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Fuzzy Order Convergence of Double Sequences in Fuzzy Riesz Spaces

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Abstract

In this paper, we introduce the concepts of fuzzy order convergence of double sequence, also define a new type of convergence, that is, $FR(O_n) -$ Convergence and $FR(O_n) -$ Cauchy of double sequence in fuzzy Riesz space and then prove some interesting results related to these notions.

Keywords: Fuzzy Riesz space, fuzzy order convergence, $FR(O_n) -$ Convergence $FR(O_n) -$ Cauchy, Banach fuzzy Riesz space.

1. Introduction.

The notion of Riesz space, (or called vector lattice) was initiated in [1]. Since then many others have developed the subject. Most of the spaces encountered in analysis are Riesz spaces. They play an important role in optimization, problems of Banach spaces, measure theory and operator. In 1965, Zadeh [2], introduced the concept of theory of fuzzy mathematics and the fuzzy order relation was first defined in [3, 4, 5]. In 1992, the notion of fuzzy order relation has been extended to fuzzy ordered sets paralleling that of classical partially ordered sets in [6]. Also the notion of fuzzy Riesz space was introduced by Ismat Beg and Misbah UI Islam [7] and [8]. Actually [5] had heard about this concept from [8]. But in fact [6] have the idea order convergence and proved theorems on this. So, In [9] fuzzy order convergence in fuzzy Riesz spaces is defined for unbounded fuzzy order convergence.
In this work, is devoted to defining and studying the fuzzy order convergence of double sequences, $FR(O_n)$ −Convergence and $FR(O_n) -$ Cauchy of double sequence in fuzzy Riesz spaces after provided the preliminary concepts necessary to understand the work. Finally, prove some interesting results related to these notions .

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2. Preliminaries

This section consists of a collection of known notions, and facts related to the theory of fuzzy Riesz spaces.

***Definition* 2.1[6].** Let X be a crisp set. A fuzzy order on X is a fuzzy subset of $X \times X$ such that the following conditions are satisfied.

i- For all $x \in X$, $\dot{\mu}(x,x) = 1$ (reflexivity).

ii- For $x, y \in X$, $\dot{\mu}(x,y) + \dot{\mu}(y,x) > 1$ implies, $x = y$ (antisymmetry).

iii- For $x, z \in X$, $\dot{\mu}(x,z) \geq \bigvee_{y \in X} [\dot{\mu}(x,y) \wedge \dot{\mu}(y,z)]$

Where, $\dot{\mu} : X \times X \longrightarrow [0,1]$ is the membership function of the fuzzy subset of $X \times X$.

A set with a fuzzy order defined on it is called a fuzzy ordered set (foset, for short).

***Definition* 2.2[8].** A (real) linear space X is said to be a fuzzy ordered linear space if X is a foset and further X satisfies the following conditions.

i- If, $x_1, x_2 \in X$ such that. $\dot{\mu}(x_1, x_2) > \frac{1}{2}$ then,

$\dot{\mu}(x_1, x_2) \leq \dot{\mu}(x_1 + x, x_2 + x)$ for all, $x \in X$,

ii- If , $x_1, x_2 \in X$ such that, $\dot{\mu}(x_1, x_2) > \frac{1}{2}$ then ,

$\dot{\mu}(x_1, x_2) \leq \dot{\mu}(\alpha x_1, \alpha x_2)$ for every $0 \leq \alpha \in \mathbb{R}$.

***Remark* 2.3[8].** From Condition (i) of the definition of the fuzzy ordered linear space, it follows that if $\dot{\mu}(x_1, x_2) > \frac{1}{2}$ and $\dot{\mu}(x_3, x_4) > \frac{1}{2}$ afterward, $\dot{\mu}(x_1 + x_3, x_2 + x_4) > \frac{1}{2}$.

***Proposition* 2.4[8].** Let X be a fuzzy ordered linear space, $x, x_1, x_2 \in X$ and α, β be real numbers, then.

i- If $\dot{\mu}(0,x_1) > \frac{1}{2}$ and $\dot{\mu}(0,x_2) > \frac{1}{2}$ then $\dot{\mu}(0,x_1+x_2) > \frac{1}{2}$.

ii- If $\dot{\mu}(0,x) > \frac{1}{2}$ and $\dot{\mu}(0,-x) > \frac{1}{2}$ then $x = 0$.

iii- If $\dot{\mu}(0,x) > \frac{1}{2}$ and $\alpha \geq 0$ then' $\dot{\mu}(0,\alpha x) > \frac{1}{2}$.

iv- If $\dot{\mu}(x_1,x_2) > \frac{1}{2}$ and $\alpha \leq 0$ then. $\dot{\mu}(\alpha x_2,\alpha x_1) > \frac{1}{2}$.

v- If $\dot{\mu}(0,x_1) > \frac{1}{2}$ and $\alpha \leq \beta$ then; $\dot{\mu}(\alpha x,\beta x) > \frac{1}{2}$.

***Definition* 2.5[6].** A foset X is called a fuzzy lattice if all finite subsets of X has sups and infs. A fuzzy lattice X is said to be complete if every subset of X has a supremum and an infimum.

***Definition* 2.6.** A subset M of a lattice X is a sublattice if $x, y \in$ M implies that $x \vee y, x \wedge y \in$ M, where these lattice operations are computed in X. A vector sublattice of a vector lattice is simply a vector subspace in addition a sublattice.

***Definition* 2.7 [7].** Let x be an element of X . Then the positive part of x is, the element $x_+ = x \vee 0$, the negative part $x_- = (-x) \vee 0$, the absolute value of x is the element $|x| = x \vee (-x)$.

***Proposition* 2.8 [7].** For any $x \in X$, the elements x_+, x_- and $|x|$ are positive. Also, the following equalities hold,

i- $x = x_+ - x_-$.

ii- $|x| = x_+ + x_-$.

***Definition* 2.9 [7].** Let X be a fuzzy Riesz space. Two elements $x_1, x_2 \in X$ are said to be orthogonal if $|x_1| \wedge |x_2| = 0$ and are written as $x_1 \perp x_2$. The definition can be extended to the subsets of X . Two subsets A_1 and A_2 are said to be orthogonal if $x_1 \perp x_2$ for any $x_1 \in A_1$ and $x_2 \in A_2$.

3. Main Results

In this section, we introduce the notion of fuzzy order convergence of double sequences in Riesz spaces and prove some basic results.

Definition 3.1.

Let X be a fuzzy Riesz space.

i- A double sequence $(x_{jk})_{j,k\in\mathbb{N}}$ of points in X is said to be Fuzzy Increasing (in short, FRi) if $(j,k) \leq (m,n)$ in $\mathbb{N}\times\mathbb{N}$ implies that $\dot{\mu}(x_{j,k}, x_{m,n}) > \frac{1}{2}$. We denoted this by $x_{j,k} \uparrow^{FR}$. Now if $\wp = sup_{j,k\in\mathbb{N}} x_{j,k}$ exists for some $\wp \in X$. Then we write, $x_{j,k} \uparrow^{FR} \wp$.

ii- A double sequence $(x_{jk})_{j,k\in\mathbb{N}}$ of points in X is said to be Fuzzy Decreasing (in short, FRd) if $(j,k) \leq (m,n)$ in $\mathbb{N}\times\mathbb{N}$ implies that, $\dot{\mu}(x_{m,n}, x_{j,k}) > \frac{1}{2}$. We denoted by $x_{j,k} \downarrow_{FR}$. Now if $\wp = inf_{j,k\in\mathbb{N}} x_{j,k}$ exists for some $\wp \in X$. Then we write $x_{j,k} \downarrow_{FR} \wp$.

***Definition* 3.2.** Let X be a fuzzy Riesz space. Then a double sequence, $(x_{jk})_{j,k\in\mathbb{N}}$ is said to be $FR(O)-$Convergent to an element $\wp$ of X provided that there exists a double sequences $(y_{jk})_{j,k\in\mathbb{N}}$ and $(z_{jk})_{j,k\in\mathbb{N}}$ of points of X such that.

i- $\dot{\mu}(y_{j,k}, x_{j,k}) > \frac{1}{2}$ and $\dot{\mu}(x_{j,k}, z_{j,k}) > \frac{1}{2}$ for every $(j,k) \in \mathbb{N}\times\mathbb{N}$.

ii- $y_{j,k} \uparrow^{FR} \wp$ and $z_{j,k} \downarrow_{FR} \wp$.

In this case, we writes

$FR(O) - \lim_{j,k\to\infty} x_{j,k} = \wp$ or $x_{j,k} \xrightarrow{FR(O)} \wp$ ***Definition* 3.3.** Let X be a fuzzy Riesz space, equipped with a fuzzy norm $\|\cdot\|$. The fu-zzy norm on X is called a Fuzzy Riesz norm if for every $x, y \in \mathbb{R}$,

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$\dot{\mu}(|x|,|y|) > \frac{1}{2}$ implies $\dot{\mu}(\|x\|,\|y\|) > \frac{1}{2}$ Any fuzzy Riesz space equipped with a fuzzy Riesz norm is called a fuzzy normed Riesz space.

A fuzzy normed Riesz space which is also a Banach space in termed a Banach fuzzy Riesz space.

***Definition* 3.4.** A sequence $(\sigma_p)_{p\in\mathbb{N}}$ in a fuzzy Riesz space X is an $(FO)-$sequence iff it is decreasing and $\wedge_{p\in\mathbb{N}}\sigma_p = 0$.

***Definition* 3.5.** Let X be a fuzzy normed Riesz space. A double sequence $(x_{jk})_{j,k\in\mathbb{N}}$ in X is $FR(O_n)-$Convergent to an element x_0 of X if there is an $(FO)-$sequence$(\sigma_p)_{p\in\mathbb{N}}$ such that for every $p\in\mathbb{N}$ with the property that $\dot{\mu}(\|x_{j,k}-x_0\|,\sigma_p) > \frac{1}{2}$, where $(j,k)\in\mathbb{N}\times\mathbb{N}$. In this case, we will write $x_{j,k}\xrightarrow{FR(O_n)} x_0$.

***Definition* 3.6.** A double sequence $(x_{jk})_{j,k\in\mathbb{N}}$ is said to be fuzzy bounded if there exists a real number $M>0$ such, $\dot{\mu}(|x_{j,k}|,M) > \frac{1}{2}$, for every $(j,k)\in\mathbb{N}\times\mathbb{N}$.

***Definition* 3.7[8].**

i- An fuzzy- lattice X is said to be complete (relatively complete) if every (bounded) subset of X has a supremum and an infimum.

ii- An fuzzy- lattice X is said to be $\sigma-$ complete (relatively $\sigma-$ complete) if every countable (countable bounded) subset has a suppermum and an infimum.

***Definition* 3.8.** A real double sequence $(x_{jk})_{j,k\in\mathbb{N}}$ in a fuzzy normed Riesz space X is said to be $FR(O_n)-$Cauchy if there is an $(FO)-$ sequence $(\sigma_p)_{p\in\mathbb{N}}$ such that for every $p\in\mathbb{N}$ there corresponds $(j,k),\ (m,n)\in\mathbb{N}\times\mathbb{N}$ with $\dot{\mu}(\|x_{j,k}-x_{m,n}\|,\sigma_p) > \frac{1}{2}$

***Definition* 3.9.** Let X be a fuzzy normed Riesz space. We will say that the fuzzy norm $\|\cdot\|$ in X is fuzzy order continuous if there exist a double sequence $(x_{jk})_{j,k\in\mathbb{N}}$ of points in X, such that $x_{j,k}\downarrow_{FR} 0$, we have $\mathrm{FR}(\mathrm{O}_n)-\lim_{j,k\to\infty}\|x_{j,k}\| = 0$.

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***Example* 3.10.** Let $L=\{m^2,n^2:m,n\in N\}$. For each $j,k\in\mathbb{N}$, so will define $\phi_{j,k}\in C[0,1]$ by

$$\phi_{j,k}=\begin{cases}1, & j,k\in L\\ h_{j,k}, & j,k\notin L\end{cases}$$

Where $h_{j,k}(x)=x^{jk}$, $x\in[0,1]$, j, k=1,2, … . Obviously, $h_{j,k}\downarrow 0$ and $\phi_{j,k}\xrightarrow{FR(O_n)}0$.

***Theorem* 3.11.** Let X be a fuzzy Riesz space and $(x_{j,k})_{j,k\in\mathbb{N}}$ be a double sequence in X. Then we have the following:

i- If $x_{j,k}\uparrow^{FR}$ ($x_{j,k}\downarrow_{FR}$), then $x_{j,k}\xrightarrow{FR(O)}\wp$ if and only if $x_{j,k}\uparrow^{FR}\wp$ ($x_{j,k}\downarrow_{FR}\wp$, respectively).

ii- Any $FR(O)-$Convergent double sequence is bounded.

Proof.

i- Suppose that $x_{j,k}\uparrow^{FR}$ and from the definition of $FR(O)-$ Convergent for double

sequence implies there exists double sequences $(y_{j,k})_{j,k\in\mathbb{N}}$ and $(z_{j,k})_{j,k\in\mathbb{N}}$ in X such that $\dot{\mu}(y_{j,k},x_{j,k})>\frac{1}{2}$ and $\dot{\mu}(x_{j,k},z_{j,k})>\frac{1}{2}$ for every $(j,k)\in\mathbb{N}\times\mathbb{N}$. And so $z_{j,k}\downarrow_{FR}\wp$, therefore $\wp\in\{z_{j,k}:j,k\in\mathbb{N}\}$. From this we have

$(\downarrow z_{j,k})(\wp)=\dot{\mu}(\wp,z_{j,k})>\frac{1}{2}$ for every $(j,k)\in\mathbb{N}\times\mathbb{N}$. At that time for any $(j,k)\le(m,n)$, where $(m,n)\in\mathbb{N}\times\mathbb{N}$ is fixed, we have $\dot{\mu}(z_{m,n},z_{j,k})>\frac{1}{2}$, $(z_{m,n})_{m,n\in\mathbb{N}}$ is FRd double sequence in X. So $\dot{\mu}(x_{m,n},z_{m,n})>\frac{1}{2}$, which shows that $\dot{\mu}(x_{m,n},z_{j,k})>\frac{1}{2}$ for every $(j,k)\le(m,n)$. Now if $(j,k)>(m,n)$, followed by under the hypothesis $(x_{j,k})_{j,k\in\mathbb{N}}$ is FRi double sequence, thereby we can inscribe $\dot{\mu}(x_{m,n},x_{j,k})>\frac{1}{2}$ and from the condition (i) of definition $FR(O)-$

Convergent, which $\dot{\mu}(x_{j,k}, z_{j,k}) > \frac{1}{2}$ can conclude that $\dot{\mu}(x_{m,n}, z_{j,k}) > \frac{1}{2}$. Now choose $(m,n) \in \mathbb{N} \times \mathbb{N}$, such that $\dot{\mu}(x_{m,n}, z_{j,k}) > \frac{1}{2}$ for every (j,k), $(m,n) \in \mathbb{N} \times \mathbb{N}$. Consequently, for any $(m,n) \in \mathbb{N} \times \mathbb{N}$, $x_{m,n} \in \boldsymbol{L}(\{z_{j,k}\colon (j,k) \in \mathbb{N} \times \mathbb{N}\})$. But $\wp = inf_{j,k\in\mathbb{N}} z_{j,k}$, and thus $x_{m,n} \in \boldsymbol{L}(\wp)$ for every $(m,n) \in \mathbb{N} \times \mathbb{N}$, which shows that $\dot{\mu}(x_{m,n}, \wp) > \frac{1}{2}$ for every $(m,n) \in \mathbb{N} \times \mathbb{N}$. Hence $\wp \in \mathbf{U}(\{x_{m,n}\colon (m,n) \in \mathbb{N} \times \mathbb{N}\}$. Continuing in this way, we presume that there is another fuzzy lower bound of $(x_{m,n})_{m,n\in\mathbb{N}}$, say $\wp^* \in X$. Namely, let $\wp^* \in \mathbf{U}(\{x_{m,n}\colon (m,n) \in \mathbb{N} \times \mathbb{N}$. Then we can find an, $\dot{\mu}(x_{m,n}, \wp^*) > \frac{1}{2}$ for every $(m,n) \in \mathbb{N} \times \mathbb{N}$, also $\dot{\mu}(y_{m,n}, x_{m,n}) > \frac{1}{2}$ for every $(m,n) \in \mathbb{N} \times \mathbb{N}$. Therefore we can write $\dot{\mu}(y_{m,n}, \wp^*) > \frac{1}{2}$ for every $(m,n) \in \mathbb{N} \times \mathbb{N}$. Thus $\wp^* \in \mathbf{U}(\{y_{m,n}\colon (m,n) \in \mathbb{N} \times \mathbb{N}\})$ implies that $\wp^* \in \mathbf{U}(\wp)$ which as $\wp = sup_{j,k\in\mathbb{N}} x_{j,k}$ or $x_{j,k} \uparrow^{FR} \wp$.

Conversely, assume that $x_{j,k} \uparrow^{FR} \wp$. Now let us consider $y_{j,k} = x_{j,k}$ and $z_{j,k} = \wp$ for every $(j,k) \in \mathbb{N} \times \mathbb{N}$. Then we have a double sequences $(y_{j,k})_{j,k\in\mathbb{N}}$ is FRi and $(z_{j,k})_{j,k\in\mathbb{N}}$ is FRd. Furthermore $y_{j,k} \uparrow^{FR} \wp$ and $z_{j,k} \downarrow_{FR} \wp$. By using hypothesis in above, we have $\dot{\mu}(y_{j,k}, x_{j,k}) > \frac{1}{2}$ and $\dot{\mu}(x_{j,k}, z_{j,k}) = \dot{\mu}(x_{j,k}, \wp) > \frac{1}{2}$, then $\dot{\mu}(y_{j,k}, x_{j,k}) > \frac{1}{2}$ and $\dot{\mu}(x_{j,k}, z_{j,k}) > \frac{1}{2}$ for every $(j,k) \in \mathbb{N} \times \mathbb{N}$.

Which shows that $x_{j,k} \xrightarrow{FR(O)} \wp$.

With a similar argument we can provide a proof for the case fuzzy decreasing double sequence.

ii- assume that $(x_{j,k})_{j,k\in\mathbb{N}}$ be a double sequence of points in X and

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$x_{j,k} \xrightarrow{FR(O)} \wp$, after that have a double sequences $(y_{j,k})_{j,k\in\mathbb{N}}$ and $(z_{j,k})_{j,k\in\mathbb{N}}$ such that $\dot{\mu}(y_{j,k}, x_{j,k}) > \frac{1}{2}$ and $\dot{\mu}(x_{j,k}, z_{j,k}) > \frac{1}{2}$ for e very $(j,k) \in \mathbb{N}\times\mathbb{N}$ and thus $(y_{j,k})_{j,k\in\mathbb{N}}$ is FRi and $(z_{j,k})_{j,k\in\mathbb{N}}$ is FRd. Now since $y_{j,k} \uparrow^{FR}$ thereby, we see that $\dot{\mu}(y_{m,n}, y_{j,k}) > \frac{1}{2}$ for every $(m,n) \le (j,k) \in \mathbb{N}\times\mathbb{N}$. Thus $y_{m,n} \in \boldsymbol{L}(\{y_{j,k}\colon (j,k) \in \mathbb{N}\times\mathbb{N}\})$. Also $\dot{\mu}(y_{j,k}, x_{j,k}) > \frac{1}{2}$ which shows that $\dot{\mu}(y_{m,n}, x_{j,k}) > \frac{1}{2}$ for every $(m,n) \le (j,k) \in \mathbb{N}\times\mathbb{N}$. Thereby we see that $y_{m,n} \in \boldsymbol{L}(\{x_{j,k}\colon (j,k) \in \mathbb{N}\times\mathbb{N}\})$ and thus $(x_{j,k})_{j,k\in\mathbb{N}}$ is bounded from below. Now it follows that $z_{j,k} \downarrow_{FR}$. So in situation we find that

$z_{m,n} \in \mathbf{U}(\{z_{j,k}\colon (j,k) \in \mathbb{N}\times\mathbb{N}\})$ must hold and hence $\dot{\mu}(z_{j,k}, z_{m,n}) > \frac{1}{2}$ for every $(m,n) \le (j,k) \in \mathbb{N}\times\mathbb{N}$. We know from Definition 3.2 that. Therefore $\dot{\mu}(x_{j,k}, z_{m,n}) > \frac{1}{2}$ for every $(m,n) \le (j,k) \in \mathbb{N}\times\mathbb{N}$ must hold, which means that $z_{m,n} \in \mathbf{U}(\{x_{j,k}\colon (j,k) \in \mathbb{N}\times\mathbb{N}\})$. We conclude that $(x_{j,k})_{j,k\in\mathbb{N}}$ is bounded from above. Hence $(x_{j,k})_{j,k\in\mathbb{N}}$ is bounded. Then the proof is finished.

***Theorem* 3.12.** In a fuzzy Riesz space X , we have the following.

i- If $(x_{j,k})_{j,k\in\mathbb{N}}$ and $(z_{j,k})_{j,k\in\mathbb{N}}$ be a double sequences thus,

$\dot{\mu}(x_{j,k}, z_{j,k}) > \frac{1}{2}$ for every $(j,k) \in \mathbb{N}\times\mathbb{N}$ and

$\mathrm{FR(O)} - \lim_{j,k\to\infty} x_{j,k} = \wp$, $\mathrm{FR(O)} - \lim_{j,k\to\infty} z_{j,k} = \wp^*$, then

$\dot{\mu}(\wp, \wp^*) > \frac{1}{2}$.

ii- Every $FR(O)-$Convergent double sequence in X has only one limit.

iii- If $x_{j,k} \xrightarrow{FR(O)} \wp$, then any double sequence of the double sequence$(x_{j,k})_{j,k\in\mathbb{N}}$, is $FR(O)-$Convergent to the same limit.

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Proof.

i- From the definition of fuzzy order convergence for double sequence, it follows that there exists a double sequence $(y_{j,k})_{j,k\in\mathbb{N}}$ of points in X such that $y_{j,k} \uparrow^{FR} \wp$ and $\dot{\mu}(y_{j,k}, x_{j,k}) > \frac{1}{2}$ for each $(j,k) \in \mathbb{N}\times\mathbb{N}$. Clearly $\dot{\mu}(x_{j,k}, z_{j,k}) > \frac{1}{2}$ implies that, $\dot{\mu}(y_{j,k}, z_{j,k}) > \frac{1}{2}$ for every $(j,k) \in \mathbb{N}\times\mathbb{N}$, and by duality, $z_{j,k} \xrightarrow{FR(O)} \wp^*$, there exists a double sequence $(s_{j,k})_{j,k\in\mathbb{N}}$ of points in X such that $s_{j,k} \downarrow_{FR} \wp^*$ and $\dot{\mu}(z_{j,k}, s_{j,k}) > \frac{1}{2}$ for every $(j,k) \in \mathbb{N}\times\mathbb{N}$. Now let $(m,n) \in \mathbb{N}\times\mathbb{N}$ be a fixed such that $(j,k) \le (m,n)$, since $(y_{j,k})_{j,k\in\mathbb{N}}$ is an fuzzy increasing double sequence then we see that $\dot{\mu}(y_{j,k}, y_{m,n}) > \frac{1}{2}$ and $\dot{\mu}(y_{m,n}, z_{m,n}) > \frac{1}{2}$ for every $(m,n) \in \mathbb{N}\times\mathbb{N}$.Thereby by transitivity $\dot{\mu}(y_{j,k}, z_{m,n}) > \frac{1}{2}$ for every $(j,k) \le (m,n)$. But $\dot{\mu}(z_{m,n}, s_{m,n}) > \frac{1}{2}$, Whence $\dot{\mu}(y_{j,k}, s_{m,n}) > \frac{1}{2}$ for every $(j,k) \le (m,n)$.

On the other hand, if $(j,k)>(m,n)$, so we now $s_{j,k} \downarrow_{FR}$, then $\dot{\mu}(s_{j,k}, s_{m,n}) > \frac{1}{2}$ and from $\dot{\mu}(z_{j,k}, s_{j,k}) > \frac{1}{2}$ for every $(j,k) \in \mathbb{N}\times\mathbb{N}$, then by transitivity we can write, $\dot{\mu}(z_{j,k}, s_{m,n}) > \frac{1}{2}$ for every $(j,k)>(m,n)$. Further $\dot{\mu}(y_{j,k}, z_{j,k}) > \frac{1}{2}$ thereby we obtain $\dot{\mu}(y_{j,k}, s_{m,n}) > \frac{1}{2}$ for every $(j,k)>(m,n)$, which means that

$s_{m,n} \in \mathbf{U}(\{y_{j,k}\colon (j,k) \in \mathbb{N}\times\mathbb{N}\})$. But we know from definition $FR(O)-$ Convergent

$\wp = sup_{j,k\in\mathbb{N}} y_{j,k}$, therefore $s_{m,n} \in \mathbf{U}(\wp)$. Since $(m,n) \in \mathbb{N}\times\mathbb{N}$ was arbitrary fixed, consequently we have $\dot{\mu}(\wp, s_{m,n}) > \frac{1}{2}$ for every $(m,n) \in \mathbb{N}\times\mathbb{N}$. Which is $\wp \in \mathbf{L}(\{s_{m,n}\colon (m,n) \in \mathbb{N}\times\mathbb{N}\})$. But we know that by definition of $FR(O)-$ Convergent of

$(z_{j,k})_{j,k\in\mathbb{N}}$ that $\wp^* = inf_{j,k\in\mathbb{N}} s_{j,k}$. It follows that $\wp \in L(\wp^*)$ must hold, from this we obtain that $\dot{\mu}(\wp,\wp^*) > \frac{1}{2}$.

ii- It is clear that $\dot{\mu}(x_{j,k}, x_{j,k}) > \frac{1}{2}$ for every $(j,k) \in \mathbb{N}\times\mathbb{N}$. Then by applying (i), $\dot{\mu}(\wp,\wp^*) > \frac{1}{2}$. So let us assume that $\dot{\mu}(\wp^*,\wp) > \frac{1}{2}$. Then we can write $\dot{\mu}(\wp,\wp^*) + \dot{\mu}(\wp^*,\wp) > 1$, and from the condition antisymmetry of fuzzy order relation, from this we can conclude that $\wp = \wp^*$.

iii- presume $x_{j,k} \xrightarrow{FR(O)} \wp$, From this have, a double sequences $(y_{j,k})_{j,k\in\mathbb{N}}$ and $(z_{j,k})_{j,k\in\mathbb{N}}$ in X such that $y_{j,k} \uparrow^{FR} \wp$ and $z_{j,k} \downarrow_{FR} \wp$. Consequently $\dot{\mu}(y_{j,k}, x_{j,k}) > \frac{1}{2}$ and $\dot{\mu}(x_{j,k}, z_{j,k}) > \frac{1}{2}$ for every $(j,k) \in \mathbb{N}\times\mathbb{N}$. Now let us consider $(x_{j_r,k_s})_{r,s\in\mathbb{N}}$ be any subsequence of $(x_{j,k})_{j,k\in\mathbb{N}}$. From now get,

$\dot{\mu}(y_{j_r,k_s}, x_{j_r,k_s}) > \frac{1}{2}$ and $\dot{\mu}(x_{j_r,k_s}, z_{j_r,k_s}) > \frac{1}{2}$ for all $(j_r,k_s) \in \mathbb{N}\times\mathbb{N}$ where $(y_{j_r,k_s})_{r,s\in\mathbb{N}}$ is FRi and $(z_{j_r,k_s})_{r,s\in\mathbb{N}}$ is FRd. We need to show that $\wp = sup_{r,s\in\mathbb{N}} y_{j_r,k_s}$. To see this, consider $\dot{\mu}(y_{j_r,k_s}, \wp) > \frac{1}{2}$ for every $(j_r,k_s) \in \mathbb{N}\times\mathbb{N}$, where $\wp \in \mathbf{U}(\{y_{j_r,k_s}\colon (j_r,k_s) \in \mathbb{N}\times\mathbb{N}\})$. Now assume that there is another fuzzy upper bound of $(y_{j_r,k_s})_{r,s\in\mathbb{N}}$, say, $\wp^* \in X$ such that $\wp^* \in \mathbf{U}(\{y_{j_r,k_s}\colon (j_r,k_s) \in \mathbb{N}\times\mathbb{N}\})$. Hence we get $\dot{\mu}(y_{j_r,k_s}, \wp^*) > \frac{1}{2}$ for every $(j_r,k_s) \in \mathbb{N}\times\mathbb{N}$. In fact, we can write $\dot{\mu}(y_{j,k}, \wp^*) > \frac{1}{2}$ for every $(j,k) \in \mathbb{N}\times\mathbb{N}$. Since $y_{j,k} \uparrow^{FR} \wp$, the supremum of a double sequence $(y_{j,k})_{j,k\in\mathbb{N}}$ is $\wp$. Hence we get $\dot{\mu}(\wp,\wp^*) > \frac{1}{2}$ or $\wp^* \in \mathbf{U}(\wp)$. It follows that $\wp = sup_{r,s\in\mathbb{N}} y_{j_r,k_s}$. Therefore part $z_{j_r,k_s} \downarrow_{FR} \wp$ can be proved similarly. This shows that $x_{j_r,k_s} \xrightarrow{FR(O)} \wp$.

***Theorem* 3.13.**

Let X be a fuzzy Riesz space and let $(x_{j,k})_{j,k\in\mathbb{N}}$, $(r_{j,k})_{j,k\in\mathbb{N}}$ and $(s_{j,k})_{j,k\in\mathbb{N}}$ be three double sequences of points in X such that.

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i- $\dot{\mu}(r_{j,k}, x_{j,k}) > \frac{1}{2}$ and $\dot{\mu}(x_{j,k}, s_{j,k}) > \frac{1}{2}$ for every $(j,k) \in \mathbb{N} \times \mathbb{N}$.

ii- $\mathrm{FR(O)} - \lim_{j,k \to \infty} r_{j,k} = \wp = \mathrm{FR(O)} - \lim_{j,k \to \infty} s_{j,k}$. Then

iii- $\mathrm{FR(O)} - \lim_{j,k \to \infty} x_{j,k} = \wp$.

Proof.

Assume that the given conditions (i) and (ii) hold for the double sequences $(x_{j,k})_{j,k\in\mathbb{N}}$, $(r_{j,k})_{j,k\in\mathbb{N}}$ and $(s_{j,k})_{j,k\in\mathbb{N}}$. Suppose $(y_{j,k})_{j,k\in\mathbb{N}}$ and $(z_{j,k})_{j,k\in\mathbb{N}}$ such that $y_{j,k} \uparrow^{FR} \wp$ and $z_{j,k} \downarrow_{FR} \wp$, from condition (ii), we have $\dot{\mu}(y_{j,k}, r_{j,k}) > \frac{1}{2}$ and $\dot{\mu}(s_{j,k}, z_{j,k}) > \frac{1}{2}$ for every $(j,k) \in \mathbb{N} \times \mathbb{N}$. Now $\dot{\mu}(y_{j,k}, r_{j,k}) > \frac{1}{2}$ and from (i) we have $\dot{\mu}(y_{j,k}, x_{j,k}) > \frac{1}{2}$ for every $(j,k) \in \mathbb{N} \times \mathbb{N}$. So from $\dot{\mu}(s_{j,k}, z_{j,k}) > \frac{1}{2}$ and $\dot{\mu}(x_{j,k}, s_{j,k}) > \frac{1}{2}$ for every $(j,k) \in \mathbb{N} \times \mathbb{N}$, we can write $\dot{\mu}(x_{j,k}, z_{j,k}) > \frac{1}{2}$ for every $(j,k) \in \mathbb{N} \times \mathbb{N}$. Thus, $\dot{\mu}(y_{j,k}, x_{j,k}) > \frac{1}{2}$ and $\dot{\mu}(x_{j,k}, z_{j,k}) > \frac{1}{2}$ for every $(j,k) \in \mathbb{N} \times \mathbb{N}$, means implies that $\mathrm{FR(O)} - \lim_{j,k \to \infty} x_{j,k} = \wp$. This completes the proof of the theorem.

***Theorem* 3.14.** Let X be a fuzzy Riesz space, $(x_{j,k})_{j,k\in\mathbb{N}}$ and $(y_{m,n})_{m,n\in\mathbb{N}}$ be two double sequences of points in X and $\bigvee_{j,k\in\mathbb{N}} x_{j,k}$, $\bigwedge_{j,k\in\mathbb{N}} x_{j,k}$, $\bigvee_{m,n\in\mathbb{N}} y_{m,n}$ and $\bigwedge_{m,n\in\mathbb{N}} y_{m,n}$ exist.

i. $\bigvee_{j,k\in\mathbb{N}} x_{j,k} = -\bigwedge_{j,k\in\mathbb{N}}(-x_{j,k})$ and $\bigwedge_{j,k\in\mathbb{N}} x_{j,k} = -\bigvee_{j,k\in\mathbb{N}}(-x_{j,k})$.

ii. $x \wedge (\bigvee_{m,n\in\mathbb{N}} y_{m,n}) = \bigvee_{m,n\in\mathbb{N}}(x \wedge y_{m,n})$ and $x \vee (\bigwedge_{m,n\in\mathbb{N}} y_{m,n}) = \bigwedge_{m,n\in\mathbb{N}}(x \vee y_{m,n})$, for any $x \in X$.

iii. $\bigvee_{j,k,m,n\in\mathbb{N}}(x_{j,k} \vee y_{m,n}) = \bigvee_{j,k\in\mathbb{N}} x_{j,k} \vee \bigvee_{m,n\in\mathbb{N}} y_{m,n}$ and $\bigwedge_{j,k,m,n\in\mathbb{N}}(x_{j,k} \wedge y_{m,n}) = \bigwedge_{j,k\in\mathbb{N}} x_{j,k} \wedge \bigwedge_{m,n\in\mathbb{N}} y_{m,n}$.

iv. $\bigvee_{j,k\in\mathbb{N}}(\alpha x_{j,k}) = \alpha(\bigvee_{j,k\in\mathbb{N}} x_{j,k})$ and $\bigwedge_{j,k\in\mathbb{N}}(\alpha x_{j,k}) = \alpha(\bigwedge_{j,k\in\mathbb{N}} x_{j,k})$, $\alpha \in \mathbb{R}$.

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Proof. We show its completely analogous to the proof of corresponding results for sequences in [5] and [6].

***Theorem* 3.15.** Let X be a fuzzy Riesz space and let $(x_{j,k})_{j,k\in\mathbb{N}}$ and $(s_{j,k})_{j,k\in\mathbb{N}}$ be two double sequences of points in X . Then

i- If $\mathrm{FR(O)}-\lim_{j,k\to\infty} x_{j,k} = \wp$ and $\mathrm{FR(O)}-\lim_{j,k\to\infty} s_{j,k} = \wp^*$, then

$\mathrm{FR(O)}-\lim_{j,k\to\infty}(x_{j,k}\vee s_{j,k}) = \wp\vee\wp^*$ and

$\mathrm{FR(O)}-\lim_{j,k\to\infty}(x_{j,k}\wedge s_{j,k}) = \wp\wedge\wp^*$.

ii- If $\mathrm{FR(O)}-\lim_{j,k\to\infty} x_{j,k} = \wp$, then

$\mathrm{FR(O)}-\lim_{j,k\to\infty} \alpha x_{j,k} = \alpha\wp$ for $0\le\alpha\in\mathbb{R}$.

iii- If $\mathrm{FR(O)}-\lim_{j,k\to\infty}|x_{j,k}| = 0$, then also

$\mathrm{FR(O)}-\lim_{j,k\to\infty} x_{j,k} = 0$.

***Proof*.**

i- To prove the first assertion, by the definition of the $\mathrm{FR(O)}-$Convergent, is encompass a double sequences $y_{j,k}\uparrow^{FR}\wp$, $z_{j,k}\downarrow_{FR}\wp$, $y'_{j,k}\uparrow^{FR}\wp^*$ and $z'_{j,k}\downarrow_{FR}\wp^*$ such that $\dot{\mu}(y_{j,k},x_{j,k}) > \frac{1}{2}$, $\dot{\mu}(x_{j,k},z_{j,k}) > \frac{1}{2}$

$\dot{\mu}(y'_{j,k},s_{j,k}) > \frac{1}{2}$ and

$\dot{\mu}(s_{j,k},z'_{j,k}) > \frac{1}{2}$ for every $(j,k)\in\mathbb{N}\times\mathbb{N}$. Then

$\dot{\mu}(y_{j,k}\vee y'_{j,k},x_{j,k}\vee s_{j,k}) > \frac{1}{2}$ and $\dot{\mu}(x_{j,k}\vee s_{j,k},z_{j,k}\vee z'_{j,k}) > \frac{1}{2}$

for every $(j,k)\in\mathbb{N}\times\mathbb{N}$, and moreover $(y_{j,k}\vee y'_{j,k})_{j,k\in\mathbb{N}}$ is FRi double sequence and $(z_{j,k}\vee z'_{j,k})_{j,k\in\mathbb{N}}$ is FRd double sequence. In order to show that $\wp\vee\wp^* = \sup_{j,k\in\mathbb{N}}(y_{j,k}\vee y'_{j,k})_{j,k\in\mathbb{N}}$ and $\wp\vee\wp^* = \inf_{j,k\in\mathbb{N}}(z_{j,k}\vee z'_{j,k})_{j,k\in\mathbb{N}}$. It is clear that

$\dot{\mu}(y_{j,k},\wp) > \frac{1}{2}$ and $\dot{\mu}(y'_{j,k},\wp^*) > \frac{1}{2}$ holds. Consequently

$\dot{\mu}(y_{j,k}\vee y'_{j,k},\wp\vee\wp^*) > \frac{1}{2}$ for every $(j,k)\in\mathbb{N}\times\mathbb{N}$. From this it follows,

$\wp \vee \wp^* \in \mathbf{U}(\{y_{j,k} \vee y'_{j,k} : (j,k) \in \mathbb{N}\times\mathbb{N}\})$. Now suppose that there is another upper bound of $\left(y_{j,k} \vee y'_{j,k}\right)_{j,k\in\mathbb{N}}$, say, $\wp^{**} \in X$ such that $\wp^{**} \in \mathbf{U}(\{y_{j,k} \vee y'_{j,k} : (j,k) \in \mathbb{N}\times\mathbb{N}\})$ then $\dot{\mu}(y_{j,k} \vee y'_{j,k}, \wp^{**}) > \frac{1}{2}$ for every $(j,k) \in \mathbb{N}\times\mathbb{N}$. Now we want to show that $\dot{\mu}(y_{m,n} \vee y'_{j,k}, \wp^{**}) > \frac{1}{2}$ for every $(m,n),(j,k) \in \mathbb{N}\times\mathbb{N}$. To see this, consider (j,k) is fixed, taking $(m,n) \le (j,k)$, we have that $\dot{\mu}(y_{m,n}, y_{j,k}) > \frac{1}{2}$, consequently $\dot{\mu}(y_{m,n} \vee y'_{j,k}, y_{j,k} \vee y'_{j,k}) > \frac{1}{2}$ for every $(m,n),(j,k) \in \mathbb{N}\times\mathbb{N}$. Also $\dot{\mu}(y_{j,k} \vee y'_{j,k}, \wp^{**}) > \frac{1}{2}$. Thereby for every $(m,n) \le (j,k)$ we can write $\dot{\mu}(y_{m,n} \vee y'_{j,k}, \wp^{**}) > \frac{1}{2}$. On the other hand if $(m,n) > (j,k)$ after that $\left(y'_{j,k}\right)_{j,k\in\mathbb{N}}$ is increasing, therefore write this as, $\dot{\mu}(y'_{j,k}, y'_{m,n}) > \frac{1}{2}$. Which implies that $\dot{\mu}(y_{m,n} \vee y'_{j,k}, y_{m,n} \vee y'_{m,n}) > \frac{1}{2}$, accordingly we know that $\dot{\mu}(y_{m,n} \vee y'_{m,n}, \wp^{**}) > \frac{1}{2}$ so we get $\dot{\mu}(y_{m,n} \vee y'_{j,k}, \wp^{**}) > \frac{1}{2}$ for all $(m,n),(j,k) \in \mathbb{N}\times\mathbb{N}$. Since (j,k) was arbitrary fixed, therefore $\dot{\mu}(y_{m,n} \vee y'_{j,k}, \wp^{**}) > \frac{1}{2}$ for each $(m,n),(j,k) \in \mathbb{N}\times\mathbb{N}$. It implies that $\wp^{**} \in \mathbf{U}(\{y_{m,n} \vee y'_{j,k} : (m,n),(j,k) \in \mathbb{N}\times\mathbb{N}\})$ and so, $\wp^{**} \in \mathbf{U}(\bigvee_{j,k,m,n\in\mathbb{N}}(y_{m,n} \vee y'_{j,k}))$. Thus we know $\bigvee_{m,n,j,k}\left(y_{m,n} \vee y'_{j,k}\right)$ exist because both $\bigvee_{m,n\in\mathbb{N}} y_{m,n}$ and $\bigvee_{j,k\in\mathbb{N}} y'_{j,k}$ exist and by Theorem 3.13, the following equality holds

$$\bigvee_{m,n,j,k}\left(y_{m,n} \vee y'_{j,k}\right) = \bigvee_{m,n\in\mathbb{N}} y_{m,n} \vee \bigvee_{j,k\in\mathbb{N}} y'_{j,k} = \wp \vee \wp^*.$$

Thus we have that $\wp^{**} \in \mathbf{U}(\wp \vee \wp^*)$. From this we obtain that $\wp \vee \wp^* = \bigvee_{j,k\in\mathbb{N}}\left(y_{j,k} \vee y'_{j,k}\right)_{j,k\in\mathbb{N}}$. With a similar argument we can provide a proof for the case $\wp \vee \wp^* = \bigwedge_{j,k\in\mathbb{N}}(z_{j,k} \vee z'_{j,k})_{j,k\in\mathbb{N}}$. Hence we get

$$\mathrm{FR(O)} - \lim_{j,k\to\infty}(x_{j,k} \vee s_{j,k}) = \wp \vee \wp^*.$$

Similarly, we get $FR(O)-\lim_{j,k\to\infty}(x_{j,k}\wedge s_{j,k})=\wp\wedge\wp^{*}$

ii- Suppose that $FR(O)-\lim_{j,k\to\infty}x_{j,k}=\wp$. Then we can find a double sequences $(y_{j,k})_{j,k\in\mathbb{N}}$ and $(z_{j,k})_{j,k\in\mathbb{N}}$ such that $\dot{\mu}(y_{j,k},x_{j,k})>\frac{1}{2}$ and $\dot{\mu}(x_{j,k},z_{j,k})>\frac{1}{2}$ for every $(j,k)\in\mathbb{N}\times\mathbb{N}$ where $y_{j,k}\uparrow^{FR}\wp$ and $z_{j,k}\downarrow_{FR}\wp$. Assume first that $\alpha>0$. Then $\dot{\mu}(\alpha y_{j,k},\alpha x_{j,k})>\frac{1}{2}$ and $\dot{\mu}(\alpha x_{j,k},\alpha z_{j,k})>\frac{1}{2}$ for every $(j,k)\in\mathbb{N}\times\mathbb{N}$. Since $(y_{j,k})_{j,k\in\mathbb{N}}$ be FRi, thereby if $(m,n)\le(j,k)$ have that, $\dot{\mu}(y_{m,n},y_{j,k})>\frac{1}{2}$. Consequently $\dot{\mu}(\alpha y_{m,n},\alpha y_{j,k})>\frac{1}{2}$. From this we have $(\alpha y_{j,k})_{j,k\in\mathbb{N}}$ be a FRi double sequence. With a similar argument we get $(\alpha z_{j,k})_{j,k\in\mathbb{N}}$ be a FRd double sequence, and moreover from Theorem 3.13, we see that $\vee_{j,k}(\alpha y_{j,k})=\alpha(\vee_{j,k}y_{j,k})=\alpha\wp$ and $\bigwedge_{j,k\in\mathbb{N}}(\alpha z_{j,k})=\alpha(\bigwedge_{j,k\in\mathbb{N}}z_{j,k})=\alpha\wp$. Then we have $FR(O)-\lim_{j,k\to\infty}\alpha x_{j,k}=\alpha\wp$. And if $\alpha<0$, then from $\dot{\mu}(y_{j,k},x_{j,k})>\frac{1}{2}$ with $\dot{\mu}(x_{j,k},z_{j,k})>\frac{1}{2}$ for each $(j,k)\in\mathbb{N}\times\mathbb{N}$. We have that $\dot{\mu}(\alpha z_{j,k},\alpha x_{j,k})>\frac{1}{2}$ along with $\dot{\mu}(\alpha x_{j,k},\alpha y_{j,k})>\frac{1}{2}$ for every $(j,k)\in\mathbb{N}\times\mathbb{N}$. Since $(y_{j,k})_{j,k\in\mathbb{N}}$ is FRi double sequence and $(z_{j,k})_{j,k\in\mathbb{N}}$ is FRd double sequence, it follows that $(\alpha y_{j,k})_{j,k\in\mathbb{N}}$ be a FRd double sequence and $(\alpha z_{j,k})_{j,k\in\mathbb{N}}$ be a FRi double sequence. By using Theorem 3.13 $\bigvee_{j,k\in\mathbb{N}}(\alpha z_{j,k})=\alpha(\bigwedge_{j,k\in\mathbb{N}}z_{j,k})=\alpha\wp$ and $\bigwedge_{j,k\in\mathbb{N}}(\alpha y_{j,k})=\alpha(\bigvee_{j,k\in\mathbb{N}}y_{j,k})=\alpha\wp$. Consequently, also in this case $FR(O)-\lim_{j,k\to\infty}\alpha x_{j,k}=\alpha\wp$. But if $\alpha=0$, then there is nothing to prove. This completes the proof of the theorem.

iii- Assume that $FR(O)-\lim_{j,k\to\infty}|x_{j,k}|=0$, then there exists a double sequence $y_{j,k}\downarrow_{FR}0$ such that $\dot{\mu}(|x_{j,k}|,y_{j,k})>\frac{1}{2}$ for every $(j,k)\in\mathbb{N}\times\mathbb{N}$. Also $\dot{\mu}(x_{j,k},|x_{j,k}|)>\frac{1}{2}$, therefore by transitivity we have $\dot{\mu}(x_{j,k},y_{j,k})>\frac{1}{2}$

for every $(j,k) \in \mathbb{N}\times\mathbb{N}$. Since $\bigwedge_{j,k\in\mathbb{N}} y_{j,k} = 0$. At that moment $0 \in (\{y_{j,k}: (j,k) \in \mathbb{N}\times\mathbb{N}\})$ or $\dot{\mu}(0, y_{j,k}) > \frac{1}{2}$ for every $(j,k) \in \mathbb{N}\times\mathbb{N}$. Consequently, we see that $\dot{\mu}(-y_{j,k}, 0) > \frac{1}{2}$ for every $(j,k) \in \mathbb{N}\times\mathbb{N}$. So we have $0 = \bigvee_{j,k\in\mathbb{N}}(-y_{j,k})$. From this we can write $-y_{j,k} \uparrow^{FR} 0$. Since we have that $|x_{j,k}| \xrightarrow{FR(O)} 0$, then we obtain for every $(j,k) \in \mathbb{N}\times\mathbb{N}$ that $\dot{\mu}(-y_{j,k}, -|x_{j,k}|) > \frac{1}{2}$ and we know $\dot{\mu}(-|x_{j,k}|, x_{j,k}) > \frac{1}{2}$. By transitivity we can note down $\dot{\mu}(-y_{j,k}, x_{j,k}) > \frac{1}{2}$. This proves that $\mathrm{FR(O)} - \lim_{j,k\to\infty} x_{j,k} = 0$.

***Theorem* 3.16.** Let X be a fuzzy Riesz space and let $(x_{j,k})_{j,k\in\mathbb{N}}$ and $(s_{j,k})_{j,k\in\mathbb{N}}$ be two double sequences in X. Then the following hold:

i- If $\mathrm{FR(O)} - \lim_{j,k\to\infty} x_{j,k} = \wp$ and $\mathrm{FR(O)} - \lim_{j,k\to\infty} s_{j,k} = \wp^*$, then $\mathrm{FR(O)} - \lim_{j,k\to\infty} (x_{j,k} + s_{j,k}) = \wp + \wp^*$.

ii- If $\mathrm{FR(O)} - \lim_{j,k\to\infty} x_{j,k} = \wp$, then $\mathrm{FR(O)} - \lim_{j,k\to\infty} x^+_{j,k} = \wp^+$, $\mathrm{FR(O)} - \lim_{j,k\to\infty} x^-_{j,k} = \wp^-$ and $\mathrm{FR(O)} - \lim_{j,k\to\infty} |x_{j,k}| = |\wp|$.

iii- If $\mathrm{FR(O)} - \lim_{j,k\to\infty} x_{j,k} = \wp$, $\mathrm{FR(O)} - \lim_{j,k\to\infty} s_{j,k} = \wp^*$ and $x_{j,k} \perp s_{j,k}$ for every $(j,k) \in \mathbb{N}\times\mathbb{N}$, then $\wp \perp \wp^*$.

Proof.

i- We can easily prove independently all these properties of $FR(O)-$Convergence.

ii- Assume that $\mathrm{FR(O)} - \lim_{j,k\to\infty} x_{j,k} = \wp$. Now to prove the first assertion.

Since we can inscribe $x^+_{j,k} = x_{j,k} \vee 0 \xrightarrow{FR(O)} \wp \vee 0 = \wp^+$, then we have

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$x^{+}_{j,k} \xrightarrow{FR(O)} \wp^{+}$. So to have the second assertion, we will write

$x^{-}_{j,k} = (-x_{j,k})^{+} \xrightarrow{FR(O)} (-\wp)^{+} = \wp^{-}$.

Finally, using the arguments just above. To see this, note down,

$\left|x_{j,k}\right| = x^{+}_{j,k} + x^{-}_{j,k} \xrightarrow{FR(O)} \wp^{+} + \wp^{-} = \left|\wp\right|$.

iii- Assume that, $\mathrm{FR(O)} - \lim_{j,k\to\infty} s_{j,k} = \wp^{*}$ as well as $x_{j,k} \perp s_{j,k}$. Is include $|x_{j,k}| \wedge |s_{j,k}| = 0$ for every $j,k \in \mathbb{N}$ and by (ii), we get $\left|x_{j,k}\right| \xrightarrow{FR(O)} \left|\wp\right|$ and $\left|s_{j,k}\right| \xrightarrow{FR(O)} \left|\wp^{*}\right|$ and thus $\left|x_{j,k}\right| \wedge \left|s_{j,k}\right| \xrightarrow{FR(O)} \left|\wp\right| \wedge \left|\wp^{*}\right|$ by Theorem 3.14. Hence that $\left|\wp\right| \wedge \left|\wp^{*}\right| = 0$ i.e., $\wp \perp \wp^{*}$.

***Theorem* 3.17.** Let X be a real fuzzy normed Riesz space. Then the subsequent statements are equivalent.

i- If X is a $\sigma-$ Complete fuzzy Riesz space and the fuzzy norm is fuzzy order continuous.

ii- Every fuzzy increasing and bounded above double sequence is $\mathrm{FR(O}_n) -$ Convergent.

Proof.

(i $\rightarrow$ ii) Suppose that $(x_{j,k})_{j,k\in\mathbb{N}}$ be a double sequence in X, let X be a $\sigma-$ Complete fuzzy Riesz space. Therefore there exists a real number $M > 0$ such that $\dot{\mu}(\left|x_{j,k}\right|, M) > \frac{1}{2}$, for every $(j,k) \in \mathbb{N}\times\mathbb{N}$ also since fuzzy norm is fuzzy order continuous, therefore $\mathrm{FR(O}_n) - \lim_{j,k\to\infty} \left\|x_{j,k}\right\| = 0$ from this there $(FO)-$Sequence $(\sigma_p)_{p\in\mathbb{N}}$ such that for every $p \in \mathbb{N}$, be able to , $\dot{\mu}(\left\|\left\|x_{j,k}\right\| - 0\right\|, \sigma_p) > \frac{1}{2}$, for every $(j,k) \in \mathbb{N}\times\mathbb{N}$.

This shows that a double sequence $(x_{j,k})_{j,k\in\mathbb{N}}$ be $\mathrm{FR(O}_n)-$Convergent.

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(ii $\rightarrow$ i) Let $(x_{j,k})_{j,k\in\mathbb{N}}$ be a double sequence in X, $(x_{j,k})_{j,k\in\mathbb{N}}$ be fuzzy increasing and bounded above. Then we have $\dot{\mu}(x_{j,k}, y_{m,n}) > \frac{1}{2}$ for every $(j,k) \le (m,n)$ in $\mathbb{N}\times\mathbb{N}$, such that $y_{m,n} = \bigvee_{j,k\in\mathbb{N}} x_{j,k}$. By the hypothesis (ii), there exists $x_0 \in X$ such that $y_{m,n} \xrightarrow{FR(O_n)} x_0$. From this there is $(FO)-$ Sequence $(\sigma_p)_{p\in\mathbb{N}}$ such that $\dot{\mu}(\|y_{m,n} - x_0\|, \sigma_p) > \frac{1}{2}$, for every $p\in\mathbb{N}$ because $(y_{m,n})_{m,n\in\mathbb{N}}$ be fuzzy increasing double sequence. But from above $\dot{\mu}(x_{j,k}, y_{m,n}) > \frac{1}{2}$ for every (j,k) $\le$ (m,n) therefore by transitivity implies that $\dot{\mu}(\|x_{j,k} - x_0\|, \sigma_p) > \frac{1}{2}$, for every $p\in\mathbb{N}$. Thereby we have $x_0 = \bigvee_{m,n\in\mathbb{N}} y_{m,n} = \bigvee_{j,k\in\mathbb{N}} x_{j,k}$.

Hence the fuzzy Riesz space X is $\sigma-$ Complete fuzzy Riesz space. It remains to show that fuzzy norm $\|\cdot\|$ is fuzzy order continuous. Assume that $(s_{j,k})_{j,k\in\mathbb{N}}$ be a double sequence in X and $s_{j,k} \downarrow_{FR} 0$, then $\dot{\mu}(0, s_{j,k}) > \frac{1}{2}$, for each $(j,k)\in\mathbb{N}\times\mathbb{N}$. Therefore $\dot{\mu}(-s_{j,k}, 0) > \frac{1}{2}$ for every $(j,k)\in\mathbb{N}\times\mathbb{N}$. By using (i) of the Definition **2.2** implies $\dot{\mu}(s_{1,1} - s_{j,k}, s_{1,1}) > \frac{1}{2}$. Consequently, $s_{1,1} - s_{j,k} \uparrow^{FR} s_{1,1}$, from condition (ii) we have that be an $(FO)-$ Sequence $(\sigma_p)_{p\in\mathbb{N}}$ such that for every $p\in\mathbb{N}$ $\dot{\mu}(\|s_{1,1} - s_{j,k} - s_{1,1}\|, \sigma_p) > \frac{1}{2}$, where $(j,k)\in\mathbb{N}\times\mathbb{N}$, implies $\dot{\mu}(\|s_{j,k}\|, \sigma_p) > \frac{1}{2}$. Hence we obtain $\mathrm{FR}(\mathrm{O}_n) - \lim_{j,k\to\infty} \|s_{j,k}\| = 0$, i.e., a fuzzy norm $\|\cdot\|$ be fuzzy order continuous.

***Theorem* 3.18.** Let X be a fuzzy normed Riesz space. Then every $\mathrm{FR}(\mathrm{O}_n)-$ Convergent double sequence $(x_{j,k})_{j,k\in\mathbb{N}}$ is $FR(O_n)-$Cauchy.

Proof.

Suppose that $(x_{j,k})_{j,k\in\mathbb{N}}$ be $FR(O_n)-$Convergent to an element x_0 of X. Then there exists an, $(FO)-$ Sequence $(\sigma_p)_{p\in\mathbb{N}}$ such that for $(j,k)\in\mathbb{N}\times\mathbb{N}$, where $\dot{\mu}(\|x_{j,k}-x_0\|,\sigma_p)>\frac{1}{2}$, for every $p\in\mathbb{N}$. If $(j,k),(m,n)$ in $\mathbb{N}\times\mathbb{N}$, then

$$\|x_{j,k}-x_{m,n}\|\le\|x_{j,k}-x_0\|+\|x_{m,n}-x_0\|\le 2\sigma_p$$

$\dot{\mu}(\|x_{j,k}-x_{m,n}\|,\|x_{j,k}-x_0\|+\|x_{m,n}-x_0\|)>\frac{1}{2}$, also, we know $\dot{\mu}(\|x_{j,k}-x_0\|+\|x_{m,n}-x_0\|,2\sigma_p)>\frac{1}{2}$ for every $p\in\mathbb{N}$. Therefore $\dot{\mu}(\|x_{j,k}-x_{m,n}\|,2\sigma_p)>\frac{1}{2}$ for every $p\in\mathbb{N}$. Hence we have a double sequence $(x_{j,k})_{j,k\in\mathbb{N}}$ is $FR(O_n)-$Cauchy.

***Theorem* 3.19.** X is Banach fuzzy Riesz space if every $FR(O_n)-$Cauchy double sequence is $FR(O_n)-$Convergent double sequence to an element $x_0\in X$.

Proof.

Assume that a double sequence $(x_{j,k})_{j,k\in\mathbb{N}}$ be a $FR(O_n)-$Cauchy but not $FR(O_n)-$ Convergent. Then there exist, $(FO)-$ Sequence $(\sigma_p)_{p\in\mathbb{N}}$ and for $(j,k)\in\mathbb{N}\times\mathbb{N}$, we get $\dot{\mu}(\|x_{j,k}-x_{m,n}\|,\sigma_p)>\frac{1}{2}$ for every $p\in\mathbb{N}$.

In particular, we can write for, $(j,k),(m,n)$ in $\mathbb{N}\times\mathbb{N}$

$$\dot{\mu}(\|x_{j,k}-x_{m,n}\|,2\|x_{j,k}-x_0\|)>\tfrac{1}{2}$$
…………………………………………….(*)

Since $(x_{j,k})_{j,k\in\mathbb{N}}$ is not $FR(O_n)-$Convergent, i.e., $\dot{\mu}(\sigma_p,\|x_{j,k}-x_0\|)>\frac{1}{2}$ where $p\in\mathbb{N}$, $(j,k)\in\mathbb{N}\times\mathbb{N}$. Therefore by (*), there exist $(FO)-$ Sequence $(\sigma_p)_{p\in\mathbb{N}}$ and for any $p\in\mathbb{N}$, we get for $(j,k),(m,n)$ in $\mathbb{N}\times\mathbb{N}$, $\dot{\mu}(\sigma_p,\|x_{j,k}-x_{m,n}\|)>\frac{1}{2}$. It's contradiction. Hence a double sequence $(x_{j,k})_{j,k\in\mathbb{N}}$ is $FR(O_n)-$Convergent.

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***Theorem* 3.20.** Let X be a fuzzy normed Riesz space, $(x_{j,k})_{j,k\in\mathbb{N}}$ be an increasing (decreasing) double sequence in X and $(x_{j,k})_{j,k\in\mathbb{N}}$ is $\mathrm{FR}(\mathrm{O}_n)-$Convergent to $x_0 \in X$. Then $x_0 = \bigvee_{j,k\in\mathbb{N}} x_{j,k}$ ($x_0 = \bigwedge_{j,k\in\mathbb{N}}(x_{j,k})$ respectively).

Proof.

Suppose that $(x_{j,k})_{j,k\in\mathbb{N}}$ be an increasing double sequence and $(j,k) \le (m,n)$ in $\mathbb{N}\times\mathbb{N}$. It implies that $\dot{\mu}(x_{j,k} - x_0 \wedge x_{j,k}, x_{j,k} \wedge x_{m,n} - x_{j,k} \wedge x_0) > \frac{1}{2}$ But we know $\dot{\mu}(x_{j,k} \wedge x_{m,n} - x_{j,k} \wedge x_0, |x_{m,n} - x_0|) > \frac{1}{2}$ therefore by transitivity we see that $\dot{\mu}(x_{j,k} - x_0 \wedge x_{j,k}, |x_{m,n} - x_0|) > \frac{1}{2}$. Consequently, for $(j,k),(m,n)$ in $\mathbb{N}\times\mathbb{N}$, $\dot{\mu}(\|x_{j,k} - x_0 \wedge x_{j,k}\|, \|x_{m,n} - x_0\|) > \frac{1}{2}$. Since $(x_{m,n})_{m,n\in\mathbb{N}}$ be an $\mathrm{FR}(\mathrm{O}_n)-$ Convergent to $x_0 \in X$, then. $(FO)-$ Sequence $(\sigma_p)_{p\in\mathbb{N}}$ such that, $\dot{\mu}(\|x_{m,n} - x_0\|, \sigma_p) > \frac{1}{2}$, where (m,n) in $\mathbb{N}\times\mathbb{N}$. By transitivity we write $\dot{\mu}(\|x_{j,k} - x_0 \wedge x_{j,k}\|, \sigma_p) > \frac{1}{2}$ for every $p \in \mathbb{N}$ means that $x_{j,k} = x_0 \wedge x_{j,k}$ and we have that $\dot{\mu}(x_{j,k}, x_0) > \frac{1}{2}$ for each $(j,k), \in \mathbb{N}\times\mathbb{N}$. Thus $x_0 \in \mathbf{U}(\{x_{j,k}: (j,k) \in \mathbb{N}\times\mathbb{N}\})$. It remains to show that $x_0 = \bigvee_{j,k\in\mathbb{N}} x_{j,k}$. Now if $y_0 \in X$ and $y_0 \in \mathbf{U}(\{x_{j,k}: (j,k) \in \mathbb{N}\times\mathbb{N}\})$. Then $\dot{\mu}(x_0 \vee y_0 - y_0, x_0 \vee y_0 - x_{j,k} \vee y_0) > \frac{1}{2}$ But $\dot{\mu}(x_0 \vee y_0 - x_{j,k} \vee y_0, |x_0 - x_{j,k}|) > \frac{1}{2}$ Transitivity implies $\dot{\mu}(x_0 \vee y_0 - y_0, |x_0 - x_{j,k}|) > \frac{1}{2}$. By using the same arguments as above, we see that $\dot{\mu}(x_0 \vee y_0, y_0) > \frac{1}{2}$. Consequently, we can write $\dot{\mu}(x_0, y_0) > \frac{1}{2}$. Hence $x_0 = \bigvee_{j,k\in\mathbb{N}} x_{j,k}$.

If $(x_{j,k})_{j,k\in\mathbb{N}}$ be an decreasing double sequence, then $x_0 = -\bigvee_{j,k\in\mathbb{N}}(-x_{j,k}) = \bigwedge_{j,k\in\mathbb{N}} x_{j,k}$.

Conflict of Interests.
There are non-conflicts of interest .

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الخلاصة

في هذا البحث قدمنا بعض المفاهيم حول التقارب الرتيب الضبابي للمتتاليات المزدوجة في فضاءات ريسز الضبابية, وكذلك عرفنا نوع جديد من التقارب وهو تقارب $FR(O_n)-$ وتقارب كوشي $FR(O_n)-$ للمتتاليات المزدوجة في فضاء ريسز الضبابي , ومن ثم برهنا بعض النتائج المهمة التي تتعلق في هذه المفاهيم.