Periodicals of Engineering and Natural Sciences ISSN 2303-4521
Vol. 8, No. 3, August 2020, pp.1786-1794

Bayesian and non-Bayesian estimation of the Lomax model based on upper record values under weighted LINEX loss function

Fuad S. Al-Duais [1, 2], Munaf Yousif Hmood [3]
[1]Mathematics Department, College of Humanities and Science, Al Aflaj, Prince Sattam Bin Abdulaziz University
[2]Administration Department, Administrative Science College, Thamar University
[3]Department of Statistics, College of Administration and Economics, University of Baghdad

ABSTRACT

In this article, we developed a new loss function, as the simplification of linear exponential loss function (LINEX) by weighting LINEX function. We derive a scale parameter, reliability and the hazard functions in accordance with upper record values of the Lomax distribution (LD). To study a small sample behavior performance of the proposed loss function using a Monte Carlo simulation, we make a comparison among maximum likelihood estimator, Bayesian estimator by means of LINEX loss function and Bayesian estimator using square error loss (SE) function. The consequences have shown that a modified method is the finest for valuing a scale parameter, reliability and hazard functions.

Keywords: Bayesian estimation, upper record values, Lomax distribution, Weighted LINEX, reliability function, hazard function

Corresponding Author:

Mathematics Department,
College of Humanities and Science, Al Aflaj, Prince Sattam Bin Abdulaziz University,
Al Aflaj,
Saudia Arabia,
Email: F.alduais@psau.edu.sa

1. Introduction

Record values along with the related statistics stand for the relevant and imperative issues in the real-world applications. Records have been very prevalent in fields of sporting, climatology, and financing or insurance. In terms of mathematics Chandler in 1952, has explained the investigation of record values and recognized numerous elementary features of records. Several scholars have deliberated inference under record values by means of diverse lifetime distributions as in [1-8].

The Lomax distribution (LD) is a very popular statistical model in reliability modelling and life testing, with extended applications in economic sciences, actuarial modeling, queuing problems, and organic disciplines. Several authors have addressed inferential issues for the Lomax distribution in accordance with record values as in [9-16].

We derive, in this study, Bayes estimator based on weighted linear exponential (WLINEX) loss function for estimating the scale parameter θ , reliability function $R(t)$ and hazard function $H(t)$ of the LD using upper record values. The aim is to compare a Maximum likelihood estimator, Bayesian estimator using LINEX loss function, Bayesian estimator using square error loss (SE) function with Bayesian estimator using Weighted Linear Exponential Loss Function (WLINEX).

Let $X_1, X_2, X_3, \cdots\cdots$ be a series of independent and identically distributed (i.i.d) random variables with (C.D.F.) F(x) along with (P.D.F.) $f(x)$. Set $Y_n = max(X_1, X_2, X_3, \dots, X_n), n \geq 1$, X_j stands for the upper record symbolized by $X_{U(i)}$ if $Y_j > Y_{j-1}, j > 1$.

Supposing that $X_{U(1)}, X_{U(2)}, X_{U(3)}, \cdots\cdots X_{U(n)}$ are the 1st n upper record values resultant from *a* sequence $\{X_i\}$ of (i.i.d), Lomax variables with density function (P.D.F.):

$$f(x;\ \theta,\delta) = \begin{cases} \theta\ \delta^{\theta}\ (\delta+\theta)^{-\theta-1} & x \geq 0\ ;\ \delta,\theta > 0 \\ 0 & o.w. \end{cases} \quad (1)$$

With cumulative distribution function (C.D.F.), we get:

$$F(x) = 1 - \delta^{\theta}(x+\delta)^{-\theta} \qquad x \geq 0\ ;\ \delta,\theta > 0 \quad (2)$$

Here, θ and δ stand for scale and shape parameters correspondingly. In addition, a reliability function $R(t)$, in addition to the hazard (instantaneous failure rate) function $H(t)$ at mission time t for the Lomax distribution have specified as follows:

$$R(t) = \delta^{\theta}(t+\delta)^{-\theta} \quad ;\ \ t \geq 0\ ;\ \delta,\theta > 0 \quad (3)$$

$$H(t) = \theta(t+\delta)^{-1} \qquad ;\ t \geq 0 \quad (4)$$

2. Maximum likelihood estimation (ML)

We discuss, in this section, a maximum likelihood estimation of a scale parameter, reliability and hazard functions of LD specified in (1) as the available data stand for record values.

We assume that a shape parameter δ stands for known and the scale parameter θ is unidentified. We notice that the 1st n upper record values has the LD whose P.D.F. and C.D.F have specified by (1) and (2) in that order. In accordance with those upper record values, for notation simplicity, we will adopt x_i instead of $X_{U(i)}$. We have the joint density function of the first n upper record values $\underline{x} \equiv x_{U(1)}, x_{U(2)}, x_{U(3)}, \cdots\cdots x_{U(n)}$ has specified according to Arnold et al. (1998) by

$$f_{1,2,3\cdots\cdots n}\left(X_{U(1)}, X_{U(2)}, X_{U(3)}, \cdots\cdots X_{U(n)}\right) = f\left(x_{U(n)}\right)\prod_{i=1}^{n-1}\frac{f\left(x_{U(i)}\right)}{1-F\left(x_{U(i)}\right)}, \quad (5)$$
$$0 \leq X_{U(1)} < X_{U(2)} < X_{U(3)}, \cdots\cdots < X_{U(n)} < \infty,$$

where $f\ (\cdot)$, as well as $F(\cdot)$ are given, one-to-one, through (1) and (2) after substituting x by $x_{U(i)}$.

A likelihood function in accordance with the n upper record values x has specified by:

$$\ell\left(\theta,\delta\,\middle|\,\underline{x}\right) = (\theta)^{n}\ \delta^{\theta}u\ \ \left(x_{u(n)}+\delta\right)^{-\theta}, \qquad u = \prod_{i=1}^{n}\ \left(x_{u(n)}+\delta\right)^{-1} \quad (6)$$

Based on (6), the natural logarithm of the likelihood function has written as:

$$\ell\left(\theta,\delta\,\middle|\,\underline{x}\right) \equiv Ln(\ell) \ = n\ Ln(\theta) + \theta Ln(\delta) - \theta\ Ln\left(x_{U(n)}+\delta\right) - \ \sum_{i=1}^{n} Ln\left(x_{U(i)}+\delta\right) \quad (7)$$

When a shape parameter δ has been identified and a scale parameter θ has been unidentified, a maximum likelihood estimates of θ is computed based on (7) as:

$$\hat{\theta}_{ML} = \frac{n}{Ln\left(x_{U(n)}+\delta\right) - Ln(\delta)} \quad (8)$$

The invariance property of MLEs enables us to obtain the MLEs $\hat{R}(t)_{ML}$ and $\hat{H}(t)_{ML}$ of $R(t)$ and $H(t)$ is specified by equations (3) and (4) after substituting θ by $\hat{\theta}_{ML}$.

$$\hat{R}(t)_{ML} = \delta^{\hat{\theta}_{ML}}(t+\delta)^{-\hat{\theta}_{ML}} \quad ;\ \ t \geq 0 \quad (9)$$

$$\hat{H}(t)_{ML} = \hat{\theta}_{ML}(t+\delta)^{-1} \qquad ;\ t \geq 0 \quad (10)$$

3. Loss functions

Based on Bayesian viewpoint, the selecton of loss function stands for the vital part in estimating and predicting problems. In this work, we use three types of loss function including squared error loss function (SE), Linear Exponential Loss Function (LINEX) along with Weighted Linear Exponential Loss Function (WLINEX)

3.1. Squared error loss function (SE)

This type has been categorized as a symmetric loss function and deems identical importance to the losses for overestimation and underestimation of identical magnitude. Squared error loss function is computed as follows:

$$L\left(\hat{\lambda},\lambda\right) = \left(\hat{\lambda}-\lambda\right)^{2} \quad (11)$$

The Bayes estimator of λ using this loss function, denoted by $\hat{\lambda}_{SE}$, is calculated by:

$$\hat{\lambda}_{SE} = E_{\pi}(\lambda|\underline{x}) \tag{12}$$

3.2 Linear exponential loss function (LINEX)

LINEX loss function stands for asymmetric loss function, which was firstly given by Varian in [17]. Assuming minimal loss happens at $\hat{\lambda} = (\hat{\theta}\ , \hat{R}(T) and\ \hat{H}(t))$, the LINEX loss function for $\lambda = (\theta\ , R(t)\ and\ H(t))$ can be expressed as

$$L(\Delta) \propto [exp[c\Delta] - c\Delta - 1] \qquad ; c \neq 0 \tag{13}$$

where $\Delta = (\hat{\lambda} - \lambda)$ and $\hat{\lambda}$ is an estimate of λ . The Bayesian estimator of λ in accordance with this loss function, denoted by $\hat{\lambda}_{LINEX}$, is gotten by:

$$\hat{\lambda}_{LINEX} = -\frac{1}{c} Ln[E_{\lambda} exp[-c\ \lambda]] \tag{14}$$

Provided that $E_{\lambda} = (e^{-c\lambda})$ exist and finite, where E_{λ} denotes the expected value.

3.3. Weighted linear exponential loss function

The researchers in the literature based on weighted loss function (LINEX) proposed this function as follows:

$$L^{\star}(\hat{\lambda} - \lambda) = w(\lambda)[exp[c\Delta] - c\Delta - 1] \qquad ; c \neq 0 \tag{15}$$

Where $\hat{\lambda}$ signifies an estimated parameter that creates an expectation of loss function (Equation (13)) as minimum as feasible. While, $w(\lambda)$ characterizes a projected weighted function that equals to:

$$w(\lambda) = exp[-z\lambda] \tag{16}$$

Subject to a posterior distribution of the parameter λ, and through a proposed weighted function as in Equation (15), we can acquire an estimated weighted Bayes of the parameter λ as follows:

$$EL^{\star}{}_{w}(\hat{\lambda}, \lambda) = \int_{\forall\lambda} L_{w}(\hat{\lambda}, \lambda) f(\lambda|\underline{x}) d\lambda$$

$$= \int_{\forall\lambda} w(\lambda) \left[exp[c(\hat{\lambda} - \lambda)] - c(\hat{\lambda} - \lambda) - 1\right] f(\lambda\,|\,\underline{x}) d\lambda$$

$$= \int_{\forall\lambda} exp[-z\lambda]\, exp\, [\, c(\hat{\lambda} - \lambda)]\, f(\lambda\ |\,\underline{x}) d\lambda - \int_{\forall\lambda} exp[-z\lambda\ ]\ \{c(\hat{\lambda} - \lambda)\, f(\lambda\ |\,\underline{x}) d\lambda$$

$$-\int_{\forall\lambda} exp[-z\lambda\ ]\ f(\lambda\,|\,\underline{x})$$

$$= exp[c\hat{\lambda}] \int_{\forall\lambda} exp\,[\,-\lambda(c + z)]\, f(\lambda|\underline{x}) d\lambda - c\hat{\lambda} \int_{\forall\lambda} exp[-z\lambda]\, f(\lambda\,|\,\underline{x})\, d\lambda$$

$$+c \int_{\forall\lambda} \lambda\, exp[-z\lambda]\, f(\lambda\,|\,\underline{x}) d\lambda - \int_{\forall\lambda} exp[-z\lambda]\, f(\lambda\,|\,\underline{x}) d\lambda$$

$$= exp[c\hat{\lambda}]\ E_{\lambda}(exp[-\lambda(z + c)\ |\ \underline{x}]) - c\hat{\lambda} E_{\lambda}(exp[-z\lambda\,|\,\underline{x}]) + c\ E_{\lambda}(\lambda\, exp[-z\lambda\,|\,\underline{x}]) \ - E_{\lambda}(exp[-z\lambda\,|\,\underline{x}])$$

$$\frac{\partial\ E_{\lambda} L^{\star}{}_{w}(\hat{\lambda}, \lambda)}{\hat{\lambda}} = \ c\ \ exp[c\hat{\lambda}]\, E_{\lambda}(exp[-\lambda(z + c)\ |\,\underline{x}]) - c\ E_{\lambda}(exp[-z\lambda\,|\,\underline{x}]) = 0$$

So, we can find that

$$c\ exp[c\hat{\lambda}]\, E_{\lambda}(exp[-\lambda(z + c)|\,\underline{x}]) = \ cE_{\lambda}\,(exp[-z\lambda\,|\,\underline{x}\,])$$

Consequently, the Bayesian estimation of the parameter λ using WLINEX will be

$$\hat{\lambda}_{WBL} \ = \frac{1}{c}\ Ln\left[\frac{E_{\lambda}(exp\ [(-z\lambda\,|\,\underline{x})]}{E_{\lambda}(exp\ [(-\lambda(z + c)\ |\,\underline{x})]}\right] \qquad ;\ z + c \neq 0 \tag{17}$$

Note that, WLINEX loss function is a generalizing of LINEX loss function, where LINEX is a special case of WLINEX when $z = 0$ in Equation (17).

4. Bayes estimators

This section explains the derivation of Bayes estimates for a scale parameter θ, a reliability function $R(t)$ and a hazard function $H(t)$ for LD. We employ three various loss functions involving the squared error loss function (SE) , the LINEX loss functions, and the weighted LINEX loss functions. Furthermore, we assume gamma $(\eta\ , a)$ be a conjugate prior distribution for θ as follow

$$g(\theta) = \frac{\eta^{a}}{\Gamma(a)} \theta^{a-1} exp[-\eta\ \theta] \qquad ;\ \ \eta > 0\ , \qquad \theta > 0 \tag{18}$$

Via Bayes theorem, joining the likelihood function in Equation (6) with the prior pdf of θ in Equation (18), a posterior distribution of θ can be gotten as

$$\pi(\theta \mid \underline{x}) = \frac{L(\theta, \underline{x}) g(\theta)}{\int_0^\infty L(\theta, \underline{x}) g(\theta) d\theta} = \frac{\mathcal{A}^{(n+a)} \theta^{n+a-1} exp[-\theta \mathcal{A}]}{\Gamma(n+a)} \tag{19}$$

where $\mathcal{A} = \eta + Ln\{ x_{u(n)} + \delta\}$

4.1. Bayes estimator based on squared error loss function (SE)

Using squared error loss function, Bayesian estimator $\hat{\theta}_{SE}$ for θ using Equation (12) is:

$$\begin{aligned} \hat{\theta}_{SE} &= E(\theta \mid \underline{x}) = \int_0^\infty \theta\, \pi(\theta \mid \underline{x})\, d\theta \\ &= \int_0^\infty \theta \frac{\mathcal{A}^{(n+a)} \theta^{n+a-1} exp[-\theta \mathcal{A}]}{\Gamma(n+a)} d\theta \\ &= \frac{n+a}{\mathcal{A}} \end{aligned} \tag{20}$$

Correspondingly, Bayesian estimator based on reliability function $R(t)$ with fixed $t \geq 0$ is be expressed as:

$$\begin{aligned} \hat{R}_{SE}(t) &= E(R(t) \mid \underline{x}) = \int_0^\infty \delta^\theta (t+\delta)^{-\theta}\, \pi(\theta \mid \underline{x})\, d\theta \\ &= \int_0^\infty \delta^\theta (t+\delta)^{-\theta} \frac{\mathcal{A}^{(n+a)} \theta^{n+a-1} exp[-\theta \mathcal{A}]}{\Gamma(n+a)}\, d\theta \\ &= \left[1 + \frac{Ln\left(1 + \frac{t}{\delta}\right)}{\mathcal{A}}\right] \end{aligned} \tag{21}$$

While for a hazard function, $H(t)$ is expressed as

$$\begin{aligned} \hat{H}_{SE}(t) &= E(H(t) \mid \underline{x}) = \int_0^\infty \theta (t+\delta)^{-1}\, \pi(\theta \mid \underline{x})\, d\theta \\ &= \int_0^\infty \theta (t+\delta)^{-1} \frac{\mathcal{A}^{(n+a)} \theta^{n+a-1} exp[-\theta \mathcal{A}]}{\Gamma(n+a)} d\theta \\ &= \left[1 + \frac{n+a}{\mathcal{A}(t+\delta)}\right] \end{aligned} \tag{22}$$

4.2. Bayes estimator using squared error loss function (SE)

Based on LINEX loss function, via using Equation (14), Bayesian estimator $\hat{\theta}_{LINEX}$ for θ, can be specified by:

$$\begin{aligned} \hat{\theta}_{LINEX} &= -\frac{1}{c} Ln\left[E_\theta\left(exp[-c\,\theta] \mid \underline{x}\right)\right] \\ &= -\frac{1}{c} Ln\left[\int_0^\infty \{exp[-c\,\theta]\}\, \pi(\theta \mid \underline{x})\, d\theta\right] \\ &= -\frac{1}{c} Ln \int_0^\infty \{exp[-c\,\theta]\} \frac{\mathcal{A}^{(n+a)} \theta^{n+a-1} exp[-\theta \mathcal{A}]}{\Gamma(n+a)}\, d\theta \\ &= -\frac{1}{c}\, Ln\left(1 + \frac{c}{\mathcal{A}}\right)^{-(n+a)} \end{aligned} \tag{23}$$

Bayes estimators for $R(t)$ as well as $H(t)$ using LINEX loss function have specified by:

$$\hat{R}_{LINEX}(t) = -\frac{1}{c} Ln\left[E_\theta\left(exp[-c\,R(t)] \mid \underline{x}\right)\right]$$
$$= -\frac{1}{c} Ln\left[\int_0^\infty \{exp[-c\,R(t)]\}\,\pi\left(\theta \mid \underline{x}\right) d\theta\right]$$
$$= -\frac{1}{c} Ln \int_0^\infty \{exp[-c\,R(t)]\} \frac{\mathcal{A}^{(n+a)}\theta^{n+a-1} exp[-\theta\mathcal{A}]}{\Gamma(n+a)} d\theta \tag{24}$$
$$= -\frac{1}{c}\, Ln\left[\sum_{i=0}^{\infty} \frac{(-c)^i}{\mathrm{i}!}\left(1 + \frac{i\,Ln\left(1+\frac{t}{\delta}\right)}{\mathcal{A}}\right)^{-(n+a)}\right]$$

and

$$\hat{H}_{LINEX}(t) = -\frac{1}{c} Ln\left[E_\theta\left(exp[-c\,H(t)] \mid \underline{x}\right)\right]$$
$$= -\frac{1}{c} Ln\left[\int_0^\infty \{exp[-c\,H(t)]\}\,\pi\left(\theta \mid \underline{x}\right) d\theta\right]$$
$$= -\frac{1}{c} Ln \int_0^\infty \{exp[-c\,H(t)]\} \frac{\mathcal{A}^{(n+a)}\theta^{n+a-1} exp[-\theta\mathcal{A}]}{\Gamma(n+a)} d\theta \tag{25}$$
$$= -\frac{1}{c}\, Ln\left[1 + \frac{c}{\mathcal{A}(t+\delta)}\right]^{-(n+a)}$$

4.3. Bayes estimator using weighted LINEX loss function (WLINEX)

Based on weighted LINEX loss function, through using (17), the Bayes estimator of θ, $R(t)$ and $H(t)$ are feasibly derived, respectively, as

$$\hat{\theta}_{WLINEX} = \frac{1}{c}\, Ln\left[\frac{E_\theta\left(exp[-z\theta] \mid \underline{x}\right)}{E_\theta\left(exp[-(z+c)\theta] \mid \underline{x}\right)}\right]$$
$$= \frac{1}{c}\, Ln\left[\frac{I_1}{I_2}\right] \tag{26}$$

Where,

$$I_1 = E_\theta\left(exp\left[\left(-z\theta \mid \underline{x}\right)\right]\right] = \int_0^\infty \{exp[-z\,\theta]\}\,\pi\left(\theta \mid \underline{x}\right) d\theta$$
$$= \int_0^\infty \{exp[-z\,\theta]\} \frac{\mathcal{A}^{(n+a)}\theta^{n+a-1} exp[-\theta\mathcal{A}]}{\Gamma(n+a)} d\theta \tag{27}$$
$$= \left(1 + \frac{z}{\mathcal{A}}\right)^{-(n+a)}$$

and

$$I_2 = E_\theta\left(exp\left[\left(-(z+c)\theta \mid \underline{x}\right)\right]\right] = \int_0^\infty \{exp[-(z+c)\,\theta]\}\,\pi\left(\theta \mid \underline{x}\right) d\theta$$
$$= \int_0^\infty \{exp[-(z+c)\,\theta]\} \frac{\mathcal{A}^{(n+a)}\theta^{n+a-1} exp[-\theta\mathcal{A}]}{\Gamma(n+a)} d\theta \tag{28}$$
$$= \left(1 + \frac{z+c}{\mathcal{A}}\right)^{-(n+a)}$$

and the Bayes estimator for $R(t)$ is given by

$$\begin{aligned}\hat{R}_{WLINEX}(t) &= \frac{1}{c}\, Ln\left[\frac{E_\theta\left(exp[-zR(t)] \mid \underline{x}\right)}{E_\theta\left(exp[-(z+c)R(t)] \mid \underline{x}\right)}\right] \\ &= \frac{1}{c}\, Ln\left[\frac{I_3}{I_4}\right]\end{aligned} \tag{29}$$

Where,

$$\begin{aligned} I_{\,3} = E_\theta\left(exp\left[\left(-zR(t) \mid \underline{x}\right)\right]\right] &= \int_0^\infty \{exp[-z\,R(t)]\}\,\pi\left(\theta \mid \underline{x}\right) d\theta \\ &= \int_0^\infty \{exp[-z\,R(t)]\}\frac{\mathcal{A}^{(n+a)}\theta^{n+a-1}exp[-\theta\mathcal{A}]}{\Gamma(n+a)}\, d\theta \\ &= \sum_{i=0}^{\infty}\frac{(-z)^i}{\text{i!}}\left(1+\frac{i\,Ln\left(1+\frac{t}{\delta}\right)}{\mathcal{A}}\right)^{-(n+a)}\end{aligned} \tag{30}$$

and

$$\begin{aligned} I_{\,4} &= E_\theta\left(exp\left[\left(-(z+c)R(t) \mid \underline{x}\right)\right]\right] \\ &= \int_0^\infty \{exp[-(z+c\,)R(t)]\}\,\pi\left(\theta \mid \underline{x}\right) d\theta \\ &= \int_0^\infty \{exp[-(z+c)\,R(t)]\}\frac{\mathcal{A}^{(n+a)}\theta^{n+a-1}exp[-\theta\mathcal{A}]}{\Gamma(n+a)}\, d\theta \\ &= \sum_{i=0}^{\infty}\frac{(-(c+z))^i}{\text{i!}}\left(1+\frac{i\,Ln\left(1+\frac{t}{\delta}\right)}{\mathcal{A}}\right)^{-(n+a)}\end{aligned} \tag{31}$$

Bayes estimators for $H(t)$ has specified by:

$$\begin{aligned}\hat{H}_{WLINEX} &= \frac{1}{c}\, Ln\left[\frac{E_\theta\left(exp[-zH(t)] \mid \underline{x}\right)}{E_\theta\left(exp[-(z+c)H(t)] \mid \underline{x}\right)}\right] \\ &= \frac{1}{c}\, Ln\left[\frac{I_5}{I_6}\right]\end{aligned} \tag{32}$$

Where,

$$\begin{aligned} I_5 &= E_\theta\left(exp[-c\,H(t)] \mid \underline{x}\right) \\ &= \int_0^\infty \{exp[-c\,H(t)]\}\,\pi\left(\theta \mid \underline{x}\right) d\theta \\ &= \int_0^\infty \{exp[-c\,H(t)]\}\frac{\mathcal{A}^{(n+a)}\theta^{n+a-1}exp[-\theta\mathcal{A}]}{\Gamma(n+a)}\, d\theta \\ &= \left[1+\frac{z}{\mathcal{A}(t+\delta)}\right]^{-(n+a)}\end{aligned} \tag{33}$$

and

$$I_6 = E_\theta\big(exp[-c\,H(t)]\,\big|\,\underline{x}\big)$$
$$= \int_0^\infty \{exp[-(c+z)\,H(t)]\}\,\pi\big(\theta\,\big|\,\underline{x}\big)\,d\theta$$
$$= \int_0^\infty \{exp[-(c+z\,)H(t)]\}\frac{\mathcal{A}^{(n+a)}\theta^{n+a-1}exp[-\theta\mathcal{A}]}{\Gamma(n+a)}\,d\theta \quad (34)$$
$$= \left[1+\frac{z+c}{\mathcal{A}(t+\delta)}\right]^{-(n+a)}$$

5. Empirical example

In this section, Monte Carlo simulation has been implemented for investigating the performance of suggested estimate compared with MLE and Bayes estimate under SE, LINEX and WLINEX loss functions to estimate the scale parameter, hazard function and reliability function of LD when the shape parameter is known. We supposed some parameters including c and z. The values of (c) are equal to -0.7 , 0.0001 and 2.The positive and negative values were selected to represent both cases of overestimate and underestimate, respectively, while the values of z were 0.0001 and 3.

The simulation consists of the subsequent steps:

1. For the specified values of prior parameters $(\eta = 2\,, a = 1)$,we generate value $\theta = 1.383$ from the Gamma prior pdf in Equation (18)
2. Based on the used value $\theta = 1.383$ form Step 1, with $\delta = 1$, $n, (n = 4{,}5{,}6\ and\ 7)$ for upper record values from Lomax distribution whose P.D.F. is given by Equation (1) are generated.
3. The different estimates of $\theta\,, R(T)$ along with $H(T)$ at time t (chosen to be 4) have calculated.
4. Steps 1 to 3 have been repetitive for 10,000 times, while the mean squared error (MSE) for every estimate (say $\hat{\lambda}$) has computed by:

$$MSE(\hat{\lambda}) = \frac{1}{10000}\sum_{i=1}^{10000}\left(\hat{\lambda}_i - \lambda\right)^2 \quad (35)$$

where λ can be $\theta\,, R(t$ or $H(t)$ and $\hat{\lambda}_i$ is the estimate at the i^{th} run.

5. The results are listed in Tables 1-2.

Table 1. MSE of the estimates of $\theta, R(t)$ and $H(t)$ when $z = 3$ and $t = 4$

Parameters	n	(.)ML	(.) SE	(.) LINEX			WLINEX		
	c			−0.7	0.0001	2	−0.7	0.0001	2
θ	4	4.0792	1.2587	2.6411	1.2586	0.2515	0.1293	0.1675	0.2831
	5	3.2044	1.1839	2.2693	1.1838	0.2861	0.1132	0.1358	0.2256
	6	2.5398	1.1128	1.8683	1.1127	0.3064	0.0993	0.1155	0.1836
	7	1.8430	0.9034	1.5117	0.9034	0.2782	0.0894	0.0998	0.1544
$R(t)$	4	0.0088	0.0047	0.0048	0.0047	0.0044	0.0043	0.0043	0.0043
	5	0.0076	0.0044	0.0046	0.0044	0.0043	0.0041	0.0041	0.0041
	6	0.0064	0.0041	0.0041	0.0041	0.0039	0.0038	0.0038	0.0037
	7	0.0052	0.0034	0.0034	0.0034	0.0033	0.0033	0.0034	0.0034
$H(t)$	4	0.1632	0.0505	0.0575	0.0503	0.0351	0.0188	0.0164	0.0113

	5	0.1282	0.0474	0.0538	0.0474	0.0340	0.0202	0.0182	0.0129
	6	0.1016	0.0437	0.0475	0.0445	0.0326	0.0205	0.0185	0.0138
	7	0.0737	0.0369	0.0405	0.0361	0.0273	0.0187	0.0172	0.0130

Table 2. MSE of the estimates of $\theta, R(t)$ and $H(t)$ when $z = 0.0001$ and $t = 4$

Parameters	n	$(.)$ML	$(.)$ SE	$(.)$ LINEX			WLINEX		
	c			-0.7	0.0001	2	-0.7	0.0001	2
θ	4	4.0792	1.2587	2.6411	1.2586	0.2515	2.6405	1.2583	0.2514
	5	3.2044	1.1839	2.2693	1.1838	0.2861	2.2689	1.1836	0.2861
	6	2.5398	1.1128	1.8683	1.1127	0.3064	1.8680	1.1126	0.3064
	7	1.8430	0.9034	1.5117	0.9034	0.2782	1.5115	0.9033	0.2782
$R(t)$	4	0.0088	0.0047	0.0048	0.0047	0.0044	0.0048	0.0047	0.0044
	5	0.0076	0.0044	0.0046	0.0044	0.0043	0.0046	0.0044	0.0043
	6	0.0064	0.0041	0.0041	0.0041	0.0039	0.0041	0.0041	0.0039
	7	0.0052	0.0034	0.0034	0.0034	0.0033	0.0034	0.0034	0.0033
$H(t)$	4	0.1632	0.0505	0.0575	0.0503	0.0351	0.0575	0.0503	0.0351
	5	0.1282	0.0474	0.0538	0.0474	0.0340	0.0538	0.0474	0.0340
	6	0.1016	0.0437	0.0475	0.0445	0.0326	0.0475	0.0445	0.0326
	7	0.0737	0.0369	0.0405	0.0361	0.0273	0.0405	0.0361	0.0273

6. Conclusion

From the results in the above tables (1) and (2) we can state the following points:

1. The Bayesian estimator under WLINEX loss function has a minimum MSE's than the estimators using LINEX loss function, SE Loss Function, or MLE's followed by the Bayesian estimator under LINEX Loss Function
2. Aimed at estimation of θ for small record sample size, the use of Bayesian method in estimation has recommended in this study.
3. To draw a conclusion about the effect of a shape parameter (c) for asymmetric loss Function, whenever (c) has been convergent to zero then the Bayesian estimates under LINEX loss function are nearly the same as the SE estimate. For that reason, we can say that stands for one of the advantageous features of asymmetric loss functions.
4. Also, to draw a conclusion about the effect of (z), we examined different values of z whenever (z) it is close to zero then the Bayesian estimates under WLINEX loss function are almost the same as the Bayesian estimates under LINEX. Therefore, it stands for one of advantageous features of working with proposed loss functions.

Acknowledgement

The Deanship of Scientific Research at Prince Sattam bin Abdul-Aziz University has supported this study and all thanks and appreciation to them by the authors of this study.

References

[1] M. Ahsanullah, Record values, The Exponential Distribution: Theory, Methods and Applications, eds. N. Balakrishnan and A. P. Basu, Gordon and Breach Publishers, Newark, New Jersey, 1995.

[2] F. Al-Duais, and M. Alhagyan, "Bayesian Estimates Based On Record Values Under Weighted LINEX Loss Function", Pakistan Journal of Statistics and Operation Research, pp.11-19, 2020.

[3] A. Soliman, E. A. Amin, and A. A. Abd-El Aziz, "Estimation and prediction from inverse Rayleigh distribution based on lower record values", *Applied Mathematical Sciences*, Vol.4, No.62, pp.3057-3066, 2010.

[4] Z. Vidović, "On MLEs of the parameters of a modified Weibull distribution based on record values", *Journal of Applied Statistics*, Vol.46, No.4, pp.715-724, 2019.

[5] A. Pak, and S. Dey, "Statistical inference for the power Lindley model based on record values and inter-record times", *Journal of Computational and Applied Mathematics*, Vol.347, pp.156-172, 2019.

[6] A. S. Hassan, M. Abd-Allah, and H. F. Nagy, " Bayesian analysis of record statistics based on generalized inverted exponential model", *Int J Adv Sci Eng Inform Technol*, Vol.8, No.2, pp.323-335, 2018.

[7] S. Singh, Y. Mani Tripathi, and S. J. Wu, “Bayesian estimation and prediction based on lognormal record values", *Journal of Applied Statistics*, Vol.44, No.5, pp.916-940, 2017.

[8] M. M. Badr, “Estimations and prediction for the compound Rayleigh distribution based on upper record values", *Journal of Computational and Theoretical Nanoscience*, Vol.15, No.2, pp.711-718, 2018.

[9] M. Ahsanullah," Record values of the Lomax distribution". *Statistica Neerlandica*, Vol.45, No.1, pp.21-29, 1991.

[10] B. C. Arnold, N. Balakrishnan, and H. N. Nagaraja, Records. John Wiley&Sons. *New York*.

[11] P. Nasiri, and S. Hosseini, " Statistical inferences for Lomax distribution based on record values (Bayesian and classical)", *Journal of Modern Applied Statistical Methods*, Vol.11, No.1, p.15, 1998.

[12] M. A. Mahmoud, R. M. El-Sagheer, A. A. Soliman, and A. H. Abd Ellah, "Bayesian estimation of P [Y< X] Based on Record Values from the Lomax Distribution and MCMC Technique", *Journal of Modern Applied Statistical Methods*, Vol. 15, No.1, p.25, 2016.

[13] H. Ellah, "Comparison of estimates using record statistics from Lomax model: Bayesian and non-Bayesian approaches", *Journal of Statistical Research of Iran JSRI*, *3*(2), 139-158, 2007.

[14] G. Prakash, "Empirical Bayes Procedures under Lomax Record Data", *Journal of Applied Mathematics*, Vol.3, No.4, pp.182-197, 2016.

[15] A. Ahmad, S. P. Ahmad, and A. Ahmed, "Bayesian Analysis of Lomax Distribution under Asymmetric Loss Functions", *Journal of Statistics Applications and Probability Letters*, Vol.*3*, No.1, pp.35-44, 2016.

[16] G. Yari, "Some statistical inferences on the upper record of Lomax distribution", *International Journal of Nonlinear Analysis and Applications*, Vol,10, No.2, pp.35-46, 2019.

[17] H. R. Varian, "A Bayesian approach to real estate assessment", Studies in Bayesian econometric and statistics in Honor of Leonard J. Savage, pp.195-208, 1975.