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Investigating First Returns: The Effect of Multicolored Vectors

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Cover Page Footnote

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Investigating First Returns: The Effect of Multicolored Vectors

By *Shakuan Frankson* and *Myka Terry*

Abstract. By definition, a first return is the immediate moment that a path, using vectors in the Cartesian plane, touches the x-axis after leaving it previously from a given point; the initial point is often the origin. In this case, using certain diagonal and horizontal vectors while restricting the movements to the first quadrant will cause almost every first return to end at the point $(2n,0)$, where $2n$ counts the equal number of up and down steps in a path. Using the first returns of Catalan, Schröder, and Motzkin numbers, which resulted from the lattice paths formed using a combination of diagonal and/or horizontal vectors, we then investigated the effect that coloring select vectors will have on each of the original generating functions.

1 Introduction

The main focus of the paper is to analyze the effect of multicoloring certain vectors used to produce the Catalan, Schröder, and Motzkin paths. In the process of multicoloring the vectors, we set an arbitrary variable a to represent the number of colors assigned to each vector, where each color will stand as an option for a path to take. Further explanation will be provided in Section 3.

Before going into direct detail about the first returns of the Catalan, Schröder, and Motzkin paths, a synopsis will also be provided below to give additional background.

1.1 Catalan Numbers

The original Dyck path uses the vectors $(1,1)$, referred to as the up vector, and $(1,-1)$, referred to as the down vector, remaining above the x-axis and extending from $(0,0)$ to $(2n,0)$, where n now represents the semi-length. The Catalan numbers, whose generating function is $\mathrm{C}(x)$, are the values enumerated along the bottom row of the x-axis using those vectors. The following vectors are visually represented in the lattice path as:

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Below is the graph of the first return of the Catalan path using the vectors provided:

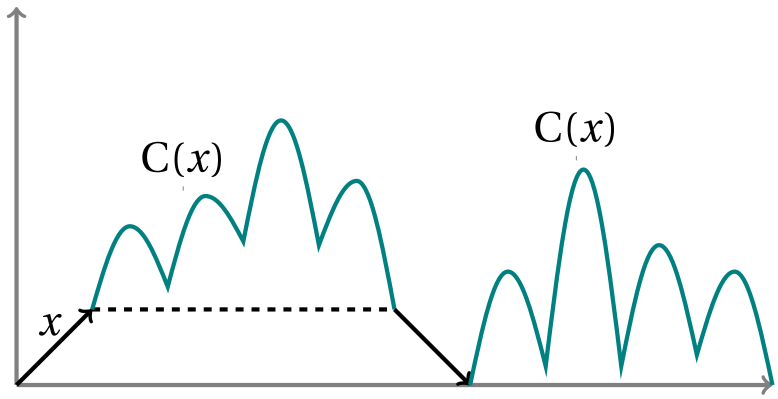


Figure 1: Example of First Return - $C(x)$

To produce the first return for $C(x)$, movement from the origin begins with two options: (1) remaining at the origin and not using the provided vectors, or (2) following the $(1,1)$ up vector, where we attain an x, continuing along a random path of $C(x)$, and returning to the x-axis using the $(1,-1)$ down vector, ending with an arbitrary infinite path of $C(x)$. There is no x on the down vector since we don't want to aerate $C(x)$, i.e. add zeroes between the terms of the sequence corresponding to the generating function $C(x)$. This notion will be more evident in the first equation. From the movements of the first return, we note that option (1) has the resulting value of 1 since there is only one way to remain at the origin, and option (2) has a value of $xC(x)\cdot C(x)$. Combining the two possibilities, the first full return is notated as the following:

$$C(x) = 1 + xC(x)\cdot C(x) = 1 + xC^2(x)$$

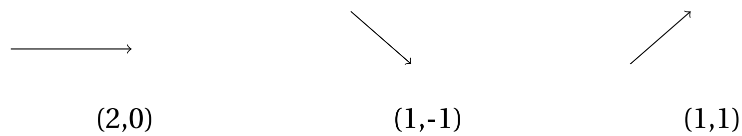
Algebraically, the first return of $C(x)$ is a generalization of the possible movements using the given up and down vectors. The number of possible paths to a point $(2n,0)$, $n \geq 0$, on the x-axis is a term of the sequence $C(x)$.

The generating function for $C(x)$ follows from the quadratic formula and yields Equation 1:

$$C(x) = \frac{1-\sqrt{1-4x}}{2x} \tag{1}$$

1.2 Schröder Numbers

Schröder numbers follow a similar pattern to the Catalan numbers while also including the horizontal vector, $(2,0)$. The path is referred to as a Schröder path. There are two distinct types of Schröder numbers seen in combinatorics: large Schröder numbers, with generating function denoted by $\mathrm{S}(x)$, and small Schröder numbers, with generating function denoted by $s(x)$. The following vectors are visually represented in the lattice path as:



These vectors create unique Schröder paths for the large Schröder and small Schröder numbers, respectively. The sole distinction between them is that $s(x)$ does not allow the $(2,0)$ along the x-axis, while $\mathrm{S}(x)$ does.

Below is the graph of the first return of the large Schröder path using the vectors provided:

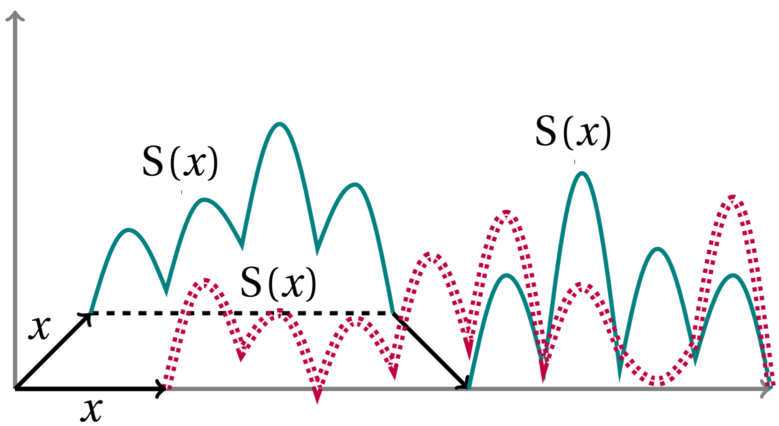


Figure 2: Example of First Return - $\mathrm{S}(x)$

To produce the first return for $\mathrm{S}(x)$, movement from the origin begins with 3 options: (1) remaining at the origin and not using the provided vectors, (2) following the $(1,1)$ up vector, where we attain an x, continuing along a random path of $\mathrm{S}(x)$, and returning to the x-axis using the $(1,-1)$ down vector, ending with an arbitrary infinite path of $\mathrm{S}(x)$, or (3) following the $(2,0)$ horizontal vector, where we attain an x, and all of the possibilities of $\mathrm{S}(x)$. Similar to the first return of $\mathrm{C}(x)$, there is no x on the down vector since we don't want to aerate $S(x)$. Also, once the path returns back to the x-axis, we want the powers of x to match regardless of when we use the given up, down, or horizontal vectors. In other words, since the horizontal vector is $(2,0)$ for large Schröder paths, the x for 1 horizontal step will bring you to the same point on the x-axis as using an up step and then a down step, which will also produce x. Thus, from the movements of the first return, we note that option (1) has the resulting value of 1 since there is only one way to remain at the origin, option (2) has a value of $xS(x) \cdot S(x)$, and option (3) has the value of $xS(x)$. Combining the 3 possibilities, the first full return is notated as the following:

$$S(x) = 1 + xS(x) + xS(x)S(x)$$

The generating function follows from the quadratic formula and yields Equation 2:

$$S(x) = \frac{1 - x - \sqrt{1 - 6x + x^2}}{2x} \tag{2}$$

Since a small Schröder path does not include the horizontal vector, $(2,0)$, the first return appears similar to the first return of the Catalan path. Below is the graph for the first return of small Schröder paths:

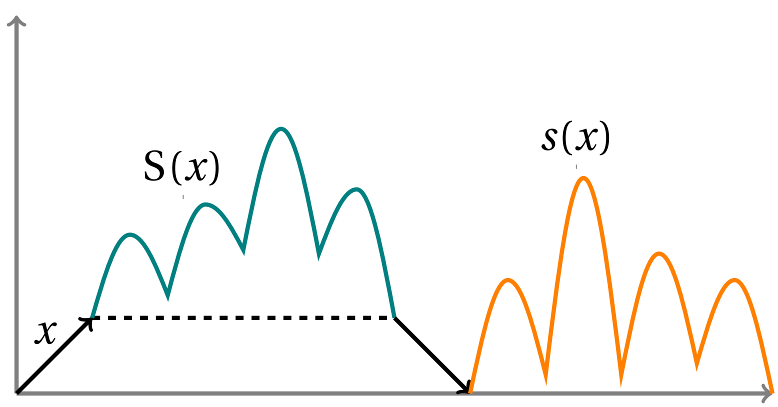


Figure 3: Example of First Return - $s(x)$

The small Schröder numbers have a slightly different pattern than the large Schröder numbers because they do not have a horizontal vector on the x-axis, as it affects the first return and therefore its generating function.

To produce the first return for $s(x)$, movement from the origin begins with two options: (1) remaining at the origin and not using the provided vectors, or (2) following the $(1,1)$ up vector, where we attain an x, continuing along a random path of $S(x)$, and returning to the x-axis using the $(1,-1)$ down vector, ending then with an arbitrary infinite path of $s(x)$. From the movements of the first return, we note that option (1) has the resulting value of 1 since there is only one way to remain at the origin, and option (2) has a value of $xS(x) \cdot s(x)$. Combining these possibilities, the first full return of $s(x)$ is notated as the following:

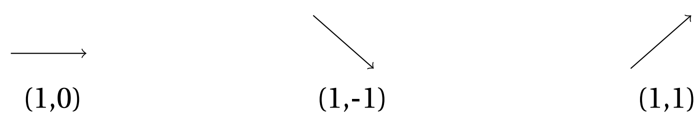
$$s(x) = 1 + xS(x) \cdot s(x)$$

Substituting the generating function of $S(x)$ into the equation above and simplifying, we get the generating function for the small Schröder numbers, denoted as Equation 3:

$$s(x) = \frac{1 + x - \sqrt{1 - 6x + x^2}}{4x} \tag{3}$$

1.3 Motzkin Numbers

Motzkin paths, denoted by the generating function $M(x)$, are similar in nature to the previous paths, but the Motzkin numbers have the smaller horizontal vector of $(1,0)$ and remain above the x-axis from $(0,0)$ to $(n,0)$. The following vectors are visually represented in the lattice path as:



Because the length of the horizontal vector is only half that of the $(2,0)$ horizontal vector found in the Schröder numbers, the terms of the sequence $M(x)$ increase rapidly. For example, the 9th term of $S(x)$ is 45 in comparison to 323 for $M(x)$. Even though there are twice as many horizontal vectors in Motzkin paths as in the Schröder paths, it is possible to use an odd number of these vectors to get a term of the sequence $M(x)$ (on the x-axis).

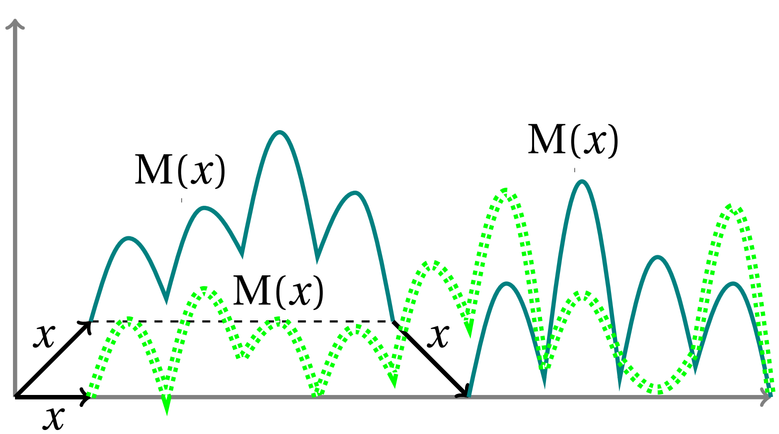


Figure 4: Example of First Return - $\mathrm{M}(x)$

To produce the first return for $\mathrm{M}(x)$, movement from the origin begins with 3 options: (1) remaining at the origin and not using the provided vectors, (2) following the $(1,1)$ up vector, where we attain an x, continuing along a random path of $\mathrm{M}(x)$, and returning to the x-axis using the $(1,-1)$ down vector, where we attain another x, ending with an arbitrary infinite path of $\mathrm{M}(x)$, or (3) following the $(1,0)$ horizontal vector, where we attain an x, and all of the possibilities of $\mathrm{M}(x)$. Unlike the first returns of $\mathrm{C}(x)$ and $\mathrm{S}(x)$, there is an x on the down vector since we want the powers of x to match regardless of when we use the given up, down, or horizontal vectors. In other words, since the horizontal vector is $(1,0)$ for Motzkin paths, the $x \cdot x$ for 2 horizontal steps will bring you to the same point on the x-axis as using an up step and then a down step, which will also produce $x \cdot x$. Thus, from the movements of the first return, we note that option (1) has the resulting value of 1 since there is only one way to remain at the origin, option (2) has a value of $x^2\mathrm{M}(x) \cdot \mathrm{M}(x)$, and option (3) has the value of $x\mathrm{M}(x)$. Combining these three possibilities, the movements of $\mathrm{M}(x)$ based on the first movement yield the equation:

$$\mathrm{M}(x) = 1 + x\mathrm{M}(x) + x^2\mathrm{M}^2(x)$$

$\mathrm{M}(x)$ can be subtracted from both sides, resulting in the equation $0 = 1 + (x-1)\mathrm{M}(x) + x^2\mathrm{M}^2(x)$ which can be solved using the quadratic formula, yielding Equation 4:

$$\mathrm{M}(x) = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2} \tag{4}$$

2 Effect of Multicolored Vectors on First Returns

The concept of multicoloring vectors was inspired by a paper called *Zebra Trees* written by Davenport, Shapiro, and Woodson, see [2]. Zebra trees are ordered trees that have a choice of 2 colors, black and white, at each edge of odd height, and the coloring of each edge affects the generating function.

Expanding off of their research, we decided to investigate the effect of assigning a colors to the down vectors and b colors to the horizontal vectors for each of the lattice paths mentioned in Section 1. Catalan, Schröder, and Motzkin numbers have generating functions that also represent the presence of one color at each vector, which also counts as one choice for a path to take during a return. By finding the generalized generating functions for a-colored down vectors, b-colored horizontal vectors, or a combination of both, we will find an efficient method for quickly calculating the values produced by the lattice paths without counting out each individual path one-by-one.

2.1 Catalan Numbers

According to Section 1, the first return for a Catalan path is represented by:

$$C(x) = 1 + 1 \cdot xC^2(x)$$

where the x is attained from the up vector, and 1 (and hence $1 \cdot x = x$) indicates that there is one color assigned to the down vector.

Since the Catalan path only uses the $(1,1)$ up vector and the $(1,-1)$ down vector, then we are only able to find the generating function for $C_a(x)$ or C_a, where C_a denotes the Catalan path with a-colored $(1,-1)$ down vectors. The path for C_a will be represented by:

$$C_a = 1 + xC_a \cdot aC_a = 1 + ax\, C_a^2 \tag{5}$$

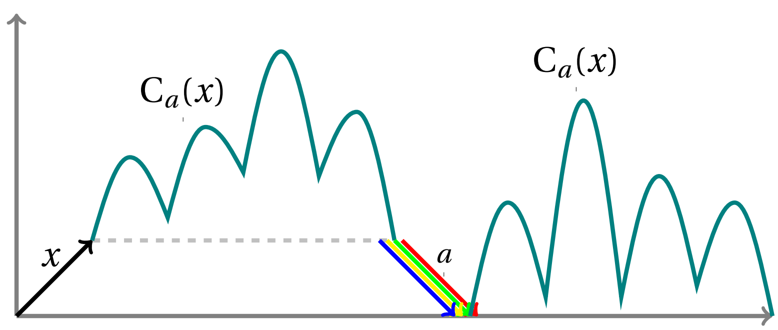


Figure 5: Example of First Return - $C_a(x)$

where, now, the x is attained from the up vector, and a (and hence $a \cdot x = ax$) represents that there are a colors assigned to the down vector.

After using the quadratic formula, the generating function yielded for C_a is:

$$C_a = \frac{1 - \sqrt{1 - 4ax}}{2ax} \tag{6}$$

The generating function for the Catalan path (when $a = 1$) is stated below for comparison purposes:

$$C(x) = \frac{1 - \sqrt{1 - 4x}}{2x} \tag{1}$$

2.2 Large Schröder Numbers

According to Section 1, the first return for a large Schröder path is represented by:

$$S(x) = 1 + xS(x) + xS^2(x)$$

Large Schröder paths use the $(1,1)$ up vector, $(1,-1)$ down vector, and $(2,0)$ horizontal vector. Hence, we are able to find the generating functions for $S_a(x)$ or S_a, where S_a denotes the large Schröder path with a-colored $(1,-1)$ down vectors, and $S_b(x)$ or S_b, where S_b denotes the large Schröder path with b-colored $(2,0)$ horizontal vectors. The path for S_a will be represented by:

$$S_a = 1 + xS_a + xS_a \cdot aS_a = 1 + xS_a + ax\,S_a^2 \tag{7}$$

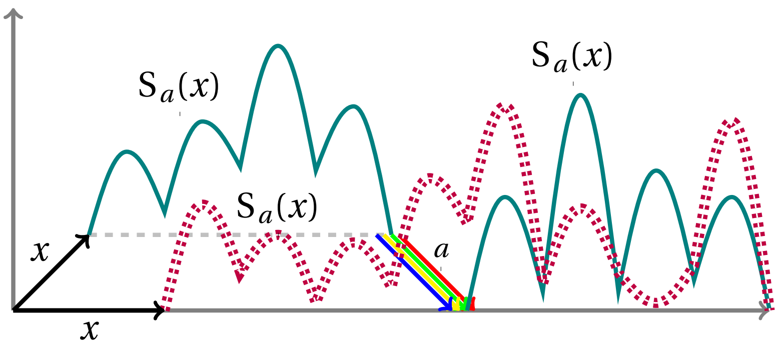


Figure 6: Example of First Return - $S_a(x)$

and the path for S_b will be represented by:

$$S_b = 1 + bxS_b + x\,S_b^2 \tag{8}$$

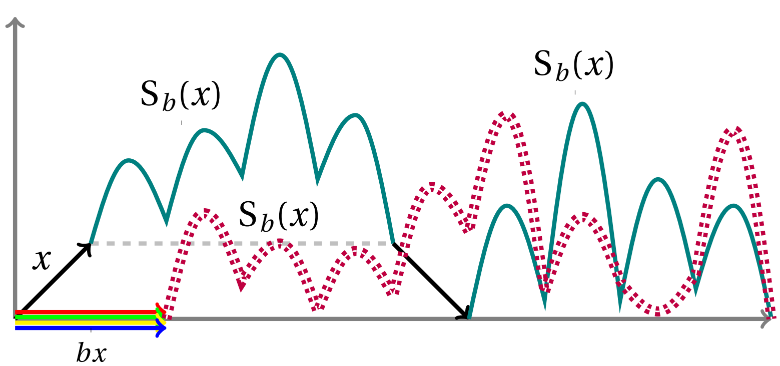


Figure 7: Example of First Return - $S_b(x)$

The path $S_{a,b}$, which denotes the large Schröder path with both a-colored down vectors and b-colored horizontal vectors, will be represented by:

$$S_{a,b} = 1 + bx\, S_{a,b} + ax\, S^2_{a,b} \tag{9}$$

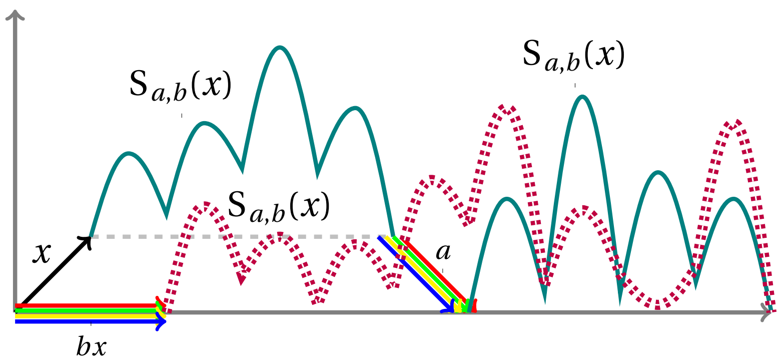


Figure 8: Example of First Return - $S_{a,b}(x)$

After using the quadratic formula, the respective generating functions yielded for S_a, S_b, and $S_{a,b}$ are:

$$S_a = \frac{1 - x - \sqrt{1 - 2x(1+2a) + x^2}}{2ax} \tag{10}$$

$$S_b = \frac{1 - bx - \sqrt{1 - 2x(b+2) + x^2 b^2}}{2x} \tag{11}$$

$$S_{a,b} = \frac{1 - bx - \sqrt{1 - 2x(b+2a) + x^2 b^2}}{2ax} \tag{12}$$

The generating function for the large Schröder path is stated below for comparison purposes:

$$S(x) = \frac{1 - x - \sqrt{1 - 6x + x^2}}{2x} \tag{2}$$

2.3 Small Schröder Numbers

According to Section 1, the first return for a small Schröder path is represented by:

$$s(x) = 1 + xs(x) \cdot \mathrm{S}(x)$$

The small Schröder path uses the $(1,1)$ up vector, $(1,-1)$ down vector, and the $(2,0)$ horizontal vector, but there are no horizontal vectors on the x-axis. Thus, we are only able to find the generating function for $s_a(x)$ or s_a, where s_a denotes the small Schröder path with a-colored $(1,-1)$ down vectors. The path for s_a will be represented by:

$$s_a = 1 + x\mathrm{S}_a \cdot as_a = 1 + ax\mathrm{S}_a s_a \tag{13}$$

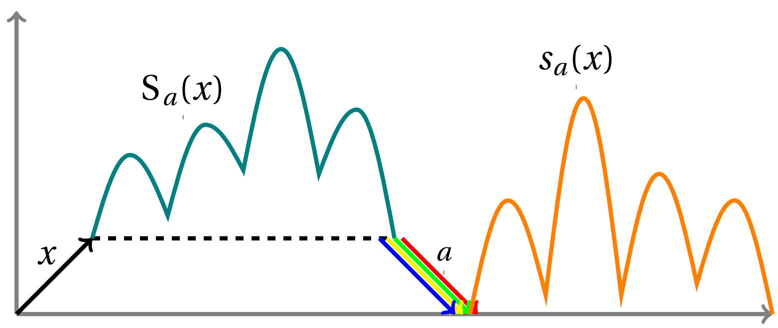


Figure 9: Example of First Return - $s_a(x)$

After substituting in the generating function for S_a in (13), we get:

$$s_a = 1 + s_a \cdot ax\left(\frac{1-x-\sqrt{1-2x(1+2a)+x^2}}{2ax}\right) = 1 + s_a\left(\frac{1-x-\sqrt{1-2x(1+2a)+x^2}}{2}\right)$$

$$\implies -1 = s_a\left(\frac{-1-x-\sqrt{1-2x(1+2a)+x^2}}{2}\right) \implies s_a = \left(\frac{-2}{-1-x-\sqrt{1-2x(1+2a)+x^2}}\right)$$

$$s_a = \frac{-2(-1-x+\sqrt{1-2x(1+2a)+x^2})}{2x(2+2a)}$$

Thus, the generating function yielded for s_a is:

$$s_a = \frac{1+x-\sqrt{1-2x(1+2a)+x^2}}{2x(1+a)} \tag{14}$$

The generating function for the small Schröder path is stated below for comparison purposes:

$$s(x) = \frac{1+x-\sqrt{1-6x+x^2}}{4x} \tag{3}$$

2.4 Motzkin Numbers

According to Section 1, the first return for a Motzkin path is represented by:

$$M(x) = 1 + xM(x) + x^2M^2(x)$$

Motzkin paths use the $(1,1)$ up vector, $(1,-1)$ down vector, and $(1,0)$ horizontal vector. Hence, we are able to find the generating functions for $M_a(x)$ or M_a, where M_a denotes the large Schröder path with a-colored $(1,-1)$ down vectors, and $M_b(x)$ or M_b, where M_b denotes the large Schröder path with b-colored $(1,0)$ horizontal vectors. The path for M_a will be represented by:

$$M_a = 1 + xM_a + xM_a \cdot axM_a = 1 + xM_a + ax^2M_a^2 \tag{15}$$

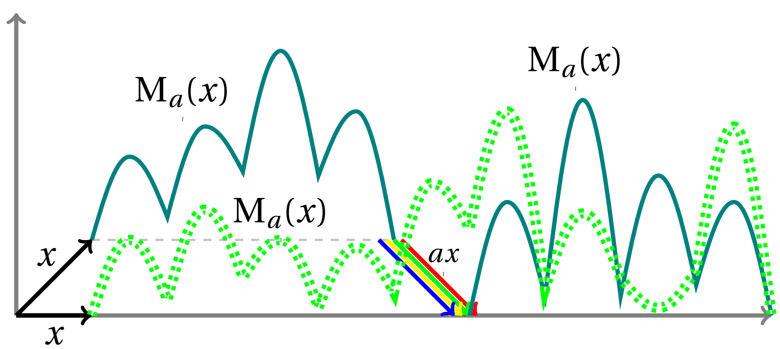


Figure 10: Example of First Return - $M_a(x)$

and the path for M_b will be represented by:

$$M_b = 1 + bxM_b + x^2\, M_b^2 \tag{16}$$

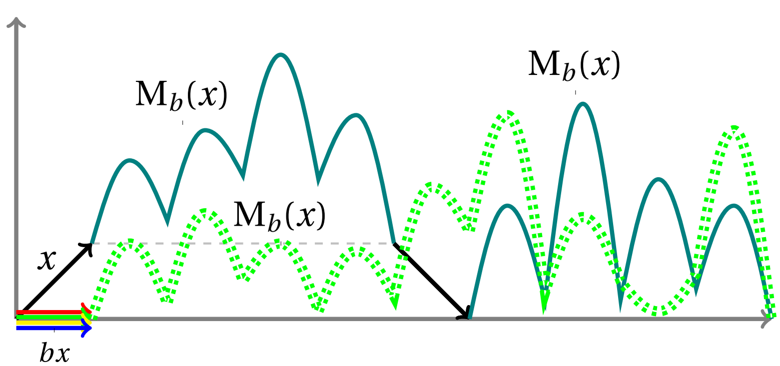


Figure 11: Example of First Return - $\mathrm{M}_b(x)$

The path $\mathrm{M}_{a,b}$, which denotes the Motzkin path with both a-colored down vectors and b-colored horizontal vectors, will be represented by:

$$\mathrm{M}_{a,b} = 1 + bx\,\mathrm{M}_{a,b} + ax^2\,\mathrm{M}^2_{a,b} \tag{17}$$

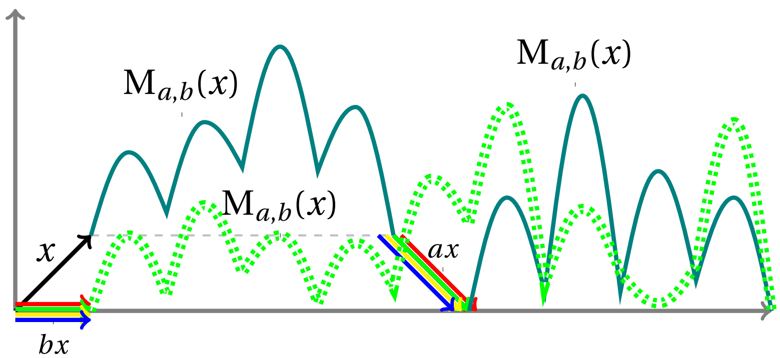


Figure 12: Example of First Return - $\mathrm{M}_{a,b}(x)$

After using the quadratic formula, the respective generating functions yielded for M_a, M_b, and $\mathrm{M}_{a,b}$ are:

$$\mathrm{M}_a = \frac{1 - x - \sqrt{1 - 2x + x^2(1-4a)}}{2ax^2} \tag{18}$$

$$\mathrm{M}_b = \frac{1 - bx - \sqrt{1 - 2bx + x^2(b^2-4)}}{2x^2} \tag{19}$$

$$\mathrm{M}_{a,b} = \frac{1 - bx - \sqrt{1 - 2bx + x^2(b^2-4a)}}{2ax^2} \tag{20}$$

The generating function for the Motzkin path is stated below for comparison purposes:

$$\mathrm{M}(x) = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2} \tag{4}$$

3 Conclusion

From the technical paper from our SPIRAL (Summer Program in Research and Learning) REU project, we found that there was a relationship found between different random walks and Pascal's Triangle. From investigating first returns and generating functions, we discovered patterns that led us to more questions, which we were able to answer in this paper. An open question that remains is how to make our results applicable to real-world problems.

4 Acknowledgements

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