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Optimal strategies for selecting coordinators[☆]

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ABSTRACT

We study optimal election sequences for repeatedly selecting a (very) small group of leaders among a set of participants (players) with publicly known unique ids. In every time slot, every player has to select exactly one player that it considers to be the current leader, oblivious to the selection of the other players, but with the overarching goal of maximizing a given parameterized global ("social") payoff function in the limit. We consider a quite generic model, where the local payoff achieved by a given player depends, weighted by some arbitrary but fixed real parameter, on the number of different leaders chosen in a round, the number of players that choose the given player as the leader, and whether the chosen leader has changed w.r.t. the previous round or not. The social payoff can be the maximum, average or minimum local payoff of the players. Possible applications include quite diverse examples such as rotating coordinator-based distributed algorithms and long-haul formation flying of social birds. Depending on the weights and the particular social payoff, optimal sequences can be very different, from simple round-robin where all players chose the same leader alternatingly every time slot to very exotic patterns, where a small group of leaders (at most 2) is elected in every time slot. Moreover, we study the question if and when a single player would not benefit w.r.t. its local payoff when deviating from the given optimal sequence, i.e., when our optimal sequences are Nash equilibria in the restricted strategy space of oblivious strategies. As this is the case for many parameterizations of our model, our results reveal that no punishment is needed to make it rational for the players to optimize the social payoff.

1. Introduction

There are many instances of both man-made and evolutionary developed biological distributed systems that verify the utility of using a leader for solving certain problems. A well-known example is the rotating coordinator approach, which is e.g. used for enforcing a univalent system state in the Byzantine fault-tolerant Phase Queen and Phase King consensus algorithms [7]. An interesting example from biology is long-haul V-formation flying of social birds like geese and pelicans, where all birds except the leader benefit energetically from the uplift of their neighbor ahead [4].

The latter immediately raises the question of why any individual should take over the exhausting role of the flock leader at all? As argued in [4], *reciprocation* and *kin selection* might play a role here: In social[1] groups (where individuals

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1 In fact, less social birds or larger populations do not use V-formation flying.

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know each other), free-loaders do not go unnoticed and thus take the risk of being mobbed in the air or at a stopover. Hence, there is pressure on every (healthy and adult) individual to take turns in leadership. Kin selection may be the motivation for certain individuals, in particular, parents, to take over the lead for gaining inclusive fitness of their whole family.

Obviously, taking turns in the role of the leader can be seen as a *strategy* of the individual birds for solving a *repeated leader election problem*: Every bird chooses a leader candidate according to this strategy, independently of the choices of the other birds, which nevertheless allows to easily determine a single leader most of the times. The birds' strategy may of course be seen as the evolutionary result of the strive for maximum fitness. A good *election sequence*, comprising the choices of all birds, should (i) reach agreement on the next leader (to make the actual leader determination easy, thereby avoiding turbulences at the formation head), (ii) encourage fair leader changes (to avoid exhaustion of the leader), but (iii) penalize too frequent changes (to minimize the cumulated adverse effects of the leader changes during the flight).

Similar considerations can be applied for rotating coordinator algorithms in distributed computing [6,16], which actually involve (a simple form of) repeated leader election. In particular, the Phase Queen consensus algorithm [7] for synchronous byzantine fault-tolerant distributed systems [13] operates in phases consisting of two rounds each. In the first round, a full message exchange is used to get the proposal values of all participating processes. If this leads to a univalent system configuration (an overwhelming majority for some value), every process can decide. In the second round of a phase, only the coordinator process sends its new proposal value, which is adopted by every still undecided receiver. If the coordinator is correct, this leads to a univalent system configuration. If the coordinator is faulty or not unique, the system may still be in a bivalent configuration at the end of the phase, so further phases are needed until the system can become univalent.

In the original Phase Queen algorithm, the process with id equal to the phase number modulo n is used as the coordinator. It is not difficult to see, however, that the Phase Queen algorithm can be adapted to work with every strategy that (i) sufficiently often reaches agreement on the coordinator (to force the system configuration to become univalent), (ii) enforces reasonably fair leader changes (to eventually choose a non-faulty coordinator and distribute the coordinator-load evenly among all processes), and (iii) to stimulate sufficiently frequent leader changes (for fast termination and, again, to avoid putting too much coordinator load on a single node in successive rounds). In order not to increase the adverse power of byzantine faulty nodes, however, every process should choose the coordinator in a given round independently of the choices of the other processes, albeit in a way that guarantees agreement on a single coordinator sufficiently often.

Main contributions: In this paper, we completely characterize election sequences L for oblivious repeated leader election in synchronous systems that optimize the global payoff ("social payoff") in the limit. We consider a system of n participants (for compatibility with game theory, we call them players), where every player $i \in \Pi = \{1, \ldots, n\}$ has to choose a player $L_i(r)$ it considers to be the leader of the current round $r \in \{1, 2, \ldots\}$. We will restrict our attention to the restricted strategy space of *oblivious strategies*, where, in round r, the players are completely unaware of the choices of the other players in rounds $1, \ldots, r$. Note that we borrowed the term "oblivious" from [22], where it is used with a slightly less restrictive meaning. If $L_i(r) = j$, player i votes for j to act as the leader in round r. Note carefully, though, that different players could vote for different leaders in a given round as well. The collection of all the players votes in round r is denoted $L(r)$, and the election sequence $L = (L(r))_{r\geq 1}$ consists of all the collective votes over time.

In order to quantitatively assess the quality of a particular election sequence, we first define a *round r-payoff* $u_i^{(r)}(L) = |\mathcal{G}_i^{(r)}(L)| + c \cdot |\mathcal{F}_i^{(r)}(L)| + c'\delta_{L_i(r-1)}(L_i(r))$ for every player $i \in \Pi$. Note carefully that player i's local payoff also depends on the leader choices of all other processes, and is hence *not* known by player i locally:

- $\mathcal{F}_i^{(r)}(L)$ is the set of players that voted for i. If this quantity is large, many players want i to be leader, and depending on the sign of the parameter c, the local payoff of i gets proportionally larger or smaller. It therefore allows to model the benefit resp. loss for being elected leader.
- $\mathcal{G}_i^{(r)}(L)$ is the set of players that voted for the same player as process i. This quantity, the parameter of which has been normalized to 1, hence models the common strive to elect a small group of leaders, ideally only one.
- $\delta_{L_i(r-1)}(L_i(r)) = 1$ if $L_i(r-1) = L_i(r)$ (i.e., the chosen leader did not change) and 0 otherwise. This quantity, along with its parameter c', allows to incorporate the strive for leader stability vs. leader changes. Note that we are aware of the limitations of this somewhat simplistic modeling, which has been chosen also for tractability reasons (see Section 6).

The (local) *payoff* of player i is then defined as $u_i(L) = \liminf_{r\to\infty} \frac{1}{r}\sum_{k=1}^{r} u_i^{(k)}(L)$, and the (global) *social payoff* is either $\operatorname{avg}\{u_1(L), \ldots, u_n(L)\}$, $\min\{u_1(L), \ldots, u_n(L)\}$ or $\max\{u_1(L), \ldots, u_n(L)\}$. We addressed these different social payoffs in order to maximize the applicability of our results: Besides the quite natural average, it may be interesting in some applications to know the smallest (resp. largest) payoff of some fixed player, which is provided by the min (resp. max) social payoff. The goal of this paper is to precisely characterize election sequences that maximize these social payoffs.

The real-valued parameters c, c' can be chosen arbitrarily to model different applications. For instance, in our V-formation flight example, $c > 0$ represents the importance of reciprocation for every individual and hence assures fair turns, whereas c' controls the frequency of the leader changes; $c' > 0$ encourages infrequent leader changes and thus minimizes the disturbances of the flock at the cost of exhaustion of the leader, whereas $c' < 0$ encourages frequent leader changes. For the social payoff, avg and min seem natural candidates, but even max may make sense here: It may be in the interest of the entire family to maximize the payoff of some (strongest) individual (in return of its parental care efforts), albeit one should probably choose $c' < 0$ in this case, to guarantee some leader changes that avoid the exhaustion of a single individual that always acts as the leader. In our Phase Queen algorithm example, $c > 0$ represents the importance of fairly choosing every process, whereas $c' < 0$ again controls the leader change frequency. For the social payoff, min is arguably the most interesting social payoff, as no process should experience an excessively low payoff in order to ensure that all processes take turns in acting as the coordinator.

We need to stress, however, that the problem studied in this paper is a global optimization problem, rather than a problem in game theory. More specifically, we do not assume that a player has a strategy to choose its leader $L_i(r)$ locally, e.g., based on local payoff maximization and/or some information on the past choices of the other players. Rather, we just assume that some "decision-maker" gives the election sequence L to the players a priori somehow, and all players faithfully choose according to L.

This does not mean, however, that our results are irrelevant with respect to game theory. First of all, it would actually be very interesting to study our social payoffs in a setting where the players choose their leader proposal using information collected in the past, possibly even in an evolutionary setting. In order to be able to compute the "price of anarchy", as e.g. in Schmid et al. in a virus inoculation game setting [17], i.e., the ratio between locally computed election sequences and the globally optimal ones, one obviously needs our results.

In view of this application of our results, it is also of interest whether and when an optimal sequence L is a Nash equilibrium, i.e., whether and when it does not pay off for a single player to deviate from L: we determine the range of parameters c, c', for which any player i's sequence of local choices $L_i(r)$ is a best response strategy to the case where all other players keep acting according to L. Note carefully that this defines a Nash equilibrium in a restricted strategy space only, namely, of those strategies where players choose leaders oblivious w.r.t. the choices of the other players in all previous rounds. This restriction is necessary, since a player is unable to detect and possibly punish the deviation of some other player i as part of its own strategy. Anyway, if an optimal strategy for some social payoff can be shown to be a Nash equilibrium in the above sense, it is rational for the players not to deviate in order to also maximize their local payoff, which in turn maximizes the social payoff. As a consequence, no punishment is needed to enforce this behavior. Note that this is in stark contrast to "folk theorems" for repeated games [11,23], where punishments are used to achieve this goal.

Paper organization. After a short overview of related work in Section 2, we introduce the cornerstones of the underlying model in Section 3. In Section 4, we characterize the optimal election sequences for maximizing the different variants of our social payoffs, for all possible parameter choices for c, c'. In Section 5, we investigate the conditions under which our optimal sequences are Nash equilibria. A discussion of the consequences of our results in Section 6 and some directions of future work in Section 7 conclude our paper.

2. Related work

One-shot leader election is a well-studied problem in distributed computing, where it requires all the processes in the system to agree on a single leader process. For systems without failures, both synchronous and asynchronous leader election algorithms can be found in any good textbook [6,16], and thanks to its close relation to distributed consensus, the same can be claimed even for (Byzantine) fault-tolerant distributed systems [10,13]. The challenge lies in the fact that the processes do not a priori know the ids of all the processes in the system, so need to reach agreement via a protocol.

One-shot leader election has also been studied in game-theoretic settings. For rational players, [2] studied Nash equilibria and even k-resilient Nash equilibria [1,12] for coalitions of a minority of deviating players. A k-resilient equilibrium ($k = 1$ in the case of a standard Nash equilibrium) provides a strategy, i.e., a protocol, which is optimal w.r.t. coalitions of k deviating players. The challenge answered in [2] is to ensure fair equilibria, where every player has the same probability of being elected as the leader. Whereas acquiescent players (that faithfully follow the protocol) are allowed here, Byzantine faulty players, as also foreseen in more general models like the BAR model [3], are forbidden: after all, electing a Byzantine player cannot be ruled out in leader election.

The challenge in one-shot leader election is to ensure that exactly one leader is elected, despite selfish interests, and without a priori common knowledge of the ids of all players. In stark contrast, in this paper, we consider the problem of (infinitely) *repeated* leader election in systems, where agreeing on a single leader is usually desirable but not mandatory. There is a reasonably rich literature on repeated games, which goes back at least to John Nash's PhD thesis. Most existing work focuses on 2-player scenarios, typically of the iterated Prisoner's Dilemma, albeit there are also results on general and multi-player repeated games [11,23]. Note that leader election for two players can be viewed as an instance of the well-known coordination game *Battle of Sexes* [15].

For infinitely repeated games, "folk theorems" have been established [11,23], which essentially show that any feasible payoff of the one-shot game underlying an infinitely repeated game can be the discounted average payoff, for a discount factor sufficiently close to 1, of a Nash equilibrium of the infinitely repeated game. However, those results depend on the fact that a deviating player can be punished by the other players, so that she has no incentive to deviate. Note carefully, however, that folk theorems are not applicable if such punishment if prohibited, e.g., when the other players are oblivious to a possible deviation of some player.

By contrast, we consider Nash equilibria in restricted strategy spaces, which have been studied in different applications in the past. The seminal work of Simon [18] proposed bounded rationality (i.e., bounded resources) as a natural way to restrict strategy spaces, and this has been studied in different contexts in economics [19,20] as well as reactive synthesis [8]. Oblivious strategy spaces have been introduced in [22], where the decision making of a player is confined to its local state (plus some long-term average information on the global system state). In the particular context of Nash equilibria, considering restricted strategy spaces is also known from literature: (a) stationary strategy equilibria, which are restricted to strategies that do not remember past information, have been studied in [9,21]; (b) since dealing with irrational probabilities is complex even in one-shot games, simple uniform strategies equilibria have been considered in [14]. In all these contexts, including ours, as argued in Section 1, strategy space restrictions are natural, and so are Nash equilibria with respect to restricted strategy spaces.

Nevertheless, our work differs significantly from all the related work mentioned above, in several aspects: Rather than the "weak" utility used in [2], where a player never considers an outcome of the leader election without a leader better than one with a leader, we consider a much richer utility function. Moreover, we consider infinitely repeated leader election with more then 2 players. And last but not least, rather than considering players who try to maximize the individual local payoff, we are looking for election sequences (that can be viewed as optimal strategies) that maximize some form of social payoff. Similar to [5], where the authors study a one-shot inoculation game that models virus infection/protection in networks, we also address the question of whether globally optimal sequences are also locally optimal, i.e., Nash equilibria. However, unlike [5], we neither characterize locally optimal strategies nor do we compute the "price of anarchy" as Schmid et al. do in [17].

3. Preliminaries

In this section, we will introduce the formal model and basic concepts used in our paper, as well as some pivotal technical definitions.

3.1. The model

Let n be a positive integer. Denote by $\Pi = \{1, 2, \dots, n\}$ the set of *players* and by $\mathcal{V} \subseteq \mathbb{N}$ the set of possible votes. A *vote* is an element of $\mathcal{V}$. An *election* is a collection of votes, indexed by Π. In this paper, we will always assume that[2] $\mathcal{V} = \Pi$.

An *election sequence* $L = (L(r))_{r\geq 1}$ is a sequence of elections, which is given to the players a priori by some "decision-maker". Given an election sequence L, we say that player i voted for player j in round r in L if $L_i(r) = j$ and call j the vote of player i in round r. Player i considers itself *leader* in round r of L if $L_i(r) = i$. Note that more than one process may consider itself leader in the same round r.

For an election sequence L, we define $\mathcal{F}_i^{(r)}(L)$ as the set of players that voted for i in round r in L. We further define $\mathcal{G}_i^{(r)}(L)$ as the set of players that voted for the same player as process i in round r in L. Formally, we have

$$\mathcal{F}_i^{(r)}(L) = \{j \in \Pi :\ L_j(r) = i\},$$
$$\mathcal{G}_i^{(r)}(L) = \{j \in \Pi :\ L_j(r) = L_i(r)\}.$$

Definition 3.1 (*Payoffs*)**.** Let c and c' be real numbers. The *payoff* of player i in round r is defined as

$$u_i^{(r)}(L) = |\mathcal{G}_i^{(r)}(L)| + c \cdot |\mathcal{F}_i^{(r)}(L)| + c' \delta_{L_i(r-1)}(L_i(r)), \tag{1}$$

where $\delta_{L_i(r-1)}(L_i(r)) = 1$ if $L_i(r-1) = L_i(r)$ and 0 otherwise, and the *payoff* of player i is

$$u_i(L) = \liminf_{r\to\infty} \frac{1}{r} \sum_{k=1}^{r} u_i^{(k)}(L).$$

Example 3.2. We illustrate these definitions by means of a short example. Assume that we have five players with the sequence of votes specified in columns:

Player	$r = 1$	2	3	4	5	6	…
1	1	1	1	1	1	1	…
2	2	1	1	2	1	1	…
3	4	4	1	4	4	1	…
4	3	4	1	3	4	1	…
5	5	5	1	5	5	1	…

The following table gives the individual payoffs in the first four rounds, as well as the (limit) payoff of each player if the sequence of the first three rounds is repeated forever.

[2] For further research, we have some generalizations in mind, where the set of possible votes can differ from the set of players. For example, one can assign (possibly identical) labels to the players.

Player	$r=1$	2	3	4	limit payoff
1	$1+c$	$2+2c+c'$	$5+5c+c'$	$1+c+c'$	$\frac{8}{3}+\frac{8}{3}c+c'$
2	$1+c$	2	$5+c'$	$1+c$	$\frac{8}{3}+\frac{1}{3}c+\frac{1}{3}c'$
3	$1+c$	$2+c'$	5	$1+c$	$\frac{8}{3}+\frac{1}{3}c+\frac{1}{3}c'$
4	$1+c$	$2+2c$	5	$1+c$	$\frac{8}{3}+c$
5	$1+c$	$1+c+c'$	5	$1+c$	$\frac{7}{3}+\frac{2}{3}c+\frac{1}{3}c'$

Based on the payoffs of the individual players according to Definition 3.1, we can define several different payoffs for the whole group of players:

Definition 3.3 (*Social Payoffs*)**.** An *ethics* is a function $\mathbb{R}^n \to \mathbb{R}$. In particular, we will consider the following ethics:

$$\min(x_1, x_2, \dots, x_n) = \min\{x_i \mid 1 \leqslant i \leqslant n\}$$
$$\max(x_1, x_2, \dots, x_n) = \max\{x_i \mid 1 \leqslant i \leqslant n\}$$
$$\mathrm{avg}(x_1, x_2, \dots, x_n) = \frac{1}{n}\sum_{i=1}^{n} x_i$$

For a election sequence L and an ethics e, we define the *e-social payoff* by $\mathfrak{u}_e(L) = e(u_1(L), \dots, u_n(L))$.

Our goal in this paper is to characterize optimal election sequences in case $\mathcal{V} = \Pi$ for the three ethics given above, in dependence on the choice of c and c'.

Definition 3.4 (*Optimal Election Sequence*)**.** An election sequence L is called *optimal* for an e-social payoff $\mathfrak{u}_e$, if

$$\mathfrak{u}_e(L) \geq \mathfrak{u}_e(L')$$

for all election sequences L'.

3.2. Technical definitions

All our characterizations of optimal election sequences will rely on the sets of round numbers given in Definition 3.5. For simplicity, we will often write just A_i instead of $A_i(L)$ and so on when using these sets.

Definition 3.5 (*Characterization Sets*)**.** For an election sequence L, we define the following sets:

$$\begin{aligned}
A_i(L) &= \{r \mid \text{all players vote for player } i \text{ in round } r\} \\
&= \{r \mid L_j(r) = i \text{ for all } j \in \Pi\}, \\
B_i(L) &= \{r \mid \text{in round } r \text{ all players vote for the same player, but not for } i\} \\
&= \{r \mid \exists\, k \neq i:\ L_j(r) = k \text{ for all } j \in \Pi\}, \\
E_i(L) &= \{r \mid \text{in round } r \text{ all players except player } i \text{ vote for } i,\ \text{player } i \text{ votes for a} \\
&\qquad \text{player } j \neq i\} \\
&= \{r \mid \exists\, k \neq i:\ L_j(r) = i \text{ for all } j \in \Pi \setminus \{i\},\ L_i(r) = k\}, \\
H_i(L) &= \{r \mid \text{in round } r \text{ player } i \text{ votes for a player } j \neq i, \text{ all the other players} \\
&\qquad \text{vote for } i \text{ or } j\} \\
&= \{r \mid \exists\, k \neq i:\ L_i(r) = k,\ L_j(r) \in \{i, k\} \text{ for all } j \in \Pi \setminus \{i\}\}, \\
A_i^*(L) &= \{r \mid \text{all players vote for player } i \text{ in round } r,\ \text{same} \\
&\qquad \text{pattern in round } r-1\} \\
&= \{r \mid L_j(r-1) = L_j(r) = i \text{ for all } j \in \Pi\}, \\
A_i^{**}(L) &= \{r \mid \text{all players vote for player } i \text{ in round } r,\ \text{same} \\
&\qquad \text{pattern in rounds } r-1 \text{ and } r-2\} \\
&= \{r \mid L_j(r-2) = L_j(r-1) = L_j(r) = i \text{ for all } j \in \Pi\},
\end{aligned}$$

$$\begin{aligned}
B_i^*(L) = \{r|\ &\text{in round } r \text{ all players vote for the same player, but not for } i,\ \text{same}\\
&\text{pattern in round } r-1\}\\
= \{r|\ &\exists\, k_1 \neq i,\ k_2 \neq i:\ L_j(r) = k_1 \text{ for all } j \in \Pi,\\
&L_j(r-1) = k_2 \text{ for all } j \in \Pi\},\\
E_i^*(L) = \{r|\ &\text{in round } r \text{ all players except player } i \text{ vote for } i,\ \text{player } i \text{ votes for a}\\
&\text{player } j \neq i,\ \text{same pattern in round } r-1\}\\
= \{r|\ &\exists\, k_1 \neq i,\ k_2 \neq i:\ L_j(r-1) = L_j(r) = i \text{ for all } j \in \Pi \setminus \{i\},\\
&L_i(r) = k_1,\ L_i(r-1) = k_2\},\\
H_i^*(L) = \{r|\ &\text{in round } r \text{ player } i \text{ votes for a player } j \neq i, \text{ all the other players}\\
&\text{vote for } i \text{ or } j,\ \text{same pattern in round } r-1\}\\
= \{r|\ &\exists\, k_1 \neq i,\ k_2 \neq i:\ L_i(r) = k_1,\ L_j(r) \in \{i, k_1\} \text{ for all } j \in \Pi \setminus \{i\},\\
&L_i(r-1) = k_2,\ L_j(r-1) \in \{i, k_2\} \text{ for all } j \in \Pi \setminus \{i\}\},
\end{aligned}$$

and

$$\begin{aligned}
D(i) &= \{r|\ \delta_{L_i(r-1)}(L_i(r)) = 1\},\\
D_0 &= \{r|\ \delta_{L_i(r-1)}(L_i(r)) = 0 \text{ for all } i\},\\
D_1 &= \{r|\ \delta_{L_i(r-1)}(L_i(r)) = 1 \text{ for all } i\}.
\end{aligned}$$

Example 3.6. We also illustrate those definitions by means of a short example. Consider the following votes of five players in the first nine rounds:

Player	r=1	2	3	4	5	6	7	8	9	…
1	2	3	3	4	4	4	5	2	4	…
2	2	3	1	4	4	4	5	1	1	…
3	2	1	1	4	4	4	5	1	1	…
4	2	3	3	4	4	4	5	1	1	…
5	2	3	1	4	4	4	5	1	1	…

The sets defined above (restricted to the first nine rounds) are as follows:

Player	A_i	B_i	E_i	H_i
1	$\emptyset$	$\{1,4,5,6,7\}$	$\{8,9\}$	$\{1,2,3,4,5,6,7,8,9\}$
2	$\{1\}$	$\{4,5,6,7\}$	$\emptyset$	$\{4,5,6,7,8\}$
3	$\emptyset$	$\{1,4,5,6,7\}$	$\{2\}$	$\{1,2,3,4,5,6,7\}$
4	$\{4,5,6\}$	$\{1,7\}$	$\emptyset$	$\{1,7,9\}$
5	$\{7\}$	$\{1,4,5,6\}$	$\emptyset$	$\{1,4,5,6\}$

Player	A_i^*	A_i^{**}	B_i^*	E_i^*	H_i^*	$D(i)$
1	$\emptyset$	$\emptyset$	$\{5,6,7\}$	$\{9\}$	$\{2,3,4,5,6,7,8,9\}$	$\{3,5,6\}$
2	$\emptyset$	$\emptyset$	$\{5,6,7\}$	$\emptyset$	$\{5,6,7,8\}$	$\{5,6,9\}$
3	$\emptyset$	$\emptyset$	$\{5,6,7\}$	$\emptyset$	$\{2,3,4,5,6,7\}$	$\{3,5,6,9\}$
4	$\{5,6\}$	$\{6\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\{3,5,6,9\}$
5	$\emptyset$	$\emptyset$	$\{5,6\}$	$\emptyset$	$\{5,6\}$	$\{5,6,9\}$

Moreover, $D_0 = \{2,4,7,8\}$ and $D_1 = \{5,6\}$.

Throughout our paper, we will employ the well-known concept of the density of a subset of the natural numbers $A \subseteq \mathbb{N}$ and its complement $A^C = \mathbb{N} \setminus A$ to describe the properties of infinite characterization sets according to Definition 3.5.

Definition 3.7 (*Density of a Set*)**.** For a set $A \subseteq \mathbb{N}$ we define the lower and upper density as

$$\underline{d}(A) = \liminf_{n\to\infty} \frac{|[1,n] \cap A|}{n} \qquad \text{and} \qquad \overline{d}(A) = \limsup_{n\to\infty} \frac{|[1,n] \cap A|}{n}$$

with $[1,n] = \{1,\ldots,n\}$. If $\underline{d}(A) = \overline{d}(A)$ (i.e., the limit $\lim_{n\to\infty} \frac{|[1,n]\cap A|}{n}$ exists), we denote this value by $d(A)$ and call it the density of A.

4. Characterization of optimal election sequences

The goal of this section is to give a characterization of optimal election sequences in the case of $\Pi = \mathcal{V}$. We will start with two lemmas giving upper and lower bounds for the payoff in a single round. Afterwards, we establish lower and upper bounds for all our social payoffs (Corollary 4.4 and Lemmas 4.5–4.11). The main results of this section are Theorems 4.13, 4.16, and 4.18, which precisely characterize the optimal election sequences for every social payoff: A sequence is optimal for the respective social payoff, in some parameter range, if and only if it satisfies the given condition.

Our first lemma gives bounds on the total payoff in a single round:

Lemma 4.1. *For every election sequence L and every $r \geqslant 1$, we have:*

$$\begin{cases} nc' & \text{if } c' \le 0 \\ 0 & \text{if } c' \ge 0 \end{cases} + n + cn \le \sum_{i=1}^{n} u_i^{(r)}(L) \le n^2 + cn + \begin{cases} 0 & \text{if } c' \le 0 \\ nc' & \text{if } c' \ge 0 \end{cases}$$

Proof. Replacing $u_i^{(r)}(L)$ by its definition yields:

$$\sum_{i=1}^{n} u_i^{(r)}(L) = \sum_{i=1}^{n} |\mathcal{G}_i^{(r)}(L)| + c \sum_{i=1}^{n} |\mathcal{F}_i^{(r)}(L)| + c' \sum_{i=1}^{n} \delta_{L_i(r-1)}(L_i(r)). \tag{2}$$

Of course, $1 \le |\mathcal{G}_i^{(r)}(L)| \le n$ and $0 \le \delta_{L_i(r-1)}(L_i(r)) \le 1$. Because every player has exactly one vote in round r, we obtain

$$\sum_{i=1}^{n} |\mathcal{F}_i^{(r)}(L)| = \left| \bigcup_{i=1}^{n} \mathcal{F}_i^{(r)}(L) \right| = n.$$

Combining this with (2) concludes the proof. $\square$

Remark 4.2. To achieve the maximum value $n^2 + nc$ in round r, all players have to vote for the same player i. In case $c' < 0$, no player has voted for this player i in round $(r-1)$; in case $c' > 0$, all players must have voted for player i in round $(r-1)$ too. In case $c' = 0$, round r is independent from round $(r-1)$. On the other hand, the minimum is obtained iff each player is voted for exactly once in round r. In case $c' < 0$, this must also be true in round $(r-1)$ and every player votes for the same player in both rounds; in case $c' > 0$ every player has to choose a vote different from his vote in round $(r-1)$. In case $c' = 0$, round r is again independent from round $(r-1)$.

The following lemma bounds the payoff of a single player in a single round.

Lemma 4.3. *For all $n \ge 2$,*

$$u_i^{(r)}(L) \le \begin{cases} n + cn & \text{if } 0 \le c \\ n & \text{if } c \le 0 \end{cases} + \begin{cases} 0 & \text{if } c' \le 0 \\ c' & \text{if } c' \ge 0 \end{cases}$$

and, for $n \ge 3$,

$$u_i^{(r)}(L) \ge \begin{cases} 1 & \text{if } 0 \le c \\ 1 + c(n-1) & \text{if } 1 - n \le c \le 0 \\ n + cn & \text{if } c \le 1 - n \end{cases} + \begin{cases} c' & \text{if } c' \le 0 \\ 0 & \text{if } c' \ge 0 \end{cases}$$

whereas

$$\begin{aligned} u_i^{(r)}(L) &\ge \min(2, 2 + 2c, 1 + c) + \min(0, c') \\ &= \begin{cases} 2 & \text{if } 1 \le c \\ 1 + c & \text{if } -1 \le c \le 1 \\ 2 + 2c & \text{if } c \le -1 \end{cases} + \begin{cases} c' & \text{if } c' \le 0 \\ 0 & \text{if } c' \ge 0 \end{cases} \end{aligned}$$

for $n = 2$.

Proof. For simplicity, we will write f instead of $|\mathcal{F}_i^{(r)}(L)|$ and g instead of $|\mathcal{G}_i^{(r)}(L)|$. The upper bounds are easy to see: In case $c \ge 0$, the payoff $u_i^{(r)}(L)$ is increasing in f and g, so the maximum is obtained for $f = g = n$, i.e., all players vote for player i. On the other hand, if $c \le 0$, $u_i^{(r)}(L)$ is increasing in g and decreasing in f, so we can choose $g = n$ and $f = 0$; this corresponds to the situation where all players vote for a player $j \ne i$. Together with $c'\delta_{L_i(r-1)}(L_i(r)) \le \max(0, c')$ this gives the upper bounds.

For $n \ge 3$ the lower bound for $c \ge 0$ is an immediate consequence of $f \ge 0$ and $g \ge 1$. In case $c \le 0$, we have to be more careful and consider two cases: Firstly, if player i votes for some player $j \ne i$, we have $u_i^{(r)}(L) \ge 1 + c(n-1) + c'\delta$

(with $\delta = \delta_{L_i(r-1)}(L_i(r))$). Secondly, if player i votes for himself, then

$$u_i^{(r)}(L) = g + cg + c'\delta = (1+c)g + c'\delta \geq \begin{cases} 1 + c + c'\delta & \text{if } -1 \leq c \leq 0 \\ n(1+c) + c'\delta & \text{if } c \leq -1. \end{cases}$$

So, for $-1 \leq c \leq 0$, we have $1 + c(n-1) \leq 1 + c$. For $c \leq -1$ we have $1 + c(n-1) \leq n(1+c)$ iff $1 - n \leq c$. With $c'\delta_{L_i(r-1)}(L_i(r)) \geq \min(c', 0)$ the bounds follow immediately. The case $n = 2$ is trivial. □

These bounds on the individual payoff in a single round immediately yield a lower bound on the social payoff:

Corollary 4.4 (*Lower Bounds for Social Payoffs*)**.** *For every election sequence L and ethics $e \in \{\min, \mathrm{avg}, \max\}$, it holds that:*

$$\mathfrak{u}_e(L) \geq \begin{cases} 1 & \text{if } 0 \leq c \\ 1 + c(n-1) & \text{if } 1 - n \leq c \leq 0 \\ n + cn & \text{if } c \leq 1 - n \end{cases} + \begin{cases} c' & \text{if } c' \leq 0 \\ 0 & \text{if } c' \geq 0 \end{cases}$$

for $n \geq 3$, whereas

$$\mathfrak{u}_e(L) \geq \begin{cases} 2 & \text{if } 1 \leq c \\ 1 + c & \text{if } -1 \leq c \leq 1 \\ 2 + 2c & \text{if } c \leq -1 \end{cases} + \begin{cases} c' & \text{if } c' \leq 0 \\ 0 & \text{if } c' \geq 0 \end{cases} \qquad \text{for } n = 2.$$

The following lemmas are devoted to the study of upper bounds of the social payoff.

Lemma 4.5 (*Upper Bound for* avg*-social Payoff*)**.** *For every election sequence L, it holds that:*

$$\mathfrak{u}_{\mathrm{avg}}(L) \leq n + c + \begin{cases} 0 & \text{if } c' \leq 0 \\ c' & \text{if } c' \geq 0. \end{cases}$$

Proof. Lemma 4.1 implies

$$\frac{1}{r}\sum_{k=1}^{r}\frac{1}{n}\sum_{i=1}^{n} u_i^{(k)}(L) \leq n + c + \begin{cases} 0 & \text{if } c' \leq 0 \\ c' & \text{if } c' \geq 0. \end{cases}$$

Exchanging the two sums and noting $\liminf \alpha_r + \liminf \beta_r \leq \liminf(\alpha_r + \beta_r)$ yields

$$\mathfrak{u}_{\mathrm{avg}}(L) = \frac{1}{n}\sum_{i=1}^{n}\liminf_{r\to\infty}\frac{1}{r}\sum_{k=1}^{r} u_i^{(k)}(L) \leq n + c + \begin{cases} 0 & \text{if } c' \leq 0 \\ c' & \text{if } c' \geq 0. \end{cases}$$

This concludes the proof. □

Lemma 4.6 (*Upper Bound for* min*-social Payoff*)**.** *For every election sequence L, it holds that:*

$$\mathfrak{u}_{\min}(L) \leq n + c + \begin{cases} 0 & \text{if } c' \leq 0 \\ c' & \text{if } c' \geq 0. \end{cases}$$

Proof. The lemma follows directly from Lemma 4.5, because we always have the inequality $\min(x_1, \ldots, x_n) \leq \mathrm{avg}(x_1, \ldots, x_n)$. □

Lemma 4.7 (*Upper Bound for* max*-social Payoff*)**.** *For every election sequence L, it holds that:*

$$\mathfrak{u}_{\max}(L) \leq \begin{cases} n + cn & \text{if } 0 \leq c \\ n & \text{if } c \leq 0 \end{cases} + \begin{cases} 0 & \text{if } c' \leq 0 \\ c' & \text{if } c' \geq 0. \end{cases}$$

Proof. This is an immediate consequence of Lemma 4.3. □

Whereas the upper bounds for $\mathfrak{u}_{\mathrm{avg}}(L)$ and $\mathfrak{u}_{\min}(L)$ given in Lemmas 4.5 and 4.6 are tight bounds, the upper bound for $\mathfrak{u}_{\max}(L)$ is not in general: To maximize $u_i(L)$ for a single player i in case $c \geq 0$, all players would have to vote for player i in (almost) all rounds. But on the other hand, if $c' \leq 0$, they get a penalty for choosing the same player as in the round before. So always choosing the same leader could be bad. Lemma 4.11 will state tight bounds in this case.

For this purpose, we need the following proposition, which asserts monotonicity of lim inf of a sequence obtained from another one by replacing a fixed subsequence.

Proposition 4.8. *Let $u = (u^{(k)})_{k\geq 1}$ be a bounded sequence of real numbers and $v = (v_1, \dots, v_\ell)$ and $\tilde{v} = (\tilde{v}_1, \dots, \tilde{v}_\ell)$ two finite sequences of length ℓ with $\sum_{j=1}^{\ell} v_j \leq \sum_{j=1}^{\ell} \tilde{v}_j$. Moreover, define the set R by the following: For $r_1 = \min\{k : u^{(k+j-1)} = v_j \text{ for all } 1 \leq j \leq \ell\}$ and $r_i = \min\{k : k \geq r_{i-1} + \ell, u^{(k+j-1)} = v_j \text{ for all } 1 \leq j \leq \ell\}$ let $R = \bigcup_{i\geq 1}\{r_i\}$. Additionally, for $R' \subseteq R$ let $R_\ell = \{r + j : r \in R', 0 \leq j \leq \ell - 1\}$.*

Construct a sequence $\tilde{u} = (\tilde{u}^{(k)})_{k\geq 1}$ according to:

- *$\tilde{u}^{(k)} = u^{(k)}$ for all $k \in \mathbb{N} \setminus R_\ell$ and*
- *$\tilde{u}^{(r+j)} = \tilde{v}^{(j+1)}$ for all $r \in R'$ and $0 \leq j \leq \ell - 1$,*

i.e., we replace some (possibly all) occurrences of v in u by $\tilde{v}$.

Then $\liminf_{r\to\infty} \frac{1}{r} \sum_{k=1}^{r} u^{(k)} \leq \liminf_{r\to\infty} \frac{1}{r} \sum_{k=1}^{r} \tilde{u}^{(k)}$.

Proof. For $r \notin R_{\ell-1}$ we have

$$\sum_{k=1}^{r} u^{(k)} = \sum_{k:\ k\in[1,r]\cap(\mathbb{N}\setminus R_\ell)} u^{(k)} + \sum_{k\in[1,r]\cap R'} \sum_{j=1}^{\ell} v_j$$
$$\leq \sum_{k:\ k\in[1,r]\cap(\mathbb{N}\setminus R_\ell)} \tilde{u}^{(k)} + \sum_{k\in[1,r]\cap R'} \sum_{j=1}^{\ell} \tilde{v}_j = \sum_{k=1}^{r} \tilde{u}^{(k)}.$$

If $r \in R'$ and $0 \leq j \leq \ell - 2$, this result implies

$$\liminf_{r\to\infty} \frac{1}{r+j} \sum_{k=1}^{r+j} u^{(k)} = \liminf_{r\to\infty} \frac{1}{r+j} \sum_{k=1}^{r-1} u^{(k)}$$
$$\leq \liminf_{r\to\infty} \frac{1}{r+j} \sum_{k=1}^{r-1} \tilde{u}^{(k)} = \liminf_{r\to\infty} \frac{1}{r+j} \sum_{k=1}^{r+j} \tilde{u}^{(k)},$$

which concludes this proof. □

Remark 4.9. Note that Proposition 4.8 still remains true if we replace finitely many finite sequences $v^{[1]}, \dots, v^{[m]}$ by $\tilde{v}^{[1]}, \dots, \tilde{v}^{[m]}$ if for all those finite sequences $\sum_{j=1}^{\ell_t} v_j^{[t]} \leq \sum_{j=1}^{\ell_t} \tilde{v}_j^{[t]}$ holds for $1 \leq t \leq m$.

Example 4.10. Let the first 24 rounds of u be given as below, and let $v = (-1, -1, 1, 1)$ and $\tilde{v} = (-5, -5, 5, 6)$. Replacing all occurrences of v by $\tilde{v}$ and using the notation $c_r(u) = 1/r \cdot \sum_{k=1}^{r} u^{(k)}$ yields:

	r = 1	2	3	4	5	6	7	8	9	10	11	12
$u^{(r)}$	0	0	-1	-1	1	1	0	0	-1	-1	1	1
$c_r(u)$	0	0	$-\frac{1}{3}$	$-\frac{1}{2}$	$-\frac{1}{5}$	0	0	0	$-\frac{1}{9}$	$-\frac{1}{5}$	$-\frac{1}{11}$	0
$\tilde{u}^{(r)}$	0	0	-5	-5	5	6	0	0	-5	-5	5	6
$c_r(\tilde{u})$	0	0	$-\frac{5}{3}$	$-\frac{5}{2}$	-1	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$	$-\frac{4}{9}$	$-\frac{9}{10}$	$-\frac{4}{11}$	$\frac{1}{6}$

	13	14	15	16	17	18	19	20	21	22	23	24	…
$u^{(r)}$	0	0	-1	-1	1	1	0	0	-1	-1	1	1	…
$c_r(u)$	0	0	$-\frac{1}{15}$	$-\frac{1}{8}$	$-\frac{1}{17}$	0	0	0	$-\frac{1}{21}$	$-\frac{1}{11}$	$-\frac{1}{23}$	0	…
$\tilde{u}^{(r)}$	0	0	-5	-5	5	6	0	0	-5	-5	5	6	…
$c_r(\tilde{u})$	$\frac{2}{13}$	$\frac{2}{14}$	$-\frac{3}{15}$	$-\frac{8}{16}$	$-\frac{3}{17}$	$\frac{1}{6}$	$\frac{3}{19}$	$\frac{3}{20}$	$-\frac{2}{21}$	$-\frac{7}{22}$	$-\frac{2}{23}$	$\frac{1}{6}$	…

Assuming that the first 24 rounds are perpetually repeated, it is easy to see that $\liminf \frac{1}{r} \sum_{k=1}^{r} u^{(k)} = \lim \frac{1}{r} \sum_{k=1}^{r} u^{(k)} = 0 < \frac{1}{6} = \lim \frac{1}{r} \sum_{k=1}^{r} \tilde{u}^{(k)} = \liminf \frac{1}{r} \sum_{k=1}^{r} \tilde{u}^{(k)}$.

Now we are ready to prove tight bounds on $\mathfrak{u}_{\max}(L)$. The main idea is to manipulate any given election sequence L step by step (only depending on the parameters c and c') in such a way that $\mathfrak{u}_{\max}$ increases in every step. The resulting sequence L' is of a particular simple shape and hence $\mathfrak{u}_{\max}(L')$ can be determined easily.

Lemma 4.11 (*Upper Bound for* max*-social Payoff*)**.** *For $n \geq 3$, if $c \geq 0$ and $c' \leq 0$, then every election sequence L satisfies*

$$\mathfrak{u}_{\max}(L) \leq \begin{cases} \frac{1}{2}n + cn + \frac{1}{2} - \frac{1}{2}c & \text{if } c \geq 1,\ c' \leq \frac{1-c-n}{2} \\ n + \frac{1}{2}cn & \text{if } c \leq 1,\ c' \leq \frac{-cn}{2} \\ n + cn + c' & \text{ow.} \end{cases}$$

Moreover, every optimal sequence L can be changed, by a sequence of replacements in accordance with Proposition 4.8, to an optimal sequence L' of the following form:

- *$c \leq 1, c' \leq \frac{-cn}{2}$: there exists a player i with $L_k(2r) = i$ and $L_k(2r+1) = j_{2r+1}$ with $j_{2r+1} \neq i$ for all $k \in \Pi$ (or with even/odd rounds changed),*
- *$c \geq 1, c' \leq \frac{1-c-n}{2}$: there exists a player i with $L_i(2r) = i$ and $L_i(2r+1) = j_{2r+1}$ with $j_{2r+1} \in \Pi \setminus \{i\}$ and $L_k(r) = i$ for all $k \in \Pi \setminus \{i\}$ for all r (or with even/odd rounds changed),*
- *ow.: there exists a player i with $L_j(r) = i$ for all $j \in \Pi$ and all $r \geq 1$.*

Proof. To prove this lemma, we will construct optimal election sequences. Let an election sequence L be given, and fix a player i with $u_i(L) = \mathfrak{u}_{\max}(L)$. Recall that $L_i = (L_i(r))_r$ is the sequence of votes of player i and $(u_i^{(r)}(L))_r$ is the sequence of utilities of player i. Given this sequence L_i, we will define an election sequence L' with $\mathfrak{u}_{\max}(L') \geq \mathfrak{u}_{\max}(L)$. For this purpose, we maximize $u_i^{(r)}(L')$ for every round, but we keep L_i fixed, i.e., $L_i = L'_i$. If $L_i(r) = i$, then in L' all players vote for player i in round r (i.e., $L'_j(r) = i$ for all $j \in \Pi$). Then $u_i^{(r)}(L') = n + cn + c'\delta \geq u_i^{(r)}(L)$ (note that δ remains unchanged). On the other hand, if $L_i(r) = j \neq i$, then in L' all the other players vote for player j too if $c \leq 1$ (i.e., $L'_k(r) = j$ for all $k \in \Pi$) or for player i if $c \geq 1$ (i.e., $L_k(r) = i$ for all $k \in \Pi \setminus \{i\}$). Then,

$$u_i^{(r)}(L') = \begin{cases} 1 + c(n-1) + c'\delta & \text{if } c \geq 1 \\ n + c'\delta & \text{if } c \leq 1. \end{cases}$$

Since the choices in L cannot lead to a larger payoff, we get $u_i^{(r)}(L') \geq u_i^{(r)}(L)$. Thus, $\mathfrak{u}_{\max}(L') \geq \mathfrak{u}_{\max}(L)$.

Thus, it is sufficient to consider election sequences in which all players vote for one fixed player in each round (in case $c \leq 1$) or in which all players vote for one fixed player i in each round or all players except player i vote for i and player i votes for player $j(r) \neq i$ (in case $c \geq 1$).

In the following, let L be an election sequence from this set and let player i be one player with maximal payoff and L_i his sequence of votes that induces the votes of all the other players. We will now show that, by replacing certain 3-blocks (the leader choice for 3 consecutive rounds) in L_i by some other 3-block, the maximum payoff in the resulting L' cannot decrease.

For a subsequence of L_i, the total payoff of a subsequence is denoted by $U_i(L_i(r), \dots, L_i(r+k)) = \sum_{\ell=0}^{k} u_i^{(r+\ell)}(L)$. In the next step, we will compare the subsequences (of L_i) j_1, j_2, j_3 with j_1, i, j_3 and i, i, i with i, j, i (with $j \neq i \neq j_\ell$). Define

$$a = \begin{cases} 1 + c(n-1) & \text{if } c \geq 1, \\ n & \text{if } c \leq 1. \end{cases}$$

So we want to compare $U_i(j_1 j_2 j_3)$ with $U_i(j_1 i j_3)$. Start with case $c \leq 1$ first. Then $j_1 j_2 j_3$ means that in these rounds every player votes for j_1, j_2 and j_3. Thus,

$$U_i(j_1 j_2 j_3) = (n + c'\delta) + (n + c'\delta_{j_1}(j_2)) + (n + c'\delta_{j_2}(j_3)) \leq (n + c'\delta) + n + n,$$

whereas

$$U_i(j_1 i j_3) = (n + c'\delta) + (n + cn) + n.$$

Similarly, if $c \geq 1$,

$$\begin{aligned} &U_i(j_1 j_2 j_3) \\ &\quad = (1 + c(n-1) + c'\delta) + (1 + c(n-1) + c'\delta_{j_1}(j_2)) + (1 + c(n-1) + c'\delta_{j_2}(j_3)) \\ &\quad \leq (1 + c(n-1) + c'\delta) + (1 + c(n-1)) + (1 + c(n-1)) \end{aligned}$$

since if i votes for j_ℓ all other players vote for i, whereas

$$U_i(j_1 i j_3) = (1 + c(n-1) + c'\delta) + (n + cn) + (1 + c(n-1)).$$

Then, $U_i(j_1 j_2 j_3) \leq U_i(j_1 i j_3)$ iff $2a \leq a + nc + n$. Since $a \leq n + nc$, this is always fulfilled. So it is sufficient to consider only election sequences where js can only occur isolated or in pairs.

On the other hand, $U_i(iii) \leq U_i(iji)$ iff $2n + 2nc + 2c' \leq a + nc + n$ iff $n + nc + 2c' \leq a$ iff $c' \leq (a - n - nc)/2$ iff

$$c' \leq b := \frac{a - n - nc}{2} = \begin{cases} \frac{1-c-n}{2} & \text{if } c \geq 1, \\ \frac{-cn}{2} & \text{if } c \leq 1. \end{cases} \tag{3}$$

Thus, in case $c' > b$, we only need to consider sequences with subsequences i^k of arbitrary length and pairs of j, in case $c' < b$ we only consider sequences in which both i and j occur isolated (i.e., a 1-block i like in $j_1 i j_2$, resp. a 1-block j like in iji) or in pairs (i.e., a 2-block ii like in $j_1 ii j_2$, resp. a 2-block $j_1 j_2$ like in $i j_1 j_2 i$).

Now we will show that we can eliminate pairs of j in an election sequence: Firstly, if $c' \geq b$, then $U_i(iiii) \geq U_i(i j_1 j_2 i)$ iff $3n + 3nc + 3c' \geq 2a + n + nc + c'\delta_{j_1}(j_2)$ iff $n + nc + (3 - \delta_{j_1}(j_2))\frac{c'}{2} \geq a$ iff $c' \geq \frac{4}{3-\delta_{j_1}(j_2)} b$. Since $b < 0$, this is always true. Consequently, if $c' \geq b$, the election sequence $L_k(r) = i$ for all $k \in \Pi$ and all rounds r is an optimal sequence with payoff $n + cn + c'$ (note that all the other players have payoff $n + c'$).

Secondly, if $c' < b$ and if r is an index with $L_i(r) \neq L_i(r+1)$, define $I_1(r)$ as the number of 1-blocks of i in L_i up to round r, $I_2(r)$ as the number of 2-blocks of i, $J_1(r)$ the number of 1-blocks of j, and $J_2(r)$ the number of 2-blocks of j (for simplicity, we may drop the index r). Moreover,

$$\begin{aligned} U(r) &:= \sum_{\ell=1}^{r} u_i^{(\ell)}(L) \leq I_1 \cdot (n+nc) + I_2 \cdot (2n+2nc+c') + J_1 \cdot a + J_2 \cdot 2a \qquad (4) \\ &= (I_1+2I_2)n + (I_1+2I_2)nc + I_2c' + (J_1+2J_2)a \\ &= \begin{cases} rn + (I_1+2I_2)nc + I_2c' & \text{if } c \leq 1, \\ (I_1+2I_2)n + rnc + I_2c' + (J_1+2J_2)(1-c) & \text{if } c \geq 1. \end{cases} \qquad (5) \end{aligned}$$

with (i) $I_1+2I_2+J_1+2J_2 = r$ and (ii) $|I_1+I_2-J_1-J_2| \leq 1$. Note that Eq. (4) holds also for $j_1 = j_2$ in a 2-block of j, as $c' \leq 0$. Now we show that we can restrict ourselves to election sequences with $J_2 = 0$: If L contains only finitely many pairs of j then define L' as the sequence which starts after the last pair; the limiting payoff does not change. On the other hand, if L contains infinitely many pairs of j, we construct a sequence L' as follows. Let r be an index such that $J_2(r)$ is even. Change the prefix of L_i according to $L_i'(1) = L_i(1)$, $L_i'(r) = L_i(r)$, $I_1' = I_1 + J_2/2$, $I_2' = I_2$, $J_1' = J_1 + 3J_2/2$, and $J_2' = 0$. More specifically, for any subsequence starting and ending with a pair of j, retain the first and the last j, delete the second j of the starting pair and shift the original sequence left by one index, and insert i before the second j in the ending pair.

We illustrate the above construction by a short example:

Example 4.12. Let the following part of L_i be given:

$$\begin{array}{llllllllllllllll} L_i\colon & \dots & \mathit{j} & \cancel{\mathrm{j}} & \mathrm{i} & \mathrm{i} & \mathrm{j} & \mathrm{i} & \mathrm{j} & \mathrm{i} & \mathrm{i} & \mathrm{j} & \mathrm{i} & \mathrm{j} & \mathit{j} & \dots \\ L_i'\colon & \dots & \mathit{j} & \mathrm{i} & \mathrm{i} & \mathrm{j} & \mathrm{i} & \mathrm{j} & \mathrm{i} & \mathrm{i} & \mathrm{j} & \mathrm{i} & \mathrm{j} & \underline{\mathrm{i}} & \mathit{j} & \dots \end{array}$$

In the construction of L_i', the blue italic js (the first and the last one) remain fixed, the red striked out j is dropped and the black sequence is shifted to the left and copied down, and the green underlined i is inserted.

It follows that the new subsequence has a total payoff greater than or equal to the total payoff of the original subsequence: For $c \leq 1$, we have

$$U'(r) = rn + (I_1' + 2I_2')nc + I_2'c' = rn + \left(I_1 + \frac{J_2}{2} + 2I_2\right) nc + I_2c'$$

and for $c \geq 1$

$$\begin{aligned} U'(r) &= (I_1' + 2I_2')n + rnc + I_2'c' + (J_1' + 2J_2')(1-c) \\ &= \left(I_1 + \frac{J_2}{2} + 2I_2\right) n + rnc + I_2c' + \left(J_1 + \frac{3J_2}{2}\right)(1-c). \end{aligned}$$

It is easy to see that $U'(r)$ is greater than or equal to the formulas given in (5). In fact, since $c' < b < 0$ and $J_2 \geq 2$, computing the actual difference using Eq. (3) reveals even

$$U'(r) - U(r) + 2c' \geq \begin{cases} \frac{J_2nc}{2} + 2c' \geq nc - cn \geq 0 & \text{if } c \leq 1, \\ \frac{J_2n}{2} - \frac{J_2(1-c)}{2} + 2c' \geq n - (1-c) + 1 - c - n \geq 0 & \text{if } c \geq 1. \end{cases}$$

The entire election sequence L' is obtained by repeating the above construction starting at round $(r+1)$.

Whereas this suggests that the total payoff in L' is larger that the total payoff in L, the above construction does not allow us to directly apply Proposition 4.8. The reason is that the length (r) of the modified subsequences as described above need not be bounded, i.e., the distance between two pairs of js can increase unboundedly. Nevertheless, we can decompose the parts between two consecutive pairs of js into three blocks of length 2 and 3, to which we can apply Proposition 4.8 simultaneously: Take Example 4.12 from above and decompose it as follows:

$$\begin{array}{llllllllllllllll} L_i\colon & \dots & \mathit{j} & \mathrm{j} & \mathrm{i} & \mathrm{i} & \mathrm{j} & \mathrm{i} & \mathrm{j} & \mathrm{i} & \mathrm{i} & \mathrm{j} & \mathrm{i} & \mathrm{j} & \mathit{j} & \dots \\ L_i'\colon & \dots & \mathit{j} & \mathrm{i} & \mathrm{i} & \mathrm{j} & \mathrm{i} & \mathrm{j} & \mathrm{i} & \mathrm{i} & \mathrm{j} & \mathrm{i} & \mathrm{j} & \mathrm{i} & \mathit{j} & \dots \end{array}$$

The block *jii* changes to *iij*; *ji* (if this is followed by *j*) to *ij*; and the final pair *jj* to *ij*. It is easy to check that the payoff of each block of the resulting sequence L' is the same or larger than the payoff of the corresponding block of the original sequence L. This construction does not only hold for this example, but also in general. Hence, we can deduce that L' has indeed a payoff greater than or equal to the payoff of L.

In final step, we also eliminate all pairs of i by the analogous procedure: If there are only finitely many, we start after the last occurrence. In case of infinitely many pairs, choose an index r such that I_2 is even and change the prefix of L_i according to $L_i'(1) = L_i(1)$, $L_i'(r) = L_i(r)$, $I_1' = I_1 + 3I_2/2$, $I_2' = 0$, and $J_1' = J_1 + I_2/2$. In particular, for any subsequence starting and ending with a pair of i, retain the first and the last i, delete the second i of the starting pair and shift the original sequence left by one index, and insert j before the second i in the ending pair. The same reasoning as before shows that the payoff of the resulting election sequence L' is greater than or equal to the payoff of L.

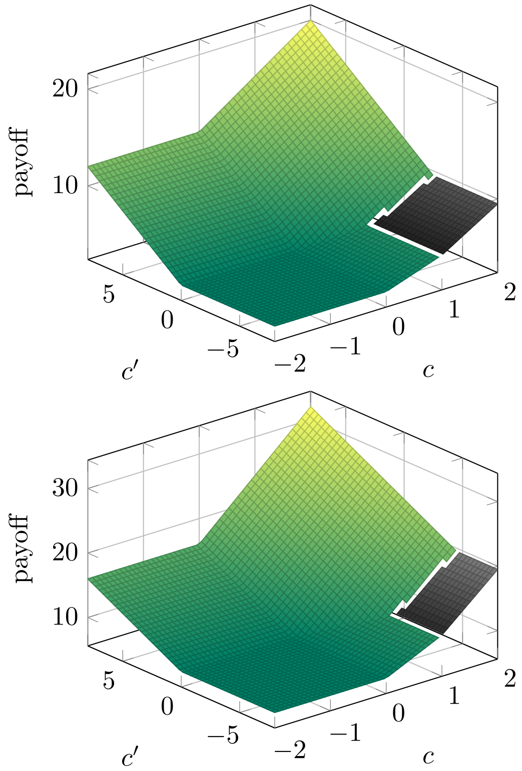

Fig. 1. Visualization of the results of Theorem 4.13 for $n = 4$ (top) and $n = 8$ (bottom). The z-axis shows the payoff for the given choice of c (x-axis) and c' (y-axis). The region in the graphs that is shaded gray depicts the range of parameters where no consensus on the leader is achieved.

Consequently, the election sequence "all players vote for $i, j, i, j, i, \ldots$" (with possibly varying j's) is an optimal election sequence with payoff $n + \frac{1}{2}cn$ in case of $c \leq 1$ (it is easy to check that player i has maximal payoff). In case of $c \geq 1$, the election sequence "all players $\neq i$ always vote for i and player i votes for $i, j, i, j, i, \ldots$" is an optimal election sequence with payoff $\frac{1}{2}n + cn + \frac{1}{2} - \frac{1}{2}c$. Again, player i has the maximal payoff. Note that it can be shown that the payoff of any other player is less than or equal to $n + \frac{c'}{2}$ here. As these are the optimal sequences stated in Lemma 4.11, we are done. □

With these preparations, we can state our first main result: the exact characterization of the optimal election sequences for $\mathfrak{u}_{\max}$. It causes the most varied behaviors of all social payoffs studied in this paper, as we need to distinguish 6 different major parameter ranges (i)–(vi), some of which with several sub-ranges. Fig. 1 in Section 6 shows the payoff as a function of c and c' for $n = 4$ and $n = 8$, also highlighting the regions where the players do not agree on a single leader. Quite different proof techniques, from tight lower- and upper-enclosing to exchange arguments to suitable payoff sequence partitioning are required to prove that the given conditions indeed lead to the maximum social payoff.

Theorem 4.13 (*Optimal Sequences for* max-*social Payoff*)**.** *The optimal election sequences for $\mathfrak{u}_{\max}$ can be characterized as follows: An election sequence L is optimal (for $n \geq 3$)*

(i) for $c > 0$, $c' \geq 0$ iff there exists an i with $d(A_i) = 1$ (with payoff $n + cn + c'$),
(ii) for $c > 0$, $c' < 0$ iff there exists an i with

- $c > 1,\ c' < \frac{1-c-n}{2}$: $d(A_i) = d(E_i) = \frac{1}{2}$, $d(D(i)) = 0$ (with payoff $\frac{1}{2}n + cn + \frac{1}{2} - \frac{1}{2}c$),
- $c > 1,\ c' = \frac{1-c-n}{2}$: $d(A_i \cup E_i) = 1$, $d(E_i^*) = 0$ (with payoff $\frac{1}{2}n + cn + \frac{1}{2} - \frac{1}{2}c$),
- $c < 1,\ c' < \frac{-cn}{2}$: $d(A_i) = d(B_i) = \frac{1}{2}$, $d(D(i)) = 0$ (with payoff $n + \frac{1}{2}cn$),
- $c < 1,\ c' = \frac{-cn}{2}$: $d(A_i \cup B_i) = 1$, $d(B_i^*) = 0$ (with payoff $n + \frac{1}{2}cn$),
- $c = 1,\ c' < \frac{-n}{2}$: $d(A_i \cup H_i) = 1$, $d(D(i)) = 0$ (with payoff $\frac{3n}{2}$),
- $c = 1,\ c' = \frac{-n}{2}$: $d(A_i \cup H_i) = 1$, $d(H_i^*) = 0$ (with payoff $\frac{3n}{2}$),
- *ow.*: $d(A_i) = 1$ (this implies $d(D(i)) = 1$) (with payoff $n + cn + c'$),

(iii) for $c < 0$, $c' > 0$ iff there exists an i with $d(B_i) = d(D(i)) = 1$ (with payoff $n + c'$),

(iv) for $c < 0$, $c' = 0$ iff there exists an i with $d(B_i) = 1$ (with payoff n),
(v) for $c < 0$, $c' < 0$ iff there exists an i with $d(B_i) = 1$ and $d(D(i)) = 0$ (with payoff n),
(vi) for $c = 0$ iff $d\left(\bigcup_i A_i\right) = 1$ and

- *$c' > 0$: $d(D_1) = 1$ (with payoff $n + c'$),*
- *$c' < 0$: $d(D_0) = 1$ (with payoff n).*

Proof. To prove (i) and $(iii) - (vi)$, we define the set C_i as

$$C_i = \{r |\ u_i^{(r)}(L) = m\},$$

where m is the maximal value for $u_i^{(r)}(L)$ from Lemma 4.3 (which defines the respective payoff given in our theorem). If there is a player i with $d(C_i) = 1$ for an election sequence L, then this election sequence is optimal:

$$\begin{aligned} m &\geq \max_{j=1\ldots n} \liminf_{r\to\infty} \frac{1}{r} \sum_{k=1}^{r} u_j^{(k)}(L) \geq \liminf_{r\to\infty} \frac{1}{r} \sum_{k=1}^{r} u_i^{(k)}(L) \\ &\geq \liminf_{r\to\infty} \frac{1}{r} \sum_{k\leq r:\ k\in C_i} u_i^{(k)}(L) + \liminf_{r\to\infty} \frac{1}{r} \sum_{k\leq r:\ k\notin C_i} u_i^{(k)}(L) \\ &\geq \liminf_{r\to\infty} \frac{|C_i \cap [1,r]|}{r} m + \liminf_{r\to\infty} \frac{r - |C_i \cap [1,r]|}{r} d \\ &= d(C_i)m + (1 - d(C_i))d = m, \end{aligned}$$

where d is a lower bound for $u_i^{(r)}(L)$ (can be obtained from Lemma 4.3, e.g., $d = 1 + \min(0, c)n + \max(0, c')$). Hence the election sequence is optimal.

Conversely, assume that an optimal election sequence L that leads to $\mathfrak{u}_{\max}(L) = m$ is given. Then, there exists a player i such that

$$m = \max_{j=1\ldots n} \liminf_{r\to\infty} \frac{1}{r} \sum_{k=1}^{r} u_j^{(k)}(L) = \liminf_{r\to\infty} \frac{1}{r} \sum_{k=1}^{r} u_i^{(k)}(L) \leq m$$

since, due to Lemma 4.3, $u_i^{(r)}(L) \leq m$. Hence, the lim inf is in fact a limit, i.e.,

$$\lim_{r\to\infty} \frac{1}{r} \sum_{k=1}^{r} u_i^{(k)}(L) = m.$$

Thus,

$$m = \lim_{r\to\infty} \frac{1}{r} \sum_{k=1}^{r} u_i^{(k)}(L) \leq \lim_{r\to\infty} \frac{1}{r} |C_i \cap [1,r]| \cdot m + \frac{1}{r}(r - |C_i \cap [1,r]|) \cdot m' \leq m,$$

where m' is the largest possible value of $u_i^{(r)}(L)$ smaller than m, which is independent of r. Defining $e := m - m'$, we get

$$\begin{aligned} m &= \lim_{r\to\infty} \frac{1}{r} |C_i \cap [1,r]| \cdot m + \frac{1}{r}(r - |C_i \cap [1,r]|) \cdot (m - e) \\ &= \lim_{r\to\infty} m - e\frac{1}{r}(r - |C_i \cap [1,r]|) = m - e \lim_{r\to\infty} \frac{1}{r}(r - |C_i \cap [1,r]|) \\ &= m - e + e \lim_{r\to\infty} \frac{|C_i \cap [1,r]|}{r}. \end{aligned}$$

So the remaining limit must exist be equal to 1, i.e., we have to obtain the maximal value for $u_i^{(r)}(L)$ in almost every round. These election sequences are exactly those stated in (i) and $(iii) - (vi)$.

Now let us turn to case (ii). Firstly, we will prove that the given election sequences are optimal. We will do this calculation only for case $c > 1$, $c' < \frac{1-c-n}{2}$, as the proof of the other cases runs along the same lines. Define $m = \frac{1}{2}n + cn + \frac{1}{2} - \frac{1}{2}c$. Then, by using Lemma 4.11, we have

$$\begin{aligned} m &\geq \max_{j=1\ldots n} \liminf_{r\to\infty} \frac{1}{r} \sum_{k=1}^{r} u_j^{(k)}(L) \geq \liminf_{r\to\infty} \frac{1}{r} \sum_{k=1}^{r} u_i^{(k)}(L) \\ &= \liminf_{r\to\infty} \frac{1}{r} \sum_{k=1}^{r} \left(|\mathcal{G}_i^{(k)}(L)| + c \cdot |\mathcal{F}_i^{(k)}(L)| + c'\delta_{L_i(k-1)}(L_i(k)) \right) \\ &\geq \liminf_{r\to\infty} \frac{1}{r} \sum_{k=1}^{r} \left(|\mathcal{G}_i^{(k)}(L)| + c \cdot |\mathcal{F}_i^{(k)}(L)| \right) + c' \limsup_{r\to\infty} \frac{1}{r} \sum_{k=1}^{r} \delta_{L_i(k-1)}(L_i(k)) \end{aligned}$$

Since $d(D(i)) = 0$, we have $\delta_{L_i(k-1)}(L_i(k)) = 0$ for almost all rounds, and so

$$\begin{aligned}
&\geq \liminf_{r\to\infty} \frac{1}{r} \sum_{k\leq r:\ k\in A_i} \left(|\mathcal{G}_i^{(k)}(L)| + c\cdot|\mathcal{F}_i^{(k)}(L)|\right)\\
&\quad + \liminf_{r\to\infty} \frac{1}{r} \sum_{k\leq r:\ k\in E_i} \left(|\mathcal{G}_i^{(k)}(L)| + c\cdot|\mathcal{F}_i^{(k)}(L)|\right)\\
&\quad + \liminf_{r\to\infty} \frac{1}{r} \sum_{k\leq r:\ k\notin A_i\cup E_i} \left(|\mathcal{G}_i^{(k)}(L)| + c\cdot|\mathcal{F}_i^{(k)}(L)|\right) + 0\cdot c'\\
&\geq \liminf_{r\to\infty} \frac{1}{r} \sum_{k\leq r:\ k\in A_i} (n+nc) + \liminf_{r\to\infty} \frac{1}{r} \sum_{k\leq r:\ k\in E_i} (1+c(n-1)) + 0\\
&= \liminf_{r\to\infty} \frac{|A_i\cap[1,r]|}{r}(n+cn) + \liminf_{r\to\infty} \frac{|E_i\cap[1,r]|}{r}(1+c(n-1))\\
&= \frac{n+cn}{2} + \frac{1+c(n-1)}{2} = \frac{n}{2} + cn + \frac{1}{2} - \frac{c}{2} = m,
\end{aligned}$$

hence the election sequence is optimal.

Secondly, to prove that all optimal election sequences are of the stated forms, we start with the following two propositions. The first proposition ensures the existence of the limit in the definition of the payoff of a single player if the payoff is 'optimal'.

Proposition 4.14. *Let $u = (u^{(k)})_{k\geq 0}$ be a sequence of payoffs of an individual player and let $U(r) = \frac{1}{r}\sum_{k=1}^{r} u^{(k)}$. If $\liminf_{r\to\infty} U(r)$ is optimal, i.e., $\liminf_{r\to\infty} U(r) \geq \liminf_{r\to\infty} \tilde{U}(r)$ for all other possible payoff-sequences $(\tilde{u}^{(k)})_{k\geq 0}$, then we have $\liminf_{r\to\infty} U(r) = \limsup_{r\to\infty} U(r)$, i.e., $\lim_{r\to\infty} U(r)$ exists.*

Proof. The rough idea of the proof is to construct from u (under the assumption $a := \liminf U(r) < \limsup U(r) =: b$) a new sequence $\hat{u}$ that consists only of those elements $u^{(t)}$ with $U(t) \geq (a+b)/2$ and to show that $\liminf \hat{U}(r) > \liminf U(r)$, which is a contradiction.

So assume $a := \liminf U(r) < \limsup U(r) =: b$ and choose ε small. Let T_0 be the first time with $U(T_0) > b-\varepsilon$, $T_0' < T_0$ the closest time before with $U(T_0') < (a+b)/2$, and T_0'' the first time after T_0 with $U(T_0'') < (a+b)/2$. Inductively, let T_{i+1} be the first time after T_i'' with $U(T_{i+1}) > b-\varepsilon$, T_{i+1}' the closest time before T_{i+1} with $U(T_{i+1}') > (a+b)/2$, and T_{i+1}'' the first time after T_{i+1} with $U(T_{i+1}'') < (a+b)/2$.

Now define a sequence $\tilde{u} = (\tilde{u}^{(k)})_{k\geq 0}$ by

$$\tilde{u} = u^{(T_0'+1)}\cdots u^{(T_0''-1)}u^{(T_1'+1)}\cdots u^{(T_1''-1)}u^{(T_2'+1)}\cdots.$$

By construction, $\liminf \tilde{U}(r) \geq (a+b)/2 > a$.

If we would consider u just as a sequence of real numbers, $\tilde{u}$ would be a sequence with $\liminf \tilde{U}(r) > \liminf U(r)$ and this would be a contradiction to the optimality of $U(r)$. But since u is a payoff-sequence, it is dependent on an underlying election sequence, in particular, the values $u^{(k)}$ depend on the vote of round $(k-1)$ through the term $c'\delta_{L_i(k-1)}(L_i(k))$. Hence, $\tilde{u}$ might not be a valid payoff-sequence since $u^{(T_i'+1)} = \tilde{u}^{(\sum_{k=0}^{i-1}(T_k''-T_k'-1))}$, and $u^{(T_i'+1)}$ depends on the vote in round T_i', thus $\tilde{u}^{(\sum_{k=0}^{i-1}(T_k''-T_k'-1))}$ depends on the 'original' vote in round T_i' and not on the 'new' vote of round $\sum_{k=0}^{i-1}(T_k''-T_k'-1)-1$. Hence, to make $\tilde{u}$ a valid payoff-sequence, we might have to change $\tilde{u}^{(\sum_{k=0}^{i-1}(T_k''-T_k'-1))}$ $(i\geq 1)$ by adding/deleting c'.

Because of this, we will now show that the set $A = \{\sum_{k=0}^{i-1}(T_k''-T_k'-1) :\ i\geq 1\}$ (i.e., the set of indices where we glue together parts of u) has density 0. If this is true, the change of $\tilde{u}^{(k)}$ $(k\in A)$ by $\pm c'$ does not change the value of $\liminf \tilde{U}$. So, for this purpose, observe the following: To increase $U(T_\ell')$ to at least $(b-\varepsilon)$ as fast as possible, we could choose $u^{(k)} = \beta$, with $\beta \geq b$ the maximum value of the sequence, i.e., $\beta = \max_k u^{(k)}$. (Recall that u is a sequence of payoffs, and consequently there are only finitely many possible values of $u^{(k)}$; since $\limsup U(r) = b$, we have $\beta \geq b$.) Then,

$$b-\varepsilon \leq U(T_\ell) = \frac{1}{T_\ell}\left(\sum_{k=1}^{T_\ell'} u^{(k)} + \sum_{k=T_\ell'+1}^{T_\ell} u^{(k)}\right) \leq \frac{1}{T_\ell}\left(T_\ell'\frac{a+b}{2} + (T_\ell - T_\ell')\beta\right)$$

$$\iff T_\ell'\left(\frac{a+b}{2} - \beta\right) \geq T_\ell(b-\varepsilon-\beta)$$

$$\iff T'_\ell\left(\beta-\frac{a+b}{2}\right)\le T_\ell\left(\beta+\varepsilon-b\right)$$

$$\iff T_\ell\ge T'_\ell\frac{\beta-\frac{a+b}{2}}{\beta+\varepsilon-b}.$$

A similar calculation shows for the time to decrease $U(T_\ell)$ to $(a+b)/2$

$$T''_\ell\ge T_\ell\frac{\alpha-b+\varepsilon}{\alpha-\frac{a+b}{2}},$$

with $\alpha\le a$ the minimum of the sequence, i.e., $\alpha=\min_k u^{(k)}$. Combining both inequalities yields

$$T''_\ell\ge T'_\ell\underbrace{\frac{\left(\beta-\frac{a+b}{2}\right)(\alpha-b+\varepsilon)}{(\beta-b+\varepsilon)\left(\alpha-\frac{a+b}{2}\right)}}_{=:q>1}.$$

So, $T''_\ell\ge T'_0q^\ell$ and

$$T''_\ell-T'_\ell\ge T'_\ell(q-1)\ge T''_{\ell-1}(q-1)\ge T_0q^{\ell-1}(q-1).$$

Thus, the differences of consecutive elements of A grow (exponentially), hence this set has density 0. Consequently, if $\hat{u}$ is constructed from $\tilde{u}$ by just changing the values of $\tilde{u}^{(\sum_{k=0}^{i-1}(T''_k-T'_k-1))}$, $\hat{u}$ differs from $\tilde{u}$ only on a 0-set and hence $\liminf\hat{U}(r)=\liminf\tilde{U}(r)>\liminf U(r)$, which is a contradiction. □

The next proposition states that if we start with an optimal sequence and replace a fixed subsequence v by a subsequence $\tilde{v}$ with larger payoff, then the set of occurrences of v must be a set of density 0.

Proposition 4.15. *Let $u=(u^{(k)})_{k\ge0}$ and $\tilde{u}=(\tilde{u}^{(k)})_{k\ge0}$ two sequences like in Proposition 4.8 with the additional assumption $\gamma:=\sum_{j=0}^{\ell-1}\tilde{v}_{j+1}-v_{j+1}>0$.*
If $\liminf_{r\to\infty}U(r)$ is optimal, then $d(R)=0$.

Proof. We will use notation and definitions from Proposition 4.8. If R is finite we have trivially $d(R)=0$. Assume $|R|=\infty$. Since $U(r)$ is optimal, $\tilde{U}(r)$ is optimal, too. Hence, by Proposition 4.14, the limits of $U(r)$ and $\tilde{U}(r)$ exist and

$$\lim_{r\to\infty}\tilde{U}(r)=\lim_{\substack{r\to\infty\\r\in R}}\tilde{U}_{r+\ell-1}=\lim_{\substack{r\to\infty\\r\in R}}(U(r+\ell-1)+\frac{1}{r+\ell-1}\sum_{k\in R\cap[1,r]}\sum_{j=0}^{\ell-1}\Delta u^{(k+j)}),$$

where $\Delta u^{(k+j)}=\tilde{u}^{(k+j)}-u^{(k+j)}=\tilde{v}_{j+1}-v_{j+1}$ for $k\in R$ and $0\le j\le\ell-1$. This implies

$$\begin{aligned}0&=\lim_{\substack{r\to\infty\\r\in R}}\tilde{U}(r)-\lim_{\substack{r\to\infty\\r\in R}}U(r)=\lim_{\substack{r\to\infty\\r\in R}}\frac{1}{r+\ell-1}\sum_{k\in R\cap[1,r]}\gamma\\&=\gamma\lim_{\substack{r\to\infty\\r\in R}}\frac{1}{r+\ell-1}|R\cap[1,r]|\end{aligned}$$

but we want to get rid of the condition $r\in R$ in the limit. For this purpose note that the term $|R\cap[1,r]|$ does not change between to consecutive elements of R, but $\frac{1}{r+\ell-1}$ decreases. Hence,

$$\ge\gamma\limsup_{r\to\infty}\frac{1}{r+\ell-1}|R\cap[1,r]|\ge\liminf_{r\to\infty}\frac{1}{r+\ell-1}|R\cap[1,r]|\ge0.$$

Hence, $d(R)$ exists and equals 0. □

With this proposition, the characterization of the optimal election sequences given in (*ii*) follows from Lemma 4.11, which stated that every optimal election sequence can be transformed, in a way compatible with Proposition 4.8, to one of the following optimal sequences:

- $c\le1, c'\le\frac{1-c-n}{2}$: there exists a player i with $L_k(2r)=i$ and $L_k(2r+1)=j_{2r+1}$ with $j_{2r+1}\ne i$ for all $k\in\Pi$ (or with even/odd rounds changed),
- $c\ge1, c'\le\frac{-cn}{2}$: there exists a player i with $L_i(2r)=i$ and $L_i(2r+1)=j_{2r+1}$ with $j_{2r+1}\in\Pi\setminus\{i\}$ and $L_k(r)=i$ for all $j\in\Pi$ for all r (or with even/odd rounds changed),
- ow.: there exists a player i with $L_j(r)=i$ for all $j\in\Pi$ and all $r\ge1$.

Due to Proposition 4.15, the required transformation steps occur only on a set of density 0, which reveals that the characterization of the above sequences applies to every optimal sequence. This completes the proof of Theorem 4.13. □

Our next main result characterizes the optimal election sequences for the avg-social payoff. Compared to Theorem 4.13, this social payoff causes substantially less varied behavior. Tight lower- and upper-enclosing is the major proof technique used here. Recall that, for $x \in \{0, 1\}$, the set of rounds where no (i.e., in case $x = 1$) resp. all ($x = 0$) players change their votes is

$$D_x = \{r \mid \delta_{L_i(r-1)}(L_i(r)) = x \text{ for all players } i \in \Pi\}.$$

Theorem 4.16 (*Optimal Sequences for* avg-*social Payoff*)**.** *In case $c \neq 0$, an election sequence is optimal for u_{avg} iff $d(A_i)$ exists for all players, $d\left(\bigcup A_i\right) = 1$ and*

$$1 = \begin{cases} d(D_1) & \text{if } c' > 0 \\ d(D_0) & \text{if } c' < 0. \end{cases}$$

The optimal payoff is $n + c + c'$ in case $c' > 0$ and $n + c$ otherwise.

In case $c = 0$, an election sequence is optimal for u_{avg} iff $d\left(\bigcup A_i\right) = 1$ and

$$1 = \begin{cases} d(D_1) & \text{if } c' > 0 \\ d(D_0) & \text{if } c' < 0. \end{cases}$$

The optimal payoff is $n + c'$ in case $c' > 0$ and n otherwise.

Proof. Under our assumptions on the densities, we have by Lemma 4.1

$$\begin{aligned} n^2 + nc + \max(nc', 0) &\geq \sum_{j=1}^{n} \liminf_{r\to\infty} \frac{1}{r} \sum_{k=1}^{r} u_j^{(k)}(L) \\ &\geq \sum_{j=1}^{n} \liminf_{r\to\infty} \frac{1}{r} \left| [1, r] \cap \bigcup_i A_i \right| \cdot n + \sum_{j=1}^{n} \liminf_{r\to\infty} \frac{1}{r} |A_j \cap [1, r]| \cdot cn \\ &\quad + \sum_{j=1}^{n} \liminf_{r\to\infty} \frac{1}{r} \left| [1, r] \cap \left(\bigcup_i A_i \right)^C \right| \cdot d + \sum_{j=1}^{n} \liminf_{r\to\infty} \frac{1}{r} |[1, r] \cap D(j)| \cdot c', \end{aligned}$$

where d is a lower bound on $|\mathcal{F}_i^{(r)}(L)| + c|\mathcal{G}_i^{(r)}(L)|$, e.g., $d = 1 + \min(0, c)n$, see Lemma 4.3. Since the densities exist, the limits inferior are limits and due to $d(D_x) = 1$ (where $x = 1$ if $c' > 0$ and $x = 0$ otherwise) we have $d(D(j)) = x$ for all players j, and so

$$\begin{aligned} &= \sum_{j=1}^{n} d\left(\bigcup_i A_i \right) n + \sum_{j=1}^{n} d(A_j) nc + 0 \cdot d + xnc' \\ &= n^2 + cn + xnc' \end{aligned}$$

and hence this is an optimal election sequence by Lemma 4.1.

On the other hand, if we have an optimal election sequence L, then

$$\begin{aligned} n^2 + nc + \max(nc', 0) &= \sum_{j=1}^{n} \liminf_{r\to\infty} \frac{1}{r} \sum_{k=1}^{r} u_j^{(k)}(L) \leq \liminf_{r\to\infty} \frac{1}{r} \sum_{j=1}^{n} \sum_{k=1}^{r} u_j^{(k)}(L) \\ &\leq \limsup_{r\to\infty} \frac{1}{r} \sum_{k=1}^{r} \sum_{j=1}^{n} u_j^{(k)}(L) \leq n^2 + nc + \max(nc', 0). \end{aligned}$$

Similar as in the proof of Theorem 4.13, we conclude

$$\begin{aligned} n^2 + nc + \max(nc', 0) &= \lim_{r\to\infty} \frac{1}{r} \sum_{j=1}^{n} \sum_{k=1}^{r} u_j^{(k)}(L) \\ &= \lim_{r\to\infty} \sum_{i=nc+n+\min(nc',0)}^{n^2+cn+\max(nc',0)} \frac{1}{r} \cdot i \cdot \left| \{k \leq r \mid \sum_{j=1}^{n} u_j^{(k)}(L) = i\} \right| \\ &\leq \lim_{r\to\infty} \frac{1}{r} (n^2 + nc + \max(nc', 0)) \\ &\qquad \cdot \left| \{k \leq r \mid \sum_{j=1}^{n} u_j^{(k)}(L) = n^2 + nc + \max(nc', 0)\} \right| \end{aligned}$$

$$+\frac{1}{r}(n^2+nc+\max(nc',0)-\tilde{e}) \cdot \left(r-\left|\{k\le r \mid \sum_{j=1}^{n} u_j^{(k)}(L)=n^2+nc+\max(nc',0)\}\right|\right) \le n^2+nc+\max(nc',0),$$

where $\tilde{e}>0$ is the difference between $n^2+nc+\max(nc',0)$ and the second largest possible value of $\sum_{j=1}^{n} u_j^{(r)}(L)$ (analogous to the proof of Theorem 4.13). As a consequence,

$$d\left(\left\{r \mid \sum_{j=1}^{n} u_j^{(r)}(L)=n^2+nc+\max(nc',0)\right\}\right)=1; \tag{6}$$

this implies $d(\bigcup A_i)=1$ and $d(D_x)=1$. Hence the proof is finished in case $c=0$. For $c\neq 0$, note that $d(\bigcup A_i)=1$ implies that in an optimal election sequence L the set

$$U_j=\{r \mid u_j^{(r)}(L)=n+\max(c',0) \text{ or } u_j^{(r)}(L)=n+nc+\max(c',0)\}$$

has density 1 for every player j. Hence, by Lemma 4.3,

$$\begin{aligned}
&\liminf_{r\to\infty}\frac{1}{r}\sum_{k=1}^{r} u_j^{(k)}(L)\\
&=\liminf_{r\to\infty}\frac{1}{r}(n+\max(c',0))\left|\left\{k\le r \mid u_j^{(k)}(L)=n+\max(c',0)\right\}\right|\\
&\quad+\frac{1}{r}(n+nc+\max(c',0))\left|\left\{k\le r \mid u_j^{(k)}(L)=n+nc+\max(c',0)\right\}\right|\\
&\quad+\sum_{\substack{i\neq n+\max(c',0)\\ i\neq n+nc+\max(c',0)}}\frac{1}{r}\cdot i\cdot\left|\{k\le r \mid u_j^{(k)}(L)=i\}\right|\\
&=n+\max(c',0)+nc\liminf_{r\to\infty}\frac{|A_j\cap[1,r]|}{r}+0=n+\max(c',0)+\underline{d}(A_j)nc.
\end{aligned}$$

Summing up for $j=1,\ldots,n$ and comparison with (6) reveals

$$\sum_{j=1}^{n}\underline{d}(A_j)=1.$$

The following proposition ensures that this already implies the existence of the densities.

Proposition 4.17. *Let $A_1,\ldots,A_n\subseteq\mathbb{N}$ with $d\left(\bigcup A_i\right)=1$, $\sum_{i=1}^{n}\underline{d}(A_i)=1$, and $d(A_i\cap A_j)=0$ for all $i\neq j$. Then $d(A_i)$ exists for all i, $1\le i\le n$.*

Proof. Define $A_{(i)}=\bigcup_{j\neq i}A_j=\bigcup A_j\setminus A_i$. Since $d\left(\mathbb{N}\setminus\bigcup A_i\right)=0$, we have

$$\underline{d}(A_i)=1-\overline{d}(A_{(i)}) \qquad\text{and}\qquad \overline{d}(A_i)=1-\underline{d}(A_{(i)}).$$

Due to $d(A_i\cap A_j)=0$, the inequality $\sum_{j\in J}\underline{d}(A_j)\le d(\bigcup_{j\in J}A_j)$ holds for $J\subseteq[1,n]$. Thus,

$$0=\sum_{i=1}^{n}\underline{d}(A_i)-1\le\underline{d}(A_i)+\underline{d}(A_{(i)})-1=1-\overline{d}(A_i)-\overline{d}(A_{(i)})\le 1-\overline{d}\left(\bigcup A_i\right)=0.$$

Consequently,

$$0=\underline{d}(A_i)+\underline{d}(A_{(i)})-1=\underline{d}(A_i)+\left(1-\overline{d}(A_i)\right)-1=\underline{d}(A_i)-\overline{d}(A_i)$$

and therefore $d(A_i)$ exists. □

So the proof of Theorem 4.16 is complete. □

Finally, we give a characterization of $\mathfrak{u}_{\min}$-optimal election sequences.

Theorem 4.18 (*Optimal Sequences for* min-*Social Payoff*)**.** *In case $c \neq 0$, an election sequence is optimal for* $\mathrm{u}_{\min}$ *iff* $d(A_i) = \frac{1}{n}$ *for all players and*

$$1 = \begin{cases} d(D_1) & \text{if } c' > 0 \\ d(D_0) & \text{if } c' < 0. \end{cases}$$

The optimal payoff is $n + c + c'$ in case $c' > 0$ and $n + c$ otherwise.

In case $c = 0$, an election sequence is optimal for $\mathrm{u}_{\min}$ *iff $d(\bigcup_i A_i) = 1$ and*

$$1 = \begin{cases} d(D_1) & \text{if } c' > 0 \\ d(D_0) & \text{if } c' < 0. \end{cases}$$

The optimal payoff is $n + c'$ in case $c' > 0$ and n otherwise.

Proof. Let $c \neq 0$. If $d(A_i) = \frac{1}{n}$ and $d(D_x) = 1$ for an election sequence L, then

$$\begin{aligned} u_i(L) &= \liminf_{r\to\infty} \frac{1}{r} \sum_{k=1}^{r} u_i^{(k)}(L) \\ &= \liminf_{r\to\infty} \left(\frac{|A_i \cap D_x \cap [1, r]|}{r}(n + nc + \max(c', 0)) \right. \\ &\qquad \left. + \frac{1}{r} \left| [1, r] \cap D_x \cap \bigcup_{j\neq i} A_j \right| (n + \max(c', 0)) + \frac{1}{r} \sum_{\substack{1\le k\le r \\ k\notin(\bigcup A_j)\cap D_x}} u_i^{(k)}(L) \right) \\ &\geq \liminf_{r\to\infty} \frac{1}{r} \left| [1, r] \cap D_x \cap \bigcup_{j} A_j \right| (n + \max(c', 0)) \\ &\qquad + \liminf_{r\to\infty} \frac{|A_i \cap D_x \cap [1, r]|}{r} nc + \liminf_{r\to\infty} \frac{1}{r} \left| [1, r] \cap D_x \cap \left(\bigcup_{j} A_j \right)^C \right| d, \end{aligned}$$

where d is a lower bound on $u_i^{(k)}(L)$. Since all terms have a limit, this leads to

$$= n + \max(c', 0) + \frac{nc}{n} + 0 = c + n + \max(c', 0). \tag{7}$$

Hence $\mathrm{u}_{\min} = c + n + \max(c', 0)$, i.e., optimal. In case $c = 0$ it is the same proof, just omitting all terms involving c.

Conversely, if an election sequence is $\mathrm{u}_{\min}$-optimal, then it is $\mathrm{u}_{\mathrm{avg}}$-optimal as well, recall Lemma 4.6. Thus, the sets A_i, $1 \leq i \leq n$ have densities which sum up to 1. To see that $d(A_i) = \frac{1}{n}$ in case $c \neq 0$, we can reuse the computation above with arbitrary densities in (7), which yields $\mathrm{u}_{\min} = \min d(A_i)nc + n + \max(c', 0)$. So the only possible choice is $d(A_i) = \frac{1}{n}$ for all i. □

5. Characterization of (restricted) Nash equilibria

The goal of this section is to characterize those optimal sequences L that are Nash-equilibria in the restricted strategy space of oblivious strategies, i.e., where no individual player can increase his own payoff by deviating from L, for all types of social payoffs studied in this paper.

Unlike in the previous sections, where we solely considered an optimization problem, we are now touching upon the game-theoretic formulation. We will therefore establish the required notation first. Recall that an election is a collection of votes indexed by the set of players Π. A (r-round-)*history* is an r-tuple of elections, and a (round r-)*decision* is a function mapping $(r-1)$-round histories to votes of a single player. A *strategy* is a collection of functions of round r-decisions; a *strategy profile* is a collection of strategies indexed by Π.

For a strategy profile $S = \left(S_i^{(r)}\right)_{i\in\Pi}$, we define $P(S)$, the *play* corresponding to S, as the sequence $\left(P^{(r)}\right)_{r\geq 1}$ of elections inductively defined by the votes

$$P_i^{(r)} = S_i^{(r)}\left(P^{(1)}, P^{(2)}, \ldots, P^{(r-1)}\right) .$$

Given a play P, we say that player i *voted* for player j in round r in play P if $P_i^{(r)} = j$ and call j the *vote* of player i in round r in play P.

With these preparations, we can now formulate Nash equilibria in the restricted strategy space of oblivious strategies, where the choice of any player in round r cannot depend on the choices of the other players in rounds $1, \ldots, r$:

Definition 5.1. Given some (optimal) election sequence L, let S be a strategy profile leading to a play $P[j]$ that is identical to L for all players $\ell \in \Pi$, $\ell \neq j$, whereas the elections of $j \in \Pi$ in $P[j]$ may be arbitrarily different from its elections in L. The election sequence L is called a *Nash equilibrium in the restricted strategy space of oblivious strategies* (restricted[3] Nash equilibrium for short) if $u_i(L) \geq u_i(P[j])$ for all players $j \in \Pi$ and all plays $P[j]$.

The following Theorems 5.2, 5.3 and 5.5 characterize the restricted Nash equilibria for all our social payoffs. Again, the most varied one will be the maximum social payoff. The major proof technique used in this section is to start out from an optimal sequence, and to use the conditions given in the characterization theorems developed in Section 4 to infer what will happen if some player j deviates. Unfortunately, there is usually no alternative but to exhaustively explore all possibilities of how it could deviate in order to find out whether and when a Nash equilibrium exists.

Theorem 5.2 (*Characterization of Nash Equilibria for Max-social Payoffs*)**.** *An optimal election sequence as given in Theorem 4.13 is a Nash equilibrium (assuming $n \geq 3$ as long as not stated otherwise):*

(i) for $c > 0$, $c' \geq 0$ iff $n \geq 1 + c$,
(ii) for $c > 0$, $c' < 0$:

- *$c > 1$, $c' < \frac{1-c-n}{2}$: never,*
- *$c > 1$, $c' = \frac{1-c-n}{2}$: never,*
- *$c < 1$, $c' < \frac{-cn}{2}$: always,*
- *$c < 1$, $c' = \frac{-cn}{2}$: for $n \geq 4$: iff $n \geq (1+c)/(1-c)$ or $d(A_i^{**}) = 0$ (player i the maximum player),*
- *$c = 1$, $c' < \frac{-n}{2}$: iff $d(D(j)) = 0$ for all players j (essentially, $H_i = B_i$ in this case),*
- *$c = 1$, $c' = \frac{-n}{2}$: iff $d(A_i^*) = 0$ and $d(D(j)) = 0$ for all players j (essentially, $H_i = B_i$ in this case),*
- *ow.: iff $n + 2c' \geq 1 + c$ in case $c \leq 1$ and $2n + 2c' \geq \max(n + 1 + c, 2 + 2c + c')$ in case $c > 1$.*

(iii) for $c < 0$, $c' > 0$ iff $n \geq 1 - c$,
(iv) for $c < 0$, $c' = 0$ iff $n \geq 1 - c$,
(v) for $c < 0$, $c' < 0$ iff $n \geq 1 - c$,
(vi) for $c = 0$ always.

Proof. In our proof, we will use the notation $U_j^\ell(r) = \sum_{k=r}^{r+\ell-1} u_j^{(k)}(L)$, $U_j^\ell(r)' = \sum_{k=r}^{r+\ell-1} u_j^{(k)}(P[j])$, and $U_j^\ell(r)'' = \sum_{k=r}^{r+\ell-1} u_j^{(k)}(P[j]')$ ($P[j]'$ will be defined later on).

Case (i): Since there exists a player i with $d(A_i) = 1$ according to Theorem 4.13.(i), we have (for almost all rounds r)

$$u_j^{(r)}(L) = \begin{cases} n + nc + c' & \text{if } j = i, \\ n + c' & \text{ow.} \end{cases}$$

By Lemma 4.3, player i cannot increase his payoff, so he will not deviate. On the other hand, if a single player $j \neq i$ changes his vote in $P[j]'$ compared to L, by voting always for himself, then his payoff changes to $u_j^{(r)} = 1 + c + c'$. So L is an equilibrium iff $1 + c \leq n$.

Case (ii): When comparing payoffs, we will only consider players $j \neq i$ (i is the player whose payoff has the maximal value), since if player i could increase his payoff, L would not be optimal.

- $c > 1$, $c' < \frac{1-c-n}{2}$: Since $d(A_i) = d(E_i) = \frac{1}{2}$, $d(D(i)) = 0$ according to Theorem 4.13.(ii) in this case, we have for almost all rounds for every player $j \neq i$

$$u_j^{(r)}(L) = \begin{cases} n + c' & \text{if } r \in A_i, \\ n - 1 + c + c' & \text{if } L_i(r) = j, \text{ i.e., } r \in E_i(L), \\ n - 1 + c' & \text{if } L_i(r) = k \neq j, \text{ i.e., } r \in E_i(L). \end{cases}$$

Note that, due to $d(D(i)) = 0$, rounds with $r \in A_i$ and $r \in E_i$ alternate (with a possible exception of a 0-set). Hence, if $u_j^{(r)}(L) = n - 1 + c + c'$, then $u_j^{(r+1)}(L) = n + c'$. If player j changes his vote in round r from $L_j(r) = i$ to $P[j]_j^{(r)} = j$, we have $u_j^{(r)}(P[j]) = 2 + 2c$ and $u_j^{(r+1)}(P[j]) = n$. Thus,

$$U_j^2(r) = 2n - 1 + c + 2c' < n < 2 + 2c + n = U_j^2(r)',$$

[3] Since we only consider restricted Nash equilibria in this paper, we usually drop the attribute "restricted" for conciseness as well.

so player j gains a benefit by deviating from L. Similarly, if $u_j^{(r)}(L) = n - 1 + c'$, then $u_j^{(r+1)}(L) = n + c'$. If j changes his vote in round r from i to j, we have $u_j^{(r)}(P[j]) = 1 + c$ and $u_j^{(r+1)}(P[j]) = n$. Hence,

$$U_j^2(r) = 2n - 1 + 2c' < n - c < 1 + c + n = U_j^2(r)',$$

so deviating increases player j's payoff again. Consequently L cannot be an equilibrium.

- $c > 1,\ c' = \frac{1-c-n}{2}$: Additionally to the cases above, the set A_i can contain (arbitrarily long) sequences of consecutive rounds. If we have sequences of length at least 3, we have $u_j^{(r-1)}(L) = u_j^{(r)}(L) = u_j^{(r+1)}(L) = n + c'$. If player j changes his vote in round r from i to himself, then $u_j^{(r)}(P[j]) = 1 + c$ and $u_j^{(r+1)}(P[j]) = n$. Hence,

$$U_j^2(r) = 2n + 2c' < n + 1 - c < 1 + c + n = U_j^2(r)',$$

i.e., deviating is again good for player j. Hence L is not an equilibrium.

- $c < 1,\ c' < \frac{-cn}{2}$: Due to $d(A_i) = d(B_i) = \frac{1}{2}$, $d(D(i)) = 0$, we have for almost all rounds for every player $j \neq i$

$$u_j^{(r)}(L) = \begin{cases} n & \text{if } L_j(r) \neq j,\ \text{i.e., } r \in B_j(L), \\ n + nc & \text{if } L_j(r) = j,\ \text{i.e., } r \notin B_j(L). \end{cases}$$

If player j deviates from L, his new payoff equals

$$u_j^{(r)}(P[j]) = \begin{cases} 1 & \text{if } r \in B_j(L),\ P[j]_j^{(r)} \neq j,\ P[j]_j^{(r)} \neq L_j(r), \\ 1 + c & \text{if } r \in B_j(L),\ P[j]_j^{(r)} = j, \\ 1 + (n-1)c & \text{if } r \notin B_j(L),\ P[j]_j^{(r)} \neq j. \end{cases}$$

Obviously, changing his vote only decreases his individual payoff, so L is an equilibrium.

- $c < 1,\ c' = \frac{-cn}{2}$: Since $d(A_i \cup B_i) = 1$, $d(B_i^*) = 0$, it holds for almost all rounds that

$$u_j^{(r)}(L) = \begin{cases} n + c'\delta & \text{if } L_j(r) \neq j, \text{ where } \delta = \delta_{L_j(r-1)}(L_j(r)), \\ n + nc & \text{if } L_j(r) = j, \end{cases}$$

or in more detail,

$$u_j^{(r)}(L) = \begin{cases} n & \text{if } r \in A_i,\ r - 1 \notin A_i, \\ n + c' & \text{if } r \in A_i,\ r - 1 \in A_i, \\ n & \text{if } r \in B_k,\ k \neq j, \\ n + nc & \text{if } L_j(r) = j,\ \text{i.e., } r \in B_j. \end{cases}$$

We have to distinguish 3 different cases here:
Firstly, note that if there are no rounds r with $r \in A_i$ and $r+1 \in A_i$, then player j cannot increase his payoff: If $r \in A_i$ or $r \in B_k$ and he decides to vote for himself (or another player) then $u_j^{(r)}(P[j]) \leq 1 + c < n = u_j^{(r)}(L)$ and $u_j^{(r+1)}$ does not change. On the other hand, if $r \in B_j$, then $u_j^{(r)} = n + nc$ is already the maximal possible payoff for a single round, so changing his vote just decreases his payoff for this round.
Secondly, if there are rounds r with $r - 1 \notin A_i$, $r \in A_i$, $r + 1 \in A_i$ and $r + 2 \notin A_i$, then $U_j^2(r + 1)$ is still the maximal possible: If $r + 2 \in B_k$, then $U_j^2(r + 1) = n + c' + n$, whereas changing his vote from i to j gives $U_j^2(r + 1)' = 1 + c + n$. If $n \geq 4$, then $U_j^2(r + 1) \geq U_j^2(r + 1)'$. On the other hand, if $r + 2 \in B_j$, then $U_j^2(r + 1) = n + c' + n + nc$. Now we have to consider three different possibilities: (i) If player j decides to vote for himself in round $(r + 1)$ then $U_j^2(r + 1)' = 1 + c + n + nc + c'$, but since $c < 1$ we always have $U_j^2(r + 1)' < U_j^2(r + 1)$. (ii) If player j votes for some player k, $i \neq k \neq j$, then $U_j^2(r + 1)' = 1 + n + nc$, but this is smaller than $U_j^2(r + 1)$ for $n \geq 4$ again. (iii) If player j decides already to deviate in round r, then the original payoff $U_j^3(r) = n + n + c' + n + nc$, whereas $U_j^3(r)' \leq 1 + c + n + n + nc$ and hence $U_j^3(r)' \leq U_j^3(r)$ for $n \geq 4$.
Thirdly, assume that there exist maximal sequences R of length greater than or equal to 3 of consecutive rounds in A_i, starting at round r. If the length of R is odd, let player j change his vote from i to j every two rounds, starting with i (we will call this the *alternating strategy*). Then, the original payoff is $U_j^R(r) = n + (R - 1)(n + c')$, whereas $U_j^R(r)' = n + \frac{R-1}{2}(1 + c + n)$. This leads to $U_j^R(r) \geq U_j^R(r)'$ iff $n(1 - c) \geq (1 + c)$. If the length of R is even, we have to consider two different cases. (i) Assume that the round $\tilde{r}$ after the end of R is in B_k. The original payoff is $U_j^R(r) = n + (R - 1)(n + c')$. If player j uses the alternating strategy, we obtain $U_j^R(r)' = n + \frac{R}{2}(1 + c) + \frac{R-2}{2}n$. If player j chooses the *shifted alternating strategy* $P[j]'$ instead of $P[j]$, where it votes for i in round r and then starts the alternating strategy as before, we get $U_j^R(r)'' = n + n + c' + \frac{R-2}{2}(1 + c) + \frac{R-2}{2}n$. Since $1 + c \leq n + c'$ for $n \geq 4$, we have $U_j^R(r)' \leq U_j^R(r)''$, and as before we have $U_j^R(r)'' \leq U_j^R(r)$ iff $n(1 - c) \geq (1 + c)$. (ii) If

$\tilde{r}$ in B_j, then $U_j^{R+1}(r) = n + (R-1)(n+c') + n + nc$. If player j applies the alternating strategy, we obtain $U_j^{R+1}(r)' = n + \frac{R}{2}(1+c) + \frac{R-2}{2}n + n + nc + c'$. On the other hand, using the shifted alternating strategy yields $U_j^{R+1}(r)'' = n+n+c'++\frac{R-2}{2}(1+c)+\frac{R-2}{2}n+n+nc$. Obviously, $U_j^{R+1}(r)' \le U_j^{R+1}(r)''$, and again, $U_j^{R+1}(r)'' \le U_j^{R+1}(r)$ iff $n(1-c) \ge (1+c)$.
We hence conclude with observing that, as long as there are no sequences of consecutive rounds of length greater than or equal to 3 in A_i contributing to the lim inf, then L is always an equilibrium. This is exactly the case if and only if $d(A_i^{**}) = 0$. On the other hand, if $\overline{d}(A_i^{**}) > 0$, then we have to require $n(1-c) \ge (1+c)$ to ensure an equilibrium.

- $c = 1,\ c' < \frac{-n}{2}$: Since $d(A_i \cup H_i) = 1$ and $d(D(i)) = 0$, we have for almost all rounds

$$u_j^{(r)}(L) = \begin{cases} n + c'\delta & \text{if } L_k(r) = i \text{ for all } k \in \Pi \text{, i.e., } r \in A_i \\ s + (n-s) + c' = n + c' & \text{if } L_j(r) = i,\ L_i(r) = j,\ L_k(r) \in \{i,j\} \\ & \quad \text{for all } k \in \Pi \setminus \{i,j\}, \\ & \quad \text{if } (s-1) \text{ other players vote for } i \text{, too,} \\ 2(n-s) & \text{if } L_j(r) = j,\ L_i(r) = j,\ L_k(r) \in \{i,j\} \\ & \quad \text{for all } k \in \Pi \setminus \{i,j\}, \\ & \quad \text{if } s \text{ players vote for } i, \\ s + c' & \text{if } L_j(r) = i,\ L_i(r) = \ell,\ L_k(r) \in \{i,\ell\} \\ & \quad \text{for all } k \in \Pi \setminus \{i,j\}, \\ & \quad \text{if } (s-1) \text{ other players vote for } i \text{, too,} \\ n - s & \text{if } L_j(r) = \ell,\ L_i(r) = \ell,\ L_k(r) \in \{i,\ell\} \\ & \quad \text{for all } k \in \Pi \setminus \{i,j\}, \\ & \quad \text{if } s \text{ players vote for } i. \end{cases}$$

Note that, due to $d(D(i)) = 0$, rounds with $r \in A_i$ and $r \in H_i$ alternate (with a possible exception of a 0-set). Hence, if $u_j^{(r)}(L) = s + (n-s) + c' = n + c'$, then $u_j^{(r+1)}(L) = n + c'$. We distinguish the following cases: (i) If player j changes his vote in round r from i to j, we have $u_j^{(r)}(P[j]) = 2(n-s+1)$ and $u_j^{(r+1)}(P[j]) = n$. Thus,

$$U_j^2(r) = 2n + 2c' < n < 2(n-s+1) + n = U_j^2(r)', \tag{8}$$

i.e., deviating leads to a higher payoff for player j. (ii) If player j would vote for a player $\ell \notin \{i,j\}$, we have $u_j^{(r)}(P[j]) = 1$ and $u_j^{(r+1)}(P[j]) = n$, which is again better than the original payoff (albeit not as good as changing his vote to j).
Completely analogously, in case $u_j^{(r)} = s + c'$, (i) changing player's j vote from i to ℓ leads to a benefit, namely $u_j^{(r)}(P[j]) = n - s + 1$, so

$$U_j^2(r) = s + c' + n + c' < s < n - s + 1 + n = U_j^2(r)'.$$

(ii) If player j would change from i to j, his new payoff in round r would be $u_j^{(r)}(P[j]) = 1 + c = 2$, and this is not larger than his payoff $n-s+1$ obtained in (i) since $s \ge 1$. (iii) If j would change its vote from i to a player $p \notin \{i,j,\ell\}$, then $u_j^{(r)}(P[j]) = 1$, which is an even smaller payoff than in (i) and (ii).
In case $u_j^{(r)}(L) = 2(n-s)$, however, player j does not want to deviate: If he would vote for some player $\ell \notin \{i,j\}$, his payoff would be only 1, and if he would vote for player i, his payoff would decrease due to Eq. (8).
Analogously, in case $u_j^{(r)}(L) = n - s$, player j cannot increase his payoff by deviating from L: Changing his vote to himself gives a new payoff of 2, and since at most $(s-2)$ players vote for player i, we have $2 \le n - s$. Voting for player i would lead to case $u_j^{(r)}(P[j]) = s + c'$, which we showed above to be suboptimal.
So player j gains a benefit by deviating from L iff $L_j(r) = i$ ($r \notin A_i$). Hence he will not deviate iff $d(D(j)) = 0$. Recall that an optimal election sequence L is an equilibrium if there is no possibility for any player to increase his payoff by deviating from L. Hence, L is an optimal election sequence iff $d(D(j)) = 0$ for all $j \in \Pi$.

- $c = 1,\ c' = \frac{-n}{2}$: All of the computations of the previous case remain true, but since we only require $d(H_i^*) = 0$ instead of $d(D(i)) = 0$, it may happen that all players vote for player i in consecutive rounds. In this case $u_j^{(r)}(L) = \frac{n}{2}$ for all players $j \ne i$ (except for the first round of this subsequence). But if player j decides to vote for himself in round r, and is still voting for i in round $(r+1)$, then $u_j^{(r)}(P[j]) = 2$ and $u_j^{(r+1)}(P[j]) = n$. Thus,

$$U_j^2(r) = n < 2 + n = U_j^2(r)',$$

i.e., player j would benefit from deviating from the optimal sequence. Hence, we have to additionally require $d(A_i^*) = 0$ to secure an equilibrium.

- Otherwise: Due to $d(A_i) = 1$, the individual payoff of player $j \ne i$ is $u_j^{(r)}(L) = n + c'$ for almost all rounds r. If player j wants to deviate from L, he can (a) either vote for himself or (b) for another player in round r, and in the following round $(r+1)$ he can (i) switch back to L, (ii) vote again for himself, or (iii) vote for another player. Comparing $U_j^2(r)'$ for these variants shows that the highest payoff will be achieved in case (a.i) with payoff $(1+c+n)$ or in case (a.ii) with payoff $(2+2c+c')$. The first case can lead to a higher payoff for arbitrary c, whereas the latter case is only interesting for $c > 1$. Comparing these payoffs with the payoff $U_j^2(r) = 2n + 2c'$ directly leads to the stated conditions.

Case (*iii*): Theorem 4.13(iii) reveals that there is a player i with $d(B_i) = d(D(i)) = 1$, which implies $d(D_1) = 1$. Hence, for almost all rounds r,

$$u_j^{(r)}(L) = \begin{cases} n + c' & \text{if } L_j(r) \neq j, \\ n + nc + c' & \text{if } L_j(r) = j. \end{cases}$$

In either case, if j deviates to a fixed player j' (in all such rounds), he has a new payoff $u_j^{(r)}(P[j]) = 1 + (n-1)c + c'$. The critical case for the equilibrium is the one with lower original payoff, which is $u_j^{(r)} = n + nc + c'$ since $c < 0$ and $c' > 0$ here. Comparing the payoffs, it turns out that L is an equilibrium iff $1 - c \leq n$.

Case (*iv*): Analogous to (*iii*) for $c' = 0$.

Case (*v*): The existence of a player i with $d(B_i) = 1$ and $d(D(i)) = 0$ implies $d(D_0) = 1$. Hence, for almost all rounds r,

$$u_j^{(r)}(L) = \begin{cases} n & \text{if } L_j(r) \neq j, \\ n + nc & \text{if } L_j(r) = j. \end{cases}$$

Similar to case (*iii*), if j deviates to a fixed player j' (in all such rounds), he has a new payoff $u_j^{(r)}(P[j]) = 1 + (n-1)c$. The critical case for the equilibrium is the one with lower original payoff, which is $u_j^{(r)} = n + nc$ since $c < 0$ and $c' < 0$ here. Comparing the payoffs, it turns out again that L is an equilibrium iff $1 - c \leq n$.

Case (*vi*): In this case, we have $u_j^{(r)}(L) = n + \max(0, c')$ for all players j and almost all rounds r. If a player deviates from L, then his best possible payoff is $u_j^{(r)}(P[j]) = 1 + \max(0, c')$, which is strictly smaller than $u_j^{(r)}(L)$. So L is always an equilibrium. □

The next main result characterizes whether and when an optimal election sequence for the average social payoff is a Nash equilibrium. Again, it is much less varied than Theorem 5.2.

Theorem 5.3 (*Characterization of Nash Equilibria for Avg-social Payoffs*)**.** *An optimal election sequence as given in Theorem 4.16 is a Nash equilibrium iff*

$$\begin{cases} \text{for } n \geq 1 - c & \text{if } c \leq 0, \\ \text{for } n \geq 1 + c & \text{if } c \geq 0. \end{cases}$$

Remark 5.4. Note that in case $c = 0$ the optimal election sequences are always an equilibrium, as no player has an incentive to become elected.

Proof. Let us start with case $c \leq 0$. Firstly, if $c' \geq 0$, it holds for almost all rounds r that

$$u_i^{(r)}(L) = \begin{cases} n + c' & \text{if } r \notin A_i, \\ n + nc + c' & \text{if } r \in A_i. \end{cases}$$

If player i wants to deviate from the optimal election sequence L in round r for $r \notin A_i$, then his payoff would change to $u_i^{(r)} = 1 + c'$ or $u_i^{(r)} = 1 + c + c'$ depending on his new vote. If $c < 0$ and $n \geq 1$, he gains no benefit by deviating. On the other hand, if $r \in A_i$ and player i decides to deviate and votes for some player $j \neq i$, then his payoff changes to $1 + (n-1)c + c'$. But

$$1 + (n-1)c + c' \leq n + nc + c' \qquad \Leftrightarrow \qquad 1 - c \leq n,$$

so we are done in case $c' \geq 0$.

Secondly, if $c' \leq 0$, we have for almost all rounds r,

$$u_i^{(r)}(L) = \begin{cases} n & \text{if } r \notin A_i, \\ n + nc & \text{if } r \in A_i. \end{cases}$$

Analogous arguments as above lead again to the condition $1 - c \leq n$.

The case $c > 0$ is very similar: Firstly, if $c' \geq 0$, it holds for almost all rounds r that

$$u_i^{(r)}(L) = \begin{cases} n + c' & \text{if } r \notin A_i, \\ n + nc + c' & \text{if } r \in A_i. \end{cases}$$

If player i wants to deviate from the optimal election sequence L in round r for $r \notin A_i$, then his payoff would change to $u_i^{(r)} = 1 + c'$ or $u_i^{(r)} = 1 + c + c'$ depending on his new vote. Since $c > 0$, his change does not lead to a larger payoff iff

$$1 + c + c' \leq n + c' \qquad \Leftrightarrow \qquad 1 + c \leq n.$$

If $r \in A_i$, then changing his vote leads to the inequality $1 - c \leq n$ as above, but this inequality is always true since $c > 0$.

Secondly, case $c' \leq 0$ can be treated analogously. □

Since the optimal election sequences for $\mathfrak{u}_{\min}$ are a subset of those for $\mathfrak{u}_{\text{avg}}$, we immediately conclude the following third main result of this section:

Theorem 5.5 (*Characterization of Nash Equilibria for Min-social Payoffs*)**.** *An optimal election sequence as given in Theorem 4.18 is a Nash equilibrium iff*

$$\begin{cases} \text{for } n \geq 1 - c & \text{if } c \leq 0, \\ \text{for } n \geq 1 + c & \text{if } c \geq 0. \end{cases}$$

6. Discussion

Thanks to the complete characterization of the optimal social payoffs for any choice of the parameters c, c' in Theorems 4.13, 4.16 and 4.18, we have developed an important ingredient for the game-theoretic analysis of repeated leader election *algorithms* that compute election sequences on-line. Like in [5], one may want to characterize locally optimal strategies, and may even want to compute the associated "price of anarchy" as in [17]. In order to accomplish this, the knowledge of globally optimal election sequences is mandatory.

Our results also allow us to draw some interesting conclusions on the role and interplay of c and c'.

First and foremost, as expected, $c' > 0$ favors always electing the same leader i, whereas $c' < 0$ stimulates leader changes. Still, for $\mathfrak{u}_{\max}$, c' must be a surprisingly large negative value (like $c' < \frac{1-c-n}{2}$) to really stimulate leader changes. Overall, this just confirms the fact that our generic model is a bit simplistic in this regard. In fact, c' parameterizes the event "leader change", rather than the *time* some player has been continuously elected leader already. Unfortunately, we were not yet able to come up with a refined model that is also mathematically tractable: what one loses immediately is locality of the local payoffs $u_i^{(r)}(L)$, which now depend only on the elections $L(r)$ and $L(r-1)$.

In addition, for $\mathfrak{u}_{\max}$, the results of Theorem 4.13 (visualized in Fig. 1) reveal several interesting facts:

1. Sign and size of c dramatically change the optimal sequence for $\mathfrak{u}_{\max}$. Quite obviously, $c < 0$ makes it optimal to choose a unique leader $j \neq i$ (where i is the player earning the maximal benefit). For $c > 0$, the optimal strategies vary considerably depending on the relation between c and c'. Note that, for sufficiently small c', like for $c > 1$ and $c' < \frac{1-c-n}{2}$, not having a unique leader, i.e., some players choosing different leaders in some rounds, becomes optimal. Note, however, that at most two leaders are elected in the limit in all optimal sequences.
2. Forcing leader changes, i.e., $c' < 0$, decreases the maximum payoff considerably. For $c > 1$, the payoff is reduced from $n + cn + c'$ to $n/2 + cn + (1 - c)/2$. Interestingly, the reduction is less dramatically for $c < 1$, where it goes down to $n + cn/2$ only. For $c \leq 0$, the maximum payoff is around n, without much dependency on c'.

For $\mathfrak{u}_{\text{avg}}$ and $\mathfrak{u}_{\min}$, we note the following:

1. There is not much dependency of $\mathfrak{u}_{\text{avg}}$ and $\mathfrak{u}_{\min}$ on the parameter c. More specifically, $c \neq$ requires the individual densities $d(A_i)$ (of the sets A_i where every player chooses i) to exist, and sum to 1 for $\mathfrak{u}_{\text{avg}}$ and being equal to $1/n$ for $\mathfrak{u}_{\min}$, which essentially means uneven/even alternation of the unique leader. For $c = 0$, all that is needed is $d(\bigcup A_i) = 1$, which amounts to unfair alternation. Unlike for $\mathfrak{u}_{\max}$, not having all players choosing the same leader in some rounds is always sub-optimal here.
2. For $\mathfrak{u}_{\text{avg}}$ and $\mathfrak{u}_{\min}$, in case $c' < 0$, round-robin is an optimal election sequence. For $c' > 0$, however, this is not the case, since round-robin causes $d(D_1) = 0$. However, an optimal election sequence for two players would be 12112211122211112222 . . ., as the time between changes increases here.
3. Neither $\mathfrak{u}_{\text{avg}}$ nor $\mathfrak{u}_{\min}$ depend much on frequent ($n + c + c'$ in the case $c' < 0$) vs. infrequent ($n + c'$ for $c' > 0$) leader changes.

So, overall, it is fair to say that $\mathfrak{u}_{\text{avg}}$ nor $\mathfrak{u}_{\min}$ are much more robust w.r.t. peculiarities of the chosen sequence: As long as sufficient fairness is guaranteed, they guarantee their optimal payoff. This payoff is, however, typically considerably smaller than the optimal payoff for $\mathfrak{u}_{\max}$.

Regarding Nash equilibria, we observe the following from Theorems 5.2, 5.3 and 5.5:

1. For $\mathfrak{u}_{\text{avg}}$ nor $\mathfrak{u}_{\min}$, Nash equilibria only depend on the relation between n and c, depending on whether $c \leq 0$ or $c \geq 0$: Too large a $|c|$ is prohibitive. The same is true for many cases of $\mathfrak{u}_{\max}$, in particular, (*iii*)–(*v*).

2. As for the optimal sequences, the situation is very different for case (*ii*) of $\mathfrak{u}_{\max}$, where $c > 0$ and $c' < 0$: Whether the optimal sequences are Nash equilibria or not depends heavily and non-trivially on the sign and value of c and c'. This is particularly true for the range $0 < c \leq 1$ and $-n/2 \leq c' < 0$.

So, overall, it is apparent that local and social payoffs go hand-in-hand for $\mathfrak{u}_{\text{avg}}$ and $\mathfrak{u}_{\min}$, provided the critical relation between n and c is respected.

7. Conclusions

We exhaustively characterized optimal sequences for repeated leader election. We restrict our attention to strategies where every player chooses the preferred leader oblivious w.r.t. the other players choice in every round, but subject to a parameterized local payoff function for player i that takes into account how many players vote for the same leader as i, how many players vote for i, and whether i chooses the same leader as in the previous round or not. The limiting social payoffs considered are the maximum, average and minimum of the players' local payoffs. In the case of the maximum social payoff, a surprisingly rich set of optimal sequences was found for some parameter ranges. In most other cases, however, there is not much variation. We also discovered that, depending on the relation of certain parameters, the optimal sequences are often Nash equilibria in the restricted strategy space of oblivious strategies. As a consequence, no rational player will deviate from the globally optimal sequence in order to improve its local payoff, without any need for punishment.

Part of our current work is devoted to a more realistic modeling of leader persistence. In some future work, we will use our results as the basis for computing the price of anarchy in evolutionary games, where players do not know an optimal oblivious strategy a priori but try to converge towards a good strategy based on locally available information.

CRediT authorship contribution statement

Martin Zeiner: conceptualization, Methodology, Formal analysis, Writing - original draft. **Ulrich Schmid:** conceptualization, Writing - review & editing. **Krishnendu Chatterjee:** validation, Writing - review & editing.

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