On a conjecture of Widom[*]

Vilmos Totik[†] and Peter Yuditskii[‡]

*Dedicated to the memory of two outstanding mathematicians,
Andrei Aleksandrovich Gonchar and Herbert Stahl*

Abstract

In 1969 Harold Widom published his seminal paper [21] which gave a complete description of orthogonal and Chebyshev polynomials on a system of smooth Jordan curves. When there were Jordan arcs present the theory of orthogonal polynomials turned out to be just the same, but for Chebyshev polynomials Widom's approach proved only an upper estimate, which he conjectured to be the correct asymptotic behavior. In this note we make some clarifications which will show that the situation is more complicated.

1 Widom's problem for Chebyshev polynomials

This paper uses some basic facts from logarithmic potential theory, see the books [6], [7] or [15] for the concepts used.

Let E be the union of finitely many disjoint Jordan curves or arcs E_k that lie in the exterior of each other and that satisfy some smoothness condition. Recall that a Jordan curve is a homeomorphic image of the unit circle, while a Jordan arc is a homeomorphic image of a segment. The Chebyshev polynomial of degree n associated with E is the unique polynomial $T_n(z) = z^n + \cdots$ which minimizes the supremum norm

$$\|T_n\|_E = \sup_{z \in E} |T_n(z)|.$$

If the minimal norm is denoted by M_n, then it is known (see e.g. [15, Theorem 5.5.4]) that

$$M_n \geq \mathrm{C}(E)^n, \qquad n = 1, 2, \ldots, \tag{1}$$

where $\mathrm{C}(E)$ denotes logarithmic capacity, and $M_n^{1/n} \to \mathrm{C}(E)$ as $n \to \infty$ ([15, Corollary 5.5.5]). It is a delicate problem how close M_n can get to the theoretical

---

lower bound $C(E)^n$; the questions we are dealing with in this paper are also connected with that problem.

It is easy to see that if E is the unit circle then $M_n = 1 = C(E)^n$, and in general, Faber proved [5] that for a single Jordan curve $M_n \sim C(E)^n$, where $\sim$ means that the ratio of the two sides tends to 1. The original problem that was considered by Chebyshev was for $E = [-1, 1]$, in which case $M_n = 2 \cdot 2^{-n} = 2 \cdot C(E)^n$, twice $C(E)^n$.

If ρ is a nonnegative weight function on E then one can similarly define the weighted Chebyshev polynomials $T_{n,\rho}$ and the weighted Chebyshev numbers $M_{n,\rho}$. If we use the square of the $L^2(\rho)$-norm instead of $L^\infty(\rho)$ (i.e. consider minimizing

$$\int_E |Q_n(z)|^2 \rho(z) d|z|$$

instead of

$$\sup_E |Q_n(z)| \rho(z)),$$

then one obtains the quantities $m_{n,\rho}$ and the extremal polynomials $P_{n,\rho}(z) = z^n + \cdots$, and here $P_{n,\rho}$ turns out to be the n-th monic othogonal polynomial with respect to ρ and $P_n(z)/\sqrt{m_{n,\rho}}$ is the orthonormal one. With these notations (under suitable conditions on E and ρ) in the paper [21] Harold Widom gave a full description of the quantities $M_{n,\rho}$, $m_{n,\rho}$, and also determined the precise behavior of the Chebyshev and orthogonal polynomials themselves away from the set E, provided E consists of Jordan curves. The description was in turn of some associated Green and Neumann functions. When some of the components of E were Jordan arcs then he also proved the same behavior for the orthogonal polynomials $P_{n,\rho}$ and for $m_{n,\rho}$, but his theory was not complete in that case for the Chebyshev polynomials $T_{n,\rho}$ and the Chebyshev numbers $M_{n,\rho}$. He wrote in discussing the interval case: "Thus M_n is asymptotically twice as large for an interval as for a closed curve of the same capacity. We conjecture that this is true generally: that is, if at least one of the E_k is an arc then the asymptotic formula for $M_{n,\rho}$ given in Theorem 8.3 must be multiplied by 2. ... Unfortunately we cannot prove these statements and so they are nothing but conjectures". Widom himself showed that his conjecture is true if E lies on the real line, i.e. it consists of a finite number of intervals, and he also showed that the asymptotics for $M_{n,\rho}$ is *at most twice* as large as the asymptotics in the curve case.

In particular, for $\rho \equiv 1$ the conjecture would imply that if E has an arc-component, then

$$\liminf_{n \to \infty} \frac{M_n}{C(E)^n} \geq 2, \tag{2}$$

since (1) is true for any set E.

Widom's paper had a huge impact on the theory of extremal polynomials, in particularly on the theory of orthogonal polynomials. It is impossible to list all further contributions, for orientation see the papers [3]–[4], [8], [9]–[13]. See also [17] for a lower bound for the Chebyshev constants for sets on the real line, and the paper [16] for the various connections of the Chebyshev problem.

The aim of this note is to make some simple clarifications in connection with Widom's conjecture. Strictly speaking the conjecture is not true in the stated form, but we shall see that Widom was absolutely right that arcs change the asymptotics when compared to asymptotics on curves. Originally the authors had some ideas indicating that the situation was more complex than how Widom conjectured, but then they realized that more than what they wanted to say can be deduced rather easily from Widom's work itself, so this note follows the setup and the arguments in [21] very closely.

For the case when E consisted purely of Jordan curves the asymptotics of Widom was in the form

$$M_{n,\rho} \sim C(E)^n \mu(\rho, \Gamma_n) \tag{3}$$

with some rather explicitly given quantity $\mu(\rho, \Gamma_n)$; see the next section. In the general case when E may have curve and arc components the following holds.

Theorem 1 *If there is at least one Jordan curve in E, then*

$$\limsup_{n \to \infty} \frac{M_{n,\rho}}{C(E)^n \mu(\rho, \Gamma_n)} \leq \theta < 2, \tag{4}$$

where θ depends only on E.

This of course disproves (2), however (2) is partially true: it was proved in [19, Theorem 1] that if a general compact E contains an arc on its outer boundary, then there is a $\beta > 0$ such that

$$\liminf_{n \to \infty} \frac{M_n}{C(E)^n} \geq 1 + \beta.$$

In general one cannot say much more than that. In fact, it was proven by Thiran and Detaille [18, Section 5] that if E is a subarc on the unit circle of central angle 2α, then

$$T_n \sim C(E)^n 2 \cos^2 \alpha/4.$$

The factor $2 \cos^2 \alpha/4$ on the right is always smaller than 2 and is as close to 1 as one wishes if α is close to π.

As for asymptotics for $M_{n,\rho}$, we shall prove in Section 3 the following. Let ρ^* be equal to ρ on the curve components of E and equal to 2ρ on the arc components. With this (4) is a consequence of

$$\limsup_{n \to \infty} \frac{M_{n,\rho}}{C(E)^n \mu(\rho^*, \Gamma_n)} \leq 1, \tag{5}$$

to be proven in the next section (see (11)). The next theorem shows that this estimate is exact when the set is symmetric with respect to the real line.

Theorem 2 *If E consists of real intervals and of Jordan curves symmetric with respect to the real line, then*

$$M_{n,\rho} \sim C(E)^n \mu(\rho^*, \Gamma_n).$$

3

It is known that if $g(z, \infty)$ is Green's function of the outer component of E with pole at infinity, then g has $p-1$ critical points which we denote as $z_1^*, \ldots, z_{p-1}^*$. Furthermore, let ν_E denote the equilibrium measure of E, and let E_{arc} be the union of the arc-components of E.

Theorem 2 gives the following bound for the Chebyshev constants.

Corollary 3 *Under the conditions of Theorem 2 the limit points of the sequence $\{M_n/C(E)^n\}$ lie in the interval*

$$\left[2^{\nu_E(E_{\mathrm{arc}})}, 2^{\nu_E(E_{\mathrm{arc}})} \exp\left\{ \sum_{j=1}^{p-1} g(z_j^*) \right\} \right]. \tag{6}$$

Furthermore, if

$$\nu_E(E_1), \ldots, \nu_E(E_p)$$

are rationally independent, then the set of limit points is precisely the interval (6).

In particular, if $E = E_{\mathrm{arc}}$, i.e. E is a subset of the real line, then

$$\liminf_{n \to \infty} \frac{M_n}{C(E)^n} \geq 2,$$

as was proved by Widom, see also [17]. When E consists of Jordan curves we have $\nu_E(E_{\mathrm{arc}}) = 0$, and so the corresponding interval is

$$\left[1, \exp\left\{ \sum_{j=1}^{p-1} g(z_j^*) \right\} \right],$$

see [21, Theorem 8.4]. On the other hand, if E contains both Jordan curves and arcs then both endpoints of the closed interval in (6) lie in the open interval $(1, 2)$ (see also Theorem 1), and the asymptotic lower bound $2^{\nu_E(E_{\mathrm{arc}})}$ for $\{M_n/C(E)^n\}$ can be as close to 1 or 2 as we wish if $\nu_E(E_{\mathrm{arc}})$ is close to 0 or 1, respectively. This is the case for example, if E consists of the unit circle and of the interval $[2, 2+\beta]$, and $\beta > 0$ is small, respectively large.

In the next two sections we show how to prove Theorems 1 and 2 using the setup and reasonings of Widom's paper [21]. In the last section we give an explicit formula (22) for the asymptotics $M_{n,1}$ in the elliptic case, that is, in the case that the boundary of Ω consists of a real interval $[\alpha, \beta]$ and a symmetric curve. The ratio $M_{n,1}/C(E)^n$ behaves in n as an almost periodic (or periodic) function, depending on the modulus of the domain and on the harmonic measure of the interval evaluated at infinity (which is the same as the mass of the equilibrium measure carried by the interval). Let us mention that generally research in this direction was started in [1].

2 Widom's theory and Theorem 1

We shall need to briefly describe Widom's paper [21].

Let $E = \cup_{k=1}^{p} E_k$ be a finite family of Jordan curves and arcs lying exterior to one another. The smoothness assumptions on E we take the assumptions of Widom's paper [21], C^{2+} will certainly suffice. We shall denote by E_{arc} the union of the arc components of E. Let further ρ be a weight on E, of which we assume for simplicity that it is positive and satisfies a Lipshitz condition, i.e. it is of class C^{+}.

The weighted Chebyshev numbers with respect to ρ will be denoted by $M_{n,\rho}$, i.e.

$$M_{n,\rho} = \inf \|\rho(\zeta)(\zeta^n + \cdots)\|_E,$$

where $\|\cdot\|_E$ denotes the supremum norm on E and where the infimum is taken for all monic polynomial of degree n.

Ω denotes the outer domain, i.e. the unbounded component of $\overline{\mathbf{C}} \setminus E$, $g(z,w)$ is its Green's function with pole at w, and $\Phi(z,w) = \exp(g(z,w) + i\tilde{g}(z,w))$ is the so called complex Green's function with the normalization that $\Phi(\infty, w) > 0$ if w is finite and $\Phi(z,\infty)/z \to c > 0$ as $z \to \infty$ if $w = \infty$. Here $c = 1/C(E)$, the reciprocal of the logarithmic capacity $C(E)$ of E. The Φ is a multi-valued analytic function, but $|\Phi|$ is single-valued. Consider multi-valued bounded functions F in Ω for which $|F|$ is single-valued and which have only finitely many zeros in Ω. Each such F has boundary values on E almost everywhere. If we define

$$\gamma_k = \gamma_k(F) = \frac{1}{2\pi}\Delta_{E_k} \arg F, \qquad k = 1, \ldots, p,$$

where the total change $\Delta_{E_k} \arg F$ of the argument of F around E_k is taken on some positively oriented curve in Ω lying close to E_k, then $\Gamma(F) = (\gamma_1, \ldots, \gamma_p)$ is called the class of F. The class of Φ^{-n} is denoted by Γ_n, $n = 1, 2, \ldots$.

For a given class $\Gamma = (\gamma_1, \ldots, \gamma_p)$ let $\mu(\rho, \Gamma)$ be the minimum of the norms $\sup_E \rho|F|$, where F runs through all functions in the class Γ with the property that $F(\infty) = 1$. There is a unique extremal function minimizing this norm, and it is of the form (see [21, Theorem 5.4])

$$F_{\rho,\Gamma}(z) = \mu(\rho, \Gamma) R^{-1}(z) \prod \Phi(z, z_j)^{-1}, \tag{7}$$

where $z_1, \ldots z_q$ are some points in Ω, their number q is at most $p - 1$, and where $R(z) = R_\rho(z)$ is the outer function in Ω with boundary values ρ, i.e. R is the multi-valued analytic function in Ω which is positive at ∞, and for which $\log |R(z)|$ is single-valued and has boundary values ρ on E. The choice of the points $z_1, \ldots, z_q$ is such that F is of class Γ, and with them we have for the extremal constant the expression

$$\mu(\rho, \Gamma) = |R(\infty)| \exp\left\{ \sum_{j=1}^{q} g(z_j) \right\}. \tag{8}$$

We shall also need the harmonic measures $\omega_k(z)$ in Ω associated with E_k, i.e. $\omega_k(t)$ is the harmonic function in Ω which has boundary value 1 on E_k and 0 on the rest of E. If $\tilde{\omega}_k$ denotes its analytic conjugate with some normalization, then $\exp(\omega_k(z)+i\tilde{\omega}_k(z))$ is again a multi-valued analytic function in Ω for which its absolute value is single-valued. When doing asymptotics, it is desirable not to have the x_j's from (7) lie too close to E, and in that case Widom changed $F_{\rho,\Gamma}$ from (7) to (see [21, p. 215-216])

$$F_\Gamma(z) = F_{(\rho,\Gamma)} = \mu(\rho,\Gamma)V_{\Gamma'}^{-1}(\infty)R^{-1}(z)\prod_1 \Phi(z,z_j)^{-1}V_{\Gamma'}(z), \qquad (9)$$

where in $\prod_1$ those $\Phi(z,z_j)^{-1}$ are kept for which z_j are of distance $\geq \delta$ from E for some small fixed δ, and

$$V_{\Gamma'}(z) = \exp\left\{\sum_{k=1}^p \lambda_k(\omega_k(z) + i\tilde{\omega}_k(z))\right\}$$

with some appropriate λ_k that ensures that F_Γ is still of class Γ. This can be done for all Γ uniformly, and although this F_Γ is no longer extremal for $\mu(\rho,\Gamma)$, it is a nice smooth function on E (uniformly in Γ), and the norm $\sup_E \rho|F_\Gamma|$ is close to $\mu(\rho,\Gamma)$ (again uniformly in Γ). In fact, λ_k are small if $\delta > 0$ is small, and $\rho(\zeta)|F_\Gamma(\zeta)|/\mu(\rho,\Gamma)$ is uniformly as close to 1 as we wish if $\delta > 0$ is sufficiently small.

Define the weight ρ^* equal to 2ρ on E_{arc} and equal to ρ on the rest of E (i.e. on the curve components of E), and consider the functions just introduced, but for ρ^* rather than for ρ. In particular, consider $F_{(\rho^*,\Gamma)}$ from (9) with ρ replaced by ρ^*. In what follows let Γ_n be the class of $\Phi^{-n}(z)$, and consider, as Widom did,

$$Q(z) = \frac{1}{2\pi i}\int_C R_{\rho^*}^{-1}(z)\prod_1 \Phi(z,z_j)^{-1}V_{\Gamma_n'}(z)\Phi(\zeta)^n \frac{d\zeta}{\zeta - z},$$

where C is a large circle about the origin (described once counterclockwise) containing E and z in its interior. We emphasize that the integrand is single-valued in Ω, and in the expression in the integrand in front of $\Phi(\zeta)^n$ we have the function from (9) (modulo some constants) made for ρ^* and for the class Γ_n. Now this Q is a polynomial of degree n, for which Widom proved in Theorem 11.4 (when ρ is replaced by ρ^*) the asymptotic formula as $n \to \infty$ (see middle of [21, p. 210])

$$Q(\zeta) = B(\zeta) + o(1), \qquad \zeta \in E,$$

where $B(\zeta)$ is the limiting value of

$$R_{\rho^*}^{-1}(z)\prod_1 \Phi(z,z_j)^{-1}V_{\Gamma'}(z)\Phi(z)^n \qquad (10)$$

on the closed curves of E, and the sum of the two limiting values on the arcs of E. Here the second and last factors are of absolute value 1 on E while the first

6

one is $1/\rho^*(\zeta)$ and the third one is as close to 1 as we wish, say $|V_{\Gamma'}(\zeta)| \le e^{2\varepsilon}$ for any given $\varepsilon > 0$ provided the $\delta > 0$ above is sufficiently small (and fixed for all n). This gives (see [21, p. 210])

$$\sup_{E \setminus E_{\mathrm{arc}}} |Q(\zeta)|\rho(\zeta) \le e^{2\varepsilon}$$

and

$$\sup_{E_{\mathrm{arc}}} |Q(\zeta)|\rho(\zeta) \le e^{2\varepsilon} 2 \sup_{E_{\mathrm{arc}}} \frac{1}{|R_{\rho^*}(\zeta)|} \rho(\zeta) = e^{2\varepsilon} 2 \sup_{E_{\mathrm{arc}}} \frac{1}{2\rho(\zeta)} \rho(\zeta) \le e^{2\varepsilon}.$$

The absolute value of the leading coefficient is at least (see [21, p. 210 and the proof of Theorem 8.3])

$$e^{-\varepsilon} C(E)^{-n} \mu(\rho^*, \Gamma_n),$$

and hence

$$M_{n,\rho} \le e^{3\varepsilon} C(E)^n \mu(\rho^*, \Gamma_n). \tag{11}$$

Let $\theta = \mu(\tau, 0)$, where $\tau(\zeta) = 2$ on E_{arc} and $\tau(\zeta) = 1$ on $E \setminus E_{\mathrm{arc}}$, and where in $\mu(\tau, 0)$ we take the 0-class of analytic functions in Ω. In other words, θ is the infimum of the norms $\sup_E \nu(\zeta)|h(\zeta)|$ for all h which is bounded and analytic in Ω and equals 1 at infinity. Then $\theta < 2$ provided there is at least one curve component of E. This follows from the fact that clearly $\mu(\tau, 0) \le 2$ if we use $h(z) \equiv 1$ as a test function, and that function is not extremal, since for the extremal function (7) the product $\nu(\zeta)|h(\zeta)|$ is constant on E (see [21, Theorem 5.4]). Now the definition of $\mu(\rho, \Gamma_n)$ implies that $\mu(\rho^*, \Gamma_n) \le \theta \mu(\rho, \Gamma_n)$, and hence (4) follows from (11) (because $\varepsilon > 0$ is arbitrary).

■

3 Widom's theory and Theorem 2

In this section we assume, as in Theorem 2, that E consists of intervals on the real line and of Jordan curves that are symmetric with respect to the real line. We show that a simple modification of some of Widom's argument gives the asymptotic formula in Theorem 2.

The upper estimate is in (5) (see (11)), so we shall only deal with the lower estimate of $M_{n,\rho}$.

Let $z_1 = z_{1,n}, \ldots, z_{q_n,n}$ be the points from (7) for $\Gamma = \Gamma_n$, both their number $q = q_n$ and the points themselves depend on n, but we shall suppress this dependence.

Let K be the set that is enclosed by the curves in E, i.e. K is the union of E with the bounded components of the complement $\overline{\mathbf{C}} \setminus E$ of E. The intersection $K \cap \mathbf{R}$ consist of some intervals $[\alpha_k, \beta_k]$, $\alpha_1 < \beta_1 < \alpha_2 < \cdots < \alpha_p < \beta_p$. We call (β_k, α_{k+1}) the contiguous intervals to E. First we show that the points z_j belong to the contiguous intervals, and each contiguous interval contains at most one of them. To this end note that Ω can be mapped by some conformal map φ onto some set $\overline{\mathbf{C}} \setminus \cup_{k=1}^p [\alpha'_k, \beta'_k]$ such that (β_k, α_{k+1}) is mapped into $(\beta'_k, \alpha'_{k+1})$.

Indeed, just take a conformal map of $\Omega \cap \mathbf{C}_+$ (where $\mathbf{C}_+$ is the upper half plane) onto $\mathbf{C}_+$, and extend it across $\mathbf{R} \setminus \cup_{k=1}^{p}[\alpha_k, \beta_k]$ by Schwarz reflection. Now the extremal problem described in the preceding section is conformally invariant, so $\varphi(z_i)$ are mapped into the corresponding z_j''s for the set $E' = \cup_{k=1}^{p}[\alpha_k', \beta_k']$ and for the function $\varphi^{-1}(\rho)$ keeping the same class Γ. But simple variation argument (see [21, p. 211]) shows that if E' lies on the real line then the correspond z_j' lie in the contiguous intervals to E' and each of these intervals may contain at most one of the z_j', which proves the claim above.

Let

$$H(z) = R_{\rho^*}^{-1}(z) \prod_1 \Phi(z, z_j)^{-1} V_{\Gamma'}(z) \qquad (12)$$

be the expression in (10) in front of $\Phi(z)^n$. Note that in (12) not all z_j appear, only those that are of distance $\geq \delta$ from E. Let $l_k = 1$ if the interval (β_k, α_{k+1}) contains a z_j appearing in (12), and let otherwise $l_k = 0$, $1 \leq k \leq p - 1$.

Now consider the proof of [21, Theorem 11.5, pp. 212-214]. As there, we cut Ω along all contiguous intervals and also along (β_p, ∞), and consider the argument of $H_\pm(\zeta)\Phi_\pm(\zeta)^n$. Take the conjugate functions so that the argument of $H(z)\Phi^n(z)$ is 0 at α_1. Then it is 0 on all $(-\infty, \alpha_1)$, and $H(z)\Phi^n(z)$ is symmetric with respect to the real line. If E_1 is an interval $([\alpha_1, \beta_1])$, then we get exactly as on p. 213 of [21] from the single-valuedness of $H(z)\Phi(z)^n$ that

$$\arg H_-(\beta_1)\Phi_-^n(\beta_1) = m_1 \pi \qquad (13)$$

$$\arg H_+(\beta_1)\Phi_+^n(\beta_1) = -m_1 \pi \qquad (14)$$

with some integer m_1. On the other hand, if E_1 is a Jordan curve, then again the symmetry of E and the single-valuedness of $H(z)\Phi(z)^n$ imply (13)–(14). The number m_1 is positive for large n (actually tends to infinity as n tends to infinity). This follows from the Cauchy-Riemann equations on E_1 (more precisely on some smooth curve lying close to E_1) and from the fact that the normal derivative of $\log|H(z)|$ in the direction of the inner normal to Ω is bounded, while the normal derivative of $\log|\Phi(z)|$ is positive on E_1.

If $l_1 = 0$, then the argument at α_2 is the same, but if $l_1 = 1$, then

$$\arg H_-(\alpha_2)\Phi_-^n(\alpha_2) = (m_1 + 1)\pi$$

$$\arg H_+(\alpha_2)\Phi_+^n(\alpha_2) = -(m_1 + 1)\pi.$$

Proceeding this way, we get exactly as Widom that

$$\arg H_\pm(\beta_k)\Phi_\pm^n(\beta_k) = \mp\left(\sum_{r=1}^{k} m_r + \sum_{r=1}^{k-1} l_k \pi\right)\pi$$

$$\arg H_\pm(\alpha_k)\Phi_\pm^n(\alpha_k) = \mp\left(\sum_{r=1}^{k-1} m_r + \sum_{r=1}^{k-1} l_k \pi\right)\pi$$

with some positive integers $m_1, \ldots, m_p$.

On the other hand, it is also true (see p. 214 in [21]) that on the arc components of E

$$|H_\pm(\zeta)\Phi_\pm(\zeta)^n|\rho^*(\zeta) = |H_\pm(\zeta)|2\rho(\zeta) = 1 + O(\varepsilon), \tag{15}$$

while on the curve components we have similarly

$$|H(\zeta)\Phi(\zeta)^n|\rho^*(\zeta) = |H(\zeta)|\rho(\zeta) = 1 + O(\varepsilon) \tag{16}$$

(where $\varepsilon > 0$ depends on the δ that we used in selecting or deleting the terms $\Phi(z, z_j)$ in (12)). At the same time the polynomial

$$Q(z) = \int_C \frac{H(\xi)\Phi(\xi)^n}{\xi - z} d\xi$$

constructed from $H(z)\Phi(z)^n$ satisfies (see [21, Lemma 11.2])

$$Q(\zeta) = H_+(\zeta)\Phi_+(\zeta)^n + H_-(\zeta)\Phi_-(\zeta)^n + o(1)$$

on every arc-component of E and

$$Q(\zeta) = H(\zeta)\Phi(\zeta)^n + o(1) \tag{17}$$

on every curve-component of E. Hence, if

$$\psi(\zeta) = \arg H_-(\zeta)\Phi_-(\zeta)^n,$$

then on any arc-component (cf. also (15))

$$Q(\zeta)\rho(z) = \frac{1}{2}Q(\zeta)\rho^*(z) = \frac{1}{2}(2\cos\psi(\zeta) + O(\varepsilon)) = \cos\psi(\zeta) + O(\varepsilon), \tag{18}$$

while on the curve-components (17) is true, which gives, in view of (16) that

$$|Q(\zeta)|\rho(\zeta) = 1 + O(\varepsilon) \tag{19}$$

there. We also get from (17) and from the argument principle that if E_k is a curve-component of E, then Q has m_k zeros inside E_k. Since on any arc-component $[\alpha_k, \beta_k]$ the function ψ is changing from some $a\pi$ with some integer a to $a\pi + m_k\pi$, we get from (18) that there are $m_1 + 1$ points $\alpha_k \le x_{1,k} < \cdots < x_{m_k+1,k} \le \beta_k$ where $Q(\zeta)\rho(\zeta)$ alternatively takes the values $\pm(1-\eta)$, where η is some small number that can be as small as we wish if $\varepsilon > 0$ is sufficiently small. In addition, if $l_k = 1$, then there is an additional sign change along (β_k, α_{k+1}), i.e. $Q(\beta_k)\rho(\beta_k)$ and $Q(\alpha_{k+1})\rho(\alpha_{k+1})$ are of opposite sign and are close to 1, say $\ge 1 - \eta$ in absolute value. We may also assume that the $O(\varepsilon)$ in (19) is $\le \eta$ in absolute value.

Let $L = \sum_{k=1}^{p-1} l_k$ be the total number of the z_j's in (12), which is the number of zeros of $H(z)$ in Ω. Since the change of the argument of $H(z)\Phi^n(z)$ around a large circle is $2n\pi$, it follows from the argument principle that the total change of the argument of $H(z)\Phi^n(z)$ around E is $2(n-L)\pi$. On the other hand, we have

calculated the total change of the argument around E to be $2\pi(m_1 + \cdots + m_p)$, hence $n = L + \sum_k m_k$.

Now these easily imply that if P is any n-th degree polynomial with the same leading coefficient as Q, then

$$\max_E |P(\zeta)|\rho(\zeta) \geq 1 - \eta.$$

Indeed, otherwise $Q - P$ would have alternating signs at the $x_{i,k}$'s, giving m_k zeros on every arc-component $[\alpha_k, \beta_k]$. With the same reasoning $Q - P$ has a zero on every contiguous interval (β_k, α_{k+1}) for which $l_k \neq 0$. By (19) and by the indirect assumption $\rho(\zeta)|Q(\zeta)| < 1 - \eta$ we get from Rouche's theorem that $Q - P$ has the same number of zeros inside any curve-component E_k as Q has there, i.e. m_k. Thus, altogether we would get $\sum_k m_k + L = n$ zeros for $Q - P$, which is impossible since $Q - P$ is of degree $< n$.

As a consequence, we obtain that if $Q(z) = \kappa_n z^n + \cdots$, then

$$M_{n,\rho} \geq \frac{1 - \eta}{|\kappa_n|}. \tag{20}$$

By the definition of the extremal quantity $\mu(\rho^*, \Gamma_n)$ we have

$$\sup_E \rho^*(\zeta)|H(\zeta)/H(\infty)| \geq \mu(\rho^*, \Gamma_n),$$

and since $\rho^*(\zeta)|H(\zeta)| = 1 + O(\varepsilon)$ (see (15)–(16)), it follows that

$$\frac{1}{|H(\infty)|} \geq e^{-\varepsilon}\mu(\rho^*, \Gamma_n)$$

provided $\delta > 0$ is sufficiently small. Therefore,

$$\frac{1}{|\kappa_n|} = \frac{1}{|H(\infty)|C(E)^{-n}} \geq C(E)^n e^{-\varepsilon}\mu(\rho^*, \Gamma_n),$$

and the lower estimate

$$\liminf \frac{M_{n,\rho}}{C(E)^n \mu(\rho^*, \Gamma_n)} \geq 1$$

follows from (20).

$\blacksquare$

Corollary 3 follows easily. Indeed, (8) shows that

$$R_{\rho^*}(\infty) \leq \mu(\rho^*, \Gamma_n) \leq R_{\rho^*}(\infty) \exp\left\{\sum_{j=1}^{p-1} g(z_j^*)\right\}. \tag{21}$$

10

When $\rho = 1$ we have

$$|R_{\rho^*}(z)| = \exp\{\omega_{E_{\mathrm{arc}}}(z) \ln 2\}$$

where $\omega_{E_{\mathrm{arc}}}$ is the harmonic measure in Ω corresponding to E_{arc}, and it is well known (see e.g. [15, Theorem 4.3.14]) that

$$\omega_{E_{\mathrm{arc}}}(\infty) = \nu_E(E_{\mathrm{arc}})$$

where ν_E is the equilibrium measure of E. Now Corollary 3 is a consequence of (21) and Theorem 2.

For the last statement in the corollary follow the proof of [21, Theorem 8.4].

■

4   Elliptic case

In this section we give an explicit formula for the asymptotics of $M_{n,1}$ assuming that E consists of an interval $[\alpha_1, \beta_1]$ and a symmetric Jordan curve E_2. In this case the domain Ω is conformally equivalent to an annulus $\{z : r_1 < |z| < r_2\}$. The ratio r_2/r_1 is a *conformal invariant of the domain* and the expression $\frac{1}{2\pi} \ln r_2/r_1$ is called the *modulus* of Ω (it is the extremal length of curves in Ω that connect the two boundary components of Ω). Following [2, §55] we use the notation

$$\tau := \frac{i}{\pi} \ln \frac{r_2}{r_1} = 2i \operatorname{mod}(\Omega).$$

Theorem 4 *Let $\omega(\infty)$ be the harmonic measure of the interval $[\alpha_1, \beta_1]$ in Ω evaluated at infinity. Then*

$$M_{n,1} \sim C(E)^n 2^{\omega(\infty)} \left| \frac{\vartheta_0 \left(\frac{\{n\omega(\infty) + |\tau'|\frac{\ln 2}{\pi}\} + \omega(\infty)}{2} \mid \tau' \right)}{\vartheta_0 \left(\frac{\{n\omega(\infty) + |\tau'|\frac{\ln 2}{\pi}\} - \omega(\infty)}{2} \mid \tau' \right)} \right|, \tag{22}$$

where $\tau' = -1/\tau$, $\{x\}$ denotes the fractional part of a real number x, and

$$\vartheta_0(t|\tau') = 1 - 2h\cos 2\pi t + 2h^4 \cos 4\pi t - 2h^9 \cos 6\pi t + \dots, \quad h = e^{\pi i \tau'},$$

is the theta-function.

Recall that here $\omega(\infty) = \nu_E([\alpha_1, \beta_1])$, where ν_E is the equilibrium measure.

Proof.   Using the conformal map φ from Section 3 that maps Ω onto some $\mathbf{C} \setminus ([\alpha_1', \beta_1'] \cup [\alpha_2', \beta_2'])$, we have the conformal mapping

$$u = u(z) = \int_{\beta_2'}^{\varphi(z)} \frac{d\xi}{\sqrt{(\xi - \alpha_1')(\xi - \beta_1')(\xi - \alpha_2')(\xi - \beta_2')}}$$

11

of Ω (which we cut along the contiguous closed interval $[\beta_1, \alpha_2]$) onto the rectangle with the vertices $-iK', K - iK', K + iK', iK'$,

$$K = u(\alpha_1), \quad K + iK' = u(\beta_1).$$

Therefore

$$z \mapsto e^{-\frac{u(z)}{K'}\pi}$$

maps conformally Ω onto the annulus $e^{-K\pi/K'} < |w| < 1$, in particular $\tau = i\frac{K}{K'}$.

In this notations for the harmonic measure $\omega(z) = \omega(z, [\alpha_1, \beta_1])$ we have

$$\omega(z) = \frac{1}{K}\Re u(z),$$

and therefore

$$\frac{1}{2\pi}\Delta_{E_1}\arg\frac{1}{\Phi} = \omega(\infty) = \frac{1}{K}\Re u(\infty), \quad \frac{1}{2\pi}\Delta_{E_1}\arg\frac{1}{\Phi(z, z_1)} = \omega(z_1) = \frac{1}{K}\Re u(z_1).$$

Further, $R(z) = \exp(\ln 2\frac{u(z)}{K})$. That is,

$$\Delta_{E_1}\arg R = \frac{2K'}{K}\ln 2 \quad \text{and} \quad R(\infty) = 2^{\omega(\infty)}. \tag{23}$$

Since the product $R^{-1}(z)\Phi(z, z_1)^{-1}\Phi(z)^n$ is single-valued, we obtain the following expression for the real part of $u(z_1)$

$$\frac{1}{K}\Re u(z_1) = \omega(z_1) = \frac{K'}{\pi K}\ln 2 + n\omega(\infty) \bmod 1.$$

Since $0 \le \omega(z_1) \le 1$ we get

$$\frac{1}{K}\Re u(z_1) = \left\{\frac{K'}{\pi K}\ln 2 + n\omega(\infty)\right\},$$

and therefore

$$u(z_1) = \left\{\frac{|\tau'|}{\pi}\ln 2 + n\omega(\infty)\right\}K + iK'. \tag{24}$$

Finally, with this notation we have the following expression for the complex Green's function, see [2, §55, eq. (4)],

$$\Phi(z) = \frac{\vartheta_1\left(\frac{u(z)+u(\infty)}{2K} \mid -\frac{1}{\tau}\right)}{\vartheta_1\left(\frac{u(z)-u(\infty)}{2K} \mid -\frac{1}{\tau}\right)}, \tag{25}$$

where ϑ_1 is the theta-function, which argument is shifted, comparably to ϑ_0, by a half-period. To be precise, see Table VIII in [2],

$$\vartheta_1(v + \frac{\tau}{2}|\tau) = ie^{\frac{-\pi i\tau}{4}}e^{-\pi iv}\vartheta_0(v|\tau).$$

Thus, due to (24) and (25), we have

$$e^{g(z_1)} = |\Phi(z_1)| = \left| \frac{\vartheta_1 \left(\frac{\{n\omega(\infty)+|\tau'|\frac{\ln 2}{\pi}\}+\omega(\infty)}{2} - \frac{1}{2\tau} \,\middle|\, -\frac{1}{\tau} \right)}{\vartheta_1 \left(\frac{\{n\omega(\infty)+|\tau'|\frac{\ln 2}{\pi}\}-\omega(\infty)}{2} - \frac{1}{2\tau} \,\middle|\, -\frac{1}{\tau} \right)} \right|.$$

Using reduction by a half period we get

$$e^{g(z_1)} = \left| \frac{\vartheta_0 \left(\frac{\{n\omega(\infty)+|\tau'|\frac{\ln 2}{\pi}\}+\omega(\infty)}{2} \,\middle|\, \tau' \right)}{\vartheta_0 \left(\frac{\{n\omega(\infty)+|\tau'|\frac{\ln 2}{\pi}\}-\omega(\infty)}{2} \,\middle|\, \tau' \right)} \right|.$$

By $\mu(1^*, \Gamma_n) = R(\infty)e^{g(z_1)}$, in combination with the second expression in (23), we obtain (22).

∎

****

References

[1] N. I. Akhiezer, Über einige Funktionen welche in zwei gegebenen Intervallen am wenigsten von Null abweichen, I–III, *Izv. AN SSSR*, **9**(1932), 1163–1202; **4**(1933), 309–344, 499–536. (German) Russian translation: N. I. Akhiezer, *Selected works in Function theory and mathematical physics*, vol. 1, Akta, Kharkiv, 2001.

[2] N.I. Akhiezer, *Elements of the Theory of Elliptic Functions*, Amer. Math. Soc., Providence, RI, 1990.

[3] A. I. Aptekarev, Asymptotic behavior of polynomials of simultaneous orthogonality and of a system of extremal problems for analytic functions, *Akad. Nauk SSSR Inst. Prikl. Mat.*, Preprint 1987, no. **168**, 29 pp. (Russian)

[4] A. I. Aptekarev, Asymptotic properties of polynomials orthogonal on a system of contours, and periodic motions of Toda chains, *Mat. Sb. (N.S.)*, **125**(167) (1984), no. 2, 231–258. (Russian)

[5] G. Faber, Über Tschebyscheffsche Polynome, *J. Reine Angew. Math.*, **150**(1919), 79–106.

[6] D. H. Armitage and S. J. Gardiner, *Classical Potential Theory*, Springer Verlag, Berlin, Heidelberg, New York, 2002.

[7] J. B. Garnett and D. E. Marshall, *Harmonic measure*, Cambridge University Press, New mathematical monographs, Cambridge, New York, 2005.

[8] V. A. Kaliaguine, A note on the asymptotics of orthogonal polynomials on a complex arc: the case of a measure with a discrete part, *J. Approx. Theory*, **80**(1995), 138–145.

[9] F. Peherstorfer, Orthogonal and extremal polynomials on several intervals, *J. Comp. Applied Math.*, **48**(1993), 187–205.

[10] F. Peherstorfer, Elliptic orthogonal and extremal polynomials, *J. London Math. Soc.*, **70**(1995), 605–624.

[11] F. Peherstorfer and K. Schiefermayr, Theoretical and numerical description of extremal polynomials on several intervals I, *Acta Math. Hungar.*, **83**(1999), 27–58.

[12] F. Peherstorfer and K. Schiefermayr, Description of extremal polynomials on several intervals and their computation, I. II, *Acta Math. Hungar.*, **83**(1999), 71–102, 103–128.

[13] F. Peherstorfer, Deformation of minimizing polynomials and approximation of several intervals by an inverse polynomial mapping, *J. Approx. Theory*, **111**(2001), 180–195.

[14] F. Peherstorfer and P. Yuditskii, Asymptotic behavior of polynomials orthonormal on a homogeneous set, *J. Anal. Math.*, **89**(2003), 113–154.

[15] T. Ransford, *Potential Theory in the Complex Plane*, Cambridge University Press, Cambridge, 1995.

[16] M. L. Sodin and P. M. Yuditskii, Functions least deviating from zero on closed subsets of the real axis, *Algebra i Analiz*, **4**(1992), 1-61; English transl. in *St. Petersburg Math. J.*, **4**(1993), 201–249.

[17] K. Schiefermayr, A lower bound for the minimum deviation of the Chebyshev polynomial on a compact real set, *East J. Approx.*, **14**(2008), 65–75.

[18] J.-P. Thiran and C. Detaille, Chebyshev polynomials on circular arcs and in the complex plane, *Progress in Approximation Theory*, 771–786. Academic Press, Boston, MA, 1991.

[19] V. Totik, Chebyshev polynomials on compact sets, *Potential Analysis*, 2013, DOI: 10.1007/s11118-013-9357-6.

[20] J. L. Walsh, *Interpolation and Approximation by Rational Functions in the Complex Domain*, third edition, Amer. Math. Soc. Colloquium Publications, **XX**, Amer. Math. Soc., Providence, 1960.

[21] H. Widom, Extremal polynomials associated with a system of curves in the complex plane, *Adv. Math.*, **3**(1969), 127–232.

Vilmos Totik

MTA-SZTE Analysis and Stochastics Research Group

Bolyai Institute

University of Szeged

Szeged

Aradi v. tere 1, 6720, Hungary

and

Department of Mathematics and Statistics

University of South Florida

4202 E. Fowler Ave, CMC342

Tampa, FL 33620-5700, USA

totik@mail.usf.edu

Peter Yuditskii

Johannes Kepler University Linz

Abteilung fur Dynamische Systeme und Approximationstheorie

A-4040 Linz, Austria.

petro.yudytskiy@jku.at