The mean value property

Vilmos Totik

"...what mathematics really consists of
is problems and solutions." (Paul Halmos)

1 Some problem challenges

I like problems and completely agree with Paul Halmos that they are from the heart of mathematics ([7]). This one I heard when I was in high school.

Problem 1. *Show that if numbers in between* 0 *and* 1 *are written into the squares of the integer lattice on the plane in such a way that each number is the average of the four neighboring numbers, then all the numbers must be the same.*

Little did I know at that time that this problem has many things to do with random walks, the fundamental theorem of algebra, harmonic functions, the Dirichlet problem or with the shape of soap films.

A somewhat more difficult version is

Problem 2. *Prove the same if the numbers are assumed only to be nonnegative.*

Since 1949 every fall there is a unique mathematical contest in Hungary named after Miklós Schweitzer, a young mathematician perished during the siege of Budapest in 1945. It is for university students without age groups, and about 10-12 problems from various fields of mathematics are posted for 10 days during which the students can use any tools and literature they want.[1] I proposed the following continuous variant of Problem 1 for the 1983 competition ([11, p. 34]).

Problem 3. *Show that if a bounded continuous function on the plane has the property that its average over every circle of radius 1 equals its value at the center of the circle, then it is constant.*

When "boundedness" is replaced here by "one-sided boundedness", say positivity, the claim is still true, but the problem gets considerably tougher.

Problem 4. *The boundedness in Problem 3 can be replaced by positivity.*

[1] The problems and solutions up to 1991 can be found in he two volumes [3] and [11]

We shall solve these problems and discuss their various connections. Although the first problem follows from the second one, we shall first solve Problem 1 because its solution will guide us in the solution of the stronger statement.

It will be clear that there is nothing special about the plane, the claims are true in any dimension.

- *If nonnegative numbers are written into every box of the integer lattice in* $\mathbf{R}^d$ *in such a way that each number is the average of the* $2d$ *neighboring numbers, then all the numbers are the same.*
- *If a nonnegative continuous function in* $\mathbf{R}^d$ *has the property that its average over every sphere of radius 1 equals its value at the center of the sphere, then it is constant.*

It will also be clear that similar statements are true for other averages (like the one taken for the 9 touching squares instead of the 4 adjacent ones).

Label a square of the integer lattice by its lower-left vertex, and let $f(i,j)$ be the number we write into the (i,j) square. So Problem 1 asks for proving that if $f(i,j) \in [0,1]$ and for all i,j we have

$$f(i,j) = \frac{1}{4}\Big(f(i-1,j) + f(i+1,j) + f(i,j-1) + f(i,j+1)\Big), \tag{1}$$

then all $f(i,j)$ are the same. Note that some kind of limitations like boundedness or one-sided boundedness is needed, for, in general, functions with the property (1) need not be constant, consider e.g. $f(i,j) = i$. (1) is called the discrete mean value property for f. First we discuss some of its consequences.

2 The maximum principle

Assume that f satisfies (1) and it is nonnegative. Notice that if, say, f takes the value 0 at an (i,j), then it must be zero everywhere. Indeed, then (1) gives that $f(i-1,j)$, $f(i+1,j)$, $f(i,j-1)$ and $f(i,j+1)$ all must be also 0, i.e. all the neighboring values must be zero. Repeating this we can get that all values of f must be 0. The same argument works if f takes its largest value at some point, so we have

Theorem 1 (Minimum/maximum principle) *If a function with the discrete mean value property on the integer lattice attains somewhere its smallest/largest value, then it must be constant.*

In particular, this implies a solution to Problem 1 if we assume that f has a limit at infinity (i.e. $f(i,j) \to \alpha$ for some α as $i^2 + j^2 \to \infty$). Unfortunately, in Problem 1 we do not know in advance that the function has a limit at infinity, so this is not a solution.

Call a subset G of the squares of the integer lattice a region if every square in G can be reached from every other square of G by moving always inside G to neighboring cells. The boundary ∂G of G is the set of squares that are not in

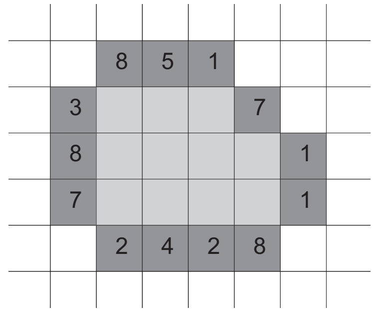

Figure 1:

G but which are neighboring to G. See Figure 1 for a typical bounded region, where the boundary consists of the darker shaded squares.

Suppose each boundary square contains a number like in Figure 1. Consider the number filling problem:

- Can the squares of G be filled in with numbers so that the discrete mean value property is true for all squares in G?
- In how many ways can such a filling be done?

This problem is called the discrete Dirichlet problem, we shall see its connection with the classical Dirichlet problem later.

The unicity of the solution is easy to get. Indeed, it is clear that the maximum/minimum principle holds (with the proof given above) also on finite regions:

Theorem 2 (Maximum principle) *Let f be a function with the discrete mean value property on a finite region, and let M be its largest value on $G \cup \partial G$. If f attains M somewhere in G, then f is the constant function.*

This gives that if a function with the mean value property is zero on the boundary, then it must be zero everywhere, and from here the unicity of the solution to the discrete Dirichlet problem follows (just take the difference of two possible solutions).

Perhaps the most natural approach to the existence part of the number filling problem is to consider the numbers to be filled in as unknowns, to write up a system of equations for them which describes the discrete mean value property and the boundary properties of f, and to solve that system. It can be readily shown that this linear system of equations is always solvable. But there is a better way to show existence that also works on unbounded regions.

3 Random walks

Consider a random walk on the squares of the integer lattice, which means that if at a moment we are in the lattice square (i, j), then we can move to any one of the neighboring squares $(i-1, j)$, $(i+1, j)$, $(i, j-1)$ or $(i, j+1)$. Which one we choose depends on some random event, like throw two fair coins, and if the result is "Head-Head" then move to $(i-1, j)$, if it is "Head-Tail" then move to $(i+1, j)$, etc.

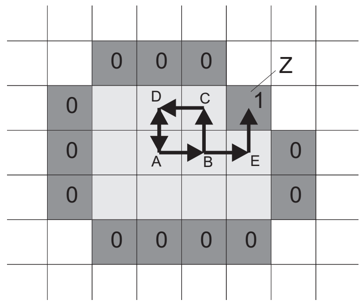

Figure 2: A sample random walk starting at $P = A$ and terminating at Z: ABCDADABEZ

We would like to find the unique value $f(P)$ of the square-filling problem at a point P of the domain G. Start a random walk from P which stops when it hits the boundary of G. Where it stops there is a prescribed number of the boundary, and since it is a random event which boundary point the walk hits first, that boundary number is also random. Now $f(P)$ is the expected value of that boundary number. Indeed, from P the walk moves to either of the four neighboring squares P_-, P_+, P^- and P^+ with probability 1/4–1/4, and then it continues as if it was started from there. So the just introduced expected value for P will be the average of the expected values for P_-, P_+, P^- and P^+. Hence the mean value property is satisfied.

Unfortunately, it is not easy to calculate the hitting probabilities and the aforementioned expected value, but the connection with the discrete mean value property is notable. Furthermore, the order can be reversed, and this connection can be used to calculate certain probabilities. Consider the following question.

Problem 5. *Two players, say* **H** *and* **T**, *where* **T** *is a dealer, repeatedly place 1–1 dollar on the table, flip a coin, and if it is Head, then* **H** *gets both notes, while if it is Tail, then* **T** *gets both of them. Suppose* **H** *starts with 30\$, and wants to know her chance of having 100\$ at some stage, when she quits the game.*

Direct calculation of the probability of success for **H** is non-trivial and rather tedious. However, using the connection between random walks and functions with the discrete mean value property we can easily show that the answer is

3/10. To this end, let $f(i)$ be the probability of success for **H** (i.e. reaching 100$) when she starts with i dollars. After the first play **H** will have either $i-1$ or $i+1$ dollars with probability 1/2–1/2 (therefore, the fortune of **H** makes a random walk on the integer lattice), and from there the play goes on as if **H** started with $i-1$ resp. $i+1$ dollars. Therefore, f satisfies the mean value property

$$f(i) = \frac{1}{2}(f(i-1) + f(i+1)), \quad i = 1, \ldots, 99, \quad f(0) = 0,\ f(100) = 1. \tag{2}$$

What we have shown in dimension 2 remains true without any change in other dimensions, in particular, the discrete Dirichlet problem (2) has one and only one solution. But $f(i) = i/100$ is clearly a solution to (2), so $f(30) = 30/100$ as was stated above. In general, if **H** starts with k dollars and her goal is to reach K dollars, then her chance of success is k/K.

For more on discrete random walks and their connection with electrical circuits see the wonderful monograph [6]. We shall return to random walks in Section 9.

4 An iteration process

The two solutions to the discrete dirichlet problems discussed so far (solving linear systems or using random walks) are not too practical. Now we discuss a fast and simple method for approximating the solution.

Let G be a bounded region and let f_0 be a given function on the boundary ∂G. We need to find a function f on $G \cup \partial G$ which agrees with f_0 on the boundary and satisfies the discrete mean value property in G. Define for any g given on $G \cup \partial G$ the function Tg the following way: for (i, j) in G let

$$Tg(i,j) = \frac{1}{4}\Big(g(i-1,j) + g(i+1,j) + g(i,j-1) + g(i,j+1)\Big), \tag{3}$$

while on the boundary set $Tg(Q) = g(Q)$. Note that we are looking for an f that satisfies $Tf = f$, so we are looking for a fixed point of the "operation" T. Fixed points are often found by iteration: let g_0 be arbitrary and form $Tg_0, T^2g_0, \ldots$. If this happens to converge, then the limit f is a fixed point.

To start the iteration, let g_0 be the function which agrees with f_0 on the boundary and which is 0 on G. Form $T^k g_0$, $k = 1, 2, \ldots$. It is a simple exercise to show that the iterates $T^k g_0$ converge (necessarily to the solution of the discrete Dirichlet problem), and the speed of convergence is geometrically fast.

5 Solution to Problem 1

Let f be a function on the lattice squares of the plain such that it has the discrete mean value property and its values lie in $[0, 1]$. We know from the maximum principle that if f assumes somewhere an extremal (largest or smallest) value, then f is constant.

The solution uses a similar idea, by considering the set $\mathcal{F}$ of all such functions and considering

$$\alpha = \sup_{f \in \mathcal{F}} (f(1,0) - f(0,0)). \tag{4}$$

Since the translation of any $f \in \mathcal{F}$ by any vector (i,j) is again in $\mathcal{F}$, and so is the rotation of f by 90 degrees, it immediately follows that α is actually the supremum of the differences of all possible values $f(P) - f(P')$ for neighboring squares P, P' and for $f \in \mathcal{F}$. Thus, Problem 1 amounts the same as showing $\alpha = 0$ (necessarily $\alpha \geq 0$).

First of all, note that there is an $f \in \mathcal{F}$ for which

$$\alpha = f(1,0) - f(0,0). \tag{5}$$

Indeed, by the definition of α, for every n there is an $f_n \in \mathcal{F}$ for which

$$f_n(1,0) - f_n(0,0) > \alpha - \frac{1}{n}.$$

By selecting repeatedly subsequences we get a subsequence f_{n_k} for which the sequences $\{f_{n_k}(i,j)\}_{k=1}^{\infty}$ converge for all (i,j). Now if

$$f(i,j) = \lim_{k \to \infty} f_{n_k}(i,j),$$

then clearly $f \in \mathcal{F}$ and (5) holds.

The function $g(i,j) = f(i+1,j) - f(i,j)$ has again the discrete mean value property, and, according to what was said before, we have $g(i,j) \leq \alpha$ and $g(0,0) = \alpha$. Thus, we get from the maximum principle that g is constant, and the constant then must be α. In particular, $f(i+1,0) - f(i,0) = g(i,0) = \alpha$ for all i. Adding these for $i = 0, 1, \ldots, m-1$ we obtain $f(m,0) - f(0,0) = m\alpha$, which is possible for large m only if $\alpha = 0$, since $f(m,0), f(0,0) \in [0,1]$. Hence, $\alpha = 0$, as was claimed.

6 Sketch of the solution to Problem 2

Let now f be a positive function on the lattice squares of the plain such that it has the discrete mean value property. Without loss of generality assume $f(0,0) = 1$, and let $\mathcal{G}$ be the family of all such f's. Then the positivity of f yields that $f(0,-1), f(0,1), f(1,0), f(-1,0) \leq 4$, and repeating this argument it follows that $0 < f(i,j) \leq 4^{|i|+|j|}$ for all $f \in \mathcal{G}$ and for all i,j. Hence, the selection process in the preceding section can be carried out without any change in the family $\mathcal{G}$. Now consider

$$\beta := \sup_{f \in \mathcal{G}} f(1,0) = \sup_{f \in \mathcal{G}} \frac{f(1,0)}{f(0,0)}. \tag{6}$$

As before, β turns out to be the supremum of the ratios $f(P')/f(P)$ for all neighboring squares P, P' and for all $f \in \mathcal{G}$, therefore Problem 2 asks for showing that $\beta = 1$ (clearly $\beta \geq 1$).

Let $f \in \mathcal{G}$ be a function for which equality is assumed in (6) (the existence of f follows from the selection process). Then

$$\beta = \frac{f(1,0)}{f(0,0)} = \frac{f(2,0) + f(1,1) + f(1,-1) + f(0,0)}{f(1,0) + f(0,1) + f(0,-1) + f(-1,0)}, \tag{7}$$

which can only be true if each one of the upper terms equals β times the term below it (since each term in the numerator is at most β times the term right below it). In particular, $f(2,0)/f(1,0) = \beta$ and $f(0,0)/f(-1,0) = \beta$. Repeat the previous argument to conclude that $f(k,0)/f(k-1,0) = \beta$ for all $k = 0, \pm 1, \pm 2, \ldots$. Thus, there is some constant $\gamma_0 > 0$ such that $f(k,0) = \gamma_0 \beta^k$ for all k. Since $f(1,1)/f(0,1) = \beta$ and $f(1,-1)/f(0,-1) = \beta$ are also true, it follows as before that $f(k, \pm 1) = \gamma_{\pm 1} \beta^k$ for all k with some $\gamma_{\pm 1} > 0$. Repeating again this argument we finally conclude that there are positive numbers γ_j such that $f(i,j) = \gamma_j \beta^i$ is true for all i, j.

Apply now the discrete mean value property:

$$\gamma_j = f(0,j) = \frac{1}{4}\Big(\gamma_j \frac{1}{\beta} + \gamma_j \beta + \gamma_{j-1} + \gamma_{j+1}\Big). \tag{8}$$

Since $\beta + 1/\beta \geq 2$, this implies $2\gamma_j \geq \gamma_{j-1} + \gamma_{j+1}$, i.e. the concavity of the sequence $\{\gamma_j\}_{-\infty}^{\infty}$. But a positive sequence on the integers can be concave only if it is constant. Thus, all the γ_j's are the same, and then (8) cannot be true if $\beta > 1$, hence $\beta = 1$ as claimed.

7 The continuous mean value property and harmonic functions

Assume that $G \subset \mathbf{R}^2$ is a domain (a connected open set) on the plane, and $f : G \to \mathbf{R}$ is a continuous real-valued function defined on G. We say that f has the mean value property in G if for every circle C which lies in G together with its interior we have

$$f(P) = \frac{1}{|C|} \int_C f, \tag{9}$$

where P is the center of C and $|C|$ denotes the length of C. (9) means that the average of f over the circle C coincides with the function value at the center of C.

Functions with this mean value property are called harmonic, and they play a fundamental role in mathematical analysis. For example, if f is the real part of a complex differentiable (so called analytic) function, then f is harmonic. The converse is also true in simply connected domains (domains without holes): if f has the mean value property (harmonic), then it is the real part of a differentiable complex function. So there is an abundance of harmonic functions, e.g. the real part of any polynomial is harmonic, say

$$f(x,y) = \Re z^n = x^n - \binom{n}{2} x^{n-2}y^2 + \binom{n}{4} x^{n-4}y^4 - \cdots$$

are all harmonic.

Although we shall not use it, we mention that the standard (but equivalent) definition of harmonicity is $f_{xx} + f_{yy} = 0$, where f_{xx} and f_{yy} denote the second partial derivatives of f with respect to x and y. We shall stay with our geometric definition.

Simple consequence of the mean value property is the maximum principle:

Theorem 3 (Maximum principle) *If a harmonic function on a domain G attains somewhere its largest value, then it must be constant.*

The reader can easily modify the argument given for Theorem 1 to verify this version.

A basic fact concerning harmonic functions is that a bounded harmonic function on the whole plane must be constant (Liouville's theorem). I heard the following proof from Paul Halmos. Suppose f satisfies (9) on the whole plane and it is bounded. Let $D_R(P)$ be the disk of radius R about some point P. Since the integral over $D_R(P)$ can be obtained by first integrating on circles C_r of radius r about P and then integrating these integrals with respect to r (from 0 to R), it easily follows that f also has the area-mean value property:

$$f(P) = \frac{1}{R^2\pi} \int_{D_R(P)} f. \tag{10}$$

If Q is another points, then the same formula holds for $f(Q)$ with $D_R(P)$ replaced by $D_R(Q)$. Now for very large R the disks $D_R(P)$ and $D_R(Q)$ are "almost the same" in the sense that outside their common part there are only two small regions in them the area of which is negligible compared to the area of the disks. So the averages of f over $D_R(P)$ and $D_R(Q)$ are practically the same (by the boundedness of f), and for $R \to \infty$ we get that in the limit the averages, and hence also the function values at P and Q are the same.

From here the fundamental theorem of algebra ("every polynomial has a zero on the complex plane") is a standard consequence (if the polynomial $\mathcal{P}$ did not vanish anywhere, then the real and imaginary parts of $1/\mathcal{P}$ would be bounded harmonic functions on the plane, hence they would be constant, which is not the case).

Note that Problem 3 claims more than Liouville's theorem, since in it the mean value property is requested only for circles with a fixed radius. In general, if we know the mean value property (9) for all circles of a fixed radius $C = C_{r_0}$, then it does not follow that f is harmonic. However, by a result of Jean Delsarte if (9) is true for all circles of radii equal to some r_0 or r_1 and r_0/r_1 does not lie in a finite exceptional set (consisting of the ratios of solutions of an equation involving a Bessel function), then f must be harmonic. In $\mathbf{R}^3$ this exceptional set is empty, and it is conjectured that it is empty in all $\mathbf{R}^d$, $d > 2$. See the most interesting paper [13], as well as the extended literature on Pompeiu's problem in [14]–[15].

8 The Dirichlet problem and soap films

The Dirichlet problem in the continuous case is the following: suppose Ω is a (bounded) domain with boundary $\partial\Omega$, and there is a continuous function g_0 given on the boundary $\partial\Omega$. Can this g_0 be (continuously) extended to G to a harmonic functions, i.e. we want to extend g_0 inside G so that it has the mean value property there.

There is a simple way to visualize the solution. Suppose that the boundary of the domain Ω is a simple closed curve γ with parametrization $\gamma(t) = (\gamma_1(t), \gamma_2(t))$. Consider the given function g_0 on $\partial\Omega = \gamma$, and with its help lift up γ into 3 dimensions: $\Gamma(t) := (\gamma_1(t), \gamma_2(t), g_0(\gamma(t)))$ is a 3 dimensional curve above γ. Now Γ can be thought of as a wire, and stretch an elastic rubber sheet (or a soap film) over Γ (see Figure 3). When in rest, the rubber sheet gives a surface over the domain Ω, which is the graph of a function f. Now it turns out that this f is automatically harmonic in Ω, and since it agrees with g_0 on the boundary (the wire is fixed), it solves the Dirichlet problem in Ω.

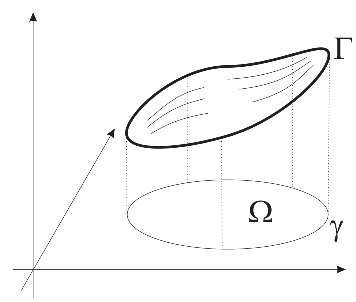

Figure 3: The plane curve γ, its lift-up Γ, and the soap film stretched to it

The unicity of the solution to the Dirichlet problem follows from the maximum principle in the same fashion that was done in the discrete case. The existence requires additional assumptions, for example if Ω is the punctured disk $\{z \mid 0 < |z| < 1\}$ and we set $g_0(z) = 0$ for $|z| = 1$ while $g_0(0) = 1$, then there is no harmonic function in Ω which continuously extends g_0. But this example is pathological (0 is an isolated point on the boundary), and it can be shown that in most cases the Dirichlet problem can be solved. In what follows we sketch how.

9 Discretization and Brownian motions

Is there a connection between the discrete and continuous Dirichlet problems discussed in Sections 2 and 8? There is indeed, and it is of practical importance.

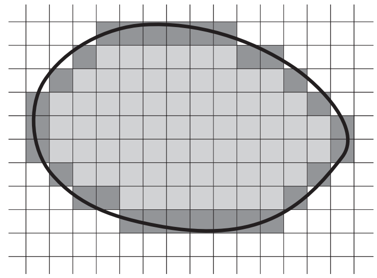

Figure 4: The domain Ω enclosed by the closed curve and the region G of squares lying inside Ω (with darker shaded boundary squares)

Let Ω be a domain as in the preceding section, and let g_0 be a continuous function on the boundary of Ω. Consider a square lattice on the plane with small mesh size, say consisting of $\tau \times \tau$ size squares with small τ. Form a region G_τ (see Figure 4) on this lattice by considering those squares in the lattice which lie in Ω (there may be a slight technical trouble that the union of these squares may not be connected, in that case let G_τ be the union of all squares that can be reached from a square containing a fixed point of Ω). We are going to consider the discrete Dirichlet problem on G_τ, the solution of which will be close to the solution of the original continuous Dirichlet problem. To this end define a boundary function on the boundary squares ∂G_τ in our lattice: if P is a boundary square, then P must intersect the boundary $\partial\Omega$, and if $z \in P \cap \partial\Omega$ is any point, then set $f_{0,\tau}(P) = g_0(z)$. Now solve this discrete Dirichlet problem on G_τ (with boundary numbers given by $f_{0,\tau}$) with the iteration technique of Section 4. Note that the iteration in Section 4 is computationally very simple and quite fast, since all one needs to do is to calculate averages of 4–4 numbers. Besides that, the convergence of the iterants to the solution is geometrically fast. Let f_τ be the solution, and we can imagine that f_τ gives us a function F_τ on the union of the squares belonging to G_τ: on every square P the value of this F_τ is identically equal to the number $f_\tau(P)$. Now if τ is small, then this function F_τ will be close on G_τ to the solution g of the continuous Dirichlet problem we are looking for.

There is yet another connection between the discrete and the continuous Dirichlet problems. We have seen in Section 3 that the discrete Dirichlet problem can be solved via random walks on the squares of the integer lattice. Now consider the just-introduced square lattice with small mesh size, and make a random walk on that lattice. If the mesh size is getting smaller then the lattice is getting denser (alternatively look at the square lattice from a far distance).

To compensate for having more and more squares, speed up the random walk. If this speeding-up is done properly, then in the limit we get a random motion on the plane, the Brownian motion. In a Brownian motion a particle moves in such a way that it continuously and randomly changes its direction.

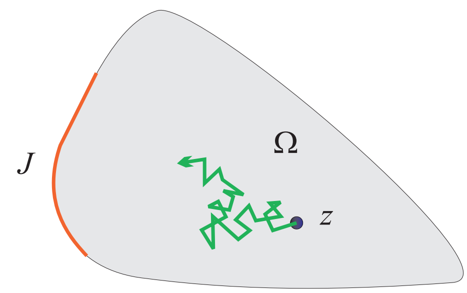

Figure 5: A Brownian motion

Let Ω have smooth boundary, and let J be an arc on that boundary, see Figure 5. Start a Brownian motion at a point $z \in \Omega$, and stop it when it hits the boundary of Ω, and let $f_J(z)$ be the probability that it hits the boundary in a point of J. This $f_J(z)$ (which is called the harmonic measure of z with respect to Ω and J) has the mean value property. Indeed, consider a circle C about the point z that lies inside Ω together with its interior. During the motion of the particle there is a fist time when the particle hits C at a point $Z \in C$. Then it continues as if it started in Z, and then the probability that it hits the boundary $\partial\Omega$ in a point of J is $f_J(Z)$. Because of the circular symmetry of C, all $Z \in C$ play equal roles, and we can conclude (at least heuristically), that the hitting probability $f_J(z)$ is the average of the hitting probabilities $f_J(Z)$, $Z \in C$, which is precisely the mean value property for f_J. It is also clear that if $z \in \Omega$ is close to a point Q on the boundary of Ω, then it is likely that the Brownian motion starting in z will hit the boundary $\partial\Omega$ close to Q. Therefore, if $Q \in J$ (except when Q is one of the endpoints of J) the probability $f_J(z)$ gets higher and higher, eventually converging to 1 as $z \to Q$, while in the case when $Q \notin J$, the probability $f_J(z)$ gets smaller and smaller, eventually converging to 0.

What we have shown is that f_J is a harmonic function in Ω which extends continuously to the boundary to 1 on the inner part of the arc J and it extends continuously to 0 on the outer part of J (therefore, at the endpoints f_J cannot have a continuous extension). In other words, not worrying about continuity at the endpoints of the arc J, we have solved the Dirichlet problem for the characteristic function

$$\chi_J(z) = \begin{cases} 1 & \text{if } z \in J \\ 0 & \text{if } z \notin J. \end{cases}$$

Now if f_0 is a continuous function on the boundary $\partial\Omega$, then to f_0 there is arbitrarily close a function of the form $h = \sum c_j \chi_{J_j}$ with a finite sum, and then $f_h := \sum c_j f_{J_j}(z)$ is a harmonic function in Ω which is close to f_0 on the boundary. Using the maximum principle it follows that, as $h \to f_0$, the functions f_h converge uniformly on $\Omega \cup \partial\Omega$ to a function f which has the mean value property in Ω (since all f_h had it) and which agrees with f_0 on $\partial\Omega$. Therefore, this f solves the Dirichlet problem.

We refer the interested reader to the book [10] for further reading concerning the mean value property and the Dirichlet problem. Robert Brown in "Brownian motion" was a Scottish botanist who, in 1827, observed in a microscope that pollen particles in suspension make an irregular, zigzag motion. The rigorous mathematical foundation of Brownian motion was made by Norbert Wiener in 1923 ([12]). The connection to the Dirichlet problem was first observed by Shizuo Kakutani [8]. This had a huge impact on further developments; there are many works that discuss the relation between random walks and problems (like the Dirichlet problem) in potential theory, see e.g. [9] or the very extensive [5].

10 Solution to Problem 4

In this proof we shall be brief, since some of the arguments have already been met before.

First of all, seemingly nothing prevents an f as in Problem 4 behave wildly, and first we "tame" these functions. Let $\mathcal{F}$ be the collection of all positive functions on the plane with the mean value property (9), and for some $\delta > 0$ let $\mathcal{F}_\delta$ be the collection of all the functions

$$f_\delta(z) = \frac{1}{\delta^2\pi}\int_{D_\delta(z)} f(u)du \tag{11}$$

for $f \in \mathcal{F}$, where $D_\delta(z)$ denotes the disk of radius δ about the point z. If we can show that

$$\beta_\delta := \sup_{g\in\mathcal{F}_\delta,\ z\in\mathbf{C},\ |\theta|=1} g(z+\theta)/g(z)$$

is 1, then we are done. Indeed, then $g(z+\theta) \le g(z)$ holds for all $g \in \mathcal{F}_\delta$, $z \in \mathbf{C}$ and any θ with $|\theta| = 1$, which actually implies $g(z+\theta) = g(z)$ (just apply the inequality to $z+\theta$ and to $-\theta$). Hence, since any two points on the plane can be connected by a polygonal line consisting of segments of length 1, every $g = f_\delta \in \mathcal{F}_\delta$ is constant, and then letting $\delta \to 0$ we get that every f in $\mathcal{F}$ is constant, as Problem 4 claims.

First we show that β_δ is finite. From the mean value property (9) we have for $f \in \mathcal{F}$

$$\begin{aligned} f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z+e^{it})dt &= \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2\pi}\int_0^{2\pi} f(z+e^{it}+e^{iu})dudt \\ &= \int_{D_2(0)} f(z+w)A(w)d|w|, \end{aligned} \tag{12}$$

where $D_2(0)$ is the disk of radius 2 about the origin, $d|w|$ denotes area-integral and $A(w)$ is a function on $D_2(0)$ that is continuous and positive for $0 < |w| < 1$. Thus, if S is the ring $1/2 \le |w| \le 3/2$ and $a > 0$ is the minimum of $A(w)$ on that ring, then

$$f(z) \ge a \int_{z+S} f.$$

Since for $|\theta| = 1$ and for $|z' - z| \le \delta \le 1/4$ the ring $z' + S$ contains the disk $D_\delta(z+\theta)$, it follows that

$$f(z') \ge a \int_{D_\delta(z+\theta)} f,$$

and upon taking the average for $z' \in D_\delta(z)$, the inequality $f_\delta(z) \ge (a\delta^2\pi) f_\delta(z+\theta)$ follows. Hence, β_δ is finite.

From the finiteness of β_δ it follows that if $R > 0$ is given and $|u| \le R$, then $a_R \le g(z+u)/g(z) \le A_R$ for all $g \in \mathcal{F}_\delta$ and all $z \in \mathbf{C}$, where the constants $a_R, A_R > 0$ depend only on R. Let $\mathcal{F}_{\delta\delta}$ be the collection of all g_δ with $g \in \mathcal{F}_\delta$. Then $\mathcal{F}_{\delta\delta} \subseteq \mathcal{F}_\delta$, and $\beta_\delta = 1$ follows from $\beta = 1$ (to be proven in a moment), where

$$\beta := \sup_{h \in \mathcal{F}_{\delta\delta},\ z \in \mathbf{C},\ |\theta|=1} h(z+\theta)/h(z).$$

Since $\mathcal{F}_{\delta\delta}$ is translation- and rotation-invariant, it is clear that

$$1 \le \beta = \sup_{h \in \mathcal{F}_{\delta\delta}, h(0)=1} h(1). \tag{13}$$

But the collection $h \in \mathcal{F}_{\delta\delta}$ with $h(0) = 1$ consists of functions that are uniformly bounded and uniformly equicontinuous on all disks $D_R(0)$, $R > 0$, hence from every sequence of such functions one can select a subsequence that converges uniformly on all the disks $D_R(0)$, $R > 0$,. Therefore, the supremum in (13) is attained, and there is an extremal function $h \in \mathcal{F}_{\delta\delta}$ with $h(1) = \beta h(0)$. Suppose that $h(z+1) = \beta h(z)$ holds for some z (we have just seen that $z = 0$ is such a value). From (12) it follows then that

$$\int_{D_2(0)} h(z+1+w)A(w)d|w| = \beta \int_{D_2(0)} h(z+w)A(w)d|w|,$$

and since here, by the definition of β, $h(z+1+w) \le \beta h(z+w)$ for all w, we can conclude that $h(z+1+w) = \beta h(z+w)$ must be true for all $|w| \le 2$. Thus, $h(1) = \beta h(0)$ implies $h(w+1) = \beta h(w)$ for all $|w| \le 2$, and repeated application of this step gives that $h(z+1) = \beta h(z)$ holds for all z.

Now let $\mathcal{F}'$ be the collection of all $h \in \mathcal{F}_{\delta\delta}$ that satisfies the just established functional equation $h(z+1) = \beta h(z)$, and let

$$\gamma := \sup_{f \in \mathcal{F}',\ z \in \mathbf{C}} h(z+i)/h(z).$$

Since $\mathcal{F}'$ is closed for translation and the operation $z \to \overline{z}$ (complex conjugation) taken in the argument, it follows that $\gamma \ge 1$, and the reasoning we just gave for

β yields that there is an extremal function $h' \in \mathcal{F}'$ such that $h'(i) = \gamma h'(0)$, and for this extremal function we have the functional equation $h'(z+i) = \gamma h'(z)$ for all $z \in \mathbf{C}$. Thus, h' satisfies both equations $h'(z+1) = \beta h'(z)$, $h'(z+i) = \gamma h'(z)$, from which it follows that if m is the minimum of h' on the unit square, then $h'(z) \geq m\beta^i\gamma^j$ at the integer lattice cell with lower left corner at (i, j).

Suppose now to the contrary that $\beta > 1$. Then the preceding estimate shows that $h'(z) \to \infty$ as the real part of z tends to infinity and the imaginary part stays nonnegative (then $i \to \infty$, $j \geq 0$). Thus, if $h''(z) = h'(z) + h'(\overline{z})$, then $h''(z) \to \infty$ as the real part of z tends to ∞, and then

$$h'''(z) = h''(z) + h''(zi) + h''(zi^2) + h''(zi^3)$$

is a function with the mean value property which tends to infinity as $z \to \infty$ (note that h' is positive, so h'', h''' are larger than any of the terms on the right of their definitions). But this contradicts the maximum/minimum principle, and that contradiction proves that, indeed, $\beta = 1$.

11 The Krein-Milman theorem

Although all the proofs we gave were elementary, one should be aware of a general principle about extremal points that lies behind these problems. Recall that in a linear space a point $P \in K$ is called an extremal point of a convex set K if P does not lie inside any segment joining two points of K.

A linear topological space is called locally convex if the origin has a neighborhood basis consisting of convex sets. For example, L^p-spaces are locally convex precisely for $p \geq 1$. Now a theorem of Mark Krein and David Milman says that if K is a compact convex set in a locally compact topological space, then K is the closure of the convex hull of its extremal points.

A point P is an extremal point for a convex set K precisely if it has the property that if P lies in the convex hull of a set $S \subset K$, then P must be one of the points of S. Now functions with a mean value property similar to those we considered in this article form a convex set K (in the locally convex topological space of continuous or discrete functions), and the mean value property itself means that each such function lies in the convex hull of some of its translates. Therefore, such a function can be an extremal point for K only if it agrees with all those translates, which means that it is constant. Now if the extremal points in K are constants, then so are all functions in K provided we can apply the Krein-Milman theorem. Hence, the crux of the matter is to prove that the additional boundedness or one-sided boundedness hypotheses set forth in our problems imply that K is compact; then the Krein-Milman theorem finishes the job. In our proofs we faced the same problem: we needed the existence of extremal functions in (4), (6), (13), for which we needed to prove some kind of compactness.

In conclusion we mentioned that the problems that have been discussed in this paper are special cases of the Choquet-Deny convolution equation first discussed by Gustave Choquet and Jaques Deny in 1960, which has applications in probability theory and far reaching generalizations in various groups/spaces. See [1], [4] and the extended list of references in [2].

Acknowledgement. Part of this work was presented in 2011 at the Fazekas Mihály Gimnázium, Budapest, Hungary as a lecture for high school students. The author is thankful to András Hraskó and János Pataki for their comments regarding the presentation. The preparation of this paper was partially supported by NSF DMS-1265375. The author also thanks the anonymous referee for her/his helpful suggestions.

References

[1] G. Choquet and J. Deny, Sur l´équation de convolution $\mu = \mu * \sigma$, *C.R. Acad. Sc. Paris*, **250**(1960), 779–801.

[2] C. Chu and T. Hilberdink, The convolution equation of Choquet and Deny on nilpotent groups, *Integr. Equat. Oper. Th.*, **26**(1996), 1–13.

[3] *Contests in Higher Mathematics, 1949-1961*, Akadémiai Kiadó, Budapest, 1968.

[4] J. Deny, Sur l´équation de convolution $\mu = \mu * \sigma$, *Sémin. Théor. Potentiel de M. Brelot*, Paris 1960.

[5] J. L. Doob, *Classical potential theory and its probabilistic counterpart*, Grundlehren der Mathematischen Wissenschaften, **262**, Springer-Verlag, New York, 1984.

[6] P. G. Doyle and J. L. Snell, *Random walks and electric networks*, Carus Mathematical Monographs, **22**. Mathematical Association of America, Washington, DC, 1984.

[7] P. Halmos, The heart of mathematics, *Amer. Math. Monthly*, **87**(1980), 519–524.

[8] S. Kakutani, Two-dimensional Brownian motion and harmonic functions, *Proc. Imp. Acad.*, **20**(1944), 706–714.

[9] S. C. Port and C. J. Stone, *Brownian motion and classical potential theory*, Probability and Mathematical Statistics, Academic Press, New York-London, 1978.

[10] T. Ransford, *Potential Theory in the Complex Plane*, Cambridge University Press, Cambridge, 1995

[11] G. Székely (editor), *Contests in Higher Mathematics*, Problem Books in Mathematics, Springer Verlag, New York, 1995.

[12] N. Wiener, Differential space, *Journal of Mathematical Physics*, **2**(1923) 131–174.

[13] L. Zalcman, Offbeat integral geometry, *Amer. Math. Monthly*, **87**(1980), 161–175.

[14] L. Zalcman, A Bibliographic Survey of the Pompeiu Problem, Approximation by Solutions of Partial Differential Equations (Hanstholm, 1991), *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, **365**, Kluwer Acad. Publ., Dordrecht, 1992, 185–194.

[15] L. Zalcman, Supplementary bibliography to: "A bibliographic survey of the Pompeiu problem" [in Approximation by solutions of partial differential equations (Hanstholm, 1991), 185–194, Kluwer Acad. Publ., Dordrecht, 1992], Radon transforms and tomography (South Hadley, MA, 2000), *Contemp. Math.*, **278**, Amer. Math. Soc., Providence, RI, 2001, 69–74.

Bolyai Institute
MTA-SZTE Analysis and Stochastics Research Group
University of Szeged
Szeged
Aradi v. tere 1, 6720, Hungary

totik@mail.usf.edu

and

Department of Mathematics and Statistics
University of South Florida
4202 E. Fowler Ave, CMC342
Tampa, FL 33620-5700, USA