Application of the Moore-Penrose Inverse Matrix in Image Deblurring

Igor Stojanović, Zoran Zlatev
Department of Computer Engineering
Faculty of Computer Science, Goce Delcev University
Stip, Macedonia
igor.stojanovik@ugd.edu.mk, zoran.zlatev@ugd.edu.mk

Predrag Stanimirović, Marko Miladonović
Department of Computer Science
Faculty of Sciences and Mathematics, University of Niš
Niš, Serbia
pecko@pmf.ni.ac.rs, markomiladinovic@gmail.com

***Abstract*—This paper presents an image deblurring method that finds application in a broad scientific field such as image deblurring. A method for image deblurring, based on the pseudo-inverse matrix is applied for removal of blur in an image caused by linear motion. This method assumes that linear motion corresponds to an integer number of pixels. Compared to other classical methods, this method attains higher values of the Improvement in Signal to Noise Ratio (ISNR) and the Peak Signal to Noise Ratio (PSNR) parameter. The values for the Mean Square Error (MSE) is lower and computational time has been decreased considerably with respect to the other methods. The presented experimental results are implemented in MATLAB.**

Keywords—deblurring; image restoration; inverse matrix; matrix equation

I. INTRODUCTION

Blurring is a form of bandwidth reduction of an ideal image owing to the imperfect image formation process [1-3]. It can be caused by relative motion between the camera and the original scene, or by an optical system that is out of focus. When aerial photographs are produced for remote sensing purposes, blurs are introduced by atmospheric turbulence, aberrations in the optical system, and relative motion between the camera and the ground. The field of image restoration is concerned with the reconstruction or estimation of the uncorrupted image from a blurred one. In the use of image restoration methods, the characteristics of the degrading system are assumed to be known a priori. The method, based on Moore-Penrose inverse matrix, is applied for the removal of blur in an image caused by linear motion. For comparison, we used two commonly used filters from the collection of least-squares filters, namely Wiener filter and the constrained least-squares filter [2]. Also we used in comparison the iterative nonlinear restoration based on the Lucy-Richardson algorithm [3].

This paper is organized as follows. In the second section we present process of image formation and problem formulation. In Section 3 we describe a method for the restoration of the blurred image. We observe certain enhancement in the parameters: *ISNR*, *MSE* and *PSNR*, compared with other standard methods for image restoration, which is confirmed by the numerical examples reported in the last section.

II. IMAGE FORMATION PROCESS

We assume that the blurring function acts as a convolution kernel or point-spread function $h(n_1, n_2)$ and the image restoration methods that are described here fall under the class of linear spatially invariant restoration filters. It is also assumed that the statistical properties (mean and correlation function) of the image do not change spatially. Under these conditions the restoration process can be carried out by means of a linear filter of which the point-spread function (PSF) is spatially invariant. If we denote by $f(n_1, n_2)$ the desired ideal spatially discrete image that does not contain any blur or noise, then the recorded image $g(n_1, n_2)$ is modeled as [2]:

$$\begin{aligned} g(n_1,n_2) &= h(n_1,n_2) * f(n_1,n_2) \\ &= \sum_{k_1=0}^{N-1}\sum_{k_2=0}^{M-1} h(k_1,k_2)\, f(n_1-k_1, n_2-k_2). \end{aligned} \tag{1}$$

The objective of the image restoration is to make an estimate $f(n_1, n_2)$ of the ideal image, under the assumption that only the degraded image $g(n_1, n_2)$ and the blurring function $h(n_1, n_2)$ are given. The problem can be summarized as follows: let *H* be a $m \times n$ real matrix. Equations of the form:

$$g = Hf,\ g \in \Re^m;\ f \in \Re^n;\ H \in \Re^{m\times n} \tag{2}$$

describe an underdetermined system of *m* simultaneous equations (one for each element of vector *g*) and $n = m + l - 1$ unknowns (one for each element of vector *f*). Here the index *l* indicates horizontal linear motion blur in pixels. The problem of restoring an image that has been blurred by linear motion, usually results of camera panning or fast object motion, consists of solving the underdetermined system (2). A blurred image can be expressed as:

$$\begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ \vdots \\ g_n \end{bmatrix} = \begin{bmatrix} h_1 & \cdots & h_l & 0 & 0 & 0 & 0 \\ 0 & h_1 & \cdots & h_l & 0 & 0 & 0 \\ 0 & 0 & h_1 & \cdots & h_l & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & h_1 & \cdots & h_l \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_m \end{bmatrix}. \quad (3)$$

The elements of matrix H are defined as: $h_i = 1/l$ for i=1, 2,..., l. The objective is to estimate an original row per row f (contained in the vector f^T), given each row of a blurred g (contained in the vector g^T) and a priori knowledge of the degradation phenomenon H. We define the matrix F as the deterministic original image, its picture elements are F_{ij} for i=1,…, r and for j=1,…, n, the matrix G as the simulated blurred can be calculated as follows:

$$G_{ij} = \frac{1}{l}\sum_{k=0}^{l-1} F_{i,j+k}, i = 1,\cdots,r,\ j = 1,\cdots,m \quad (4)$$

with $n = m + l - 1$, where l is the linear motion blur in pixels. Equation (4) can be written in matrix form of the process of *horizontal* blurring as:

$$G = \left(HF^T\right)^T = FH^T. \quad (5)$$

Since there is an infinite number of exact solutions for f or F in the sense that satisfy the equation $g = Hf$ or $G = FH^T$, an additional criterion that find a sharp restored matrix is required.

The process of blurring with vertical motion is with the form:

$$g = Hf,\ g \in \Re^m;\ f \in \Re^r;\ H \in \Re^{m\times r} \quad (6)$$

where $r = m + l - 1$, and l is linear vertical motion blur in pixels. The matrix H is Toeplitz matrix as the matrix given in (3), but with other dimensions. The matrix form of the process of *vertical* blurring of the images is:

$$G = HF, G \in \Re^{m\times n}; H \in \Re^{m\times r}; F \in \Re^{r\times n}. \quad (7)$$

Let us first consider a case where the blurring of the columns in the image is independent of the blurring of the rows - s*eparable two-dimensional blur*. When this is the case, then there exist two matrices H_c and H_r, such that we can express the relation between the original and blurred images as:

$$G = H_c F H_r^T, G \in \Re^{m_1\times m_2}; \quad H_c \in \Re^{m_1\times r}; F \in \Re^{r\times n}; H_r \in \Re^{m_2\times n}. \quad (8)$$

where $n = m_2 + l_1 - 1$, $r = m_1 + l_2 - 1$, l_1 is linear horizontal motion blur in pixels and l_2 is linear vertical motion blur in pixels.

III. Method for image deblurring

The notion of Moore-Penrose inverse (pseudoverse) matrix of square or rectangular pattern is introduced by H. Moore in 1920 and again from R. Penrose in 1955, who was not aware of the work of Moore. Let T is real matrix with dimension $m \times n$ and $\Re(T)$ is the range of T. The relation of the form:

$$Tx = b, T \in R^{m\times n}, b \in R^m, \quad (9)$$

are obtained in the analysis and modeling of many practical problems. It is known that when T is a singular matrix, its unique Moore-Penrose inverse matrix is defined. In case when T is real matrix with dimension $m \times n$, Moore and Penrose proved that Moore-Penrose inverse matrix $T^\dagger$ is a unique matrix that satisfies the following four relations:

- $TT^\dagger T = T$;
- $T^\dagger TT^\dagger = T^\dagger$;
- $(TT^\dagger)^T = TT^\dagger$;
- $(T^\dagger T)^T = T^\dagger T$.

We will use the following proposition from [5]:

Let $T \in R^{m\times n}, b \in R^m, b \notin \Re(T)$ and we have a relationship $Tx = b$, then we have $T^\dagger b = u$, where u is the minimal norm solution and $T^\dagger$ is the Moore-Penrose inverse matrix of T.

Since relation (2) has infinitely many exact solutions for f, we need an additional criterion for finding the necessary vector for restoration. The criterion that we use for the restoration of blurred image is the minimum distance between the measured data:

$$\min(\|\hat{f} - g\|) \quad (10)$$

where $\hat{f}$ are the first m elements of the unknown image f, which is necessary to restore, with the following constraint:

$$\|Hf - g\| = 0. \quad (11)$$

Following the above proposal, only one solution of the relation $g = Hf$ minimizes the norm $\|Hf - g\|$. If this solution is marked by $\hat{f}$, then for it is true:

$$\hat{f} = H^{\dagger} g \,. \tag{12}$$

Taking into account the relations of horizontal blurring (2) and (5), and relation (12) solution for the restored image is:

$$\hat{F} = G(H^T)^{\dagger} = G(H^{\dagger})^T \,. \tag{13}$$

In the case of process of *vertical blurring* solution for the restored image, taking into account equations (6), (7) and (12), is:

$$\hat{F} = H^{\dagger} G \,. \tag{14}$$

When we have a *separable two-dimensional blurring* process, the restored image is given by:

$$\hat{F} = H_c^{\dagger} G (H_r^{\dagger})^T \,. \tag{15}$$

IV. Numerical Results

In this section we have tested the method based on Moore-Penrose inverse matrix (or generalized inverse - GIM method) of images and present numerical results and compare with two standard methods for image restoration called least-squares filters: Wiener filter and constrained least-squares filter and the iterative method called Lucy-Richardson algorithm. The experiments have been performed using Matlab programming language on an Intel(R) Core(TM) i5 CPU M430 @ 2.27 GHz 64/32-bit system with 4 GB of RAM memory running on the Windows 7 Ultimate Operating System.

In image restoration the improvement in quality of the restored image over the recorded blurred one is measured by the signal-to-noise ratio (*SNR*) improvement [6].

The simplest and most widely used full-reference quality metric is the mean squared error (*MSE*) [6, 7], along with the related quantity of peak signal-to-noise ratio (*PSNR*). The advantages of *MSE* and *PSNR* are that they are very fast and easy to implement. With *PSNR* greater values indicate greater image similarity, while with *MSE* greater values indicate lower image similarity.

A. Horizontal motion

Fig. 1, Original Image, shows a deterministic original standard Matlab image Barbara. Fig. 1, Degraded Image, presents the degraded Camera image for l=30. Finally, from Fig. 1, GIM Restored Image, Wiener Restored Image, Constrained LS Restored Image and Lucy-Richardson Restored Image, it is clearly seen that the details of the original image have been recovered.

Fig. 1. Restoration in simulated degraded Barbara image for length of the horizontal blurring process, l=20.

The difference in quality of restored images can hardly be seen by human eye. For this reason, the *ISNR* and *MSE* have been chosen in order to compare the restored images. Fig. 2 – 4 shows the corresponding *ISNR*, *MSE* and *PSNR* values. The figures illustrate that the quality of the restoration is as satisfactory as the classical methods or better (l<100 pixels). Regarding these three parameters *ISNR*, *MSE*, *PSNR* similar results we present in [8] for other standard Matlab image called Cameraman.

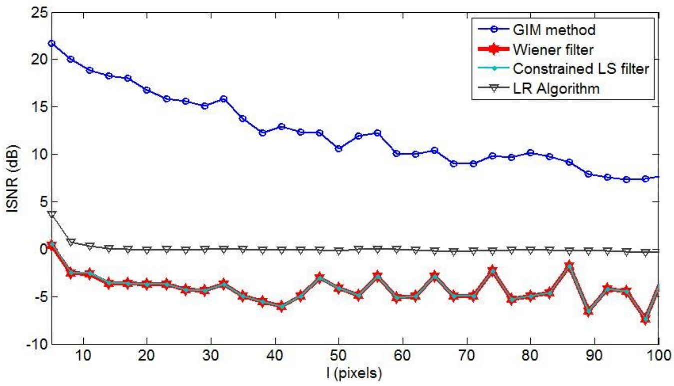

Fig. 2. Improvement in signal-to-noise-ratio vs. length of the blurring process in pixels.

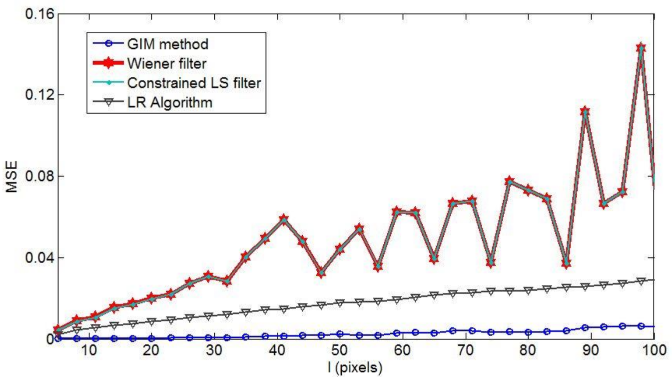

Fig. 3. Mean squared error vs. length of the blurring process in pixels.

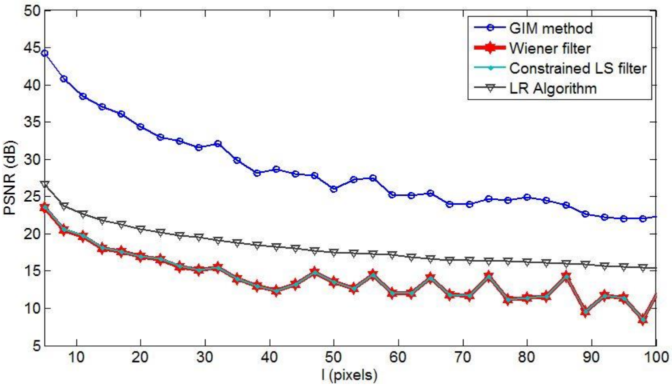

Fig. 4. Peak signal-to-noise-ratio vs. length of the blurring process in pixels.

B. *Vertical motion*

Obviously the method is not restricted to restoration of images blurred from horizontal motion. The results present in Fig. 5 – 8 refer when we have vertical blurring process.

Fig. 5. Restoration in simulated vertical degraded image for length of the blurring process, l=20.

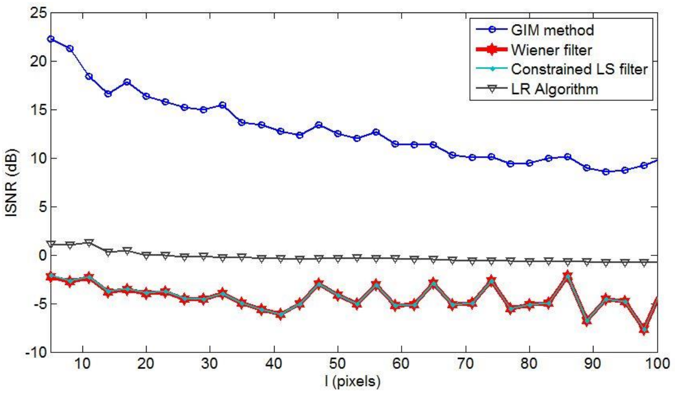

Fig. 6. Improvement in signal-to-noise-ratio vs. length of the blurring process in pixels.

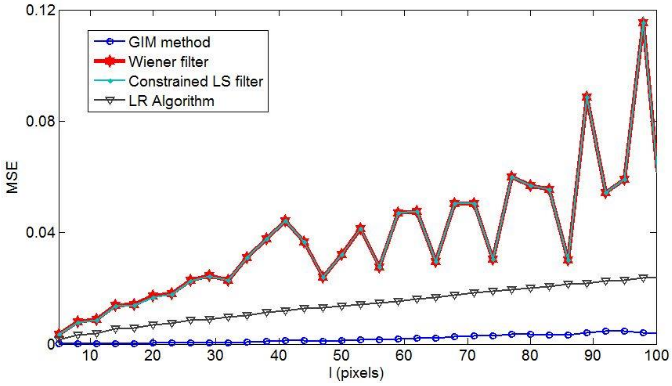

Fig. 7. Mean squared error vs. length of the blurring process in pixels.

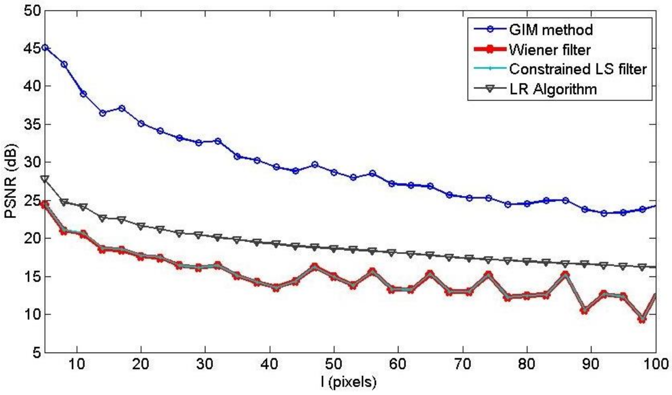

Fig. 8. Peak signal-to-noise-ratio vs. length of the blurring process in pixels.

C. *Separable two-dimensional blur*

The results for the standard Matlab image Barbara in case of separable two-dimensional blur are given on Fig. 9 – 12.

Fig. 9. Restoration in simulated degraded Barbara image for length of the blurring process l_1=15 and l_2=20.

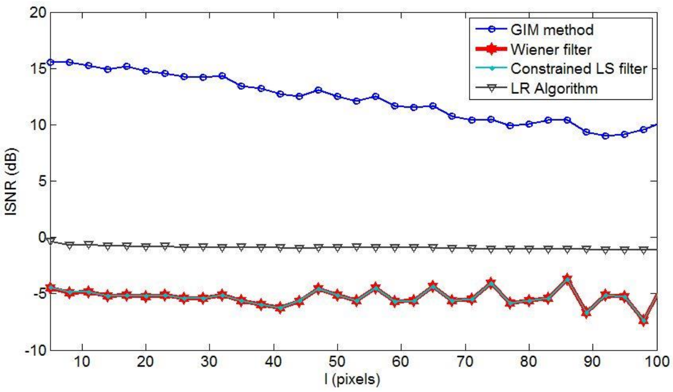

Fig. 10. Improvement in signal-to-noise-ratio vs. length of the blurring process in pixels.

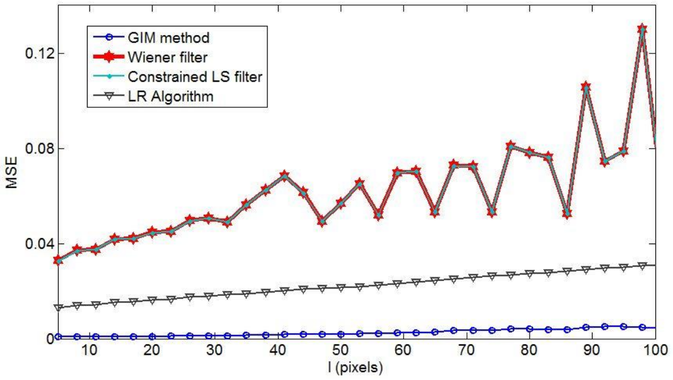

Fig. 11. Mean squared error vs. length of the blurring process in pixels

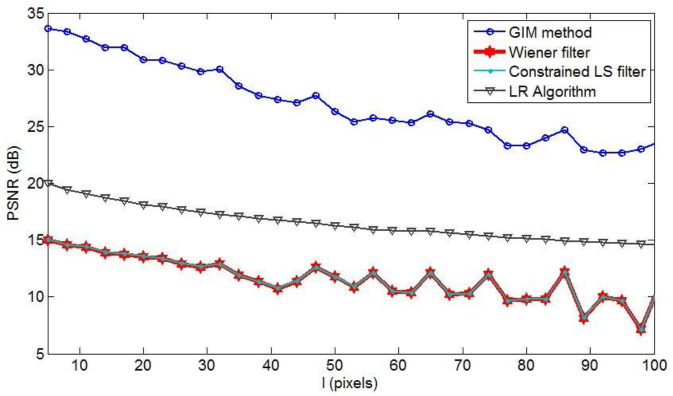

Fig. 12. Peak signal-to-noise-ratio vs. length of the blurring process in pixels.

D. Time consuming

Tastings are made for different values of dimensions of the image $r \times n$, while parameter *1* takes values from 5 to 101 with step 3. The results presented in the next pictures refer to the time of calculation for getting of the restored picture in seconds (*t*(sec)) as a function of $l \leq 101$. Comparison is performed for the presented model for restoration (GIM1 Method) [5], the method from [9, 10] called LM method and the standard methods, for different random matrices with dimensions: 600×600, 800×600 and 1000×600.

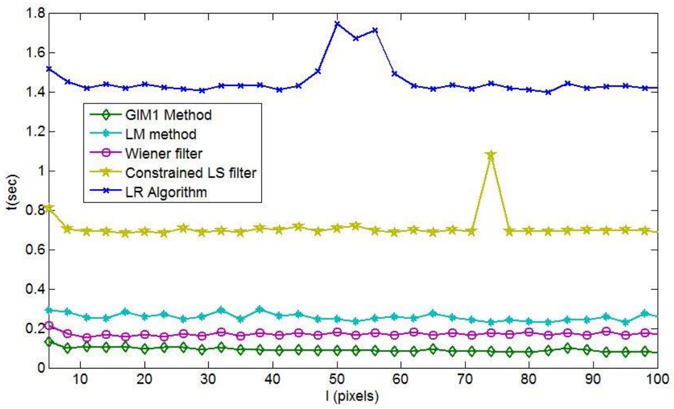

Fig. 13. Computational time vs. length of the blurring procces in pixels for *r*=600, *n*=600.

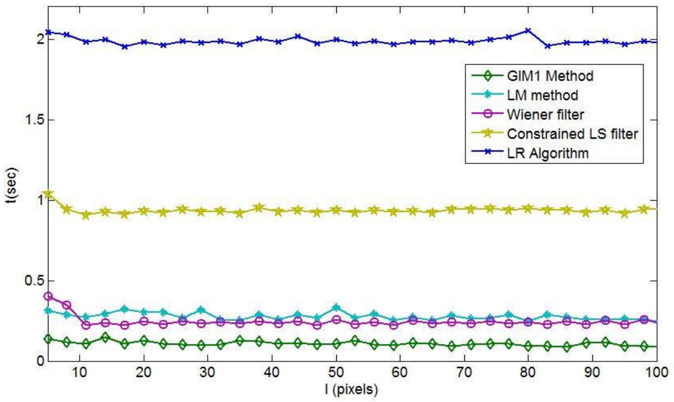

Fig. 14. Computational time vs. length of the blurring procces in pixels for *r*=800, *n*=600.

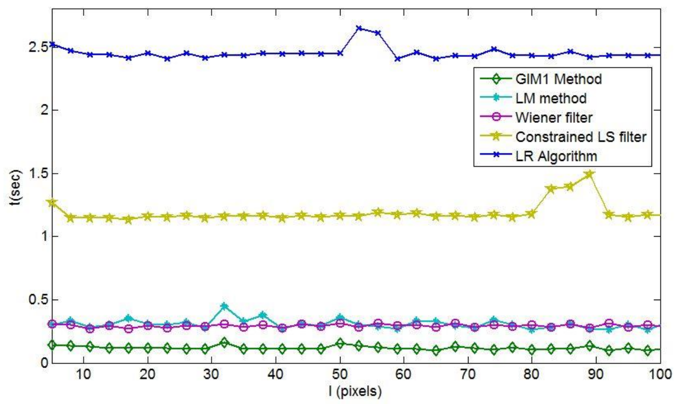

Fig. 15. Computational time vs. length of the blurring procces in pixels for *r*=1000, *n*=600.

With using of the proposed method, the resolution of the restored image stays at very high level, and another advantage of these methods is also in the required computational time which is very close or smaller, compared with other methods and techniques. Improving of the standard methods for image restoration are presented in [11, 12].

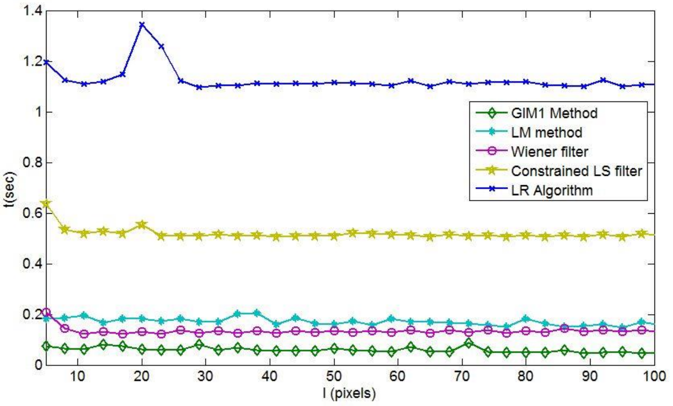

Fig. 16. Computational time vs. length of the blurring procces in pixels for Barbara.

In Fig. 16 the results for standard image Barbara is shown. The times of the computational calculation that correspond to our method are almost independent in matter of the increasing of the parameter *l*.

V. CONCLUSIONS

We introduce a computational method, based on the Moore-Penrose inverse matrix, to restore an image that has been blurred by linear motion.

We are motivated by the problem of restoring blurry images via the well-developed mathematical methods and techniques based on Moore-Penrose inverse matrix in order to obtain an approximation of the original image.

We present the results by comparing our method and that of the Wiener filter, Constrained least-squares filter and Lucy-Richardson algorithm, well-established restoration methods. Results are shown concerning when we have horizontal, vertical and separable two-dimensional blurring of the image.

In the method we studied, the resolution of the restored image remains at a very high level, yet the *ISNR* and *PSNR* is considerably higher. Although the computational efficiency is improved in comparison to other methods and techniques.

REFERENCES

[1] J. Biemond, R. L. Lagendijk, and R. M. Mersereau, "Iterative methods for image deblurring", Proc. IEEE, 78(5):856–883, 1990.

[2] Al Bovik, The essential guide to the image processing, Academic Press, 2009.

[3] Rafael C. Gonzalez and Richard E. Woods, Digital Image Processing, 2nd Edition, Prentice-Hall, 2002.

[4] Rafael C. Gonzalez, Richard E. Woods, and Steven L. Eddins, Digital Image Processing Using MATLAB, Prentice-Hall, 2003.

[5] Spiros Chountasis, Vasilios N. Katsikis, and Dimitrios Pappas, "Applications of the Moore-Penrose Inverse in Digital Image Restoration", Mathematical Problems in Engineering Volume 2009 (2009).

[6] Ahmet M. Eskicioglu and Paul S. Fisher, "Image Quality Measures and Their Performance", IEEE Transactions on Communications, vol. 43, pp. 2959-2965, Dec. 1995.

[7] Zhou Wang and Alan C. Bovik, "Mean Squared Error: Love It or Leave It? A New Look at Signal Fidelity Measures", IEEE Signal Processing Magazine, vol. 26, no. 1, pp. 98-117, Jan. 2009.

[8] I. Stojanovic, P. Stanimirovic, M. Miladinovic and D. Stojanovic, "Application of Non-Iterative Method in Image Deblurring", Journal of Computer Science and Control Systems (JCSCS), ISSN 1844-6043, pp. 99-102, Vol. 5, Nr. 1, 2012.

[9] I. Stojanovic, P. Stanimirovic and M. Miladinovic, "Applying the Algorithm of Lagrange Multipliers in Digital Image Restoration.", FACTA UNIVERSITATIS, Series Mathematics and Informatics, ISSN 0352-9665, Vol. 27, No 1, pp. 41-54, 2012.

[10] I. Stojanovic, I. Kraljevski and S. Chungurski, "Using of the Algorithm of Lagrange Multipliers in Image Restoration", Journal of Electrical and Electronics Engineering (JEEE), ISSN 1844-6035, pp. 203-206, Vol. 3, Nr. 2, 2010.

[11] P. Stanimirovic, S. Chountasis, D. Pappas, I. Stojanovic, "Removal of blur in images based on least squares solutions", *Mathematical Methods in the Applied Sciences*, Print ISSN: 0170-4214, Online ISSN: 1099-1476, DOI:10.1002/mma.2751.

[12] P. Stanimirovic, I. Stojanovic, S. Chountasis, D. Pappas, "Image Deblurring Process Based on Separable Restoration Methods", *Computational and Applied Mathematics*, ISSN: 0101-8205 (Print) 1807-0302 (Online), DOI: 10.1007/s40314-013-0062-2.