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Cover Page Footnote

I would like to thank my advisor, Dr. Anita O'Mellan, for her guidance and support during the composition of this paper. I would also like to thank Dr. Alexis Byers for introducing me to the TikZ package inside LaTeX, through which the digital construction of these graphs was made possible. Last, but not least, I would like to thank the YSU Department of Mathematics and Statistics for the opportunity and the resources to explore this intriguing topic.

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On the Equitable Total $(k+1)$-Coloring of k-Regular Graphs

By *Bryson Stemock*

Abstract. A graph is considered to be totally colored when one color is assigned to each vertex and to each edge so that no adjacent or incident vertices or edges bear the same color. The *total chromatic number* of a graph is the least number of colors required to totally color a graph. This paper focuses on k-regular graphs, whose symmetry and regularity allow for a closer look at general total coloring strategies. Such graphs include the previously defined Möbius ladder, which has a total chromatic number of 5, as well as the newly defined bird's nest, which is shown to have a total chromatic number of 4. Furthermore, a total 4-coloring of the Petersen graph is examined and the total $(k+1)$-colorings of k-regular graphs is discussed. More specifically, it is proposed that any $(k+1)$-coloring of any k-regular graph is inherently equitable for all $3 \le k \le 5$ given a bound on the order of the graph. That is to say that every color is used no more than one time more than any other color when totally coloring the graph.

1 Introduction

A popular topic in graph theory is graph coloring, generally proper vertex or edge coloring. One can take this a step further, though, to totally color a graph. That is to say that a graph is totally colored when a color is assigned to each vertex and to each edge such that no adjacent or incident vertex or edge bears the same color [1]. When k colors are used, we call this a **total k-coloring**. Let c_i be the number of times the color i is used in the k-coloring. A k-coloring (total or otherwise) is considered to be **equitable** if $|c_i - c_j| \le 1$ for all $i, j \in \{1, 2, \ldots, k\}$ [4, 8]. For the sake of brevity and because this paper focuses exclusively on total colorings, the term "total coloring" is used interchangeably with the term "coloring" throughout the paper.

The least number of colors required to totally color a graph, G, is known as the **total chromatic number**, denoted as $\chi^{\mathrm{T}}(\mathrm{G})$, while the least number of colors required to *equitably* totally color that graph is known as the **equitable total chromatic number**,

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denoted as $\chi^{T}_{=}(G)$. A commonly cited proposed property of a graph's total chromatic number is Mehdi Behzad's Total Coloring Conjecture, or TCC, which proposes that for any graph, G, the following holds:

$$\Delta(G)+1 \le \chi^{T}(G) \le \Delta(G)+2$$

where $\Delta(G)$ represents the maximum degree of the graph [1]. The first inequality is obvious and will be used at multiple junctures of this paper. The vertex with maximum degree is incident to $\Delta(G)$ edges, which must all bear a different color since they are incident to each other as well. Then, by adding in the additional color of the vertex, which must also be different from any colors used on the edges, one automatically requires at least $\Delta(G)+1$ colors to totally color G.

In this paper, we will look at k-regular graphs, which means that each vertex is adjacent to k other vertices. Hence, we will always require at least $k+1$ colors to totally color our graphs. The number of vertices in a graph is called the **order** of the graph. We will explore a new 3-regular graph, the bird's nest, and a previously defined 3-regular graph, the Möbius ladder, their total colorability, and the equitability of these total colorings. We will also consider the popular Petersen graph, which is 3-regular, and discuss the equitability of $(k+1)$-colorings of k-regular graphs in general.

2 The bird's nest

Define the bird's nest, B_n, as a cycle with n vertices, C_n, with added edges as follows. Let the vertex $v_i \in V(B_n) = \{v_1, v_2, ..., v_n\}$. If $i \equiv 1 \mod 4$ or $i \equiv 2 \mod 4$, then an edge is added connecting v_i to v_{i+2}. Therefore, this graph is 3-regular. So that this description is well-defined, we require that $4 \mid n$. Note that if $n = 4$, we have the complete graph K_4, which is not 4-colorable, so we will also require that $n \ge 8$.

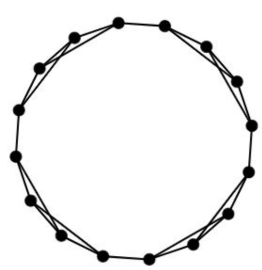


Figure 1: B_{16} is given as a visual example of a bird's nest.

Rather than using the general definition for the bird's nest, it is clearer to discuss the bird's nest in terms of a subgraph, shown in Figure 2 and henceforth referred to as "subgraph H", which is the subgraph induced by vertices v_i, v_{i+1}, v_{i+2}, and v_{i+3} for $i \equiv 1 \mod 4$. Now we can define the bird's nest as the amalgamation of two or more

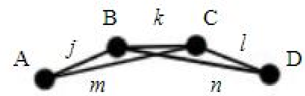


Figure 2: The subgraph H of the bird's nest.

copies of subgraph H, joined to each other by an edge connecting the vertices of degree 2, resulting in a 3-regular graph. Since subgraph H contains 4 vertices, we still get $4 \mid n$ for every bird's nest. To discuss the colorability of B_n, we will first look a bit more closely at the colorability of subgraph H. Define the coloring $\psi : V(B_n) \cup E(B_n) \to \{1,2,3,4\}$.

Lemma 2.1. *There are exactly two unique total 4-colorings of subgraph H.*

Proof. First, we will examine how many colors can be used on the vertices. Since vertices A, B, and C are all adjacent to each other, we need at least 3 colors and, since there are only 4 vertices, we can use at most 4 colors. Suppose we use four distinct colors on the vertices, which gives us $\psi(A) = 1$, $\psi(B) = 2$, $\psi(C) = 3$, and $\psi(D) = 4$ without loss of generality. Then we must have $\psi(j) \in \{3,4\}$ and $\psi(l) \in \{1,2\}$ leaving four ways to color these edges. If $\psi(j) = 3$ and $\psi(l) = 1$, we must have $\psi(k) = 4$ which leaves no possible value for $\psi(n)$. If $\psi(j) = 3$ and $\psi(l) = 2$, we must have $\psi(m) = 4$ and $\psi(n) = 1$, leaving no possible value for $\psi(k)$. If $\psi(j) = 4$ and $\psi(l) = 1$, there is no available value for $\psi(k)$. Finally, if $\psi(j) = 4$ and $\psi(l) = 2$, then there is no possible value for $\psi(m)$. For clarity, all four possible colorings are shown in Figure 3 where the chosen colors for the outer side edges are in bold. Since the vertex colors are all different, the vertex colors are arbitrary.

Hence, there must be exactly three colors used on the four vertices. Now we must have $\psi(A) = \psi(D)$ since those are the only pairwise nonadjacent vertices. Consider our new view of how to color subgraph H: Now we have $\psi(l) \in \{2,4\}$. If $\psi(l) = 2$, then we must have $\psi(m) = 4$ which means we get $\psi(k) = 1$. If $\psi(l) = 4$, then $\psi(k) = 1$ is forced. Thus, we must have $\psi(k) = 1$. Figure 5, then, presents what we have so far.

There are two ways to color this now. We know $\psi(n) \in \{3,4\}$. If $\psi(n) = 3$, we must have $\psi(j) = 4$, which means we must have $\psi(m) = 2$ and $\psi(l) = 4$. If $\psi(n) = 4$, we must have $\psi(l) = 2$ and $\psi(j) = 3$, which means that $\psi(m) = 4$. Thus, we have exactly two unique 4-colorings of subgraph H and we have found them, shown in Figure 6. They are arbitrarily named μ and ϵ. □

Now, since B_n is simply a collection of multiples of subgraph H strung together end to end and since we know how to color subgraph H, we can discuss how to color B_n. It turns out that which coloring scheme we use, μ or ϵ, to color each subgraph H in B_n is

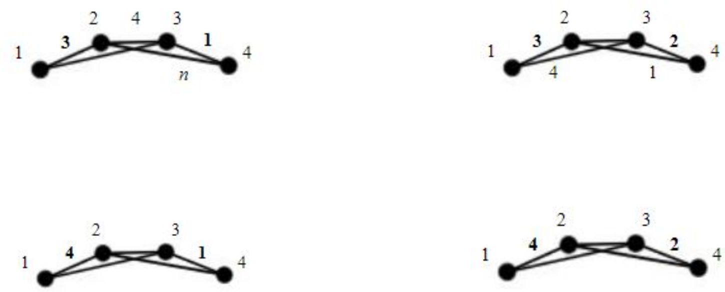


Figure 3: You cannot use a distinct color for each vertex in subgraph H.

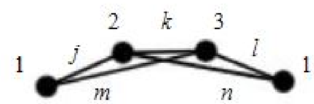


Figure 4: Our current understanding of the 4-coloring of subgraph H.

irrelevant. It only matters that one of them is used since the only difference is that we swap the inner and outer edges (swapping j with m and l with n in Figure 5). That is to say that they are interchangeable.

Theorem 2.2. *The total chromatic number of* B_n *is 4. That is,* $\chi^{\mathrm{T}}(\mathrm{B}_n) = 4$.

Proof. Since we have a vertex of degree 3, we know $\chi^{\mathrm{T}}(\mathrm{B}_n) \geq 4$. Because the bird's nest is an accumulation of copies of subgraph H's, we should consider whether we have an even or an odd number of copies in our bird's nest. First, suppose we have an even number of copies. Then $8 \mid n$. Take, for example, the smallest possible bird's nest where this happens, B_8. A total 4-coloring of B_8 is shown in figure 7.

To 4-color B_n for $n > 8$ such that $8 \mid n$, we repeat the coloring of B_8. That is, color vertices $v_1, v_2, \ldots, v_8$ using the colors 1, 4, 3, 1, 2, 3, 4, 2 consecutively and then color v_i for $i > 8$ so that $\psi(v_i) = \psi(v_j)$ for $i \equiv j \mod 8$ and $\psi(v_i) = 2$ for all i where $8 \mid i$. Next, color the first eight edges of the n-cycle $v_1v_2, v_2v_3, \ldots, v_8v_9$ with 3, 1, 4, 3, 4, 2, 3, 4 and repeat this color pattern on the remaining edges of the cycle. Color inner edges 2 for each instance of subgraph H with vertices colored 1, 4, 3, 1, and color the inner edges 1 for each instance of subgraph H with vertices colored 2, 3, 4, 2.

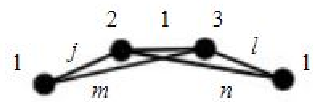


Figure 5: Our updated understanding of the 4-coloring of subgraph H.

(a) Coloring μ

(b) Coloring ϵ

Figure 6: The two unique colorings for our subgraph of B_n.

A 4-coloring of B_{16} is shown in Figure 8 following the process described above. Note that the 4-coloring of B_8, which is the template for this process, utilizes each color exactly 5 times. Therefore, in the coloring of B_n where $8 \mid n$, each color will be used exactly $5(\frac{n}{8})$ times.

Now suppose we have an odd number of copies of subgraph H in B_n. Then $8 \nmid n$. We can apply the same process from the first part of the proof, we just need a different base case now. This time, we will start with the 4-colored B_{12} (since the only 3-regular graph with four vertices is K_4, which is not 4-colorable, as discussed above), as shown in Figure 9.

To 4-color B_n for $n > 12$ such that $8 \nmid n$, we begin with the coloring of B_{12} and utilize our coloring scheme for B_8 to build upon it. That is, color vertices v_1, v_2, ..., v_{12} using the colors 3, 4, 2, 3, 1, 2, 3, 1, 2, 3, 4, 2 consecutively and then color vertices v_{13}, v_{14}, ..., v_{20} using the colors 1, 4, 3, 1, 2, 3, 4, 2. Now, color v_i for $i > 20$ so that $\psi(v_i) = \psi(v_{i-8})$. Next, color the first twelve edges of the n-cycle v_1v_2, v_2v_3, ..., $v_{12}v_{13}$ with 2, 3, 4, 2, 3, 1, 2, 3, 4, 2, 3, 4. Now, color the next eight edges of the n-cycle, $v_{13}v_{14}$, $v_{14}v_{15}$, ..., $v_{20}v_{21}$ (for the case of B_{20}, the last edge is $v_{20}v_1$) with 3, 1, 4, 3, 4, 2, 3, 4 and repeat this pattern of 8 colors on the remaining edges of the cycle. Color remaining inner edges of each H subgraph with the color not used on the four vertices of that subgraph. A 4-coloring of B_{20} is shown in Figure 10 using the process described above. Note that the 4-coloring of B_{12} described above and shown in Figure 9 utilizes the colors 1 and 4 exactly 7 times each and the colors 2 and 3 exactly 8 time each. Then, for every set of 8 vertices added to B_{12} and the associated edges, each color is used exactly 5 times. Hence, in the coloring

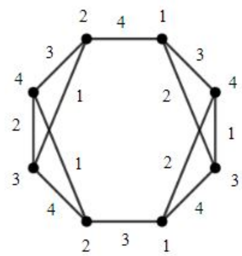


Figure 7: A 4-coloring of B_8.

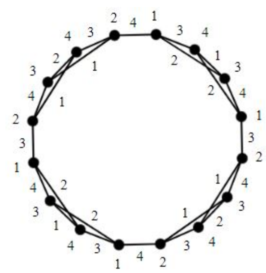


Figure 8: A 4-coloring of B_{16}.

of B_n where $8 \nmid n$, the colors 1 and 4 will be used exactly $7+5(\frac{n-12}{8})$ times and the colors 2 and 3 will be used exactly $8+5(\frac{n-12}{8})$ times. Thus, for all $n \equiv 0 \mod 4$, B_n has a total 4-coloring, so $\chi^T(B_n) = 4$. □

Corollary 2.3. *Every bird's nest can be equitably 4-colored. That is to say that* $\chi^T_=(B_n) = 4$.

Proof. This follows from the counts given in **theorem 2.2**. □

3 The Möbius ladder

Define the Möbius ladder, M_n, as a cycle with n vertices, C_n, with added edges connecting vertex v_i with vertex $v_{i+\frac{n}{2}}$ for all $i \in \{1,2,\ldots,\frac{n}{2}\}$ [5]. Hence, the graph is 3-regular. These added edges will be referred to as the "rungs" of the Möbius ladder. By this definition, n must always be even. Guy and Harary (1967) also defined this type of graph for odd n, but we will restrict our analysis to even n for our purposes. In 1988, Chetwynd and Hilton used the idea of "conformable vertex colorings" to prove that $\chi^T(M_n) = 5$ [2]. Here, I provide a constructive proof of the same result as well as a discussion on the equitable total chromatic number of M_n. Define the coloring $\xi : V(M_n) \cup E(M_n) \to \{1,2,3,4,5\}$

In order to establish a lower bound for $\chi^T(M_n)$, we must attempt to 4-color M_n. One of the most common techniques used to begin attempting to color a graph, especially a graph with as much symmetry as M_n, is to use a pattern. One might wonder, then, if

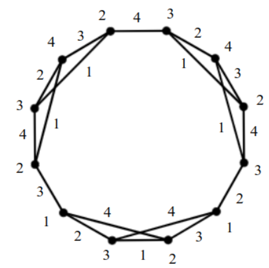


Figure 9: The 4-colored B_{12}, our base case.

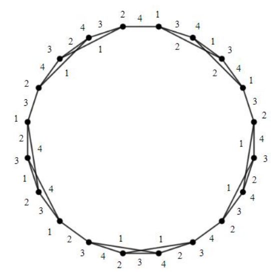


Figure 10: A 4-coloring of B_{20}.

M_n can be 4-colored by applying the same color to each rung and simply using a "1-2-3" pattern on the outer cycle. This, however, causes problems.

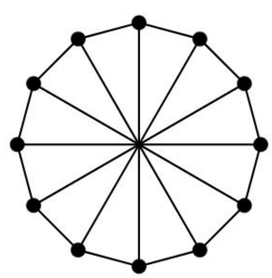


Figure 11: M_{12} is given as a visual example of a Möbius ladder.

Lemma 3.1. *More than one color is required to color the rungs of* M_n *if only 4 total colors are used.*

Proof. Suppose we use the color 4 or every rung in M_n. Then, on the outer cycle, a pattern of some sort of permutation of 1, 2, and 3 must be used since each vertex and each outer edge will then be adjacent to one of the rungs, which bears the color 4. The reason that a pattern is required is because, after choosing only two colors on the outside, a pattern is forced. Take the example in Figure 12 where only relevant parts of

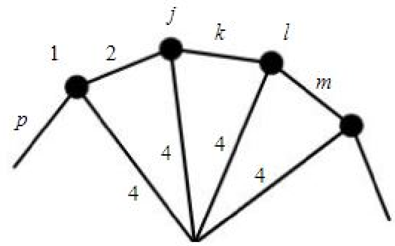


Figure 12: Using the same color for each rung.

a subgraph, not an entire subgraph, are shown (i.e. this figure contains pendant edges without showing end vertices).

If the first vertex is colored with 1 and the clockwise adjacent edge is colored with 2, then we have already forced $\xi(j) = 3$, which means $\xi(k) = 1$, then $\xi(l) = 2$, so $\xi(m) = 3$, and so on and so forth. It is also worth noting that we must have $\xi(p) = 3$. This means that our pattern repeats after every three vertices.

If we can't fit in a whole number of copies of our pattern, we will run into issues connecting the end to the beginning, so this pattern will clearly not work if $3 \nmid n$. Suppose, then, that $3 \mid n$. With this pattern, we know that $\xi(v_i) = \xi(v_{i+3})$ for all $i \in \{1,2,...,n-3\}$. That means that $\xi(v_i) = \xi(v_j)$ for each $i \equiv q \mod 3$ and $j \equiv q \mod 3$ for some $q \in \{0,1,2\}$ and for all $i, j \in \{1,2,...,n\}$. We know from the definition of M_n that vertex v_i is adjacent to vertex $v_{i+\frac{n}{2}}$. We also know that, since $3 \mid n$ and n is even, we must have $3 \mid \frac{n}{2}$. Let $i \in \{1,2,..., \frac{n}{2}\}$. Then there exists a $q_0 \in \{0,1,2\}$ such that $i \equiv q_0 \mod 3$. Furthermore, since $3 \mid \frac{n}{2}$, we know that $i + \frac{n}{2} \equiv q_0 \mod 3$ so $\xi(v_i) = \xi(v_{i+\frac{n}{2}})$. But v_i is adjacent to $v_{i+\frac{n}{2}}$, a contradiction. Thus, more than one color must be used to color the rungs of M_n if only four colors are used. □

Since the pattern did not work, let us instead try to break down the problem into smaller pieces. This leads us to our second lemma, which involves the subgraph that is partially displayed in Figure 13 (we will call it "subgraph I" due to its shape). Henceforth, for the sake of simplicity, only the relevant parts of subgraph I will be displayed and/or discussed, though it will continue to be referred to as "subgraph I".

Lemma 3.2. *There are only two unique total 4-colorings of the subgraph I of* M_n *shown in Figure 13.*

Proof. This proof requires that we consider how many colors must be used to color the two upper and two lower edges. Clearly, we cannot use only one color. Suppose we used only two colors, 1 and 2, shown in Figure 14. Then the vertices must be colored with 3 and 4, leaving no color available for the rung. Hence, we cannot use only two colors. Now suppose that we use all four colors for these edges, displayed in Figure 15. Now,

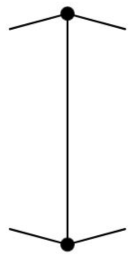


Figure 13: The relevant parts of subgraph I, an important subgraph of M_n.

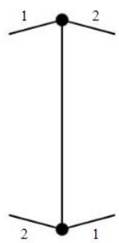


Figure 14: Using 2 colors on the upper and lower edges of subgraph I.

once again, we don't have any available colors for the rung and so we cannot use all four colors. Thus, since we are attempting to 4-color the Möbius ladder, we must use exactly three colors for the upper and lower edges of each subgraph I contained in M_n. Since the upper two colors can be chosen arbitrarily and since we must have a different color at the bottom with one repeated color, we are left with only two permutations of these colors on the bottom, in the arbitrarily named colorings α and γ in Figure 16. Therefore, we have not only proven that there are two unique total colorings of subgraph I, but we have found these colorings since the chosen permutations of the lower edges force the colors of the vertices and of the rung. □

Lemma 3.3. *The coloring* α *cannot appear anywhere in any 4-coloring of the Möbius ladder.*

Proof. Suppose we have the coloring α of a given subgraph I contained in M_n. Then we have the setup in Figure 17, where we must either have $\xi(X) = 3$ or $\xi(X) = 4$. Let $\xi(X) = 3$. Then we have two options to color the edge clockwise of this vertex (1 or 4), shown in Figure 18.

If we choose to color this clockwise edge with 1, then we must have $\xi(j) = 4$, forcing $\xi(k) = 1$, which in turn means that $\xi(l) = 3$. This can be seen in Figure 19, where all bold colors are forced by our choice. Note that the vertices labeled P and Q must be colored 2 and 4 (one way or another). Thus, no color is left for the rung connecting P and Q and so proceeding with the color 1 from our coloring α where $\xi(X) = 3$ is forbidden. Now our

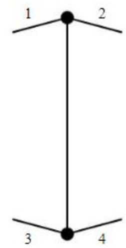
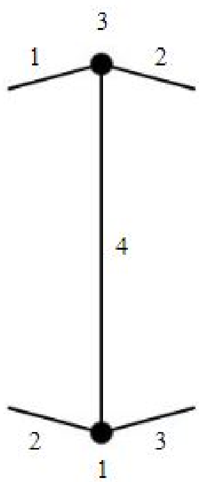
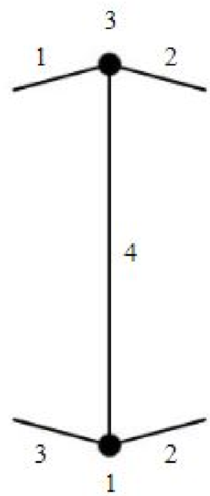


Figure 15: Using 4 colors on the upper and lower edges of subgraph I.



(a) Coloring α



(b) Coloring γ

Figure 16: The two unique colorings for our subgraph of M_n.

clockwise edge must be colored with 4 if $\xi(X) = 3$. This time, we need $\xi(j) = 1$, which requires $\xi(k) = 4$, so now we must have $\xi(l) = 3$. The result is displayed in Figure 20 where all colors forced by this decision are in bold.

But then P and Q must be colored with 1 and 2, leaving no color for the rung in between them. Therefore, we cannot proceed with the color 4 when $\xi(X) = 3$ and so $\xi(X) \neq 3$.

Suppose then that $\xi(X) = 4$. Now our clockwise edge can hold either 1 or 3 as its color. Using 1, we get the result in Figure 21. Hence, we need $\xi(j) = 3$, which forces $\xi(k) = 1$, which means we must have $\xi(l) = 4$. This can be seen in Figure 22 with the forced colors in bold. Since vertices P and Q must be colored with 2 and 3, we cannot color the rung between them and so we cannot proceed with the color 1. Let's proceed, then, with the color 3. We know that we must have $\xi(k) = 1$, which means that there is no possible value for $\xi(j)$. Hence, we cannot proceed with the color 3 and so $\xi(X) \neq 4$. But we said $\xi(X) \in \{3,4\}$ and showed that $\xi(X) \neq 3$ and $\xi(X) \neq 4$, a contradiction. Thus, the coloring α

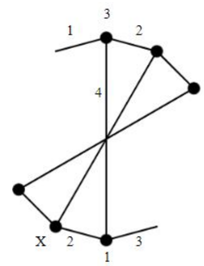


Figure 17: The α coloring of subgraph I within M_n.

Figure 18: Allowing $\xi(X) = 3$.

has no place in any 4-coloring of a Möbius ladder. $\square$

This means that for every subgraph I in M_n, each subgraph **must** be colored with the coloring γ. We can push this a little further, though. As was the case with subgraph I, we will now utilize a "subgraph I+", which itself is technically identical to subgraph I. However, an additional vertex is displayed moving forward, which is now relevant to the proof. Therefore, only the relevant parts of subgraph I+ will be shown, as was the case with subgraph I, though it will still be referenced as "subgraph I+".

Lemma 3.4. *Let the subgraph I+ be the subgraph I of* M_n *with an additional vertex,* p*, shown in Figure 24. There exists a unique coloring for subgraph I+.*

Proof. Apply the coloring γ to subgraph I+, demonstrated in Figure 24. Clearly, $\xi(p) \in \{2,4\}$. If $\xi(p) = 4$, we get the result presented in Figure 25. If $\xi(m) = 1$, then there is no allowable value for $\xi(j)$. If $\xi(m) = 2$, then we must have $\xi(j) = 1$, which leaves no allowable value for $\xi(k)$. Therefore, we must have $\xi(p) = 2$, which means that the coloring γ can be extended to a unique, more strict form which we will call Γ. $\square$

It turns out that this is enough to prove our next theorem.

Theorem 3.5. *The total chromatic number of the Möbius ladder is 5. That is,* $\chi^T(M_n) = 5$.

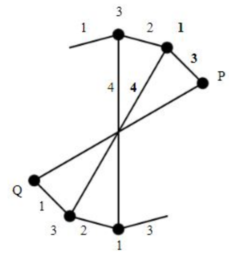


Figure 19: Proceeding with the color 1 when $\xi(X) = 3$.

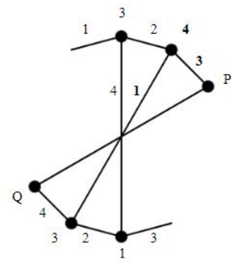


Figure 20: Proceeding with the color 4 when $\xi(X) = 3$.

Proof. M_n has a maximum degree of 3, so we know $\chi^T(M_n) \geq 4$. Suppose the Möbius ladder, M_n, is totally 4-colorable. Then, by **lemma 3.4**, the coloring Γ must be applied to every subgraph I+ within M_n. Consider the subgraph of the Möbius ladder in Figure 27. Note the positions of the color 3, which have been circled. These are the same positions that correspond to the boxes placed around the color 2 and around X. Hence, since the coloring Γ must be applied to *every* subgraph I+ in M_n, we must have $\xi(X) = 2$. But the edge X is already adjacent to an edge with color 2, a contradiction. Thus, the Möbius ladder is not totally 4-colorable and so $\chi^T(M_n) \geq 5$.

Moving forward, note that since there are $\frac{n}{2}$ copies of subgraph I in M_n, we get $\frac{n}{2}$ rungs, all of which can be assigned the color 5. Note also that there are n vertices and n outer edges in M_n as well. Consider the ladder in Figure 28. For the sake of simplicity, let $\xi(v_i) = \xi(e_i) = p$, where $i \equiv p \mod 4$ if $p \in \{1,2,3\}$ and $\xi(v_i) = \xi(e_i) = 4$ if $i \equiv 0 \mod 4$, where $i \in \{1,2,\ldots,\frac{n}{2}\}$. To fully examine this coloring, we must break things down into two cases.

First, suppose $8 \mid n$. Now, since n is even by the definition of the Möbius ladder, we know that $4 \mid \frac{n}{2}$. Furthermore, we know by definition that vertex v_i is adjacent to vertex $v_{i+\frac{n}{2}}$. But since $4 \mid \frac{n}{2}$, if we continued our coloring pattern for the first half of the cycle we would get that there exists a $q \in \{0,1,2,3\}$ such that $i \equiv q \mod 4$ and $i + \frac{n}{2} \equiv q \mod 4$. That is, $\xi(v_i) = \xi(v_{i+\frac{n}{2}})$, which we cannot allow. Instead, we can use the following

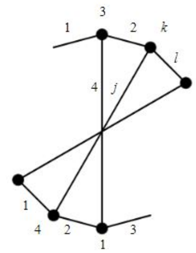


Figure 21: Proceeding with the color 1 when ξ(X) = 4.

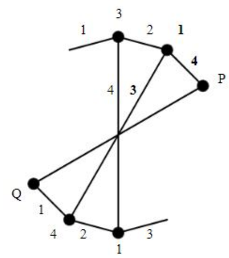


Figure 22: Proceeding with the color 1 when ξ(X) = 4.

algorithm for all $i \in \{\frac{n}{2}+1, \frac{n}{2}+2, ..., n\}$:

$$\xi(v_i) = \xi(e_i) = \begin{cases} 1 & \text{if } i \equiv 2 \mod 4 \\ 2 & \text{if } i \equiv 1 \mod 4 \\ 3 & \text{if } i \equiv 0 \mod 4 \\ 4 & \text{if } i \equiv 3 \mod 4 \end{cases}$$

In essence, we are flipping our 1's with our 2's and our 3's with our 4's on the second half of the cycle.

Now suppose $8 \nmid n$. There are two sub-cases now.

Take the case where $4 \mid n$ (while still requiring that $8 \nmid n$). This time, it is *not* the case that $4 \mid \frac{n}{2}$, which means that we can extend our pattern. That is, let $\xi(v_i) = \xi(e_i) = p$, where $i \equiv p \mod 4$, if $p \in \{1,2,3\}$ and $\xi(v_i) = \xi(e_i) = 4$ if $i \equiv 0 \mod 4$, where $i \in \{1,2,...,n\}$. This time, using our coloring pattern, vertices $\xi(v_i) \neq \xi(v_{i+\frac{n}{2}})$, so we have a successful 5-coloring. M_{12} is shown in Figure 30 as a visual example. Next, take the case where $4 \nmid n$. Let $v_i \in V(M_n) = \{v_1, v_2, ..., v_n\}$. Then there exists a $q \in \{0,1,2,3\}$ such that $i \equiv q \mod 4$. Since $4 \nmid n$ we know that $4 \nmid \frac{n}{2}$ which means it is not the case that $i + \frac{n}{2} \equiv q \mod 4$. Hence, $\xi(v_i) \neq \xi(v_{i+\frac{n}{2}})$ which means that we can extend our pattern, though only to a certain point. Let's take M_{14} as an example. Since our pattern repeats after every four vertices, we are really only concerned with how the "end" of the cycle connects to the first vertex we began to color. That is to say, we do not lose generality by choosing M_{14} as opposed

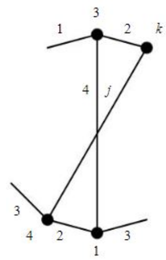


Figure 23: Proceeding with the color 3 when $\xi(X) = 4$.

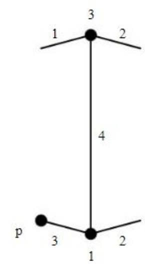


Figure 24: The coloring γ of subgraph I shown on subgraph I+.

to M_{274}. We can see now that our pattern cannot continue past the vertex v_{n-2} and the edge e_{n-2}. However, this can quickly be fixed. For clarity, we should acknowledge the extension of our coloring scheme first. That is, let $\xi(v_i) = \xi(e_i) = p$, where $i \equiv p \mod 4$, if $p \in \{1,2,3\}$ and $\xi(v_i) = \xi(e_i) = 4$ if $i \equiv 0 \mod 4$, where $i \in \{1,2,\ldots,n-2\}$. Now let $\xi(v_{n-1}) = 1$, $\xi(v_n) = 2$, $\xi(e_{n-1}) = 3$, and $\xi(e_n) = 4$. We are rewarded with a successful 5-coloring of M_{14}. The colors of the "end" vertices and edges, v_{n-1}, v_n, e_{n-1}, and e_n, are in bold for clarity in Figure 32.

Hence, we have a 5-coloring for all Möbius ladders so $\chi^T(M_n) = 5$. □

Corollary 3.6. *Each Möbius ladder can be equitably 5-colored, so the equitable total chromatic number of* M_n *is 5 for all even n. That is to say that* $\chi^T_=(M_n) = 5$.

Proof. Since $\chi^T(M_n) = 5$, we know that $\chi^T_=(M_n) \geq 5$. For this proof, we need only examine our coloring methods used to prove **theorem 3.5** a bit further. First, suppose $8 \mid n$. We already found an algorithm to color such Möbius ladders. This algorithm, though, actually yields an equitable 5-coloring. Since we have n vertices and n edges on the outer cycle and since the colors 1 through 4 are distributed evenly on the outer cycle, we have $\frac{n}{4}$ vertices with the color m and $\frac{n}{4}$ edges with the color m which means we used the color m a total of $\frac{n}{4} + \frac{n}{4} = \frac{n}{2}$ times for all $m \in \{1,2,3,4\}$. We also colored each rung with the color 5 and there are $\frac{n}{2}$ rungs, which means that each color was used $\frac{n}{2}$ times, giving us an equitable 5-coloring.

Now suppose $8 \nmid n$. If $4 \mid n$, our pattern uses each color 1, 2, 3, and 4 a total of $\frac{n}{2}$ times

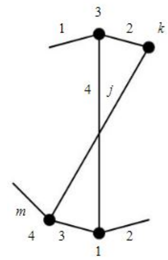


Figure 25: The coloring γ of subgraph I+ where $\xi(p) = 4$.

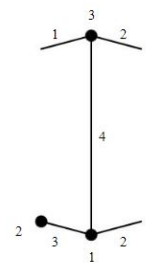


Figure 26: The coloring Γ, which must be applied to every subgraph I+ in M_n.

($\frac{n}{4}$ times each on the vertices and $\frac{n}{4}$ times each on the edges) and the color 5 on each of the $\frac{n}{2}$ rungs, giving us an equitable 5-coloring.

If $4 \nmid n$, we can use the same method outlined in the proof for **theorem 3.5** to color M_n where $4 \nmid n$. Once again, we have $\frac{n}{2}$ rungs, all with the color 5. Since we used our pattern for all vertices v_i and for all edges e_i where $i \in \{1,2,...,n-2\}$, we know that the color m was used $\frac{(n-2)+(n-2)}{4} = \frac{n}{2} - 1$ times for every $m \in \{1,2,3,4\}$, plus one additional time as a color for one of the "end" vertices or edges. That means that, in total, the color m was used $\frac{n}{2} - 1 + 1 = \frac{n}{2}$ times. Thus, all five colors were used $\frac{n}{2}$ times and we have an equitable 5-coloring, so $\chi^T_=(M_n) = 5$. □

4 Equitable, total 4-colorings of other 3-regular graphs

It wouldn't do to talk about 3-regular graphs without mentioning the Petersen graph. An equitable 4-coloring is shown in Figure 33. In 1995, E. S. Mahmoodian and M. A. Shokrollahi conjectured that the only uniquely total colorable graphs are paths, cycles, and empty graphs [7]. Hence, there is very likely another total 4-coloring of the Petersen graph. My claim is that any other total 4-coloring must also be equitable.

Proposition 4.1. *All total 4-colorings of the Petersen graph are equitable.*

Proof. With ten vertices and fifteen edges, any equitable total 4-coloring of the Petersen graph must utilize one color seven times and the remaining three must be used six

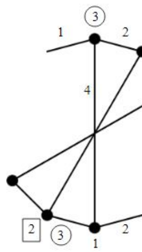


Figure 27: The coloring Γ on an extension of subgraph I+.

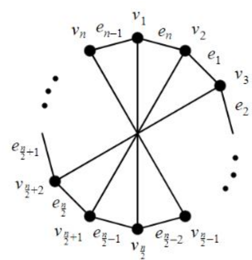


Figure 28: Labeling the vertices and edges of M_n.

times each. Therefore, to generate a non-equitable total coloring, either one color must be used eight times or two colors must be used seven times each, leaving one of the remaining colors to be used only five times.

First, we must consider on how many edges a single color can be used. Suppose we color four of the fifteen edges with the color 1. We know we cannot use the color 1 on the vertices incident with these edges, which means that there remain only two vertices out of the ten that can be colored with 1. Thus, if we want to explore the use of a single color seven or more times, that color can be used on at most three edges.

Now, we have to consider how many times a color can be used on the vertices of this graph. The Petersen graph contains two disjoint 5-cycles, one "inside" and one "outside" as it is arranged in Figure 33. Clearly, the same color can only be used at most two times on each 5-cycle which means that a single color can be used on at most four vertices. Hence, since a single color can be used on at most four vertices and on at most three edges, no color can be used more than seven times. This means that the only way to generate a non-equitable total 4-coloring for the Petersen graph would be to use two colors seven times each.

To exhaust this final plausible case, we need only investigate the placement of these colors on the vertices of the graph. We know each color must be used on four vertices each. We will use the colors 1 and 2 to explore this possibility. As I mentioned above, the same color can only be used twice on the "inner" 5-cycle and twice on the "outer" 5-cycle of this arrangement of the Petersen graph. Thus, the only unique way to use a

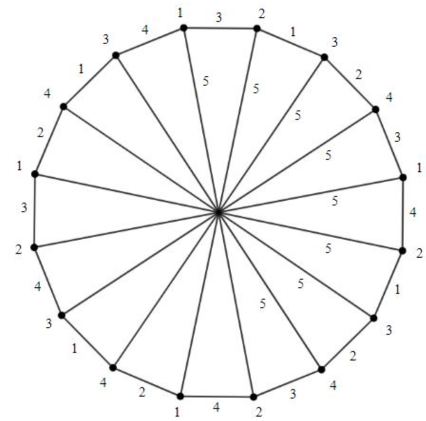


Figure 29: M_{16}, colored using our algorithm, provided as a visual example.

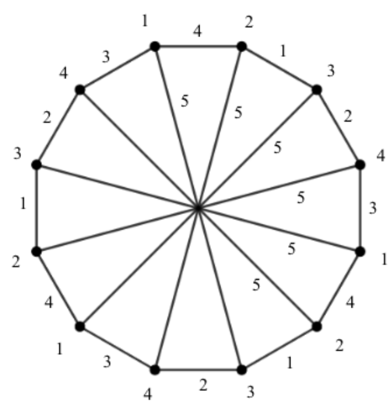


Figure 30: M_{12} provided as a visual example.

color on four vertices can be seen in Figure 34.

Now we can only use the color 2 on vertices k **or** m, on vertices j **or** n, and on vertices p **or** q. This means that the color 2 can be used on at most three vertices and, hence, cannot be used seven times in total. Therefore, two colors cannot both be used seven times on the Petersen graph and no color can be used more than seven times, either. This means that the only way to totally 4-color the Petersen graph is by using one color seven times and the other colors six times each. Thus, every total 4-coloring of the Petersen graph must be equitable. □

When 4-coloring the Petersen graph and the bird's nest, the main goal was to obtain a 4-coloring and to then go back and see if an equitable 4-coloring existed. However, each successful 4-coloring was inherently equitable. Initially, this lead me to believe that all total 4-colorings of 3-regular graphs would be equitable. However, Dantas et al. (2016) found a 3-regular graph of order 20 whose total chromatic number is 4 and

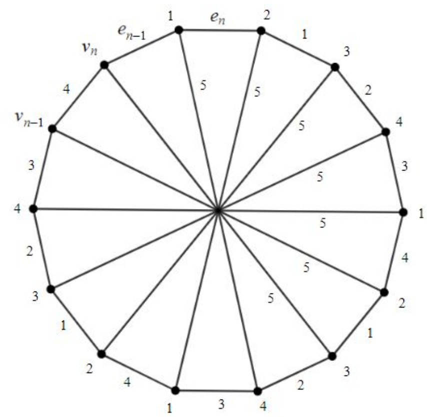


Figure 31: M_{14} provided as a visual example.

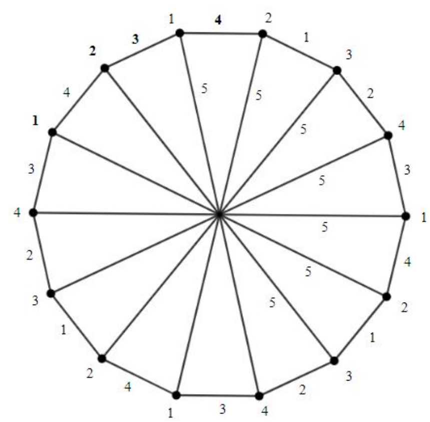


Figure 32: The 5-coloring of M_{14}.

whose equitable total chromatic number is 5 [3]. Due to the nature of how this graph was constructed, though, I strongly believe that this is the smallest (referring to the graph's order) example of this type of graph.

Conjecture 4.2. Every total 4-coloring of a 3-regular graph, G is equitable, given that the order of G is less than 20.

There seems to be something about the "crowded" nature of these 3-regular graphs that forces equitability in all 4-colorings for graphs that are indeed able to be 4-colored. It is as though the choices for the color of each vertex and of each edge become forced very quickly as one moves through the process of assigning colors. The main reason that this conjecture is suspected to be true is as was stated above that, when 4-coloring these 3-regular graphs, no effort toward equitable coloring was made. Rather, it appeared as a characteristic of each 4-coloring of each 3-regular graph.

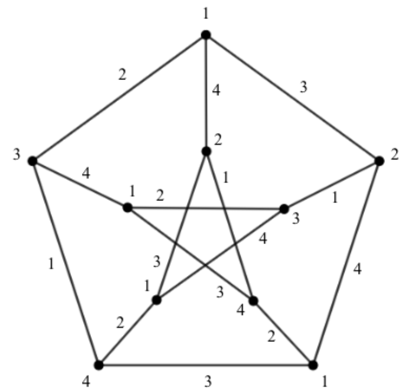


Figure 33: An equitable 4-coloring of the Petersen graph.

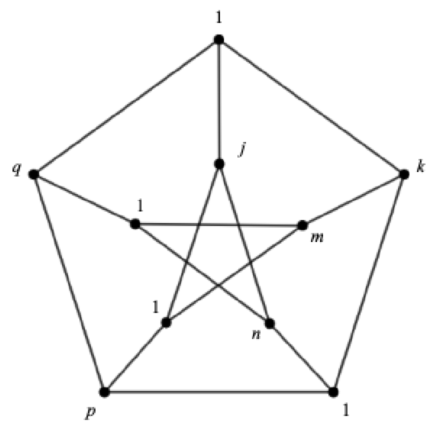


Figure 34: Using the same color on four vertices of the Petersen graph.

5 Equitable, total $(k+1)$-colorings of k-regular graphs

While in the preliminary stages of researching this topic, I came across Koester's graph [6], the 4-regular graph displayed in Figure 35. My first instinct was, of course, to attempt to totally 5-color it, which I did. You can find this total 5-coloring in the appendix. As you will see, it is a bit too busy to properly represent without enlarging it to the size of a full page. Upon finishing, I noticed that even though I wasn't trying to equitably color the graph, the coloring was equitable anyways (all five colors were used twenty-four times each).

As I mentioned in the discussion of **conjecture 4.2**, a 3-regular graph with 20 vertices was presented by Dantas et al. (2016) whose total chromatic number is 4 but whose equitable total chromatic number is 5. Furthermore, in 1994 Hung-Lin found a 6-regular graph on 11 vertices with a total chromatic number of 7 but no equitable total 7-coloring.

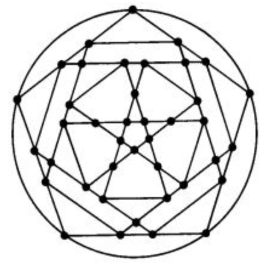


Figure 35: The 4-regular Koester's graph.

It is my belief that for each k, there exists some upper bound on the order of a k-regular graph such that below this bound, every total $(k+1)$-coloring must be equitable.

Conjecture 5.1. Every total $(k+1)$-coloring of a k-regular graph, G, is equitable given some bound on the order of G.

The main characteristic of these k-regular graphs that leads me to this conclusion is how quickly one is forced into using certain colors in certain positions. For example, take the proof of **lemma 3.3**. Notice how few choices are required to ascertain each contradiction. In addition, for odd n, K_n is $(n-1)$-regular, totally n-colorable, and all such total n-colorings are equitable. It's a bit like dominoes, really. Once the first one or two go down, the rest are quick to follow. It is possible that higher graph orders introduce a certain amount of chaos into graphs which allow the freedom to obtain non-equitable total $(k+1)$-colorings for k-regular graphs.

6 Conclusions and Future Research

In summary, we have found the total chromatic numbers of the Möbius ladder and of the bird's nest to be 5 and 4, respectively, for all such graphs with more than four vertices. This last stipulation is, of course, because any 3-regular graph with 4 vertices is isomorphic to the complete graph with four vertices, K_4. In addition, we have proposed that every total $(k+1)$-coloring of every k-regular graph is inherently equitable for all k given some upper bound on the order of the graph.

Future research may include total 5-colorings of the various 3-regular graphs here, including discussions on the presence of vertex and edge equitability "inside" of equitable total 5-colorings. A new concept called "set equitability" is also under construction, which may be used alongside the classic vertex, edge, and total equitability definitions. Lastly, the upper bound on graph order has yet to be discovered for 4-regular and 5-regular graphs among others. More research is required to discover these bounds and to test those already in place.

7 Appendix

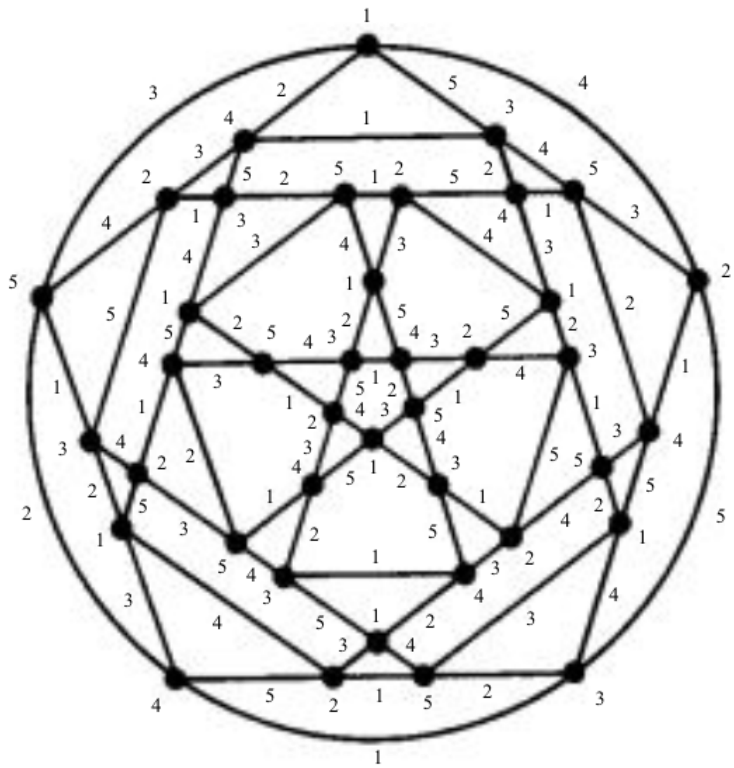


Figure 36: The equitable total 5-coloring of the 4-regular Koester's graph.

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