{E}(U_{k-1}^8)} = \mathrm{O}(k^6)$ and $\mathbb{E}(\|\boldsymbol{M}_k\|^4 V_{k-1}^4) \leqslant \sqrt{\mathbb{E}(\|\boldsymbol{M}_k\|^8)\,\mathbb{E}(V_{k-1}^8)} = \mathrm{O}(k^4)$ by Corollary B.5. This yields (5.23). □

We call the attention that our moment conditions (2.3) with $q = 8$ are used for applying Corollaries B.5 and B.6.

6 Proof of Theorem 4.2

The proof of the second convergence in Theorem 4.2 is similar to the proof of Theorem 4.1. Consider the sequence of stochastic processes

$$\boldsymbol{\mathcal{Z}}_t^{(n)} := \begin{bmatrix} \boldsymbol{\mathcal{M}}_t^{(n)} \\ \boldsymbol{\mathcal{N}}_t^{(n)} \\ \mathcal{P}_t^{(n)} \end{bmatrix} := \sum_{k=1}^{\lfloor nt\rfloor} \boldsymbol{Z}_k^{(n)} \qquad \text{with} \qquad \boldsymbol{Z}_k^{(n)} := \begin{bmatrix} n^{-1}\boldsymbol{M}_k \\ n^{-2}\boldsymbol{M}_k U_{k-1} \\ n^{-1/2}\langle \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{M}_k\rangle V_{k-1} \end{bmatrix}$$

for $t \in \mathbb{R}_+$ and $k, n \in \mathbb{N}$.

6.1 Theorem. *Suppose that the assumptions of Theorem 3.1 hold. If* $\langle \widetilde{\boldsymbol{C}}\boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}}\rangle = 0$, *then*

$$\boldsymbol{\mathcal{Z}}^{(n)} \stackrel{\mathcal{D}}{\longrightarrow} \boldsymbol{\mathcal{Z}} \qquad \text{as} \quad n \to \infty, \tag{6.1}$$

where the process $(\boldsymbol{\mathcal{Z}}_t)_{t\in\mathbb{R}_+}$ *with values in* $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}$ *is the pathwise unique strong solution of the SDE*

$$\mathrm{d}\boldsymbol{\mathcal{Z}}_t = \gamma(t, \boldsymbol{\mathcal{Z}}_t) \begin{bmatrix} \mathrm{d}\boldsymbol{\mathcal{W}}_t \\ \mathrm{d}\widetilde{\mathcal{W}}_t \end{bmatrix}, \qquad t \in \mathbb{R}_+, \tag{6.2}$$

with initial value $\boldsymbol{\mathcal{Z}}_0 = \mathbf{0}$, *where* $(\boldsymbol{\mathcal{W}}_t)_{t\in\mathbb{R}_+}$ *and* $(\widetilde{\mathcal{W}}_t)_{t\in\mathbb{R}_+}$ *are independent standard Wiener processes of dimension 2 and 1, respectively, and* $\gamma : \mathbb{R}_+ \times (\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}) \to \mathbb{R}^{5\times 3}$ *is defined by*

$$\gamma(t, \boldsymbol{x}) := \begin{bmatrix} (\langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{x}_1 + t\widetilde{\boldsymbol{\beta}}\rangle^+)^{1/2}\, \widetilde{\boldsymbol{C}}^{1/2} & 0 \\ (\langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{x}_1 + t\widetilde{\boldsymbol{\beta}}\rangle^+)^{3/2}\, \widetilde{\boldsymbol{C}}^{1/2} & 0 \\ \mathbf{0} & \langle \boldsymbol{V}_0\boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}}\rangle^{1/2} M^{1/2} \end{bmatrix}$$

for $t \in \mathbb{R}_+$ *and* $\boldsymbol{x} = (\boldsymbol{x}_1^\top, \boldsymbol{x}_2^\top, x_3)^\top \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}$.

As in the case of the SDE (5.2), the SDE (6.2) has a pathwise unique strong solution with initial value $\boldsymbol{\mathcal{Z}}_0 = \mathbf{0}$, for which we have

$$\boldsymbol{\mathcal{Z}}_t = \begin{bmatrix} \boldsymbol{\mathcal{M}}_t \\ \boldsymbol{\mathcal{N}}_t \\ \mathcal{P}_t \end{bmatrix} = \begin{bmatrix} \int_0^t \mathcal{Y}_s^{1/2}\, \widetilde{\boldsymbol{C}}^{1/2}\, \mathrm{d}\boldsymbol{\mathcal{W}}_s \\ \int_0^t \mathcal{Y}_s \, \mathrm{d}\boldsymbol{\mathcal{M}}_s \\ \langle \boldsymbol{V}_0\boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}}\rangle^{1/2} M^{1/2} \widetilde{\mathcal{W}}_t \end{bmatrix}, \qquad t \in \mathbb{R}_+.$$

One can again easily derive

$$\begin{bmatrix} \boldsymbol{\mathcal{X}}^{(n)} \\ \boldsymbol{\mathcal{Z}}^{(n)} \end{bmatrix} \stackrel{\mathcal{D}}{\longrightarrow} \begin{bmatrix} \boldsymbol{\mathcal{X}} \\ \boldsymbol{\mathcal{Z}} \end{bmatrix} \qquad \text{as } n \to \infty, \tag{6.3}$$

where

$$\boldsymbol{\mathcal{X}}_t^{(n)} = n^{-1} \boldsymbol{X}_{\lfloor nt \rfloor}, \qquad \boldsymbol{\mathcal{X}}_t = \mathcal{Y}_t \boldsymbol{u}_{\mathrm{right}}, \qquad t \in \mathbb{R}_+, \qquad n \in \mathbb{N}.$$

Next, by Lemma D.3, convergence (6.3) and Lemma C.2 imply

$$\sum_{k=1}^n \begin{bmatrix} n^{-3} U_{k-1}^2 \\ n^{-2} \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{M}_k \rangle U_{k-1} \\ n^{-1/2} \langle \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{M}_k \rangle V_{k-1} \end{bmatrix} \stackrel{\mathcal{D}}{\longrightarrow} \begin{bmatrix} \int_0^1 \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{\mathcal{X}}_t \rangle^2 \, \mathrm{d}t \\ \int_0^1 \mathcal{Y}_t \, \mathrm{d}\langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{\mathcal{M}}_t \rangle \\ \langle \boldsymbol{V}_0\boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}}\rangle^{1/2} M^{1/2} \widetilde{\mathcal{W}}_1 \end{bmatrix} \qquad \text{as } n \to \infty.$$

Note that this convergence holds even in case $M = 0$. The limiting random vector can be written in the form as given in Theorem 4.2, since $\langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{\mathcal{X}}_t \rangle = \mathcal{Y}_t$ and $\langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{\mathcal{M}}_t \rangle = \mathcal{Y}_t - \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}}\, t \rangle$ for all $t \in \mathbb{R}_+$.

One can prove Theorem 6.1 similarly to Theorem 5.1, for a detailed discussion, see Barczy et al. [8].

7 Proof of Theorem 4.3

The first convergence in Theorem 4.3 follows from the following approximation.

7.1 Lemma. *Suppose that the assumptions of Theorem 3.1 hold. If* $\langle \widetilde{\boldsymbol{C}} \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{u}_{\mathrm{left}} \rangle = 0$, *then for each* $T > 0$,

$$\sup_{t\in[0,T]} \left| \frac{1}{n^3} \sum_{k=1}^{\lfloor nt \rfloor} U_{k-1}^2 - \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle^2 \frac{t^3}{3} \right| \stackrel{\mathbb{P}}{\longrightarrow} 0, \qquad \text{as } n \to \infty. \tag{7.1}$$

Proof. We have

$$\left| \frac{1}{n^3} \sum_{k=1}^{\lfloor nt \rfloor} U_{k-1}^2 - \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle^2 \frac{t^3}{3} \right| \leqslant \frac{1}{n^3} \sum_{k=1}^{\lfloor nt \rfloor} \left| U_{k-1}^2 - \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle^2 (k-1)^2 \right| + \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle^2 \left| \frac{1}{n^3} \sum_{k=1}^{\lfloor nt \rfloor} (k-1)^2 - \frac{t^3}{3} \right|,$$

where

$$\sup_{t\in[0,T]} \left| \frac{1}{n^3} \sum_{k=1}^{\lfloor nt \rfloor} (k-1)^2 - \frac{t^3}{3} \right| \to 0, \qquad \text{as } n \to \infty,$$

hence, in order to show (7.1), it suffices to prove

$$\frac{1}{n^3} \sum_{k=1}^{\lfloor nT \rfloor} \left| U_k^2 - \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle^2 k^2 \right| \stackrel{\mathbb{P}}{\longrightarrow} 0, \qquad \text{as } n \to \infty. \tag{7.2}$$

Recursion (4.1) yields $\mathbb{E}(U_k) = \mathbb{E}(U_{k-1}) + \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle$, $k \in \mathbb{N}$, with intital value $\mathbb{E}(U_0) = 0$, hence $\mathbb{E}(U_k) = \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle k$, $k \in \mathbb{N}$. For the sequence

$$\widetilde{U}_k := U_k - \mathbb{E}(U_k) = U_k - \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle k, \qquad k \in \mathbb{N}, \tag{7.3}$$

by (4.1), we get a recursion $\widetilde{U}_k = \widetilde{U}_{k-1} + \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{M}_k \rangle$, $k \in \mathbb{N}$, with intital value $\widetilde{U}_0 = 0$. Applying Doob's maximal inequality (see, e.g., Revuz and Yor [24, Chapter II, Theorem 1.7]) for the martingale $\widetilde{U}_n = \sum_{k=1}^n \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{M}_k \rangle$, $n \in \mathbb{N}$,

$$\mathbb{E}\left(\sup_{t\in[0,T]} \left| \sum_{k=1}^{\lfloor nt \rfloor} \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{M}_k \rangle \right|^2 \right) \leqslant 4\, \mathbb{E}\left(\left| \sum_{k=1}^{\lfloor nT \rfloor} \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{M}_k \rangle \right|^2 \right) = 4 \sum_{k=1}^{\lfloor nT \rfloor} \mathbb{E}\big(\langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{M}_k \rangle^2 \big) = \mathrm{O}(n),$$

where we applied Corollary B.5. Consequently,

$$n^{-1} \max_{k\in\{1,\ldots,\lfloor nT \rfloor\}} |U_k - \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle k| = n^{-1} \max_{k\in\{1,\ldots,\lfloor nT \rfloor\}} |\widetilde{U}_k| \stackrel{\mathbb{P}}{\longrightarrow} 0 \qquad \text{as } n \to \infty. \tag{7.4}$$

We have

$$|U_k^2 - k^2 \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle^2| \leqslant |U_k - k \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle|^2 + 2k \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle |U_k - k \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle|,$$

hence

$$n^{-2} \max_{k\in\{1,\ldots,\lfloor nT \rfloor\}} |U_k^2 - k^2 \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle^2| \leqslant \Big(n^{-1} \max_{k\in\{1,\ldots,\lfloor nT \rfloor\}} |U_k - k \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle| \Big)^2 + \frac{2\lfloor nT \rfloor}{n^2} \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle \max_{k\in\{1,\ldots,\lfloor nT \rfloor\}} |U_k - k \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle| \stackrel{\mathbb{P}}{\longrightarrow} 0,$$

as $n \to \infty$. Thus,

$$\frac{1}{n^3} \sum_{k=1}^{\lfloor nT \rfloor} \big| U_k^2 - k^2 \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle^2 \big| \leqslant \frac{\lfloor nT \rfloor}{n^3} \max_{k \in \{1, \dots, \lfloor nT \rfloor\}} \big| U_k^2 - k^2 \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle^2 \big| \xrightarrow{\mathbb{P}} 0,$$

as $n \to \infty$, thus we conclude (7.2), and hence (7.1). □

The second convergence in Theorem 4.3 follows from Lemma C.2, since assumption $\langle \widetilde{\boldsymbol{C}} \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{u}_{\mathrm{left}} \rangle = 0$ implies $\langle \widetilde{\boldsymbol{C}} \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}} \rangle = 0$ (see the beginning of the proof of Theorem 3.5). For the last convergence in Theorem 4.3 we need the following approximation.

7.2 Lemma. *Suppose that the assumptions of Theorem 3.1 hold. If* $\langle \widetilde{\boldsymbol{C}} \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{u}_{\mathrm{left}} \rangle = 0$, *then for each* $T > 0$,

$$\sup_{t \in [0,T]} \left| \frac{1}{n^2} \sum_{k=1}^{\lfloor nt \rfloor} U_k V_k - \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle \langle \boldsymbol{v}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle \frac{\widetilde{\delta} t^2}{2(1-\delta)} \right| \xrightarrow{\mathbb{P}} 0 \qquad \text{as } n \to \infty.$$

Proof. First we show, by the method of the proof of Lemma 7.1, convergence

$$\sup_{t \in [0,T]} \left| \frac{1}{n^2} \sum_{k=1}^{\lfloor nt \rfloor} U_k - \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle \frac{t^2}{2} \right| \xrightarrow{\mathbb{P}} 0 \qquad \text{as } n \to \infty \tag{7.5}$$

for each $T > 0$. We have

$$\left| \frac{1}{n^2} \sum_{k=1}^{\lfloor nt \rfloor} U_{k-1} - \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle \frac{t^2}{2} \right| \leqslant \frac{1}{n^2} \sum_{k=1}^{\lfloor nt \rfloor} \big| U_{k-1} - \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle (k-1) \big|$$
$$+ \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle \left| \frac{1}{n^2} \sum_{k=1}^{\lfloor nt \rfloor} (k-1) - \frac{t^2}{2} \right|,$$

where

$$\sup_{t \in [0,T]} \left| \frac{1}{n^2} \sum_{k=1}^{\lfloor nt \rfloor} (k-1) - \frac{t^2}{2} \right| \to 0, \qquad \text{as } n \to \infty,$$

hence, in order to show (7.5), it suffices to prove

$$\frac{1}{n^2} \sum_{k=1}^{\lfloor nT \rfloor} \big| U_k - \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle k \big| \xrightarrow{\mathbb{P}} 0, \qquad \text{as } n \to \infty. \tag{7.6}$$

Using (7.4), we obtain

$$\frac{1}{n^2} \sum_{k=1}^{\lfloor nT \rfloor} \big| U_k - \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle k \big| \leqslant \frac{\lfloor nT \rfloor}{n^2} \max_{k \in \{1, \dots, \lfloor nT \rfloor\}} \big| U_k - \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle k \big| \xrightarrow{\mathbb{P}} 0,$$

as $n \to \infty$, thus we conclude (7.6), and hence (7.5).

In order to prove the statement of the lemma, we derive a decomposition of $\sum_{k=1}^{\lfloor nt\rfloor} U_k V_k$. Using recursions (4.1) and (4.3), we obtain

$$\begin{aligned}\mathbb{E}(U_k V_k \,|\, \mathcal{F}_{k-1}) &= \mathbb{E}\Big[\big(U_{k-1} + \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{M}_k + \widetilde{\boldsymbol{\beta}}\rangle\big)\big(\delta V_{k-1} + \langle \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{M}_k + \widetilde{\delta}\,\widetilde{\boldsymbol{\beta}}\rangle\big) \,\Big|\, \mathcal{F}_{k-1}\Big] \\ &= \delta U_{k-1}V_{k-1} + \widetilde{\delta}\langle \boldsymbol{v}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}}\rangle U_{k-1} + \delta\langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}}\rangle V_{k-1} + \widetilde{\delta}\langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}}\rangle\langle \boldsymbol{v}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}}\rangle + \langle \boldsymbol{V}_0\boldsymbol{u}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}}\rangle,\end{aligned}$$

since, by $\langle \widetilde{\boldsymbol{C}}\boldsymbol{u}_{\mathrm{left}}, \boldsymbol{u}_{\mathrm{left}}\rangle = 0$ and $\widetilde{\boldsymbol{C}} = (\boldsymbol{V}_1 + \boldsymbol{V}_2)/2$, we conclude $\langle \boldsymbol{V}_i\boldsymbol{u}_{\mathrm{left}}, \boldsymbol{u}_{\mathrm{left}}\rangle = 0$, $i \in \{1, 2\}$, thus by (5.14),

$$\mathbb{E}(\langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{M}_k\rangle\langle \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{M}_k\rangle \,|\, \mathcal{F}_{k-1}) = \boldsymbol{u}_{\mathrm{left}}^\top\, \mathbb{E}(\boldsymbol{M}_k\boldsymbol{M}_k^\top \,|\, \mathcal{F}_{k-1})\boldsymbol{v}_{\mathrm{left}} = \langle \boldsymbol{V}_0\boldsymbol{u}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}}\rangle.$$

Consequently,

$$\begin{aligned}\sum_{k=1}^{\lfloor nt\rfloor} U_k V_k &= \sum_{k=1}^{\lfloor nt\rfloor}\big[U_kV_k - \mathbb{E}(U_kV_k \,|\, \mathcal{F}_{k-1})\big] + \sum_{k=1}^{\lfloor nt\rfloor}\mathbb{E}(U_kV_k \,|\, \mathcal{F}_{k-1}) \\ &= \sum_{k=1}^{\lfloor nt\rfloor}\big[U_kV_k - \mathbb{E}(U_kV_k \,|\, \mathcal{F}_{k-1})\big] + \delta\sum_{k=1}^{\lfloor nt\rfloor} U_{k-1}V_{k-1} + \widetilde{\delta}\langle \boldsymbol{v}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}}\rangle\sum_{k=1}^{\lfloor nt\rfloor} U_{k-1} \\ &\quad + \delta\langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}}\rangle\sum_{k=1}^{\lfloor nt\rfloor} V_{k-1} + \widetilde{\delta}\langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}}\rangle\langle \boldsymbol{v}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}}\rangle\lfloor nt\rfloor + \langle \boldsymbol{V}_0\boldsymbol{u}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}}\rangle\lfloor nt\rfloor,\end{aligned}$$

and we obtain

$$\begin{aligned}\sum_{k=1}^{\lfloor nt\rfloor} U_k V_k &= \frac{1}{1-\delta}\sum_{k=1}^{\lfloor nt\rfloor}\big[U_kV_k - \mathbb{E}(U_kV_k \,|\, \mathcal{F}_{k-1})\big] - \frac{\delta}{1-\delta}U_{\lfloor nt\rfloor}V_{\lfloor nt\rfloor} + \frac{\widetilde{\delta}\langle \boldsymbol{v}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}}\rangle}{1-\delta}\sum_{k=1}^{\lfloor nt\rfloor} U_{k-1} \\ &\quad + \frac{\delta\langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}}\rangle}{1-\delta}\sum_{k=1}^{\lfloor nt\rfloor} V_{k-1} + \frac{\widetilde{\delta}\langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}}\rangle\langle \boldsymbol{v}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}}\rangle + \langle \boldsymbol{V}_0\boldsymbol{u}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}}\rangle}{1-\delta}\lfloor nt\rfloor.\end{aligned}$$

Using (B.8) with $(\ell, i, j) = (4, 1, 1)$ we obtain

$$n^{-2}\sup_{t\in[0,T]}\left|\sum_{k=1}^{\lfloor nt\rfloor}\big[U_kV_k - \mathbb{E}(U_kV_k \,|\, \mathcal{F}_{k-1})\big]\right| \stackrel{\mathbb{P}}{\longrightarrow} 0 \qquad \text{as } n \to \infty.$$

Using (B.7) with $(\ell, i, j) = (3, 1, 1)$ we obtain $n^{-2}\sup_{t\in[0,T]}|U_{\lfloor nt\rfloor}V_{\lfloor nt\rfloor}| \stackrel{\mathbb{P}}{\longrightarrow} 0$ as $n \to \infty$. The assumption $\langle \widetilde{\boldsymbol{C}}\boldsymbol{u}_{\mathrm{left}}, \boldsymbol{u}_{\mathrm{left}}\rangle = 0$ implies $\langle \widetilde{\boldsymbol{C}}\boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}}\rangle = 0$, hence, by Barczy et al. [8, formula (C.2)], we obtain

$$n^{-2}\sup_{t\in[0,T]}\left|\sum_{k=1}^{\lfloor nt\rfloor} V_{k-1}\right| \stackrel{\mathbb{P}}{\longrightarrow} 0.$$

Consequently,

$$n^{-2}\sup_{t\in[0,T]}\left|\sum_{k=1}^{\lfloor nt\rfloor} U_kV_k - \frac{\widetilde{\delta}\langle \boldsymbol{v}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}}\rangle}{1-\delta}\sum_{k=1}^{\lfloor nt\rfloor} U_{k-1}\right| \stackrel{\mathbb{P}}{\longrightarrow} 0$$

as $n \to \infty$. Using (B.7) with $(\ell, i, j) = (2, 1, 0)$ we obtain $n^{-2} \sup_{t \in [0,T]} U_{\lfloor nt \rfloor} \xrightarrow{\mathbb{P}} 0$ as $n \to \infty$. Thus, by (7.5), we conclude the statement of the lemma. □

The proof of Theorem 4.3 is similar to the proof of Theorems 4.1 and 4.2. Consider the sequence of stochastic processes

$$\boldsymbol{\mathcal{Z}}_t^{(n)} := \begin{bmatrix} \mathcal{N}_t^{(n)} \\ \mathcal{P}_t^{(n)} \end{bmatrix} := \sum_{k=1}^{\lfloor nt \rfloor} \boldsymbol{Z}_k^{(n)} \qquad \text{with} \qquad \boldsymbol{Z}_k^{(n)} := \begin{bmatrix} n^{-3/2} \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{M}_k \rangle U_{k-1} \\ n^{-1/2} \langle \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{M}_k \rangle V_{k-1} \end{bmatrix}$$

for $t \in \mathbb{R}_+$ and $k, n \in \mathbb{N}$. Theorem 4.3 follows from Lemmas 7.1 and C.2, and the following theorem.

7.3 Theorem. *If* $\langle \widetilde{\boldsymbol{C}} \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{u}_{\mathrm{left}} \rangle = 0$ *then*

$$\boldsymbol{\mathcal{Z}}^{(n)} \xrightarrow{\mathcal{D}} \boldsymbol{\mathcal{Z}}, \qquad \text{as } n \to \infty, \tag{7.7}$$

where the process $(\boldsymbol{\mathcal{Z}}_t)_{t \in \mathbb{R}_+}$ *with values in* $\mathbb{R}^2$ *is the pathwise unique strong solution of the SDE*

$$\mathrm{d}\boldsymbol{\mathcal{Z}}_t = \gamma(t)\, \mathrm{d}\widetilde{\boldsymbol{\mathcal{W}}}_t, \qquad t \in \mathbb{R}_+, \tag{7.8}$$

with initial value $\boldsymbol{\mathcal{Z}}_0 = \boldsymbol{0}$, *where* $(\widetilde{\boldsymbol{\mathcal{W}}}_t)_{t \in \mathbb{R}_+}$ *is a 2-dimensional standard Wiener process, and* $\gamma : \mathbb{R}_+ \to \mathbb{R}^{2 \times 2}$ *is defined by*

$$\gamma(t) := \begin{bmatrix} \langle \boldsymbol{V}_0 \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{u}_{\mathrm{left}} \rangle \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle^2 \, t^2 & \frac{\widetilde{\delta} \langle \boldsymbol{V}_0 \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}} \rangle \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle \langle \boldsymbol{v}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle \, t}{1 - \delta} \\ \frac{\widetilde{\delta} \langle \boldsymbol{V}_0 \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}} \rangle \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle \langle \boldsymbol{v}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle \, t}{1 - \delta} & \langle \boldsymbol{V}_0 \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}} \rangle \, M \end{bmatrix}^{1/2}$$

for $t \in \mathbb{R}_+$.

The SDE (7.8) has a pathwise unique strong solution with initial value $\boldsymbol{\mathcal{Z}}_0 = \boldsymbol{0}$, for which we have

$$\boldsymbol{\mathcal{Z}}_t =: \begin{bmatrix} \mathcal{N}_t \\ \mathcal{P}_t \end{bmatrix} = \int_0^t \begin{bmatrix} \langle \boldsymbol{V}_0 \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{u}_{\mathrm{left}} \rangle \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle^2 \, s^2 & \frac{\widetilde{\delta} \langle \boldsymbol{V}_0 \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}} \rangle \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle \langle \boldsymbol{v}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle \, s}{1 - \delta} \\ \frac{\widetilde{\delta} \langle \boldsymbol{V}_0 \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}} \rangle \langle \boldsymbol{u}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle \langle \boldsymbol{v}_{\mathrm{left}}, \widetilde{\boldsymbol{\beta}} \rangle \, s}{1 - \delta} & \langle \boldsymbol{V}_0 \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}} \rangle \, M \end{bmatrix}^{1/2} \mathrm{d}\widetilde{\boldsymbol{\mathcal{W}}}_s$$

for $t \in \mathbb{R}_+$.

The proof of Theorem 7.3 can be found in Barczy et al. [8].

Appendices

A SDE for multi-type CBI processes

For handling $\boldsymbol{M}_k$, $k \in \mathbb{N}$, we need a representation of multi-type CBI processes as pathwise unique strong solutions of certain SDEs with jumps. In what follows we recall some notations and results from Barczy et al. [5].

Let $\mathcal{R} := \bigcup_{j=0}^d \mathcal{R}_j$, where $\mathcal{R}_j$, $j \in \{0, 1, \ldots, d\}$, are disjoint sets given by

$$\mathcal{R}_0 := \mathcal{U}_d \times \{(\mathbf{0}, 0)\}^d \subset \mathbb{R}_+^d \times (\mathbb{R}_+^d \times \mathbb{R}_+)^d,$$

and

$$\mathcal{R}_j := \{\mathbf{0}\} \times \mathcal{H}_{j,1} \times \cdots \times \mathcal{H}_{j,d} \subset \mathbb{R}_+^d \times (\mathbb{R}_+^d \times \mathbb{R}_+)^d, \qquad j \in \{1, \ldots, d\},$$

where

$$\mathcal{H}_{j,i} := \begin{cases} \mathcal{U}_d \times \mathcal{U}_1 & \text{if } i = j, \\ \{(\mathbf{0}, 0)\} & \text{if } i \neq j. \end{cases}$$

(Recall that $\mathcal{U}_1 = \mathbb{R}_{++}$.) Let m be the uniquely defined measure on $\mathcal{V} := \mathbb{R}_+^d \times (\mathbb{R}_+^d \times \mathbb{R}_+)^d$ such that $m(\mathcal{V} \setminus \mathcal{R}) = 0$ and its restrictions on $\mathcal{R}_j$, $j \in \{0, 1, \ldots, d\}$, are

$$m|_{\mathcal{R}_0}(\mathrm{d}\boldsymbol{r}) = \nu(\mathrm{d}\boldsymbol{r}), \qquad m|_{\mathcal{R}_j}(\mathrm{d}\boldsymbol{z}, \mathrm{d}u) = \mu_j(\mathrm{d}\boldsymbol{z})\,\mathrm{d}u, \quad j \in \{1, \ldots, d\}, \tag{A.1}$$

where we identify $\mathcal{R}_0$ with $\mathcal{U}_d$ and $\mathcal{R}_1, \ldots, \mathcal{R}_d$ with $\mathcal{U}_d \times \mathcal{U}_1$ in a natural way. Using again this identification, let $f : \mathbb{R}^d \times \mathcal{V} \to \mathbb{R}_+^d$, and $g : \mathbb{R}^d \times \mathcal{V} \to \mathbb{R}_+^d$, be defined by

$$f(\boldsymbol{x}, \boldsymbol{r}) := \begin{cases} \boldsymbol{z} \mathbb{1}_{\{\|\boldsymbol{z}\| < 1\}} \mathbb{1}_{\{u \leqslant x_j\}}, & \text{if } \boldsymbol{x} = (x_1, \ldots, x_d)^\top \in \mathbb{R}^d, \ \boldsymbol{r} = (\boldsymbol{z}, u) \in \mathcal{R}_j, \ j \in \{1, \ldots, d\}, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

$$g(\boldsymbol{x}, \boldsymbol{r}) := \begin{cases} \boldsymbol{r}, & \text{if } \boldsymbol{x} \in \mathbb{R}^d, \ \boldsymbol{r} \in \mathcal{R}_0, \\ \boldsymbol{z} \mathbb{1}_{\{\|\boldsymbol{z}\| \geqslant 1\}} \mathbb{1}_{\{u \leqslant x_j\}}, & \text{if } \boldsymbol{x} = (x_1, \ldots, x_d)^\top \in \mathbb{R}^d, \ \boldsymbol{r} = (\boldsymbol{z}, u) \in \mathcal{R}_j, \ j \in \{1, \ldots, d\}, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

Consider the disjoint decomposition $\mathcal{R} = \mathcal{V}_0 \cup \mathcal{V}_1$, where $\mathcal{V}_0 := \bigcup_{j=1}^d \mathcal{R}_{j,0}$ and $\mathcal{V}_1 := \mathcal{R}_0 \cup \left(\bigcup_{j=1}^d \mathcal{R}_{j,1}\right)$ are disjoint decompositions with $\mathcal{R}_{j,k} := \{\mathbf{0}\} \times \mathcal{H}_{j,1,k} \times \cdots \times \mathcal{H}_{j,d,k}$, $j \in \{1, \ldots, d\}$, $k \in \{0, 1\}$, and

$$\mathcal{H}_{j,i,k} := \begin{cases} \mathcal{U}_{d,k} \times \mathcal{U}_1 & \text{if } i = j, \\ \{(\mathbf{0}, 0)\} & \text{if } i \neq j, \end{cases} \qquad \mathcal{U}_{d,k} := \begin{cases} \{\boldsymbol{z} \in \mathcal{U}_d : \|\boldsymbol{z}\| < 1\} & \text{if } k = 0, \\ \{\boldsymbol{z} \in \mathcal{U}_d : \|\boldsymbol{z}\| \geqslant 1\} & \text{if } k = 1. \end{cases}$$

Note that $f(\boldsymbol{x}, \boldsymbol{r}) = \mathbf{0}$ if $\boldsymbol{r} \in \mathcal{V}_1$, $g(\boldsymbol{x}, \boldsymbol{r}) = \mathbf{0}$ if $\boldsymbol{r} \in \mathcal{V}_0$, hence $\boldsymbol{e}_i^\top f(\boldsymbol{x}, \boldsymbol{r}) g(\boldsymbol{x}, \boldsymbol{r}) \boldsymbol{e}_j = 0$ for all $(\boldsymbol{x}, \boldsymbol{r}) \in \mathbb{R}^d \times \mathcal{V}$ and $i, j \in \{1, \ldots, d\}$.

Consider the following objects:

(E1) a probability space $(\Omega, \mathcal{F}, \mathbb{P})$;

(E2) a d-dimensional standard Brownian motion $(\boldsymbol{W}_t)_{t \in \mathbb{R}_+}$;

(E3) a stationary Poisson point process p on $\mathcal{V}$ with characteristic measure m;

(E4) a random vector $\boldsymbol{\xi}$ with values in $\mathbb{R}_+^d$, independent of $\boldsymbol{W}$ and p.

A.1 Remark. Note that if objects (E1)–(E4) are given, then $\boldsymbol{\xi}$, $\boldsymbol{W}$ and p are automatically mutually independent according to Remark 3.4 in Barczy et al. [4]. For a short review on point measures and point processes needed for this paper, see, e.g., Barczy et al. [4, Section 2]. □

Provided that the objects (E1)–(E4) are given, let $(\mathcal{F}_t^{\boldsymbol{\xi},\boldsymbol{W},p})_{t\in\mathbb{R}_+}$ denote the augmented filtration generated by $\boldsymbol{\xi}$, $\boldsymbol{W}$ and p, see Barczy et al. [4].

Let us consider the d-dimensional SDE

$$\begin{aligned}\boldsymbol{X}_t &= \boldsymbol{X}_0 + \int_0^t (\boldsymbol{\beta} + \boldsymbol{D}\boldsymbol{X}_s)\,\mathrm{d}s + \sum_{i=1}^d \boldsymbol{e}_i \int_0^t \sqrt{2c_i X_{s,i}^+}\,\mathrm{d}W_{s,i} \\ &\quad + \int_0^t \int_{\mathcal{V}_0} f(\boldsymbol{X}_{s-}, \boldsymbol{r})\,\widetilde{N}(\mathrm{d}s, \mathrm{d}\boldsymbol{r}) + \int_0^t \int_{\mathcal{V}_1} g(\boldsymbol{X}_{s-}, \boldsymbol{r})\,N(\mathrm{d}s, \mathrm{d}\boldsymbol{r}), \qquad t \in \mathbb{R}_+,\end{aligned} \tag{A.2}$$

where $\boldsymbol{X}_t = (X_{t,1}, \ldots, X_{t,d})^\top$, $\boldsymbol{D} := (d_{i,j})_{i,j\in\{1,\ldots,d\}}$ given by

$$d_{i,j} := \widetilde{b}_{i,j} - \int_{\mathcal{U}_d} z_i \mathbb{1}_{\{\|\boldsymbol{z}\|\geqslant 1\}}\,\mu_j(\mathrm{d}\boldsymbol{z}),$$

$N(\mathrm{d}s, \mathrm{d}\boldsymbol{r})$ is the counting measure of p on $\mathbb{R}_{++}\times\mathcal{V}$, and $\widetilde{N}(\mathrm{d}s, \mathrm{d}\boldsymbol{r}) := N(\mathrm{d}s, \mathrm{d}\boldsymbol{r}) - \mathrm{d}s\,m(\mathrm{d}\boldsymbol{r})$.

A.2 Definition. *Suppose that the objects* (E1)–(E4) *are given. An* $\mathbb{R}_+^d$*-valued strong solution of the SDE* (A.2) *on* $(\Omega, \mathcal{F}, \mathbb{P})$ *and with respect to the standard Brownian motion* $\boldsymbol{W}$*, the stationary Poisson point process* p *and initial value* $\boldsymbol{\xi}$*, is an* $\mathbb{R}_+^d$*-valued* $(\mathcal{F}_t^{\boldsymbol{\xi},\boldsymbol{W},p})_{t\in\mathbb{R}_+}$*-adapted càdlàg process* $(\boldsymbol{X}_t)_{t\in\mathbb{R}_+}$ *such that* $\mathbb{P}(\boldsymbol{X}_0 = \boldsymbol{\xi}) = 1$,

$$\mathbb{P}\left(\int_0^t \int_{\mathcal{V}_0} \|f(\boldsymbol{X}_s, \boldsymbol{r})\|^2\,\mathrm{d}s\,m(\mathrm{d}\boldsymbol{r}) < \infty\right) = 1, \qquad \mathbb{P}\left(\int_0^t \int_{\mathcal{V}_1} \|g(\boldsymbol{X}_{s-}, \boldsymbol{r})\|\,N(\mathrm{d}s, \mathrm{d}\boldsymbol{r}) < \infty\right) = 1$$

for all $t \in \mathbb{R}_+$, *and equation* (A.2) *holds* $\mathbb{P}$*-a.s.*

Further, note that the integrals $\int_0^t (\boldsymbol{\beta} + \boldsymbol{D}\boldsymbol{X}_s)\,\mathrm{d}s$ and $\int_0^t \sqrt{2c_i X_{s,i}^+}\,\mathrm{d}W_{s,i}$, $i \in \{1, \ldots, d\}$, exist, since $\boldsymbol{X}$ is càdlàg. For the following result, see Theorem 4.6 in Barczy et al. [5].

A.3 Theorem. *Let* $(d, \boldsymbol{c}, \boldsymbol{\beta}, \boldsymbol{B}, \nu, \boldsymbol{\mu})$ *be a set of admissible parameters such that the moment conditions* (2.3) *hold with* $q = 1$. *Suppose that objects* (E1)–(E4) *are given. If* $\mathbb{E}(\|\boldsymbol{\xi}\|) < \infty$, *then there is a pathwise unique* $\mathbb{R}_+^d$*-valued strong solution to the SDE* (A.2) *with initial value* $\boldsymbol{\xi}$*, and the solution is a multi-type CBI process with parameters* $(d, \boldsymbol{c}, \boldsymbol{\beta}, \boldsymbol{B}, \nu, \boldsymbol{\mu})$.

We note that the SDE (A.2) can be written in other forms, see Barczy et al. [5, Section 5] for $d \in \{1, 2\}$ or (1.2) for $d = 2$.

Further, one can rewrite the SDE (A.2) in a form which does not contain integrals with respect to non-compensated Poisson random measures, and then one can perform a linear transformation in order to remove randomness from the drift as follows, see Lemma 4.1 in Barczy et al. [6]. This form is very useful for handling $\boldsymbol{M}_k$, $k \in \mathbb{N}$.

A.4 Lemma. *Let* $(d, \boldsymbol{c}, \boldsymbol{\beta}, \boldsymbol{B}, \nu, \boldsymbol{\mu})$ *be a set of admissible parameters such that the moment conditions* (2.3) *hold with* $q = 1$. *Suppose that objects* (E1)–(E4) *are given with* $\mathbb{E}(\|\boldsymbol{\xi}\|) < \infty$. *Let* $(\boldsymbol{X}_t)_{t\in\mathbb{R}_+}$ *be a pathwise unique* $\mathbb{R}_+^d$*-valued strong solution to the SDE* (A.2) *with initial value* $\boldsymbol{\xi}$. *Then*

$$\mathrm{e}^{-t\widetilde{\boldsymbol{B}}}\boldsymbol{X}_t = \boldsymbol{X}_0 + \int_0^t \mathrm{e}^{-s\widetilde{\boldsymbol{B}}}\widetilde{\boldsymbol{\beta}}\,\mathrm{d}s + \sum_{\ell=1}^d \int_0^t \mathrm{e}^{-s\widetilde{\boldsymbol{B}}}\boldsymbol{e}_\ell\sqrt{2c_\ell X_{s,\ell}}\,\mathrm{d}W_{s,\ell} + \int_0^t\int_{\mathcal{V}} \mathrm{e}^{-s\widetilde{\boldsymbol{B}}} h(\boldsymbol{X}_{s-}, \boldsymbol{r})\,\widetilde{N}(\mathrm{d}s, \mathrm{d}\boldsymbol{r})$$

for all $t \in \mathbb{R}_+$, *where the function* $h : \mathbb{R}^d \times \mathcal{V} \to \mathbb{R}^d$ *is defined by* $h := f + g$, *hence*

$$\boldsymbol{X}_t = \mathrm{e}^{(t-s)\widetilde{\boldsymbol{B}}}\boldsymbol{X}_s + \int_s^t \mathrm{e}^{(t-u)\widetilde{\boldsymbol{B}}}\widetilde{\boldsymbol{\beta}}\,\mathrm{d}u + \sum_{\ell=1}^d \int_s^t \mathrm{e}^{(t-u)\widetilde{\boldsymbol{B}}}\boldsymbol{e}_\ell\sqrt{2c_\ell X_{u,\ell}}\,\mathrm{d}W_{u,\ell}$$
$$+ \int_s^t\int_{\mathcal{V}} \mathrm{e}^{(t-u)\widetilde{\boldsymbol{B}}} h(\boldsymbol{X}_{u-}, \boldsymbol{r})\,\widetilde{N}(\mathrm{d}u, \mathrm{d}\boldsymbol{r})$$

for all $s, t \in \mathbb{R}_+$, *with* $s \leqslant t$. *Consequently,*

$$\boldsymbol{M}_k = \sum_{\ell=1}^d \int_{k-1}^k \mathrm{e}^{(k-u)\widetilde{\boldsymbol{B}}}\boldsymbol{e}_\ell\sqrt{2c_\ell X_{u,\ell}}\,\mathrm{d}W_{u,\ell} + \int_{k-1}^k\int_{\mathcal{V}} \mathrm{e}^{(k-u)\widetilde{\boldsymbol{B}}} h(\boldsymbol{X}_{u-}, \boldsymbol{r})\,\widetilde{N}(\mathrm{d}u, \mathrm{d}\boldsymbol{r})$$

for all $k \in \mathbb{N}$.

Proof. The last statement follows from (3.4), since $\int_{k-1}^k \mathrm{e}^{(k-u)\widetilde{\boldsymbol{B}}}\widetilde{\boldsymbol{\beta}}\,\mathrm{d}u = \int_0^1 \mathrm{e}^{(1-u)\widetilde{\boldsymbol{B}}}\widetilde{\boldsymbol{\beta}}\,\mathrm{d}u = \overline{\boldsymbol{\beta}}$. □

Note that the formulas for $(\boldsymbol{X}_t)_{t\in\mathbb{R}_+}$ and $(\boldsymbol{M}_k)_{k\in\mathbb{N}}$ in Lemma A.4 are generalizations of formulas (3.1) and (3.3) in Xu [25], the first displayed formula in the proof of Lemma 2.1 in Huang et al. [11], and formulas (1.5) and (1.7) in Li and Ma [21], respectively.

A.5 Lemma. *Let* $(\boldsymbol{X}_t)_{t\in\mathbb{R}_+}$ *be a 2-type CBI process with parameters* $(2, \boldsymbol{c}, \boldsymbol{\beta}, \boldsymbol{B}, \nu, \boldsymbol{\mu})$ *such that* $\boldsymbol{X}_0 = \boldsymbol{0}$, $\boldsymbol{\beta} \neq \boldsymbol{0}$ *or* $\nu \neq 0$, *and* (3.1) *holds with some* $\gamma \in \mathbb{R}$ *and* $\kappa \in \mathbb{R}_{++}$ *such that* $s = \gamma + \kappa = 0$ *(hence it is irreducible and critical). Suppose that the moment conditions* (2.3) *hold with* $q = 2$.

If, in addition, $\langle \widetilde{\boldsymbol{C}}\boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}}\rangle = 0$, *then* $\langle \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{M}_k\rangle \stackrel{\text{a.s.}}{=} \langle \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{\eta}_k\rangle$, $k \in \mathbb{N}$, *with*

$$\boldsymbol{\eta}_k := \int_{k-1}^k\int_{\mathcal{R}_0} \mathrm{e}^{(k-s)\widetilde{\boldsymbol{B}}}\boldsymbol{r}\,\widetilde{N}(\mathrm{d}s, \mathrm{d}\boldsymbol{r}), \qquad k \in \mathbb{N}.$$

If, in addition, $\langle \widetilde{\boldsymbol{C}}\boldsymbol{u}_{\mathrm{left}}, \boldsymbol{u}_{\mathrm{left}}\rangle = 0$, *then* $\langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{M}_k\rangle \stackrel{\text{a.s.}}{=} \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{\eta}_k\rangle$, $k \in \mathbb{N}$.

The sequence $(\boldsymbol{\eta}_k)_{k\in\mathbb{N}}$ *consists of independent and identically distributed random vectors.*

Proof. The assumption $\langle \widetilde{\boldsymbol{C}}\boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}}\rangle = 0$ implies $c_\ell = 0$ for each $\ell \in \{1, 2\}$ (see the beginning of the proof of Theorem 3.5), thus

$$\langle \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{M}_k\rangle = \int_{k-1}^k\int_{\mathcal{V}} \langle \boldsymbol{v}_{\mathrm{left}}, \mathrm{e}^{(k-s)\widetilde{\boldsymbol{B}}} h(\boldsymbol{X}_{s-}, \boldsymbol{r})\rangle\,\widetilde{N}(\mathrm{d}s, \mathrm{d}\boldsymbol{r}) = \langle \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{\eta}_k\rangle + \zeta_{k,1} + \zeta_{k,2}$$

with

$$\zeta_{k,j} := \int_{k-1}^k \int_{\mathcal{R}_j} \langle \boldsymbol{v}_{\mathrm{left}}, \mathrm{e}^{(k-s)\widetilde{\boldsymbol{B}}} \boldsymbol{z} \rangle \mathbb{1}_{\{u \leqslant X_{s-,j}\}} \, \widetilde{N}(\mathrm{d}s, \mathrm{d}\boldsymbol{r}), \qquad k \in \mathbb{N}, \quad j \in \{1, 2\}.$$

We have $\mathrm{e}^{(k-s)\widetilde{\boldsymbol{B}}^\top} \boldsymbol{v}_{\mathrm{left}} = \mathrm{e}^{(\gamma-\kappa)(k-s)} \boldsymbol{v}_{\mathrm{left}}$, since $\boldsymbol{v}_{\mathrm{left}}$ is a left eigenvector of $\mathrm{e}^{(k-s)\widetilde{\boldsymbol{B}}}$ belonging to the eigenvalue $\mathrm{e}^{(\gamma-\kappa)(k-s)}$, hence

$$\zeta_{k,j} = \int_{k-1}^k \int_{\mathcal{R}_j} \mathrm{e}^{(\gamma-\kappa)(k-s)} \langle \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{z} \rangle \mathbb{1}_{\{u \leqslant X_{s-,j}\}} \, \widetilde{N}(\mathrm{d}s, \mathrm{d}\boldsymbol{r}), \qquad k \in \mathbb{N}, \quad j \in \{1, 2\}.$$

We have $\zeta_{k,j} = I_{k,j} - I_{k-1,j}$, $k \in \mathbb{N}$, with $I_{t,j} := \int_0^t \int_{\mathcal{R}_j} \mathrm{e}^{(\gamma-\kappa)(k-s)} \langle \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{z} \rangle \mathbb{1}_{\{u \leqslant X_{s-,j}\}} \, \widetilde{N}(\mathrm{d}s, \mathrm{d}\boldsymbol{r})$, $t \in \mathbb{R}_+$. The process $(I_{t,j})_{t \in \mathbb{R}_+}$ is a martingale, since

$$\begin{aligned}
&\mathbb{E}\bigg(\int_{k-1}^k \int_{\mathcal{U}_2} \int_{\mathcal{U}_1} |\mathrm{e}^{(\gamma-\kappa)(k-s)} \langle \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{z} \rangle \mathbb{1}_{\{u \leqslant X_{s-,j}\}}|^2 \, \mathrm{d}s \, \mu_j(\mathrm{d}\boldsymbol{z}) \, \mathrm{d}u \bigg) \\
&\qquad = \int_{k-1}^k \mathrm{e}^{2(\gamma-\kappa)(k-s)} \, \mathbb{E}(X_{s,j}) \, \mathrm{d}s \int_{\mathcal{U}_2} |\langle \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{z} \rangle|^2 \, \mu_j(\mathrm{d}\boldsymbol{z}) \\
&\qquad \leqslant \|\boldsymbol{v}_{\mathrm{left}}\|^2 \int_{k-1}^k \mathrm{e}^{2(\gamma-\kappa)(k-s)} \, \mathbb{E}(X_{s,j}) \, \mathrm{d}s \int_{\mathcal{U}_2} \|\boldsymbol{z}\|^2 \, \mu_j(\mathrm{d}\boldsymbol{z}) < \infty,
\end{aligned}$$

see Ikeda and Watanabe [12, Chapter II, page 62], formula (2.11) in Barczy et al. [5] and moment condition (2.3) with $q = 2$. Consequently, for each $k \in \mathbb{N}$ and $j \in \{1, 2\}$, we have $\mathbb{E}(\zeta_{k,j}) = 0$.

Moreover, the assumption $\langle \widetilde{\boldsymbol{C}} \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}} \rangle = 0$ implies $\int_{\mathcal{U}_2} \langle \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{z} \rangle^2 \, \mu_\ell(\mathrm{d}\boldsymbol{z}) = 0$ for each $\ell \in \{1, 2\}$ (see the beginning of the proof of Theorem 3.5), thus

$$\begin{aligned}
\mathbb{E}(\zeta_{k,j}^2) &= \mathbb{E}\bigg(\int_{k-1}^k \int_{\mathcal{U}_2} \int_{\mathcal{U}_1} \mathrm{e}^{2(\gamma-\kappa)(k-s)} \langle \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{z} \rangle^2 \mathbb{1}_{\{u \leqslant X_{s-,j}\}} \, \mathrm{d}s \, \mu_j(\mathrm{d}\boldsymbol{z}) \, \mathrm{d}u \bigg) \\
&= \int_{k-1}^k \mathrm{e}^{2(\gamma-\kappa)(k-s)} \, \mathbb{E}(X_{s,j}) \, \mathrm{d}s \int_{\mathcal{U}_2} \langle \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{z} \rangle^2 \, \mu_j(\mathrm{d}\boldsymbol{z}) = 0
\end{aligned}$$

by Ikeda and Watanabe [12, Chapter II, Proposition 2.2]. Consequently, $\zeta_{k,j} \stackrel{\mathrm{a.s.}}{=} 0$, and we obtain $\langle \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{M}_k \rangle \stackrel{\mathrm{a.s.}}{=} \langle \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{\eta}_k \rangle$, $k \in \mathbb{N}$.

In a similar way, $\langle \widetilde{\boldsymbol{C}} \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{u}_{\mathrm{left}} \rangle = 0$ implies $\langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{M}_k \rangle \stackrel{\mathrm{a.s.}}{=} \langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{\eta}_k \rangle$, $k \in \mathbb{N}$.

The Poisson point process p admits independent increments, hence $\boldsymbol{\eta}_k$, $k \in \mathbb{N}$, are independent.

For each $k \in \mathbb{N}$, the Laplace transform of the random vector $\boldsymbol{\eta}_k$ has the form

$$\begin{aligned}
\mathbb{E}(\mathrm{e}^{-\langle \boldsymbol{\theta}, \boldsymbol{\eta}_k \rangle}) &= \exp\bigg\{ -\int_{k-1}^k \int_{\mathcal{U}_2} \Big(1 - \mathrm{e}^{-\langle \boldsymbol{\theta}, \mathrm{e}^{(k-s)\widetilde{\boldsymbol{B}}} \boldsymbol{r} \rangle} \Big) \, \mathrm{d}s \, \nu(\boldsymbol{r}) \bigg\} \\
&= \exp\bigg\{ -\int_0^1 \int_{\mathcal{U}_2} \Big(1 - \mathrm{e}^{-\langle \boldsymbol{\theta}, \mathrm{e}^{(1-u)\widetilde{\boldsymbol{B}}} \boldsymbol{r} \rangle} \Big) \, \mathrm{d}u \, \nu(\boldsymbol{r}) \bigg\} = \mathbb{E}(\mathrm{e}^{-\langle \boldsymbol{\theta}, \boldsymbol{\eta}_1 \rangle})
\end{aligned}$$

for all $\boldsymbol{\theta} \in \mathbb{R}_+^2$, see, i.e., Kyprianou [18, page 44], hence $\boldsymbol{\eta}_k$, $k \in \mathbb{N}$, are identically distributed. □

B On moments of multi-type CBI processes

In the proof of Theorem 3.1, good bounds for moments of the random vectors and variables $(\boldsymbol{M}_k)_{k\in\mathbb{Z}_+}$, $(\boldsymbol{X}_k)_{k\in\mathbb{Z}_+}$, $(U_k)_{k\in\mathbb{Z}_+}$ and $(V_k)_{k\in\mathbb{Z}_+}$ are extensively used. The following estimates are proved in Barczy and Pap [7, Lemmas B.2 and B.3].

B.1 Lemma. *Let* $(\boldsymbol{X}_t)_{t\in\mathbb{R}_+}$ *be a multi-type CBI process with parameters* $(d, \boldsymbol{c}, \boldsymbol{\beta}, \boldsymbol{B}, \nu, \boldsymbol{\mu})$ *such that* $\mathbb{E}(\|\boldsymbol{X}_0\|^q) < \infty$ *and the moment conditions* (2.3) *hold with some* $q \in \mathbb{N}$. *Suppose that* $(\boldsymbol{X}_t)_{t\in\mathbb{R}_+}$ *is irreducible and critical. Then*

$$\sup_{t\in\mathbb{R}_+} \frac{\mathbb{E}(\|\boldsymbol{X}_t\|^q)}{(1+t)^q} < \infty. \tag{B.1}$$

In particular, $\mathbb{E}(\|\boldsymbol{X}_t\|^q) = \mathrm{O}(t^q)$ *as* $t \to \infty$ *in the sense that* $\limsup_{t\to\infty} t^{-q}\,\mathbb{E}(\|\boldsymbol{X}_t\|^q) < \infty$.

B.2 Lemma. *Let* $(\boldsymbol{X}_t)_{t\in\mathbb{R}_+}$ *be a multi-type CBI process with parameters* $(d, \boldsymbol{c}, \boldsymbol{\beta}, \boldsymbol{B}, \nu, \boldsymbol{\mu})$ *such that* $\mathbb{E}(\|\boldsymbol{X}_0\|^q) < \infty$ *and the moment conditions* (2.3) *hold, where* $q = 2p$ *with some* $p \in \mathbb{N}$. *Suppose that* $(\boldsymbol{X}_t)_{t\in\mathbb{R}_+}$ *is irreducible and critical. Then, for the martingale differences* $\boldsymbol{M}_n = \boldsymbol{X}_n - \mathbb{E}(\boldsymbol{X}_n \,|\, \boldsymbol{X}_{n-1})$, $n \in \mathbb{N}$, *we have* $\mathbb{E}(\|\boldsymbol{M}_n\|^{2p}) = \mathrm{O}(n^p)$ *as* $n \to \infty$ *that is,* $\sup_{n\in\mathbb{N}} n^{-p}\,\mathbb{E}(\|\boldsymbol{M}_n\|^{2p}) < \infty$.

We have $\mathrm{Var}(\boldsymbol{M}_k \,|\, \mathcal{F}_{k-1}) = \mathrm{Var}(\boldsymbol{X}_k \,|\, \boldsymbol{X}_{k-1})$ and $\mathrm{Var}(\boldsymbol{X}_k \,|\, \boldsymbol{X}_{k-1} = \boldsymbol{x}) = \mathrm{Var}(\boldsymbol{X}_1 \,|\, \boldsymbol{X}_0 = \boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}_+^d$, since $(\boldsymbol{X}_t)_{t\in\mathbb{R}_+}$ is a time-homogeneous Markov process. Hence Lemma 4.4 in Barczy et al. [6] implies the following formula for $\mathrm{Var}(\boldsymbol{M}_k \,|\, \mathcal{F}_{k-1})$.

B.3 Proposition. *Let* $(\boldsymbol{X}_t)_{t\in\mathbb{R}_+}$ *be a multi-type CBI process with parameters* $(d, \boldsymbol{c}, \boldsymbol{\beta}, \boldsymbol{B}, \nu, \boldsymbol{\mu})$ *such that* $\mathbb{E}(\|\boldsymbol{X}_0\|^2) < \infty$ *and the moment conditions* (2.3) *hold with* $q = 2$. *Then for all* $k \in \mathbb{N}$, *we have*

$$\mathrm{Var}(\boldsymbol{M}_k \,|\, \mathcal{F}_{k-1}) = \sum_{i=1}^d (\boldsymbol{e}_i^\top \boldsymbol{X}_{k-1}) \boldsymbol{V}_i + \boldsymbol{V}_0,$$

where

$$\boldsymbol{V}_i := \sum_{\ell=1}^d \int_0^1 \langle \mathrm{e}^{(1-u)\widetilde{\boldsymbol{B}}} \boldsymbol{e}_i, \boldsymbol{e}_\ell \rangle \mathrm{e}^{u\widetilde{\boldsymbol{B}}} \boldsymbol{C}_\ell \mathrm{e}^{u\widetilde{\boldsymbol{B}}^\top} \,\mathrm{d}u, \qquad i \in \{1, \ldots, d\},$$

$$\boldsymbol{V}_0 := \int_0^1 \mathrm{e}^{u\widetilde{\boldsymbol{B}}} \left(\int_{\mathcal{U}_d} \boldsymbol{z}\boldsymbol{z}^\top \nu(\mathrm{d}\boldsymbol{z}) \right) \mathrm{e}^{u\widetilde{\boldsymbol{B}}^\top} \,\mathrm{d}u + \sum_{\ell=1}^d \int_0^1 \left(\int_0^{1-u} \langle \mathrm{e}^{v\widetilde{\boldsymbol{B}}} \widetilde{\boldsymbol{\beta}}, \boldsymbol{e}_\ell \rangle \,\mathrm{d}v \right) \mathrm{e}^{u\widetilde{\boldsymbol{B}}} \boldsymbol{C}_\ell \mathrm{e}^{u\widetilde{\boldsymbol{B}}^\top} \,\mathrm{d}u.$$

Note that $\boldsymbol{V}_0 = \mathrm{Var}(\boldsymbol{X}_1 \,|\, \boldsymbol{X}_0 = \boldsymbol{0})$.

B.4 Proposition. *Let* $(\boldsymbol{X}_t)_{t\in\mathbb{R}_+}$ *be a multi-type CBI process with parameters* $(d, \boldsymbol{c}, \boldsymbol{\beta}, \boldsymbol{B}, \nu, \boldsymbol{\mu})$ *such that* $\mathbb{E}(\|\boldsymbol{X}_0\|^q) < \infty$ *and the moment conditions* (2.3) *hold with some* $q \in \mathbb{N}$. *Then* *for all* $j \in \{1, \ldots, q\}$ *and* $i_1, \ldots i_j \in \{1, \ldots, d\}$, *there exists a polynomial* $P_{j,i_1,\ldots,i_j} : \mathbb{R}^d \to \mathbb{R}$ *having degree at most* $\lfloor j/2 \rfloor$, *such that*

$$\mathbb{E}\left(M_{k,i_1} \cdots M_{k,i_j} \,|\, \mathcal{F}_{k-1}\right) = P_{j,i_1,\ldots,i_j}(\boldsymbol{X}_{k-1}), \qquad k \in \mathbb{N}, \tag{B.2}$$

where $\boldsymbol{M}_k =: (M_{k,1}, \ldots, M_{k,d})^\top$. *The coefficients of the polynomial* $P_{j,i_1,\ldots,i_j}$ *depends on* d, $\boldsymbol{c}$, $\boldsymbol{\beta}$, $\boldsymbol{B}$, ν, μ_1, $\ldots$, μ_d.

The proof of Proposition B.4 can be found in Barczy et al. [8].

B.5 Corollary. *Let* $(\boldsymbol{X}_t)_{t\in\mathbb{R}_+}$ *be a 2-type CBI process with parameters* $(2, \boldsymbol{c}, \boldsymbol{\beta}, \boldsymbol{B}, \nu, \boldsymbol{\mu})$ *such that* $\boldsymbol{X}_0 = \boldsymbol{0}$, $\boldsymbol{\beta} \neq \boldsymbol{0}$ *or* $\nu \neq 0$, *and* (3.1) *holds with some* $\gamma \in \mathbb{R}$ *and* $\kappa \in \mathbb{R}_{++}$ *such that* $s = \gamma + \kappa = 0$ *(hence it is irreducible and critical). Suppose that the moment conditions* (2.3) *hold with some* $q \in \mathbb{N}$. *Then*

$$\mathbb{E}(\|\boldsymbol{X}_k\|^i) = \mathrm{O}(k^i), \qquad \mathbb{E}(\|\boldsymbol{M}_k\|^{2j}) = \mathrm{O}(k^j), \qquad \mathbb{E}(U_k^i) = \mathrm{O}(k^i), \qquad \mathbb{E}(V_k^{2j}) = \mathrm{O}(k^j)$$

for $i, j \in \mathbb{Z}_+$ *with* $i \leqslant q$ *and* $2j \leqslant q$.

If, in addition, $\langle \widetilde{\boldsymbol{C}} \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}} \rangle = 0$, *then*

$$\mathbb{E}(|\langle \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{M}_k \rangle|^i) = \mathrm{O}(1), \qquad \mathbb{E}(V_k^{2j}) = \mathrm{O}(1)$$

for $i, j \in \mathbb{Z}_+$ *with* $i \leqslant q$ *and* $2j \leqslant q$.

If, in addition, $\langle \widetilde{\boldsymbol{C}} \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{u}_{\mathrm{left}} \rangle = 0$, *then*

$$\mathbb{E}(|\langle \boldsymbol{u}_{\mathrm{left}}, \boldsymbol{M}_k \rangle|^i) = \mathrm{O}(1)$$

for $i \in \mathbb{Z}_+$ *with* $i \leqslant q$.

The proof of Corollary B.5 can be found in Barczy et al. [8].

B.6 Corollary. *Let* $(\boldsymbol{X}_t)_{t\in\mathbb{R}_+}$ *be a 2-type CBI process with parameters* $(2, \boldsymbol{c}, \boldsymbol{\beta}, \boldsymbol{B}, \nu, \boldsymbol{\mu})$ *such that* $\boldsymbol{X}_0 = \boldsymbol{0}$ $\boldsymbol{\beta} \neq \boldsymbol{0}$ *or* $\nu \neq 0$, *and* (3.1) *holds with some* $\gamma \in \mathbb{R}$ *and* $\kappa \in \mathbb{R}_{++}$ *such that* $s = \gamma + \kappa = 0$ *(hence it is irreducible and critical). Suppose that the moment conditions* (2.3) *hold with some* $\ell \in \mathbb{N}$. *Then*

(i) *for all* $i, j \in \mathbb{Z}_+$ *with* $\max\{i, j\} \leqslant \lfloor \ell/2 \rfloor$, *and for all* $\theta > i + \frac{j}{2} + 1$, *we have*

$$n^{-\theta} \sum_{k=1}^{n} |U_k^i V_k^j| \stackrel{\mathbb{P}}{\longrightarrow} 0 \qquad \text{as } n \to \infty, \tag{B.3}$$

(ii) *for all* $i, j \in \mathbb{Z}_+$ *with* $\max\{i, j\} \leqslant \ell$, *for all* $T > 0$, *and for all* $\theta > i + \frac{j}{2} + \frac{i+j}{\ell}$, *we have*

$$n^{-\theta} \sup_{t\in[0,T]} |U_{\lfloor nt \rfloor}^i V_{\lfloor nt \rfloor}^j| \stackrel{\mathbb{P}}{\longrightarrow} 0 \qquad \text{as } n \to \infty, \tag{B.4}$$

(iii) *for all* $i, j \in \mathbb{Z}_+$ *with* $\max\{i, j\} \leqslant \lfloor \ell/4 \rfloor$, *for all* $T > 0$, *and for all* $\theta > i + \frac{j}{2} + \frac{1}{2}$, *we have*

$$n^{-\theta} \sup_{t \in [0,T]} \left| \sum_{k=1}^{\lfloor nt \rfloor} [U_k^i V_k^j - \mathbb{E}(U_k^i V_k^j \mid \mathcal{F}_{k-1})] \right| \xrightarrow{\mathbb{P}} 0 \qquad \text{as } n \to \infty. \tag{B.5}$$

If, in addition, $\langle \widetilde{\boldsymbol{C}} \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}} \rangle = 0$, *then*

(iv) *for all* $i, j \in \mathbb{Z}_+$ *with* $\max\{i, j\} \leqslant \lfloor \ell/2 \rfloor$, *and for all* $\theta > i + 1$, *we have*

$$n^{-\theta} \sum_{k=1}^{n} |U_k^i V_k^j| \xrightarrow{\mathbb{P}} 0 \qquad \text{as } n \to \infty, \tag{B.6}$$

(v) *for all* $i, j \in \mathbb{Z}_+$ *with* $\max\{i, j\} \leqslant \ell$, *for all* $T > 0$, *and for all* $\theta > i + \frac{i+j}{\ell}$, *we have*

$$n^{-\theta} \sup_{t \in [0,T]} |U_{\lfloor nt \rfloor}^i V_{\lfloor nt \rfloor}^j| \xrightarrow{\mathbb{P}} 0 \qquad \text{as } n \to \infty, \tag{B.7}$$

(vi) *for all* $i, j \in \mathbb{Z}_+$ *with* $\max\{i, j\} \leqslant \lfloor \ell/4 \rfloor$, *for all* $T > 0$, *and for all* $\theta > i + \frac{1}{2}$, *we have*

$$n^{-\theta} \sup_{t \in [0,T]} \left| \sum_{k=1}^{\lfloor nt \rfloor} [U_k^i V_k^j - \mathbb{E}(U_k^i V_k^j \mid \mathcal{F}_{k-1})] \right| \xrightarrow{\mathbb{P}} 0 \qquad \text{as } n \to \infty, \tag{B.8}$$

(vii) *for all* $j \in \mathbb{Z}_+$ *with* $j \leqslant \lfloor \ell/2 \rfloor$, *for all* $T > 0$, *and for all* $\theta > \frac{1}{2}$, *we have*

$$n^{-\theta} \sup_{t \in [0,T]} \left| \sum_{k=1}^{\lfloor nt \rfloor} [V_k^j - \mathbb{E}(V_k^j \mid \mathcal{F}_{k-1})] \right| \xrightarrow{\mathbb{P}} 0 \qquad \text{as } n \to \infty. \tag{B.9}$$

The proof of Corollary B.6 can be found in Barczy et al. [8].

C CLS estimators

For the existence of CLS estimators we need the following approximations.

C.1 Lemma. *Suppose that the assumptions of Theorem 3.1 hold. For each* $T > 0$, *we have*

$$n^{-2} \sup_{t \in [0,T]} \left| \sum_{k=1}^{\lfloor nt \rfloor} V_k^2 - \frac{\langle \widetilde{\boldsymbol{C}}\, \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}} \rangle}{1 - \delta^2} \sum_{k=1}^{\lfloor nt \rfloor} U_{k-1} \right| \xrightarrow{\mathbb{P}} 0, \qquad \text{as } n \to \infty.$$

The proof of Lemma C.1 can be found in Barczy et al. [8].

C.2 Lemma. *Suppose that the assumptions of Theorem 3.1 hold. If* $\langle \widetilde{\boldsymbol{C}} \boldsymbol{v}_{\mathrm{left}}, \boldsymbol{v}_{\mathrm{left}} \rangle = 0$, *then for each* $T > 0$,

$$\sup_{t \in [0,T]} \left| \frac{1}{n} \sum_{k=1}^{\lfloor nt \rfloor} V_k^2 - Mt \right| \stackrel{\mathbb{P}}{\longrightarrow} 0 \qquad \text{as } n \to \infty,$$

where M *is defined in* (3.12).

Moreover, $M = 0$ *if and only if* $(\widetilde{\beta}_1 - \widetilde{\beta}_2)^2 + \int_{\mathcal{U}_2} (z_1 - z_2)^2 \, \nu(\mathrm{d}\boldsymbol{z}) = 0$, *which is equivalent to* $X_{k,1} \stackrel{\text{a.s.}}{=} X_{k,2}$ *for all* $k \in \mathbb{N}$.

The proof of Lemma C.2 can be found in Barczy et al. [8].

C.3 Lemma. *If* $(\boldsymbol{X}_t)_{t \in \mathbb{R}_+}$ *is a 2-type CBI process with parameters* $(2, \boldsymbol{c}, \boldsymbol{\beta}, \boldsymbol{B}, \nu, \boldsymbol{\mu})$ *such that* (3.1) *holds with some* $\gamma \in \mathbb{R}$ *and* $\kappa \in \mathbb{R}_{++}$ *such that* $s = \gamma + \kappa = 0$ *(hence it is irreducible and critical),* $\mathbb{E}(\|\boldsymbol{X}_0\|) < \infty$, *and the moment conditions* (2.3) *hold with* $q = 1$, *then* $\mathbb{P}(H_n) \to 1$ *as* $n \to \infty$, *and hence, the probability of the existence of a unique CLS estimator* $\widehat{\varrho}_n$ *converges to 1 as* $n \to \infty$, *and this CLS estimator has the form given in* (3.6) *on the event* H_n.

If, in addition, $\|\boldsymbol{c}\|^2 + \sum_{i=1}^2 \int_{\mathcal{U}_2} (z_1 - z_2)^2 \, \mu_i(\mathrm{d}\boldsymbol{z}) > 0$ *or* $(\widetilde{\beta}_1 - \widetilde{\beta}_2)^2 + \int_{\mathcal{U}_2} (z_1 - z_2)^2 \, \nu(\mathrm{d}\boldsymbol{z}) > 0$, *then* $\mathbb{P}(\widetilde{H}_n) \to 1$ *as* $n \to \infty$, *and hence the probability of the existence of unique CLS estimator* $\widehat{\delta}_n$ *converges to 1 as* $n \to \infty$. *The CLS estimator* $\widehat{\delta}_n$ *has the form given in* (3.6) *on the event* $\widetilde{H}_n$.

The proof of Lemma C.3 can be found in Barczy et al. [8].

D A version of the continuous mapping theorem

The following version of continuous mapping theorem can be found for example in Kallenberg [17, Theorem 3.27].

D.1 Lemma. *Let* (S, d_S) *and* (T, d_T) *be metric spaces and* $(\xi_n)_{n \in \mathbb{N}}$, ξ *be random elements with values in* S *such that* $\xi_n \stackrel{\mathcal{D}}{\longrightarrow} \xi$ *as* $n \to \infty$. *Let* $f : S \to T$ *and* $f_n : S \to T$, $n \in \mathbb{N}$, *be measurable mappings and* $C \in \mathcal{B}(S)$ *such that* $\mathbb{P}(\xi \in C) = 1$ *and* $\lim_{n \to \infty} d_T(f_n(s_n), f(s)) = 0$ *if* $\lim_{n \to \infty} d_S(s_n, s) = 0$ *and* $s \in C$. *Then* $f_n(\xi_n) \stackrel{\mathcal{D}}{\longrightarrow} f(\xi)$ *as* $n \to \infty$.

For the case $S = \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ and $T = \mathbb{R}^q$ (or $T = \mathbb{D}(\mathbb{R}_+, \mathbb{R}^q)$), where d, $q \in \mathbb{N}$, we formulate a consequence of Lemma D.1.

For functions f and f_n, $n \in \mathbb{N}$, in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$, we write $f_n \stackrel{\mathrm{lu}}{\longrightarrow} f$ if $(f_n)_{n \in \mathbb{N}}$ converges to f locally uniformly, that is, if $\sup_{t \in [0,T]} \|f_n(t) - f(t)\| \to 0$ as $n \to \infty$ for all $T > 0$. For measurable mappings $\Phi : \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d) \to \mathbb{R}^q$ (or $\Phi : \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d) \to \mathbb{D}(\mathbb{R}_+, \mathbb{R}^q)$) and $\Phi_n : \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d) \to \mathbb{R}^q$ (or $\Phi_n : \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d) \to \mathbb{D}(\mathbb{R}_+, \mathbb{R}^q)$), $n \in \mathbb{N}$, we will denote by $C_{\Phi, (\Phi_n)_{n \in \mathbb{N}}}$

the set of all functions $f \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}^d)$ such that $\Phi_n(f_n) \to \Phi(f)$ (or $\Phi_n(f_n) \xrightarrow{\mathrm{lu}} \Phi(f)$) whenever $f_n \xrightarrow{\mathrm{lu}} f$ with $f_n \in \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$, $n \in \mathbb{N}$.

We will use the following version of the continuous mapping theorem several times, see, e.g., Barczy et al. [2, Lemma 4.2] and Ispány and Pap [14, Lemma 3.1].

D.2 Lemma. *Let $d, q \in \mathbb{N}$, and $(\boldsymbol{\mathcal{U}}_t)_{t\in\mathbb{R}_+}$ and $(\boldsymbol{\mathcal{U}}_t^{(n)})_{t\in\mathbb{R}_+}$, $n \in \mathbb{N}$, be $\mathbb{R}^d$-valued stochastic processes with càdlàg paths such that $\boldsymbol{\mathcal{U}}^{(n)} \xrightarrow{\mathcal{D}} \boldsymbol{\mathcal{U}}$. Let $\Phi : \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d) \to \mathbb{R}^q$ (or $\Phi : \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d) \to \mathbb{D}(\mathbb{R}_+, \mathbb{R}^q)$) and $\Phi_n : \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d) \to \mathbb{R}^q$ (or $\Phi_n : \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d) \to \mathbb{D}(\mathbb{R}_+, \mathbb{R}^q)$), $n \in \mathbb{N}$, be measurable mappings such that there exists $C \subset C_{\Phi,(\Phi_n)_{n\in\mathbb{N}}}$ with $C \in \mathcal{D}_\infty(\mathbb{R}_+, \mathbb{R}^d)$ and $\mathbb{P}(\boldsymbol{\mathcal{U}} \in C) = 1$. Then $\Phi_n(\boldsymbol{\mathcal{U}}^{(n)}) \xrightarrow{\mathcal{D}} \Phi(\boldsymbol{\mathcal{U}})$.*

In order to apply Lemma D.2, we will use the following statement several times, see Barczy et al. [3, Lemma B.3].

D.3 Lemma. *Let $d, p, q \in \mathbb{N}$, $h : \mathbb{R}^d \to \mathbb{R}^q$ be a continuous function and $K : [0, 1] \times \mathbb{R}^{2d} \to \mathbb{R}^p$ be a function such that for all $R > 0$ there exists $C_R > 0$ such that*

$$\|K(s, x) - K(t, y)\| \leqslant C_R (|t - s| + \|x - y\|) \tag{D.1}$$

for all $s, t \in [0, 1]$ and $x, y \in \mathbb{R}^{2d}$ with $\|x\| \leqslant R$ and $\|y\| \leqslant R$. Moreover, let us define the mappings $\Phi, \Phi_n : \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d) \to \mathbb{R}^{q+p}$, $n \in \mathbb{N}$, by

$$\Phi_n(f) := \left(h(f(1)), \frac{1}{n} \sum_{k=1}^{n} K\left(\frac{k}{n}, f\left(\frac{k}{n}\right), f\left(\frac{k-1}{n}\right) \right) \right),$$

$$\Phi(f) := \left(h(f(1)), \int_0^1 K(u, f(u), f(u)) \, \mathrm{d}u \right)$$

for all $f \in \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$. Then the mappings Φ and Φ_n, $n \in \mathbb{N}$, are measurable, and $C_{\Phi,(\Phi_n)_{n\in\mathbb{N}}} = \mathbb{C}(\mathbb{R}_+, \mathbb{R}^d) \in \mathcal{D}_\infty(\mathbb{R}_+, \mathbb{R}^d)$.

E Convergence of random step processes

We recall a result about convergence of random step processes towards a diffusion process, see Ispány and Pap [14]. This result is used for the proof of convergence (5.1).

E.1 Theorem. *Let $\boldsymbol{\gamma} : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d\times r}$ be a continuous function. Assume that uniqueness in the sense of probability law holds for the SDE*

$$\mathrm{d}\boldsymbol{\mathcal{U}}_t = \gamma(t, \boldsymbol{\mathcal{U}}_t) \, \mathrm{d}\boldsymbol{\mathcal{W}}_t, \qquad t \in \mathbb{R}_+, \tag{E.1}$$

with initial value $\boldsymbol{\mathcal{U}}_0 = \boldsymbol{u}_0$ for all $\boldsymbol{u}_0 \in \mathbb{R}^d$, where $(\boldsymbol{\mathcal{W}}_t)_{t\in\mathbb{R}_+}$ is an r-dimensional standard Wiener process. Let $(\boldsymbol{\mathcal{U}}_t)_{t\in\mathbb{R}_+}$ be a solution of (E.1) with initial value $\boldsymbol{\mathcal{U}}_0 = \boldsymbol{0} \in \mathbb{R}^d$.

For each $n \in \mathbb{N}$, *let* $(\boldsymbol{U}_k^{(n)})_{k\in\mathbb{N}}$ *be a sequence of* d*-dimensional martingale differences with respect to a filtration* $(\mathcal{F}_k^{(n)})_{k\in\mathbb{Z}_+}$, *that is,* $\mathbb{E}(\boldsymbol{U}_k^{(n)} \,|\, \mathcal{F}_{k-1}^{(n)}) = 0$, $n \in \mathbb{N}$, $k \in \mathbb{N}$. *Let*

$$\boldsymbol{\mathcal{U}}_t^{(n)} := \sum_{k=1}^{\lfloor nt \rfloor} \boldsymbol{U}_k^{(n)}, \qquad t \in \mathbb{R}_+, \quad n \in \mathbb{N}.$$

Suppose that $\mathbb{E}\left(\|\boldsymbol{U}_k^{(n)}\|^2\right) < \infty$ *for all* $n, k \in \mathbb{N}$. *Suppose that for each* $T > 0$,

(i) $\sup\limits_{t\in[0,T]} \left\| \sum\limits_{k=1}^{\lfloor nt \rfloor} \operatorname{Var}\big(\boldsymbol{U}_k^{(n)} \,|\, \mathcal{F}_{k-1}^{(n)}\big) - \int_0^t \boldsymbol{\gamma}(s, \boldsymbol{\mathcal{U}}_s^{(n)}) \boldsymbol{\gamma}(s, \boldsymbol{\mathcal{U}}_s^{(n)})^\top \mathrm{d}s \right\| \stackrel{\mathbb{P}}{\longrightarrow} 0,$

(ii) $\sum\limits_{k=1}^{\lfloor nT \rfloor} \mathbb{E}\left(\|\boldsymbol{U}_k^{(n)}\|^2 \mathbb{1}_{\{\|\boldsymbol{U}_k^{(n)}\| > \theta\}} \,\big|\, \mathcal{F}_{k-1}^{(n)}\right) \stackrel{\mathbb{P}}{\longrightarrow} 0$ *for all* $\theta > 0$,

where $\stackrel{\mathbb{P}}{\longrightarrow}$ *denotes convergence in probability. Then* $\boldsymbol{\mathcal{U}}^{(n)} \stackrel{\mathcal{D}}{\longrightarrow} \boldsymbol{\mathcal{U}}$ *as* $n \to \infty$.

Note that in (i) of Theorem E.1, $\|\cdot\|$ denotes a matrix norm, while in (ii) it denotes a vector norm.

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