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VIEWS OF TEACHERS AND 7TH GRADERS ON AN ENRICHED LEARNING ENVIRONMENT DESIGNED FOR IMPROVING MATHEMATICAL REASONING[i]

Emrullah Erdem[1ii],
Yasin Soylu[2]
[1]Assoc. Prof. Dr.,
Adıyaman University,
Faculty of Education,
Adıyaman, Turkey
[2]Prof. Dr.,
Atatürk University,
Kazım Karabekir Faculty of Education,
Erzurum, Turkey

Abstract:
The purpose of the present study is to reveal the views of middle school mathematics teachers and students regarding a learning environment designed around different methods for improving Mathematical Reasoning (MR). The study was conducted with 27 seventh grade students studying at a randomly selected state school in a city center in Turkey, the teacher who instructs these students' mathematics courses, and another mathematics teacher employed at the school. For eight weeks (32 class hours in total), the students were instructed on the subjects of fractions and integers through educational games, concrete materials, cartoons, computer-assisted applications, association with daily life, and collaborative group argumentation. Data for the study were collected through interviews conducted with participants, observations of the teachers, and diaries kept by the students throughout the implementation process. The data were analyzed using a content analysis technique. Evidence was found that this environment improved the MR of students, provided effective and permanent learning, and increased attendance at lessons, but the teachers and the students also agreed that it had different effects in terms of classroom management such as noise, control over the classroom, lack of exam marks, not taking notes, time problems, and effects on students' success in their central exams. It is also possible to say that, with the help of open-ended high-level problems, students tried to provide solutions instead of focusing on choices for answers, and thus they used more MR skill.

[i] This study is a part of the doctoral thesis prepared by Emrullah Erdem, "*The effect of enriched learning environment on mathematical reasoning and attitude*", Turkey, 2015.
[ii] Correspondence: email eerdem@outlook.com

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1. Introduction

Recently, it has become a necessity to use multiple methods (separately or in combination) to teach concepts effectively to students all over the world. Designing learning environments where the teaching methods that activate multiple senses are used together will increase the effectiveness of teaching. As it is known, individuals learn 83% of what they know by sight, 11% by hearing, 3.5% by smell, 1.5% by touch, and 1% by taste. Additionally, keeping the time constant, people remember 10% of what they read, 20% of what they hear, 30% of what they see, 50% of what they both see and hear, 70% of what they say, and 90% of what they both do and say (Çilenti, 1988). When such supportive learning environments are provided, all students can make inferences, refute these inferences, and employ required Mathematical Reasoning (MR) (Yackel & Hanna, 2003).

2. Mathematical Reasoning

Mathematical Reasoning (MR) can be defined as an individual culture that helps to make sense of what is happening in the world through the questioning of "Why" and "How" by looking from the math window, and that enables to reach correct decisions as a result of this interpretation. Previous studies link students' difficulties in mathematics to superficial reasoning and rote learning strategies that are employed in more traditional learning environments (Bergqvist & Lithner, 2012; Lithner, 2003; 2017). MR enables students to go beyond the routine use of procedures, moving them towards understanding concepts, properties, and procedures as being logical, interrelated, and coherent aspects of mathematics (Mata-Pereira & da Ponte, 2017, p. 170). As it may be understood from these explanations, MR skill may be considered a prerequisite for making sense of and explaining knowledge. This is because a thought cannot be considered reasoning if it is not based on knowledge, it is not justified, or it does not contain logical approaches (Umay, 2003).

Developing MR in classrooms is a core aspect of teaching and learning mathematics (Ball & Bass, 2003). The literature features several cases that demonstrate the development of MR. For example, Francisco & Maher (2005) found that encouraging students to keep track of their own mathematical activities, using complex tasks where relevant problems are, allowing students to work collaboratively, and expecting them to justify their opinions all aid in MR improvement. Umay (2003) pointed out that student-centered learning environments, where all students are able to actively participate and develop awareness of their own reasoning skills, are ideal places for developing MR. Moreover, in order to improve reasoning, it is recommended to use problem situations that deploy technology through group projects that attract students, as well as to increase the complexity of these situations by including statistics (NCTM, 1989). Introducing students to different types of reasoning also plays a role in the development of reasoning (NCTM, 2000). The literature also details how MR is improved through social interactions, games, commercial activities, and constructive debates among individuals (Schliemann & Carraher, 2002). Mata-Pereira & da Ponte (2017) suggested that teachers need to provide challenging learning environments, rather than just classes where students solve exercises using well-known procedures, in order to promote MR. In the study by Erdem, Fırat & Gürbüz (2019), it was found that an enriched learning environment through educational games, computer-aided applications, open-ended high-level problems and constructive discussions in collaborative groups improved middle school students' MR.

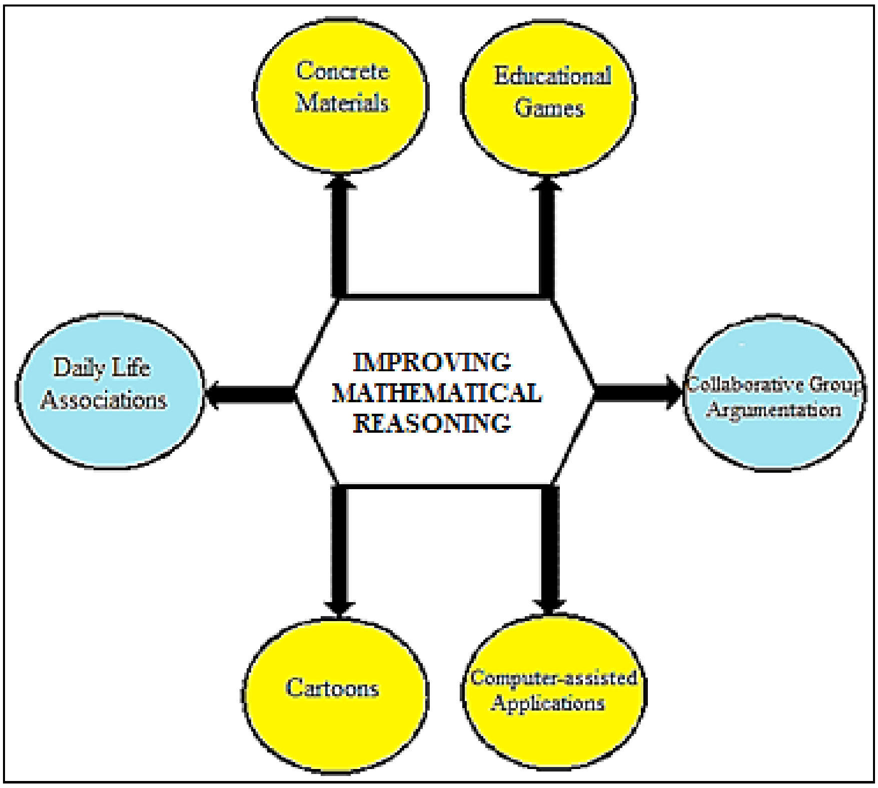


Figure 1: Methods that were used in the learning environment

2.1 Literature Framework of the Methods Used

Figure 1 shows the methods that were used in the learning environment designed to improve MR in the present study. These methods are explained as follows:

1) **Collaborative Group Argumentation:** Classrooms typically contain students with different success levels, skills, and backgrounds. This is why students in collaborative learning environments need to interact in small, heterogeneous groups in order learn most successfully (Slavin, 1987). A setting where students interact with each other and can comfortably share their mathematical ideas is an ideal setting for improving MR (Mata-Pereira & da Ponte, 2017; Yankelewitz, Mueller & Maher, 2010). Such argumentation environments develop pupils'

reasoning (Kuhn, Shaw & Felton, 1997) and problem-solving skills (Ellis, Özgür & Reiten, 2018). Vygotsky (1978) also reported that the reasoning of children develops best in environments where they live with their peers and have frequent social interactions. In such an environment, each individual has the opportunity to be affected by the reasoning of the others (Maher & Davis, 1995).

2) **Association with Daily Life:** An individual's cultural acquisitions and reasoning affects their usage of mathematics and mathematical comprehension (Schliemann & Carraher, 2002). Considering the abstract structure of mathematics, it becomes necessary to associate mathematics with real life in order to make it more concrete and readily applicable to children. This is because the students assign importance to mathematics that they can find in real life, and only then they can they make sense of mathematics' abstract representations. It is emphasized that ensuring that students deal with situations that they can relate to the real world and make sense of these in their minds may improve their reality-association skills (Inoue, 2008). Considering that MR is developed by dealing with extraordinary real problems and having experiences that connect to those problems, the importance of associating mathematics with daily life is evident here.
3) **Concrete Materials:** Students may experience difficulty when learning abstract concepts as they transition from concrete operations to abstract operations in their later schooling. It is believed that using concrete materials can help with overcoming this difficulty, therefore it may be possible to transfer knowledge from what is concrete to what is abstract. Materials for teaching that visualize and clearly present abstract mathematical expressions can activate students' senses (Moyer, 2001), facilitating the transition from concrete mathematics to abstract mathematics (Moyer, Bolyard & Spikell, 2002). Previous research has demonstrated that concrete materials increase students' use and enjoyment of mathematics by providing student-centered and rich learning opportunities (Raphael & Wahlstrom, 1989), improve the motivations of students by making mathematics learning fun (McNeil & Jarvin, 2007; Moyer, 2001), contribute to effective and permanent learning (Castro, 1998; Thompson, 1992), and help students think creatively and engage their imagination (Moyer et al., 2002).
4) **Computer-Assisted Applications:** International reform studies on mathematics education (NCTM, 2000) reported that, with the help of technological tools, students make more effective decisions, improve their reasoning effectiveness, and focus on problem-solving better. Pratt (2000) suggested that mathematics instruction with computer-assisted applications was more effective in teaching mathematics concepts. McCoy (1996) reported that computer-assisted instruction made learning concepts easier.
5) **Educational Games:** It is highly important to make learning environments attractive, particularly for younger age groups. Prensky (2001) suggests that there is problem-solving in games, and that this triggers creativity. Since winning the game is students' main goal, they have to pay more mental effort to the task by

thinking along the lines of "*what should we do?*", "*this is more logical,*" "*this would increase our probability of winning,*" and "*if …, then …, otherwise, …*", and this requires more imagination and more reasoning. Studies report that games encourage students to think logically (Kamii & Rummelsburg, 2008), contribute to structuring of information (Booker, 2000), increase the attention and motivations of students (Bragg, 2007), and improve the amount of knowledge of younger children in particular (Kumar & Lightner, 2007).

6) **Cartoons:** Visualization makes it possible to understand complex and abstract mathematical subjects. Fischbein (1987) highlighted that a visual object is not only a meaningful structuring of a desired outcome in a mathematics problem, but it also is an important factor in explaining the analytical development of the solution. Some of the most effective tools used in visualization and materialization of mathematical concepts are *cartoons*. Cartoons may play an important role in providing students with the skills to better understand their environment and the society they are living in, as well as to improve their approach on social issues and more easily establish a causal relationship among events by advancing their reasoning skills (Şengül & Dereli, 2013).

2.2 Relationship Between MR and Problem Solving

It may be argued that one of the processes where MR is most frequently observed is problem-solving. Many studies report that MR is a basic skill used effectively in problem-solving (Briscoe & Stout, 2001; Diezmann & English, 2001; English, 1998; Lithner, 2008; Schoenfeld, 1985). Therefore, during the entire implementation process of the present study, students were told to solve open-ended problems under supervision of the researcher in order to evaluate their reasoning skills and improve those skills in collaborative groups. Open-ended problems should be included in mathematics classes more often because 1) they can be solved by using multiple strategies, and 2) different results may be obtained in such problems. Thus, reasoning skill should be assessed by employing open-ended tasks following certain criteria, rather than using true-false or multiple-choice questions (Erdem, 2011; Suzuki, 1997). Frederiksen (1984) pointed out that the answers to open-ended problems cannot be formulated right away—there is no fixed method for their solutions—therefore they require reasoning on the part of the person who solves them. Lithner (2008) showed the role of MR in the problem-solving process as seen in Figure 2:

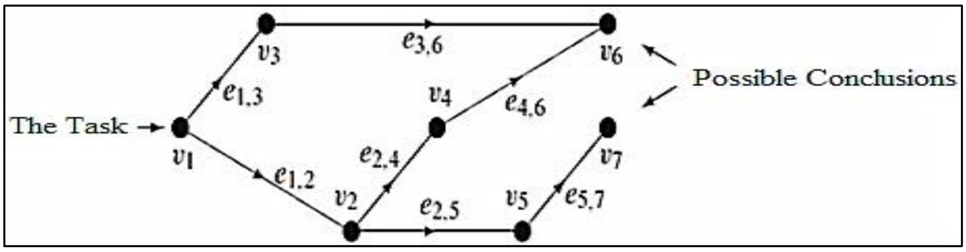


Figure 2: Reasoning in the problem-solving process

> *"A vertex v_n represents both a momentary state of knowledge and of the (sub)task. The reasoner makes a strategy choice among the edges leading from v_n. The strategy implementation is represented by a transition edge $e_{n,m}$. Here knowledge not already accessed in vn is recalled or constructed and added up to form the new knowledge state in v_m, where the task is partially resolved and therefore a new task state is formulated. A reason is the motivation supporting transitions between vertices"* (Lithner, 2008, p. 257).

Here, Lithner states that the knowledge or conclusion that is reached by using correct reasoning and selecting a suitable strategy may, in fact, turn into a new problem situation. This new situation may require a new reasoning process and selection of new strategies. However, while different strategies may be used to reach the correct result, it should be kept in mind that there is a possibility to reach different results based on this repeated reasoning and strategy selection process. Thus, correct reasoning and selection of suitable strategies are important to reach the correct results in the problem-solving process.

2.3 Significance of the Current Study

There is a clear prejudice in society surrounding mathematics. This prejudice is reflected in learning environments and it leads students to dislike or even fear mathematics. Designing learning environments by using new approaches is important to improve MR skills, which allow critical, logical and in-depth thinking that are all absolutely required for doing mathematics. In this context, it is believed that MR will be improved with the help of *educational games*, *computer-assisted applications*, *cartoons*, *concrete materials*, *association with daily life*, *constructive group argumentation*, and *open-ended problems* that allow higher-level thinking. Determining the views of the teachers (who are the main facilitators of mathematics learning) and students (who are the subjects of this learning environment) is important in order to determine the effectiveness of this process. The objective of this study is to design a new learning environment by using different methods to improve MR, and to determine the views of middle school mathematics teachers and seventh-grade students regarding this environment.

3. Method

3.1 Research Design

Because this study examined the views of participants in detail, it employed a case study methodology. In such studies, generally more than one data collection method is employed, because these studies aim to gather a rich diversity of data types that support each other (Yin, 2011). Through this type of triangulation, a case is examined in depth and within its own setting in a multidimensional way.

3.2 Subjects

The participants in this research were 27 seventh-grade students studying at a randomly selected state middle school in a city center in Turkey, as well as the mathematics teacher who taught these students mathematics course and another mathematics teacher employed at the school. This study used seventh-graders as participants because 1) these students are more familiar with middle school level mathematics content (in comparison with 5^{th}- and 6^{th}-graders); 2) 7^{th}-graders do not have concerns such as the Examination for Transition to High School from Middle School (in comparison with 8^{th}-graders); and 3) the subject of integers is taught in the 7^{th} grade. During the process of instruction, the students involved in the study sat in groups of four. Students were arranged in heterogeneous groups in terms of mathematics achievement, according to the math scores on the students' report cards. In this way, the seating arrangement was designed for the students to learn from their friends through collaborative argumentation that would arise throughout the instruction process. To keep the identities of the students anonymous, their names were coded as "Student A," "Student B," "Student C," and so on. To keep the identities of the teachers hidden, the teacher with 9 years of professional experience was coded as "Teacher A," and the teacher with 20 years of professional experience was coded as "Teacher B." The study aimed to help both teachers not only realize the importance of the use of new and different methods, but also to contribute to their profession through their participation in this research.

3.3 Data Collection

Data collection tools for this study included interviews with the students and the teachers about the implementation process, teacher reflections recorded on an observation form, and diaries kept by the students. In their interviews, the teachers were asked questions such as "Does the designed learning environment help students improve their mathematical reasoning?", "Did the instruction in the designed learning environment make any changes in the roles of the teacher and the students? If so, could you explain?", and "How does the designed learning environment affect the students' learning?". The students were asked questions such as "What do you think about the designed learning environment?", "How did the designed learning environment contribute to you?", "What are the differences that separate the practices in the designed learning environment from the practices in your previous classes?", and "Did the instruction in the designed learning environment make any changes in the roles of the teacher and the students? If so, could you explain?". The teachers made observations throughout the study by using the semi-structured observation form prepared by the researchers. Additionally, throughout the whole process and at the end of the instruction each week, the students kept diaries where they reflected on their experiences. The researcher inferred that, with these diaries, more information could be obtained about the designed learning environment through the eyes of the students.

3.4 Data Analysis

The data were analyzed using content analysis methods. In content analysis, the obtained data are examined in detail and unnoticed themes are discovered through a descriptive approach (Yin, 2011). During this process, at the end of each week the teacher opinions, student opinions, and student diaries were examined separately. The similar and different aspects of the data collected with each data collection tool were determined. After this, the process of coding started. In the coding process, the data were examined, divided into meaningful sections, and one word or sentence was used to demonstrate what each section represents. Categories were created by gathering the concepts or codes that had common aspects. In order to achieve coding reliability, Miles & Huberman's (1994) reliability formula reliability = agreement / (agreement + disagreement) was utilized. The reliability coefficient of the coding made independently by the two researchers, who are experts in mathematics instruction, was in the range of .85–.90. Additionally, when presenting findings from the data, the researchers determined that presenting direct quotes was an ideal approach for reflecting the views of individual research participants. As such, unique responses to the questions by the teachers, noteworthy or striking statements from students during interviews, and interesting reflections provided in the student diaries are included directly in the data set without any amendments or alterations.

3.5 Procedure

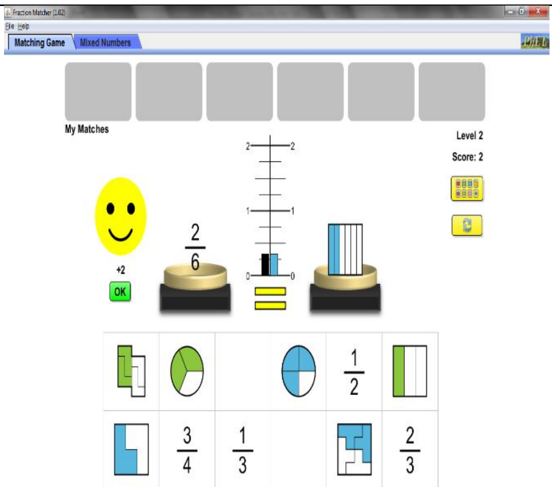
Using the teaching methods mentioned previously, the researcher carried out instruction for four hours a week over the course of eight weeks (for 32 class hours in total) based on the learning outcomes for the subjects of integers and fractions that are included in the Middle School Mathematics Curriculum. Some scenes from this instruction process are presented in Figure 3. In the Turkish curriculum, 29 class hours are allocated to meet the nine outcomes for the subject of fractions for the 5th grade, 24 class hours to fractions and 16 class hours to six outcomes for the subject of integers for the 6th grade, and 12 class hours to three outcomes for integers for the 7th grade (MNE, 2013). Since the study combined the similar outcomes for the subjects of fractions and integers, 32 class hours were found suitable to reach these 19 outcomes. The subjects of integers and fractions were selected as the topic of this study by considering that (a) the literature reported that integers (e.g. Fischbein, 1987; Hativa & Cohen, 1995) and fractions (e.g. Lee, 2017; Tirosh, 2000) are among subjects that students find difficult to learn; (b) they are sub-areas of learning under the area of "Numbers and Operations," which is the most comprehensive learning area of the Middle School Mathematics Curriculum (covering 5th, 6th, 7th, and 8th grades) (MNE, 2013); and (c) there is a need for integers and fractions in daily life. Moreover, the study focused on two topics because there was a concern that MR may not develop through the instruction of a single topic. The researcher believed that, because instruction on two topics would last longer than instruction on a single topic, it would be easier to determine the improvement of the students' reasoning skills.

The computer-assisted applications used in the learning environment were named "Air Balloon and Diver," "Take the Squirrel to the Exit," and "Identify, Model, and Compare Fractions." First, these software tools were projected on screen and explained to the students. After the applications were explained, the students were given an opportunity to observe the computer screen in their collaborative teams. Under the supervision of the researcher, the students learned by interacting with these applications and having constructive debates on their subjects and the questions that they posed. It can be asserted that these implementations are instructive (because they were designed to meet the targeted outcomes through their content delivery), entertaining (because they are projected on a computer screen), and thought-provoking (because of their use of visual questions). One of these applications is described below.

Air Balloon and Diver: This application was prepared for students to reach the outcomes of "interprets integers and shows them on the number line" and "determines the absolute value of an integer and makes sense of it." The application was designed by using the NetBeans editor and Java programming language. When the students click the "Start" button on the computer screen once, the balloon rises by two meters, while the diver dives deeper by two meters. This process continues this way with each click. With this application, the students could associate the concept of a positive integer with the balloon's rise above the ground level (at zero meters) and the concept of a negative integer with the diver reaching the depths of the sea. For example, the students would notice that the expression "-2 meters" refers to 2 meters of depth. Additionally, the students could make sense of the concept of absolute value with the help of this. For example, students would notice that the distances from the balloon and the diver's positions to the ground level are equal when the balloon goes up by 2 meters and the diver dives by 2 meters at the same time. Here, they will see what the equality $|-2|=|+2|=2$ *means. Furthermore, with this application—which has a similar logic to that of a number line—the students would develop an understanding of the number line, distribution of integers on the number line, and their signs. Figure 3-d provide interfaces from the application.*

The educational games used in the learning environment were named "Get the Highest Score by Shooting the Target," "Hit the Largest Integer," and "Find the Equivalent." The students played all these games in groups of four. To increase the students' motivation, awards were given to the group that won each game. All students in each group were encouraged to participate. For each game, the scores of all the groups were written on the board and all the students were allowed to see these scores. To maintain students' focus on the instructional aspects of the games and prevent them from getting distracted, they were asked constructive questions such as "Why did you think this way?", "Which probability is higher?", and "How else could this be?". One of these games is described below.

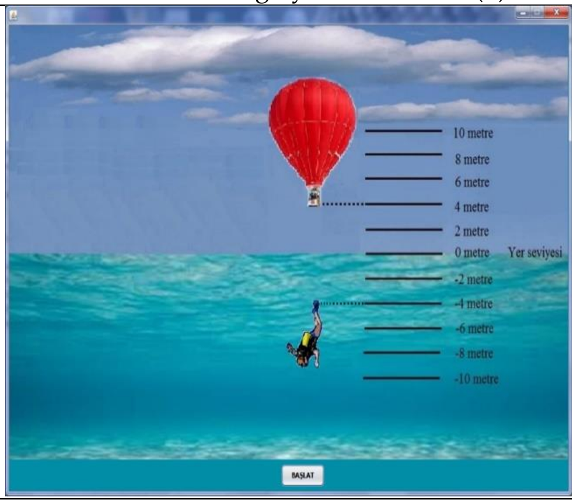
Find the Equivalent: This game was designed to help students reach the outcome of "Understands that simplification or expansion will not change the value of the fraction and forms fractions that are the equivalents of a certain fraction." This game used a board divided into sections, where the fractions $\frac{1}{3}, \frac{1}{4}, \frac{3}{4}, \frac{4}{8}, \frac{2}{5}, \frac{2}{2}, \frac{3}{7}, \frac{6}{11}, \frac{5}{10}, \frac{1}{1}, \frac{1}{9}, \frac{4}{7}, \frac{8}{4}, \frac{3}{2}, \frac{2}{1}$, *and* $\frac{1}{2}$ *were randomly placed into slots. Groups take turns opening one slot, and the group that finds a fraction that is equivalent to* *the fraction $\frac{1}{2}$ wins the game. This game developed students' understanding of the concepts of equivalence, simplification, and expansion in fractions. An image of the learning environment where the game "Find the Equivalent" was implemented is provided in Figure 3-e.*



A computer-assisted application (a)

Problem-solving by collaboration (b)

A cartoon (c)



An interface from Air Balloon and Diver (d)

Materials from the "Find the Equivalent" educational game (e)

Figure 3: Reflections from the learning environment (from the thesis by Erdem, 2015)

Concrete materials were designed for the students to reach outcomes related to integers and fractions, and collaborative student groups used these materials in games in the classroom. Concrete materials facilitate effective learning because they attract attention, are visual, and are used by students directly. Figures 3-e show concrete materials and an image of a game that uses these materials.

Throughout the entire implementation process, students were asked open-ended questions to shed light on their reasoning skills, as well as to further develop these skills. Questions such as "Why do you think this way?", "How did you reach this conclusion?", "Why?", and "How else could it be?" allowed them to express their views in the learning environment. Additionally, the students discussed and solved open-ended problems that required reasoning related to fractions and integers each week under supervision of the researcher. In this way, the students tried to provide a solution instead of focusing on choices for answers, therefore using more mathematical reasoning than they would in a traditional learning environment. Examples of such open-ended problems are provided in the "Findings" section.

4. Findings

Table 1: The Main and Sub-Categories Reached through the Analysis Process

Main Categories	Sub-Categories
Effective and Permanent Learning	▪ Visualization ▪ Concretization ▪ Constructive Discussions ▪ Using Different Methods ▪ Providing Effective Feedback ▪ Association with Daily Life
Improvement of Reasoning	▪ Using Open-ended Problems ▪ Allowing Students to Explain ▪ Guiding towards the Correct from the Incorrect ▪ Establishing Causality ▪ Encouraging Different Solution Strategies ▪ Problem-Solving as Groups
Increasing Participation in the Classroom	▪ Making the Process Fun with o Educational Games o Cartoons o Computer-Assisted Applications o Concrete Materials ▪ Using Rewards ▪ Considering Individual Differences
Classroom Management	▪ Noise ▪ Not Taking Notes ▪ Time Problems ▪ Control Over the Classroom ▪ Lack of Exam Marks ▪ Effects in Central Examinations

The content analysis process yielded both main categories and sub-categories. Student views (*from interviews and diaries*) and teacher views (*from interviews and their observations*) regarding each sub-category were included and interpreted in the analysis process. The main and sub-categories are presented in Table 1.

4.1 Effective and Permanent Learning

To illustrate applications of the general category of *Effective and Permanent Learning*, teacher and student views demonstrating each sub-category are provided below. Additionally, observations and diaries that support student and teacher views, if any, are also covered in the relevant sections.

4.1.1 Visualization

The students and teachers stated that visuals were prominent in the designed learning environment, and that this facilitated permanent learning. For example, Teacher B stated:

> *"Visual figures were used sometimes on a computer and sometimes on paper for all targeted outcomes. The students comprehended the subjects with their visual, auditory and other senses with the help of the caricatures and computer images shown to them as seeing is more effective than hearing in learning, I believe this led the students to learn better*."
>
> Additional student reflections on this sub-category include:
>
> *"… It is more permanent as we have the class visually. For example, the balloon and diver pictures on the computer that I saw three weeks ago are still on my mind…"* (Student J)

4.1.2 Concretization

The students and the teachers stated that concretization provided fun and permanent learning. Teacher A:

> *"The researcher used fraction tiles in the beginning of the class. The characteristic of the fraction tiles was that it represented the concept of a fraction and the halves and the quarters could be interchanged. With this material, the students have seen the answers to the questions in their mind like how many pieces a whole was divided into or what the fraction represented by a shaded area was. Additionally, the ranking of fractions with the same numerator could be understood as the numerators in the fraction files were equal. In the meantime, the students comprehended the subject with the prepared caricatures and activity sheets. Moreover, with the help of the dart board which included the pre-formed fractions, the students not only played a fun game, but also learned how to rank fractions*."
>
> *"…With the fraction tiles material, we have learned equivalence, unit fraction, and ranking in fractions and integers with the dart material…"* (Student M)

4.1.3 Constructive Discussions

The students and the teachers stated that, with the help of the constructive discussions in the heterogeneously organized groups, the students learned more actively by interacting with each other, and helped each other solve problems. Teacher A:

> *"Passive students became more active by interactions with their peers in the group. The students also felt that they had the right to speak equally and take chances to speak. They shared their correct or incorrect information in the group without hesitation and received help from their friends when they had problems."*

> *"…We were solving the problems that we could not previously solve with our friends. Our friends who used to not ask about what they did not understand in previous classes were able to ask their friends without hesitation…"* (Student D)

4.1.4 Using Different Methods

The students and the teachers stated that the different instruction methods facilitated more effective and permanent learning by appealing to more senses. Teacher B:

> *"The students comprehended the subject visually and by hearing with the help of caricatures and pictures on the computer that were provided to them. The dialogues in the given caricatures made the subject more attractive for the children. The role of the activity sheets was important in adding and subtracting positive and negative numbers. Addition and subtraction operations were introduced to the students with the help of the activity sheets. Work on the number line model, expression of four operations, and additionally, computer-assisted modelling work doubled the interest of the students in the class. Later, there were areas on the darts game material prepared by our teacher to represent positive and negative numbers, and zero. This was effective in introducing integers and four operations on integers."*

> *"…First of all, it was not like any of our previous classes. They tried to teach us by entertaining and different methods. For example, they used caricatures. We both laughed and reinforced our knowledge. We shared our knowledge by solving problems with our friend and showed complete teamwork. Then we played games and had much fun, but unfortunately, we lost. We really had fun today"* (Student J)

4.1.5 Providing Effective Feedback

The students and the teachers stated that the feedback provided to the students helped them notice and correct their mistakes. Teacher A:

> *"I noticed that the students occasionally got stuck at some point while solving problems. The teacher's interventions and feedback for the groups were guiding for them to solve the problem."*

"…We solved problems by reasoning as a group. Our teacher was helping in questions we could not solve or those we found difficult as a group, but they were not providing the correct answer, only clues…" (Student P)

4.1.7 Association with Daily Life

The students and the teachers stated that instruction by association with daily life seemed familiar for the students and thus achieved effective learning. Teacher B:

"For example, say you related negative integers to temperatures. This will be noticed by the students while they are watching weather reports, or they will understand the concept in the classroom better because they have seen it on TV as it will be more familiar for them. For example, saying the temperature is -10 C in Erzurum (a city) and 10 C in Adıyaman (a city)… Values below and above zero… This will lead the concept of cold to be associated with negative numbers for Erzurum which is colder."

"…Especially caricatures can be understood later, while they sometimes change our views. The shapes and people inside them, animals, numbers, for example, the elevator example which I remember…" (Student G)

4.2 Improvement of Reasoning

To illustrate applications of the general category of *Improvement of Reasoning*, teacher and student views demonstrating each sub-category are provided below. Additionally, observations and diaries that support student and teacher views, if any, are also covered in the relevant sections.

4.2.1 Using Open-ended Problems

The students and teachers described how the open-ended, the problems required reasoning, which highly contributed to the improvement of MR. Teacher B:

"They asked Einstein, if we gave you a problem and one hour to solve it, what would you do? He said he would think about the ways to solve it for 50 min, think about which of these was to use for 9 min, and solve it in 1 min. That is, the issue is to be able to understand the problem by reasoning. You can solve a problem if you understand it. Old-timers say this, too. Understanding the problem is half of the solution. I saw that students had difficulties in these open-ended problems that required reasoning about integers and fractions. They could not solve them right away. They had long discussions with their friends. They needed more reasoning. I could say that this process will contribute to improvement of their reasoning skills."

"…We solved a lot of difficult problems with our friends. Even the most hardworking person in our classroom could not solve some problems. The problems were not

immediately understood. We were understanding their logic with the help of our teacher after thinking on it for a long time. This led us to think more…" (Student N)

4.2.2 Allowing Students to Explain

The students and teachers emphasized how encouraging students to explain their work and make justifications for their choices contributed to the improvement of their MR. Teacher B:

> *"Encouraging students to explain the ways of solving the problems led them to express themselves comfortably. Even more passive students started to spend effort when we went to them and listened to their explanations. That, is when we asked how they thought in person, they tried to work on the problems more. I think this led to more correct and logical thinking."*

> *"…When our teacher said we should always express our opinions without hesitation and approached our friends who made explanations approvingly (even if they made mistakes), all our friends started to speak up without hesitation. By discussing together this way, we not only though more on the problems with our friends but also took part in reasoning more…"* (Student N)

4.2.3 Guiding Towards the Correct from the Incorrect

The students and the teachers explained that students determining a correct answer based on their understandings of incorrect answers contributed to the improvement of their MR. Teacher A:

> *"The teacher should be like a guide who helps the process and directs students, so that the student can make interpretations and decisions. It should be aimed to lead the student to find the solution to the problem. Instead of directly telling them, leading them to find this solution by themselves would lead the students to think more, therefore helping improve their reasoning skills. I observed such a process in this environment. You helped the students fix their own mistakes."*

> *"…They made us find it ourselves. This is why we had difficulty in solving the problems. However, we found it easier to solve them later…"* (Student I)

4.2.4 Establishing Causality

The students and the teachers implied that the instruction in the learning environment that was carried out based on a causal relationship contributed to the improvement of students' MR skills. Teacher B:

> *"If the children cannot make interpretations, what can they learn? If they do not have interpretation skills, they will just memorize. However, if instruction is based on causality relationships as you showed here, their interpretation skills will improve."*

> *"…In these classes, questions like Why do you think so? or Why? were asked frequently. We had to think a lot to answer these questions, but this was useful for us. It provided us with the skills to interpret what we did…"* (Student M)

4.2.5 Encouraging Different Solution Strategies

The students and teachers stated that encouraging students to use different solution strategies while solving problems contributed to their MR development. Teacher B:

> *"We solve problems in our classes by using a single solution strategy, and unfortunately, lead the students towards this direction. I saw in these classes that the teacher encouraged the students to develop different solution strategies."*

> *"…I was afraid of difficult problems before, but in these classes, I am not able to solve these problems better thanks to the different ways shown to us by our friends and our researcher teacher. I just need to think more…"* (Student G)

4.2.6 Problem-solving as Groups

The students and the teachers stated that problem-solving as a group contributed to the development of the students' MR skills. Teacher A:

> *"Providing the problems on a single sheet and the students in the group working on the problem together only on this sheet helped them learn from each other. Four people gathering, focusing on the question and solving it contributes to their thinking."*

> *"…We solved the problems as a group. We verified our solutions on the board. We were able to solve the problems we did not know about by asking for help from each other of from our teacher…"* (Student H)

4.3 Increasing Participation in the Classroom

To illustrate applications of the general category of *Increasing Participation in the Classroom*, teacher and student views demonstrating each sub-category are provided below. Additionally, observations and diaries that support student and teacher views, if any, are also covered in the relevant sections.

4.3.1 Making the Process Fun

The students and teachers described that the instruction in the learning environment that used different methods was fun, thus it increased participation in the classroom. Teacher B:

"The teacher's management of the class by making presentations on the computer and with various games led the students to understand the subject better. The class became more fun with the games. The students not only had fun, but they also understood the subject better. The students who were passive in the classroom became more active with the games played in the setting. As the students liked the games, they asked for more games and longer playing times in the classroom."

"…Our activities were nice. I think it is better to learn mathematics this way. Such activities are needed for mathematics to be nicer. The caricatures were beautiful. We played games by throwing darts. The class was very entertaining…" (Student A)

4.3.2 Using Rewards

The students and the teachers explained that using rewards increased participation in the classroom. Teacher A:

"It is a known fact that rewards increase motivation. When one uses rewards, the students participate in the classroom to receive the reward, so, they become more motivated. You gave some awards to the students who won the games or solved the problems correctly, and this increase participation."

"…We worked a lot to get the award, but the other group won. We will work harder next week and be the group that gets the award…" (Student I)

4.3.3 Considering Individual Differences

The students and teachers emphasized that considering individual differences increased participation in the mathematics classroom. Teacher B:

"Some students learn better by hearing, some by seeing and some by doing. I think this environment was useful in this sense. With this method, every student understands that you find value in each of them and pays importance to both you and the class. For example, we have to satisfy Student B visually, or else they would not understand the problem. Student K listens to the comment and explanation of the teacher carefully, follows it letter by letter, during the break, they tell them 'professor, you used this word incorrectly.' Student R, on the other hand, asks 'should the correct thing here or there be this?' That is, the level of perception for every student is different, and their perception may be improved in different ways."

"…Personally, I am interested in computers. I liked that mathematic subjects were taught on the computer. If only this could be the case for every time…" (Student Y)

4.4 Classroom Management

To illustrate applications of the general category of *Classroom Management*, teacher and student views demonstrating each sub-category are provided below. Additionally, observations and diaries that support student and teacher views, if any, are also covered in the relevant sections.

4.4.1 Noise

The students and teachers stated that the noise that arose in the learning environment posed a problem for classroom management. Teacher A:

> *"There was a problem about listening to the class. As the students in the group tried to solve the problems before other groups, there was noise. Additionally, the children became competitive for winning while playing games, and this led to too much noise."*

> *"…On the other hand, there was noise as everyone wanted to participate in the games. There was also noise when the groups competed for winning awards…"* (Student O)

4.4.2 Not Taking Notes

The students and teachers explained that they found it strange that note taking was not involved in the learning environment for this study. Teacher A:

> *"Everything is fine, but if we did not ask the students to take note like you did, they would have problems in written examinations. Besides, how is assessment possible without taking notes?"*

> *"…We took a lot of notes in our previous classes, not only did we not take much notes in these classes, but the focus was more on instruction…"* (Student P)

4.4.3 Time problems

The teachers stated that the type of instruction carried out in the learning environment requires much effort and thus is very time-consuming, and that the curriculum does not allocate this much time for these targeted outcomes. Teacher A:

> *"We cannot provide instruction materials, and we cannot carry out activities or games in the classes as they take much time and effort to prepare."*

The students also agreed that the instruction in this learning environment took a large amount of time.

> *"…The process took a lot of time when the subject was taught in a different way, via caricatures, games, and computers…"* (Student C)

4.4.4 Control over the Classroom

The teachers described issues in the learning environment related to control over the classroom. Teacher A:

> "*The students did not have grade concerns in these classes, so, they sometimes made too much noise, especially, to participate in the games. The teacher found it difficult to establish control over the classroom*."

> This concern was echoed by the students as well.

> *"…Our friends made too much noise in the classroom as they would not be graded. The teacher sometimes found it difficult to establish silence in the classroom…"* (Student A)

4.4.5 Lack of Exam Marks

The teachers stated that the students demonstrated a disregard for grades in the study learning environment. Teacher B:

> "*I saw that grading is highly effective for the students and for establishing control over the classroom, because the teacher found it difficult to establish control as the students did not have concerns about their grades*." The students confirmed that they did not have concerns about grades in this learning environment.

> *"…If only we could do mathematics this way without fears of bad grades. It is very comfortable…"* (Student H)

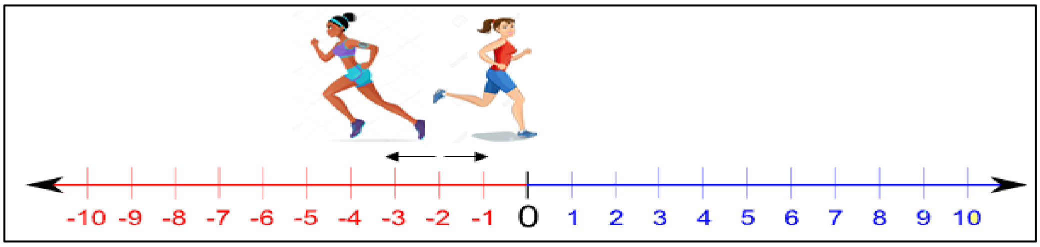
4.4.6 Effects on the Central Examinations

The teachers surmised that the instruction in the learning environment could have questionable effects on students' central examination performance. Teacher B: "You solved open-ended problems in general with the students and focused on subject instruction very much, but the system wants us to turn students into machines for solving multiple-choice questions (because of the SBS system). Families assess the success of the teacher and the school based on the success in such examinations. We focus less on instruction and more on solving problems. We solve a lot of multiple-choice questions and provide the students with the practical logic of the multiple-choice format." The students also held this view.

> *"…The subjects were taught in different ways. We had much fun in this process, but we should have solved more multiple-choice tests to prepare for the central exam 'SBS'…"* (Student O)

To illustrate the effects of solving open-ended problems in collaborative groups in more detail, constructive argumentations on two different open-ended problems from two different student groups are provided below.

Problem 1:



Merve and Pelin start to run from the point -2 on the number line in different directions, at the same time and at the same speed. When the distance between them becomes 10 units, what is the sum of the integers shown at the points they are standing on? Explain.
Student H: *The distance between them will be 10 units…* (thinking)
Student P: *How will get away, how much?*
Student J: *They will run the same.*
Student K: *One will be at +5 on the right and the other will be at -5 on the left.*
They look at each other (for confirmation).
Student H: *No. It is not like that. They are not starting at point 0.*
Student J: *Yes, they are starting at point -2.*
Student H: *They will run towards left and right from -2.*
Student P: *Correct.*
Student H: *For Merve; -2 +(-5)=-7 and for Pelin; -2+(+5)=+3.*
Student K: *…but there is no -7 on the number line.*
Student J: *You will assume (it is there) or draw it.*
Student J: *Pelin is at +3, correct. Is Merve… at -7?*
Student H: *Alright, J, let us draw 5 steps towards the left from -2 on the number line.*
Student H draws the steps one by one on the number line.
Student H: *We will draw the point -7 ourselves.*
Student J: *Okay, okay.*
Student P: *I understood better when we drew it.*
Student K: *Mee, too.*
Student J: *So, -7+3=-4.*
Student H: *Yes, I also found it as -4.*

Student H, whose MR was at a high level, led the group. They did not solve the problem by themselves, however, but rather they made correct interventions in their friends' mistakes. For example, they indicated on the number line that Student K's inference that "*one will be at +5 on the right and the other will be at -5 on the left*" was incorrect, which led K to notice their mistake. Moreover, H expressed that Merve would arrive at the number -7, helping eliminate the confusion in their friends in the group. Student P even stated that "*I understood better when we drew it*" after H made an explanation on the number line.

As it may be understood from the group discussions shown above, the collaborative argumentation carried out by the students to solve the problems not only achieved solidarity in the group, but also helped group members to be useful for each other. In these discussions, it was observed that students with higher MR led the groups in general and helped their friends with parts that they found difficult or did not understand. Not only did the students with low mathematics performance capitalize on this collaboration, but the high-performance students also utilized and adapted their friends' approaches when needed. In summary, with the help of the instruction carried out by forming groups of students with different mathematics performance levels, the students were able to engage in constructive argumentation, mitigate each other's weaknesses, and reach higher levels of MR towards the end of the implementation process.

5. Discussion

The results of the study showed that the enriched learning environment improved MR, provided effective and permanent learning, and increased attendance at lessons, but the teachers and the students agreed that it also had potentially detrimental effects on classroom management aspects such as noise, control over the classroom, lack of concerns for examination grades, not taking notes, time issues, and a negative impact on the students' success in the central examination system. The conclusion that MR levels can improve with the help of different teaching methods employed in the designed environment is supported by the results of other studies in the literature. Previous research has shown that MR skill can be effectively improved by organizing students in collaborative groups and facilitating learning by discussion among students (Erdem et al., 2019; Mata-Pereira & da Ponte, 2017; Mueller & Yankelewitz, 2014), using concrete materials (Pijls, Dekker & Van Hout-Wolters, 2007), integrating technology-assisted (computer-assisted) instruction (Erdem et al., 2019; Kramarski & Zeichner, 2001), incorporating instruction by educational games (Erdem et al., 2019; Lach & Sakshaug, 2004; Olson, 2007), associating mathematics lessons with daily life (Erdem, 2011; Erdem, Gürbüz & Duran, 2011), allowing students to express their views on the reasoning activities of their peers (Artzt & Yaloz-Femia, 1999; Pape, Bell & Yetkin, 2003), leading students to take notes of strategies they use while solving problems and allowing them to discuss these strategies with the teacher and other students (Ellis et al., 2018; Pape et al., 2003), and posing open-ended problems that are not immediately solved and require complex reasoning instead of multiple-choice questions (Erdem, 2011; Erdem & Gürbüz, 2015).

The present study determined that the environment provided effective and permanent learning due to its emphasis on *visualization, concretization, constructive* *discussions, usage of different instruction methods, providing effective feedback,* and *association with daily life*. This result supports those of studies that reported the importance of concrete materials in the learning environment (Castro; 1998; Moyer, 2001; Thompson, 1992), constructive argumentation (Cobb, Yackel & Wood, 1992; Mueller & Yankelewitz, 2014; Yackel, Cobb & Wood 1999), and feedback (Akkuzu, 2014; Bragg et al. 2016) in achieving permanent learning.

This study revealed that the environment increased participation in the classroom primarily through *making the process fun (by educational games, cartoons, computer-assisted applications and concrete materials), using reward* and *considering individual differences*. This result is consistent with those of studies that reported increases in the activeness of students and participation in the classroom based on tools that appeal to different senses (Gardner, 1997), appropriately delivered rewards (Ross & Braden, 1991), and consideration of learners' individual differences (Kezar, 2001). It was also discovered that the designed learning environment had mixed effects on learning management aspects like *noise, control over the classroom, disregard for grades, not taking notes, time issues* and *effects on success in the central examination*. This result supports previous studies that report that noise affects learning negatively (Yaman, 2009).

Teachers in the study reported that it was strange for them not to take notes in the learning environment, and this also provided a new perspective for the students. It is believed that the teacher should leave note-taking to the student's discretion, and that making students take rote notes (for example through copying content from the textbook into their notebook) may make the class boring, as well as lead the student to study based on memorization (an ineffective learning strategy). In this context, it could be argued that it is more beneficial for students to willingly write down things they find important, which they will be interested in and thus will remember while reading later. Additionally, the study found that students in this learning environment had fewer concerns about their grades, which led them to make more noise than usual. This problem could be overcome if the instruction were carried out over a longer time period, since the students would get used to the teacher and their style of instruction.

Both the teachers and students involved in this study stated that the instruction carried out in the learning environment required much effort and time. It is reasonable to say that the instruction itself took time, and that it also took significant effort to prepare the different instruction tools. However, one might also argue that, when considering the Middle School Mathematics Curriculum (for 5^{th}, 6^{th}, 7^{th}, and 8^{th} grades), there is ample time allocated for the implementation of such different instruction methods for teaching the subjects of fractions and integers. Özpolat (2013) revealed in a previous study that Turkish teachers are highly dependent on the national curriculum, and that existing central examinations negatively affect the adoption of student-centered education interventions. The current study also found that the instruction carried out in the designed learning environment may have questionable effects on the success in the central examination. Earlier studies (Black & Wiliam, 1998; Crook, 1988) reported that central examinations have a distinct effect on learning environments, because these examinations encourage examination-oriented instruction. Considering that the open-ended, high-level questions used in this study required more reasoning in comparison to multiple-choice questions, it may be argued that students' experiences with such problems will positively impact their performance in the central examination.

Another important issue that this study emphasizes is the necessity of using open-ended problems to assess and improve MR. In the light of the entire research process, it can be stated that with the help of the open-ended problems, the students tried to present solutions instead of focusing on choices for answers, as well as explaining their solutions, therefore engaging in more mathematical reasoning. In parallel, it is known that for students to improve their reasoning, this reasoning needs to be made visible and/or audible through oral and/or written explanations (Swan, 2011). The need to use open-ended problems intensively in assessment of MR has been reported in studies by Erdem (2011), Erdem & Gürbüz (2015), Frederiksen (1984), Henningsen & Stein (1997), and Suzuki (1997). As students solved open-ended problems in collaborative groups, they were asked questions to provide insight into their MR levels, such as "Why do you think this way?", "How?", "How did you reach this conclusion?", and "How else could you solve it?" Such guidance allows students to express their thoughts, contributing to their expression of themselves through the language of mathematics. Howell and Wilson (2014) also recommended using similar guidance to assess how students engage in MR. When students are interacting with problems, the objective—in addition to reaching the correct solution—should be to reveal the reasoning they used in reaching their conclusion, and consequently to improve that reasoning. After many years spent observing and designing mathematics classes, Swan (2011) explained most of the attention in the classroom is focused on "answer-getting," not on improving the quality of the reasoning. As soon as students obtain the answer, they are moved to a fresh problem, reflecting little (if at all) on the process they went through. Thus, reasoning does not improve because it does not become the object of attention.

6. Conclusions and Implementations

Based on the results of the present research, it is possible to sum up the effects of the methods employed in the designed environment as follows:

- **Educational game:** It has attracted the attention of students, provided fun learning, and increased participation in the classroom.
- **Computer-assisted application:** It has provided the students with the opportunity to use technology and learn by visualization.
- **Cartoon**: It has added humor to learning and provided an entertaining environment for students.
- **Concrete material:** It has provided learning by feeling and doing, and helped students to complete activities.
- **Association with daily life:** It has provided the opportunity to learn from real life examples and has aroused curiosity because it connects with students' own lives.

- **Constructive group argumentation:** It has given students the opportunity to learn from each other and helped them to correct each other's deficiencies and misconceptions. This has been one of the most effective contributions to the development of MR.
- **Open-ended problems:** It has led to different forms of problem-solving by requiring high-level thinking and strategy development. This is another method to help the improvement of MR.

In contrast to these improvements, the study also found that the learning environment had potentially negative effects such as noise, decreased control over the classroom, a lack of exam marks, not taking notes, time problems, and unknown effects on the central exams.

The following recommendations may be made in the light of the results reached in this study: (1) In order to assess and improve MR, incorporation of open-ended problems that cannot be solved right away should be increased; (2) Involvement of cartoons where humor is present in mathematics instruction should be increased; (3) The effects of enriched learning environments on subjects other than fractions and integers should be researched to add to the existing literature; (4) The effects of learning environments using different instruction tools should be investigated through longitudinal studies, in order to determine how these environments impact students' success in their central examinations.

References

Akkuzu, N. (2014). The role of different types of feedback in the reciprocal interaction of teaching performance and self-efficacy belief. *Australian Journal of Teacher Education, 39*(3), 36-66.

Artzt, A. & Yaloz-Femia, S. (1999). Mathematical reasoning during small-group problem-solving. In L. Stiff and F. Curio (eds.), *Developing Mathematical Reasoning in Grades K-12*: 1999 Yearbook (pp. 115–126), National Council of Teachers of Mathematics, Reston, VA.

Ball, D., & Bass, H. (2003). Making mathematics reasonable in school. In J. Kilpatrick, W. Martin, & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 27–44). Reston, VA: NCTM.

Bergqvist, T., & Lithner, J. (2012). Mathematical reasoning in teachers' presentations. *The Journal of Mathematical Behavior*, *31*(2), 252-269.

Black, P. & Wiliam, D. (1998). Assessment and classroom learning. *Assessment in Education*, *5*(1), 7-74.

Booker, G. (2000). *The maths game: using instructional games to teach mathematics*. Wellington, NZ: New Zealand Council for Educational Research.

Bragg, L. (2007). Students' conflicting attitudes towards games as a vehicle for learning mathematics: A methodological dilemma. *Mathematics Education Research Journal, 19*(1), 29–44.

Bragg, L. A., Herbert, S., Loong, E. Y. K., Vale, C., & Widjaja, W. (2016). Primary teachers notice the impact of language on children's mathematical reasoning. *Mathematics Education Research Journal*, *28*(4), 523-544.

Briscoe, C. & Stout, D. (2001). Prospective elementary teachers' use of mathematical reasoning in solving a lever mechanics problem. *School Science and Mathematics*, *101*(5), 228-235.

Castro, C. S. (1998). Teaching probability for conceptual change. *Educational Studies in Mathematics, 35*, 233-254.

Cobb, P., Yackel, E., & Wood, T. (1992). Interaction and learning in mathematics classroom situations. *Educational Studies in Mathematics*, *23*, 99 -122.

Crook, T. J. (1988). The impact of classroom evaluation practice on student. *Review of Educational Research*, *58*(4), 438-481.

Çilenti, K. (1988). *Eğitim teknolojisi ve öğretim.* Ankara: Kadıoğlu Matbaası.

Diezmann, C. & English, L. D. (2001). Developing young children's mathematical power. *Roeper Review*, *24*(1), 11-13.

Ellis, A., Özgür, Z. & Reiten, L. (2018). Teacher moves for supporting student reasoning. *Mathematics Education Research Journal,* https://doi.org/10.1007/s13394-018-0246-6

English, L. D. (1998). Reasoning by analogy in solving comparison problems. *Mathematical Cognition*, *4*(2), 125-146.

Erdem, E. (2011). *An investigation of the seventh grade students' mathematical and probabilistic reasoning skills* (MA Thesis). Adıyaman University, Adıyaman, Turkey.

Erdem, E., Gürbüz, R., & Duran, H. (2011). An investigation of mathematics used in daily life from past to present: theory out practice in. *Turkish Journal of Computer and Mathematics Education, 2*, 232–246.

Erdem, E. & Gürbüz, R. (2015). An analysis of seventh-grade students' mathematical reasoning. *Çukurova University Faculty of Education Journal, 45*(1), 123-142.

Erdem, E. (2015). *The effect of enriched learning environment on mathematical reasoning and attitude* (Doctoral dissertation). Ataturk University, Erzurum, Turkey.

Erdem, E., Fırat, T., & Gürbüz, R. (2019). Improving mathematical reasoning and mathematics attitude of disadvantaged children in rural regions. *Journal of Computer and Education Research, 7*(14), 673-697.

Fischbein, E. (1987). *Intuition in science and mathematics: An educational approach*. Dordrecht: Reidel.

Francisco, J. M. & Maher, C. A. (2005). Conditions for promoting reasoning in problem-solving: Insights from a longitudinal study. *Journal of Mathematical Behavior*, *24*, 361–372.

Frederiksen, N. (1984). Implications of cognitive theory for instruction in problem-solving. *Review of Educational Research*, *54*, 363-407.

Gardner, H. (1997). Multiple intelligences as a partner in school improvement. *Educational Leadership, 55*(1), 20-21.

Hativa, N. & Cohen, D. (1995). Self-learning of negative number concepts by lower division elementary students through solving computer-provided numerical problems. *Educational Studies in Mathematics, 28*(2), 401-431.

Henningsen, M. & Stein, M. K. (1997). Mathematical tasks and student cognition: classroom-based factors that support and inhibit high-level mathematical thinking and reasoning. *Journal for Research in Mathematics Education, 28*(5), 524-549.

Howell, T. H. & Wilson, P. H. (2014). *The role of teachers' questions in support of students' articulation of their mathematical reasoning*. Paper presented in proceedings of the 41the Annual Meeting of the Research Council on Mathematics Learning (pp. 105-112).

Inoue, N. (2008). Minimalism as a guiding principle: Linking mathematical learning to everyday knowledge. *Mathematical Thinking and Learning*, *10*(1), 36-67.

Kamii, C. & Rummelsburg, J. (2008). Arithmetic for first graders lacking number concepts. *Teaching Children Mathematics, 14*(7), 389–394.

Kezar, A. (2001). Theory of multiple intelligences: implications for higher education, *Innovative Higher Education*, *26*(2), 141-154.

Kramarski, B. & Zeichner, O. (2001). Using technology to enhance mathematical reasoning: Effects of feedback and self-regulation learning. *Educational Media International*, *38*(2-3), 77-82.

Kuhn, D., Shaw, V., & Felton, M. (1997). Effects of dyadic interaction on argumentative reasoning. *Cognition and Instruction, 15,* 287–315.

Kumar, R. & Lightner, R. (2007). Games as an interactive classroom technique: perceptions of corporate trainers, college instructors and students. *International Journal of Teaching and Learning in Higher Education, 19*(1), 53–63.

Lach, T. & Sakshaug, L. (2004). The role of playing games in developing algebraic reasoning, spatial sense and problem-solving. *Focus on Learning Problems in Mathematics*, *26*(1), 34-42.

Lee, M. Y. (2017). Pre-service teachers' flexibility with referent units in solving a fraction division problem. *Educational Studies in Mathematics*, *96*(3), 327-348.

Lithner, J. (2003). Students' mathematical reasoning in university textbook exercises. *Educational Studies in Mathematics*, *52*, 29–55.

Lithner, J. (2008). A research framework for creative and imitative reasoning. *Educational Studies in Mathematics, 67*, 255-276.

Lithner, J. (2017). Principles for designing mathematical tasks that enhance imitative and creative reasoning. *ZDM*, *49*(6), 937-949.

Ma, L. (1999). *Knowing and teaching elementary mathematics: teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Erlbaum.

Maher, C. A. & Davis, R. B. (1995). *Children's explorations leading to proof*. In C. Hoyles and L. Healy (Eds.), Justifying and proving in school mathematics (pp. 87-105).

Mathematical Sciences Group, Institute of Education, University of London, London.

Mata-Pereira, J., & da Ponte, J. P. (2017). Enhancing students' mathematical reasoning in the classroom: teacher actions facilitating generalization and justification. *Educational Studies in Mathematics*, *96*(2), 169-186.

McCoy, L. P. (1996). Computer-based mathematics learning. *Journal of Research Computing in Education*, *28*(4), 438-460.

McNeil, N. M. & Jarvin, L. (2007). When theories don't add up: Disentangling the manipulatives debate. *Theory into Practice*, *46*(4), 309-316.

Ministry of National Education (MNE). (2009). *Middle school mathematics 6–8. classes teaching program.* Ankara, Turkey: Head Council of Education and Morality.

Ministry of National Education (MNE). (2013). *Middle school mathematics 5–8. classes teaching program.* Ankara, Turkey: Head Council of Education and Morality.

Miles, M. B. & Huberman, A. M. (1994). *An expanded sourcebook: qualitative data analysis (2nd Editon).* Thousand Oaks, CA: Sage.

Moyer, P. S. (2001). Are we having fun yet? How teachers use manipulatives to teach mathematics. *Educational Studies in Mathematics*, 47, 175-197.

Moyer, P. S., Bolyard, J. J., & Spikell, M. M. (2002). What are virtual manipulatives? *Teaching Children Mathematics*, *8*, 372-377.

Mueller, M. and Yankelewitz, D. (2014). Fallacious argumentation in student reasoning: Are there benefits? *European Journal of Science and Mathematics Education, 2(1), 27-38.*

National Council of Teachers of Mathematics [NCTM] (1989). *Curriculum and evaluation standards for school mathematics*. Reston: Virginia.

National Council of Teachers of Mathematics [NCTM] (2000). *Principles and standards for school mathematics.* Reston, VA.

Olson, J. (2007). Developing students' mathematical reasoning through games. *Teaching Children Mathematics*, *13*(9), 464-471.

Özdemir, E. M., Duru, A., & Akgün, L. (2005). Two or three dimensional thinking: visualization of some identities with two and three dimensional geometrical figures. *Kastamonu Education Journal, 13*(2), 527-540.

Özpolat, V. (2013). The place of student-centered approach in teachers' occupational priorities. *National Education Journal, 200*, 5-27.

Pape, S. J., Bell. C. V., & Yetkin, I. E. (2003). Developing mathematical thinking and self-regulated learning: A teaching experiment in a seventh-grade mathematics classroom. *Educational Studies in Mathematics 53,* 179-202.

Pijls, M., Dekker, R., & Van Hout-Wolters, B. (2007). Reconstruction of a collaborative mathematical learning process. *Educational Studies in Mathematics*, *65*, 309-329.

Polaki, M. V. (2002). Using instruction to identify key features of Basotho elementary students' growth in probabilistic thinking. *Mathematical Thinking and Learning*, *4*(4), 285-313.

Pratt, D. (2000). Making sense of the total of two dice. *Journal for Research in Mathematics Education*, *31*(5), 602-625.

Prensky, M. (2001). Fun, play and games: what makes games engaging. In M. Prensky (Ed.), *Digital game-based learning.* New York: McGraw-Hill

Ragasa, C. Y. (2008). A comparison of computer-assisted instruction and the traditional method of teaching basic statistics. *Journal of Statistics Education*, *16*(1), 1-10.

Raphael, D. & Wahlstrom, M. (1989). The influence of instructional aids on mathematics achievement. *Journal for Research in Mathematics Education*, *20,* 173–190.

Ross, P. A. & Braden, J. P. (1991). The effects of token reinforcement versus cognitive behavior modification on learning-disabled students' math skills. *Psychology in the Schools*, *28*(3), 247-256.

Schliemann, A. D. & Carraher, D. W. (2002). The evolution of mathematical reasoning: Everyday versus idealized understandings. *Developmental Review*, *22*(2), 242-266.

Schoenfeld, A. H. (1985). *Mathematical problem-solving*. Orlando: Academic Press.

Slavin, R. E. (1987). Cooperative learning and cooperative school. *Educational Leadership*, *45*, 7–13.

Suzuki, K. (1997). *Cognitive constructs measured in word problems: A comparison of students' responses in performance-based tasks and multiple-choice tasks for reasoning*. Paper presented at the Annual Meeting of the American Educational Research Association, Chicago

Swan, M. (2011). Improving reasoning: analysing alternative approaches. Retrieved from http://nrich.maths.org/7812/index

Şengül, S. & Dereli, M. (2013). The effects of teaching integers using cartoons on 7th grade students' achievements and retention levels. *The Journal of Academic Social Science Studies*, *6*(7), 973-1003.

Thompson, P. W. (1992). Notations, conventions and constraints: Contributions to effective uses of concrete materials in elementary mathematics. *Journal for Research in Mathematics Education, 23(*2), 123-147.

Tirosh, D. (2000). Enhancing prospective teachers' knowledge of children's conceptions: the case of division of fractions. *Journal for Research in Mathematics Education*, *31*(1), 5-25.

Umay, A. (2003). Mathematical reasoning ability. *Hacettepe University Journal of Education*, *24*, 234-243.

Vygotsky, L. S. (1978). *Mind and society: The development of higher mental processes.* Cambridge, MA: Harvard University Press.

Yackel, E. Cobb, P., & Wood, T. (1999). The interactive constitution of mathematical meaning in one second grade classroom: An illustrative example. *Journal of Mathematical Behavior, 17*(4), 469-488.

Yackel, E. & Hanna, G. (2003). Reasoning and proof. In J. Kilpatrick, G. Martin and D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 227–236). Reston, VA: National Council of Teachers of Mathematics.

Yaman, H. (2009). Teachers' views on the applicability of the Turkish course curriculum in crowded primary classrooms. *Educational Sciences: Theory & Practice*, *9*(1), 349-359.

Yankelewitz, D., Mueller, M., & Maher, C. A. (2010). A task that elicits reasoning: A dual analysis. *The Journal of Mathematical Behavior, 29*, 76-85.

Yin, R. K. (2011). *Qualitative research from start to finish*. New York: The Guilford Press.