LE MATEMATICHE
Vol. LXXV (2020) – Issue II, pp. 411–424
doi: 10.4418/2020.75.2.1

# TWENTY-SEVEN QUESTIONS ABOUT THE CUBIC SURFACE

KRISTIAN RANESTAD - BERND STURMFELS

We present a collection of research questions on cubic surfaces in 3-space. These questions inspired the collection of papers published in this special issue of the Le Matematiche, so this article serves as an introduction. The number of questions is meant to match the number of lines on a cubic surface. We end with a list of problems that are open.

## 1. Introduction

One of most prominent results in classical algebraic geometry, derived two centuries ago by Cayley and Salmon, states that every smooth cubic surface in complex projective 3-space $\mathbb{P}^3$ contains 27 straight lines. This theorem has inspired generations of mathematicians, and it is still of importance today. The advance of tropical geometry and computational methods, and the strong current interest in applications of algebraic geometry, led us to revisit the cubic surface.

Section 3 of this article gives a brief exposition of the history of that classical subject and how it developed. We offer a number of references that can serve as first entry points for students of algebraic geometry who wish to learn more.

In December 2018, the second author compiled the list of 27 question. His original text was slightly edited and it now constitutes our Section 2 below. These questions were circulated, and they were studied by various groups of young mathematicians, in particular in Leipzig and Berlin. In May 2019, a one-day seminar on cubic surfaces was held in Oslo. Different teams worked on the

Received on September 12, 2019

questions and they made excellent progress. It was then decided to call for a special issue of *Le Matematiche*, with submission deadline in September 2019.

This call resulted in the 14 articles listed as [48]–[61]. In Section 4 we give an overview of this collection, and we briefly discuss the contribution of each article. For each article, we identify which of the 27 question it refers to. We also highlight ten key problems that remain open. These make it clear that there is still a lot of research to be done, and our readers are invited to join the effort.

## 2. Questions

The text that follows in this section was written and circulated by the second author in December 2018. The list of 27 questions was conceived as a working document that evolves over time. It was initially not meant to be published. It simply represents what the second author wanted to know, but did not know back then. The current status of the 27 questions will be our topic in Section 4.

In spite of two centuries of research on cubic surfaces, it appears that there are still many unresolved questions, especially when it comes to computational, tropical and applied aspects. Please feel free to circulate this text. It is aimed at stimulating further work on cubic surfaces, or equivalently, on symmetric $4\times4\times4$ tensors. Your feedback and comments will be greatly appreciated.

A cubic surface in $\mathbb{P}^3$ is the zero set of a homogeneous polynomial

$$\begin{aligned} f \;=\; & c_1x^3+c_2y^3+c_3z^3+c_4w^3+c_5x^2y+c_6x^2z+c_7x^2w+c_8xy^2 \\ & +c_9y^2z+c_{10}y^2w+c_{11}xz^2+c_{12}yz^2+c_{13}z^2w+c_{14}xw^2 \\ & +c_{15}yw^2+c_{16}zw^2+c_{17}xyz+c_{18}xyw+c_{19}xzw+c_{20}yzw. \end{aligned} \tag{1}$$

We work over a field $K$ of characteristic 0, such as $\mathbb{Q}$, $\mathbb{Q}_p$, $\mathbb{Q}(t)$, $\mathbb{R}$, $\mathbb{C}$, $\overline{\mathbb{Q}(t)}$, or $\mathbb{C}\{\{t\}\}$. The 15-dimensional group $\mathrm{PGL}(4)$ acts naturally on the projective space $\mathbb{P}^{19} = \mathbb{P}(\mathrm{Sym}_3(K^4))$ whose coordinates are $(c_1 : c_2 : \cdots : c_{20})$. Our first question concerns the orbits of that action. Here the point of departure would be Kazarnovskii's general formula, found in [15, 29], for degrees of orbits.

**Question 1.** Given a generic homogeneous cubic $f$ in $x,y,z,w$, what can we say about the orbit closure $\overline{\mathrm{PGL}(4)\cdot f}$? What is the degree of this variety in $\mathbb{P}^{19}$? Can we determine some of its defining polynomial equations?

The geometric invariant theory of cubic surfaces is well understood. In his 1861 article [38], Salmon found six fundamental invariants. Their degrees are 8, 16, 24, 32, 40 and 100. The square of the last one is a polynomial in the first five. Over a century later, Beklemishev [5] proved that Salmon's list is complete.

**Question 2.** How to evaluate the six invariants? Same question for their tropicalizations. (See the textbook [31] for basics on tropical geometry).

One obvious invariant is the discriminant of $f$. This is a polynomial of degree 32 in the coefficients $c_1, \dots, c_{20}$. Edge [20] corrects a formula written in fundamental invariants due to Salmon, apparently also repeated by Clebsch.

The next question assumes familiarity with the combinatorial theory of discriminants that was developed by Gel'fand, Kapranov and Zelevinsky [22].

**Question 3.** How many monomials does the discriminant have? How many vertices does its Newton polytope have, i.e. how many D-equivalence classes of regular triangulations are there?

The discriminant of the cubic $f$ equals the resultant of the quadrics $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}, \frac{\partial f}{\partial w}$. We therefore seek formulas for the resultant of four quadrics in $x, y, z, w$.

**Question 4.** Can we write the resultant of four quadratic surfaces in $\mathbb{P}^3$ as the determinant of an $8 \times 8$-matrix whose entries are linear forms in the brackets? This resultant is the Chow form of the Veronese embedding of $\mathbb{P}^3$ into $\mathbb{P}^9$. Such a formula would be derived from a nice Ulrich sheaf on that Veronese threefold.

Assuming the answer to Question 4 to be affirmative, we specialize to get an $8 \times 8$-matrix whose entries are quartics in $c_1, \dots, c_{20}$ and whose determinant equals the discriminant of $f$. Note that the discriminant has degree 32 in the $c_i$.

**Question 5.** Which varieties in $\mathbb{P}^{19}$ arise by imposing rank conditions on the $8\times8$-matrix in Question 4?

In his 1899 article [33], E.J. Nansen writes the above resultant as the determinant of a $20 \times 20$-matrix. We can again specialize this to a matrix whose determinant is the discriminant of $f$.

**Question 6.** Which varieties in $\mathbb{P}^{19}$ arise by imposing rank conditions on this $20\times20$-matrix?

It seems reasonable to surmise that the loci in Questions 5 and 6 are cubic surfaces with prescribed types of singularities. The simplest scenario is the occurrence of simple nodes as the only singularities. Then these are at most 4.

**Question 7.** For $k = 2, 3, 4$, the variety of $k$-nodal cubics is irreducible of codimension $k$ in $\mathbb{P}^{19}$. Using his numerical software [7], Sascha Timme computed that the degrees of these varieties are 280, 800 and 305 respectively. Later we learned that these degrees, and many more, had been found by Vainsencher [46]. Can we find explicit low-degree polynomials that vanish on these varieties?

**Question 8.** Can we find 17 real points in $\mathbb{P}^3$ such that all 280 of the 2-nodal cubics through these points are real? Can we find 16 real points in $\mathbb{P}^3$ such that all 800 of the 3-nodal cubics through these points are real? Also, are there configurations such that no such cubic is real?

The 4-nodal cubics are Cayley symmetroids. These arise in convex optimization, as boundaries of feasible regions in semidefinite programming.

**Question 9.** Can we find 15 real points in $\mathbb{P}^3$ that lie on 305 real Cayley symmetroids?

The following question arose from a conversation with Hannah Markwig.

**Question 10.** Can the numbers 280, 800 and 305 be derived tropically?

The next question refers to the classical construction of cubic surfaces by blowing up the projective plane $\mathbb{P}^2$ in six points.

**Question 11.** How to construct six points with integer coordinates in $\mathbb{P}^2$, and a basis for the space of cubics through these points, such that the resulting cubic surface in $\mathbb{P}^3$ has a smooth tropical surface for its 2-adic tropicalization? Which unimodular triangulations of the Newton polytope $3\Delta_3$ of a dense cubic $f$ arise?

Up to symmetry, there are 14373645 unimodular triangulations of the triple tetrahedron $3\Delta_3$. This number was reported recently in [28, Theorem 3.1].

**Question 12.** Given a cubic surface over a valued field $K$, how to decide whether its tropicalization is smooth after some linear change of coordinates? How to search $\mathrm{PGL}(4,K)$?

Salmon's invariant of degree 100 vanishes precisely when the cubic surface has an Eckardt point, that is, a point common to three of its 27 lines. This invariant deserves further study.

**Question 13.** What is the singular locus of the Eckardt hypersurface of degree 100 in $\mathbb{P}^{19}$?

A homogeneous cubic polynomial $f$ in $x,y,z,w$ can be interpreted as a symmetric tensor of format $4\times4\times4$. A typical tensor has complex rank 5, but its real rank becomes 6 as one crosses the *real rank boundary*. This is a hypersurface of degree 40 in $\mathbb{P}^{19}$ studied by Michałek and Moon [32, Proposition 3.4].

**Question 14.** What can be said about the tropicalization of the Michałek–Moon hypersurface?

Seigal [43, Proposition 2.6] identifies the Hessian discriminant as the locus where the complex rank of cubics $f$ jumps from 5 to 6. She points out that this discriminant has degree $\leq 120$ and is invariant under the action of $\mathrm{PGL}(4, K)$.

**Question 15.** How to write the Hessian discriminant in terms of Salmon's six fundamental invariants of the cubic surface?

The *eigenpoints* of $f$ are the fixed points of the gradient map $\nabla f : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$. This was studied by Abo et al. in [1]. For generic cubics $f$, there are 15 eigenpoints. They form the *eigenconfiguration* of the $4 \times 4 \times 4$ tensor $f$.

**Question 16.** Which configurations of 15 points in $\mathbb{P}^3$ arise as eigenpoints of a cubic surface?

The *eigendiscriminant* is a hypersurface of degree 96 in $\mathbb{P}^{19}$. This object and its degree are studied in [1, Section 4]. It represents cubic surfaces that possess an eigenpoint of multiplicity $\geq 2$. This hypersuface deserves further study.

**Question 17.** Does there exists a compact determinantal formula for the eigendiscriminant of the cubic surface?

We learned from Bruin and Sertöz [8] that there are 255 Cayley symmetroids containing a general complete intersection $(2,3)$-curve in $\mathbb{P}^3$, one for each 2-torsion point on the Jacobian of such a genus 4 curve. This is less than the number 305 of Cayley symmetroids found in a general $\mathbb{P}^4$ in $\mathbb{P}^{19}$; cf. Question 9.

**Question 18.** What explains the drop from 305 to 255 when we count Cayley symmetroids that lie in the special 4-plane in $\mathbb{P}^{19}$ of all cubic surfaces containing a given space sextic?

The following question paraphrases Problem 5.4 in [45]. It was studied by Bernal et al. [6], but the authors of that article left it largely unresolved.

**Question 19.** What are all toric degenerations of Cox rings of cubic surfaces?

The following question paraphrases Conjecture 5.3 in [37].

**Question 20.** Can we identify a tropical basis for the universal Cox ideal of cubic surfaces?

In tropical geometry, it is a big challenge to relate intrinsic Del Pezzo geometry to embedded geometry in $\mathbb{P}^3$. This is reminiscent from the curve case.

**Question 21.** There are two generic types of tropical del Pezzo surfaces of degree 3, characterized by the tree arrangements in [37, Figures 4 and 5]. Can we identify cubics $f$ that realize these two types by looking at the valuations of the six invariants in Question 2?

The lines in $\mathbb{P}^3$ are points $p = (p_{12} : p_{13} : p_{14} : p_{23} : p_{24} : p_{34})$ in the Grassmannian $\mathrm{Gr}(2,4) \subset \mathbb{P}^5$. The *universal Fano variety* in $\mathbb{P}^{19} \times \mathbb{P}^5$ parametrizes lines on cubic surfaces. Its ideal is generated modulo the Plücker quadric by 20 polynomials of degree $(3,1)$ in $(p,c)$. These are derived in [28, Section 6].

**Question 22.** Can we find an explicit tropical basis for universal Fano variety?

A real cubic surface in $\mathbb{P}^3_{\mathbb{R}}$ has either one or two connected components. In the latter case, the cubic is *hyperbolic*. It bounds a convex body that is of interest in optimization. For some background on real cubics we refer to [35].

**Question 23.** Can we find a semialgebraic description for the set of smooth hyperbolic cubics in $\mathbb{P}^{19}_{\mathbb{R}}$? How to express this case distinction in terms of the six fundamental invariants?

Every cubic $f$ whose Hessian discriminant (in Question 15) is non-zero has a unique representation as a sum of five third powers of linear forms, $f = \ell_1^3 + \ell_2^3 + \ell_3^3 + \ell_4^3 + \ell_5^3$. This is Sylvester's Pentahedral Theorem. Salmon [38] uses this to write the invariants.

**Question 24.** Can we find explicit linear forms $\ell_i \in \mathbb{Z}[x,y,z,w]$ such that the 2-adic tropicalization of $V(f)$ is tropically smooth. Which unimodular triangulations of the triple tetrahedron $3\Delta_3$ arise?

If we project a cubic surface $V(f)$ from a general point $p$ on that surface, then we get a double-covering of $\mathbb{P}^2$ branched along a quartic curve. The 28 bitangents of that curve are the images of the 27 lines on $V(f)$ plus one more line which is the exceptional divisor over $p$.

**Question 25.** Can this correspondence from 27 to 28 be understood in tropical geometry? In particular, can we see the seven 4-tuples of tropical bitangents already in the tropical cubic surface $\mathrm{Trop}(V(f))$?

The seven 4-tuples of bitangents of a tropical plane quartic are studied by Chan and Jiradilok in [11]. For cubic surfaces, Brundu and Logar [9] offer a computational study of $f$ via the following alternative normal form:

$$\begin{aligned} f \;=\; & a_1(2x^2y - 2xy^2 + xz^2 - xzw - yw^2 + yzw) + a_2(x-w)(xz+yw) + \\ & a_3(z+w)(yw-xz) + a_4(y-z)(xz+yw) + a_5(x-y)(yw-xz). \end{aligned}$$

This amounts to fixing an *L-set*, i.e. a special configuration of five lines on $V(f)$.

**Question 26.** How to compute the Brundu–Logar normal form in practice? Can we write $a_1, a_2, a_3, a_4, a_5$ as rational functions in $c_1, c_2, \ldots, c_{20}$? What does this tell us tropically?

Here is another interesting normal form, called the *Cayley–Salmon form* of $f$ by Dolgachev [17]. A general cubic surface has 120 distinct representations

$$f \;=\; \ell_1\ell_2\ell_3 + m_1m_2m_3, \tag{2}$$

where the $\ell_i$ and $m_j$ are linear forms. We learned this from Buckley and Kośir [10] who derived it from the classical construction of *Steiner sets*. Steiner called the two triples in (2) a *Triederpaar*, and he showed that there are 120 such pairs.

**Question 27.** How to compute the 120 Cayley–Salmon representations (2) of a given cubic surface in practice? Can this be tropicalized?

## 3. History

This section offers a historical introduction to the cubic surface and its 27 lines. It is aimed at students and other non-experts. One goal is to introduce accessible sources for further study of this beautiful theme in classical algebraic geometry. Those who seek a textbook introduction are referred to the books by Dolgachev [17, Chapter 9] and Reid [36, Chapter 7]. In applications, one considers cubic surfaces over the real numbers, and for those we refer to the book by Segre [41]

Arthur Cayley first showed that there are 27 lines on a general complex cubic surface, and then Salmon showed that there are 27 lines on every smooth complex cubic surface, both in the middle of the 19th century. Their results are remarkable given that nonsingular cubic surfaces are not all projectively equivalent. In fact, by a simple parameter count, there is a 4-dimensional family of isomorphism classes of cubic surfaces. Salmon found six fundamental invariants for cubic surfaces. Their degrees are 8, 16, 24, 32, 40 and 100, and the square of the last one is a polynomial in the other five. These invariants are homogeneous polynomials in the 20 coefficients $c_i$ of the cubic form $f$. They parameterize naturally the four-dimensional family of projective equivalence classes of cubic surfaces. For a concrete example, we computed the full expansion of Salmon's invariant of degree 8. It is a sum of 7261 terms, which looks like this:

$$\begin{gathered}192c_1^2c_{10}^3c_{13}^3 - 864c_1^2c_{10}^3c_{13}c_{16}c_3 + 2592c_1^2c_{10}^3c_3^2c_4 + 1728c_1^2c_{10}^2c_{12}^2c_{13}c_4\\ -648c_1^2c_{10}^2c_{12}^2c_{16}^2 - 288c_1^2c_{10}^2c_{12}c_{13}^2c_{15} + 432c_1^2c_{10}^2c_{12}c_{13}c_{16}c_{20} + \cdots\\ \cdots + 1728c_3c_4^2c_5^2c_6c_9^2 + 1728c_3c_4^2c_6^2c_8^2c_9 + 384c_3c_4c_7^3c_9^3 + 192c_4^2c_6^3c_9^3.\end{gathered}$$

A crucial ingredient in the investigations of Cayley and Salmon was the use of normal forms for $f$. The two most important ones were the Cayley-Salmon form and the Sylvester form. The Cayley-Salmon form, i.e. general $f$ is the sum of two triple products of linear forms, gives a nice access to the combinatorial structure of the 27 lines. Steiner [44] called the pair of triples of planes defined by the triple products a *Triederpaar*, and showed that a general cubic surface has 120 such Triederpaare. Sylvester's pentahedral form, i.e. a general $f$ is the sum of five cubes of linear forms, is the key to Salmon's computation of invariants.

Schläfli exploited the combinatorics of the 27 lines in [39]. In particular, he described the 36 double-sixes formed by the 27 lines. As was shown later by C. Segre in [42] (see also [16]), the double-sixes correspond one to one to the equivalence classes of linear determinantal presentation of $f$. Subsequently Schläfli, in [40], classified irreducible cubic surfaces according to singularities into 22 different types, and the number of real lines on real smooth cubic surfaces. A double-six of lines determines a pair of birational morphisms of the cubic surface to the plane $\mathbb{P}^2$. Each morphism contracts one set of six lines in the double-six to points in $\mathbb{P}^2$, and it maps the other set to conics in $\mathbb{P}^2$. This correspondence, showing that a smooth cubic surface is isomorphic to the projective plane blown up in six points, goes back to Clebsch [12].

The incidence graph of the 27 lines has a large automorphism group, identified with the Weyl group $E_6$ by Jordan [27]. The automorphism group of the surface is naturally a subgroup of $E_6$. The general nonsingular cubic surface has however no nontrivial automorphisms. The complex surfaces with nontrivial automorphisms are the ones with at least one Eckardt point, see [19]. These are points contained in three of the lines. A surface with an Eckardt point admits a projective involution that fixes the three lines pointwise.

In his paper [21] in the first volume of Mathematische Annalen, Geiser noted the relation between the 27 lines on a cubic surface and 28 bitangents to a plane quartic curve. The curve is the branch locus of the projection of the surface away from a general point on the surface. The birational automorphism defined by interchanging the sheets of this double cover is called the *Geiser involution*.

The early results by Schäfli on real cubic surfaces inspired the production of models of cubic surfaces with up to 27 real lines. The culmination of the work of this period is found in B. Segre's book [41]. This book is a comprehensive investigation into the five different kinds of real smooth cubic surfaces. To this date, the book [41] is the best source on cubic surfaces over the real numbers.

Well into the 20th century, algebraic geometry textbooks featured the smooth cubic surfaces. Baker's book *Principles of Geometry* [4] is an example. This changed in the second half of the 20th century. Sheaves and schemes now provided better tools for moduli spaces, like Hilbert schemes and geometric invariant theory (GIT). Dolgachev, van Geemen and Kondo used GIT to describe a 4-dimensional proper moduli space of $k$-nodal cubic surfaces [18] with $k = 0$ or $k = 1, \dots, 4$. Allcock, Carlson and Toledo [2] constructed a GIT model of the moduli space as quotient of a complex 4-dimensional ball by a reflection group.

The textbooks on abstract algebraic geometry by Hartshorne [26] and Griffiths and Harris [23] introduce smooth cubic surfaces as embeddings of the plane blown up in six points. Miles Reid's undergraduate textbook [36] gives a more elementary introduction to smooth cubic surfaces and its 27 lines, without reference to the isomorphism with the plane blown up in six points. A comprehensive treatment starting with the 27 lines and the group $E_6$, covering determinantal representations, power sum presentations and automorphisms, is given by Dolgachev [17]. It includes also a very nice historical overview on cubic surfaces. A nice overview of real cubic surfaces from a modern minimal program perspective is found in Kollár's lecture notes [30] on real algebraic surfaces.

Tropical geometry started at the beginning of the 21st century and it offers a new perspective on classical algebraic geometry. The book by Maclagan and Sturmfels offers an introduction which includes tropical surfaces in 3-space in [31, Section 4.5]. These are the natural tropicalizations of affine surfaces in the complement of the coordinate hyperplanes in $\mathbb{P}^3$. The relationship between complex and tropical geometry is an active area of research. Smooth tropical cubic surfaces may have infinitely many lines [34, 47]. Recent work in [14, 28, 37] concerns different representations of the cubic surface, aimed at revealing all the tropical lines in the tropicalization. The tropicalization of the complement of a triangle in a cubic surface has been considered by Gross et al. [24] to explain mirror symmetry phenomena of Calabi-Yau varieties.

## 4. Progress

Considerable progress on the 27 questions was made between January 2019 and September 2019. This resulted in 14 articles, written for the special volume of *Le Matematiche*. Here we introduce these papers, with emphasis on how they address the prompts in Section 2. Naturally, many questions remain unanswered. We conclude with a list of ten open problems, extracted from the 27 questions. We consider these to be especially interesting for further research.

The first three papers concern the group action of $\mathrm{PGL}(4)$ on the projective space $\mathbb{P}^{19}$ of cubic surfaces. Brustenga, Timme and Weinstein [49] use numerical algebraic geometry to answer Question 1. The general orbit in $\mathbb{P}^{19}$ is a 15-dimensional variety of degree 96120. Cazzador and Skauli [51] develop the intersection-theoretic approach to this problem. It rests on the solution by Aluffi and Faber for the same problem concerning $\mathrm{PGL}(3)$-orbits of plane curves. Elsenhans and Jahnel [55] resolve the first part of Question 2, by giving a practical algorithm for evaluating invariants and covariants of cubic surfaces. Based on the classical method of transvections, it is implemented in `Magma`.

We next come to the discriminant of the cubic surface. Kastner and Löwe [58] present a computational solution to Question 3: the Newton polytope has 166104 vertices. The paper by Bunnett and Keneshlou [50] addresses Questions 4, 6 and 7. The answer to Question 4 is “no” since there is no rank 1 Ulrich sheaf on the Veronese surface in $\mathbb{P}^9$. The authors determine a rank 2 Ulrich sheaf, they construct a Pfaffian representation of size $16 \times 16$ for the discriminant, and they examine the rank strata of Nansen’s $20 \times 20$ matrix. Keneshlou [59] identifies the singular locus of the Eckardt hypersurface. This solves Question 13.

Normal forms of cubic surfaces are important for many applications. Panizzut, Sertöz and Sturmfels [60] introduce a new normal form which works well for tropical geometry. They also answer most of Question 11. Donten-Bury, Görlach and Wrobel [54] resolve Question 19 by describing a classification of all toric degenerations of cubic surfaces. Their approach rests on Khovanskii bases of Cox rings. Hahn, Lamboglia and Vargas [57] address Question 27. They describe two methods for computing the 120 Cayley-Salmon equations.

Cubic surfaces correspond to symmetric tensors of format $4 \times 4 \times 4$. Seigal and Sukarto [61] investigate how their tensor rank is reflected in the singularity structure of the surface. An important player in their paper is the Hessian discriminant. This invariant is the object studied by Dinu and Seynnaeve [53], who present the answer to Question 15. The article by Çelik, Galuppi, Kulkarni and Sorea [52] studies the spectral theory of symmetric $4 \times 4 \times 4$ tensors. Their parametrization of the eigenpoints furnishes a partial answer to Question 16.

Two articles examine cubic surfaces via tropical geometry. Brandt and Geiger [48] give a partial answer to Question 10. They develop a tropical theory of binodal cubics through 17 given points in $\mathbb{P}^3$. In that setting, the classical count of 280 drops to 214. Geiger [56] studies the combinatorics of lines on tropical cubic surfaces. She proves that the Brundu-Logar normal form gives surfaces that are not tropically smooth, thus answering part two of Question 27.

We also note that an observant referee proposed an answer for Question 18. The following explanation for the drop was given. Consider the special 4-space of cubic surfaces that contain a $(2,3)$-curve $C$. This 4-space contains a $\mathbb{P}^3$ of reducible symmetroids, all with the unique quadric surface $Q$ that contains $C$, as a component. This $\mathbb{P}^3$ is an excess intersection that counts as 50 in the 305.

The 14 articles offer considerable new insights on cubic surfaces. However, some of the harder questions still remain unanswered. We conclude with a reprise of Section 2, in the form of a list of **ten open problems** on cubic surfaces.

1. Determine the prime ideal of the generic $\mathrm{PGL}(4)$ orbit on $\mathbb{P}^{19}$. (Q1)
2. How to evaluate the six fundamental invariants for a tropical cubic? (Q2)
3. Identify all rank varieties for the discriminant matrices in [50]. (Q5-6)

4. Determine the prime ideals of the varieties of $k$-nodal cubics in $\mathbb{P}^{19}$. (Q7)
5. Find $19-k$ points whose interpolating $k$-nodal surfaces are all real. (Q8-9)
6. Prove [60, Conjecture 4.1]: *Smooth tropical cubics have* 27 *lines*. (Q11)
7. Which cubics are tropically smooth after a coordinate change? (Q12)
8. Find determinantal or pfaffian formulas for eigendiscriminants. (Q17)
9. What is the correct tropicalization of Sylvester's Pentahedral Form? (Q24)
10. Relate the 27 lines to the 28 bitangents in the tropical setting. (Q25)

These are our favorites among problems extracted from the 27 questions. They underscore our belief that, even two centuries after Cayley and Salmon, the investigation of cubic surfaces will continue to be an active area of research.

## REFERENCES


[1] H. Abo, A. Seigal and B. Sturmfels: *Eigenconfigurations of tensors*, Algebraic and Geometric Methods in Discrete Mathematics, Contemporary Mathematics 685, Amer. Math. Soc., Providence, RI (2017) 1–25.

[2] D. Allcock, J. A. Carlson, D. Toledo: *The complex hyperbolic geometry of the moduli space of cubic surfaces*, J. Algebraic Geom. **11** (2002), 659-724.

[3] D. Allcock, J.A. Carlson, D. Toledo: *Hyperbolic geometry and moduli of real cubic surfaces*. Ann. Sci. Éc. Norm. Supér. (4) 43 (2010), no. 1, 69–115.

[4] H. Baker: *Principles of Geometry*, vols.1-6., Cambridge University Press, 1922.

[5] N.D. Beklemishev: *Invariants of cubic forms of four variables*, Vestnik Moscow Univ. Ser. I Mat. Mekh. **2** (1982) 42–49.

[6] M. Bernal, D. Corey, M. Donton-Bury, N. Fujita and G. Merz: *Khovanskii bases of Cox-Nagata rings and tropical geometry*, Combinatorial Algebraic Geometry, 133–157, Fields Inst. Communications **80**, Springer, New York.

[7] P. Breiding and S. Timme: *HomotopyContinuation.jl: A package for homotopy continuation in Julia*, Lecture Notes in Computer Science **10931**, 458-465, 2018.

[8] N. Bruin and E. Sertöz: *Prym varieties of genus four curves*, Transactions of the American Mathematical Society (2020), `arXiv:1808.07881`.

[9] M. Brundu and A. Logar: *Parametrization of the orbits of cubic surfaces*, Transformation Groups **3** (1998) 209–239.

[10] A. Buckley and T. Košir: *Determinantal representations of smooth cubic surfaces* **125** (2007) 115–140.

[11] M. Chan and P. Jiradilok: *Theta characteristics of tropical $K_4$-curves*, Combinatorial Algebraic Geometry, 65–86, Fields Inst. Comm. **80**, Springer, New York.

[12] A. Clebsch: *Die Geometrie auf den Flächen dritter Ordnung*, J. Reine Angew. Math. **65** (1866) 359-380.

[13] L. Cremona: *Mémoire de géométrie pure sur les surfaces du troisiéme ordre*, Journal des Math. pures et appl. **68** (1868)

[14] M.A. Cueto, A. Deopurkar: *Anticanonical tropical cubic del Pezzos contain exactly 27 lines*, `arXiv:1906.08196`

[15] H. Derksen and G. Kemper: *Computational Invariant Theory*, Encyclopedia of Mathematical Sciences, vol 130, Springer-Verlag, Berlin 2002.

[16] I. V. Dolgachev: *Luigi Cremona and cubic surfaces*. In Luigi Cremona (1830-1903). Convegno di Studi matematici, Istituto Lombardo, Accademia di Scienze e Lettere, Milano (2005), 55-70

[17] I.V. Dolgachev: *Classical Algebraic Geometry, A Modern View*, Cambridge University Press, 2012.

[18] I.V. Dolgachev, B. van Geemen, S. Kondo: *A complex ball uniformization of the moduli space of cubic surfaces via periods of K3 surfaces*, J. Reine Angew. Math. **588** (2005), 99–148.

[19] F.E. Eckardt: *Uber diejenigen Flächen dritten Grades, auf denen sich drei gerade Linien in einem Punkte schneiden*, Mathematische Annalen **10** (1876) 227-272

[20] W.L. Edge: *The discriminant of a cubic surface*, Proceedings of the Royal Irish Academy Ser. A **80A** (1980) 75–78.

[21] C. Geiser: *Über die Doppeltangenten einer ebenen Curve vierten Grades*, Mathematische Annalen **1** (1860) 129-138.

[22] I. M. Gel'fand, M. Kapranov and A. Zelevinsky: *Discriminants, Resultants, and Multidimensional Determinants*, Birkhäuser, Boston, MA, 1994.

[23] P. Griffiths and J. Harris: *Principles of Algebraic Geometry*, Wiley & Sons, 1978.

[24] M. Gross, P. Hacking, S. Keel, B. Siebert: *The mirror of the cubic surface*, `arXiv:1910.08427`

[25] P. Hacking, S. Keel and J. Tevelev: *Stable pair, tropical, and log canonical compactifications of moduli spaces of del Pezzo surfaces*, Invent. Math. **178** (2009) 173–227.

[26] R. Hartshorne: *Algebraic Geoometry*, GTM **52**, Springer (1977)

[27] C. Jordan: *Traité des substitutions et équations algébriques*, Paris, Gauthier- Villars, 1870.

[28] M. Joswig, M. Panizzut and B. Sturmfels: *The Schläfli fan*, `arXiv:1905.11951`.

[29] B.Ya. Kazarnovskii: *Newton polyhedra and the Bézout formula for matrix-valued functions of finite-dimensional representations*, Functional Analysis and its Applications **21** (1987) 319–321.

[30] J. Kollár: *Real algebraic surfaces*, Lecture notes of Trento summer school, September 1997, `arXiv:alg-geom/9712003`.

[31] D. Maclagan and B. Sturmfels: *Introduction to Tropical Geometry*, Graduate Studies in Mathematics, Vol 161, American Mathematical Society, 2015.

[32] M. Michałek and H. Moon: *Spaces of sums of powers and real rank boundaries*, Beiträge zur Algebra und Geometrie **59** (2018) 645–663.

[33] E. J. Nansen: *On the eliminant of a set of quadrics, ternary or quaternary*, Proceedings of the Royal Society of Edinburgh **22** (1899) 353–358.

[34] M. Panizzut and M.D. Vigeland: *Tropical Lines on Cubic Surfaces*, `arXiv:0708.3847` (revised 2019).

[35] I. Polo-Blanco and J. Top: *Explicit real cubic surfaces*, Canadian Mathematical Bulletin **11** (2008) 125–133.

[36] M. Reid: *Undergraduate Algebraic Geometry*, LMS Student texts **12**, Cambridge University Press, 1988.

[37] Q. Ren, K. Shaw and B. Sturmfels: *Tropicalization of del Pezzo surfaces*, Advances in Mathematics **300** (2016) 156–189.

[38] G. Salmon: *On quaternary cubics*, Philosophical Transactions of the Royal Society, **150** (1861) 229–239.

[39] L. Schläfli: *An attempt to determine the twenty-seven lines upon a surface of the third order and to divide such surfaces into species in reference to the reality of the lines upon the surface*, Quart. J. Math. **2** (1858) 55–65, 110–121.

[40] L. Schläfli: *On the distribution of surfaces of the third order into species, in reference to the absence or presence of singular points, and the reality of their lines*, Phil. Trans. of Roy. Soc. London **6** (1863) 201–241.

[41] B. Segre: *The Non-singular Cubic Surfaces*, Oxford University Press, 1942

[42] C. Segre: *Sur la génération projective des surfaces cubiques*, Archiv der Math. und Phys. **10** (1906) 209–215

[43] A. Seigal: *Ranks and symmetric ranks of cubic surfaces*, `arXiv:1801.05377`.

[44] J. Steiner: *Über die Flächen dritten Grades*, Journ. für reiner und angew. Math., **53** (1856)e 133–141

[45] B. Sturmfels and Z. Xu: *Sagbi bases of Cox-Nagata rings*, Journal of the European Mathematical Society **12** (2010) 429–459.

[46] I. Vainsencher: *Hypersurfaces with up to six double points*, Communications in Algebra **31** (2003) 4107–4129.

[47] M.D. Vigeland: *Smooth tropical surfaces with infinitely many tropical lines*, Ark. Mat. 48 (2010), **1**, 177-206.

[48] M. Brandt and A. Geiger: *A tropical count of binodal cubic surfaces*, Le Matematiche, this volume.

[49] L. Brustenga, S. Timme, M. Weinstein: *96120: The degree of the linear orbit of a cubic surface*, Le Matematiche, this volume.

[50] D. Bunnett and H. Keneshlou: *Determinantal representations of the cubic discriminant*, Le Matematiche, this volume.

[51] E. Cazzador and B. Skauli: *Towards the degree of the PGL$(4)$-orbit of a cubic surface*, Le Matematiche, this volume.

[52] T. Çelik, F. Galuppi, A. Kulkarni and M.-S. Sorea: *On the eigenpoints of cubic surfaces*, Le Matematiche, this volume.

[53] R. Dinu and T. Seynnaeve: *The Hessian discriminant*, Le Matematiche, this volume.

[54] M. Donten-Bury, P. Görlach and M. Wrobel: *Towards classifying toric degenerations of cubic surfaces*, Le Matematiche, this volume.

[55] A.-S. Elsenhans and J. Jahnel: *Computing invariants of cubic surfaces*, Le Matematiche, this volume.

[56] A. Geiger: *On realizability of lines on tropical cubic surfaces and the Brundu-Logar normal form*, Le Matematiche, this volume.

[57] M. Hahn, S. Lamboglia and A. Vargas: *A short note on Cayley-Salmon equations*, Le Matematiche, this volume.

[58] K. Kastner and R. Löwe: *The Newton polytope of the discriminant of a quaternary*

*cubic form*, Le Matematiche, this volume.
[59] H. Keneshlou: *Cubic surfaces on the singular locus of the Eckardt hypersurface*, Le Matematiche, this volume.
[60] M. Panizzut, E. Sertöz and B. Sturmfels: *An octanomial model for cubic surfaces*, Le Matematiche, this volume.
[61] A. Seigal and E. Sukarto: *Ranks and singularities of cubic surfaces*, Le Matematiche, this volume.

*KRISTIAN RANESTAD*
*Department of Mathematics, University of Oslo*
*e-mail:* `ranestad@math.uio.no`

*BERND STURMFELS*
*MPI for Mathematics in the Sciences, Leipzig*
*e-mail:* `bernd@mis.mpg.edu`