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AN OCTANOMIAL MODEL FOR CUBIC SURFACES

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We present a new normal form for cubic surfaces that is well suited for p-adic geometry, as it reveals the intrinsic del Pezzo combinatorics of the 27 trees in the tropicalization. The new normal form is a polynomial with eight terms, written in moduli from the E_6 hyperplane arrangement. If such a surface is tropically smooth then its 27 tropical lines are distinct. We focus on explicit computations, both symbolic and p-adic numerical.

1. Introduction

Any configuration of six distinct points in the projective plane $\mathbb{P}^2$ lies on a cuspidal cubic. Thus, after an automorphism of $\mathbb{P}^2$, the homogeneous coordinates of the points are the columns of a 3×6 matrix that has the following special form:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ d_1 & d_2 & d_3 & d_4 & d_5 & d_6 \\ d_1^3 & d_2^3 & d_3^3 & d_4^3 & d_5^3 & d_6^3 \end{pmatrix}. \qquad (1)$$

The 3×3-minors of this matrix factor into linear factors, and so does the condition for the six points to lie on a conic. The linear forms are $d_i - d_j$ and $d_i + d_j + d_k$ and $d_1 + d_2 + d_3 + d_4 + d_5 + d_6$, for a total of $36 = 15 + 20 + 1$. These define the 36 hyperplanes in the reflection arrangement of Coxeter type E_6. The complement of this hyperplane arrangement uniformizes the 4-dimensional

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moduli space of cubic surfaces. This representation of marked del Pezzo surfaces of degree three is used in many sources, including [4, 5, 9, 18].

In this paper we study the cubic surface in $\mathbb{P}^3$ that is obtained by blowing up $\mathbb{P}^2$ at the six points in (1). We write this cubic explicitly in terms of its moduli parameters d_i. To this end, we choose the following specific basis for the four-dimensional space of ternary cubics that vanish on our six points in $\mathbb{P}^2$:

$$x = F_{12}F_{34}F_{56}\,, \quad y = F_{13}F_{25}F_{46}\,, \quad z = F_{12}F_{35}F_{46}\,, \quad w = F_{13}F_{24}F_{56}. \qquad (2)$$

The factors are linear forms that vanish on pairs of points. Explicitly, these are

$$F_{ij} \;=\; d_i d_j(d_i + d_j)\cdot X \;-\; (d_i^2 + d_i d_j + d_j^2)\cdot Y \;+\; Z \quad \text{for } 1 \leq i < j \leq 6. \qquad (3)$$

As in (2) and (3), we shall write $(X:Y:Z)$ and $(x:y:z:w)$ for the homogeneous coordinates on $\mathbb{P}^2$ and $\mathbb{P}^3$ respectively. The resulting cubic surface is defined by

$$a\cdot xyz \,+\, b\cdot xyw \,+\, c\cdot xzw \,+\, d\cdot yzw \,+\, e\cdot x^2 y \,+\, f\cdot xy^2 \,+\, g\cdot z^2 w \,+\, h\cdot zw^2. \qquad (4)$$

The coefficients $a,b,\ldots,h$ of this octanomial are quintics in the moduli parameters $d_1,\ldots,d_6$. These are displayed in Proposition 2.1. The *support* of (4) equals

$$\mathcal{A} = \big\{(1110),(1101),(1011),(0111),(2100),(1200),(0021),(0012)\big\}. \quad (5)$$

The symmetry group of the Newton polytope $\mathrm{conv}(\mathcal{A})$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$. It acts by compatibly permuting the six points in $\mathbb{P}^2$ and the coordinates in $\mathbb{P}^3$.

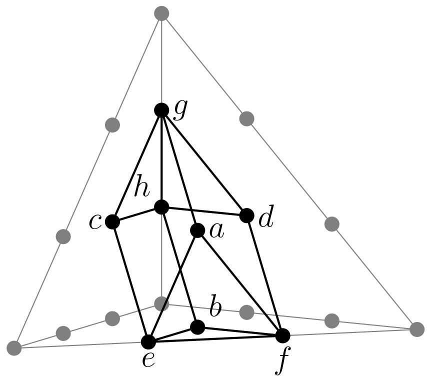

Figure 1: The Newton polytope $\mathrm{conv}(\mathcal{A})$ of the octanomial model.

We propose the octanomial (4) as a new normal form for cubic surfaces. Our study was inspired by the recent work of Cueto and Deopurkar [5]. They map cubic surfaces from $\mathbb{P}^3$ into $\mathbb{P}^{44}$ by the linear forms of the 45 tritangent planes. They prove that this embedding reveals the arrangement of 27 trees determined in [18]. This raises the following question: *Which of the coordinate projections* $\mathbb{P}^{44} \dashrightarrow \mathbb{P}^3$ *best preserve the tropical line arrangement?* We examined all $\binom{45}{4} = 148{,}995$ possibilities. Among those with smallest support, we chose the projection in (2), and we embarked on the detailed study presented here.

For our octanomial surfaces, tropical smoothness imposes a striking constraint on the field of definition of the 27 lines (Theorem 3.5). We believe that the same constraint also holds for cubics with full support (Conjecture 4.1).

The material that follows is organized into three sections. In Section 2 we write the octanomial cubic (4) in terms of the moduli $d_1, \dots, d_6$, we compute its discriminant, and we classify the unimodular triangulations of its Newton polytope (Theorem 2.5). The universal Fano scheme for the octanomial model, described in Proposition 2.8, shows that the 27 lines are grouped into 15 clusters.

Section 3 concerns the del Pezzo combinatorics of tropically smooth cubic surfaces over p-adic fields. Tropicalization takes their 27 lines to arrangements of 27 distinct trees. We study these arrangements and how they correspond to the different types of surfaces. In addition to the stable arrangements in [18], we encounter some new types. These confirm results on non-stable surfaces in [5].

Section 4 is independent of the earlier sections. Here the focus is not on the octanomial model but we study general dense cubic surfaces (14) in $\mathbb{P}^3$. Based on Theorems 4.2 and 4.3, we offer algorithms for computing their intrinsic structure as a del Pezzo surface over a field with valuation. Example 4.4 shows the construction of dense cubics that are tropically smooth, Naruki general and have their six points in $\mathbb{P}^2$ over $\mathbb{Q}$. This addresses Question 11 from the *Twenty-seven Questions about the Cubic Surface* [17]. We conclude with a method that transforms a general cubic into octanomial form (4), by identifying its moduli coordinates $d_1, \dots, d_6$.

This paper is heavily computational. It relies on numerous implementations and experiments with the software `Macaulay2` [8], `Magma` [2] and `Maple` [15]. Our codes and our data are posted at our supplementary materials website[1].

2. Algebra and Combinatorics

We begin by presenting explicit formulas for working with the octanomial model.

[1] `https://software.mis.mpg.de/octanomial`

Proposition 2.1. The eight coefficients of the cubic surface (4) defined by (2) are

$$\begin{aligned}
a = &\; d_1d_3d_2d_4(d_1+d_3-d_2-d_4)+d_2d_4d_5d_6(d_2+d_4-d_5-d_6)+d_5d_6d_1d_3(d_5+d_6-d_1-d_3)+\\
&d_5d_6(d_5{+}d_6)(d_1^2{+}d_3^2{-}d_2^2{-}d_4^2)+d_1d_3(d_1{+}d_3)(d_2^2{+}d_4^2{-}d_5^2{-}d_6^2)+d_2d_4(d_2{+}d_4)(d_5^2{+}d_6^2{-}d_1^2{-}d_3^2)\\
b = &\; d_1d_2d_3d_5(d_1+d_2-d_3-d_5)+d_3d_5d_4d_6(d_3+d_5-d_4-d_6)+d_4d_6d_1d_2(d_4+d_6-d_1-d_2)+\\
&d_4d_6(d_4{+}d_6)(d_1^2{+}d_2^2{-}d_3^2{-}d_5^2)+d_1d_2(d_1{+}d_2)(d_3^2{+}d_5^2{-}d_4^2{-}d_6^2)+d_3d_5(d_3{+}d_5)(d_4^2{+}d_6^2{-}d_1^2{-}d_2^2)\\
c = &\; d_1d_3d_2d_5(d_1+d_3-d_2-d_5)+d_2d_5d_4d_6(d_2+d_5-d_4-d_6)+d_4d_6d_1d_3(d_4+d_6-d_1-d_3)+\\
&d_4d_6(d_4{+}d_6)(d_1^2{+}d_3^2{-}d_2^2{-}d_5^2)+d_1d_3(d_1{+}d_3)(d_2^2{+}d_5^2{-}d_4^2{-}d_6^2)+d_2d_5(d_2{+}d_5)(d_4^2{+}d_6^2{-}d_1^2{-}d_3^2)\\
d = &\; d_1d_2d_3d_4(d_1+d_2-d_3-d_4)+d_3d_4d_5d_6(d_3+d_4-d_5-d_6)+d_5d_6d_1d_2(d_5+d_6-d_1-d_2)+\\
&d_5d_6(d_5{+}d_6)(d_1^2{+}d_2^2{-}d_3^2{-}d_4^2)+d_1d_2(d_1{+}d_2)(d_3^2{+}d_4^2{-}d_5^2{-}d_6^2)+d_3d_4(d_3{+}d_4)(d_5^2{+}d_6^2{-}d_1^2{-}d_2^2)\\
e = &\; -(d_1+d_3+d_5)(d_2+d_4+d_6)(d_1-d_5)(d_2-d_6)(d_3-d_4)\\
f = &\; -(d_1+d_2+d_4)(d_3+d_5+d_6)(d_1-d_4)(d_2-d_5)(d_3-d_6)\\
g = &\; -(d_1+d_3+d_4)(d_2+d_5+d_6)(d_1-d_4)(d_2-d_6)(d_3-d_5)\\
h = &\; -(d_1+d_2+d_5)(d_3+d_4+d_6)(d_1-d_5)(d_3-d_6)(d_2-d_4)
\end{aligned}$$

Proof and Discussion. This was found by a calculation in `Macaulay2` over the field $K = \mathbb{Q}(d_1,d_2,d_3,d_4,d_5,d_6)$, carried out with the help of Mike Stillman. According to the Hilbert–Burch Theorem, the cubics (2) that cut out the six points in $\mathbb{P}^2$ are maximal minors of a 3×4 matrix with entries in $K[X,Y,Z]_1$. We computed this 3×4 matrix and we rearranged it to a 3×3 matrix with entries in $K[x,y,z,w]_1$ whose determinant is the octanomial (4) with the above coefficients. This determinantal representation of our cubic, as well as explicit formulas for all 27 lines over K, are posted on our supplementary website. □

Remark 2.2. The eight coefficients in Proposition 2.1 sum to zero. The image of the parametrization above is a quartic hypersurface in the hyperplane $\mathbb{P}^6 \subset \mathbb{P}^7$ given by $a+b+c+d+e+f+g+h=0$. The equation of that hypersurface in $\mathbb{P}^6$ has 134 terms when written in the first seven coordinates. It equals $4a^3b+4a^3e+4a^3f+9a^2b^2+4a^2bc+4a^2bd+20a^2be+\cdots+13b^2e^2+\cdots+4f^4+4f^3g+4f^2g^2=0$.

Remark 2.3. The expansions of the quintics a,b,c,d in Proposition 2.1 have 36 terms in $d_1,d_2,\ldots,d_6$. These are precisely the D_4-invariant quintics that are denoted by Q in [5, Example 2.6] and denoted by F_1 in [4, Lemma 4.4].

Proposition 2.4. The discriminant of the cubic (4) is the degree 32 polynomial

$$2^{16}3^5\cdot e^2f^2g^2h^2\cdot(ac-eg)^2\cdot(ad-fg)^2\cdot(bc-eh)^2\cdot(bd-fh)^2\cdot\Delta_{\mathcal{A}}, \qquad (6)$$

where $\mathcal{A}$ is the point configuration (5) and $\Delta_{\mathcal{A}}$ is the $\mathcal{A}$-*discriminant* [6, Chapter 9]. In our case, the $\mathcal{A}$-discriminant has 49 terms of degree 8, namely

$$\begin{aligned}\Delta_{\mathcal{A}} = {} & a^4b^2h^2-2a^3b^3gh-2a^3b^2cdh-2a^3bcfh^2-2a^3bdeh^2+4a^3efh^3+a^2b^4g^2-2a^2b^3cdg\\ &+a^2b^2c^2d^2+8a^2b^2cfgh+8a^2b^2degh+4a^2bc^2dfh+4a^2bcd^2eh-6a^2befgh^2+a^2c^2f^2h^2\\ &-10a^2cdefh^2+a^2d^2e^2h^2-2ab^3cfg^2-2ab^3deg^2+4ab^2c^2dfg+4ab^2cd^2eg-6ab^2efg^2h\\ &-2abc^3d^2f-2abc^2d^3e-10abc^2f^2gh-26abcdefgh-10abd^2e^2gh-2ac^3df^2h+8ac^2d^2efh\\ &-2acd^3e^2h+18acef^2gh^2+18ade^2fgh^2+4b^3efg^3+b^2c^2f^2g^2-10b^2cdefg^2+b^2d^2e^2g^2\\ &-2bc^3df^2g+8bc^2d^2efg-2bcd^3e^2g+18bcef^2g^2h+18bde^2fg^2h+c^4d^2f^2-2c^3d^3ef\\ &+4c^3f^3gh+c^2d^4e^2-6c^2def^2gh-6cd^2e^2fgh+4d^3e^3gh-27e^2f^2g^2h^2.\end{aligned}$$

Proof. The resultant of four quaternary quadrics was written as the determinant of a 20×20 matrix by Nanson [16]. We apply Nanson's formula to the four partial derivatives of (4). The result is the expression shown in (6). The $\mathcal{A}$-discriminant is obtained by removing all factors in (6) that are supported on proper faces of the convex hull of $\mathcal{A}=\{(1110),(1101),\dots,(0012)\}$. □

The study of the *Schläfli fan* in [12] identified another family of cubic surfaces with eight terms. The next theorem is analogous to the combinatorial results reported in [12, Section 7], but now for the new octanomial model (4).

Theorem 2.5. The configuration $\mathcal{A}$ is the vertex set of a 3-dimensional lattice polytope of normalized volume 7. This polytope has 70 regular triangulations in 14 symmetry classes under the action of $(\mathbb{Z}/2\mathbb{Z})^3$. Unimodular triangulations occur for 53 of the 70 triangulations. They come in 10 symmetry classes:

Orbit size	Stanley–Reisner ideal	representative weights	GKZ vector
1	$\langle ah,bg,cf,de,eg,eh,fg,fh\rangle$	$(4,1,7,2,9,5,9,9)$	$(5,5,5,5,2,2,2,2)$
4	$\langle ab,ac,ah,cd,cf,eh,fg,fh\rangle$	$(5,2,8,4,2,9,5,9)$	$(2,5,2,5,5,2,5,2)$
4	$\langle ab,ah,bg,cf,eg,eh,fg,fh\rangle$	$(8,3,1,1,4,9,6,9)$	$(3,3,5,7,4,2,2,2)$
4	$\langle ab,ac,ah,bc,bg,cf,fg,fh,egh\rangle$	$(9,9,6,1,4,8,8,5)$	$(2,2,3,7,6,2,3,3)$
4	$\langle ab,ac,ad,ah,bc,cd,cf,fh,bfg,deh\rangle$	$(7,3,9,1,2,4,1,9)$	$(1,4,1,4,6,3,6,3)$
4	$\langle ab,ac,ad,ah,bc,bd,bg,cf,egh,fgh\rangle$	$(8,8,2,6,1,6,3,7)$	$(1,1,3,5,6,4,4,4)$
8	$\langle ab,ac,ah,bg,cf,eh,fg,fh\rangle$	$(4,2,4,1,3,8,4,4)$	$(2,3,4,7,5,2,3,2)$
8	$\langle ab,ac,ad,ah,bg,cf,eh,fh\rangle$	$(4,4,3,1,1,1,1,7)$	$(1,3,4,6,5,3,4,2)$
8	$\langle ab,ac,ad,ah,cd,cf,eh,fh,bfg\rangle$	$(9,3,8,4,3,9,2,9)$	$(1,5,2,4,5,3,6,2)$
8	$\langle ab,ac,ad,ah,bc,bg,cf,fh,egh\rangle$	$(9,9,4,3,4,7,3,8)$	$(1,2,3,6,6,3,4,3)$

Proof and explanation. The vertex set of the 3-dimensional polytope $\mathrm{conv}(\mathcal{A})$ is $\mathcal{A}$ which we identify with $\{a,b,c,d,e,f,g,h\}$. This polytope has eight facets:

$$aceg,\quad adfg,\quad bceh,\quad bdfh,\quad aef,\quad bef,\quad cgh,\quad dgh. \tag{7}$$

The derivation of its regular triangulations is a computation, using either an algebraic approach or a geometric approach. The following algebraic approach is based on [6, Chapter 10]. Namely, we compute the *principal $\mathcal{A}$-determinant*

$$E_{\mathcal{A}} = abcd\cdot e^2f^2g^2h^2\cdot(ac-eg)\cdot(ad-fg)\cdot(bc-eh)\cdot(bd-fh)\cdot\Delta_{\mathcal{A}}. \tag{8}$$

The GKZ vectors are the exponent vectors of the lowest monomials of (8) with respect to all generic weight vectors $\upsilon \in \mathbb{R}^8$. Representative weight vectors υ are shown in the third column. The associated triangulation is a simplicial complex of dimension 3 on the vertex set $\mathcal{A}$. As is customary in combinatorial commutative algebra [20], we encode this simplicial complex by its *Stanley–Reisner ideal*, shown in the second column. This squarefree monomial ideal is the initial ideal $\mathrm{in}_\upsilon(I_\mathcal{A})$ of the *toric ideal* $I_\mathcal{A}$ associated with our configuration $\mathcal{A}$. □

Remark 2.6. The toric ideal of $\mathcal{A}$ is minimally generated by eight quadrics:

$$I_\mathcal{A} = \left\langle \underline{ab}-cf\,,\underline{ac}-eg\,,\underline{ad}-fg\,,\underline{ah}-cd\,,\underline{bg}-cd\,,\underline{cf}-de\,,\underline{eh}-bc\,,\underline{fh}-bd \right\rangle. \quad (9)$$

We underlined the highest monomials with respect to the weight vector υ in row 8 of the table above. These monomials generate the initial ideal $\mathrm{in}_\upsilon(I_\mathcal{A})$. The eight generators form the reduced Gröbner basis of $I_\mathcal{A}$ with respect to υ.

Remark 2.7. When computing regular triangulations algebraically, it is important to note is that the meaning of *initial monomial* depends on the context. For a given weight vector υ, we take lowest monomials in the principal $\mathcal{A}$-determinant $E_\mathcal{A}$ in order to match taking highest monomials in the toric ideal $I_\mathcal{A}$.

Each line in $\mathbb{P}^3$ is encoded by its Plücker coordinates $(p_{01} : p_{02} : p_{03} : p_{12} : p_{13} : p_{23}) \in \mathbb{P}^5$. To be precise, the affine cone over the line is the image of the skew-symmetric 4×4-matrix (p_{ij}). The universal Fano variety of (4) lives in $\mathbb{P}^5 \times \mathbb{P}^7$. Its points are lines on octanomial cubic surfaces. We shall present two descriptions of these lines, first from the perspective of equations (Proposition 2.8), and second in terms of the intrinsic del Pezzo geometry (Proposition 3.1).

The ideal I_{ufv} of the universal Fano variety is minimally generated by $1+20$ polynomials in $\mathbb{Q}[p_{01}, p_{02}, p_{03}, p_{12}, p_{13}, p_{23}, a, b, c, d, e, f, g, h]$. This is implied by the results in [12, Section 6]. The following census of the 27 lines on our cubic surface was obtained with `Macaulay2`. It arises from the primary decomposition of I_{ufv} in the above polynomial ring with 14 variables.

Proposition 2.8. The ideal I_{ufv} of the Fano variety in $\mathbb{P}^5 \times \mathbb{P}^7$ has 15 minimal primes, $9 = 4+4+1$ of degree 1, and $6 = 2+4$ of degree 3. The degree is the size of the fiber over the $\mathbb{P}^7$ of octanomials (4). Four of the 27 lines are the coordinate lines $\langle x,z\rangle$, $\langle x,w\rangle$, $\langle y,z\rangle$, $\langle y,w\rangle$. Another four correspond to lines in coordinate planes in $\mathbb{P}^3$, namely $\langle x, dy+gz+hw\rangle$, $\langle y, cx+gz+hw\rangle$, $\langle z, ex+fy+bw\rangle$ and $\langle w, ex+fy+az\rangle$. One unique line is disjoint from the coordinate lines. That line has Plücker coordinates (10). Two triples of lines intersect pairs of coordinate lines, and four triples of lines intersect precisely one coordinate line each.

Remark 2.9. The line that is disjoint from the coordinate lines is defined by $\langle\,(bg{-}ah)y+(fh{-}bd)z+(ad{-}fg)w,\ (bg{-}ah)x+(eh{-}bc)z+(ac{-}eg)w\rangle$. It has

$$\begin{aligned}(p_{01}:p_{02}:\cdots:p_{23}) &= \big(bg{-}ah : fh{-}bd : ad{-}fg : bc{-}eh : eg{-}ac : cf{-}de\big) \\ &= \text{maximal minors of } \left(\begin{smallmatrix} a & b & e & f \\ g & h & c & d \end{smallmatrix}\right).\end{aligned} \tag{10}$$

Proposition 2.1 gives a formula for the octanomial cubic over the field $K = \mathbb{Q}(d_1,d_2,d_3,d_4,d_5,d_6)$. The six points in (1) and the cubics in (2) are also defined over K. Hence, so are the 27 lines and their 135 intersection points in $\mathbb{P}^3$. The census in Proposition 2.8 translates into formulas for these objects over K.

The middle four coordinates in (10) are products of ten linear forms. These come from the root system E_6. The first and last coordinates are products of five such linear forms with one quintic as in Remark 2.3. Here is another instance.

Example 2.10. Each line that intersects a pair of coordinate lines has two zero Plücker coordinates. The other four Plücker coordinates are products of roots. For instance, one of the three lines with $p_{02} = p_{13} = 0$ has the other coordinates

$$p_{01} = (d_5{-}d_6)(d_4{-}d_6)(d_3{-}d_5)(d_2{-}d_4), \qquad p_{03} = (d_4-d_6)^2(d_3-d_5)(d_2-d_5),$$
$$p_{12} = -(d_5-d_6)^2(d_3-d_4)(d_2-d_4), \qquad p_{23} = (d_5{-}d_6)(d_4{-}d_6)(d_3{-}d_4)(d_2{-}d_5).$$

Similarly, one of the three lines with $p_{03} = p_{12} = 0$ satisfies

$$\begin{aligned} p_{01} &= (d_3{+}d_4{+}d_5)(d_2{+}d_4{+}d_5)(d_1{+}d_3{+}d_4)(d_1{+}d_2{+}d_5),\\ p_{02} &= (d_3{+}d_4{+}d_5)^2(d_1{+}d_2{+}d_5)(d_1{+}d_2{+}d_4),\\ p_{13} &= -(d_2{+}d_4{+}d_5)^2(d_1{+}d_3{+}d_5)(d_1{+}d_3{+}d_4),\\ p_{23} &= -(d_3{+}d_4{+}d_5)(d_2{+}d_4{+}d_5)(d_1{+}d_3{+}d_5)(d_1{+}d_2{+}d_4).\end{aligned}$$

Formulas for all 27 lines and their 135 intersections are posted on our website.

3. Arrangements of Trees

In this section we study the octanomial model over a field with valuation. For concreteness, we work over the rational numbers $\mathbb{Q}$ with the p-adic valuation, for some prime $p \geq 5$. We consider the open surface obtained by removing the 27 lines on a given cubic in $\mathbb{P}^3$. Ren, Shaw and Sturmfels [18] studied the *intrinsic tropicalization* of such surfaces, and they identified two generic types of tropical surfaces. They are characterized by their structure at infinity, which is an arrangement of 27 trees with 10 leaves [18, Figures 4 and 5]. These surfaces are points in the *Naruki fan*, the tropical moduli space of cubic surfaces in [9]. See [18, Table 1] for a census of cones in this fan. Cueto and Deopurkar [5] realized the tree arrangements geometrically via a natural embedding into $\mathbb{P}^{44}$.

For the following proposition we fix the function field $K = \mathbb{Q}(d_1,\ldots,d_6)$. The cubic surface S given by (4) over the function field K is smooth in $\mathbb{P}^3$ and has no Eckhart points. The 27 lines on S are labeled $E_1,\dots,E_6$, $F_{12},\dots,F_{56}$, $G_1,\dots,G_6$, as in [5, 18]. Namely, the line E_i is the exceptional fiber over the i-th point in (1), F_{ij} is the line connecting the i-th and j-th points, and G_j is the conic through the five points other than the j-th. Each line intersects 10 other lines. For instance, E_i meets the five lines F_{ij} and the five lines G_j where $j \neq i$. These 27 labels are now matched with the census of lines in Proposition 2.8.

Proposition 3.1. The 27 lines on the cubic (4) are as follows. We first have

$$F_{12} = \{x = z = 0\}, \quad F_{13} = \{y = w = 0\}, \quad F_{46} = \{y = z = 0\}, \quad F_{56} = \{x = w = 0\}.$$

The other four lines that lie in coordinate planes are

$$F_{34} \subset \{x = 0\}, \quad F_{25} \subset \{y = 0\}, \quad F_{35} \subset \{z = 0\}, \quad F_{24} \subset \{w = 0\}.$$

Two triples of lines (in Example 2.10) intersect a pair of coordinate lines, namely

$$E_1, F_{45}, G_1 \text{ intersect } F_{12}, F_{13} \qquad \text{and} \qquad E_6, F_{23}, G_6 \text{ intersect } F_{46}, F_{56}.$$

The following four triples of lines intersect a unique coordinate line:

$$\begin{array}{ll} E_2, F_{36}, G_2 \text{ intersect } F_{12}, & E_3, F_{26}, G_3 \text{ intersect } F_{13}, \\ E_4, F_{15}, G_4 \text{ intersect } F_{46}, & E_5, F_{14}, G_5 \text{ intersect } F_{56}. \end{array} \tag{11}$$

Finally, F_{16} is the unique line (10) that does not intersect any coordinate line.

Proof. The pullback of $\{x = 0\} \cap S$ from $\mathbb{P}^3$ to $\mathbb{P}^2$ is $F_{12}F_{34}F_{56}$ and the pullback of $\{z = 0\} \cap S$ is $F_{12}F_{35}F_{46}$. The line F_{12} is common to both and thus equals $\{x = z = 0\}$. The line F_{34} appears only in $\{x = 0\}$. The lines E_1, F_{45}, G_1 all meet F_{12} and F_{13} but are not in a coordinate plane; e.g. F_{45} does not appear as a factor in (2). The argument for (11) is similar. Hence, F_{16} is the match for (10). □

Pairs of intersecting lines determine 135 points on $S \subset \mathbb{P}^3$. Using the rational formulas in Section 2, we precomputed these points over $K = \mathbb{Q}(d_1, d_2, \dots, d_6)$. This allows for rapid evaluation when the d_i are specialized to rational numbers.

We now describe a general construction of 27 metric trees which is essential in the theory of tropical del Pezzo surfaces [5, 18]. Consider a smooth cubic surface $S \subset \mathbb{P}^3$, defined over a valued field. For instance, we could take $\mathbb{Q}$ with its p-adic valuation and let S be the octanomial surface defined by $d_1, \dots, d_6 \in \mathbb{Q}$. For each of the lines on S, we fix an isomorphism with $\mathbb{P}^1$ by projecting to one of the six coordinate axes in $\mathbb{P}^3$. The 10 points on that line are now the columns of a 2×10 matrix with entries in $\mathbb{Q}$. We record the p-adic valuations of the 45 maximal minors of this matrix. This data determines a phylogenetic tree with

10 leaves as in [13, Section 4.3]. In our implementation we use the method of *Quartet Puzzling* [3] for constructing the tree from its 45 pairwise distances.

Each interior edge in one of our trees is encoded by a list of ≤ 7 splits. A *split* is a partition of the ten leaf labels into two subsets that is induced by removing an interior edge from the tree. Let s_i denote the number of splits into i leaves versus $10-i$ leaves, where $i \in \{2,3,4,5\}$. The string $[s_2s_3s_4s_5]$ is a combinatorial invariant of the tree. The *tree statistic* of our arrangement is the multiset of these 27 strings. The notation $[s_2s_3s_4s_5]^u$ means that the string $[s_2s_3s_4s_5]$ arises from u of the 27 trees. We illustrate the use of this notation with an example.

Example 3.2. Ren et al. [18] identified two types, denoted (aaaa) and (aaab), of tree arrangements that are generic in the Naruki fan. The drawings in [5, Figures 4 and 5] or in [18, Figures 4 and 5] show that their tree statistics are $\{[4021]^{24},[4020]^3\}$ for (aaaa), and $\{[2221]^{12},[4201]^{12},[4210]^3\}$ for (aaab).

We now come to the main results in this section. A choice of moduli parameters $(d_1,d_2,d_3,d_4,d_5,d_6)$ in $\mathbb{Q}^6$ is called *Naruki general* if the resulting tree arrangement is of type (aaaa) or of type (aaab). We say that the octanomial (4) is *tropically smooth* if the induced polyhedral subdivision of its Newton polytope $\mathrm{conv}(\mathcal{A})$ is one of the 53 unimodular triangulations in Theorem 2.5.

Tropical smoothness implies classical smoothness for hypersurfaces in $\mathbb{P}^n$ with full support. This can be deduced from [13, Proposition 4.5.1]. However, sparse hypersurfaces are typically singular in $\mathbb{P}^n$ even if they are tropically smooth in the sense above. It is a key feature of our octanomial model, based on delicate combinatorics, that the two notions of smoothness are compatible.

Theorem 3.3. Every tropically smooth octanomial is classically smooth in $\mathbb{P}^3$.

Proof. The secondary polytope of $\mathcal{A}$ is the Newton polytope of the principal $\mathcal{A}$-determinant, by [6, Theorem 10.1.4]. The principal $\mathcal{A}$-determinant (8) has the same (non-monomial) irreducible factors as the discriminant (6) of the cubic (4). Hence, the two polytopes have the same normal fan. This is the secondary fan of $\mathcal{A}$. Since the surface (4) is tropically smooth, the vector $\mathrm{val}(a,b,\ldots,h)$ lies in the interior of a maximal cone of this fan. This means that the initial form of the discriminant (6) picked by this weight vector is a monomial. In particular, the discriminant is nonzero, and we conclude that the surface is smooth. □

Remark 3.4. It would be interesting to identify a combinatorial characterization of all configurations $\mathcal{A}$ in $\mathbb{Z}^d$ for which the conclusion of Theorem 3.3 holds. Put differently, under which conditions on the support set $\mathcal{A}$ does the tropical smoothness of a hypersurface in $\mathbb{R}^d$ imply tropical smoothness in $\mathbb{TP}^d$?

We saw in Proposition 2.8 that eight of the 27 lines on an octanomial surface lie in coordinate planes in $\mathbb{P}^3$. Therefore, the tropicalizations of these lines are in the boundary of the tetrahedron that represents the tropical projective space $\mathbb{TP}^3$. We refer to [14, Chapter 3] for the polyhedral construction of tropical toric varieties like $\mathbb{TP}^3$. The following is the octanomial version of Conjecture 4.1.

Theorem 3.5. Fix a prime $p \geq 5$ and an octanomial cubic surface S defined over $\mathbb{Q}_p$. If S is tropically smooth then the 27 lines on S have distinct tropicalizations in $\mathbb{TP}^3$. In that case, it follows that all 27 lines on S are defined over $\mathbb{Q}_p$.

Proof. We use normalized Plücker coordinates where the first nonzero entry is 1. We claim that the coordinates of the 27 lines have distinct valuations. By Proposition 2.8, nine of the lines are identified by their zero coordinates. The other lines come in six triplets. Each triplet is identified by its zero coordinates. We must show that the tropical lines from the same triplet are distinct.

For the lines in a triplet, each nonzero Plücker coordinate p_{ij} lies in a degree three extension of $\mathbb{Q}(a,b,\dots,h)$. From the minimal primes in Proposition 2.8, we compute the minimal polynomial of p_{ij}. This gives six sets of irreducible polynomials in $\mathbb{Z}[a,b,\dots,h][t]$ that are cubic in t. The first two sets have size three. The other four sets have size four. All coefficients in $\mathbb{Z}[a,b,\dots,h]$ have their own coefficients in $\{\pm1,\pm2\}$. These scalars have valuation 0 since $p \geq 5$.

As (4) is tropically smooth, $\mathrm{val}(a,b,\dots,h) \in C$, where $C \subset \mathbb{R}^8$ is the Gröbner cone (cf. [20, Chapter 8]) for one of the 53 squarefree initial ideals of $I_{\mathcal{A}}$. This containment translates into linear inequalities among valuations. For instance, for the reduced Gröbner basis in (9), we see that $\mathrm{val}(a,b,\dots,h) \in C$ if and only if

$$\begin{array}{ll} \mathrm{val}(a)+\mathrm{val}(b) > \mathrm{val}(c)+\mathrm{val}(f), & \mathrm{val}(a)+\mathrm{val}(c) > \mathrm{val}(e)+\mathrm{val}(g), \\ \mathrm{val}(a)+\mathrm{val}(d) > \mathrm{val}(f)+\mathrm{val}(g), & \mathrm{val}(a)+\mathrm{val}(h) > \mathrm{val}(c)+\mathrm{val}(d), \\ \mathrm{val}(b)+\mathrm{val}(g) > \mathrm{val}(c)+\mathrm{val}(d), & \mathrm{val}(c)+\mathrm{val}(f) > \mathrm{val}(d)+\mathrm{val}(e), \\ \mathrm{val}(e)+\mathrm{val}(h) > \mathrm{val}(b)+\mathrm{val}(c), & \mathrm{val}(f)+\mathrm{val}(h) > \mathrm{val}(b)+\mathrm{val}(d). \end{array} \tag{12}$$

We now assume that this holds, i.e. $\mathrm{val}(a,\dots,h)$ is in one the 53 Gröbner cones.

Take one of the univariate cubics $P = c_3t^3 + c_2t^2 + c_1t^2 + c_0$ obtained above. In each $c_i \in \mathbb{Z}[a,b,\dots,h]$ we look for a monomial whose valuation equals $v_i = \mathrm{val}(c_i)$, assuming the Gröbner cone inequalities. The three roots of P give a coordinate p_{ij} for a triplet of lines. Their valuations are distinct if and only if

$$v_0 + v_2 > 2v_1 \quad \text{and} \quad v_1 + v_3 > 2v_2. \tag{13}$$

Our strategy is to show that the cone inequalities like (12) imply those in (13).

This strategy would fail for some P if one of its coefficients c_i does not have a unique monomial of minimal valuation v_i, or if the four valuations do not satisfy (13). If this happens then we must discard P and move on to the next univariate cubic in the same list (of three or four cubics). For each of the six lists, it suffices to identify one cubic P in that list having distinctly valued roots.

We applied this strategy to each of the 10 representative initial ideals in Theorem 2.5 and to each of the six lists of univariate cubics. For each triangulation and each triplet, we found that there is a cubic P that works. □

A fundamental problem regarding cubic surfaces over a valued field is to understand the relationship between their extrinsic and intrinsic tropicalizations. This is the motivation for the embedding into $\mathbb{P}^{44}$ studied in [5]. We are now aiming to take this back into $\mathbb{P}^3$. For the octanomial model, we show that genericity can occur simultaneously on both the extrinsic and the intrinsic side.

Proposition 3.6. For at least five of the ten combinatorial types in Theorem 2.5, there exists a Naruki general vector $d = (d_1, d_2, \ldots, d_6) \in \mathbb{Q}^6$ whose corresponding octanomial cubic is tropically smooth and has this combinatorial type.

At present, we do not know whether the other five types are realizable. We derived Proposition 3.6 by an extensive computation, based on sampling Naruki general points d from $\mathbb{Q}^6$. The following sampling method was used. The tropical moduli space, which is the 4-dimensional Naruki fan, is the image of a tropical linear space of dimension 5. This is the uniformization $\mathrm{Berg}(\mathrm{E}_6) \to \mathrm{trop}(\mathcal{Y}^0)$ in the second row of [18, Equation (3.1)]. The domain is the Bergman fan associated with the root system E_6. This is a 5-dimensional fan in $\mathbb{R}^{35} \simeq \mathbb{R}^{36}/\mathbb{R}\mathbf{1}$. The number of cones in these fans are reported in [18, Lemma 3.1].

Let $\mathbb{E}_6$ denote the matrix in $\{0,1\}^{6\times 36}$ whose columns are the 36 roots of E_6. The map $d \mapsto \mathrm{val}(d\cdot\mathbb{E}_6)$ takes $\mathbb{Q}^6$ onto the Bergman fan $\mathrm{Berg}(\mathrm{E}_6)$. Here we are referring to the fan structure on $\mathrm{Berg}(\mathrm{E}_6)$ described by Ardila et al. in [1]. We seek general points in the 142,560 maximal cones of $\mathrm{Berg}(\mathrm{E}_6)$. To find them, we select an ordered column basis of $\mathbb{E}_6$. We pick $e = (e_1, e_2, e_3, e_4, e_5, e_6) \in \mathbb{Q}^6$ such that $\mathrm{val}(e_1) < \mathrm{val}(e_2) < \cdots < \mathrm{val}(e_6)$. The row vector d is obtained by multiplying the inverse of the 6×6 submatrix of $\mathbb{E}_6$ on the left by e. The basis specifies a chain of flats in the matroid of E_6, and $\mathrm{val}(d\cdot\mathbb{E}_6)$ lies in the maximal cone of $\mathrm{Berg}(\mathrm{E}_6)$ indexed by that chain (cf. [13, Theorem 4.2.6]). With this choice, d is likely to land in a maximal Naruki cone under $\mathbb{Q}^6 \to \mathrm{Berg}(\mathrm{E}_6) \to \mathrm{trop}(\mathcal{Y}^0)$.

Proof of Proposition 3.6. Fix prime $p \geq 5$. The following two vectors are Naruki general of type (aaaa) and their octanomial surfaces are tropically smooth:

$$(2+p^5-p^7-p^9,\, -p^3+p^9,\, -1+p^7,\, -p^3-p^7+p^9+p^{11},\, -1+p^9,\, 1+p^3-p^9),$$
$$(1+p^3-p^9,\, p^3+p^5-p^9,\, -2+p^5+p^7+p^9-p^{11},\, 1-p^3-p^5+p^{11},\, -p^3+p^9,\, -1+p^9).$$

The resulting octanomial surfaces are tropically smooth. The corresponding triangulations appear in lines 1 and 2 in the classification in Theorem 2.5.

Likewise, the following three integer vectors $(d_1,\dots,d_6)$ are Naruki general and their tree arrangements have type (aaab):

$$\big(-1+p^7, 2+p^5-p^7-p^9, -p^3+p^9, -1+p^9, 1+p^3-p^9, 1+p^3-2p^9+p^{11}\big),$$
$$\big(-1+p^3-p^5+p^7,\ 2-p^7+p^9-p^{11},\ 2-p^3+p^5-p^7+p^9-p^{11},$$
$$-1+p^7-p^9+p^{11},\ -2+p^3+p^7-p^9+p^{11},\ -1+p^{11}\big),$$
$$\big(2-p^5-p^7+p^9, 2-p^3-p^5+p^9, -1+p^5+p^7-p^9, -1+p^3, -1+p^5, -1+p^3-p^9+p^{11}\big).$$

The resulting octanomial surfaces are tropically smooth. The corresponding triangulations appear in lines 3, 4 and 7 in the classification in Theorem 2.5.

At present, we do not know whether any tropically smooth octanomial surface of types 5, 6, 8, 9, 10 can have a Naruki general tree arrangement. □

We now come to octanomial cubics that are not Naruki general. In all examples that follow we work over $\mathbb{Q}$ with the p-adic valuation for $p=5$. We begin with cubic surfaces that are *stable*, in the sense discussed in [18, Section 5]. These correspond to the lower-dimensional cones in the Naruki fan $\mathrm{trop}(\mathcal{Y}^0)$. Their 27 trees are obtained from those in (aaaa) or (aaab) by contracting some interior edges. The various stable non-generic types are listed in [18, Table 1].

Example 3.7. The following moduli vectors $d \in \mathbb{Q}^6$ define tree arrangements that are non-generic but stable. We indicate the type as denoted in [18, Table 1].

$(d_1,d_2,d_3,d_4,d_5,d_6)$	Tree arrangement	Type
$(2377,-2375,1240,2385,2425,2625)$	$\{[2210]^1,[2220]^4,[2221]^8,$ $[4201]^{12},[4210]^2\}$	(aab)
$(-843,124,724,744,1537,844)$	$\{[2020]^1,[4020]^6,[4021]^{20}\}$	(aaa)

Our last family of instances is the most interesting and mysterious one. These $d \in \mathbb{Q}^6$ give tree arrangements that do not appear in [18, Table 1]. The corresponding cubic surfaces are *not stable*. This means that the fiber of the vertical map $\mathrm{trop}(\mathcal{G}^0) \to \mathrm{trop}(\mathcal{Y}^0)$ in [18, Equation (3.1)] has dimension 3. The tropical cubic surfaces arising from these d are contained in that fiber. The combinatorial structure is hence not revealed by the analysis in [18, Section 3].

Example 3.8. The following vectors $d \in \mathbb{Q}^6$ determine cubics that are not stable:

$(d_1,d_2,d_3,d_4,d_5,d_6)$	Tree arrangement
$(-719,1081,-359,-347,-9287,10081)$	$\{[2220]^6,[3210]^3,[3220]^6,[4201]^{12}\}$
$(120,-3099,-3095,620,-595,3100)$	$\{[2220]^2,[2221]^8,[3210]^1,[3220]^2,[4201]^{12},[4210]^2\}$
$(-6719,1248,7248,-519,481,-479)$	$\{[3020]^1,[4020]^4,[4021]^{20},[5020]^2\}$

To see that these arrangements are not stable, we note that any edge contraction in a tree would lead an entry s_i in $[s_2s_3s_4s_5]$ to decrease. Therefore the trees $[3220]$ and $[5020]$ appearing above do not arise from either (aaaa) or (aaab).

To understand examples such as these, one needs to go much beyond [18]. This was accomplished by Cueto and Deopurkar in their remarkable article [5]. In [5, Proposition 4.4], they explain that the tree arrangement is determined by the valuations of the Cross functions. Each Cross function is a difference of Yoshida functions, and it factors into four roots and a quintic as in Remark 2.3. Non-stable tree arrangements arise because of cancellations for specific $d_1,\ldots,d_6 \in \mathbb{Q}$. If the p-adic valuation of a Cross function is not predicted by combinatorics then Table 1 in [18] does not apply. The phenomenon is explained in [5, Section 10], where a detailed explanation of non-stable trees and their edge lengths is given. For example, consider the tree of type $[5,0,0,0]$ shown in [5, Figure 6]. This tree explains the initial entry "5" in the last type in Example 3.8.

We thank Angelica Cueto for explaining these results to us and for confirming the correctness of Example 3.8 by analyzing the surfaces in $\mathbb{P}^{44}$. At present we do not know which combinatorial types of non-stable tree arrangements are realizable over fields such as $\mathbb{Q}_p$. This will be the topic of a subsequent project.

4. Dense Cubics

We now turn to cubic surfaces in $\mathbb{P}^3$ whose defining polynomial has full support:

$$\begin{aligned} &c_0w^3+c_1w^2z+c_2wz^2+c_3z^3+c_4w^2y+c_5wyz+c_6yz^2\\ &+c_7wy^2+c_8y^2z+c_9y^3+c_{10}w^2x+c_{11}wxz+c_{12}xz^2+c_{13}wxy\\ &+c_{14}xyz+c_{15}xy^2+c_{16}wx^2+c_{17}x^2z+c_{18}x^2y+c_{19}x^3. \end{aligned} \tag{14}$$

This uses notation as in [12]. Our primary goal is to state a conjecture on the arithmetic of the 27 lines on a tropically smooth cubic surface over a complete valued field such as $\mathbb{Q}_p$. This generalizes Theorem 3.5. For the most part, Section 4 is independent of the previous sections. We no longer study the octanomial model. Only at the very end, we return to the title of this paper, by describing an algorithm for transforming (14) into octanomial normal form (4).

For ease of exposition we assume that the c_i are general, so no line in the cubic meets any of the six coordinate lines in $\mathbb{P}^3$. Hence, all 27 lines lie in the six standard affine charts $\{p_{ij} \neq 0\}$ of the Grassmannian. To compute the 27 lines, we substitute $w = sx+ty$ and $z = ux+vy$ into (14) where s,t,u,v are unknowns. The result is a binary cubic in x,y whose coefficients define the Fano scheme:

$$\begin{aligned} c_0t^3+c_1t^2v+c_2tv^2+c_3v^3+c_4t^2+c_5tv+c_6v^2+c_7t+c_8v+c_9 &= 0,\\ c_0s^3+c_1s^2u+c_2su^2+c_3u^3+c_{10}s^2+c_{11}su+c_{12}u^2+c_{16}s+c_{17}u+c_{19} &= 0,\\ 3c_0st^2+2c_1stv+c_2sv^2+c_1t^2u+2c_2tuv+3c_3uv^2+2c_4st+c_5sv+c_{10}t^2 &\\ +c_5tu+c_{11}tv+2c_6uv+c_{12}v^2+c_7s+c_{13}t+c_8u+c_{14}v+c_{15} &= 0,\\ 3c_0s^2t+c_1s^2v+2c_1stu+2c_2suv+c_2tu^2+3c_3u^2v+c_4s^2+2c_{10}st+c_5su &\\ +c_{11}sv+c_{11}tu+c_6u^2+2c_{12}uv+c_{13}s+c_{16}t+c_{14}u+c_{17}v+c_{18} &= 0. \end{aligned}$$

These four cubic equations in four unknowns s,t,u,v have 27 distinct solutions over the algebraic closure of $K = \mathbb{Q}(c_0, c_1, \dots, c_{19})$. One can try to solve this symbolically (e.g. in Magma), but this is unpractical for dense polynomials (14).

Using combinatorial methods [7], we can instead compute the set of p-adic valuations $\mathrm{val}(s,t,u,v) \in \mathbb{Q}^4$ of the 27 solutions $(s,t,u,v) \in \overline{\mathbb{Q}}^4$. Moreover, the p-adic series expansion of these four scalars up to some desired order can also be found. To do this, we use the implementation of p-adic arithmetic in Magma. The feasibility of this approach is underscored by Conjecture 4.1 below.

Smooth tropical surfaces, with full Newton polytope, come in 14,373,645 combinatorial types. We used the database presented in [12] to sample from tropically smooth cubics, and to conduct experiments that support Conjecture 4.1. In each case, we also computed the 135 intersection points in $\mathbb{P}^3$ over $\mathbb{Q}_p$ and we built the arrangement of 27 trees using Quartet Puzzling as in Section 3.

Conjecture 4.1. If the surface (14) over $\mathbb{Q}_p$ is tropically smooth then its lines have distinct tropicalizations. In particular, the 27 lines in $\mathbb{P}^3$, their 135 intersection points, and the 6 points in $\mathbb{P}^2$ obtained by blowing down six skew lines are all defined over the p-adic field $\mathbb{Q}_p$. The algebraic closure of $\mathbb{Q}_p$ is not needed.

Kristin Shaw announced a proof that every tropically smooth family of complex cubic surfaces contains 27 lines whose tropical limits are distinct (personal communication, 2019). The approach is based on a correspondence theorem between tropical [19] and complex intersection theories that is proved using tropical homology [11]. Shaw's result, if true in the p-adic setting, would imply Conjecture 4.1. We suspect that the conjecture holds over all complete discretely valued fields. Our Theorem 3.5 gives further evidence in this direction.

We next explain how to find six points in $\mathbb{P}^2$ and a basis for the space of cubics through these points that yields the given cubic (14). We implemented the following method in Magma. Our code runs fast because computations are done using floating point arithmetic in $\mathbb{Q}_p$. This fact is based on Conjecture 4.1.

Let $S \subset \mathbb{P}^3$ be the cubic surface in (14). Arguing as above, we find its 27 lines p-adically. Fix six pairwise skew lines $E_1, \dots, E_6 \subset S$. Label the other 21 lines on S as follows. The line F_{ij} intersects E_i and E_j but no other E_k, and G_i intersects E_j for $j \in \{1, \dots, 6\} \backslash \{i\}$. We will compute a morphism $\pi : S \to \mathbb{P}^2$ that contracts $E_1, \dots, E_6$ to points. The map π is specified uniquely by requiring

$$\pi(E_1) = (1:0:0),\ \pi(E_2) = (0:1:0),\ \pi(E_3) = (0:0:1),\ \pi(E_4) = (1:1:1).$$

For $i \neq j$, let H_{ij} be the plane spanned by G_i and E_j, and let h_{ij} be a linear form defining H_{ij}. For any point $q \in \mathbb{P}^3 \setminus G_i$, let H_{iq} be the plane spanned by G_i and q.

Theorem 4.2. Let $U_{ij} = S \setminus (G_i \cup G_j \cup F_{ij})$. Then $S = U_{12} \cup U_{13} \cup U_{23}$, and the blow-down map $\pi : S \to \mathbb{P}^2$ is given on each chart by a quadratic map as follows:

$$\begin{aligned}
\pi|_{U_{12}}(q) &= \big(u_{12} \cdot h_{12}(q)h_{23}(q) : v_{12} \cdot h_{21}(q)h_{13}(q) : w_{12} \cdot h_{12}(q)h_{21}(q)\big),\\
\pi|_{U_{13}}(q) &= \big(u_{13} \cdot h_{13}(q)h_{32}(q) : v_{13} \cdot h_{13}(q)h_{31}(q) : w_{13} \cdot h_{31}(q)h_{12}(q)\big),\\
\pi|_{U_{23}}(q) &= \big(u_{23} \cdot h_{23}(q)h_{32}(q) : v_{23} \cdot h_{23}(q)h_{31}(q) : w_{23} \cdot h_{32}(q)h_{21}(q)\big).
\end{aligned}$$

Here u_{ij}, v_{ij}, w_{ij} are nonzero constants that are determined by $\pi(E_4) = (1:1:1)$.

Proof. The equation $S = U_{12} \cup U_{13} \cup U_{23}$ is seen by examining the incidences among the lines F_{ij} and G_k that appear in the definition of the open sets U_{ij}.

We now prove the formula for π on U_{12}. The other two cases follow upon relabeling. Let $\gamma_1 : \mathbb{P}^3 \dashrightarrow F_{23}$ denote the projection from the line G_1, and let $\gamma_2 : \mathbb{P}^3 \dashrightarrow F_{13}$ denote the projection from G_2. Note that $\gamma_i(E_j)$ is a point if $j \in \{1,2,3,4,5,6\} \backslash \{i\}$. We identify F_{23} with $\mathbb{P}^1$ so that $\gamma_1(E_2) = (1:0)$, $\gamma_1(E_3) = (0:1)$ and $\gamma_1(E_4) = (1:1)$. Likewise, we identify F_{13} with $\mathbb{P}^1$ so that $\gamma_2(E_1) = (1:0)$, $\gamma_2(E_3) = (0:1)$ and $\gamma_1(E_4) = (1:1)$. This implies $\gamma_1(q) = (h_{13}(q) : \mu \cdot h_{12}(q))$ and $\gamma_2(q) = (h_{23}(q) : \nu \cdot h_{21}(q))$ where μ and ν are constants. The asserted formula for $\pi|_{U_{12}}(q)$ is obtained by fusing these two projections $S \dashrightarrow \mathbb{P}^1$.

The blow-down map $\pi : S \to \mathbb{P}^2$ has the property that $\gamma_1|_S$ is the composition of π followed by $\mathbb{P}^2 \dashrightarrow \{X = 0\} \simeq \mathbb{P}^1$, and $\gamma_2|_S$ is π followed by $\mathbb{P}^2 \dashrightarrow \{Y = 0\} \simeq \mathbb{P}^1$. We see from our construction that F_{12}, F_{13}, F_{23} are mapped to the three coordinate lines in $\mathbb{P}^2$ and that $E_1, E_2, \ldots, E_6$ are mapped to points. The images in $\mathbb{P}^2$ of the other 18 lines are determined by their intersection patterns on $S \subset \mathbb{P}^3$. In particular, the point $\pi(E_4)$ lies in $\{XYZ \neq 0\}$. This completes the proof. □

The blowup map $\varphi : \mathbb{P}^2 \dashrightarrow S \subset \mathbb{P}^3$ is defined by a tuple (g_0, g_1, g_2, g_3) of ternary cubics. The cubic curve cut out by g_i is the image under π of the intersection of S with the i-th coordinate plane in $\mathbb{P}^3$. Using the formulas for π above, we may compute the vanishing locus of g_i and therefore find a cubic $\tilde{g}_i = \lambda_i g_i$. In practice, we carry this out by interpolation using the six base points and the images of the points on S lying on coordinate lines. Since we assumed that S is tropically smooth, its points on coordinate lines are all defined over $\mathbb{Q}_p$.

To find the scalars $\lambda_0, \ldots, \lambda_3$, we consider the map $\tilde{\varphi} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^3$ defined by $(\mu_0 \tilde{g}_0, \ldots, \mu_3 \tilde{g}_3)$ with indeterminates μ_i. The additional constraint that the image of $\tilde{\varphi}$ must lie in S reveals the entries of $(\mu_0, \ldots, \mu_3) = (\lambda_0^{-1}, \ldots, \lambda_3^{-1})$.

We now turn to Question 11 from the 27 questions: *How to construct six points with integer coordinates in $\mathbb{P}^2$, and a basis for the space of cubics through these points, such that the resulting cubic surface in $\mathbb{P}^3$ has a smooth tropical surface for its p-adic tropicalization? Which unimodular triangulations arise?*

The following theorem, which is conditional on our conjecture, would give the definitive answer to Question 11. It implies that every smooth tropical cubic arises, and hence so does each of the $14,373,645$ unimodular triangulations.

Theorem 4.3. Suppose Conjecture 4.1 holds. Fix any tropically smooth cubic (14) whose coefficients c_i are p-adic numbers. Then there exist six points in $\mathbb{P}^2$ and a basis for their cubics, both defined over the rational numbers $\mathbb{Q}$, such that the resulting classical cubic surface in $\mathbb{P}^3$ has the same tropicalization as (14).

Proof. Two classical cubics in $\mathbb{P}^3$ have the same tropicalization if the valuations of their coefficients coincide. This holds if the distance between their coefficient vectors in $\mathbb{Q}_p^{20}$ is small with respect to the p-adic supremum norm.

Let S be the tropically smooth cubic in (14). Let $\mathfrak{p} = (p_1, \ldots, p_6)$ and $\mathfrak{g} = (g_0, \ldots, g_3)$ be the six points and four ternary cubics constructed by blowing down S as above. We claim that $\mathfrak{p}$ and $\mathfrak{g}$ can be approximated with rational coefficients so that the new cubic defined by these approximations is close to (14).

If Conjecture 4.1 holds then the six points $\mathfrak{p}$ are defined over $\mathbb{Q}_p$. Therefore, we can find six rational points $\mathfrak{q} = (q_1, \ldots, q_6)$ approximating $\mathfrak{p}$ to any specified degree of p-adic accuracy. Let V be the space of ternary cubics passing through $\mathfrak{q}$ and let $\tilde{\mathfrak{g}} = (\tilde{g}_0, \ldots, \tilde{g}_3)$ be orthogonal projections of the cubics in $\mathfrak{g}$ onto V. Orthogonal projection can be defined in the p-adic setting, and the distances $||g_i - \tilde{g}_i||$ are comparable to the distances $||p_i - q_i||$ in an explicit fashion. We can and do choose $\tilde{\mathfrak{g}}$ to have rational coefficients. Let $\tilde{S}$ be the resulting cubic in $\mathbb{P}^3$.

The coefficient vectors of the cubic surfaces S and $\tilde{S}$ span the kernels of the following 19×20 matrices M and $\tilde{M}$. Choose 19 general points in $\mathbb{P}^2_{\mathbb{Q}}$, evaluate the maps $\mathbb{P}^2 \to \mathbb{P}^3$ given by $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ on these points, and then evaluate all cubic monomials in x, y, z, w on the image points to get the columns of M and $\tilde{M}$.

By Cramer's rule, the coefficients $c_i, \tilde{c}_i$ of $S, \tilde{S}$ are the maximal minors of $M, \tilde{M}$. Since $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ are arbitrarily close, so are the entries of the matrices M and $\tilde{M}$. The cubic forms that define the surfaces S and $\tilde{S}$ are now arbitrarily close in the p-adic supremum norm. Hence, S and $\tilde{S}$ have the same tropicalization. □

The proof of Theorem 4.3 translates into the following algorithm for answering Question 11. We begin by choosing one of the $14,373,645$ distinguished representatives in [12] for the smooth tropical cubic surfaces. Next we choose a classical cubic S with full support (14) having that tropicalization. We then check that the 27 lines have distinct tropicalizations as asserted by Conjecture 4.1. We compute $(\mathfrak{p}, \mathfrak{g})$ and a rational approximation $(\tilde{\mathfrak{p}}, \tilde{\mathfrak{g}})$ as explained in the proof. The final step is to verify that S and $\tilde{S}$ have the same tropicalization. We implemented this algorithm in `Magma`. It is available at our website.

Example 4.4. We illustrate the algorithm for the combinatorial type # 7 in [12]. The representative coefficient vector for this type of tropical cubic surface is

$$(\mathrm{val}(c_0),\ldots,\mathrm{val}(c_{19})) = (16,7,3,11,9,2,5,3,0,0,5,0,4,3,0,2,6,5,8,15).$$

We fix $p=5$ and choose the canonical lifts $c_0 = 5^{16}$, $c_1 = 5^7$, $c_2 = 5^3, \ldots, c_{19} = 5^{15}$. We compute the 27 lines in this cubic, and we find that it is Naruki general.

We identify six skew lines $E_1,\ldots,E_6$ and write the map π as in Theorem 4.2. The six points in $\mathbb{P}^2$ are $p_i = \pi(E_i)$, with p_1,p_2,p_3,p_4 in standard position. For the other two points p_5,p_6, the first few terms in their p-adic expansions are

$$\begin{aligned} p_5 &= \big(5^{-1}\cdot(4+4\cdot 5^1+1\cdot 5^2+3\cdot 5^3+\ldots) : 5^5\cdot(4+4\cdot 5^1+4\cdot 5^2+0\cdot 5^3+\ldots) : 1\big),\\ p_6 &= \big(5^{7}\cdot(1+4\cdot 5^1+1\cdot 5^2+1\cdot 5^3+\ldots) : 5^5\cdot(4+4\cdot 5^1+4\cdot 5^2+4\cdot 5^3+\ldots) : 1\big).\end{aligned}$$

We next replace these by nearby points over $\mathbb{Q}$. We set $q_i = p_i$ for $i = 1,2,3,4$. We round p_5 and p_6 to rational points while retaining 14 digits of precision:

$$\begin{aligned} q_5 &= \big(2473616049/5 : -425393750 : 1\big),\\ q_6 &= \big(1331718750 : -2324221875 : 1\big).\end{aligned}$$

The special $\mathbb{Q}_p$-basis $\mathfrak{g} = (g_0,\ldots,g_3)$ of cubics through $\mathfrak{p} = (p_1,\ldots,p_6)$ is found as described after Theorem 4.2. Fix any $\mathbb{Q}$-basis for the space V of cubics through $\mathfrak{q} = (q_1,\ldots,q_6)$. We project each g_i into V via the non-archimedean version of the Gram–Schmidt process [21, Section 2.3]. This step is done over $\mathbb{Q}_p$. The image cubics are rounded to cubics over $\mathbb{Q}$ while staying in V and preserving the distance to $\mathfrak{g}$. We used a variation of p-adic LLL explained in [10] to find good rational approximations of p-adic vectors. The result is $\tilde{\mathfrak{g}} = (\tilde{g}_0,\ldots,\tilde{g}_3)$. The cubic surface $\tilde{S} \subset \mathbb{P}^3$ is the image of the map $\mathfrak{g}$. Its rational coefficients $\tilde{c}_i$ are very big. Their valuations match those of the c_i we started with. For instance,

$$\begin{aligned} \tilde{c}_0 = {}& 5^{16}\cdot(1+3\cdot 5^1+4\cdot 5^2+3\cdot 5^3+2\cdot 5^4+4\cdot 5^5+2\cdot 5^6+4\cdot 5^7+1\cdot 5^9+3\cdot 5^{12}\\ &+3\cdot 5^{13}+1\cdot 5^{14}+1\cdot 5^{16}+1\cdot 5^{18}+4\cdot 5^{19}+1\cdot 5^{20}+4\cdot 5^{21}+2\cdot 5^{22}+4\cdot 5^{24}+\ldots\\ &\cdots+3\cdot 5^{209}+4\cdot 5^{210}+2\cdot 5^{212}+2\cdot 5^{213}+2\cdot 5^{214}+1\cdot 5^{215}+4\cdot 5^{216}+2\cdot 5^{217}).\end{aligned}$$

This concludes our derivation of an explicit example for answering Question 11.

We close Section 4 by returning to the octanomial model. We explain how to derive, for a given cubic, the moduli coordinates d_i and thus the normal form (4). This requires us to identify a cuspidal cubic through our six points in $\mathbb{P}^2$ and to transform that cubic into the standard form $\{X^2Z = Y^3\}$. From this we can then read off the matrix (1). We carry out this computation in `Magma` as follows.

Our input is six points $p_1,\ldots,p_6$ in $\mathbb{P}^2$ over a field K. There is a web of cubic curves passing through them. We choose a seventh point p_7 to cut down the dimension and obtain a net $\mathcal{N} \simeq \mathbb{P}^2$ of cubics. If the points are general enough then $\mathcal{N}$ contains 24 cuspidal cubics. Hence, the product space $\mathcal{N}\times\mathbb{P}^2$ contains 24 pairs (f,r) where r is a cusp on the cubic $\{f=0\}$. These 24 points are defined by bihomogeneous equations represented by the gradient and the 2×2 minors of the Hessian of $f\in\mathcal{N}$. Our equations are $\nabla f(r)=0$ and $\mathrm{rank}\big(\mathrm{He}(f)(r)\big)=1$.

Extending to $\overline{K}$, we pick one solution (f,r). The cubic curve $\{f=0\}$ passes through $p_1,\dots,p_7$ and has a cusp at r. In addition, it has a unique smooth inflection point r'. Let ℓ_0,ℓ_1,ℓ_2 denote linear forms defining the cuspidal tangent, the line through r and r', and the inflection tangent at r' such that $f=\ell_1^3-\ell_0^2\ell_2$.

The triple (ℓ_0,ℓ_1,ℓ_2) defines the automorphism of $\mathbb{P}^2_{\overline{K}}$ that puts the cubic C into standard form $X^2Z=Y^3$. In particular, the E_6 moduli can now be read off:

$$d_i \;=\; \ell_1(p_i)/\ell_0(p_i) \qquad \text{for } i=1,2,\dots,6.$$

We explain how to implement this method in our setting, where the given cubic (14) has rational coefficients c_i. We first identify the points $p_1,\dots,p_6$ in $\mathbb{P}^2_K$, where $K=\mathbb{Q}_p$ as above. For computing the 24 solutions (f,r) in $\mathcal{N}\times\mathbb{P}^2_K$, one would like to use Gröbner bases. But, this requires special care because the polynomials to be solved have numerical coefficients, namely series in $\mathbb{Q}_p$.

Instead, we work with the rational approximations $q_1,\dots,q_6\in\mathbb{P}^2_{\mathbb{Q}}$ computed above. We also choose $q_7\in\mathbb{P}^2_{\mathbb{Q}}$. A cuspidal curve through $q_1,\dots,q_7$ typically does not exist over $\mathbb{Q}$. But, it often exists over $\mathbb{Q}_p$, depending on the choice of q_7. We always succeeded with this after several tries. Now, the pair (f,r) has been found over $\mathbb{Q}_p$. The rest of the computation is linear algebra over $\mathbb{Q}_p$. Namely, we compute ℓ_0 as the tangent line to f at r. The Hessian of f equals $\ell_0^2\ell_1$ times a constant, so we can find ℓ_1 up to constant. Next, the inflection point r' is found by intersecting the curve f with line ℓ_1. Finally ℓ_2 is the inflection line at r'. The constants are now found from the desired equation $f=\ell_1^3-\ell_0^2\ell_2$.

We applied this method to the rational configuration $q=(q_1,\dots,q_6)$ in Example 4.4. From the resulting cuspidal cubic f, we read the moduli parameters

$$\begin{aligned}
d_1 &= 1,\\
d_2 &= 2+2\cdot5^1+2\cdot5^2+3\cdot5^3+2\cdot5^4+1\cdot5^5+2\cdot5^6+4\cdot5^9\dots,\\
d_3 &= 2+4\cdot5^1+4\cdot5^2+4\cdot5^3+2\cdot5^4+4\cdot5^5+4\cdot5^6+2\cdot5^7+4\cdot5^8+1\cdot5^9+\dots,\\
d_4 &= 2+2\cdot5^1+2\cdot5^2+3\cdot5^3+4\cdot5^4+3\cdot5^5+4\cdot5^6+3\cdot5^9+\dots,\\
d_5 &= 1+2\cdot5^4+2\cdot5^5+1\cdot5^6+2\cdot5^7+1\cdot5^8+\dots,\\
d_6 &= 1+3\cdot5^1+2\cdot5^2+1\cdot5^3+4\cdot5^4+3\cdot5^5+4\cdot5^6+1\cdot5^7+4\cdot5^8+1\cdot5^9+\dots
\end{aligned}$$

The full details on this example, and on all others, are found on our website.

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