9-25-2020

Topological and H^q Equivalence of Cyclic n-gonal Actions on Riemann Surfaces - Part II

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September 25, 2020

Abstract

In the first paper in this two-part series we related the generating vectors of topological equivalence classes of conformal actions of a prime order cyclic group G, on compact Riemann surfaces S, to the characters of the representations of G on the spaces of holomorphic q-differentials, $\mathcal{H}^q(S)$. In turn, following the work of I. Guerrero, *Holomorphic families of compact Riemann surfaces with automorphisms*, characters of the representations on the spaces of holomorphic q-differentials can be related to the system of local rotations of the group action of G. In this follow-up paper we connect the work of Guerrero and the results of the first paper by describing in detail the rotation data system of a group action. We relate the rotation data to generating vectors and the characters of G on the spaces of holomorphic q-differentials. Though we work with a general finite group action as much as possible, our primary examples and results are about cyclic n-gonal actions.

Contents

1 Introduction

1.1 Motivation and overview

In this paper, as in its predecessor [7], we consider conformal actions of a finite group G on various compact Riemann surfaces S. This note is a follow-up to [7], taking into account the work of Guerrero [14] on the system of local rotations at the fixed points of a G action (*rotation data system*) and the representations of G on $\mathcal{H}^q(S)$, the space of q-differentials on S. In [7], the focus was on prime cyclic n-gonal actions and the comparison of topological equivalence of actions and the weaker equivalence relation derived from the representation of G on q-differentials.

The equivalence relation defined by Abelian differentials, $q = 1$, has been used to classify actions in low genus by some authors. However, this equivalence relation fails to distinguish topologically distinct actions in some cases; the smallest example is $G = \mathbb{Z}_5$ in genus 4. We call this phenomenon *conflation* of actions, a term introduced in [7]. Therefore, in [7], we considered which degrees of differentials, if any, can distinguish topologically distinct actions. Topological equivalence of actions is important since the action classes are directly linked to finite subgroups of the mapping class group in a 1-1 fashion, and strongly linked (though not 1-1) to the equisymmetric strata of the moduli space of surfaces of a given genus. See the parent paper [7] and [4] for more details.

In [14], Guerrero claims that the representations of G on q-differentials for $q = 1, 2$ contain enough information to determine the rotation data system of an action (definition in Section 2.2). In turn, the rotation data is sufficient to determine the topological equivalence class of an action, at least in the cyclic n-gonal case (Proposition 50). One of the main goals of this paper is to establish Guerrero's results in a computationally direct manner using a transition matrix. However, the development in our paper shows that his claim is not correct, the smallest counterexample occurs for $G = \mathbb{Z}_{16}$. A revised statement of Guerrero's claim and proof of the revised claim is given in Section 5.

Guerrero's main work is to introduce holomorphic families of surfaces with automorphisms. He proves that the characters of representations of G on q-differentials in a family with a connected base are constant. This result can be extended to the system of local rotations of a G action. Following up on this idea, we shall introduce another equivalence relation, *equisymmetry*, based on families, which is strongly linked to systems of local rotations.

An additional value of considering the system of local rotations is that it can be easily defined in the positive characteristic case and can be used as a classifying attribute for equisymmetry, a form of "topological equivalence" of actions, in positive characteristic (see Sections 1.6, 2.3, and 6). Though we will not make a formal effort to develop the positive characteristic case, we shall make a series of remarks on how one might extend concepts and results to the positive characteristic case, and give a few examples. We also make some suggestions for further work in the positive characteristic case in

Section 6.

Here is a brief overview of the paper. In the remainder of this section we recall the basics of a conformal action of a finite group G on a compact Riemann surface S, generating vectors for actions, the representations of G on $\mathcal{H}^q(S)$, and holomorphic families of actions, the subject of Guerrero's work. In Section 2 we carry out a detailed analysis of the local rotation constants of G, introducing the rotation data system of a G action. In Section 3 we link up rotation data systems and generating vectors, in Section 4 we examine the link rotation data systems $\rightarrow \mathcal{H}^q$ characters, and in Section 5 we analyze $\mathcal{H}^q$ conflation and explain how to recover rotation data systems from $\mathcal{H}^q$ characters. Finally, in Section 6 we discuss some ideas for further work.

We adopt all the notation and concepts in [7]. Typically, S will denote a compact, Riemann surface. We also allow for the possibility that S is a complete, smooth, irreducible curve (1-dimensional variety) over any algebraically closed field $\mathbf{k}$. In particular, a compact Riemann surface is a smooth curve over $\mathbb{C}$. A *variety* V is an algebraic set defined over $\mathbf{k}$. If we need to assume that V is irreducible we shall explicitly say so. Also, for completeness of this paper we repeat some of the most basic ideas from [7] in the following sub-sections.

Throughout, we use the cyclic n-gonal case as a clarifying example. It is the simplest and most studied case with an explicit equation. See Examples 8, 21, 46, and 47. All computer calculations were done with MAGMA [23].

1.2 Group actions and generating vectors

We are interested in constructing and analyzing conformal group actions on Riemann surfaces up to conformal or topological equivalence. In this subsection we define conformal actions and their generating vectors, introduce cyclic n-gonal actions, our primary example, in subsection 1.3, and then discuss topological and conformal equivalence of actions in subsection 1.4.

Definition 1 *The finite group G* acts (conformally) *on the Riemann surface S through a given monomorphism:*

$$\epsilon : G \rightarrow \mathrm{Aut}(S).$$

The subgroup $\epsilon(G)$ is called the image subgroup *or* action image.

To construct and analyze G actions, we shall introduce and use branched covers, monodromy epimorphisms and generating vectors, as follows. The quotient surface $S/G = T$ is a compact Riemann surface of genus τ with a unique conformal structure so that

$$\pi_\epsilon = \pi_G : S \to S/G = T \tag{1}$$

is holomorphic. (We use π_ϵ if we need to be specific about the action.) The quotient map $\pi_G : S \to T$ is ramified uniformly over a finite set $B_G = \{Q_1, \ldots, Q_t\}$ such that π_G is an unramified covering exactly over $T^\circ = T - B_G$. Let $S^\circ = \pi_G^{-1}(T^\circ)$ so that $\pi_G : S^\circ \to T^\circ$ is an unramified covering space whose group of deck transformations, $\mathrm{Gal}(S^\circ/T^\circ)$, equals $\epsilon(G)$ restricted to S°. The covering $S^\circ \to T^\circ$ determines a normal subgroup $\Pi_G = \pi_1(S^\circ) \lhd \pi_1(T^\circ)$ and an exact sequence $\Pi_G \hookrightarrow \pi_1(T^\circ) \twoheadrightarrow \epsilon(G)$ by mapping loops to deck transformations. Composing the last map with $\epsilon(G) \overset{\epsilon^{-1}}{\to} G$ we get an exact sequence

$$\Pi_\xi = \Pi_G \hookrightarrow \pi_1(T^\circ) \overset{\xi}{\twoheadrightarrow} G, \tag{2}$$

which we will call a *(regular) monodromy epimorphism* (Again, we use Π_ξ if we need to be specific about the epimorphism.)

Remark 2 *We have left out base points to simplify the exposition, and so ξ is ambiguous up to inner automorphisms, but this is inconsequential. If needed, we will fix base points $Q_0 \in T^\circ$ and $P_0 \in \pi_G^{-1}(Q_0)$.*

Remark 3 *The map from epimorphisms ξ to actions ϵ is given by*

$$\xi \to \epsilon = \widetilde{\xi^{-1}}, \tag{3}$$

where the tilde is the homomorphism $\pi_1(T^\circ) \to \pi_1(T^\circ)/\Pi_G \to \mathrm{Aut}(S)$, defined by path lifting to deck transformations.

The fundamental group $\pi_1(T^\circ)$ has the following presentation:

$$\text{generators} \;:\; \{\alpha_i, \beta_i, \gamma_j, 1 \le i \le \tau, 1 \le j \le t\}, \tag{4}$$

$$\text{relation} \;:\; \prod_{i=1}^{\tau}[\alpha_i, \beta_i] \prod_{j=1}^{t} \gamma_j = 1. \tag{5}$$

We let

$$\mathcal{G} = (\alpha_1, \ldots, \alpha_\tau, \beta_1, \ldots, \beta_\tau, \gamma_1, \ldots, \gamma_t) \tag{6}$$

denote the $2\tau+t$ tuple of generators of $\pi_1(T^\circ)$. The generating system is not unique but we fix it once T° and a base point $Q_0 \in T_0$ have been selected. Define

$$a_i = \xi(\alpha_i), b_i = \xi(\beta_i), c_j = \xi(\gamma_j). \tag{7}$$

Then, the $2\tau + t$ tuple

$$\mathcal{V} = (a_1, \ldots, a_\tau, b_1, \ldots, b_\tau, c_1, \ldots, c_t) \tag{8}$$

is called a *generating vector* for the action. We observe that

$$G = \langle a_1, \ldots, a_\tau, b_1, \ldots, b_\tau, c_1, \ldots, c_t \rangle . \tag{9}$$

Defining

$$n_j = o(c_j), \tag{10}$$

we also have

$$\prod_{i=1}^{\tau}[a_i, b_i] \prod_{j=1}^{t} c_j = c_1^{n_1} = \cdots = c_t^{n_t} = 1. \tag{11}$$

The *signature* of the action (more precisely, of the generating vector) is $(\tau; n_1, \ldots, n_t)$, In the n-gonal case with $\tau = 0$ we write $(n_1, \ldots, n_t)$. The number τ is called the *orbit genus,* namely the genus of T, and the n_j are called the *periods* of the action. The possible signatures are constrained by the Riemann Hurwitz equation

$$\frac{2\sigma - 2}{|G|} = 2\tau - 2 + \sum_{j=1}^{t}\left(1 - \frac{1}{n_j}\right). \tag{12}$$

Given a generating vector $\mathcal{V}$ satisfying equations 9 - 11 above, a regular monodromy epimorphism, as in equation 2, is determined by the equations 7. The subgroup Π_ξ may be used to construct a covering space S° whose compactification S is a Riemann surface with G action, quotient $T = S/G$, and the given generating vector and signature. The genus of S is determined by the Riemann-Hurwitz equation 12.

Remark 4 *If G is abelian, then $c_1 \cdots c_t = 1$.*

Remark 5 *The number of generating vectors for a given genus σ is finite. This follows from the Riemann Hurwitz formula 12. Both τ and t are bounded since $1 - \frac{1}{n_j} \geq \frac{1}{2}$, so the number of signatures and, hence, generating vectors is finite.*

Remark 6 *The Riemann Hurwitz formula can be rewritten*

$$\sigma - 1 = \frac{|G|}{2}\left(2\tau - 2 + \sum_{j=1}^{t}\left(1 - \frac{1}{n_j}\right)\right) \tag{13}$$

$$= |G|(\tau - 1) + \frac{1}{2}\sum_{j=1}^{t}\left(|G| - \frac{|G|}{n_j}\right), \tag{14}$$

so that if $|G|$ *is odd, then* $\sum_{j=1}^{t}\left(|G| - \frac{|G|}{n_j}\right)$ *must be an even integer. The quantity* $\frac{1}{2}\left(2\tau - 2 + \sum_{j=1}^{t}\left(1 - \frac{1}{n_j}\right)\right)$ *functions as an orbifold Euler characteristic.*

Remark 7 *For positive characteristic "conformal" simply means that* $x \to g \cdot x$ *is a birational isomorphism of* S*. The above discussion applies to the positive characteristic case, except for the use of the fundamental group, and generating vectors. The fundamental group could be replaced by the algebraic fundamental group, but that discussion is beyond the scope of this paper.*

1.3 Cyclic n-gonal actions

We now present an example of our principle object of study - a general cyclic n-gonal action. The details of our construction allow us to construct a nice family of actions in subsection 1.6.

Example 8 Cyclic n-gonal surfaces. *Let* $m_1, \ldots, m_t$*, and* n *be integers satisfying:*

1. $1 \leq m_j < n$,
2. n *divides* $m_1 + \cdots + m_t$*, and*
3. $\gcd(m_1, \ldots, m_t) = 1$.

Then the projective, possibly singular, curve, $\overline{S}$*, defined by*

$$y^n = (x - a_1)^{m_1}(x - a_2)^{m_2} \cdots (x - a_t)^{m_t}, \tag{15}$$

where the $a_1, \ldots, a_t \in \mathbb{C}$ are distinct, is called an irreducible, cyclic n-gonal curve. If $m_j > 1$ the point $(a_j, 0)$ is singular. There are $e_j = \gcd(m_j, n)$ local branches of $\overline{S}$ at $(a_j, 0)$. The normalization map $\nu : S \to \overline{S}$ resolves the singularities and e_j points lie over $(a_j, 0)$. The action of $G = \mathbb{Z}_n$ on $\overline{S}$ is defined by $(x, y) \to (x, u^k y)$ where $u = \exp(2\pi i/n)$. This action lifts to S and the quotient map $\pi_G : S \to S/G$ is given by $\pi_G : S \xrightarrow{\upsilon} \overline{S} \xrightarrow{\pi} \mathbb{P}^1$ where $\pi(x, y) = x$. The map π_G is branched over each $Q_j = a_j$, but is unbranched over ∞, by condition 2. Letting g be the automorphism $(x, y) \to (x, uy)$, we have $c_j = g^{m_j}$ and c_j fixes the e_j points lying over Q_j. The order of c_j is n/e_j so $n = n_j e_j$. The resulting surface is called a cyclic n-gonal surface. For more details see [6].

Example 9 *The preceding example can be extended to characteristic p as when p does not divide n.*

1.4 Equivalence of actions and generating vectors

We define equivalence of actions and then translate the concept to monodromy epimorphisms and generating vectors.

Definition 10 *Two conformal actions ϵ_1, ϵ_2 of G on possibly different surfaces S_1, S_2 are* topologically equivalent *if there is an intertwining, orientation-preserving homeomorphism $h : S_1 \to S_2$ and an automorphism $\omega \in \mathrm{Aut}(G)$ such that*

$$\epsilon_2(g) = h\epsilon_1(\omega(g))h^{-1}, \forall g \in G. \tag{16}$$

If h is a conformal map then we say that the actions are conformally equivalent*. If $\omega = Id_G$ then we say that ϵ_1, ϵ_2 are* topologically or conformally conjugate.

Remark 11 *For positive characteristic conformal equivalence of actions is defined. For topological equivalence we need to use equisymmetry, introduced in subsection 1.6.*

Let us rephrase topological equivalence in terms of epimorphisms. Assume we are given two regular monodromy epimorphisms

$$\Pi_{\xi_1} \hookrightarrow \pi_1(T_1^\circ) \overset{\xi_1}{\twoheadrightarrow} G \tag{17}$$

$$\Pi_{\xi_2} \hookrightarrow \pi_1(T_2^\circ) \overset{\xi_2}{\twoheadrightarrow} G \tag{18}$$

determined by actions ϵ_1 and ϵ_2 (using equation 3). Then, the two actions are topologically equivalent if and only if there is an orientation preserving homeomorphism $\overline{h} : T_1^\circ \to T_2^\circ$ and an automorphism $\omega \in \mathrm{Aut}(G)$ such that

$$\xi_2 = \omega \circ \xi_1 \circ \overline{h}_*^{-1}, \tag{19}$$

where $\overline{h}_* : \pi_1(T_1^\circ) \to \pi_1(T_2^\circ)$ is the induced map on fundamental groups. In diagram form we have

$$\begin{array}{ccc} \Pi_{\xi_1} & \stackrel{h_*}{\to} & \Pi_{\xi_2} \\ \downarrow & & \downarrow \\ \pi_1(T_1^\circ) & \stackrel{\overline{h}_*}{\to} & \pi_1(T_2^\circ) \\ \downarrow \xi_1 & & \downarrow \xi_2 \\ G & \stackrel{\omega}{\to} & G \end{array} \tag{20}$$

Geometrically, in terms of branched covers, the diagram above translates to the commutative diagram

$$\begin{array}{ccc} S_1 & \stackrel{h}{\to} & S_2 \\ \downarrow \pi_{\epsilon_1} & & \downarrow \pi_{\epsilon_2} \\ T_1 & \stackrel{\overline{h}}{\to} & T_2 \end{array}, \tag{21}$$

where the surfaces S_1 and S_2 are obtained from the kernels of ξ_1 and ξ_2. The covering map h exists by standard covering space theory and equation 19. Finally for action monomorphisms ϵ_1, ϵ_2 the formula corresponding to equation 19 is

$$\epsilon_2 = Ad_h \circ \epsilon_1 \circ \omega^{-1}, \tag{22}$$

according to equation 3. This condition is slightly different than the one given in equation 16. However, the formula in 16 is simpler to state and yields the same equivalence relation. We really only need to worry about the difference when we relate a generating vector to an action later in this subsection.

Remark 12 *In the previous discussion, if T_1° and T_2° equal a common punctured surface T°, then the possible automorphisms h_* of $\pi_1(T^\circ)$ form a subgroup $\mathcal{B} \lhd \mathrm{Aut}(\pi_1(T^\circ))$ of index 2 and the formula 19 defines an action of $\mathrm{Aut}(G) \times \mathcal{B}$ on the epimorphisms $\pi_1(T^\circ) \stackrel{\xi}{\twoheadrightarrow} G$. The orbits of this action on epimorphisms correspond to topological equivalence classes of actions. Fixing $\mathcal{G}$ in equation 6, we can transfer the action of $\mathrm{Aut}(G) \times \mathcal{B}$ to generating* *vectors, as we discuss next. The set of such orbits is finite and may be interpreted geometrically by considering, for a fixed pair* (T, B_G)*, the set of all (conformal equivalence classes of) regular branched covers* $\pi : S \to T$*, where* S *has genus* σ, $\mathrm{Gal}(S/T) \simeq G$*, and such that* π *is ramified exactly over* B_G*.*

Remark 13 *Each homeomorphism* $\overline{h}$ *of* T° *extends to* T *and determines a permutation* $\rho_{\overline{h}} \in \Sigma_t$*, defined by* $\overline{h}(Q_j) = Q_{\rho_{\overline{h}}(j)}$*. It is well known that* $\overline{h}_*\left(\gamma_j\right)$ *is conjugate to* $\gamma_{\rho_{\overline{h}}(j)}$*. We also note that we need only consider the outer automorphisms of* $\pi_1(T^\circ)$*. Since* $\xi \circ Ad_\delta = Ad_{\xi(\delta)} \circ \xi$*, for* $\delta \in \pi_1(T^\circ)$*, each inner automorphism* Ad_δ *of* $\pi_1(T^\circ)$*, can be transferred to an inner automorphism* $Ad_{\xi(\delta)}$ *of* G*.*

Example 14 Quasi-platonic actions. *A* G*-action is quasi-platonic if* T° *is a thrice punctured sphere which we assume is* $\widehat{\mathbb{C}} - \{0, 1, \infty\}$ *with branch points ordered as* $(Q_1, Q_2, Q_3) = (0, 1, \infty)$*. The conformal automorphisms of* $\widehat{\mathbb{C}} - \{0, 1, \infty\}$*, the anharmonic group, are determined by the permutation* ρ_h *induced on* $\{0, 1, \infty\}$ *:*

$\overline{h}(z)$	z	$1-z$	$\frac{1}{z}$	$\frac{z}{z-1}$	$\frac{1}{1-z}$	$\frac{z-1}{z}$
ρ_h	Id	$(0,1)$	$(0,\infty)$	$(1,\infty)$	$(0,1,\infty)$	$(1,0,\infty)$

(23)

Furthermore, any homeomorphism of T° *fixing the punctures is isotopically trivial. Therefore, the action of the outer automorphisms induced by* $\mathcal{B}$ *is carried by the anharmonic group above. It follows then, that the action on generating vectors* $\rho_{\overline{h}} : (c_1, c_2, c_3) \to (c_1', c_2', c_3')$*, determined by* $\xi \circ \overline{h}_*^{-1}$ *up to inner automorphisms, is given by:*

$\rho_{\overline{h}}$	(c_1', c_2', c_3')
Id	$\omega \cdot (c_1, c_2, c_3)$
$(0,1)$	$\omega \cdot (c_1 c_2 c_1^{-1}, c_1, c_3)$
$(0,\infty)$	$\omega \cdot (c_3, c_2, c_3 c_1 c_3^{-1})$
$(1,\infty)$	$\omega \cdot (c_1, c_2 c_3 c_2^{-1}, c_2)$
$(0,1,\infty)$	$\omega \cdot (c_3, c_1, c_2)$
$(1,0,\infty)$	$\omega \cdot (c_2, c_3, c_1)$

(24)

where $\omega \in \mathrm{Aut}(G)$ *is appropriately chosen, independently in each case. In particular, if the signature has three distinct entries, then the action of* $\mathcal{B}$ *is completely subsumed by the* $\mathrm{Aut}(G)$ *action.*

In the two following subsections we analyze separately the $\mathrm{Aut}(G)$ action and the $\mathrm{Aut}(\pi_1(T^\circ))$ action on generating vectors.

1.4.1 Image subgroups and $\mathrm{Aut}(G)$

As noted in [7], focusing on the action ϵ, instead of the *image subgroup* $\epsilon(G)$, allows us to consider the group G as the primary object as well giving us a certain preciseness and economy of exposition. For instance, for the three equivalence relations and their attributes we consider in this paper, we have these associations (not all exact):

- action $\to$ generating vector $\to$ classify topological equivalence,
- action $\to$ characters on $\mathcal{H}^q$ $\to$ classify $\mathcal{H}^q$ equivalence, and
- action $\to$ rotation data system $\to$ classify equisymmetry.

The focus on actions instead of image subgroups comes at the expense of having to take into account the various actions $\epsilon \circ \omega$ for $\omega \in \mathrm{Aut}(G)$, and how $\mathrm{Aut}(G)$ operates on the attribute of the action. Indeed, for two actions ϵ_1, ϵ_2 on the same surface, $\epsilon_1(G) = \epsilon_2(G)$ if and only if $\epsilon_2 = \epsilon_1 \circ \omega$ for some $\omega \in \mathrm{Aut}(G)$. Thus ϵ_1 and $\epsilon_1 \circ \omega$ are automatically considered equivalent when the focus is on the image subgroups. The study of subgroups of the mapping class group and the branch locus of moduli space are a particular instances where the consideration of the image subgroups is important. In some types of equivalence, such as conformal or topological equivalence, ϵ_1 and $\epsilon_1 \circ \omega$ are automatically considered equivalent as a part of the definition. Otherwise, if we wish to require that ϵ_1 and $\epsilon_1 \circ \omega$ are equivalent, under some equivalence relation, we say that the actions are *equivalent up to automorphisms.* If $\mathcal{V}$ corresponds to the action ϵ then the vector corresponding to $\epsilon \circ \omega^{-1}$ or equivalently $\omega \circ \xi$ is denoted by $\omega \cdot \mathcal{V}$ and has the formula

$$\omega \cdot \mathcal{V} = (\omega a_1, \ldots, \omega a_\tau, \omega b_1, \ldots, \omega b_\tau, \omega c_1, \ldots, \omega c_t)\,. \tag{25}$$

The new vector determines a possibly different branched cover $S' \to T$. However, the same kernel $\Pi_G = \Pi_\xi$ is determined in both equations 17 - 18 and so S and S' are conformally equivalent.

1.4.2 Conjugate equivalence of generating vectors

Let $\overline{h} : T_1^\circ \to T_2^\circ$ be a homeomorphism with cover h as given in diagram 21, and let us compute the new vector corresponding to $\xi_2 = \xi_1 \circ \overline{h}_*^{-1}$ (and hence

$\epsilon_2 = Ad_h \circ \epsilon_1$). Let

$$\mathcal{V}' = (a'_1, \ldots, a'_\tau, b'_1, \ldots, b'_\tau, c'_1, \ldots, c'_t)$$

be the generating vector determined by ξ_2. To compute $\mathcal{V}'$, we must first determine $\overline{h}$, then compute the image $\delta' = h_*^{-1}(\delta)$ of a generator δ in $\mathcal{V}$ as a word in the generator list $\mathcal{G}$, and then calculate calculate $\xi(\delta')$ from the vector $\mathcal{V}$. For example, in the n-gonal case with $T_1^\circ = T_2^\circ = T^\circ$ consider the action induced by switching exactly two of the branch points $Q_j \leftrightarrow Q_{j+1}$, even though this may alter the signature. A homeomorphism $\overline{h}$ may be found such that $\overline{h}_*^{-1}$ maps $\gamma_j \to \gamma_j \gamma_{j+1} \gamma_j^{-1}$, $\gamma_{j+1} \to \gamma_j$, and all other generators fixed. A new generating vector is determined:

$$c'_j = c_j c_{j+1} c_j^{-1}, \; c'_{j+1} = c_j, \qquad (26)$$

$$c'_j = c_{j+1}, \; c'_{j+1} = c_j, \text{ for abelian } G \qquad (27)$$

and all other elements fixed. A number of these transpositions may be combined to obtain a new generating vector with the same signature, assuming the permutation of the Q_j's preserves the n_j's. This is called the *braid action* on generating vectors. For more details and formulas for the $\overline{h}_*^{-1}$ action, see [5], [10], and [16] for example.

We summarize the foregoing in a Proposition.

Proposition 15 *Topological equivalence can be transferred to generating vectors as follows. The action of* $\overline{h}_*^{-1}$ *is described by the operation*

$$(a_1, \ldots, a_\tau, b_1, \ldots, b_\tau, c_1, \ldots, c_t) \to (a'_1, \ldots, a'_\tau, b'_1, \ldots, b'_\tau, c'_1, \ldots, c'_t),$$

where the primed generators are determined geometrically as above. The combined action of ω *and* $\overline{h}_*^{-1}$ *is*

$$(a_1, \ldots, a_\tau, b_1, \ldots, b_\tau, c_1, \ldots, c_t) \to (\omega a'_1, \ldots, \omega a'_\tau, \omega b'_1, \ldots, \omega b'_\tau, \omega c'_1, \ldots, \omega c'_t). \qquad (28)$$

The braid action equations 26 and 27 generate the topological conjugacy action for the surfaces lying over T°, *in the orbit genus zero case.*

Using the above remark we can prove the following. See also [13] and [16].

Proposition 16 *Two n-gonal actions of an abelian group given by generating vectors $(c_1, \ldots, c_t)$ and $(c'_1, \ldots, c'_t)$ are topologically equivalent if and only if there are $\omega \in \mathrm{Aut}(G)$ and a permutation ρ of $\{1, \ldots, t\}$ such that*

$$(c'_1, \ldots, c'_t) = \left(\omega c_{\rho(1)}, \ldots, \omega c_{\rho(t)}\right). \tag{29}$$

For a general group we may say something about the conjugacy classes $C_j = c_j^G$.

Proposition 17 *As in Remark 12 consider two generating vectors $\mathcal{V}, \mathcal{V}'$ for two G actions ϵ, ϵ' with branched covers $S \to T$ and $S' \to T$ and the same branch set. Let $(c_1, \ldots, c_t)$ and $(c'_1, \ldots, c'_t)$ denote the periodic parts of $\mathcal{V}$ and $\mathcal{V}'$, respectively, and let $C_j = c_j^G$ and $C'_j = \left(c'_j\right)^G$ denote the conjugacy classes determined by the c_j and c'_j, respectively. If the actions are topologically equivalent then there are $\omega \in \mathrm{Aut}(G)$ and a permutation ρ of $\{1, \ldots, t\}$ such that*

$$C'_j = \omega C_{\rho(j)}. \tag{30}$$

1.5 Representations on q-differentials

The group G acts linearly on the various spaces of holomorphic q-differentials $\mathcal{H}^q(S)$. The space $\mathcal{H}^q(S)$ is the $\mathbb{C}$-vector space of holomorphic sections of $T^*(S) \bigotimes \cdots \bigotimes T^*(S)$ (q times), where $T^*(S)$ is the cotangent bundle, a holomorphic line bundle. We are going to be "pedantically precise" about the action of G since it will be important later in our discussion of rotation constants. The group G acts on $\mathcal{H}^q(S)$ by the representation

$$H_g^q(\psi(P)) = \left(dg^{-1}\right)^* (\psi(g^{-1}P)), \tag{31}$$

where $\psi \in \mathcal{H}^q(S)$, and we use g to denote the mapping $\epsilon(g)$. The formula makes sense since

$$\begin{aligned} \psi(P) &\in T_P^*(S) \bigotimes \cdots \bigotimes T_P^*(S), \qquad (32) \\ \psi(g^{-1}P) &\in T_{g^{-1}P}^*(S) \bigotimes \cdots \bigotimes T_{g^{-1}P}^*(S) \end{aligned}$$

and

$$\left(dg^{-1}\right)^* : T_{g^{-1}P}^*(S) \bigotimes \cdots \bigotimes T_{g^{-1}P}^*(S) \to T_P^*(S) \bigotimes \cdots \bigotimes T_P^*(S). \tag{33}$$

We use g^{-1} as $\phi \to \phi^*$ is contravariant. We denote the character of the representation $\mathcal{H}^q(S)$ by

$$\mathrm{ch}_{\mathcal{H}^q(S)}(g) = Trace(H_g^q), \tag{34}$$

Definition 18 *Two actions ϵ_1, ϵ_2 of G on possibly different surfaces S_1, S_2 are called $\mathcal{H}^q$-equivalent if the representations on $\mathcal{H}^q(S_1)$ and $\mathcal{H}^q(S_2)$ are equivalent over $\mathbb{C}$.*

Remark 19 *One classification scheme of automorphism groups of Riemann surfaces S of a fixed genus σ is to determine the images of $\mathrm{Aut}(S)$ as subgroups of $GL_\sigma(\mathbb{C})$ up to conjugacy, through the representation on Abelian differentials $\mathcal{H}^1(S)$. This is the same as $\mathcal{H}^1$-equivalence up to automorphisms applied the group $\epsilon(G) = \mathrm{Aut}(S)$.*

Remark 20 *There are only finitely many $\mathcal{H}^q$-equivalence classes since there can only be finitely many inequivalent representations of a fixed degree.*

1.6 Holomorphic families of curves with G-action

1.6.1 Families of curves

In [14], Guerrero considers families of Riemann surfaces with G-action. The notion of families of curves (surfaces) as an approach to topological equivalence is useful in studying the moduli space and can be used in positive characteristic, where topological equivalence can't be usefully defined.

A *holomorphic family of surfaces (curves)* is a morphism $\pi : E \to B$ such that each $S_b = \pi^{-1}(b)$, $b \in B$ is a compact Riemann surface. We assume that E and B are smooth and that π is a proper submersion, so that the family S_b, $b \in B$ is locally trivial. As a consequence, all the fibres S_b have the same genus. Typically, we assume that E and B are a smooth, irreducible, locally closed varieties, though in [14] Guerrero assumes that E and B are connected, complex manifolds and that π is a proper, holomorphic submersion. We allow for that case also, since it is useful in studying the moduli space and Teichmüller space. In the positive characteristic case, we drop the term holomorphic and speak of a *smooth family of curves.* Of course E and B must be a smooth, irreducible, locally closed varieties, and "proper submersion" must be appropriately defined.

A *holomorphic family of actions* $\epsilon(g)$, $g \in G$ for a holomorphic family of surfaces S_b, $b \in B$ is a family of monomorphisms

$$\epsilon_b : G \to \mathrm{Aut}(S_b),\ b \in B \tag{35}$$

such that: for each $g \in G$ the map $(b, x) \to (b, \epsilon_b(g)x)$ is an automorphism of the variety $V = \{(b, x) : \pi(x) = b\}$. Moreover, we assume that ϵ_b varies holomorphically with b, namely that for each $g \in G$ the map

$$\epsilon(g) : E \to E,\ \epsilon(g)(x) = \epsilon_{\pi(x)}(g)(x) \tag{36}$$

is a holomorphic automorphism of E. Alternatively, we can describe ϵ as a G-action on E that commutes with π :

$$\begin{array}{ccc} E & \overset{\epsilon(g)}{\to} & E \\ \downarrow \pi & & \downarrow \pi \\ B & \overset{id}{\to} & B \end{array} . \tag{37}$$

In positive characteristic, the notion of *smooth family of actions* is similarly defined. In particular, each $\epsilon(g)$ should be an invertible morphism of E.

Example 21 *Cyclic n-gonal families. Continuing Example 8, consider the family of curves defined by*

$$y^n = (x - a_1)^{m_1} \cdot \cdots \cdot (x - a_t)^{m_t}, \tag{38}$$

with $(a_1, \ldots, a_t) \in \mathbb{C}^t - \Delta$, where Δ is the multi-diagonal. The family is constructed by first taking all points of the form $(x, y, a_1, \ldots, a_t) \in \mathbb{C}^{t+2}$ that satisfy 38 and then forming the closure E_1 of these points in $\mathbb{P}^2 \times (\mathbb{C}^t - \Delta)$. After normalizing E_1 we get $\pi : E \to B = \mathbb{C}^t - \Delta$ such that $\pi(x, y, a_1, \ldots, a_t) = (a_1, \ldots, a_t)$. The action ϵ_b, $b \in B$ of $G = \mathbb{Z}_n$ on E_1 is defined by $(x, y) \to (x, u^k y)$ where $u = \exp(2\pi i/n)$. The action is then lifted to E. Clearly this action is holomorphic in b.

1.6.2 Equisymmetry

Using families of curves, another type of action equivalence can be defined which serves as topological equivalence in positive characteristic. It is especially useful in considering $\mathcal{H}^q$ representations and rotational data systems.

Definition 22 *Two actions ϵ_1, ϵ_2 of G on possibly different surfaces S_1, S_2 are* (directly) equisymmetric *$\epsilon_1 \sim_D \epsilon_2$ if the following occurs. There is a family of curves $\pi : E \to B$, with a family of actions $\epsilon_b : G \to \mathrm{Aut}(\pi^{-1}(b))$, $b \in B$, such that there are $b_1, b_2 \in B$ with isomorphisms $\phi_i : \pi^{-1}(b_i) \simeq S_i$ and $\epsilon_i = \phi_i \circ \epsilon_{b_i} \circ \phi_i^{-1}$. Two actions $\epsilon_1 : G \to \mathrm{Aut}(S_1)$ and $\epsilon_m : G \to \mathrm{Aut}(S_m)$ are* equisymmetric *if there is a sequence of surfaces S_i and actions $\epsilon_i : G \to \mathrm{Aut}(S_i)$ such that $\epsilon_1 \sim_D \epsilon_2$, $\epsilon_2 \sim_D \epsilon_3, \ldots, \epsilon_{m-1} \sim_D \epsilon_m$. Typically the relations $\epsilon_i \sim_D \epsilon_{i+1}$ come from distinct families as i varies. Finally, two actions ϵ_1, ϵ_2 are equisymmetric up to automorphisms if there is an $\omega \in \mathrm{Aut}(G)$ such that $\epsilon_1 \circ \omega$ and ϵ_2 are equisymmetric.*

Remark 23 *If $\psi_i \in \mathrm{Aut}(S_i), i = 1, 2$ then ϵ_1, ϵ_2 are (directly) equisymmetric if and only if $\psi_1 \circ \epsilon_1 \circ \psi_1^{-1}$, $\psi_2 \circ \epsilon_2 \circ \psi_2^{-1}$ are (directly) equisymmetric.*

For Riemann surfaces the following hold. The second proposition follows form a result in [14] that says the $\mathcal{H}^q$ characters are constant in a family with a connected base.

Proposition 24 *Two actions ϵ_1, ϵ_2 of G, on possibly different surfaces S_1, S_2, are topologically equivalent if and only if they are equisymmetric up to automorphisms.*

Proposition 25 *Two equisymmetric actions ϵ_1, ϵ_2 of G, on possibly different surfaces S_1, S_2, have the same character for the representation on $\mathcal{H}^q(S)$.*

1.6.3 Families in positive characteristic

For a field $\mathbf{k}$ of characteristic $p > 0$, and action of G on S is *tame* if p does not divide $|G|$ otherwise it is called *wild*.

Example 26 *Example 21 extends to characteristic p as long as p does not divide n. The actions are tame.*

Example 27 *Let $\mathbf{k}$ be a field of positive characteristic p. Let $E = \mathbb{P}^1(\mathbf{k}) \times \mathbf{k}^*$, $B = \mathbf{k}^*$, and $\pi : E \to B$ be given by $\pi(x, b) = b$. Let $G = \mathbb{Z}_p$ and $\epsilon_b(j)x = x + jb$ for $j \in \mathbb{Z}_p$ and $x \in \mathbf{k} \subset \mathbb{P}^1(\mathbf{k})$. See Example 36 for the discussion of the action at ∞, including the fact that all the actions are wild.*

2 Rotation data of a G-action

2.1 Fixed points and rotation constants

We define and then organize the rotation constants at fixed points into structures to help interpret the results of Guerrero and to relate rotation constants to generating vectors. We define two types of systems, one that is well adapted to the Eichler trace formula (total rotation data) and another that is well adapted to generating vectors (primitive rotation data).

Recall that the *rotation constant* $\varepsilon(P,g)$ of g at a fixed point P is the $o(g)$th root of unity determined by the map of one dimensional complex spaces

$$\left(dg^{-1}\right)^* : T_P^*(S) \to T_P^*(S), \tag{39}$$

where dg represents the differential of the self map $\epsilon(g)$ determined by the G-action $\epsilon : G \to \mathrm{Aut}(S)$. If g does not fix P then define $\varepsilon(P,g) = 0$. The map of tangent spaces

$$dg : T_P(S) \to T_P(S), \tag{40}$$

equals $\frac{1}{\varepsilon(P,g)}$ and is called the multiplier, and $\arg(\overline{\varepsilon(P,g)})$ is the angle of rotation of g at P, in the complex case. The multiplier is a linear approximation to the map $x \to gx$ at P.

Remark 28 *Note that $\varepsilon(P,g)^{o(g)} = 1$ as $g^{o(g)} = 1$. With more work one can prove that $\varepsilon(P,g)$ is a primitive $o(g)$th root of unity. See Section 2.3. Also, the rotation constant is dependent on the action ϵ, since $(dg^{-1})^*$ in equation 39 is dependent on ϵ.*

Rotation constants satisfy these simple properties:

$$\varepsilon(P,g^k) = (\varepsilon(P,g))^k \tag{41}$$

and

$$\varepsilon(hP,hgh^{-1}) = \varepsilon(P,g). \tag{42}$$

Equation 41 is easily proven from 39. To prove equation 42 we first note that P is a fixed point of g if and only if hP is a fixed point of hgh^{-1}. From equation 33 for $q = 1$ and the contravariant functoriality of $\phi \to (d\phi)^*$ we obtain

$$\left(d\left(hgh^{-1}\right)\right)_{hP}^* = \left(dh^{-1}\right)_{hP}^* \circ (dg)_P^* \circ (dh)_P^*,$$

and

$$\left(dh^{-1}\right)^*_{hP} = \left((dh)^*_P\right)^{-1},$$

where the subscripts denote the base point of the cotangent space domain of the given differentials. Equation 42 now easily follows.

The rotation constant $\varepsilon(P,g)$ has the form $\exp\left(-\frac{2\pi ik}{o(g)}\right)$, where $1 \leq k \leq o(g)-1$ and k must be relatively prime to $o(g)$ by Remark 28. We call P a *simple fixed point* of g if $k=1$, since the angle of rotation of g at P, $\arg(\overline{\varepsilon(P,g)}) = \frac{2\pi}{o(g)}$, is as small as possible. We also observe that the element $h = g^s$ inherits all the fixed points of g, namely $gP = P$ implies $hP = P$. We call such a point an *inherited fixed point* of h if $\langle h\rangle \subsetneq \langle g\rangle$. In contrast, we call a fixed point P' of h a *primitive fixed point* if the stabilizer $G_{P'} = \langle h\rangle$. In the next subsection we shall see that a generating vector has a complicated relationship to the (total) rotation constant data. However, this relationship is simplified if we just consider primitive rotation constant data.

We finish this subsection by describing how the simple, primitive fixed points are reflected in the group structure. If P is any point fixed by an element $g \in G$ then $g \in G_P = \langle h\rangle$ for some h. Now $\varepsilon(P,h) = \exp\left(-\frac{2\pi ik}{o(h)}\right)$, for some k relatively prime to $o(h)$, we set $h' = h^{k^{-1}}$ where k^{-1} is the multiplicative inverse of k in $\mathbb{Z}^*_{o(h)}$. Then the rotation constant $\varepsilon(P,h')$ satisfies

$$\varepsilon(P,h') = \varepsilon(P,h^{k^{-1}}) = \exp\left(-\frac{2\pi ikk^{-1}}{o(h)}\right) = \exp\left(-\frac{2\pi i}{o(h)}\right),$$

so that h' is the unique element of $\langle h\rangle$ such that P is a simple, primitive fixed point of h'. In fact, if h'' is any other element in G such that P is a simple, primitive fixed point of h'' then $h'' \in G_P = \langle h\rangle$, and h'' generates $\langle h\rangle$. But h' is the only element of $\langle h\rangle$ for which P is a simple, primitive fixed point. We denote h' by h_P and call it a *distinguished stabilizer element* and define the map $\delta : P \to h_P$. We observe that

$$h_{xP} = xh_Px^{-1} \tag{43}$$

for $x \in G$. Therefore, distinguished stabilizers consist of entire conjugacy classes. If $h_{xP} = h_P$ then $xh_Px^{-1} = h_P$ so that $x \in Z_G(\langle h_P\rangle)$. It follows that

$$\left|GP \cap \delta^{-1}(h_P)\right| = \frac{|Z_G(\langle h_P\rangle)|}{o(h_P)}. \tag{44}$$

Note that the inverse point image $\delta^{-1}(h_P)$ could be split (disjointly) over several fixed point orbits if the signature has multiple entries equaling $o(h)$. Finally, the inverse point images $\delta^{-1}(h_P)$ define a useful partition on the fixed points of G on S. We summarize the foregoing in a lemma.

Lemma 29 *For a fixed point P in S let $\delta(P) = h_P$ denote the distinguished stabilizer element of P, i.e., the unique element $h \in G_P$ such that P is a simple, primitive fixed point of h. The distinguished stabilizers and the map $\delta : P \to h_P$ satisfy equations 43 and 44. The map δ also defines a partition of all fixed points in S, $F_i = \delta^{-1}(h_{P,i})$, $i = 1, \ldots, w$, where $h_{P,i}$, $i = 1, \ldots, w$ are the distinguished stabilizer elements in G. A point P is a simple fixed point of $g \in G - \{1\}$ if and only if $g = h_P^e$ where $o(g)$ divides $o(h)$ and $e = o(h)/o(g)$. Moreover, the points in the set $\delta^{-1}(h_P)$, consisting of all the simple, primitive fixed points of h_P are simple, fixed points of g, inherited if $e > 1$.*

Proof. The equations 43 and 44 have already been proven. Since δ is a function with unique images, then the inverse point images are disjoint and exhaustive, establishing that δ induces a partition. Now assume that P is a simple fixed point of g. Since h_P generates G_P then, for $g \in G_P$, $g = h_P^{ek}$, where $e = o(h_P)/o(g)$, $\gcd(k, o(g)) = 1$, and $k < o(g)$, according to Lemma 38. But,

$$\varepsilon(P, g) = \varepsilon(P, h_P^{ek}) = \exp\left(-\frac{2\pi iek}{o(h_P)}\right) = \exp\left(-\frac{2\pi ik}{o(g)}\right),$$

and we must have $k = 1$ and $g = h_P^e$. On the other hand if $g = h_P^e$ then the above equation immediately shows that P is a simple fixed point of g. If $e > 1$ then it also follows that every point of $\delta^{-1}(h_P)$ is an inherited fixed point of g. ■

2.2 Rotation data systems

We want to record the rotation data in two different ways: total rotation data and primitive rotation data. As noted previously these two structures have different applications. We also need to translate back and forth between the two structures. The translation between the two structures depends only on the group structure of G and is realized by a linear transformation. We give examples in the next subsection.

The rotation constant $\varepsilon(P,g)$ has the form $\exp\left(-\frac{2\pi ik}{o(g)}\right)$, where $1 \le k \le o(g)-1$ and k must be relatively prime to $o(g)$. For such a k let $l_k(g)$ be the number of fixed points of g with rotation constant $\exp\left(-\frac{2\pi ik}{o(g)}\right)$ counted over the fixed point set S^g :

$$l_k(g) = \left|\left\{P \in S^g : \varepsilon(P,g) = \exp\left(-\frac{2\pi ik}{o(g)}\right)\right\}\right|. \tag{45}$$

The values $l_k(g)$ satisfy

$$l_k(hgh^{-1}) = l_k(g),\ h \in G \tag{46}$$

and

$$l_{ks}(g^s) = l_k(g), \tag{47}$$

when s is relatively prime to $o(g)$. Consequently,

$$l_k(g^s) = l_{ks^{-1}}(g) \tag{48}$$

and

$$l_k(g) = l_1(g^{k^{-1}}), \tag{49}$$

where s^{-1}, k^{-1} are the multiplicative inverses of s, k in $\mathbb{Z}^*_{o(g)}$, respectively.

We can summarize the rotation constant data $\{l_k(g)\}$ of an element $g \in G$, as a vector, as follows. Define the vector

$$L(g) = \left[\begin{array}{ccccc} l_1(g) & \cdots & l_k(g) & \cdots & l_{o(g)-1}(g) \end{array}\right],\ \ \gcd(k, o(g)) = 1 \tag{50}$$

consisting of $\phi(o(g))$ integers, and call it the *rotation constant vector* of g. If g has no fixed points then $L(g)$ is the zero vector. We note immediately from equation 42 that rotation constant vectors are conjugation invariant namely,

$$L(hgh^{-1}) = L(g),\ h \in G, \tag{51}$$

so we need only compute the vectors for conjugacy classes in G. From equation 48 we also see that for $\gcd(s, o(g)) = 1$

$$L(g^s) = \left[\begin{array}{ccc} l_{s^{-1}}(g) & \cdots & l_{(o(g)-1)s^{-1}}(g) \end{array}\right]. \tag{52}$$

Collecting everything together, we define the *(total) rotation data system* (*RDS*) for a conformal action of G on S to be the set of pairs

$$\left\{(g, L(g)) : g^G \text{ is a nontrivial class of } G\right\}, \tag{53}$$

one pair for each non-trivial conjugacy class.

Remark 30 *The RDS system is dependent on the action ϵ of G because $(dg^{-1})^*$ in equation 39 is dependent on ϵ. Indeed, the RDS system is to be used to distinguish various classes of actions. The number of RDS systems for a given group G and genus σ is finite. For, given $g \in G$,*

$$|S^g| = \sum_k l_k(g)$$

and $|S^g| \leq 2\sigma + 2$. The latter equation is well known, but easily follows from the fact that the trace of g_* in the homology representation on $H_1(S)$ satisfies:

$$-2\sigma \leq tr(g_*) = 2 - |S^g|.$$

Remark 31 *The system $\{L(g^{-1}), g \in G\}$ corresponds to the λ function used in [14].*

We also consider similar structures for primitive fixed points as well. Define:

$$p_k(g) = \left| \left\{ P \in S^g : G_P = \langle g \rangle\, ,\ \varepsilon(P,g) = \exp\left(-\frac{2\pi ik}{o(g)} \right) \right\} \right| \tag{54}$$

and

$$P(g) = \left[\begin{array}{ccccc} p_1(g) & \cdots & p_k(g) & \cdots & p_{o(g)-1}(g) \end{array}\right],\ \ \gcd(k, o(g)) = 1, \tag{55}$$

again consisting of $\phi(o(g))$ integers, and call it the *primitive rotation constant vector* of g. Of course, there are analogues to equations 51 and 52, namely

$$P(hgh^{-1}) = P(g),\ \ h \in G \tag{56}$$

and

$$P(g^s) = \left[\begin{array}{ccc} p_{s^{-1}}(g) & \cdots & p_{s^{-1}o(g)-1}(g) \end{array}\right]. \tag{57}$$

We also define the *primitive rotation data system*

$$\left\{(g, P(g)) : g^G \text{ is a nontrivial class of } G\right\}.$$

In the next subsection we describe the relations among generating vectors, primitive rotation data, and the total rotation data.

Finally, if we want to compactly represent rotation data of an action, we need only account for the number of simple fixed points over all the conjugacy classes of G. So we define the *reduced rotation vector* $\overline{L}(g)$ and the *reduced* *primitive rotation vector* $\overline{P}(g)$ as follows. Let $g_1 = 1, g_2, \ldots, g_l$ be conjugacy class representatives of G then we define

$$\overline{L}_G = \left[\begin{array}{ccc} l_1(g_2) & \cdots & l_1(g_l) \end{array} \right] \tag{58}$$

and

$$\overline{P}_G = \left[\begin{array}{ccc} p_1(g_2) & \cdots & p_1(g_l) \end{array} \right]. \tag{59}$$

Remark 32 *Because of equation 49 and its analogue for $p_k(g)$, the rotation data and primitive rotation data are easily calculated from $\overline{L}_G$ and $\overline{P}_G$. Specifically if $1 = k_1 < k_2 < \cdots < k_{\phi(g)} = o(g) - 1$ are the invertible elements of $\mathbb{Z}_{o(g)}$,then*

$$L(g) = \left[\begin{array}{cccc} l_1(g) & l_1(g^{k_2^{-1}}) & \cdots & l_1(g^{-1}) \end{array} \right], \tag{60}$$

by formula 49 and since $g^{(o(g)-1)^{-1}} = g^{-1}$. Similarly,

$$P(g) = \left[\begin{array}{cccc} p_1(g) & p_1(g^{k_2^{-1}}) & \cdots & p_1(g^{-1}) \end{array} \right]. \tag{61}$$

Remark 33 *Note that, by definition,*

$$p_1(h) = \left|\delta^{-1}(h)\right|. \tag{62}$$

The reduced primitive rotation vector $\overline{P}(g)$, is an analog of a generating vector, at least in the abelian case.

Assumption 34 *It is convenient to order the l conjugacy class representatives $g_1 = 1, g_2, \ldots, g_l$, so that:*

- *$o(g_i) \leq o(g_j)$ for $i \leq j$, and*
- *conjugacy classes with representatives that are powers of each other occur in a single contiguous block.*

This listing is consistent with the Magma computations of conjugacy classes. Where warranted, we will use the alternative labelling $g_0 = 1, g_2, \ldots, g_{l-1}$ with the above restrictions, especially if we want to match up with an ordering $\chi_0, \ldots, \chi_{l-1}$ of the characters, with χ_0 the trivial character. Also, when needed, we denote g_j^G by $\mathcal{C}_j$.

Example 35 *For the cyclic group,* $G = \langle g : g^{12} = 1 \rangle$, *the conjugacy classes are singletons, listed in this order:* $g^{12}, g^6, g^4, g^8, g^3,\ g^9, g^2, g^{10}, g, g^5, g^7, g^{11}$, *with orders* $1, 2, 3, 3, 4, 4, 6,\ 6, 12, 12, 12, 12$, *respectively. Note that the exponents* r_j *in the equation* $g_j = g^{r_j}$ *have the following form:* $12 \cdot 1$, $6 \cdot 1$, $4 \cdot 1$, $4{\cdot}2$, $3{\cdot}1$, $3{\cdot}3$, $2{\cdot}1$, $2{\cdot}5$, $1{\cdot}1$, $1{\cdot}3$, $1{\cdot}5$, $1{\cdot}7$. *The first factor is* $e_j = \gcd(12, r_j)$ *and the second factor is* $k_j = r_j/e_j$. *Within a block all the* e_j*'s, as well as the orders* $d_j = o(g^{e_j k_j}) = 12/e_j$, *are equal, and the* k_j*'s are in increasing order. We call the ordering* IOIE *(increasing order, increasing exponent) since the orders tend to increase as we go left to right (i.e., don't decrease) and within a block of elements of the same order the exponents increase.*

2.3 Local rings and positive characteristic

We now look at extending the above ideas to the positive characteristic case. First let us look at local rings in the complex case. Let $\mathbb{C}(S)$ be the field of meromorphic functions on S, and let $\mathcal{O}_P(S) \subset \mathbb{C}(S)$ be the local ring of meromorphic functions on S, holomorphic at P. The maximal ideal of $\mathcal{O}_P(S)$ is $M_P(S) = \{f \in \mathcal{O}_P(S) : f(P) = 0\}$. The automorphism $\epsilon(g)$ induces an automorphism of $\mathbb{C}(S)$ via $f \to f \circ \epsilon(g)$, we denote the induced map by g^*. If g fixes P then g^* bijectively maps $\mathcal{O}_P(S)$ and $M_P^s(S)$, $s > 0$ to themselves. The quotient vector space $M_P(S)/M_P^2(S)$ is isomorphic to $\mathbb{C}$ and is canonically isomorphic to $T_P^*(S)$, via the map $f \to df$ on $\mathcal{O}_P(S)$. Thus $(g^{-1})^*$ acting on $M_P(S)/M_P^2(S)$ is the same as $(dg^{-1})^*$ acting on $T_P^*(S)$.

Using the local ring we can show that $(dg^{-1})^*$ is multiplication by ζ, a primitive $o(g)$th root of unity. Clearly $\zeta^{o(g)} = 1$, we next show that ζ must also be primitive.

Let z be a local parameter at P, namely $M_P(S) = z\mathcal{O}_P(S)$. Now g^* preserves $\mathcal{O}_P(S)$ and $M_P^s(S)$, for all $s > 0$ and hence $g \to (g^{-1})^*$defines a linear representation of the stabilizer G_P on the finite dimensional vector space $\mathcal{O}_P(S)/M_P^s(S)$ for all $s > 0$. The kernels of the representations form a decreasing sequence of subgroups in G_P. If h lies in the intersection of the kernels of all these representations then $h^*z - z \in M_P^s(S)$, for all $s > 0$. But $\cap_s M_P^s(S) = (0)$ so that $hz - z = 0$. As a consequence, h^* must be trivial on all of $\mathbb{C}(S)$ and hence h is trivial on S. It follows that for some $s > 0$ the representation on $\mathcal{O}_P(S)/M_P^s(S)$ is faithful.

Now $\mathcal{O}_P(S) \supset M_P(S) \supset M_P^2(S) \supset \cdots \supset M_P^s(S)$ is a chain of G_P invariant subspaces and z^j spans the one dimensional quotient $M_P^j(S)/M_P^{j+1}(S)$. With respect to the basis $1, z, z^2, \ldots, z^{s-1}$ the linear map $(g^{-1})^*$ has the lower triangular matrix form:

$$\left(g^{-1}\right)^{*} \to \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \zeta & 0 & \cdots & 0 \\ 0 & \alpha_{1,1} & \zeta^{2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & \alpha_{1,s-2} & \alpha_{2,s-3} & \cdots & \zeta^{s-1} \end{bmatrix}, \tag{63}$$

where

$$\left(g^{-1}\right)^{*} z^{j} = \zeta^{j} z^{j} + \sum_{i=1}^{s-1-j} \alpha_{j,i} z^{j+i} + O(z^{s}) \tag{64}$$

and $O(z^{s})$ represents a remainder term in $M_{P}^{s}(S)$. Unless ζ has order $o(g)$, then there is a power $h = g^{s}$, where h is a non-identity element, and

$$\left(h^{-1}\right)^{*} \to \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & \beta_{1,1} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & \beta_{1,s-2} & \beta_{2,s-3} & \cdots & 1 \end{bmatrix} \tag{65}$$

where the $\beta_{j,i}$ have a definition similar to $\alpha_{j,i}$ Let B be the matrix on the right in equation 65. Taking a power B^{r} multiplies the elements $\beta_{j,1}$ of the first subdiagonal by r. It follows that $o(h)\beta_{j,1} = 0$ as $B^{o(h)}$ equals the identity matrix. Thus, the first subdiagonal is zero. Continuing inductively, we show that all subdiagonals are zero, and B is the identity. It follows that h is the identity of G_{P} since s was taken large enough to guarantee that the representation was faithful on G_{P}. But this contradicts that h was a non-identity element.

Coincidentally, the argument shows that G_{P} must be cyclic, since it it isomorphic to the group of $o(g)$ roots of unity.

The discussion above is helpful in defining rotation constants for a curve over an algebraically closed field $\mathbf{k}$ of positive characteristic p. In positive characteristic a point is called wildly ramified if the rotation constant map $G_{P} \to \mathbf{k}^{*}$, $g \to \left(dg^{-1}\right)^{*}$ is not faithful. Note that the image of G_{P} in $\mathbf{k}^{*}$ must have order relatively prime to p. In the wildly ramified case the matrix in equation 65 will not be the identity for some h and some level s. Let us give a simple example of wild ramification.

Example 36 *Let $S = \mathbb{P}^1(\mathbf{k})$ and $G \subset \mathbf{k}$ be a finite additive group, e.g., a finite subfield. Define the action of G on $\mathbf{k}$, the affine part of S, by $(a, z) \to z + a$ for $a \in G, z \in \mathbf{k}$. To extend the action of G to the point at infinity we let $w = 1/z$ be a coordinate patch about ∞, with $w = 0$ corresponding to infinity and z and w defined on the common domain $\mathbf{k}^*$. In w-coordinates the action of G is given by*

$$(a, w) \to \frac{1}{\frac{1}{w} + a} = \frac{w}{1 + aw}. \tag{66}$$

Thus the G action has no fixed points on $\mathbf{k}$ and fixes ∞ in its entirety. If we compute the B matrix of equation 65 for the h corresponding to $z \to z + a$ and for level $s = 6$ we get:

$$\left(h^{-1}\right)^* \to \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & a & 1 & 0 & 0 & 0 \\ 0 & a^2 & 2a & 1 & 0 & 0 \\ 0 & a^3 & 3a^2 & 3a & 1 & 0 \\ 0 & a^4 & 4a^3 & 6a^2 & 4a & 1 \end{bmatrix}. \tag{67}$$

The matrix entries are obtained by computing power series for $\left(\frac{w}{1-aw}\right)^j$. We see that the representation is already faithful for $s = 3$, by looking at the 3×3 upper left matrix. It is a satisfying exercise to show that the matrix really defines a representation of G and that the pth power of the matrix is the identity. This action is wildly ramified and the stabilizer at infinity can be a very large elementary abelian group.

Remark 37 *To extend the definitions of $L(g)$ and $P(g)$ to fields other than $\mathbb{C}$, we need to select a system of primitive Nth roots ζ_N in $\mathbf{k}^*$ which satisfy:*

$$\zeta_N = (\zeta_{MN})^M \tag{68}$$

and then define

$$l_k(g) = \left|\left\{P \in S^g : \varepsilon(P, g) = \zeta_{o(g)}^k\right\}\right| \tag{69}$$

and

$$p_k(g) = \left|\left\{P \in S^g : G_P = \langle g \rangle\,,\ \varepsilon(P, g) = \zeta_{o(g)}^k\right\}\right|. \tag{70}$$

The rest is straight forward. The ζ_N only need to be defined for N dividing the exponent of G.

3 Rotation data and generating vectors

We are next going to show how to compute the rotation constant data for a G-action from a generating vector for the action. We will first work out the simpler cyclic case, since this is of greatest interest, and then move on to the general case. We can record rotation data and the generating action by three vectors, all of which are interrelated by invertible matrices. We will introduce these vectors and discuss their relationship in the next subsection. Before proceeding to the relation between rotation data and generating vectors, we list some well-known facts in the next four lemmas, valid for an arbitrary finite group and its action on a surface. The first two lemmas are easy statements about finite groups and the last two are about rotation numbers, normalizers, and centralizers. For later work, we also explicitly describe the $IOIE$ ordering for cyclic groups.

3.1 Group theoretic considerations

We leave the proof of the next lemma to the reader.

Lemma 38 *Suppose that G is a finite group, $g, h \in G$ and that:*

$$g = h^r, 0 \leq r < o(h)$$

and

$$o(h) = n, \ \ \gcd(r, n) = e, \ \ k = r/e.$$

Then, $g = h^{ek}$ with

$$e = o(h)/o(g)$$

and

$$k < o(g) \text{ and } \gcd(k, o(g)) = 1.$$

Lemma 39 *Let G be a finite group and $H_1, H_2 \leq G$ be conjugate subgroups such that $xH_1x^{-1} = H_2$. Then $Ad_x : y \to xyx^{-1}$ maps $N_G(H_1)$ bijectively to $N_G(H_2)$. If x' is another element such that $x'H_1x'^{-1} = H_2$ then $x^{-1}\, x' \in N_G(H_1)$ and $x'x^{-1} \in N_G(H_2)$.*

Proof. Let $y \in N_G(H_1)$ and $h_2 \in H_2$. Set $h_1 = x^{-1}h_2x \in H_1$ and $h_1' = yh_1y^{-1} \in H_1$. Then

$$xyx^{-1}h_2xy^{-1}x^{-1} = xyh_1y^{-1}x^{-1} = xh_1'x^{-1} \in H_2$$

and, hence, $xyx^{-1} \in N_G(H_2)$. So Ad_x maps $N_G(H_1)$ to $N_G(H_2)$ with inverse $Ad_{x^{-1}}$. Next observe that $Ad_{x^{-1}x'}$ maps

$$H_1 \xrightarrow{Ad_{x'}} H_2 \xrightarrow{Ad_{x^{-1}}} H_1$$

so that $x^{-1}\, x' \in N_G(H_1)$ and likewise $x'x^{-1} \in N_G(H_2)$. ■

IOIE ordering In subsequent sections, when discussing cyclic groups, we are going use the *IOIE* ordering of elements of $G = \langle g \rangle$ starting with $g_0 = 1$ as in Assumption 34 and Example 35. We will be a bit more explicit about the quantities, r_j, e_j, k_j, and d_j defined by the ordering $g_0 = 1, \ldots, g_{n-1}$ of $\langle g \rangle$. The exponents $r_0, \ldots, r_{n-1}$ are defined by

$$g_j = g^{r_j}. \tag{71}$$

The r_j, and e_j, k_j, and d_j defined from them, satisfy (see Lemma 38)

$$\begin{array}{rcl} r_j & = & e_j k_j, \\ e_j & = & \gcd(n, r_j),\ k_j = \dfrac{r_j}{e_j},\ d_j = \dfrac{n}{e_j} = o(g_j), \\ 1 & \leq & k_j \leq d_j,\ \gcd(k_j, d_j) = 1. \end{array} \tag{72}$$

Let

$$V_d = \mathbb{Z}_d^* = \{k : 1 \leq k \leq d-1,\ \gcd(k, d) = 1\} \tag{73}$$

The g_j are in blocks of non-decreasing order $o(g_j)$ (non-increasing e_j). Each block for a fixed e consists of elements of order $d = n/e$, namely

$$\mathcal{O}_d = \left\{ g^{ek} : k \in V_d \right\} \tag{74}$$

where the elements of V_d are ordered as they would be in $\mathbb{Z}$.

3.2 Fixed points, normalizers, and centralizers

Lemma 40 Fixed Points. *Let $(a_1, \ldots, a_\tau, b_1, \ldots, b_\tau, c_1, \ldots, c_t)$ be a generating vector for a G-action on a surface S and let $Q_1, \ldots, Q_t$ be the branch points of $\pi_G : S \to S/G$. Then we have the following:*

1. *There is a point P_j, lying over Q_j such that the stabilizer $G_{P_j} = \langle c_j \rangle$ and such that GP_j, the orbit of points lying over Q_j, are in 1-1 correspondence to the cosets in $G/\langle c_j \rangle$ via $h\langle c_j \rangle \leftrightarrow hP_j$. Moreover, the stabilizer of hP_j is $h\langle c_j \rangle h^{-1}$.*

2. *Every fixed point P of $g \in G - \{1\}$ lies over exactly one Q_j.*

3. *An element $g \in G - \{1\}$ fixes a point $h_j P_j$ of S lying over Q_j if and only if*
$$g = h_j c_j^{r_j} h_j^{-1}, \tag{75}$$
for some r_j. The exponent r_j factors as
$$r_j = e_j k_j, \tag{76}$$
where
$$e_j = \gcd(r_j, n_j) = \frac{o(c_j)}{o(g)} \tag{77}$$
and
$$\gcd(k_j, o(g)) = 1 \text{ and } k_j < o(g). \tag{78}$$

4. *If an element $g \in G - \{1\}$ fixes a point $h_j P_j$ of S lying over Q_j then the fixed points of g lying over Q_j are:*
$$S^g \cap GP_j = \{x h_j P_j : x \in N_G(\langle g \rangle)\} \tag{79}$$
and
$$|S^g \cap GP_j| = \frac{|N_G(\langle g \rangle)|}{o(c_j)}. \tag{80}$$

Proof. A point P_j as in statement 1 is constructed in the proof of statement 1 of Lemma 42 below. Statement 4 tells us how many choices we have for P_j. The remaining details of statement 1 and 2 are left to the reader. We also leave the proof of statement 3 to the reader noting that results of Lemma 38 are used.

For statement 4 assume that g fixes an additional point lying over Q_j. By statement 1 the point must be of the form $x h_j P_j$ and $g x h_j P_j = x h_j P_j$ or $h_j^{-1} x^{-1} g x h_j P_j = P_j$. It follows that $h_j^{-1} x^{-1} g x h_j \in \langle c_j \rangle$, or $x^{-1} g x \in h_j \langle c_j \rangle h_j^{-1}$. Since g and $x^{-1} g x$ have the same order, and both lie in the cyclic group $h_j \langle c_j \rangle h_j^{-1}$, they both generate the same subgroup of order $o(g)$. It follows that $x^{-1} g x = g^s$ for some s, so $x \in N_G(\langle g \rangle)$. If, on the other hand, $x \in N_G(\langle g \rangle)$ and $x^{-1} g x = g^s$ then, $g x h_j P_j = x x^{-1} g x h_j P_j = x g^s h_j P_j = x h_j P_j$, establishing 79. Finally, for equation 80, consider the surjective map $N_G(\langle g \rangle) \to S^g \cap \pi_G^{-1}(Q_j)$ defined by $x \to x h_j P_j$. The fibres of the map are the cosets $x h_j \langle c_j \rangle h_j^{-1}$ in $N_G(\langle g \rangle)/h_j \langle c_j \rangle h_j^{-1}$, from which 80 follows. Note that $h_j \langle c_j \rangle h_j^{-1} \subseteq N_G(\langle g \rangle)$ as follows. By Lemma 39 Ad_{h_j} maps $N_G\left(\left\langle c_j^{r_j} \right\rangle\right)$ bijectively to $N_G\left(\left\langle h_j c_j^{r_j} h_j^{-1} \right\rangle\right) = N_G(\langle g \rangle)$. But $\langle c_j \rangle \subseteq N_G\left(\left\langle c_j^{r_j} \right\rangle\right)$, and so $h_j \langle c_j \rangle h_j^{-1} \subseteq N_G(\langle g \rangle)$. ■

Remark 41 *Total number of fixed points. From Statement 1 of the preceding lemma we see that the totality of points fixed by any nontrivial g, namely* $\bigcup_{g \in G - \{1\}} S^g$, *has the following size:*

$$\left| \bigcup_{g \in G - \{1\}} S^g \right| = \sum_{j=1}^{t} \frac{|G|}{o(c_j)} = \sum_{j=1}^{t} \frac{|G|}{n_j}. \tag{81}$$

The total number of fixed points can also be computed using distinguished stabilizers and the vector $\overline{P}_G$. *For a given* $h \in G$, $\delta^{-1}(h)$ *is the set of points for which h is a distinguished stabilizer, and* $\delta^{-1}(h_1)$ *and* $\delta^{-1}(h_2)$ *are disjoint if* $h_1 \neq h_2$. *For an entire conjugacy class,* h^G, *the sets* $\delta^{-1}(h')$, $h' \in h^G$ *are all pairwise disjoint and have the same cardinality, so that, using equation 62,*

$$\left| \bigcup_{h' \in h^G} \delta^{-1}(h') \right| = \left| h^G \right| p_1(h).$$

We finally arrive at

$$\sum_{j=1}^{t} \frac{|G|}{n_j} = \left| \bigcup_{g \in G - \{1\}} S^g \right| = \sum_{j=2}^{l} |G^{g_j}| \, p_1(g_j) \tag{82}$$

expressing the number of fixed point in terms of the signature of the reduced primitive rotation vector $\overline{P}_G$. *The* g_j *are defined in Assumption 34.*

Lemma 42 *Rotation constants. Let* $g, c_j, P_j, Q_j, h_j, r_j, e_j,$ *and* k_j *be as in Lemma 40. Then we have the following:*

1. *There is a point* P_j, *(selected from the possible points* P_j *in the preceding lemma) lying over* Q_j, *such that* $G_{P_j} = \langle c_j \rangle$ *and*

$$\varepsilon(P_j, c_j) = \exp\left(-\frac{2\pi i}{o(c_j)}\right). \tag{83}$$

2. *Let $g \in G - \{1\}$, fixing $h_j P_j$ and suppose that $g = h_j c_j^{r_j} h_j^{-1}$ with $r_j = e_j k_j$ as in equation 76. Then we have*

$$\varepsilon(h_j P_j, g) = \exp\left(-\frac{2\pi i k_j}{o(g)}\right). \tag{84}$$

3. *Let $g \in G - \{1\}$ and let $N = N_G(\langle g \rangle)$ and $Z = Z_G(\langle g \rangle)$. Suppose that g fixes a point $h_j P_j$ lying over Q_j, so that 75 - 80 hold. Let*

$$U = \left\{u \in \mathbb{Z}_{o(g)}^* : g^u = xgx^{-1}, \text{ for some } x \in N_G(\langle g \rangle)\right\}.$$

Then, the rotation constants $\varepsilon(P, g)$ for P in $S^g \cap \pi_G^{-1}(Q_j)$ have the form

$$\exp\left(-\frac{2\pi i k_j u}{o(g)}\right), u \in U. \tag{85}$$

Moreover, there are $|N| / |Z|$ distinct values of such rotation constants, with $|Z| / o(c_j)$ repeats of each value.

Proof. For statement 1 we assume that the loops $\alpha_i, \beta_i, \gamma_j$ are chosen in the standard way, and that the a_i, b_i, c_j are determined as deck transformations by lifting the loops $\alpha_i, \beta_i, \gamma_j$. Since γ_j encircles Q_j exactly once in a counter-clockwise fashion the automorphism $\epsilon(c_j)$ is a local counter-clockwise rotation through $2\pi/o(c_j)$ radians around some point P_j. The points $P_1, \ldots, P_t$ can be identified by lifting $\gamma_1^{n_1}, \ldots, \gamma_t^{n_t}$.

For statement 2 observe that

$$\begin{aligned}
\varepsilon(h_j P_j, g) &= \varepsilon(h_j P_j, h_j c_j^{r_j} h_j^{-1}) \\
&= \varepsilon(P_j, c_j^{r_j}) \\
&= \exp\left(-\frac{2\pi i r_j}{o(c_j)}\right) \\
&= \exp\left(-\frac{2\pi i k_j}{o(g)}\right).
\end{aligned}$$

For statement 3 we argue as follows. By equation 79 the fixed points in $S^g \cap \pi_G^{-1}(Q_j)$ comprise the set $\{x h_j P_j : x \in N_G(\langle g \rangle)\}$. Now, letting $xgx^{-1} =$

g^u, then $xg^{u^{-1}}x^{-1} = g$ and

$$\begin{aligned} \varepsilon(xh_jP_j, g) &= \varepsilon(xh_jP_j, xg^{u^{-1}}x^{-1}) \\ &= \varepsilon(h_jP_j, g^{u^{-1}}) \\ &= \varepsilon(P_j, c_j^{r_j})^{u^{-1}} \\ &= \exp\left(-\frac{2\pi i k_j u^{-1}}{o(g)}\right). \end{aligned}$$

The integer u^{-1} runs though U as u runs though U, since U is a subgroup of $\mathbb{Z}^*_{o(g)}$. Thus we get all the rotation constants in equation 85. Since the value u is the same on each coset of $Z_G(\langle g \rangle)$ in $N_G(\langle g \rangle)$ then there are $|N| / |Z|$ distinct values. Since the total number of fixed points in $S^g \cap \pi_G^{-1}(Q_j)$ is $|N| / o(c_j)$ then there are $|Z| / o(c_j)$ repeats of each value. ■

3.3 Action vectors

To aid in our discussion in the remainder of this section and our calculation of the character $\mathrm{ch}_{\mathcal{H}^q(S)}$ in Section 4, we introduce three vectors corresponding to total rotation data, primitive rotation data, and generating vectors. We will assume that the conjugacy class representatives $g_0 = 1, \ldots, g_{l-1}$ of G are ordered as in Assumption 34. If $G = \langle g \rangle$ is cyclic then we assume $IOIE$ ordering (3.1) of $g_0, \ldots, g_{n-1}$.

We extend the vector $\overline{L}_G$ to the vector X, called the *total action vector,* to take into account the genus of S. Set

$$\begin{aligned} x_0 &= \sigma - 1,\ x_j = l_1(g_j),\ j > 0 \\ X &= \begin{bmatrix} x_0 & \cdots & x_{l-1} \end{bmatrix}^t = \begin{bmatrix} x_0 & \overline{L}_G \end{bmatrix}^t. \end{aligned} \tag{86}$$

The value x_0 corresponds to the identity element, for which l_1 is not defined, but it allows us to keep track of the genus of S.

Analogously, we extend the vector $\overline{P}_G$ to the *primitive action vector, Z,* via

$$\begin{aligned} z_0 &= \sigma - 1,\ z_j = p_1(g_j),\ j > 0 \\ Z &= \begin{bmatrix} z_0 & \cdots & z_{l-1} \end{bmatrix}^t = \begin{bmatrix} x_0 & \overline{P}_G \end{bmatrix}^t. \end{aligned} \tag{87}$$

The vectors X, Z satisfy

$$X = AZ,\ A = \begin{bmatrix} 1 & 0 \\ 0 & A' \end{bmatrix}, \tag{88}$$

where A' is determined in the next two subsections. We summarize the foregoing in a proposition, which we prove in the next two subsections.

Proposition 43 *Let G be group acting on a surface S and suppose that X and Z are the* total action vector *and* primitive action vector, *respectively. The vectors are to be computed with respect to a list of conjugacy class representatives $g_0 = 1, \ldots, g_{l-1}$ ordered according to Assumption 34 or IOIE ordering. Then, there is a matrix A as given in equation 88, such that A is an upper triangular matrix with integer entries whose inverse has integer entries. The formulas for the entries of the submatrix A' are given in equations 100 for the cyclic case and in 111 for the general case.*

Next, consider a generating vector for the G action:

$$\mathcal{V} = (a_1, \ldots, a_\tau, b_1, \ldots, b_\tau, c_1, \ldots, c_t) . \tag{89}$$

If G is abelian then the product $c_1 \cdots c_t = 1$ and the braid action (equation 27) simply permutes the c_j. The numbers

$$w_j = |\{i : c_i = g_j\}| \tag{90}$$

are invariant under the braid action and so the periodic part $(c_1, \ldots, c_t)$ of a generating vector, up to reordering, may be represented by $(w_1, \ldots, w_{l-1})$. We set

$$w_0 = \tau - 1, W = \left[\begin{array}{ccc} w_0 & \cdots & w_{l-1} \end{array} \right]^t \tag{91}$$

and call W the *generating action vector*. We will show that Z, W satisfy a matrix relation

$$Z = BW \tag{92}$$

where we determine a formula for B in the next subsection.

We can also construct a generating action vector W in the general case. The braid action permutes the conjugacy classes $C_i = c_i^G$ and hence the numbers

$$w_j = \left|\left\{i : C_i = g_j^G\right\}\right|, \ j > 0 \tag{93}$$

are also invariant under the braid action. As before, we set

$$w_0 = \tau - 1, W = \left[\begin{array}{ccc} w_0 & \cdots & w_{l-1} \end{array} \right]^t . \tag{94}$$

Formula 92 also holds where the formula for B is determined in subsection 3.5. In the abelian case, both definitions in equations 90 and 93 yield the same vector and the two formulas for B yield the same result. We summarize the foregoing in a proposition, which we prove in the next two subsections.

Proposition 44 *Let G be group acting on a surface S and suppose that Z and W are the* primitive action vector *and the* generating action vector *respectively. The vectors are to be computed with respect the generating vector $\mathcal{V}$ (equation 89) and a list of conjugacy class representatives $g_0 = 1, \ldots, g_{l-1}$, ordered according to Assumption 34 or IOIE ordering. Then, there is a matrix as given in equation 88, such that B is an invertible, upper triangular matrix with integer entries. The formulas the entries of the submatrix are given in equations 103 for the cyclic case and in 112 for the general case.*

Remark 45 *For W to be an action generating vector for a cyclic group, the following are necessary and sufficient conditions on the vector W.*

1. *The first component must be an integer ≥ -1, the remaining components are nonnegative integers.*
2. *Let $E = \left[\begin{array}{ccc} r_0 & \cdots & r_{n-1} \end{array}\right]$ be the vector determined by $g_j = g^{r_j}$. Then the dot product $E \cdot W = 0 \bmod n$*
3. *The least common multiple of the periods present in the signature must equal n or the first component must be ≥ 0.*

To see why these are necessary and sufficient conditions select a hypothesized generating vector $\mathcal{V}$ (equation 89) whose action vector is W. For statement 1 we note that the first component equals $\tau - 1$ where τ is the orbit genus. For statement 2 we see that $c_1 \cdots c_t = g^{E \cdot W}$. Since $c_1 \cdots c_t = 1$ the dot product condition is forced. Note that the orbit genus value does not affect the value of the dot products mod n as $r_0 = n$. In statement 3, if the condition on the periods holds then $G = \langle c_1, \ldots, c_t \rangle$. Otherwise, if the first component is ≥ 0 then $\tau \geq 1$ and we may take $a_1 = g$ to satisfy $G = \langle a_1, \ldots, a_\tau, b_1, \ldots, b_\tau, c_1, \ldots, c_t \rangle$.

3.4 The cyclic case

Suppose that $G = \langle h \rangle$ is cyclic, we shall compute $l_k(g)$ and $p_k(g)$ for $g \in G - \{1\}$, assuming we have a given generating vector for the action. This approach for computing $l_k(g)$ and $p_k(g)$ was used in the prime order case in the parent paper [7], and in that case $l_k(g) = p_k(g)$. We shall also show how to compute $\overline{L}_G$ and $\overline{P}_G$, a linear relation between $\overline{L}_G$ and $\overline{P}_G$, and discuss the relationship between $\overline{P}_G$ and the generating vector. These relations are summarized by equations 88 and 92 and Propositions 43 and 44. We will prove all these assertions for cyclic groups in this section.

Let $(c_1, \ldots, c_t)$ be the list of elliptic elements for a generating vector for the G-action. We first compute the rotation constants of the elements $c_j^r \in \langle c_j \rangle - \{1\}$ at the points lying over Q_j. Fix P_j as in Lemmas 40 and 42. By statement 1 of Lemma 40 every point lying over Q_j has the form $h_j P_j$ for some $h_j \in G$, and there are $u_j = o(h)/o(c_j)$ such $h_j P$. Since $c_j h_j P_j = h_j c_j P_j = h_j P_j$ then c_j fixes all these points and likewise c_j^r fixes all these points. Write $r = ek$ where $e = \gcd(r, n_j)$. Then by Lemma 38 we get $e = o(c_j)/o(c_j^r)$, and k is relatively prime to $o(c_j^r)$ and $k < o(c_j^r)$. By equation 84 we get:

$$\varepsilon(h_j P_j, c_j^r) = \exp\left(-\frac{2\pi i k}{o(c_j^r)}\right). \tag{95}$$

Thus we get u_j fixed points lying over Q_j all having the same rotation constant $\exp\left(-\frac{2\pi i k}{o(c_j^r)}\right)$.

Now fix $g \in G$ and vary the Q_j and c_j. By statement 2 of Lemma 40, any point fixed by g must lie over exactly one Q_j. By statement 3 of the same lemma every point lying over Q_j has the form $h_j P_j$, and g fixes $h_j P_j$ if and only if $g = h_j c_j^{r_j} h_j^{-1} = c_j^{r_j}$ for some r_j. Since G is cyclic $g = c_j^{r_j}$ for some r_j if and only if $o(g)$ divides $o(c_j)$. We write $r_j = e_j k_j$ where $e_j = o(c_j)/o(g)$. From equation 95 we see that we have u_j fixed points lying over Q_j with rotation constant $\exp\left(-\frac{2\pi i k_j}{o(g)}\right)$. Defining

$$\begin{aligned} l_k^j(g) &= u_j \text{ if } o(g)|o(c_j) \text{ and } g = c_j^{e_j k}, \\ l_k^j(g) &= 0 \text{ otherwise}, \end{aligned}$$

we get

$$l_k(g) = \sum_{j=1}^{t} l_k^j(g).$$

The reduced vector $\overline{L}_G$ is formed from the $l_1^j(g)$:

$$\begin{aligned} l_1^j(g) &= \frac{n}{o(c_j)} \text{ if } o(g)|o(c_j),\ g = c_j^{e_j}, e_j = o(c_j)/o(g) \\ l_1^j(g) &= 0 \text{ otherwise}, \\ l_1(g) &= \sum_{j=1}^{t} l_1^j(g). \end{aligned}$$

For primitive rotation data we set:

$$\begin{aligned} p_k^j(g) &= \frac{n}{o(c_j)} \text{ if } o(g) = o(c_j) \text{ and } g = c_j^k, \\ p_k^j(g) &= 0 \text{ otherwise,} \end{aligned}$$

and then

$$p_k(g) = \sum_{j=1}^{t} p_k^j(g).$$

The reduced vector $\overline{P}_G$ is formed from the $p_1^j(g)$:

$$\begin{aligned} p_1^j(g) &= u_j \text{ if } g = c_j, \\ p_1^j(g) &= 0 \text{ otherwise,} \\ p_1(g) &= \sum_{j=1}^{t} p_1^j(g). \end{aligned}$$

Example 46 *Consider the cyclic group $\mathbb{Z}_6$ acting on a surface of genus 6 with signature $(2,3,6,6,6)$ and a total of 8 fixed points. To be specific, select the generating vector $(3,2,1,1,5)$. Assuming a class ordering of $G = \langle h : h^6 = 1 \rangle$ given by $h^6, h^3, h^2, h^4, h, h^5$, the systems $\{L(g)\}$ and $\{P(g)\}$ are given in the table below. The column $\mathbb{Z}^*_{o(g)}$ simply records the positive integers $k < o(g)$ relatively prime to $o(g)$. The set $\mathbb{Z}^*_{o(g)}$ is the domain of $L(g)$ and $P(g)$.*

class	$o(g)$	$\mathbb{Z}^*_{o(g)}$	$L(g)$	$P(g)$
h^3	2	$\{1\}$	(6)	(3)
h^2	3	$\{1,2\}$	$(4,1)$	$(2,0)$
h^4	3	$\{1,2\}$	$(1,4)$	$(0,2)$
h	6	$\{1,5\}$	$(2,1)$	$(2,1)$
h^5	6	$\{1,5\}$	$(1,2)$	$(1,2)$

Table 1. Rotation constant data for $\mathbb{Z}_6 = \langle h : h^6 = 1 \rangle$

From the table we see that $\overline{L}_G = (6,4,1,2,1)$ and $\overline{P}_G = (3,2,0,2,1)$.

Example 47 *We consider the cyclic group $\mathbb{Z}_5$ acting on various surfaces of genus 4 with signature $(5,5,5,5)$ and a total of 4 fixed points. In the prime* *cyclic case the total number of fixed points equals the number of branch points. Since we assume that there are 4 branch points, then, up to topological equivalence, the generating vectors are* $(1,1,1,2)$, $(1,1,4,4)$ *and* $(1,2,3,4)$. *In the 2nd column we give the vectors for* $\overline{L}_G = \overline{P}_G = (p_1(g), p_1(g^2), p_1(g^3), p_1(g^4))$. *In the prime cyclic case we always have* $\overline{L}_G = \overline{P}_G$.

$\mathcal{V}$	$\overline{L}_G = \overline{P}_G$
$(1,1,1,2)$	$(3,1,0,0)$
$(1,1,4,4)$	$(2,0,0,2)$
$(1,2,3,4)$	$(1,1,1,1)$

Table 2. Rotation constant data for $\mathbb{Z}_5$*-actions*

Transition matrices for the cyclic case We now prove Propositions 43 and 44, in the cyclic case, by determining formulas and giving numerical examples for the submatrix A' and the matrix B, introduced in equations 88 and 92, respectively.

As seen in our discussion above, the vector $\overline{P}_G$ has can be directly computed from the generating vector in a simple fashion. The vector $\overline{L}_G$ takes a bit more work. However, we now show that

$$\overline{L}_G = A'\overline{P}_G \tag{96}$$

$$\overline{P}_G = (A')^{-1}\,\overline{L}_G \tag{97}$$

where A' is a square $(n-1)\times(n-1)$ invertible, upper triangular matrix of integers, determined by the group structure of G. Let $g \in G - \{1\}$ be an arbitrary element. Let $F_i = \delta^{-1}(h_{P,i})$, $i = 1, \ldots, w$ be the partition of the fixed points of the G action on S guaranteed by Lemma 29. Suppose that $P \in F_i$ is a simple, inherited fixed point of g. Then $g = h_{P,i}^{e_i}$ with $e_i = o(h_{P,i})/o(g)$ according to Lemma 29. Also by the lemma all points of F_i are simple fixed points of g. Since $|F_i| = p_1(h_{P,i})$ and the F_i are disjoint, then

$$l_1(g) = \sum_{h_{P,i}} p_1(h_{P,i}),$$

where the sum is over the set $\left\{h_{P,i} : g = h_{P,i}^{e_i}, e_i = o(h_{P,i})/o(g)\right\}$. Now if $g = h^e$, $e = o(h)/(o(g))$ and h is not one of $h_{P,i}$ then $p_1(h) = 0$ by definition. Thus we actually have:

$$l_1(g) = \sum_{h \in \mathrm{roots}_G(g)} p_1(h), \tag{98}$$

where

$$\text{roots}_G(g) = \{h \in G : g = h^e, e = o(h)/o(g)\}. \qquad (99)$$

This formula is valid for all possible cyclic G actions and so we may find a matrix A' encoding the formulas that is independent of the actual distribution of simple, primitive fixed points $\overline{P}_G$. It follows that the matrix A' of equation 96 is given by

$$\begin{aligned} A'_{g,h} &= 1, \ g = h^e, e = o(h)/o(g), \\ A'_{g,h} &= 0, \ \text{otherwise}, \end{aligned} \qquad (100)$$

where we use $g, h \in G - \{1\}$ to index the entries of A.

Remark 48 *By construction, because of the ordering assumed in Assumption 34, $A'_{g,g} = 1$, and $A'_{g,h} > 0$, $g \neq h$ implies that $o(g) < o(h)$. Therefore A' is upper triangular with 1's on the diagonal and, hence, an invertible matrix. Likewise the equation 88 holds since*

$$X = \begin{bmatrix} \sigma - 1 \\ \overline{L}_G \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & A' \end{bmatrix} \begin{bmatrix} \sigma - 1 \\ \overline{P}_G \end{bmatrix} = AZ.$$

Next we show how to compute the generating vector from rotation data, at least the elliptic elements. We note that for each Q_j there are $u_j = |G| / o(c_j)$ simple fixed points of c_j lying over Q_j. To get the elliptic part of the signature and the generating vector we write down a list of $p_1(g_1)/u_1$ periods $o(g_1)$ and $p_1(g_1)/u_1$ copies of g_1, then $p_1(g_2)/u_2$ periods $o(g_2)$ and $p_1(g_2)/u_2$ copies of g_2, etc. If indeed $\overline{P}_G$ comes from a generating vector then the product of the elements written down will be the identity. In terms of the vectors Z and W (equations 87 and 91) we have:

$$z_j = p_1(g_j) = \frac{n}{o(g_j)} w_j = u_j w_j, \ j > 0. \qquad (101)$$

For z_0 we rewrite the Riemann Hurwitz equation:

$$\begin{aligned} 2\sigma - 2 &= |G|(2\tau - 2) + \sum_{i=1}^{t} |G| \left(1 - \frac{1}{n_i}\right) \\ &= |G|(2\tau - 2) + \sum_{j=1}^{n-1} |G| w_j \left(1 - \frac{1}{o(g_j)}\right) \\ &= n(2\tau - 2) + \sum_{j=1}^{n-1} \left(n - \frac{n}{o(g_j)}\right) w_j, \\ &= n(2\tau - 2) + \sum_{j=1}^{n-1} (n - u_j) w_j, \end{aligned}$$

or

$$z_0 = \sigma - 1 = nw_0 + \sum_{j=1}^{n-1} \frac{n - u_j}{2} w_j. \tag{102}$$

The matrix B such that $Z = BW$ is, therefore, the upper triangular, invertible matrix

$$B = \begin{bmatrix} n & \frac{n-u_1}{2} & \cdots & \frac{n-u_{n-1}}{2} \\ & u_1 & \cdots & 0 \\ & & \ddots & \vdots \\ & & & u_{n-1} \end{bmatrix}. \tag{103}$$

From its structure, we see that it is invertible.

Here are some numerical examples for $G = \mathbb{Z}_6$.

Example 49 *Continuing Example 46 we get:*

$$A' = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \ (A')^{-1} = \begin{bmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

We check $\overline{L}_G - A'\overline{P}_G = 0:$

$$\begin{bmatrix} 6 \\ 4 \\ 1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The equation $X = AZ$ is also easily verified.

Next, we see that:

$$Z = \begin{bmatrix} 5 \\ 3 \\ 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \; W = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \; B = \begin{bmatrix} 6 & \frac{3}{2} & 2 & 2 & \frac{5}{2} & \frac{5}{2} \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} \frac{1}{6} & -\frac{1}{12} & -\frac{1}{6} & -\frac{1}{6} & -\frac{5}{12} & -\frac{5}{12} \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

We check that $Z - BW = 0$:

$$\begin{bmatrix} 5 \\ 3 \\ 2 \\ 0 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 6 & \frac{3}{2} & 2 & 2 & \frac{5}{2} & \frac{5}{2} \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

We conclude the discussion of cyclic n-gonal actions with a proof that the RDS system is a classifier for topological equivalence, as summarized in the proposition below. See the works of Gilman [13] and Harvey [16] for variants of this result.

Proposition 50 *Two cyclic n-gonal actions ϵ_1, ϵ_2 of $G = \mathbb{Z}_n$, on possibly different surfaces S_1, S_2, are topologically conjugate if and only if the rotational data systems are equal.*

Proof. Assume that $RDS(\epsilon_1) = RDS(\epsilon_2)$. Then $\overline{P}_G(\epsilon_1) = \overline{P}_G(\epsilon_2)$ and by the above discussion the branch points of each action may be reordered so that they have the same generating vectors with the ordering of the c_i compatible with the ordering $g_1, g_2, \ldots, g_{n-1}$. Then the actions are equivalent by Proposition 16. ■

3.5 The general case

Now let G be any group and $(a_1, \ldots, a_\tau, b_1, \ldots, b_\tau, c_1, \ldots, c_t)$ a generating vector for the G-action with signature $(\tau : n_1, \ldots, n_t)$. We repeat all the steps in the cyclic case taking into account non-commutativity. Any point fixed by a non-trivial element $g \in G$ must lie in exactly one of the orbits $GP_1, \ldots, GP_t$ where P_j is a simple, primitive fixed point of c_j described in Lemma 42. We compute the following quantities for one P_j at a time:

$$l_k^j(g) = \left| \left\{ P \in GP_j \cap S^g : \varepsilon(P, g) = \exp\left(-\frac{2\pi i k}{o(g)}\right) \right\} \right|. \quad (104)$$

We may add up the terms:

$$l_k(g) = \sum_{j=1}^{t} l_k^j(g).$$

Lemma 42 characterizes the rotation constants of the fixed points of g on GP_j. If g fixes some point $h_j P_j \in GP_j$ then $g = h_j c_j^{e_j k_j} h_j^{-1}$, where $k_j < o(g)$ is relatively prime to $o(g)$, and

$$\varepsilon(h_j P_j, g) = \exp\left(-\frac{2\pi i k_j}{o(g)}\right), \quad (105)$$

according to equation 84. According to Lemma 42

$$\begin{aligned} l_{k_j u}^j(g) &= \frac{|Z|}{o(c_j)}, \text{ for } u \in U, \\ l_{k_j u}^j(g) &= 0, \text{ otherwise,} \end{aligned}$$

where Z and U are as defined in the lemma.

For the primitive rotation data we define

$$p_k^j(g) = \left| \left\{ P \in GP_j \cap S^g : o(g) = o(c_j),\ \varepsilon(P, g) = \exp\left(-\frac{2\pi i k}{o(g)}\right) \right\} \right|. \quad (106)$$

At a primitive fixed point $P = h_j P_j$, lying over Q_j, we have $g = h_j c_j^{k_j} h_j^{-1}$ with $\gcd(k_j, o(c_j)) = 1$. According to Lemma 42:

$$\begin{aligned} p_{k_j u}^j(g) &= \frac{|Z|}{o(c_j)}, \text{ for } u \in U, \\ p_{k_j u}^j(g) &= 0, \text{ otherwise,} \end{aligned}$$

where Z and U are as defined in the lemma.

The reduced vector $\overline{L}_G$ is formed from the $l_1^j(g)$:

$$\begin{aligned} l_1^j(g) &= \frac{|Z|}{o(c_j)} \text{ if } o(g)|o(c_j),\ g = h_j c_j^{e_j k_j} h_j^{-1}, k_j^{-1} \in U, \\ l_1^j(g) &= 0 \text{ otherwise}, \\ l_1(g) &= \sum_{j=1}^{t} l_1^j(g). \end{aligned} \tag{107}$$

Likewise for $\overline{P}_G$ we get:

$$\begin{aligned} p_1^j(g) &= \frac{|Z|}{o(c_j)} \text{ if } o(g) = o(c_j),\ g = h_j c_j^{k_j} h_j^{-1}, k_j^{-1} \in U, \\ p_1^j(g) &= 0 \text{ otherwise}, \\ p_1(g) &= \sum_{j=1}^{t} p_1^j(g). \end{aligned} \tag{108}$$

Note that if $k_j^{-1} \in U$, then there is an $x \in N_G(\langle g \rangle)$ such that $xgx^{-1} = g^{k_j^{-1}} = h_j c_j^{k_j k_j^{-1}} h_j^{-1} = h_j c_j h_j^{-1}$, or $g = x^{-1} h_j c_j h_j^{-1} x$. Thus the condition $g = h_j c_j^{k_j} h_j^{-1}, k_j^{-1} \in U$ can be replaced by $g \in c_j^G$.

Example 51 *Consider the group $G = PSL_2(7)$ action on Klein's quartic of genus 3 with signature $(2, 3, 7)$ and 164 branch points total. A generating vector for G, represented as permutations of the 8 element set $\mathbb{P}^1(\mathbb{F}_7)$ is:*

$$\begin{aligned} c_1 &= (1,3)(2,6)(4,7)(5,8), \\ c_2 &= (3,6,7)(4,5,8), \\ c_3 &= (1,3,4,5,7,2,6). \end{aligned}$$

The non trivial conjugacy classes, the order of a class representative, $\mathbb{Z}_{o(g)}^*$, the orders of normalizers and centralizers, U and $L(g)$ are given in the table *below.*

class	$o(g)$	$\mathbb{Z}^*_{o(g)}$	$\|N_G(\langle g\rangle)\|$	$\|Z_G(\langle g\rangle)\|$	U	$L(g)$
$2A$	2	$\{1\}$	8	8	$\{1\}$	(4)
$3A$	3	$\{1,2\}$	6	3	$\{1,2\}$	$(1,1)$
$4A$	4	$\{1,3\}$	8	4	$\{1,3\}$	$(0,0)$
$7A$	7	$\{1,2,3,4,5,6\}$	21	7	$\{1,2,4\}$	$(1,1,0,1,0,0)$
$7B$	7	$\{1,2,3,4,5,6\}$	21	7	$\{1,2,4\}$	$(0,0,1,0,1,1)$

Table 3. Rotation Constant data for $PSL_2(7)$ Hurwitz action

We describe how to calculate $L(g)$ given the other entries. We leave the lines for 2A and 4A for the reader. For line 3A $|N_G(\langle g\rangle)|\,/\,|Z_G(\langle g\rangle)|$ has order 2 so $U=\{1,2\}$ and the multiplicity $|Z_G(\langle g\rangle)|\,/\,|\langle c_2\rangle|=1$. For the 7A line we note that c_3 lies in the 7A class. From the table, $U=\{1,2,4\}$ a subgroup $\mathbb{Z}^*_{o(g)}$ of order 3 and index 2. The multiplicity equals 1. The row for 7B is complementary, since c_3^3 lies in $7B$. Since all the elliptic elements have prime order the primitive rotation data is the same as the total rotation data. Finally, $\overline{L}_G=\overline{P}_G=(4,1,0,1,0)$. Note that if the periods of an action are all primes then $\overline{L}_G=\overline{P}_G$.

Example 52 *Consider the group $G=Alt_7$ acting on a surface S with signature $(3,4,6)$. The genus of S must be 316 and there are a total 1890 fixed points. A generating vector for G (computed randomly with MAGMA [23]) is*

$$\begin{aligned} c_1 &= (1,6,2)(3,7,4),\\ c_2 &= (1,3,6,5)(2,7),\\ c_3 &= (1,5)(2,3)(4,7,6). \end{aligned}$$

The orders of classes of G, in the standard MAGMA ordering, are given in the following table.

Class number	1	2	3	4	5	6	7	8	9
Order	1	2	3	3	4	5	6	7	7

The class number of c_1 is 4 so that class list for (c_1,c_2,c_3) is $(4,5,7)$. With this class ordering we have $\overline{L}_G=(10,6,3,1,0,2,0,0)$ and $\overline{P}_G=(0,0,3,1,0,2,0,0)$, also computed using MAGMA.

The only other class list for a $(3, 4, 6)$ *generating vector is* $(3, 5, 7)$. *By MAGMA computation there no generating vectors with a* $(3, 5, 7)$ *class list and there are three* $\mathrm{Aut}(G)$*-classes of generating vectors with class list* $(4, 5, 7)$. *By reviewing the construction of* $\overline{P}_G$ *we see that these three vectors have the same* $\overline{P}_G$. *But by example 14 the three actions are topologically inequivalent. Thus, it is possible to have topologically inequivalent actions with the same rotation data.*

The equality of rotation data for generating vectors with the same class lists is more general that the preceding example.

Proposition 53 *Suppose that two actions* ϵ, ϵ' *are given by generating vectors* $(a_1, \ldots, a_\tau, b_1, \ldots, b_\tau, c_1, \ldots, c_t)$ *and* $(a'_1, \ldots, a'_\tau, b'_1, \ldots, b'_\tau, c'_1, \ldots, c'_t)$. *Let* $\kappa : G \to K$ *be the class map. Then, if the class lists* $(\kappa(c_1), \ldots, \kappa(c_t))$ *and* $(\kappa(c'_1), \ldots, \kappa(c'_t))$ *are equal, we must have* $RDS(\epsilon) = RDS(\epsilon')$.

Proof. Going through the steps for computing the RDS we see that $l_k(g)$ only depends on $\kappa(g)$ and the $\kappa(c_j)$ not the actual values of the c_j. Also note that if the c'_j can be operated upon by braid operations so that we get the same class lists then $RDS(\epsilon) = RDS(\epsilon')$. ∎

Transition matrix for general case As in the cyclic case let us determine the transition matrix of the equation $\overline{L}_G = A'\overline{P}_G$. Computing the entries of A' is similar to the cyclic case, except that we need to account for non-commutativity. Let $g \in G - \{1\}$ be an arbitrary conjugacy class representative. We may use the argument of the cyclic case up to equation 98 to get an equation similar to equation 98.

$$l_1(g) = \sum_{h \in \mathrm{roots}_G(g)} p_1(h). \tag{109}$$

The cyclic structure of G was not used in deriving this formula or the definition of $\mathrm{roots}_G(g)$. The sum in equation 109 can be rewritten by aggregating over conjugacy classes:

$$l_1(g) = \sum_{C \in K - \{1\}} \sum_{h \in C \cap \mathrm{roots}_G(g)} p_1(h), \tag{110}$$

where K is the set of conjugacy classes of G.

Now we consider the structure of $C \cap \text{roots}_G(g)$. Suppose that $h_0 \in C \cap \text{roots}_G(g)$. A second element $h = xh_0x^{-1} \in C \cap \text{roots}_G(g)$ satisfies

$$g = (h)^e = xh_0^e x^{-1} = xgx^{-1}$$

so that $x \in Z_G(\langle g \rangle)$. Also any xh_0x^{-1} with $x \in Z_G(\langle g \rangle)$ is in $C \cap \text{roots}_G(g)$, by the same calculation. The set $C \cap \text{roots}_G(g)$ is therefore the $Z_G(\langle g \rangle)$ orbit of h_0 under conjugation and the subgroup $Z_G(\langle h_0 \rangle)$ is the stabilizer of h_0. Therefore, there are $|Z_G(\langle g \rangle)| \,/\, |Z_G(\langle h_0 \rangle)|$ group elements in $C \cap \text{roots}_G(g)$ if it is non-empty. The action of $x \in Z_G(g)$ on the simple fixed points of g arising from a chosen $h \in C \cap \text{roots}_G(g)$ is given by

$$\delta^{-1}(xhx^{-1}) = x\delta^{-1}(h),$$

and for $x \in Z_G(\langle h \rangle)$

$$x\delta^{-1}(h) = \delta^{-1}(xhx^{-1}) = \delta^{-1}(h).$$

Thus, $Z_G(\langle h \rangle)$ permutes the elements of $\delta^{-1}(h)$ and if $x, y \in Z_G(\langle g \rangle)$ have distinct images in $Z_G(\langle g \rangle)/Z_G(\langle h \rangle)$ then the sets of points $x\delta^{-1}(h)$ and $y\delta^{-1}(h)$ are disjoint. Since $p_1(h) = \left|\delta^{-1}(h)\right|$ It follows that for our chosen h

$$\sum_{h' \in C \cap \text{roots}_G(g)} p_1(h') = \frac{|Z_G(\langle g \rangle)|}{|Z_G(\langle h \rangle)|} p_1(h).$$

Note that $\frac{|Z_G(\langle g \rangle)|}{|Z_G(\langle h \rangle)|}$ depends only on the conjugacy classes of g and h. We can now determine the entries $A_{g,h}$ where $g \in C_g, h \in C_h$ are representatives of non-trivial conjugacy classes $C_g = g^G, C_h = h^G$ of G :

$$\begin{aligned} A'_{g,h} &= \frac{|Z_G(\langle g \rangle)|}{|Z_G(\langle h \rangle)|}, \; C_h \cap \text{roots}_G(g) \neq \phi \\ A'_{g,h} &= 0, \; \text{otherwise.} \end{aligned} \tag{111}$$

This equation reduces to equation 100 when G is cyclic.

Remark 54 *Determining the pairs C_g, C_h for which $C_h \cap \text{roots}_G(g) \neq \phi$ is easy as long as we have the class map $\kappa : G \to K$. For a given C_h and e dividing $o(h)$ we compute $\kappa(h^e) = C_g$. We then set $A'_{g,h} = \frac{|Z_G(g)|}{|Z_G(h)|}$. Also note that the discussion in Remark 48 also applies to the general transition matrix A'.*

Remark 55 *The map* $\rho_e : G \to G$, $\rho_e(h) = h^e$ *is an example of a* G-*equivariant map* $\rho : U \to V$ *between action spaces, namely, we have* $\rho(gu) = g\rho(u)$, $u \in U$. *In our particular case* $\rho_e(xhx^{-1}) = (xhx^{-1})^e = xh^e x^{-1} = x\rho_e(h)x^{-1}$. *It is easily shown that if* G *is transitive on* U *and* $v = \rho(u)$ *then* $\rho^{-1}(v)$ *is canonically isomorphic* G_v/G_u *via the map* $g \to gu$.

Example 56 *Continuing Example 52 we get for transition matrix.*

$$A = \begin{bmatrix} 1 & 0 & 0 & 6 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \ A^{-1} = \begin{bmatrix} 1 & 0 & 0 & -6 & 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We check that $\overline{L}_G - A\overline{P}_G = 0$.

$$\begin{bmatrix} 10 \\ 6 \\ 3 \\ 1 \\ 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 6 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 3 \\ 1 \\ 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

We finish by finding a formula for B in equation 92 for general G. Let g_j be a representative of C_j. According to equation 108 and the statement following, g_j has a simple, primitive fixed point lying over Q_i if and only if $g_j \in c_i^G$, i.e., $g_j^G = c_i^G$. In this case there are $|Z_G(g_j)|/o(c_i) = |Z_G(g_j)|/o(g_j)$ such fixed points. Thus the number of primitive fixed points of g_j is $u_j = |Z_G(g_j)|/o(g_j)$ fixed points times the number of repeats of g_j^G in the list $C_1, \ldots, C_t$. This is simply $u_j w_j$. Thus the diagonal elements $B(j,j)$, $j > 0$ are given by $B(j,j) = u_j$. there are no nonzero non-diagonal elements below the top row. The top row is computed in the same fashion as in the cyclic case using the Riemann Hurwitz formula. The final result is

$$B = \begin{bmatrix} n & \frac{1}{2}(n - v_1) & \cdots & \frac{1}{2}(n - v_{l-1}) \\ & u_1 & \cdots & 0 \\ & & \ddots & \vdots \\ & & & u_{l-1} \end{bmatrix}, \tag{112}$$

where $n = |G|$ and $v_j = |G| / o(g_j)$.

Example 57 *Continuing Example 51 we have*

$$Z = \begin{bmatrix} 2 \\ 4 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \; W = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \; B = \begin{bmatrix} 168 & 42 & 56 & 63 & 72 & 72 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} \frac{1}{168} & -\frac{1}{16} & -\frac{1}{3} & -\frac{3}{8} & -\frac{3}{7} & -\frac{3}{7} \\ 0 & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

We check that $Z - B'W = 0$:

$$\begin{bmatrix} 2 \\ 4 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 168 & 42 & 56 & 63 & 72 & 72 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

3.6 Equisymmetry and rotation data

Though we do not give a complete proof here, equisymmetric actions have the same rotation data. We formalize this in the proposition following.

Proposition 58 *Two equisymmetric actions ϵ_1, ϵ_2 of G, on possibly different surfaces S_1, S_2, have the same rotational data systems.*

Example 59 *In the family given in Example 21 the c_j are determined by the m_j and so are constant within the family. Thus the action vector W and $Z = BW$ and $X = ABW$ are constant, implying that the rotation data systems are constant.*

Example 60 *In example 52 we exhibited three topologically inequivalent actions with the same rotational data systems. Thus action with the same rotational data systems may not be equisymmetric.*

Remark 61 *As noted in Proposition 25, Guerrero proves in [14] that $\mathcal{H}^q(S_b)$ characters are constant in a holomorphic family with a connected base. By our later results in Section 5.2, rotation data can be recovered from a series of $\mathcal{H}^q$ characters for a cyclic group. Then Proposition 25 implies Proposition 58 for cyclic groups. Since the proposition is true for every cyclic subgroup of G, it is true for all of G.*

Here is a sketch of an alternative proof of the proposition that could be applied in positive characteristic. Let the family $S_b, b \in B$ be implemented by $\pi : E \to B$, where B is irreducible or connected. For $g \in G$ let $E^g = \{x \in E : gx = x\}$. For each $b \in B$, $\pi^{-1}(b) \cap E^g = S_b \cap E^g$ is the fixed point subset S_b^g of S_b. The fixed point set E^g is a smooth, closed set in E, that may not be connected, and the restricted map $\pi : E^g \to B$ is a covering space. (The work is in proving the statements of the previous sentence.) The rotation constant $\varepsilon(x, g)$ is defined at all points $x \in E^g$ and is a locally constant regular function. Indeed, $\varepsilon(x, g)$ can be constructed on E^g as a regular function by using vector fields and differential forms and the g action on them. Since $\varepsilon(x, g)$ is a regular function on E^g with only a finite number of values it must be constant on each irreducible component of E^g. Using covering space theory, the number

$$l_{k,b}(g) = \left| \left\{ P \in S_b^g : \varepsilon(P, g) = \exp\left(-\frac{2\pi i k}{o(g)}\right) \right\} \right|$$

is a constant function on B. This implies that the RDS is constant on B.

4 Rotation data and characters on $\mathcal{H}^q(S)$

In this section we are going to see how the rotation constant vectors and the characters of the $\mathcal{H}^q(S)$ representations determine each other. This extends the work in the parent paper [7] for the prime cyclic case. The Eichler trace formula in the next subsection tells us how to compute the characters of G on $\mathcal{H}^q(S)$ in terms of rotation data. Of great interest is the reverse process of computing the rotation data from characters of G on $\mathcal{H}^q(S)$. There are two possible approaches to this:

1. For a given $g \in G - \{1\}$, determine the rotation data $L(g)$ of an action from the character values $\mathrm{ch}_{\mathcal{H}^1(S)}(g), \ldots, \mathrm{ch}_{\mathcal{H}^q(S)}(g), \ldots$ The relationship is linear with the character values given by a matrix product $FL(g)$.

2. For a given $g \in G - \{1\}$, determine the vectors $L(g^s)$, $s = 1, \ldots, o(g) - 1$ from the character values $\mathrm{ch}_{\mathcal{H}^q(S)}(g^s)$, $s = 1, \ldots, o(g)$, for a small number of values of q. Here, we are trying to establish conditions under which $\mathcal{H}^q$ equivalence implies rotational equivalence, and hence topological equivalence in the cyclic n-gonal case. By Remark 32 all of the $L(g^s)$ are determined from $\overline{L}_G$. Thus, we will focus on the relation between $\overline{L}_G$ (or more precisely X) and the character values $\left\{\mathrm{ch}_{\mathcal{H}^q(S)}(g^s), s = 1, \ldots, o(g)\right\}$ for a fixed q. Again, we shall utilize a matrix product.

In the points 1 and 2 above we see that we only need to know the values of the character on the subgroup $\langle g \rangle$ for a selected $g \in G - \{1\}$.

The Eichler trace formula, discussed in the next subsection will be fundamental to our calculations. In subsection 4.2 we will extend our discussion of case 2 above to determine the vector R_q, (equation 129) which records the decomposition of $\mathrm{ch}_{\mathcal{H}^q(S)}$ into irreducibles. For cyclic G, the relationship has a matrix vector form, $R_q = \frac{1}{n} N_q X$, where N_q has a nice and easily computable format. Using this equation we can do a detailed analysis of $\mathcal{H}^q$ conflation in Section 5 and obtain a results on when we can recover the RDS from a selection of R_q's.

4.1 Eichler trace formula

We recall the Eichler trace formula as developed in [12]. Let $\mathrm{ch}_{\mathcal{H}^q(S)}$ denote the character of the representation of G on $\mathcal{H}^q(S)$. First, define the class functions $\lambda_q : G \to \mathbb{C}$, $q \geq 1$

$$\lambda_q(1) = (\sigma-1)(2q-1), \quad (113)$$

$$\lambda_q(g) = \sum_{P\in S^g} \frac{(\varepsilon(P,g))^q}{1-\varepsilon(P,g)},\ g\neq 1, \quad (114)$$

where the last sum is zero if the fixed point set S^g is empty. Then, the Eichler trace formula says that the characters $\mathrm{ch}_{\mathcal{H}^q(S)}$ are given by

$$\mathrm{ch}_{\mathcal{H}^1(S)}(g) = \lambda_1(g)+1, \quad (115)$$

$$\mathrm{ch}_{\mathcal{H}^q(S)}(g) = \lambda_q(g), q>1. \quad (116)$$

In terms of class functions, formulas 115 and 116 may be rewritten:

$$\mathrm{ch}_{\mathcal{H}^1(S)} = \lambda_1+\chi_0, \quad (117)$$

$$\mathrm{ch}_{\mathcal{H}^q(S)} = \lambda_q,\ q>1. \quad (118)$$

The formula 114 can be rewritten in terms of the rotation constant vectors as

$$\lambda_q(g) = \sum_{\gcd(k,o(g))=1} l_k(g)\frac{\exp\left(-\frac{2\pi ik}{o(g)}\right)^q}{1-\exp\left(-\frac{2\pi ik}{o(g)}\right)} = \sum_{\gcd(k,o(g))=1} l_k(g)\frac{\vartheta^{kq}}{1-\vartheta^k}, \quad (119)$$

where $\vartheta = \exp\left(-\frac{2\pi i}{o(g)}\right)$.

Remark 62 *Letting* $m = \exp(G)$, *the exponent of* G, *note that* $\lambda_{q+m}(g) = \lambda_q(g)$, *for* $g \in G-\{1\}$, *as* $\vartheta^m = 1$. *Thus, we have complete information if we know* $\mathrm{ch}_{\mathcal{H}^1(S)}(g),\ldots,\ \mathrm{ch}_{\mathcal{H}^m(S)}(g), g\in G$.

Fix g and consider all q. For a fixed $g\in G-\{1\}$ the formula 119 can be written as a matrix equation which allows us to solve for the $l_k(g)$ in terms of the $\mathrm{ch}_{\mathcal{H}^q(S)}(g)$ with q varying. For this discussion let $n = o(g), \vartheta = \exp\left(-\frac{2\pi i}{n}\right)$. Consider the $n\times\phi(n)$ matrix $F=[f_{q,k}]$ with entries

$$f_{q,k} = \frac{\vartheta^{kq}}{1-\vartheta^k},\ q=1,\ldots,n,\ \ \gcd(k,n)=1,$$

and the vectors

$$\Lambda(g) = \begin{bmatrix} \lambda_1(g) \\ \lambda_2(g) \\ \vdots \\ \lambda_n(g) \end{bmatrix}, \; L(g) = \begin{bmatrix} l_1(g) \\ \vdots \\ l_k(g) \\ \vdots \\ l_{n-1}(g) \end{bmatrix}, \; \gcd(k,n) = 1.$$

Then, from the Eichler Trace formula 115, 116, and 119, we get

$$\Lambda(g) = FL(g). \tag{120}$$

The matrix F has the factorization $F = F'D$, where the columns of F' have the form

$$\begin{bmatrix} \vartheta^{k\cdot 1} \\ \vartheta^{k\cdot 2} \\ \vdots \\ \vartheta^{k\cdot n} \end{bmatrix}, \; \gcd(k,n) = 1,$$

and

$$D = \operatorname{diag}\left(\frac{1}{1-\vartheta}, \ldots, \frac{1}{1-\vartheta^k}, \ldots, \frac{1}{1-\vartheta^{n-1}}\right), \; \gcd(k,n) = 1.$$

The columns of F', up to cyclic permutation, are columns of the Discrete Fourier Transform matrix F_n. Using the same proof that $F_n^* F_n = nI_n$ we can show that $F'^* F' = nI_{\phi(n)}$. We may now solve for $L(g)$:

$$\begin{aligned} \Lambda(g) &= F'DL(g) \\ F'^*\Lambda(g) &= F'^*FDL(g) \\ F'^*\Lambda(g) &= nDL(g) \end{aligned}$$

and finally

$$L(g) = \frac{1}{n} D^{-1} F'^* \Lambda(g). \tag{121}$$

We put the foregoing into a proposition.

Proposition 63 *Let G act conformally on the surface S. Then for each $g \in G - \{1\}$ the set of character values $\left\{\operatorname{ch}_{\mathcal{H}^q(S)}(g) : \; 1 \le q \le o(g)\right\}$ and the vector $L(g)$ determine each other.*

Proof. The two matrix formulas 120 and 121 show the mutual determination between the systems. ■

Fix q and consider all of $\langle g \rangle$ Assume that $G = \langle g \rangle$ is cyclic, $o(g) = n$, $\zeta = \exp\left(-\frac{2\pi i}{n}\right)$, and that we know all $L(h)$, $h \in G - \{1\}$ or, what amounts to the same, we know $\overline{L}_G$. We develop a matrix formula for the character values $\left\{\mathrm{ch}_{\mathcal{H}^q(S)}(h), h \in G\right\}$ using $X = \left[\begin{array}{cc} \sigma - 1 & \overline{L}_G \end{array}\right]^t$. We are going use the *IOIE* ordering of elements of G starting with $g_0 = 1$ as in Assumption 34. This will put our transformation matrix in block diagonal form. To simplify calculation, we work with the function λ_q and translate results to $\mathrm{ch}_{\mathcal{H}^q(S)}$ via equations 115 and 116.

For any function $\psi : G \to \mathbb{C}$, we associate a column vector Y_ψ

$$\psi \leftrightarrow Y_\psi = \left[\begin{array}{ccc} \psi(g_0) & \cdots & \psi(g_{n-1}) \end{array}\right]^t$$

with the *IOIE* ordering of the g_j. We are seeking a matrix M_q for which

$$Y_{\lambda_q} = M_q X. \tag{122}$$

As $\lambda_q(1) = (2q-1)(\sigma-1)$ then $M_q(0,0) = 2q-1$ and $M_q(j,0) = M_q(0,s) = 0$ for $j, s \geq 1$. Next, let us compute the elements of $M_q(j,s)$ for $o(g_j) = d > 1$. Writing $e = n/d$ then

$$g_j = g^{e_j k_j} = g^{ek_j}, \text{ for some } k_j \in V_d.$$

Noting $\exp\left(-\frac{2\pi i}{d}\right) = \zeta^e$, we get from equations 119 and 49:

$$\lambda_q(g^{ek_j}) = \sum_{\gcd(k,d)=1} l_k(g^{ek_j}) \frac{\zeta^{ekq}}{1-\zeta^{ek}} = \sum_{k \in V_d} l_1\left(g^{ek_j k^{-1}}\right) \frac{\zeta^{ekq}}{1-\zeta^{ek}}.$$

Set $u_s = k_j k^{-1}$, so that $k = k_j u_s^{-1}$, and as k runs though V_d so does u_s. Continuing,

$$\begin{aligned}
\lambda_q(g^{ek_j}) &= \sum_{k \in V_d} l_1\left(g^{ek_j k^{-1}}\right) \frac{\zeta^{ekq}}{1-\zeta^{ek}} \\
&= \sum_{u_s \in V_d} l_1(g^{eu_s}) \frac{\zeta^{ek_j u_s^{-1} q}}{1-\zeta^{ek_j u_s^{-1}}} \\
&= \sum_{u_s \in V_d} l_1(g^{eu_s}) \frac{(\zeta^{eq})^{k_j u_s^{-1}}}{1-(\zeta^e)^{k_j u_s^{-1}}}.
\end{aligned}$$

It follows then the only components of X appearing in the equation correspond to $x_s = l_1(g_s) = l_1(g^{eu_s})$. Thus, $M_q(j,s) = 0$ unless s belongs to the block for which $o(g_s) = d$. The rows j for which the equation is valid are those for which $o(g_j) = d$. Thus, the only nonzero elements with rows such that $o(g_j) = d$ and columns such $o(g_s) = d$, must have the rows and columns that satisfy both constraints, and we get a square block on the diagonal. We see that

$$M_q(j,s) = \frac{(\zeta^{eq})^{k_j u_s^{-1}}}{1 - (\zeta^e)^{k_j u_s^{-1}}} \tag{123}$$

where

$$g_j = g^{ek_j}, g_s = g^{eu_s}.$$

Example 64 *Here are the matrices* M_1 *and* M_2 *for* $n = 6$ *:*

$$M_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\zeta-2}{3} & -\frac{\zeta+1}{3} & 0 & 0 \\ 0 & 0 & -\frac{\zeta+1}{3} & \frac{\zeta-2}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \zeta-1 & -\zeta \\ 0 & 0 & 0 & 0 & -\zeta & \zeta-1 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{2\zeta-1}{3} & \frac{2\zeta-1}{3} & 0 & 0 \\ 0 & 0 & \frac{2\zeta-1}{3} & -\frac{2\zeta-1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{bmatrix}.$$

4.2 RDS to $\mathcal{H}^q(S)$ character transfer matrix for cyclic groups

We are going to analyze the distribution of eigenvalues of the $\langle g \rangle$-action on $\mathcal{H}^q(S)$, or what amounts to the same thing, the decomposition of the action of $\langle g \rangle$ on $\mathcal{H}^q(S)$ into irreducible characters. We develop a linear relation between the character decomposition R_q (defined below) and $X = \begin{bmatrix} \sigma - 1 & \overline{L}_G \end{bmatrix}^t$ and hence we may try to compute $\overline{L}_G$ linearly from the character decomposition of $\mathcal{H}^q(S)$. Guerrero asserts in [14] that knowing the characters for $q = 1, 2$ is sufficient for a complete determination of $\overline{L}_G$, however this in not supported by our analysis. The main conjecture of the parent paper [7] is that a single value of q will suffice to determine $\overline{L}_G$ for a prime cyclic group.

We denote the irreducible characters of G by χ_i. These characters have the form

$$\chi_i(g^j) = (\vartheta_i)^j \,, \tag{124}$$

where $\vartheta_0, \ldots, \vartheta_{n-1}$ is a listing of the nth roots of unity. We impose an ordering on the characters by choosing ϑ_i as follows

$$\vartheta_i = \exp\left(\frac{2\pi i r_i}{n}\right) = \zeta^{-r_i} \tag{125}$$

so that

$$\begin{aligned} \chi_i(g^j) &= \zeta^{-jr_i}, &(126)\\ \chi_i(g_j) &= \zeta^{-r_i r_j}, &(127) \end{aligned}$$

where the r_i are as given in equation 71. With this numbering χ_0 will be the trivial character, which is standard notation.

Fix q, we shall decompose the character $\mathrm{ch}_{\mathcal{H}^q(S)}$ into irreducible characters:

$$\begin{aligned} \mathrm{ch}_{\mathcal{H}^1(S)} - \chi_0 &= \lambda_1 = \mu_1^0\chi_0 + \cdots + \mu_1^{n-1}\chi_{n-1}, &(128)\\ \mathrm{ch}_{\mathcal{H}^q(S)} &= \lambda_q = \mu_q^0\chi_0 + \cdots + \mu_q^{n-1}\chi_{n-1}, \end{aligned}$$

where $\mu_q^i = \langle \lambda_q, \chi_i \rangle$ is the multiplicity inner product. We define the representation vector

$$R_q = \begin{bmatrix} \mu_q^0 \\ \vdots \\ \mu_q^{n-1} \end{bmatrix}, \tag{129}$$

a slight variation of the vector in the parent article [7]. In the prime cyclic case in [7], we were able to compute a vector R_0 and integer valued $p \times (p-1)$ matrix such that $R_q = \frac{1}{p}\left(R_0 - N\overline{P}_G\right)$. We shall find a version for cyclic groups

$$R_q = \frac{1}{n} N_q X, \tag{130}$$

where N_q is an $n \times n$ matrix such that $2N_q$ has integer entries. Since we include the genus as a part of X we do not need the vector R_0.

Remark 65 *For $q = 1$ we have:*

$$\begin{aligned}\sigma - 1 &= \dim \mathcal{H}^1(S) - 1 = \mathrm{ch}_{\mathcal{H}^1(S)}(1) - \chi_0(1)\\ &= \mu_1^0 \chi_0(1) + \cdots + \mu_1^{n-1}\chi_{n-1}(1)\\ &= \mu_1^0 + \cdots + \mu_1^{n-1}.\end{aligned}$$

and for $q > 1$

$$\begin{aligned}(2q-1)(\sigma - 1) &= \dim \mathcal{H}^q(S) = \mathrm{ch}_{\mathcal{H}^q(S)}(1)\\ &= \mu_q^0 \chi_0(1) + \cdots + \mu_q^{n-1}\chi_{n-1}(1)\\ &= \mu_q^0 + \cdots + \mu_q^{n-1}.\end{aligned}$$

Hence the column sum of the vector R_q in equation 129 is $(2q-1)(\sigma-1)$.

In addition, it is well known that quotient genus equals the dimension of the invariant 1-differentials. This dimension equals $\mu_1^0 + 1$ which is one more than the first entry of R_1.

Now assume that $G = \langle g \rangle$, $o(g) = n$. From equation 119 we have

$$\begin{aligned}\langle \lambda_q, \chi_i \rangle &= \frac{1}{n}\sum_{h\in G} \lambda_q(h)\overline{\chi_i}(h)\\ &= \frac{\lambda_q(1)}{n} + \frac{1}{n}\sum_{h\in G-\{1\}} \overline{\chi_i}(h)\left(\sum_{\gcd(k,o(h))=1} l_k(h)\frac{\zeta_h^{kq}}{1-\zeta_h^k}\right)\\ &= \frac{(2q-1)(\sigma-1)}{n} + \frac{1}{n}\sum_{h\in G-\{1\}} \overline{\chi_i}(h)\left(\sum_{\gcd(k,o(h))=1} l_k(h)\frac{\zeta_h^{kq}}{1-\zeta_h^k}\right),\end{aligned}$$

where, in the inner sum, $\zeta_h = \exp\left(-\frac{2\pi i}{o(h)}\right) = \zeta^e$ with $e = n/o(h)$. We rewrite the last equation by utilizing the disjoint union $G - \{1\} = \bigcup_{1<d|n} \mathcal{O}_d$, and using equations 73 and 74 in the inner sum. We get

$$\begin{aligned}\langle \lambda_q, \chi_i \rangle &= \frac{(2q-1)x_0}{n} + \frac{1}{n}\sum_{d>1}\sum_{h\in\mathcal{O}_d}\sum_{k\in V_d} \overline{\chi_i}(h) l_k(h)\frac{\zeta^{ekq}}{1-\zeta^{ek}}\\ &= \frac{(2q-1)x_0}{n} + \frac{1}{n}\sum_{d>1}\sum_{s\in V_d}\sum_{k\in V_d} \overline{\chi_i}(g^{es}) l_k(g^{es})\frac{\zeta^{ekq}}{1-\zeta^{ek}}.\end{aligned}$$

Now we have $\vartheta_i = \zeta^{-r_i}$ and so $\overline{\chi_i}(g^{es}) = \zeta^{esr_i}$ and

$$\langle \lambda_q, \chi_i \rangle = \frac{(2q-1)x_0}{n} + \frac{1}{n} \sum_{d>1} \sum_{s \in V_d} \sum_{k \in V_d} l_1(g^{esk^{-1}}) \frac{\zeta^{e(kq+sr_i)}}{1-\zeta^{ek}}.$$

We now concentrate on the inner double sum for a single block $\mathcal{O}_d$. Set $u = sk^{-1}$, so that $k = su^{-1}$. Then

$$\sum_{s \in V_d} \sum_{k \in V_d} l_1(g^{esk^{-1}}) \frac{\zeta^{e(kq+sr_i)}}{1-\zeta^{ek}} = \sum_{s \in V_d} \sum_{u \in V_d} l_1(g^{eu}) \frac{\zeta^{es(u^{-1}q+r_i)}}{1-\zeta^{esu^{-1}}}.$$

Now we compute the coefficient of $l_1(g_j) = l_1(g^{eu})$, using the change of variables $v = su^{-1}$, $s = uv$ to get

$$\sum_{s \in V_d} \frac{\zeta^{es(u^{-1}q+r_i)}}{1-\zeta^{esu^{-1}}} = \sum_{v \in V_d} \frac{(\zeta^{ev})^{q+ur_i}}{1-\zeta^{ev}}.$$

Write $\vartheta = \zeta^e = \exp\left(-\frac{2\pi i}{d}\right)$. Since $g^{eu} = g^{e_j k_j} \implies u = k_j$ then

$$\sum_{v \in V_d} \frac{(\zeta^{ev})^{q+ur_i}}{1-\zeta^{ev}} = \sum_{v \in V_d} \frac{(\vartheta^{v})^{q+k_j r_i}}{1-\vartheta^{v}}.$$

So, we need to know how to compute following sums

$$P(d,m) = \sum_{\vartheta} \frac{\vartheta^m}{1-\vartheta},$$

where the sum is over the primitive dth roots unity ϑ and $0 \le m < d$. We show how to calculate these sums in subsection 5.4.

Putting it all together we get see that N_q and X have the following block structure

$$N_q = \begin{bmatrix} N_q(0,0) = 2q-1 & N_q(0,j) = P(d_j, q),\ j>0 \\ N_q(i,0) = 2q-1, i>0 & N_q(i,j) = P(d_j, q+k_j r_i),\ i>0,\ j>0 \end{bmatrix} \tag{131}$$

$$X = \begin{bmatrix} \sigma - 1 \\ \overline{L}_G \end{bmatrix}. \tag{132}$$

Example 66 *Continuing Example 64, Here are the matrices N_1 and N_2 for $n = 6$,with $\frac{1}{2}$ factored out to show that the entries are integer or half integers.*

$$N_1 = \frac{1}{2}\begin{bmatrix} 2 & -1 & -2 & -2 & -2 & -2 \\ 2 & 1 & -2 & -2 & 2 & 2 \\ 2 & -1 & 2 & 0 & -2 & 4 \\ 2 & -1 & 0 & 2 & 4 & -2 \\ 2 & 1 & 0 & 2 & -4 & 2 \\ 2 & 1 & 2 & 0 & 2 & -4 \end{bmatrix},$$

$$N_2 = \frac{1}{2}\begin{bmatrix} 6 & 1 & 0 & 0 & -4 & -4 \\ 6 & -1 & 0 & 0 & 4 & 4 \\ 6 & 1 & -2 & 2 & 2 & 2 \\ 6 & 1 & 2 & -2 & 2 & 2 \\ 6 & -1 & 2 & -2 & -2 & -2 \\ 6 & -1 & -2 & 2 & -2 & -2 \end{bmatrix}.$$

For later discussion we note the form of $N_1^* N_1$

$$N_1^* N_1 = \begin{bmatrix} 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 2 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 12 & -6 \\ 0 & 0 & 0 & 0 & -6 & 12 \end{bmatrix}.$$

The blocks correspond to the group elements of the same order.

4.3 Action of Aut(G) on rotation data and representations

To relate rotational data and characters on $\mathcal{H}^q$ to topological equivalence we need to consider how Aut(G) acts upon them. This is especially important since Guerrero's and other authors' consider only the image subgroups. Based on Proposition 16, we have already dealt with the topological part of topological equivalence by imposing an order on the elements of G and by aggregating information according to conjugacy class. The action of Aut(G) on rotation data and representations begins with the action on the monomorphism $\epsilon : G \to \mathrm{Aut}(S)$ and the epimorphism $\xi : \pi_1(T^\circ, Q_0) \to G$ by means of the related formulas $\epsilon \to \epsilon \circ \omega^{-1}$, $\xi \to \omega \circ \xi$, and the assignment $\xi \to \widetilde{\xi^{-1}} = \epsilon$ given in equations 22, 19, and 3. We have the following:

Lemma 67 *Let W and R_q denote the generating action vector and representation vector determined by the action ϵ of the cyclic group G. The automorphism $\omega : x \to x^s$ of G and the IOIE ordering of G (equation 71) determine a permutation $\varrho = \varrho_\omega$ of $\{0, \ldots n-1\}$ such that $\omega(g_j) = g_j^s = g^{sr_j} = g_{\varrho(j)}$. Let P be the permutation matrix whose only non-zero entries are given by $P_{j,\varrho^{-1}(j)} = 1$ ($P_{\varrho(j),j} = 1$). Then W' and R_q', the generating action vector and representation vector, determined by $\omega \circ \xi$ and $\epsilon \circ \omega^{-1}$, respectively, satisfy*

$$W' = PW, R_q' = P^{-1}R_q. \tag{133}$$

Proof. The generating vector $(a_1', \ldots, a_\tau', b_1', \ldots, b_\tau', c_1', \ldots c_t')$ for $\omega \circ \xi$ has $c_i' = \omega c_i$ The case $w_0' = w_0$ is easy and left to the reader. For $j > 0$

$$\begin{aligned} w_j' &= |\{i : \omega c_i = g_j\}| \\ &= \left|\left\{i : c_i = \omega^{-1} g_j = g_{\varrho^{-1}(j)}\right\}\right| \\ &= w_{\varrho^{-1}(j)} \end{aligned}$$

Now $P_{j,\varrho^{-1}(j)}$ is the only non zero entry of P in the jth row of P and so PW has $w_{\varrho^{-1}(j)}$ as its jth entry.

Next we compute the action of ω on representation vectors. The rotation constants and the trace function $\lambda_q(g)$ are computed via the action map $\epsilon : G \to \mathrm{Aut}(S)$. For R_q' we use the action map $\epsilon \circ \omega^{-1}$ and hence the function $\lambda_q'(h) = \lambda_q(\omega^{-1}(h))$. The jth component of R_q' is

$$(\mu')_q^j = \left\langle \lambda_q', \chi_j \right\rangle = \frac{1}{n} \sum_{h \in G} \lambda_q(\omega^{-1}(h)) \overline{\chi}_j(h).$$

Change variables $h' = \omega^{-1}(h), h = \omega(h')$ to get:

$$\begin{aligned} (\mu')_q^j &= \frac{1}{n} \sum_{h' \in G} \lambda_q(h') \overline{\chi}_j(\omega(h')) = \frac{1}{n} \sum_{h' \in G} \lambda_q(h') \overline{\chi}_j((h')^s) \\ &= \frac{1}{n} \sum_{i=0}^{n-1} \lambda_q(g^{r_i}) \overline{\chi}_j(g^{sr_i}) = \frac{1}{n} \sum_{i=0}^{n-1} \zeta^{sr_j r_i} \lambda_q(g^{r_i}) = \\ &= \frac{1}{n} \sum_{i=0}^{n-1} \chi_{\rho(j)}(g^{r_i}) \lambda_q(g^{r_i}) = \mu_q^{\rho(j)} \end{aligned}$$

The argument used to show $W' = PW$ can be also used to conclude that $R_q' = P^{-1}R_q$. ■

5 Analysis of Conflation

We are going to analyze when $\mathcal{H}^q$ equivalent actions have the same rotation data. But, before proceeding with our development of conflation, we do a little linear algebra.

5.1 Some linear algebra

Let $A_1, \ldots, A_m$ be a list of matrices all of which have c columns. Define the stacked matrix:

$$B_s = \begin{bmatrix} A_1 \\ \vdots \\ A_s \end{bmatrix}, \tag{134}$$

and define

$$C_s = B_s^* B_s = A_1^* A_1 + \cdots + A_s^* A_s', \tag{135}$$

where * denotes the Hermitian transpose. We have the following Lemma.

Lemma 68 *Let A_s, B_s, C_s be as defined above, Then*

1. *The null space of B_s equals the null-space of C_s.*
2. *The rank of C_s equals the rank of B_s.*
3. *The null space of B_s contains the null space of B_{s+1}.*
4. $\text{nullity}\,(C_{s+1}) \leq \text{nullity}(C_s)$ *and* $\text{rank}\,(C_{s+1}) \geq \text{rank}(C_s)$ *for all s.*
5. *The matrix B_s has trivial null space if and only if C_s is invertible.*

Proof. For statement 1, let X be a vector in the null space of C_s then

$$0 = X^* C_s X = X^* B_s^* B_s X = \|B_s X\|^2$$

so that $B_s X = 0$. If $BX = 0$, then

$$\|C_s X\|^2 = X^* C_s^* C_s X = X^* C_s^* B_s^* B_s X = 0,$$

so that $C_s X = 0$. For statement 2, observe that

$$\text{rank}(B_s) = c - \text{nullity}(B_s) = c - \text{nullity}(C_s) = \text{rank}(C_s).$$

For statement 3, observe that X is in the null space of B_{s+1} if and only if $A_1 X = A_2 X = \cdots = A_{s+1} X = 0$, implying $B_s X = 0$. Statements 4 and 5 follow from the first three. ∎

5.2 $\mathcal{H}^{qL}$ equivalent actions

Suppose that two cyclic actions ϵ_a, ϵ_b are $\mathcal{H}^q$ equivalent for a selection of q values $qL = \{q_1, \ldots, q_f\}$. They will determine the same set of multiplicity vectors $R_{q_1}, \ldots, R_{q_f}$. We want to determine if the actions are also rotationally equivalent. We go through the example of $qL = \{1, 2\}$ as suggested by Guerrero's work and then give a more general statement. For convenience we concatenate the vectors $R_{q_1}, \ldots, R_{q_f}$ into a matrix

$$R_{qL} = \left[\begin{array}{ccc} R_{q_1} & \cdots & R_{q_f} \end{array} \right] \tag{136}$$

called the *representation matrix.* Let X_a, Z_a, W_a and X_b, Z_b, W_b denote the total action vector, primitive action vector, and generating action vector for ϵ_a and ϵ_b, respectively. We have these equations:

$$\begin{aligned} \frac{1}{n}N_1X_a &= R_1 = \frac{1}{n}N_1X_b, \\ \frac{1}{n}N_2X_a &= R_2 = \frac{1}{n}N_2X_b, \end{aligned} \tag{137}$$

$$\begin{aligned} \frac{1}{n}N_1AZ_a &= R_1 = \frac{1}{n}N_1AZ_b, \\ \frac{1}{n}N_2AZ_a &= R_2 = \frac{1}{n}N_2AZ_b, \end{aligned} \tag{138}$$

and

$$\begin{aligned} \frac{1}{n}N_1ABW_a &= R_1 = \frac{1}{n}N_1ABW_b, \\ \frac{1}{n}N_2ABW_a &= R_2 = \frac{1}{n}N_2ABW_b. \end{aligned} \tag{139}$$

It follows that

$$\begin{aligned} N_1(X_a - X_b) &= 0, \\ N_2(X_a - X_b) &= 0, \end{aligned}$$

i.e., $X_a - X_b$ is in the null-space of the stacked matrix

$$\left[\begin{array}{c} N_1 \\ N_2 \end{array} \right].$$

Similarly, $Z_a - Z_b$, and $W_a - W_b$ are in the null-spaces of the stacked matrices

$$\begin{bmatrix} N_1 A \\ N_2 A \end{bmatrix} \text{ and } \begin{bmatrix} N_1 AB \\ N_2 AB \end{bmatrix},$$

respectively. The equations 137 are best for computing actions from multiplicity vectors and equations 139 are best for constructing conflation examples.

We shall consider some examples of non-trivial null spaces presently, but assume for the moment that the null-spaces are trivial. How do we find X_a? Write in block matrix form:

$$\begin{bmatrix} N_1 \\ N_2 \end{bmatrix} X_a = n \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}.$$

If the null space of $\begin{bmatrix} N_1 \\ N_2 \end{bmatrix}$ is trivial, then

$$\begin{bmatrix} N_1 \\ N_2 \end{bmatrix}^* \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} N_1^* & N_2^* \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = N_1^* N_1 + N_2^* N_2$$

is an invertible matrix. So we get

$$\begin{aligned} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}^* \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} X_a &= n \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}^* \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \\ \left(N_1^* N_1 + N_2^* N_2\right) X_a &= n \left(N_1^* R_1 + N_2^* R_2\right) \\ X_a &= n \left(N_1^* N_1 + N_2^* N_2\right)^{-1} \left(N_1^* R_1 + N_2^* R_2\right). \end{aligned}$$

In the general case we have the following easily proven proposition:

Proposition 69 *Suppose that $R_q = \frac{1}{n} N_q X$ for all $q \geq 0$ and that the matrix*

$$\begin{bmatrix} N_{q_1} \\ \vdots \\ N_{q_f} \end{bmatrix}$$

has trivial null-space. Then

$$X = n \left(N_{q_1}^* N_{q_1} + \cdots + N_{q_f}^* N_{q_f}\right)^{-1} \left(N_{q_1}^* R_{q_1} + \cdots + N_{q_f}^* R_{q_f}\right). \tag{140}$$

We also have the following proposition whose proof we defer to Section 5.5.

Proposition 70 *The matrix* $N_1^* N_1 + \cdots + N_n^* N_n$ *is an invertible diagonal matrix.*

Using Lemma 68 and Propositions 69 and 70 we may prove the following.

Proposition 71 *Let the cyclic group* G *of order* n *act on an arbitrary surface* S *and let all other notation be as above. Then, there is* s_0*, depending only on* n*, such that* $1 \leq s_0 \leq n$*, and such that for* $s \geq s_0$ *the multiplicity vectors* $R_1, \ldots, R_s$ *determine the rotational data system* $L(g)$*,* $g \in G - \{1\}$ *by means of the equation*

$$[\sigma - 1, \overline{L}_G] = X = n \left(N_{_1}^* N_{_1} + \cdots + N_s^* N_s \right)^{-1} \left(N_{_1}^* R_{_1} + \cdots + N_s^* R_s \right). \quad (141)$$

The number s_0 *is independent of the action.*

Proof. By Lemma 68 and Proposition 70 there is a $s_0 \leq n$ such that $N_{_1}^* N_{_1} + \cdots + N_s^* N_s$ is invertible for all $s \geq s_0$. Then equation 141 follows from Proposition 69. Since the matrices N_q only depend on the structure of G then s_0 is independent of the action. ■

The values of s_0 can be easily determined using MAGMA. In Table 4 we give a list of values of s_0 for selected small n.

n		2	3	4	5	6	7	8	9	10
s_0		1	1	1	2	1	2	2	2	2

n	11	12	13	14	15	16	17	18	19	20
s_0	2	2	2	2	2	3	2	2	2	3

n	21	22	23	24	25	26	27	28	29	30
s_0	2	2	2	3	6	2	4	3	2	2

n	31	32	33	34	35	36	37	38	39	40
s_0	2	5	2	2	2	4	2	2	2	5

n	41	42	43	44	45	46	47	48	49	50
s_0	2	2	2	3	4	2	2	5	8	6

n	60	64
s_0	3	9

Table 4. Values of $s_0 = \min\left\{ s : \sum_{q=1}^{s} N_q^* N_q \text{ is invertible} \right\}$

It is also interesting to construct a table measuring the severity of conflation for $q = 1$ since $\mathcal{H}^1$ equivalence has been used as a classifier of actions. As we shall see shortly, the dimension of the null-space of $N_1^* N_1$ is such measure. We will call this the *degree of* $\mathcal{H}^1$ *conflation*. The values of the degree, for selected small n, are given in Table 5. We see some patterns:

- If n is a prime then the degree is $(n-3)/2$.
- If n is twice a prime then the degree is $n/2 - 3$.

n		2	3	4	5	6	7	8	9	10
degree		0	0	0	1	0	2	1	2	2

n	11	12	13	14	15	16	17	18	19	20
degree	4	1	5	4	4	4	7	4	8	5

n	21	22	23	24	25	26	27	28	29	30
degree	7	8	10	5	10	10	10	9	13	8

n	31	32	33	34	35	36	37	38	39	40
degree	14	11	13	14	14	10	17	15	16	13

n	41	42	43	44	45	46	47	48	49	50
degree	19	14	20	17	17	20	22	15	22	20

n	60	64
degree	19	26

Table 5. Degrees of $\mathcal{H}^1$ *conflation* $= \text{nullity}\,(N_1^* N_1)$

5.3 Constructing conflation pairs

We now explore the relationship between conflation and the null-space of $N_{q_1}^* N_{q_1} + \cdots + N_{q_f}^* N_{q_f}$. For a q-list $qL = \{q_1, \ldots, q_f\}$ and an action signature W of G we say that two generating action vectors W_a and W_b are qL *conflated* (or form a qL-*conflation pair*) if the multiplicity vectors $R_{q_1}, \ldots, R_{q_f}$ are the same for both vectors, namely

$$N_q ABW_a = R_q = N_q ABW_b,\ \ q \in qL. \tag{142}$$

It follows that

$$\begin{bmatrix} N_{q_1} \\ \vdots \\ N_{q_f} \end{bmatrix}^* \begin{bmatrix} N_{q_1} \\ \vdots \\ N_{q_f} \end{bmatrix} AB(W_a - W_b) = 0 \tag{143}$$

or

$$\left(N_{q_1}^* N_{q_1} + \cdots + N_{q_f}^* N_{q_f}\right) AB(W_a - W_b) = 0. \tag{144}$$

By Lemma 68 equations 142 hold if and only if equation 144 holds.

Since $\left(N_{q_1}^* N_{q_1} + \cdots + N_{q_f}^* N_{q_f}\right) AB$ has rational coefficients there is a basis of its null space with integer coefficients – use Gauss elimination and then clear denominators. From each basis vector in the null space we will construct a conflated pair. We illustrate the steps with a null-space vector for $n = 16$ and $qL = \{1, 2\}$ in Example 73. Pick any basis null vector W_0 and create two vectors

$$\begin{aligned} W_a(j) &= \max(W_0(j), 0), \\ W_b(j) &= \max(-W_0(j), 0). \end{aligned}$$

By construction, $W_0 = W_a - W_b$, the entries of W_a and W_b are nonnegative integers, and neither of W_a and W_b have a non-zero entry in the same location. For W_a and W_b to be action generating vectors, the following necessary and sufficient conditions must hold, according to Remark 45.

1. The first component must be an integer ≥ -1, and the remaining components are nonnegative integers.
2. The dot products $E \cdot W_a$ and $E \cdot W_b$ must be divisible by $n = 16$.
3. The least common multiple of the periods present in the action vector must equal $n = 16$ or the first component must be ≥ 0.

Now, by construction, the first components are integers ≥ 0 and so conditions 1 and 3 always hold. For the second condition, in the case in the example at hand, the dot products are divisible by n because of the symmetric placement of the 1's. In the general case, it is possible that condition 2 fails. However, if we multiply W_0 by a suitable divisor of n, then W_0 remains in the kernel and 2 is satisfied. We shall assume that such an adjustment has been performed, using as small a divisor as possible. Conditions 1 and 3 will still hold true. It turns out that W_a and W_b are $\text{Out}(G)$ equivalent giving a rather trivial conflation. However the pair of vectors $W_a + W_b$ and $2W_b$ will yield a truly conflated pair. The details are in Theorem 74 and its proof.

Remark 72 *In all the sample MAGMA computations of conflation pairs the produced vectors W_a and W_b were are $\text{Out}(G)$ equivalent. Presumably, this is some artifact of the production of a basis for the null-space.*

Example 73 *Let $n = 16$ and $qL = \{1, 2\}$. Then, the null-space of $N_1^* N_1 + N_2^* N_2$ is one dimensional and we have:*

$$\begin{aligned} W_0 &= \left[\begin{array}{cccccccccccccccc} 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \end{array}\right]^t \\ W_a &= \left[\begin{array}{cccccccccccccccc} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{array}\right]^t \\ W_b &= \left[\begin{array}{cccccccccccccccc} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]^t \end{aligned}$$

For clarity we have arranged the W_0, $W_a, W_b, W_a + W_b$, and $2W_b$ along with two additional rows indicating period (order) and exponent in the $IOIE$ ordering. We can more easily see how the various vectors line up with period and exponent.

$$\left[\begin{array}{c} P \\ E \\ W_0 \\ W_a \\ W_b \\ W_a + W_b \\ 2W_b \end{array}\right] = \left[\begin{array}{cccccccccccccccc} 1 & 2 & 4 & 4 & 8 & 8 & 8 & 8 & 16 & 16 & 16 & 16 & 16 & 16 & 16 & 16 \\ 16 & 8 & 4 & 12 & 2 & 6 & 10 & 14 & 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 2 \end{array}\right]$$

The representation matrix, in row format, of both vectors is given by the matrix following. The first row indicates the numbering of the characters.

$$\left[\begin{array}{c} \chi_j \\ R_1 \\ R_2 \end{array}\right] = \left[\begin{array}{cccccccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 0 & 4 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\ 12 & 16 & 18 & 18 & 18 & 18 & 18 & 18 & 16 & 16 & 16 & 16 & 16 & 16 & 16 & 16 \end{array}\right]$$

We are now ready to prove a theorem on conflation and the relation between $\mathcal{H}^q$ equivalence and rotational equivalence.

Theorem 74 *Let $qL = \{q_1, \ldots, q_f\}$ be a list of degrees. Let G be a cyclic group of order n and all other notation as above. Then, we have the following:*

1. *If $N_{q_1}^* N_{q_1} + \cdots + N_{q_f}^* N_{q_f}$ is an invertible matrix then two actions are qL-equivalent, up to automorphisms, if and only if they are rotationally equivalent up to automorphisms.*

2. *If two actions are qL-equivalent up to automorphisms if and only if they are rotationally equivalent up to automorphisms, then $N_{q_1}^* N_{q_1} + \cdots + N_{q_f}^* N_{q_f}$ is an invertible matrix.*

Proof. For statement 1, we already know from equation 140 that if $N_{q_1}^* N_{q_1} + \cdots + N_{q_f}^* N_{q_f}$ is invertible then the map

$$W \longrightarrow \left[\begin{array}{ccc} R_{q_1} & \cdots & R_{q_f} \end{array} \right] = \frac{1}{n} \left[\begin{array}{ccc} N_{q_1} ABW & \cdots & N_{q_1} ABW \end{array} \right]$$

is bijective with inversion formula

$$W = n\,(AB)^{-1} \left(N_{q_1}^* N_{q_1} + \cdots + N_{q_f}^* N_{q_f} \right)^{-1} \left(N_{q_1}^* R_{q_1} + \cdots + N_{q_f}^* R_{q_f} \right). \tag{145}$$

If W_1 and W_2 are $\mathrm{Out}(G)$-equivalent then Lemma 67 shows that the representation matrices are $\mathrm{Out}(G)$-equivalent and vice versa.

For statement 2 we only need to show that if $N_{q_1}^* N_{q_1} + \cdots + N_{q_f}^* N_{q_f}$ is not invertible then we can produce a conflation pair W_1 and W_2 which are not $\mathrm{Out}(G)$ equivalent, and such that they determine the same qL-representation matrix R_{qL}. To this end, let W_a and W_b be constructed as in the discussion above. Assume that the largest entry of W_b is greater than or equal to the largest entry of W_a, switching W_a and W_b if needed. The two vectors $W_a + W_b$ and $2W_b$ also satisfy the conditions to be generating action vectors. The largest entry of $W_a + W_b$ does not exceed the largest entry of W_b and hence the largest entry of $2W_b$ strictly exceeds the largest entry of $W_a + W_b$. It follows that the two vectors cannot be $\mathrm{Out}(G)$ equivalent. Now $W_a + W_b - 2W_b = W_a - W_b$ is the null-space of $\left(N_{q_1}^* N_{q_1} + \cdots + N_{q_f}^* N_{q_f} \right) AB$ and their representation matrices are the same. ■

5.4 Sums of roots of unity

As noted in Section 4.2 we need to be able to compute the following sums

$$P(n,m) = \sum_{\vartheta \text{ primitive}} \frac{\vartheta^m}{1-\vartheta},$$

where the sum is over the primitive nth roots unity ϑ and m is any integer. We can do this using Möbius inversion and the functions

$$S(n,m) = \sum_{\vartheta \neq 1} \frac{\vartheta^m}{1-\vartheta},$$

where the sum is over the non-trivial nth roots of unity. For convenience, in using the Möbius inversion formula, we set $P_{1,m} = S_{1,m} = 0$. Then, we easily see that

$$S(n,m) = \sum_{d|n} P_{d,m}, \tag{146}$$

by breaking the roots of unity into disjoint subsets of primitive dth roots of unity for d dividing n. The boundary definitions $P_{1,m} = S_{1,m} = 0$ allow us to include $d = 1$ in the sum. The Möbius inversion formula allows us to compute the $P_{d,m}$ in terms of the $S_{d,m}$, for d dividing n.

$$P(n,m) = \sum_{d|n} \mu\left(\frac{n}{d}\right) S(d,m), \tag{147}$$

where μ is the Möbius function.

Of course, this equation is useless unless we can get a nice formula for $S(n,m)$, which we proceed to do. Let ζ be a primitive root nth of unity, any ζ will do. Then

$$P(n,m) = \sum_{\gcd(k,n)=1}^{n-1} \frac{\zeta^{km}}{1-\zeta^k}, \tag{148}$$

$$S(n,m) = \sum_{k=1}^{n-1} \frac{\zeta^{km}}{1-\zeta^k}. \tag{149}$$

To eliminate the denominator in the second equation, write

$$\begin{aligned} S(n,m) &= \lim_{x\to 1^-} \left(\sum_{k=0}^{n-1} \frac{\zeta^{km}}{1-x\zeta^k} - \frac{1}{1-x} \right) \\ &= \lim_{x\to 1^-} \left(\sum_{k=0}^{n-1} \sum_{j=0}^{\infty} \zeta^{k(m+j)} x^j - \frac{1}{1-x} \right) \\ &= \lim_{x\to 1^-} \left(\sum_{j=0}^{\infty} \left(\sum_{k=0}^{n-1} \zeta^{k(m+j)} \right) x^j - \frac{1}{1-x} \right). \end{aligned}$$

The sum $\sum_{k=0}^{n-1} \left(\zeta^{(m+j)}\right)^k$ is non-zero if and only if $m + j = 0 \bmod n$, and when j satisfies this condition the sum is n. Let j_0 be the smallest non-negative j for which $m + j = 0 \bmod n$, namely, $j_0 = (n - m) \bmod n$ with j_0 in the range $0 \le j_0 < n$. Then,

$$
\begin{aligned}
S(n,m) &= \lim_{x\to 1^-}\left(n\sum_{j'=0}^{\infty} x^{j_0+nj'} - \frac{1}{1-x}\right)\\
&= \lim_{x\to 1^-}\left(\frac{nx^{j_0}}{1-x^n} - \frac{1}{(1-x)}\right)\\
&= \lim_{x\to 1^-}\left(\frac{nx^{j_0}(1-x)-(1-x^n)}{(1-x)\,(1-x^n)}\right).
\end{aligned}
$$

Using L'Hôpital's rule twice, we get:

$$S(n,m) = \frac{n-1-2j_0}{2}.$$

So if $1 \le m \le n$ then $j_0 = (n-m) \bmod n = n-m$ and

$$S(n,m) = \frac{2m-1-n}{2}. \tag{150}$$

For $n = 6$, the Möbius formula is $P(6,m) = S(6,m) - S(3,m) - S(2,m)$, and we have the following table:

m	$S(2,m)$	$S(3,m)$	$S(6,m)$	$P(2,m)$	$P(3,m)$	$P(6,m)$
1	$-1/2$	-1	$-5/2$	$-1/2$	-1	-1
2	$1/2$	0	$-3/2$	$1/2$	1	-2
3	$-1/2$	1	$-1/2$	$-1/2$	-1	-1
4	$1/2$	-1	$1/2$	$1/2$	1	1
5	$-1/2$	0	$3/2$	$-1/2$	-1	2
6	$1/2$	1	$5/2$	$1/2$	1	1

Table 6. Values of $S(n,m)$ and $P(n,m)$, $n = 2,3,6$

Remark 75 *Some observations:*

- *We see that both $S(n,m)$ and $P(n,m)$ satisfy*

$$P(n,m+n) = P(n,m),\ \ P(n,-m) = -P(n,m+1), \tag{151}$$

and

$$S(n,m+n) = S(n,m),\ \ S(n,-m) = -S(n,m+1). \tag{152}$$

The second formula in each equation comes from the substitution $\vartheta \to \vartheta^{-1}$.

- *The formula 150 is consistent with our definition of* $S_{1,m}$.
- *For odd* n, $S_{n,m}$ *is an integer, and for even* n, $S_{n,m}$ *is a half integer.*
- *For* $n > 2$ $P_{n,m}$ *is an integer. This is easily established from equation 147 if* n *is odd. For even* n *we also use equation 147 by taking into account the parity of the numerators of the* $S(d,m)$ *for the terms in which* $\mu\left(\frac{n}{d}\right)$ *is non-zero.*

5.5 The sums of $N_1^*N_1 + \cdots + N_n^*N_n$

We prove Proposition 70.
Proof. Our proof will depend on the factorization

$$N_q = \mathcal{F}M_q \tag{153}$$

Where $\mathcal{F}$ is a type of Fourier transform matrix. Recall that the vector of the λ_q in $IOIE$ order is given by $Y_q = Y_{\lambda_q} = M_qX$. In turn the multiplicity μ_q^i is given by

$$\begin{aligned} \mu_q^i &= \frac{1}{n}\sum_{j=0}^{n-1}\lambda_q(g_j)\overline{\chi_i(g_j)} \\ &= \frac{1}{n}\sum_{j=0}^{n-1}Y_q(j)\zeta^{r_ir_j}, \end{aligned}$$

using equation 127. It follows that

$$\frac{1}{n}N_qX = R_q = \frac{1}{n}\mathcal{F}Y_q = \frac{1}{n}\mathcal{F}M_qX,$$

where $\mathcal{F} = \left[\zeta^{r_ir_j}\right]$. Thus $N_q = \mathcal{F}M_q$ and it is easily shown that $\mathcal{F}^*\mathcal{F} = nI_n$. It follows that

$$\begin{aligned} N_1^*N_1 + \cdots + N_n^*N_n &= M_1^*\mathcal{F}^*\mathcal{F}M_1 + \cdots + M_1^*\mathcal{F}^*\mathcal{F}M_1 \\ &= n\left(M_1^*M_1 + \cdots + M_n^*M_n\right). \end{aligned}$$

The entries of $M_1^*M_1 + \cdots + M_n^*M_n$ are sums of dot products $\left(C_q^i\right)^* C_q^j$ of columns C_q^i, C_q^j, of M_q. Now M_q has a block diagonal form corresponding to the blocks of equal order in $g_0, \ldots, g_{n-1}$. If two columns correspond to different blocks then the dot product is automatically zero. Otherwise the s, t entry $d_{s,t}$ is given by

$$d_{s,t} = \sum_{q=1}^{n} \sum_{j} \overline{M_q(j, s)} M_q(j, t)$$

where j varies over the indices for the block containing s and t. Now from formula 123

$$M_q(j, s) = \frac{(\zeta^{eq})^{k_j u_s^{-1}}}{1 - (\zeta^{e})^{k_j u_s^{-1}}},$$

where

$$g_j = g^{ek_j}, g_s = g^{eu_s}.$$

Continuing, we have

$$\begin{aligned} d_{s,t} &= \sum_{q=1}^{n} \sum_{j} \overline{\frac{(\zeta^{eq})^{k_j u_s^{-1}}}{1 - (\zeta^{e})^{k_j u_s^{-1}}}} \frac{(\zeta^{eq})^{k_j u_t^{-1}}}{1 - (\zeta^{e})^{k_j u_t^{-1}}} \\ &= \sum_{q=1}^{n} \sum_{j} \overline{\frac{(\zeta^{ek_j u_s^{-1}})^q}{1 - (\zeta^{e})^{k_j u_s^{-1}}}} \frac{(\zeta^{ek_j u_t^{-1}})^q}{1 - (\zeta^{e})^{k_j u_t^{-1}}} \\ &= \sum_{q=1}^{n} \sum_{j} \frac{\left(\left(\zeta^{ek_j u_t^{-1}}\right) / \left(\zeta^{ek_j u_s^{-1}}\right)\right)^q}{\overline{\left(1 - (\zeta^{e})^{k_j u_s^{-1}}\right)} \left(1 - (\zeta^{e})^{k_j u_t^{-1}}\right)}. \end{aligned}$$

If we hold j fixed then $\zeta^{ek_j u_s^{-1}}$ runs through all the primitive n/e roots of unity as u_s^{-1} runs through the permissible values of u_s Thus unless $s = t$, $\left(\zeta^{ek_j u_t^{-1}}\right) / \left(\zeta^{ek_j u_s^{-1}}\right)$ is a non trivial nth root of unity. Now reverse the order of summation to get

$$\begin{aligned} d_{s,t} &= \sum_{j} \frac{1}{\overline{\left(1 - (\zeta^{e})^{k_j u_s^{-1}}\right)} \left(1 - (\zeta^{e})^{k_j u_t^{-1}}\right)} \sum_{q=1}^{n} \left(\left(\zeta^{ek_j u_t^{-1}}\right) / \left(\zeta^{ek_j u_s^{-1}}\right)\right)^q \\ &= n \sum_{j} \frac{1}{\overline{\left(1 - (\zeta^{e})^{k_j u_s^{-1}}\right)} \left(1 - (\zeta^{e})^{k_j u_t^{-1}}\right)}, \ s = t \\ &= 0, \ s \neq t. \end{aligned}$$

In case $s = t$ the number is non zero. ■

Example 76 *For $n = 6$ the matrix $N_1^*N_1 + \cdots + N_n^*N_n$ is given by*

$$N_1^*N_1 + \cdots + N_n^*N_n = \begin{bmatrix} 1716 & 0 & 0 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 24 & 0 & 0 & 0 \\ 0 & 0 & 0 & 24 & 0 & 0 \\ 0 & 0 & 0 & 0 & 72 & 0 \\ 0 & 0 & 0 & 0 & 0 & 72 \end{bmatrix}.$$

6 Further work

Here is a list of projects for further work.

6.1 Projects for Riemann surfaces

Project C.1 Find a nice form for the matrices M_q and N_q for a general group.

Project C.2 Carry out the conflation analysis for a general group.

Project C.3 Let the G-signature of an action be $(C_1, \ldots, C_t)$. Define

$$X(C_1, \ldots, C_t) = \{(c_1, \ldots, c_t) \in C_1 \times \cdots \times C_t) : c_1 \cdots c_t = 1\}, \quad (154)$$
$$X^\circ(C_1, \ldots, C_t) = \{(c_1, \ldots, c_t) \in X(C_1, \ldots, C_t) : G = \langle c_1, \ldots, c_t \rangle\} \quad (155)$$

What can be said about the topological equivalence classes in $X^\circ(C_1, \ldots, C_t)$?

6.2 Projects for curves in arbitrary characteristics

Project P.1 Show that the RDS is constant in a family with irreducible base.

Project P.2 Can generating vectors be defined in positive characteristics? If so can they be used to geometrically construct a surface with a G action. If not, find a tame n-gonal action on a curve such that the generating vector set $X^\circ(C_1, \ldots, C_t)$ in equation 155 is empty.

Project P.3 Can the algebraic fundamental group of T° be employed in constructing surfaces over $P^1(\mathbf{k})$ using monodromy epimorphisms? If so, how are these epimorphisms constructed? Can this be linked to Project P.2?

Project P.4 Carry out the $\mathcal{H}^q$ analysis in positive characteristic.

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