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Global in time existence of Sobolev solutions to semi-linear damped σ-evolution equations in L^q scales

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submitted by **M.Sc. Tuan Anh Dao**

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- Prof. Dr. Michael Reissig
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Contents

1. Introduction

1.1. Background

During the last decades, fractional order partial differential equations have gained a great attention from many authors because of their wide applications in various disciplines such as physics, chemistry, mechanics and so on. The fractional Laplacian $(-\Delta)^s$ not only concerns Lévy fights in physics but also arises in stochastic theory. These operators, well-known as the operators associated with symmetric $2s$-stable Lévy processes, play an essential role to explore many different subjects on partial differential equations. One of them is the study of *semi-linear damped σ-evolution equations*

$$\begin{cases} u_{tt}+(-\Delta)^\sigma u+\mu(-\Delta)^\delta u_t = f(u,u_t,|D|^a u), & x\in\mathbb{R}^n,\, t>0,\\ u(0,x)=u_0(x),\quad u_t(0,x)=u_1(x), & x\in\mathbb{R}^n, \end{cases} \tag{1.1}$$

where $\sigma \geqslant 1$, $\delta \in [0,\sigma]$, for any $\mu > 0$, the nonlinearity $f = f(u,u_t,|D|^a u)$ with $a \in (0,\sigma)$ and for the Cauchy conditions (u_0,u_1) belonging to suitable function spaces.

Let us begin introducing some previous results which state $L^p - L^q$ decay estimates for solutions to the following Cauchy problem for the *σ-evolution equations*:

$$\begin{cases} u_{tt}+(-\Delta)^\sigma u = 0, & x\in\mathbb{R}^n,\, t>0,\\ u(0,x)=u_0(x),\quad u_t(0,x)=u_1(x), & x\in\mathbb{R}^n, \end{cases} \tag{1.2}$$

for any $\sigma \geqslant 1$. Namely, one of the most typical important problems of type (1.2) with $\sigma = 1$, the so-called *classical free wave model*, arises in many fields of applied sciences such as acoustics, electromagnetics and fluid dynamics. It describes mechanical waves (e.g. vibrating string with $n = 1$, vibrating membrane with $n = 2$, or vibrating elastic solid with $n = 3$) and light waves as well. In order to derive $L^p - L^q$ estimates, the main approach is the applications of method of stationary phase. In particular, it is necessary to understand deeply about oscillating integrals with localized amplitudes in different parts of the extended phase space. Without requiring for additional regularity of the data and with considering the first data $u_0 = 0$, we want to address the readers to two pioneering papers [59] and [64] in which solutions to (1.2) satisfy the following $L^p - L^q$ decay estimate away from the conjugate line:

$$\|u(t,\cdot)\|_{L^q} \lesssim t^{1-n(\frac{1}{p}-\frac{1}{q})}\|u_1\|_{L^p},$$

where the point $(\frac{1}{p},\frac{1}{q})$ belongs to the closed triangle with vertices $P_1 = (\frac{1}{2}+\frac{1}{n+1},\frac{1}{2}-\frac{1}{n+1})$, $P_2 = (\frac{1}{2}-\frac{1}{n-1},\frac{1}{2}-\frac{1}{n-1})$ and $P_3 = (\frac{1}{2}+\frac{1}{n-1},\frac{1}{2}-\frac{1}{n-1})$. Here we notice that we define $P_2 = (0,0)$ and $P_3 = (1,1)$ in the space dimension $n = 1$ or $n = 2$. After that, we are able to extend the admissible range for (p,q) to get the $L^p - L^q$ estimates by using additional regularity of the data. The authors, for examples, in the papers [50] and [65] obtained the following $L^p - L^p$ estimate:

$$\|u(t,\cdot)\|_{L^p} \lesssim (1+t)^{(n-1)|\frac{1}{p}-\frac{1}{2}|}\|u_0\|_{H^s_p} + t(1+t)^{[(n-1)|\frac{1}{p}-\frac{1}{2}|-1]^+}\|u_1\|_{H^r_p},$$

where $p \in (1,\infty)$, $s \geqslant (n-1)|\frac{1}{p}-\frac{1}{2}|$ and $r \geqslant \big[(n-1)|\frac{1}{p}-\frac{1}{2}|-1\big]^+$. Quite recently, the authors in [25] have developed this method to study (1.2) for any $\sigma > 1$. By taking $u_0 = 0$, the following $L^p - L^q$ estimate holds:

$$\|u(t,\cdot)\|_{L^q} \lesssim t^{1-\frac{n}{\sigma}(\frac{1}{p}-\frac{1}{q})}\|u_1\|_{L^p},$$

for all $1 \leqslant p \leqslant q \leqslant \infty$, with $\frac{1}{p}+\frac{1}{q} \leqslant 1$ and $\frac{1-\sigma}{p}-\frac{1}{q} \leqslant \sigma(\frac{1}{n}-\frac{1}{2})$ or $\frac{1}{p}+\frac{1}{q} \geqslant 1$ and $\frac{1}{p}+\frac{\sigma-1}{q} \leqslant \sigma(\frac{1}{n}+\frac{1}{2})$. Under additional regularity, the estimate

$$\|u(t,\cdot)\|_{L^q} \lesssim t^{-\frac{n}{\sigma}(\frac{1}{p}-\frac{1}{q})}\big(\|u_0\|_{L^p} + t\,\|u_0\|_{H^\sigma_p} + t\,\|u_1\|_{L^p}\big)$$

was derived in [25]. Moreover, the associated semi-linear Cauchy problem for (1.2) with the power nonlinearity $|u|^p$, $p > 1$, has been widely investigated among the mathematical community (see, for example, [25, 30, 31, 33, 40, 41, 45, 66, 67, 68, 78, 80]).

A second interesting model related to (1.1), namely that with $\sigma = 1$ and $\delta = 0$, is well-known as the *classical damped wave equation*

$$\begin{cases} u_{tt} - \Delta u + u_t = 0, & x \in \mathbb{R}^n,\ t > 0, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.3}$$

By the aid of the damping term u_t, in the papers [47] and [48] the author established $L^p - L^q$ decay estimates on the conjugate line. Afterwards, the following $L^p - L^q$ decay estimates away from the conjugate line have been stated in [53]:

$$\big\|\partial_t^j \nabla^k u(t,\cdot)\big\|_{L^q} \lesssim t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-j-\frac{k}{2}}\big(\|u_0\|_{L^p} + \|u_1\|_{L^p}\big),$$

with $1 < p \leqslant q < \infty$, where $j,\ k \geqslant 0$ are integers and for all space dimensions $n \geqslant 2$. Also, in the same paper the precise interpolation of the diffusive structure as $t \to \infty$ has been discussed in the $L^p - L^q$ framework. Main goal of the cited papers is to apply the obtained $L^p - L^q$ estimates to deal with the corresponding Cauchy problem for semi-linear equations. Concerning the classical semi-linear damped wave equation with nonlinearity term $|u|^p$, the authors in [73] proved the global (in time) existence of energy solutions for $p > p_{Fuj}(n) = 1 + \frac{2}{n}$, the so-called Fujita exponent, and for $p \leqslant \frac{n}{n-2}$ if $n \geqslant 3$. Besides, they also indicated a blow-up result in the inverse case $1 < p < p_{Fuj}(n)$ which was improved for $1 < p \leqslant p_{Fuj}(n)$ in the paper [79] by using the well-known test function method so far. For the purpose of further considerations, there are numerous papers involving (1.3) with the nonlinearity term $|u|^{p-1}u$ in the place of $|u|^p$ (see more [37, 38, 44, 54, 58]).

A third remarkable model which has widely studied in several recent papers, for instance, [7, 12, 52, 57] is the Cauchy problem for *structurally damped wave equations*

$$\begin{cases} u_{tt} - \Delta u + \mu(-\Delta)^\delta u_t = 0, & x \in \mathbb{R}^n,\ t > 0, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.4}$$

with $\delta \in (0,1]$. In particular, in [57] the authors divided the phase space into two parts including sufficiently small and sufficiently large frequencies in order to study Fourier multipliers with oscillations in the representation of solutions to (1.4). More in detail, to do this, there appeared two main strategies in [57]. They applied heavily radial symmetry combined with the theory of modified Bessel functions (see also [26]) and took into considerations the connection to Fourier multipliers appearing for wave models, respectively, for small frequencies and large frequencies. Consequently, having L^1 estimates for oscillating integrals was to conclude the $L^p - L^q$ estimates away from the conjugate line for solutions to (1.4) in the distinct cases $\delta = \frac{1}{2}$, $\delta \in (0, \frac{1}{2})$ and $\delta \in (\frac{1}{2}, 1)$ as follows:

$$\|u(t,\cdot)\|_{L^q} \lesssim t^{-n(1-\frac{1}{r})}\|u_0\|_{L^p} + t^{1-n(1-\frac{1}{r})}\|u_1\|_{L^p},$$

$$\|u(t,\cdot)\|_{L^q} \lesssim \begin{cases} t^{-[\frac{n}{2}](\frac{1}{2\delta}-1)\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|u_0\|_{L^p} + t^{1-[\frac{n-2}{2}](\frac{1}{2\delta}-1)\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|u_1\|_{L^p} & \text{if } t \in (0,1], \\ t^{-\frac{n}{2(1-\delta)}(1-\frac{1}{r})}\|u_0\|_{L^p} + t^{1-\frac{n+2-4\delta}{2(1-\delta)}(1-\frac{1}{r})}\|u_1\|_{L^p} & \text{if } t \in [1,\infty), \end{cases}$$

and

$$\|u(t,\cdot)\|_{L^q} \lesssim \begin{cases} t^{-\frac{n+2-4\delta}{2\delta}(1-\frac{1}{r})}\|u_0\|_{L^p} + t^{1-\frac{n}{2\delta}(1-\frac{1}{r})}\|u_1\|_{L^p} & \text{if } t \in (0,1], \\ t^{-\frac{n+2-4\delta}{2\delta}(1-\frac{1}{r})+[\frac{n}{2}](1-\frac{1}{2\delta})\frac{1}{r}}\|u_0\|_{L^p} + t^{1-\frac{n}{2\delta}(1-\frac{1}{r})+[\frac{n}{2}-1](1-\frac{1}{2\delta})\frac{1}{r}}\|u_1\|_{L^p} & \text{if } t \in [1,\infty), \end{cases}$$

respectively, for all $1 \leqslant p \leqslant q \leqslant \infty$ and $1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{p}$. In the special case $\delta = 1$ related to visco-elastic type damping, this model was considered more in detail in [69]. The author obtained a potential decay estimate for solutions localized to low frequencies, whereas their high-frequency part decays exponentially under the requirement of a suitable regularity for the data by application of the Marcinkiewicz theorem (see, for example, [46, 75]) to related Fourier multipliers. Thereafter, considering the case of semi-linear visco-elastic or structurally damped wave models the authors in [12, 60] proved the global (in time) existence of small data energy solutions in low space dimensions by using classical energy estimates, i.e. estimates on the base of L^2 norms. In addition, in [7] some suitable high-frequency $L^q - L^q$ estimates, with $q \in (1,\infty)$, for solutions to (1.4) have been obtained for $\delta \in (0, \frac{1}{4})$. Meanwhile, in the remaining case $\sigma \in [\frac{1}{4}, 1)$ the authors developed these estimates relying on some techniques in [57]. Then, by the application of the achieved estimates some global (in time) existence results of small data Sobolev solutions were presented in [7] for "parabolic like models" corresponding to (1.4) with $\delta \in (0, \frac{1}{2})$.

More recently, based on ideas from [57] the authors in [11] considered the following linear Cauchy problem for (1.1):

$$\begin{cases} u_{tt} + (-\Delta)^\sigma u + \mu(-\Delta)^\delta u_t = 0, & x \in \mathbb{R}^n,\, t > 0, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.5}$$

and concluded the following $L^p - L^q$ estimate away from the conjugate line for solutions to (1.5) in the case $\sigma \geqslant 1$ and $\delta = \frac{\sigma}{2}$:

$$\|u(t,\cdot)\|_{L^q} \lesssim t^{-\frac{n}{\sigma}(1-\frac{1}{r})}\|u_0\|_{L^p} + t^{1-\frac{n}{\sigma}(1-\frac{1}{r})}\|u_1\|_{L^p}$$

for all $1 \leqslant p \leqslant q \leqslant \infty$ and $1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{p}$. The use of $(L^1 \cap L^q) - L^q$ estimates to (1.5) with $q \in (1,2]$, i.e. the mixing of additional L^1 regularity for the data on the basis of $L^q - L^q$ estimates was investigated in [11] to study semi-linear σ-evolution models (1.1) with the nonlinearities $|u|^p$ and $|u_t|^p$ in the case $\delta = \frac{\sigma}{2}$. The effective tools that the authors applied were results from Harmonic Analysis such as Gagliardo-Nirenberg inequality, the fractional powers rule and embeddings into L^∞ (see also [61]). Some classical versions of Gagliardo-Nirenberg inequality can be found, for example, in [12, 35, 55]. Independently from [11, 57], another approach in [9] was to derive sharp $L^p - L^q$ estimates, with $1 < p \leqslant q < \infty$, to (1.5) and some L^q estimates for solutions and some of their derivatives, with $q \in (1,\infty)$, to (1.1) in the case $\delta \in [0, \frac{\sigma}{2}]$. In particular, here the authors found an explicit way to obtain these estimates for (1.5) by using the Mikhlin-Hörmander multiplier theorem for kernels localized to high frequencies. Due to the lack of $L^1 - L^1$ estimates, they used two different strategies to look for the global (in time) existence of small data Sobolev solutions to the semi-linear models (1.1). On the one hand, they took account of additional $L^1 \cap L^\infty$ regularity in the first case with $\delta = \frac{\sigma}{2}$. Additional $L^\eta \cap L^{\bar{q}}$ regularity, on the other hand, was replaced for any small η and large $\bar{q}$ in the second case with $\delta \in (0, \frac{\sigma}{2})$. In [26] the authors mentioned some different interesting models related to (1.5), namely those with $\sigma = \delta = 2$, well-known as the visco-elastic damped plate models. Here some decay estimates of the energy and qualitative properties of energy solutions were studied as well.

Finally, let us mention briefly some known results to (1.5) with $\mu = b(t)$, the so-called *structurally damped σ-evolution equation with time-dependent dissipation*. In the PhD thesis [42], the author developed WKB analysis to derive an explicit representation formula based on Fourier multipliers for solutions. Using the obtained presentation formula, he derived $L^2 - L^2$ decay estimates for energies of higher order and $L^p - L^q$ estimates on the conjugate line as well under some "effectiveness assumptions" of the coefficient $\mu = b(t)$. Moreover, some qualitative properties of energy solutions such as parabolic effect and smoothing effect were explained in detail. In the paper [8], the authors have introduced a complete classification distinguishing between effective damping and non-effective damping for the model of interest $b(t) = \beta(1+t)^\alpha$ with $\beta > 0$ and $\alpha \in (-1,1)$. Furthermore, they have verified that in the former case the asymptotic profile of Sobolev solutions to (1.5) is the same as that to an anomalous diffusion equation under a suitable choice of data. This means there appears the diffusion phenomenon.

1.2. Main goals and structure of the thesis

In this thesis, we are going to study the following Cauchy problems for semi-linear damped σ-evolution models:

$$\begin{cases} u_{tt} + (-\Delta)^\sigma u + \mu(-\Delta)^\delta u_t = f(u, u_t, |D|^a u), & x \in \mathbb{R}^n,\, t > 0, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.6}$$

with $\sigma \geqslant 1$, $\mu > 0$ and $\delta \in [0,\sigma]$. Here the function $f(u, u_t, |D|^a u)$ stands for the power nonlinearities $|u|^p$, $|u_t|^p$ and $\big||D|^a u\big|^p$ with a given number $p > 1$ and some constant $a \in (0,\sigma)$. We are interested in exploring two main models including σ-evolution models with structural damping $\delta \in (0,\sigma)$ and those with visco-elastic damping $\delta = \sigma$ (or strong damping, see also [36, 39]) in the general cases $\sigma \geqslant 1$.

The main goal of the present thesis is to prove the global (in time) existence of small data Sobolev solutions to (1.6) from suitable function spaces basing on L^q spaces by using $(L^m \cap L^q) - L^q$ and $L^q - L^q$ estimates for solutions, with $q \in (1,\infty)$ and $m \in [1,q)$, to the corresponding linear model with vanishing right-hand side, i.e. the mixing of additional L^m regularity for the data on the basis of

$L^q - L^q$ estimates. To establish desired results, we would like to investigate L^1 estimates for oscillating integrals in the presentation of solutions to the linear problem by applying the theory of modified Bessel functions and Faà di Bruno's formula. Then, it is reasonable to derive $L^p - L^q$ estimates not necessarily on the conjugate line for solutions to the linear problem, with $1 \leqslant p \leqslant q \leqslant \infty$, in the case of structural damping $\delta \in (0, \sigma)$. Unfortunately, this strategy fails in the cases of external damping $\delta = 0$ and visco-elastic type damping $\delta = \sigma$. For this reason, we apply the Mikhlin-Hörmander multiplier theorem for kernels localized to high frequencies to obtain $L^q - L^q$ estimates, with $q \in (1, \infty)$, for solutions to the linear problem in the latter cases by assuming a suitable regularity for the data. Having $L^q - L^q$ estimates after assuming additional L^m regularity for the data and some of modern tools from Harmonic Analysis in [61] (see also [12, 35]) play a fundamental role to prove results for the global (in time) existence of small data Sobolev solutions to (1.6). Throughout this thesis, we recognize that the flexible choice of parameters σ, δ, m and q not only brings some benefits to relax the restrictions to the admissible exponents p but also affects our global (in time) existence results remarkably.

For the case $\delta \in (0, \frac{\sigma}{2})$, we want to underline that we intend to use different strategies allowing no loss of decay and some loss of decay combined with loss of regularity to deal with (1.6). *Loss of regularity* (see, for example, [4, 9, 50, 59]) is a well-known phenomenon describing the effect that the regularity of obtained solutions to semi-linear models is less than that of the initial data. This phenomenon appearing in our global (in time) existence results is due to the singular behavior of time-dependent coefficients in estimates for solutions to the linear model localized to high frequencies as $t \to +0$. However, we can compensate this difficulty by assuming higher regularity for the data. *Loss of decay* is understood when decay rates in estimates for solutions to semi-linear models are worse than those given for solutions to the corresponding linear model. Additional benefits of allowing loss of decay (see [7]) are to show how the restrictions to the admissible exponents p could be relaxed.

For the remaining case $\delta \in (\frac{\sigma}{2}, \sigma]$, compared to the regularity of the initial data we can see that a loss of regularity of the obtained Sobolev solutions to (1.6) does not happen. More precisely, a smoothing effect appears for some derivatives of solutions to the corresponding linear equation with respect to the time variable. This brings some benefits in treament of the semi-linear equations (1.6). Finally, we want to emphasize that the properties of solutions to (1.6) change completely from $(0, \frac{\sigma}{2})$ to $(\frac{\sigma}{2}, \sigma]$. Here we propose to distinguish between "parabolic like models" in the case $\delta \in (0, \frac{\sigma}{2})$ and "σ-evolution like models" in the case $\delta \in (\frac{\sigma}{2}, \sigma]$ according to expected decay estimates.

The structure of this thesis is organized as follows: In Chapter 2 we follow the same approach from the paper [57] with minor modifications in steps of proofs to conclude $L^p - L^q$ estimates not necessarily on the conjugate line for solutions to the linear problem, with $1 \leqslant p \leqslant q \leqslant \infty$, in the case $\delta = \frac{\sigma}{2}$. Then, we may obtain $(L^m \cap L^q) - L^q$ and $L^q - L^q$ estimates with $q \in (1, \infty)$ and $m \in [1, q)$. In Chapters 3 and 4 we develop some L^1 estimates relying on several techniques from [57] for oscillating integrals in the presentation of Sobolev solutions to the corresponding linear model by using the theory of modified Bessel functions combined with Faà di Bruno's formula in the cases $\delta \in (0, \frac{\sigma}{2})$ and $\delta \in (\frac{\sigma}{2}, \sigma)$, respectively. We also derive L^∞ estimates to conclude L^r estimates for all $r \in [1, \infty]$. For this reason, we state $L^p - L^q$ estimates not necessarily on the conjugate line, $(L^m \cap L^q) - L^q$ and $L^q - L^q$ estimates, with $q \in (1, \infty)$ and $m \in [1, q)$ for solutions to the linear model. The point in Chapters 3 and 4 is the application of the Mikhlin-Hörmander multiplier theorem for kernels localized to high frequencies to obtain $L^q - L^q$ estimates, with $q \in (1, \infty)$, for solutions to the linear problem by assuming a suitable regularity for the data in the cases $\delta = 0$ and $\delta = \sigma$, respectively. Afterwards, in Chapters 5, 6 and 7 we investigate the global (in time) existence of small data solutions to the semi-linear models from suitable function spaces basing on L^q spaces including energy solutions, Sobolev solutions, energy solutions with a suitable higher regularity and large regular solutions in the cases $\delta = \frac{\sigma}{2}$, $\delta \in (0, \frac{\sigma}{2})$ and $\delta \in (\frac{\sigma}{2}, \sigma]$, respectively. To do this, we apply $(L^m \cap L^q) - L^q$, $L^q - L^q$ estimates from Chapters 2, 3, 4 and some tools from Harmonic Analysis such as the fractional Gagliardo-Nirenberg inequality, the fractional Leibniz rule, the fractional chain rule, the fractional powers rule and the fractional Sobolev embedding. The emphasis in Chapter 6 is on presenting in detail two used different strategies allowing no loss of decay and some loss of decay combined with loss of regularity. In Chapter 8 we explain some qualitative properties including Gevrey smoothing and propagation of singularities for energy solutions to the linear problem. Finally, Chapter 9 is to devote to the proof of a blow-up result for (1.6) with the nonlinearity $|u|^p$, where $\sigma \geqslant 1$ and $\delta \in [0, \sigma)$ are assumed to be any fractional numbers, by the application of a modified test function method.

2. Linear structurally damped σ-evolution models with $\delta = \frac{\sigma}{2}$

The main purpose of this chapter is to study linear structurally damped σ-evolution models of the form

$$u_{tt} + (-\Delta)^\sigma u + \mu(-\Delta)^\delta u_t = 0, \quad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) \tag{2.1}$$

with $\sigma \geqslant 1$, $\mu > 0$ and $\delta = \frac{\sigma}{2}$. This is a family of structurally damped σ-evolution models in the special case $\delta = \frac{\sigma}{2}$. Our goal is to obtain $L^q - L^q$ estimates for solutions to (2.1) by assuming additional L^m regularity for the data with $m \in [1,q)$, where $q \in (1,\infty)$ is given.

To do this, let us explain our objectives and strategies as follows:

- By using the partial Fourier transformation we can reduce the partial differential equation to study an ordinary differential equation parameterized by ξ.
- Due to different asymptotic behavior of the characteristic roots depending on μ, we divide our considerations into two sub-cases: $\mu = 2$ and $\mu \neq 2$.
- The key tool is to use suitable L^p estimates with $p \in [1,\infty]$ from [57] for the oscillating integrals in the following form:

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a e^{-c|\xi|^{2\alpha}t}\big)(t,\cdot)\big\|_{L^p},$$

  and

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a e^{-c_1|\xi|^{2\alpha}t}\cos(c_2|\xi|^{2\alpha}t)\big)(t,\cdot)\big\|_{L^p}, \quad \big\|\mathfrak{F}^{-1}\big(|\xi|^a e^{-c_1|\xi|^{2\alpha}t}\sin(c_2|\xi|^{2\alpha}t)\big)(t,\cdot)\big\|_{L^p}.$$

- Then, by using Young's convolution inequality we can conclude $L^m - L^q$ estimates with $q \in [1,\infty]$ and $m \in [1,q]$, $L^m \cap L^q - L^q$ and $L^q - L^q$ estimates with $q \in (1,\infty)$ and $m \in [1,q)$ for solutions to (2.1).

2.1. A first Cauchy problem for linear structurally damped σ-evolution models

Let us consider the following family of parameter-dependent Cauchy problems:

$$u_{tt} + (-\Delta)^\sigma u + \mu(-\Delta)^{\frac{\sigma}{2}} u_t = 0, \quad u(s,x) = 0, \quad u_t(s,x) = u_1(x), \tag{2.2}$$

where $s \geqslant 0$ is a fixed non-negative real parameter, $\mu > 0$ and $\sigma \geqslant 1$. Thanks to the change of variables $t \to t - s$, we have here in mind the following Cauchy problem:

$$v_{tt} + (-\Delta)^\sigma v + \mu(-\Delta)^{\frac{\sigma}{2}} v_t = 0, \quad v(0,x) = 0, \quad v_t(0,x) = v_1(x). \tag{2.3}$$

Using partial Fourier transformation to (2.3) we obtain the Cauchy problem for $\widehat{v}(t,\xi) := \mathfrak{F}(v(t,x))$ and $\widehat{v_1}(\xi) := \mathfrak{F}(v_1(x))$

$$\widehat{v}_{tt} + \mu|\xi|^\sigma \widehat{v}_t + |\xi|^{2\sigma}\widehat{v} = 0, \quad \widehat{v}(0,\xi) = 0, \quad \widehat{v}_t(0,\xi) = \widehat{v_1}(\xi). \tag{2.4}$$

$L^m \cap L^q - L^q$ and $L^q - L^q$ estimates

In this section, we want to prove the following result.

Theorem 2.1.1. *Let $q \in (1,\infty)$ be given and $m \in [1,q)$. Then, the solutions to (2.3) satisfy the $(L^m \cap L^q) - L^q$ estimates*

$$\big\|\partial_t^j |D|^a v(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{a}{\sigma}-j}\|v_1\|_{L^m \cap H_q^{[a+(j-1)\sigma]^+}}$$

and the $L^q - L^q$ estimates

$$\big\|\partial_t^j |D|^a v(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{1-\frac{a}{\sigma}-j}\|v_1\|_{H_q^{[a+(j-1)\sigma]^+}}.$$

Consequently, the solutions to (2.2) satisfy the $(L^m \cap L^q) - L^q$ estimates

$$\big\|\partial_t^j |D|^a u(t,\cdot)\big\|_{L^q} \lesssim (1+t-s)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{a}{\sigma}-j}\|u_1\|_{L^m \cap H_q^{[a+(j-1)\sigma]^+}},$$

and the $L^q - L^q$ estimates

$$\big\|\partial_t^j |D|^a u(t,\cdot)\big\|_{L^q} \lesssim (1+t-s)^{1-\frac{a}{\sigma}-j}\|u_1\|_{H_q^{[a+(j-1)\sigma]^+}},$$

where $1+\frac{1}{q} = \frac{1}{r}+\frac{1}{m}$, $j=0,1$, for any $a \geqslant 0$ and for all dimensions $n \geqslant 1$.

Proof. Here we follow ideas from [11]. We divide our considerations into three sub-cases: $\mu = 2$, $\mu \in (2,\infty)$ and $\mu \in (0,2)$.

Special case $\mu = 2$. We have a double root $\lambda_{1,2}(\xi) = -|\xi|^\sigma$. The solutions and their derivative in time to (2.4) are, respectively,

$$\widehat{v}(t,\xi) = t\, e^{-|\xi|^\sigma t}\widehat{v_1}(\xi),$$

and

$$\widehat{v}_t(t,\xi) = (1-t\,|\xi|^\sigma)e^{-|\xi|^\sigma t}\widehat{v_1}(\xi).$$

Transforming back to $v(t,x) = \mathfrak{F}^{-1}(\widehat{v}(t,\xi))$ there appear oscillating integrals estimated by using Corollary 3 from [57].

Proposition 2.1.1. *The estimate*

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a e^{-c|\xi|^{2\alpha}t}\big)(t,\cdot)\big\|_{L^r} \lesssim t^{-\frac{n}{2\alpha}(1-\frac{1}{r})-\frac{a}{2\alpha}}$$

holds for any $\alpha \in (0,\infty)$, $r \in [1,\infty]$, $t > 0$ and for all dimensions $n \geqslant 1$. The numbers a and c are, respectively, supposed to be non-negative and positive.

Due to Proposition 2.1.1 and by virtue of Young's convolution inequality from Proposition B.1.1, we conclude

$$\begin{aligned}
\big\||D|^a v(t,\cdot)\big\|_{L^q} &= \big\|\mathfrak{F}^{-1}\big(|\xi|^a t\, e^{-|\xi|^\sigma t}\widehat{v_1}(\xi)\big)(t,\cdot)\big\|_{L^q}\\
&\lesssim t\,\big\|\mathfrak{F}^{-1}\big(|\xi|^a e^{-|\xi|^\sigma t}\big)(t,\cdot)\big\|_{L^r}\big\|\mathfrak{F}^{-1}(\widehat{v_1}(\xi))\big\|_{L^m}\\
&\lesssim t^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{a}{\sigma}}\|v_1\|_{L^m} \quad \text{for } t \in [1,\infty),
\end{aligned}$$

and

$$\begin{aligned}
\big\||D|^a v(t,\cdot)\big\|_{L^q} &\lesssim t\,\big\|\mathfrak{F}^{-1}\big(|\xi|^{\min\{a,\sigma\}}e^{-|\xi|^\sigma t}\big)(t,\cdot)\big\|_{L^1}\big\|\mathfrak{F}^{-1}(|\xi|^{[a-\sigma]^+}\widehat{v_1}(\xi))\big\|_{L^q}\\
&\lesssim t^{1-\frac{\min\{a,\sigma\}}{\sigma}}\|v_1\|_{\dot{H}_q^{[a-\sigma]^+}} \lesssim \|v_1\|_{H_q^{[a-\sigma]^+}} \quad \text{for } t \in (0,1],
\end{aligned}$$

These estimates imply

$$\big\||D|^a v(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{a}{\sigma}}\|v_1\|_{L^m \cap H_q^{[a-\sigma]^+}}.$$

Analogously, in the further considerations we are not only interested in estimates for solutions but also in their derivatives

$$\begin{aligned}
\big\||D|^a v_t(t,\cdot)\big\|_{L^q} &= \big\|\mathfrak{F}^{-1}\big(|\xi|^a \widehat{v}_t(t,\xi)\big)(t,\cdot)\big\|_{L^q} = \big\|\mathfrak{F}^{-1}\big(|\xi|^a(1-t\,|\xi|^\sigma)e^{-|\xi|^\sigma t}\widehat{v_1}(\xi)\big)(t,\cdot)\big\|_{L^q}\\
&\lesssim \big\|\mathfrak{F}^{-1}\big(|\xi|^a(1-t\,|\xi|^\sigma)e^{-|\xi|^\sigma t}\big)(t,\cdot)\big\|_{L^r}\,\big\|\mathfrak{F}^{-1}(\widehat{v_1}(\xi))\big\|_{L^m}\\
&\lesssim \Big(\big\|\mathfrak{F}^{-1}(|\xi|^a e^{-|\xi|^\sigma t})(t,\cdot)\big\|_{L^r} + t\,\big\|\mathfrak{F}^{-1}(|\xi|^{a+\sigma}e^{-|\xi|^\sigma t})(t,\cdot)\big\|_{L^r}\Big)\|v_1\|_{L^m}\\
&\lesssim t^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{a}{\sigma}}\|v_1\|_{L^m} \quad \text{for } t \in [1,\infty),
\end{aligned}$$

and

$$\begin{aligned}\big\||D|^a v_t(t,\cdot)\big\|_{L^q} &\lesssim \Big(\big\|\mathfrak{F}^{-1}(e^{-|\xi|^\sigma t})(t,\cdot)\big\|_{L^1} + t\,\big\|\mathfrak{F}^{-1}(|\xi|^\sigma e^{-|\xi|^\sigma t})(t,\cdot)\big\|_{L^1}\Big)\big\|\mathfrak{F}^{-1}(|\xi|^a \widehat{v_1}(\xi))\big\|_{L^q}\\ &\lesssim \|v_1\|_{\dot{H}^a_q} \lesssim \|v_1\|_{H^a_q} \quad \text{for } t\in(0,1].\end{aligned}$$

Hence, we derive the estimates

$$\big\||D|^a v_t(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{a}{\sigma}}\|v_1\|_{L^m\cap H^a_q}.$$

Case $\mu\in(2,\infty)$. We have the roots

$$\lambda_{1,2} = \frac{1}{2}|\xi|^\sigma\big(-\mu\pm\sqrt{\mu^2-4}\big).$$

The solutions to (2.4) are

$$w(t,\xi) = \frac{e^{\lambda_1 t}-e^{\lambda_2 t}}{\lambda_1-\lambda_2}w_1(\xi) =: \widehat{K_1}(t,\xi)\widehat{v_1}(\xi).$$

We shall estimate the two terms $\widehat{K_1}(t,\xi)$ and $\partial_t\widehat{K_1}(t,\xi)$. Taking account of $\widehat{K_1}(t,\xi)$ the Newton-Leibniz formula implies

$$\widehat{K_1}(t,\xi) = t\int_0^1 e^{\frac{1}{2}\big(-\mu+(2\theta-1)\sqrt{\mu^2-4}\big)|\xi|^\sigma t}d\theta.$$

The other interesting term is

$$\begin{aligned}\partial_t\widehat{K_1}(t,\xi) &= \frac{\lambda_1 e^{\lambda_1 t}-\lambda_2 e^{\lambda_2 t}}{\lambda_1-\lambda_2}\\ &= \frac{-\mu+\sqrt{\mu^2-4}}{2\sqrt{\mu^2-4}}e^{-\frac{1}{2}|\xi|^\sigma(\mu-\sqrt{\mu^2-4})t} + \frac{\mu+\sqrt{\mu^2-4}}{2\sqrt{\mu^2-4}}e^{-\frac{1}{2}|\xi|^\sigma(\mu+\sqrt{\mu^2-4})t}.\end{aligned}$$

The application of Proposition 2.1.1 and Young's convolution inequality from Proposition B.1.1 lead to the following estimates:

$$\begin{aligned}\big\||D|^a v(t,\cdot)\big\|_{L^q} &= \big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_1}(t,\xi)\widehat{v_1}(\xi)\big)(t,\cdot)\big\|_{L^q} \lesssim \big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_1}(t,\xi)\big)(t,\cdot)\big\|_{L^r}\big\|\mathfrak{F}^{-1}(\widehat{v_1}(\xi))\big\|_{L^m}\\ &\lesssim t\,\Big\|\mathfrak{F}^{-1}\Big(\int_0^1 |\xi|^a e^{\frac{1}{2}\big(-\mu+(2\theta-1)\sqrt{\mu^2-4}\big)|\xi|^\sigma t}d\theta\Big)(t,\cdot)\Big\|_{L^r}\|v_1\|_{L^m}\\ &\lesssim t\int_0^1 \Big\|\mathfrak{F}^{-1}\Big(|\xi|^a e^{\frac{1}{2}\big(-\mu+(2\theta-1)\sqrt{\mu^2-4}\big)|\xi|^\sigma t}\Big)(t,\cdot)\Big\|_{L^r}d\theta\,\|v_1\|_{L^m}\\ &\lesssim t\,\big\|\mathfrak{F}^{-1}(|\xi|^a e^{-c|\xi|^\sigma t})(t,\cdot)\big\|_{L^r}\|v_1\|_{L^m} \lesssim t^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{a}{\sigma}}\|v_1\|_{L^m} \quad \text{for } t\in[1,\infty),\end{aligned}$$

where

$$c := \frac{1}{2}\big(\mu-(2\theta-1)\sqrt{\mu^2-4}\big) \geqslant \frac{1}{2}\big(\mu-\sqrt{\mu^2-4}\big) \geqslant \frac{1}{\mu} > 0.$$

Moreover, we also get for $t\in(0,1]$

$$\begin{aligned}\big\||D|^a v(t,\cdot)\big\|_{L^q} &\lesssim \big\|\mathfrak{F}^{-1}\big(|\xi|^{\min\{a,\sigma\}}\widehat{K_1}(t,\xi)\big)(t,\cdot)\big\|_{L^1}\big\|\mathfrak{F}^{-1}(|\xi|^{[a-\sigma]^+}\widehat{v_1}(\xi))\big\|_{L^q}\\ &\lesssim t\,\Big\|\mathfrak{F}^{-1}\Big(\int_0^1 |\xi|^{\min\{a,\sigma\}}e^{\frac{1}{2}\big(-\mu+(2\theta-1)\sqrt{\mu^2-4}\big)|\xi|^\sigma t}d\theta\Big)(t,\cdot)\Big\|_{L^1}\|v_1\|_{\dot{H}^{[a-\sigma]^+}_q}\\ &\lesssim t\int_0^1 \Big\|\mathfrak{F}^{-1}\Big(|\xi|^{\min\{a,\sigma\}}e^{\frac{1}{2}\big(-\mu+(2\theta-1)\sqrt{\mu^2-4}\big)|\xi|^\sigma t}\Big)(t,\cdot)\Big\|_{L^1}d\theta\,\|v_1\|_{H^{[a-\sigma]^+}_q}\\ &\lesssim t\,\big\|\mathfrak{F}^{-1}(|\xi|^{\min\{a,\sigma\}}e^{-c|\xi|^\sigma t})(t,\cdot)\big\|_{L^1}\|v_1\|_{H^{[a-\sigma]^+}_q}\\ &\lesssim t^{1-\frac{\min\{a,\sigma\}}{\sigma}}\|v_1\|_{H^{[a-\sigma]^+}_q} \lesssim \|v_1\|_{H^{[a-\sigma]^+}_q}.\end{aligned}$$

These estimates imply

$$\big\||D|^a v(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{a}{\sigma}}\|v_1\|_{L^m\cap H^{[a-\sigma]^+}_q}.$$

The other estimates are

$$\begin{aligned}\big\||D|^a v_t(t,\cdot)\big\|_{L^q} &= \big\|\mathfrak{F}^{-1}\big(|\xi|^a \hat{v}_t(t,\xi)\big)(t,\cdot)\big\|_{L^q} = \big\|\mathfrak{F}^{-1}\big(|\xi|^a \partial_t \widehat{K_1}(t,\xi)\hat{v_1}(\xi)\big)(t,\cdot)\big\|_{L^q}\\ &\lesssim \big\|\mathfrak{F}^{-1}\big(|\xi|^a \partial_t \widehat{K_1}(t,\xi)\big)(t,\cdot)\big\|_{L^r}\big\|\mathfrak{F}^{-1}\big(\hat{v_1}(\xi)\big)\big\|_{L^m}\\ &\lesssim \Big\|\mathfrak{F}^{-1}\Big(|\xi|^a e^{-\frac{1}{2}|\xi|^\sigma\left(\mu-\sqrt{\mu^2-4}\right)t}\Big)(t,\cdot)\Big\|_{L^r}\|v_1\|_{L^m}\\ &\quad+\Big\|\mathfrak{F}^{-1}\Big(|\xi|^a e^{-\frac{1}{2}|\xi|^\sigma\left(\mu+\sqrt{\mu^2-4}\right)t}\Big)(t,\cdot)\Big\|_{L^r}\|v_1\|_{L^m}\\ &\lesssim \Big(\big\|\mathfrak{F}^{-1}(|\xi|^a e^{-c_1|\xi|^\sigma t})(t,\cdot)\big\|_{L^r}+\big\|\mathfrak{F}^{-1}(|\xi|^a e^{-c_2|\xi|^\sigma t})(t,\cdot)\big\|_{L^r}\Big)\|v_1\|_{L^m}\\ &\lesssim t^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{a}{\sigma}}\|v_1\|_{L^m} \quad \text{for } t\in[1,\infty),\end{aligned}$$

where

$$c_1 := \frac{1}{2}\big(\mu-\sqrt{\mu^2-4}\big)>0 \quad\text{and}\quad c_2 := \frac{1}{2}\big(\mu+\sqrt{\mu^2-4}\big)>0.$$

Moreover, we also derive for $t\in(0,1]$

$$\begin{aligned}\big\||D|^a v_t(t,\cdot)\big\|_{L^q} &\lesssim \big\|\mathfrak{F}^{-1}\big(\partial_t \widehat{K_1}(t,\xi)\big)(t,\cdot)\big\|_{L^1}\big\|\mathfrak{F}^{-1}(|\xi|^a\hat{v_1}(\xi))\big\|_{L^q}\\ &\lesssim \Big\|\mathfrak{F}^{-1}\Big(e^{-\frac{1}{2}|\xi|^\sigma\left(\mu-\sqrt{\mu^2-4}\right)t}\Big)(t,\cdot)\Big\|_{L^1}\|v_1\|_{\dot{H}^a_q}\\ &\quad+\Big\|\mathfrak{F}^{-1}\Big(e^{-\frac{1}{2}|\xi|^\sigma\left(\mu+\sqrt{\mu^2-4}\right)t}\Big)(t,\cdot)\Big\|_{L^1}\|v_1\|_{\dot{H}^a_q}\\ &\lesssim \Big(\big\|\mathfrak{F}^{-1}(e^{-c_1|\xi|^\sigma t})(t,\cdot)\big\|_{L^1}+\big\|\mathfrak{F}^{-1}(e^{-c_2|\xi|^\sigma t})(t,\cdot)\big\|_{L^1}\Big)\|v_1\|_{H^a_q}\lesssim\|v_1\|_{H^a_q}.\end{aligned}$$

Hence, we may conclude

$$\big\||D|^a v_t(t,\cdot)\big\|_{L^q}\lesssim(1+t)^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{a}{\sigma}}\|v_1\|_{L^m\cap H^a_q}.$$

Case $\mu\in(0,2)$. The characteristic roots are

$$\lambda_{1,2}=\frac{1}{2}|\xi|^\sigma\big(-\mu\pm i\sqrt{4-\mu^2}\big).$$

The solutions to (2.4) are

$$\hat{v}(t,\xi)=\frac{2}{|\xi|^\sigma\sqrt{4-\mu^2}}\sin\Big(\frac{1}{2}|\xi|^\sigma\sqrt{4-\mu^2}t\Big)e^{-\frac{1}{2}\mu|\xi|^\sigma t}\hat{v_1}(\xi).$$

The partial derivative in time is

$$\hat{v}_t(t,\xi)=\Big(\cos\Big(\frac{1}{2}|\xi|^\sigma\sqrt{4-\mu^2}t\Big)-\frac{\mu}{\sqrt{4-\mu^2}}\sin\Big(\frac{1}{2}|\xi|^\sigma\sqrt{4-\mu^2}t\Big)\Big)e^{-\frac{1}{2}\mu|\xi|^\sigma t}\hat{v_1}(\xi).$$

We apply a modification of Lemma 5 from [57] to study the oscillating integrals

$$\mathfrak{F}^{-1}\big(e^{-c_1|\eta|^\sigma}\cos(c_2|\eta|^\sigma)\big)(t,x) \quad\text{and}\quad \mathcal{F}^{-1}\big(e^{-c_1|\eta|^\sigma}\sin(c_2|\eta|^\sigma)\big)(t,x)$$

after carrying out the change of variables $\xi=\eta t^{-\frac{1}{\sigma}}$. This yields the following result.

Proposition 2.1.2. *For all positive c_1, non-negative a and real $c_2\neq 0$ the estimates*

$$\begin{aligned}\big\|\mathfrak{F}^{-1}\big(|\xi|^a e^{-c_1|\xi|^{2\alpha}t}\cos(c_2|\xi|^{2\alpha}t)\big)(t,\cdot)\big\|_{L^r}&\lesssim t^{-\frac{n}{2\alpha}(1-\frac{1}{r})-\frac{a}{2\alpha}},\\ \big\|\mathfrak{F}^{-1}\big(|\xi|^a e^{-c_1|\xi|^{2\alpha}t}\sin(c_2|\xi|^{2\alpha}t)\big)(t,\cdot)\big\|_{L^r}&\lesssim t^{-\frac{n}{2\alpha}(1-\frac{1}{r})-\frac{a}{2\alpha}},\end{aligned}$$

hold for any $\alpha\in(0,\infty)$, $r\in[1,\infty]$, $t>0$ and for all dimensions $n\geqslant 1$.

Firstly, to estimate $\widehat{K_1}(t,\xi)$ we apply

$$\widehat{K_1}(t,\xi)=t\int_0^1 e^{\frac{1}{2}\left(-\mu+i(2\theta-1)\sqrt{4-\mu^2}\right)|\xi|^\sigma t}d\theta.$$

Using this formula and Young's convolution inequality from Proposition B.1.1 we conclude

$$\begin{aligned}
\big\||D|^a v(t,\cdot)\big\|_{L^q} &= \big\|\mathfrak{F}^{-1}\big(|\xi|^a \widehat{K_1}(t,\xi)\widehat{v_1}(\xi)\big)(t,\cdot)\big\|_{L^q} \lesssim \big\|\mathfrak{F}^{-1}\big(|\xi|^a \widehat{K_1}(t,\xi)\big)(t,\cdot)\big\|_{L^r}\big\|\mathfrak{F}^{-1}(\widehat{v_1}(\xi))\big\|_{L^m}\\
&\lesssim t\Big\|\mathfrak{F}^{-1}\Big(\int_0^1 |\xi|^a e^{\frac{1}{2}\big(-\mu+i(2\theta-1)\sqrt{4-\mu^2}\big)|\xi|^\sigma t}d\theta\Big)(t,\cdot)\Big\|_{L^r}\|v_1\|_{L^m}\\
&\lesssim t\int_0^1 \Big\|\mathfrak{F}^{-1}\Big(|\xi|^a e^{\frac{1}{2}\big(-\mu+i(2\theta-1)\sqrt{4-\mu^2}\big)|\xi|^\sigma t}\Big)(t,\cdot)\Big\|_{L^r}d\theta\,\|v_1\|_{L^m}\\
&\lesssim t^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{a}{\sigma}}\|v_1\|_{L^m} \quad \text{ for } t\in[1,\infty).
\end{aligned}$$

Moreover, we also get for $t\in(0,1]$

$$\begin{aligned}
\big\||D|^a v(t,\cdot)\big\|_{L^q} &\lesssim \big\|\mathfrak{F}^{-1}\big(|\xi|^{\min\{a,\sigma\}}\widehat{K_1}(t,\xi)\big)(t,\cdot)\big\|_{L^1}\big\|\mathfrak{F}^{-1}(|\xi|^{[a-\sigma]^+}\widehat{v_1}(\xi))\big\|_{L^q}\\
&\lesssim t\Big\|\mathfrak{F}^{-1}\Big(\int_0^1 |\xi|^{\min\{a,\sigma\}}e^{\frac{1}{2}\big(-\mu+i(2\theta-1)\sqrt{4-\mu^2}\big)|\xi|^\sigma t}d\theta\Big)(t,\cdot)\Big\|_{L^1}\|v_1\|_{\dot{H}^{[a-\sigma]^+}_q}\\
&\lesssim t\int_0^1 \Big\|\mathfrak{F}^{-1}\Big(|\xi|^{\min\{a,\sigma\}}e^{\frac{1}{2}\big(-\mu+i(2\theta-1)\sqrt{4-\mu^2}\big)|\xi|^\sigma t}\Big)(t,\cdot)\Big\|_{L^1}d\theta\|v_1\|_{H^{[a-\sigma]^+}_q}\\
&\lesssim t^{1-\frac{\min\{a,\sigma\}}{\sigma}}\|v_1\|_{H^{[a-\sigma]^+}_q} \lesssim \|v_1\|_{H^{[a-\sigma]^+}_q}.
\end{aligned}$$

Therefore, we derive

$$\big\||D|^a v(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{a}{\sigma}}\|v_1\|_{L^m\cap H^{[a-\sigma]^+}_q}.$$

Secondly, we are interested in the derivatives of solutions

$$\begin{aligned}
\big\||D|^a v_t(t,\cdot)\big\|_{L^q} &= \big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{v}_t(t,\xi)\big)(t,\cdot)\big\|_{L^q}\\
&= \Big\|\mathfrak{F}^{-1}\Big(|\xi|^a\Big(\cos\Big(\frac{1}{2}|\xi|^\sigma\sqrt{4-\mu^2}t\Big)-\frac{\mu}{\sqrt{4-\mu^2}}\sin\Big(\frac{1}{2}|\xi|^\sigma\sqrt{4-\mu^2}t\Big)\Big)e^{-\frac{1}{2}\mu|\xi|^\sigma t}\widehat{v_1}(\xi)\Big)\Big\|_{L^q}\\
&\lesssim \Big\|\mathfrak{F}^{-1}\Big(|\xi|^a\Big(\cos\Big(\frac{1}{2}|\xi|^\sigma\sqrt{4-\mu^2}t\Big)-\frac{\mu}{\sqrt{4-\mu^2}}\sin\Big(\frac{1}{2}|\xi|^\sigma\sqrt{4-\mu^2}t\Big)\Big)e^{-\frac{1}{2}\mu|\xi|^\sigma t}\Big)(t,\cdot)\Big\|_{L^r}\\
&\quad\times\big\|\mathfrak{F}^{-1}(\widehat{v_1}(\xi))\big\|_{L^m}\\
&\lesssim \Big\|\mathfrak{F}^{-1}\Big(|\xi|^a\cos\Big(\frac{1}{2}|\xi|^\sigma\sqrt{4-\mu^2}t\Big)e^{-\frac{1}{2}\mu|\xi|^\sigma t}\Big)(t,\cdot)\Big\|_{L^r}\|v_1\|_{L^m}\\
&\quad+\Big\|\mathfrak{F}^{-1}\Big(|\xi|^a\sin\Big(\frac{1}{2}|\xi|^\sigma\sqrt{4-\mu^2}t\Big)e^{-\frac{1}{2}\mu|\xi|^\sigma t}\Big)(t,\cdot)\Big\|_{L^r}\|v_1\|_{L^m}\\
&\lesssim t^{-\frac{n}{\sigma}(-\frac{1}{r})-\frac{a}{\sigma}}\|v_1\|_{L^m} \quad \text{ for } t\in[1,\infty),
\end{aligned}$$

and

$$\begin{aligned}
\big\||D|^a v_t(t,\cdot)\big\|_{L^q} &\lesssim \Big\|\mathfrak{F}^{-1}\Big(\Big(\cos\Big(\frac{1}{2}|\xi|^\sigma\sqrt{4-\mu^2}t\Big)-\frac{\mu}{\sqrt{4-\mu^2}}\sin\Big(\frac{1}{2}|\xi|^\sigma\sqrt{4-\mu^2}t\Big)\Big)e^{-\frac{1}{2}\mu|\xi|^\sigma t}\Big)(t,\cdot)\Big\|_{L^1}\\
&\qquad\times\big\|\mathfrak{F}^{-1}(|\xi|^a\widehat{v_1}(\xi))\big\|_{L^q}\\
&\lesssim \Big\|\mathfrak{F}^{-1}\Big(\cos\big(\frac{1}{2}|\xi|^\sigma\sqrt{4-\mu^2}t\big)e^{-\frac{1}{2}\mu|\xi|^\sigma t}\Big)(t,\cdot)\Big\|_{L^1}\|v_1\|_{\dot{H}^a_q}\\
&\qquad+\Big\|\mathfrak{F}^{-1}\Big(\sin\big(\frac{1}{2}|\xi|^\sigma\sqrt{4-\mu^2}t\big)e^{-\frac{1}{2}\mu|\xi|^\sigma t}\Big)(t,\cdot)\Big\|_{L^1}\|v_1\|_{\dot{H}^a_q}\\
&\lesssim \|v_1\|_{H^a_q} \quad \text{ for } t\in(0,1].
\end{aligned}$$

Hence, we conclude the estimates

$$\big\||D|^a v_t(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{a}{\sigma}}\|v_1\|_{L^m\cap H^a_q}.$$

Summarizing, the proof to Theorem 2.1.1 is completed. □

$L^m - L^q$ estimates

From the proof of Theorem 2.1.1 we have the following corollary.

Corollary 2.1.1. *Let $q \in [1, \infty]$ be given and $m \in [1, q]$. Then, the solutions to (2.3) satisfy the $L^m - L^q$ estimates*

$$\big\|\partial_t^j |D|^a v(t,\cdot)\big\|_{L^q} \lesssim t^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{a}{\sigma}-j}\|v_1\|_{L^m},$$

and the solutions to (2.2) satisfy the $L^m - L^q$ estimates

$$\big\|\partial_t^j |D|^a u(t,\cdot)\big\|_{L^q} \lesssim (t-s)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{a}{\sigma}-j}\|u_1\|_{L^m},$$

where $1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{m}$, $j = 0, 1$, for any $a \geqslant 0$ and for all dimensions $n \geqslant 1$.

2.2. A second Cauchy problem for linear structurally damped σ-evolution models

Let us turn to the following Cauchy problem:

$$u_{tt} + (-\Delta)^\sigma u + \mu(-\Delta)^{\frac{\sigma}{2}} u_t = 0, \quad u(0,x) = u_0(x), \quad u_t(0,x) = 0, \tag{2.5}$$

where $\mu > 0$ and $\sigma \geqslant 1$. Applying the partial Fourier transformation to (2.5) we obtain the Cauchy problem for $\hat{u}(t,\xi) := \mathfrak{F}(u(t,x))$ and $\widehat{u_0}(\xi) := \mathfrak{F}(u_0(x))$ as follows:

$$\hat{u}_{tt} + \mu|\xi|^\sigma \hat{u}_t + |\xi|^{2\sigma}\hat{u} = 0, \quad \hat{u}(0,\xi) = \widehat{u_0}(\xi), \quad \hat{u}_t(0,\xi) = 0. \tag{2.6}$$

$L^m \cap L^q - L^q$ and $L^q - L^q$ estimates

In this section, we want to prove the following result.

Theorem 2.2.1. *Let $q \in (1, \infty)$ be given and $m \in [1, q)$. Then, the solutions to (2.5) satisfy the $(L^m \cap L^q) - L^q$ estimates*

$$\big\|\partial_t^j |D|^a u(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{a}{\sigma}-j}\|u_0\|_{L^m \cap H_q^{a+j\sigma}},$$

and the $L^q - L^q$ estimates

$$\big\|\partial_t^j |D|^a u(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{-\frac{a}{\sigma}-j}\|u_0\|_{H_q^{a+j\sigma}},$$

where $1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{m}$, $j = 0, 1$, for any $a \geqslant 0$ and for all dimensions $n \geqslant 1$.

Proof. Special case $\mu = 2$. The characteristic double root is $\lambda_{1,2} = -|\xi|^\sigma$. The representation of solutions to (2.6) and their partial derivative in time are

$$\hat{u}(t,\xi) = (1 + t\,|\xi|^\sigma)e^{-|\xi|^\sigma t}\widehat{u_0}(\xi),$$

and

$$\hat{u}_t(t,\xi) = -t\,|\xi|^{2\sigma}e^{-|\xi|^\sigma t}\widehat{u_0}(\xi),$$

respectively. Using Young's convolution inequality from Proposition B.1.1 and Proposition 2.1.1 we obtain

$$\begin{aligned}
\big\||D|^a u(t,\cdot)\big\|_{L^q} &= \big\|\mathfrak{F}^{-1}\big(|\xi|^a(1+t\,|\xi|^\sigma)e^{-|\xi|^\sigma t}\widehat{u_0}(\xi)\big)(t,\cdot)\big\|_{L^q} \\
&\lesssim \big\|\mathfrak{F}^{-1}\big(|\xi|^a(1+t\,|\xi|^\sigma)e^{-|\xi|^\sigma t}\big)(t,\cdot)\big\|_{L^r}\big\|\mathfrak{F}^{-1}(\widehat{u_0}(\xi))\big\|_{L^m} \\
&\lesssim \Big(\big\|\mathfrak{F}^{-1}(|\xi|^a e^{-|\xi|^\sigma t})(t,\cdot)\big\|_{L^r} + t\,\big\|\mathfrak{F}^{-1}(|\xi|^{a+\sigma}e^{-|\xi|^\sigma t})(t,\cdot)\big\|_{L^r}\Big)\|u_0\|_{L^m} \\
&\lesssim t^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{a}{\sigma}}\|u_0\|_{L^m} \quad \text{ for } t \in [1,\infty),
\end{aligned}$$

and

$$\begin{aligned}
\big\||D|^a u(t,\cdot)\big\|_{L^q} &\lesssim \big\|\mathfrak{F}^{-1}\big((1+t\,|\xi|^\sigma)e^{-|\xi|^\sigma t}\big)(t,\cdot)\big\|_{L^1}\big\|\mathfrak{F}^{-1}(|\xi|^a\widehat{u_0}(\xi))\big\|_{L^q} \\
&\lesssim \Big(\big\|\mathfrak{F}^{-1}(e^{-|\xi|^\sigma t})(t,\cdot)\big\|_{L^1} + t\,\big\|\mathfrak{F}^{-1}(|\xi|^\sigma e^{-|\xi|^\sigma t})(t,\cdot)\big\|_{L^1}\Big)\|u_0\|_{\dot{H}_q^a} \lesssim \|u_0\|_{H_q^a},
\end{aligned}$$

for $t \in (0,1]$. These estimates imply

$$\big\||D|^a u(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{a}{\sigma}}\|u_0\|_{L^m\cap H^a_q}.$$

In the further considerations we are not only interested in estimates for solutions, but also in those for their partial derivatives. For this reason we can proceed as follows:

$$\begin{aligned}
\big\||D|^a u_t(t,\cdot)\big\|_{L^q} &= \big\|\mathfrak{F}^{-1}\big(|\xi|^a \hat{u}_t(t,\xi)\big)(t,\cdot)\big\|_{L^q} = t\,\big\|\mathfrak{F}^{-1}\big(|\xi|^{a+2\sigma}e^{-|\xi|^\sigma t}\widehat{u_0}(\xi)\big)(t,\cdot)\big\|_{L^q}\\
&\lesssim t\,\big\|\mathfrak{F}^{-1}\big(|\xi|^{a+2\sigma}e^{-|\xi|^\sigma t}\big)(t,\cdot)\big\|_{L^r}\big\|\mathfrak{F}^{-1}(\widehat{u_0}(\xi))\big\|_{L^m}\\
&\lesssim t^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{a}{\sigma}-1}\|u_0\|_{L^m} \quad \text{for } t \in [1,\infty),
\end{aligned}$$

and

$$\begin{aligned}
\big\||D|^a u_t(t,\cdot)\big\|_{L^q} &\lesssim t\,\big\|\mathfrak{F}^{-1}\big(|\xi|^{\sigma}e^{-|\xi|^\sigma t}\big)(t,\cdot)\big\|_{L^1}\big\|\mathfrak{F}^{-1}(|\xi|^{a+\sigma}\widehat{u_0}(\xi))\big\|_{L^q}\\
&\lesssim \|u_0\|_{\dot{H}^{a+\sigma}_q} \lesssim \|u_0\|_{H^{a+\sigma}_q} \quad \text{for } t \in (0,1].
\end{aligned}$$

Hence, we derive the estimate

$$\big\||D|^a u_t(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{a}{\sigma}-1}\|u_0\|_{L^m\cap H^{a+\sigma}_q}.$$

Case $\mu \in (2,\infty)$. We have the roots

$$\lambda_{1,2} = \frac{1}{2}|\xi|^\sigma\big(-\mu \pm \sqrt{\mu^2-4}\big).$$

The solutions to (2.6) are

$$\hat{u}(t,\xi) = \frac{\lambda_1 e^{\lambda_2 t}-\lambda_2 e^{\lambda_1 t}}{\lambda_1-\lambda_2}\widehat{u_0}(\xi) =: \widehat{K_0}(t,\xi)\widehat{u_0}(\xi).$$

Taking account of the two terms $\widehat{K_0}(t,\xi)$ and $\partial_t\widehat{K_0}(t,\xi)$ we re-write them as

$$\begin{aligned}
\widehat{K_0}(t,\xi) &= \frac{\lambda_1 e^{\lambda_2 t}-\lambda_2 e^{\lambda_1 t}}{\lambda_1-\lambda_2}\\
&= \frac{-\mu+\sqrt{\mu^2-4}}{2\sqrt{\mu^2-4}}e^{-\frac{1}{2}|\xi|^\sigma\left(\mu+\sqrt{\mu^2-4}\right)t} + \frac{\mu+\sqrt{\mu^2-4}}{2\sqrt{\mu^2-4}}e^{-\frac{1}{2}|\xi|^\sigma\left(\mu-\sqrt{\mu^2-4}\right)t},\\
\partial_t\widehat{K_0}(t,\xi) &= \frac{\lambda_1\lambda_2}{\lambda_1-\lambda_2}(e^{\lambda_2 t}-e^{\lambda_1 t}) = -\lambda_1\lambda_2\widehat{K_1}(t,\xi) = -|\xi|^{2\sigma}\widehat{K_1}(t,\xi),
\end{aligned}$$

where $\widehat{K_1}(t,\xi)$ is denoted as in the treatment of the first Cauchy problem. Therefore, we get

$$\partial_t\widehat{K_0}(t,\xi) = -t\,|\xi|^{2\sigma}\int_0^1 e^{\frac{1}{2}\left(-\mu+(2\theta-1)\sqrt{\mu^2-4}\right)|\xi|^\sigma t}d\theta.$$

The application of Proposition 2.1.1 and Young's convolution inequality from Proposition B.1.1 lead to the following estimates:

$$\begin{aligned}
\big\||D|^a u(t,\cdot)\big\|_{L^q} &= \big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_0}(t,\xi)\widehat{u_0}(\xi)\big)(t,\cdot)\big\|_{L^q} \lesssim \big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_0}(t,\xi)\big)(t,\cdot)\big\|_{L^r}\big\|\mathfrak{F}^{-1}(\widehat{u_0}(\xi))\big\|_{L^m}\\
&\lesssim \Big\|\mathfrak{F}^{-1}\Big(|\xi|^a e^{-\frac{1}{2}|\xi|^\sigma\left(\mu+\sqrt{\mu^2-4}\right)t}\Big)(t,\cdot)\Big\|_{L^r}\|u_0\|_{L^m}\\
&\quad+\Big\|\mathfrak{F}^{-1}\Big(|\xi|^a e^{-\frac{1}{2}|\xi|^\sigma\left(\mu-\sqrt{\mu^2-4}\right)t}\Big)(t,\cdot)\Big\|_{L^r}\|u_0\|_{L^m}\\
&\lesssim \Big(\big\|\mathfrak{F}^{-1}(|\xi|^a e^{-c_1|\xi|^\sigma t})(t,\cdot)\big\|_{L^r} + \big\|\mathfrak{F}^{-1}(|\xi|^a e^{-c_2|\xi|^\sigma t})(t,\cdot)\big\|_{L^r}\Big)\|u_0\|_{L^m}\\
&\lesssim t^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{a}{\sigma}}\|u_0\|_{L^m} \quad \text{for } t \in [1,\infty),
\end{aligned}$$

where

$$c_1 := \frac{1}{2}\big(\mu+\sqrt{\mu^2-4}\big) > 0 \quad \text{and} \quad c_2 := \frac{1}{2}\big(\mu-\sqrt{\mu^2-4}\big) > 0.$$

Moreover, we may also derive for $t \in (0,1]$

$$\begin{aligned}
\big\||D|^a u(t,\cdot)\big\|_{L^q} &\lesssim \big\|\mathfrak{F}^{-1}\big(\widehat{K_0}(t,\xi)\big)(t,\cdot)\big\|_{L^1}\big\|\mathfrak{F}^{-1}(|\xi|^a\widehat{u_0}(\xi))\big\|_{L^q}\\
&\lesssim \Big\|\mathfrak{F}^{-1}\Big(e^{-\frac{1}{2}|\xi|^\sigma\big(\mu-\sqrt{\mu^2-4}\big)t}\Big)(t,\cdot)\Big\|_{L^1}\|u_0\|_{\dot{H}^a_q}\\
&\quad+\Big\|\mathfrak{F}^{-1}\Big(e^{-\frac{1}{2}|\xi|^\sigma\big(\mu+\sqrt{\mu^2-4}\big)t}\Big)(t,\cdot)\Big\|_{L^1}\|u_0\|_{\dot{H}^a_q}\\
&\lesssim \Big(\big\|\mathfrak{F}^{-1}(e^{-c_1|\xi|^\sigma t})(t,\cdot)\big\|_{L^1}+\big\|\mathfrak{F}^{-1}(e^{-c_2|\xi|^\sigma t})(t,\cdot)\big\|_{L^1}\Big)\|u_0\|_{H^a_q} \lesssim \|u_0\|_{H^a_q}.
\end{aligned}$$

These estimates imply

$$\big\||D|^a u(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{a}{\sigma}}\|u_0\|_{L^m\cap H^a_q}.$$

As estimates for $|D|^a u$ we obtain

$$\begin{aligned}
\big\||D|^a u_t(t,\cdot)\big\|_{L^q} &= \big\|\mathfrak{F}^{-1}\big(|\xi|^a\hat{u}_t(t,\xi)\big)(t,\cdot)\big\|_{L^q} = \big\|\mathfrak{F}^{-1}\big(|\xi|^a\partial_t\widehat{K_0}(t,\xi)\widehat{u_0}(\xi)\big)(t,\cdot)\big\|_{L^q}\\
&\lesssim \big\|\mathfrak{F}^{-1}\big(|\xi|^a\partial_t\widehat{K_0}(t,\xi)\big)(t,\cdot)\big\|_{L^r}\big\|\mathfrak{F}^{-1}(\widehat{u_0}(\xi))\big\|_{L^m}\\
&\lesssim t\Big\|\mathfrak{F}^{-1}\Big(\int_0^1|\xi|^{a+2\sigma}e^{\frac{1}{2}\big(-\mu+(2\theta-1)\sqrt{\mu^2-4}\big)|\xi|^\sigma t}d\theta\Big)(t,\cdot)\Big\|_{L^r}\|u_0\|_{L^m}\\
&\lesssim t\int_0^1\Big\|\mathfrak{F}^{-1}\Big(|\xi|^{a+2\sigma}e^{\frac{1}{2}\big(-\mu+(2\theta-1)\sqrt{\mu^2-4}\big)|\xi|^\sigma t}\Big)(t,\cdot)\Big\|_{L^r}d\theta\,\|u_0\|_{L^m}\\
&\lesssim t\big\|\mathfrak{F}^{-1}(|\xi|^{a+2\sigma}e^{-c|\xi|^\sigma t})(t,\cdot)\big\|_{L^r}\|u_0\|_{L^m} \lesssim t^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{a}{\sigma}-1}\|u_0\|_{L^m} \qquad \text{for } t\in[1,\infty),
\end{aligned}$$

where

$$c := \frac{1}{2}\big(\mu-(2\theta-1)\sqrt{\mu^2-4}\big) \geqslant \frac{1}{2}\big(\mu-\sqrt{\mu^2-4}\big) \geqslant \frac{1}{\mu} > 0.$$

Moreover, we also get for $t \in (0,1]$

$$\begin{aligned}
\big\||D|^a u_t(t,\cdot)\big\|_{L^q} &\lesssim t\Big\|\mathfrak{F}^{-1}\Big(\int_0^1|\xi|^\sigma e^{\frac{1}{2}\big(-\mu+(2\theta-1)\sqrt{\mu^2-4}\big)|\xi|^\sigma t}d\theta\Big)(t,\cdot)\Big\|_{L^1}\big\|\mathfrak{F}^{-1}(|\xi|^{a+\sigma}\widehat{u_0}(\xi))\big\|_{L^q}\\
&\lesssim t\int_0^1\Big\|\mathfrak{F}^{-1}\Big(|\xi|^\sigma e^{\frac{1}{2}\big(-\mu+(2\theta-1)\sqrt{\mu^2-4}\big)|\xi|^\sigma t}\Big)(t,\cdot)\Big\|_{L^1}d\theta\,\|u_0\|_{\dot{H}^{a+\sigma}_q}\\
&\lesssim t\big\|\mathfrak{F}^{-1}(|\xi|^\sigma e^{-c|\xi|^\sigma t})(t,\cdot)\big\|_{L^1}\|u_0\|_{H^{a+\sigma}_q} \lesssim \|u_0\|_{H^{a+\sigma}_q}.
\end{aligned}$$

Hence, we may conclude

$$\big\||D|^a u_t(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{a}{\sigma}-1}\|u_0\|_{L^m\cap H^{a+\sigma}_q}.$$

Case $\mu \in (0,2)$. The characteristic roots are

$$\lambda_{1,2} = \frac{1}{2}|\xi|^\sigma\big(-\mu \pm i\sqrt{4-\mu^2}\big).$$

The solutions to (2.6) are

$$\hat{u}(t,\xi) = \Big(\cos\Big(\frac{1}{2}|\xi|^\sigma\sqrt{4-\mu^2}t\Big)+\frac{\mu}{\sqrt{4-\mu^2}}\sin\Big(\frac{1}{2}|\xi|^\sigma\sqrt{4-\mu^2}t\Big)\Big)e^{-\frac{1}{2}\mu|\xi|^\sigma t}\widehat{u_0}(\xi).$$

The partial derivative in time is

$$\hat{u}_t(t,\xi) = -\frac{2|\xi|^\sigma}{\sqrt{4-\mu^2}}\sin\Big(\frac{1}{2}|\xi|^\sigma\sqrt{4-\mu^2}t\Big)e^{-\frac{1}{2}\mu|\xi|^\sigma t}\widehat{u_0}(\xi).$$

Applying Proposition 2.1.2 and Young's convolution inequality from Proposition B.1.1 we conclude

$$\begin{aligned}
&\big\||D|^a u(t,\cdot)\big\|_{L^q}\\
&\quad= \Big\|\mathfrak{F}^{-1}\Big(|\xi|^a\Big(\cos\Big(\frac{1}{2}|\xi|^\sigma\sqrt{4-\mu^2}t\Big)+\frac{\mu}{\sqrt{4-\mu^2}}\sin\Big(\frac{1}{2}|\xi|^\sigma\sqrt{4-\mu^2}t\Big)\Big)e^{-\frac{1}{2}\mu|\xi|^\sigma t}\widehat{u_0}(\xi)\Big)(t,\cdot)\Big\|_{L^q}\\
&\quad\lesssim \Big\|\mathfrak{F}^{-1}\Big(|\xi|^a\Big(\cos\Big(\frac{1}{2}|\xi|^\sigma\sqrt{4-\mu^2}t\Big)+\frac{\mu}{\sqrt{4-\mu^2}}\sin\Big(\frac{1}{2}|\xi|^\sigma\sqrt{4-\mu^2}t\Big)\Big)e^{-\frac{1}{2}\mu|\xi|^\sigma t}\Big)\Big\|_{L^r}\\
&\qquad\times\big\|\mathfrak{F}^{-1}(\widehat{u_0}(\xi))\big\|_{L^m}\\
&\quad\lesssim \Big\|\mathfrak{F}^{-1}\Big(|\xi|^a\cos\Big(\frac{1}{2}|\xi|^\sigma\sqrt{4-\mu^2}t\Big)e^{-\frac{1}{2}\mu|\xi|^\sigma t}\Big)(t,\cdot)\Big\|_{L^r}\|u_0\|_{L^m}\\
&\qquad+\Big\|\mathfrak{F}^{-1}\Big(|\xi|^a\sin\Big(\frac{1}{2}|\xi|^\sigma\sqrt{4-\mu^2}t\Big)e^{-\frac{1}{2}\mu|\xi|^\sigma t}\Big)(t,\cdot)\Big\|_{L^r}\|u_0\|_{L^m}\\
&\quad\lesssim t^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{a}{\sigma}}\|u_0\|_{L^m}\qquad\text{for } t\in[1,\infty),
\end{aligned}$$

and

$$\begin{aligned}
\big\||D|^a u(t,\cdot)\big\|_{L^q} &\lesssim \Big\|\mathfrak{F}^{-1}\Big(\Big(\cos\Big(\frac{1}{2}|\xi|^\sigma\sqrt{4-\mu^2}t\Big)+\frac{\mu}{\sqrt{4-\mu^2}}\sin\Big(\frac{1}{2}|\xi|^\sigma\sqrt{4-\mu^2}t\Big)\Big)e^{-\frac{1}{2}\mu|\xi|^\sigma t}\Big)(t,\cdot)\Big\|_{L^1}\\
&\qquad\times\big\|\mathfrak{F}^{-1}(|\xi|^a\widehat{u_0}(\xi))\big\|_{L^q}\\
&\lesssim \Big\|\mathfrak{F}^{-1}\Big(\cos\Big(\frac{1}{2}|\xi|^\sigma\sqrt{4-\mu^2}t\Big)e^{-\frac{1}{2}\mu|\xi|^\sigma t}\Big)(t,\cdot)\Big\|_{L^1}\|u_0\|_{\dot{H}^a_q}\\
&\qquad+\Big\|\mathfrak{F}^{-1}\Big(\sin\Big(\frac{1}{2}|\xi|^\sigma\sqrt{4-\mu^2}t\Big)e^{-\frac{1}{2}\mu|\xi|^\sigma t}\Big)(t,\cdot)\Big\|_{L^1}\|u_0\|_{\dot{H}^a_q}\\
&\lesssim \|u_0\|_{H^a_q}\qquad\text{for } t\in(0,1].
\end{aligned}$$

Therefore, we may conclude

$$\big\||D|^a u(t,\cdot)\big\|_{L^q}\lesssim(1+t)^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{a}{\sigma}}\|u_0\|_{L^m\cap H^a_q}.$$

For the partial derivatives of u_t we obtain

$$\begin{aligned}
\big\||D|^a u_t(t,\cdot)\big\|_{L^q} &= \Big\|\mathfrak{F}^{-1}\Big(|\xi|^a\frac{-2|\xi|^\sigma}{\sqrt{4-\mu^2}}\sin\Big(\frac{1}{2}|\xi|^\sigma\sqrt{4-\mu^2}t\Big)e^{-\frac{1}{2}\mu|\xi|^\sigma t}\widehat{u_0}(\xi)\Big)(t,\cdot)\Big\|_{L^q}\\
&\lesssim \Big\|\mathfrak{F}^{-1}\Big(|\xi|^{a+\sigma}\sin\Big(\frac{1}{2}|\xi|^\sigma\sqrt{4-\mu^2}t\Big)e^{-\frac{1}{2}\mu|\xi|^\sigma t}\Big)(t,\cdot)\Big\|_{L^r}\big\|\mathfrak{F}^{-1}(\widehat{u_0}(\xi))\big\|_{L^m}\\
&\lesssim t^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{a}{\sigma}-1}\|u_0\|_{L^m}\qquad\text{for } t\in[1,\infty),
\end{aligned}$$

and

$$\begin{aligned}
\big\||D|^a u_t(t,\cdot)\big\|_{L^q} &\lesssim \Big\|\mathfrak{F}^{-1}\Big(\sin\Big(\frac{1}{2}|\xi|^\sigma\sqrt{4-\mu^2}t\Big)e^{-\frac{1}{2}\mu|\xi|^\sigma t}\Big)(t,\cdot)\Big\|_{L^1}\big\|\mathfrak{F}^{-1}(|\xi|^{a+\sigma}\widehat{u_0}(\xi))\big\|_{L^q}\\
&\lesssim \|u_0\|_{\dot{H}^{a+\sigma}_q}\lesssim\|u_0\|_{H^{a+\sigma}_q}\qquad\text{for } t\in(0,1].
\end{aligned}$$

Hence, we may conclude the estimates

$$\big\||D|^a u_t(t,\cdot)\big\|_{L^q}\lesssim(1+t)^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{a}{\sigma}-1}\|u_0\|_{L^m\cap H^{a+\sigma}_q}.$$

Summarizing, the proof of Theorem 2.2.1 is completed. □

$L^m - L^q$ estimates

From the proof of Theorem 2.2.1 we have the following corollary.

Corollary 2.2.1. *Let $q\in[1,\infty]$ be given and $m\in[1,q]$. Then, the solutions to (2.5) satisfy the L^m-L^q estimates*

$$\big\|\partial_t^j|D|^a u(t,\cdot)\big\|_{L^q}\lesssim t^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{a}{\sigma}-j}\|u_0\|_{L^m},$$

where $1+\frac{1}{q}=\frac{1}{r}+\frac{1}{m}$, $j=0,1$, for any $a\geqslant 0$ and for all dimensions $n\geqslant 1$.

2.3. A third Cauchy problem for linear structurally damped σ-evolution models

In this section, let us consider the Cauchy problem for structurally damped σ-evolution equations in the form

$$u_{tt} + (-\Delta)^\sigma u + \mu(-\Delta)^{\frac{\sigma}{2}} u_t = 0, \quad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \tag{2.7}$$

with $\sigma \geqslant 1$, $\mu > 0$. We may summarize the results from Sections 2.1 and 2.2 as follows:

$L^m \cap L^q - L^q$ and $L^q - L^q$ estimates

Theorem 2.3.1. *Let $q \in (1, \infty)$ and $m \in [1, q)$. Then, the solutions to (2.7) satisfy the $(L^m \cap L^q) - L^q$ estimates*

$$\big\|\partial_t^j |D|^a u(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{a}{\sigma}-j}\|u_0\|_{L^m \cap H_q^{a+j\sigma}} + (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{a}{\sigma}-j}\|u_1\|_{L^m \cap H_q^{[a+(j-1)\sigma]^+}}$$

and the $L^q - L^q$ estimates

$$\big\|\partial_t^j |D|^a u(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{-\frac{a}{\sigma}-j}\|u_0\|_{H_q^{a+j\sigma}} + (1+t)^{1-\frac{a}{\sigma}-j}\|u_1\|_{H_q^{[a+(j-1)\sigma]^+}},$$

where $1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{m}$, $j = 0, 1$, for any $a \geqslant 0$ and for all dimensions $n \geqslant 1$.

$L^m - L^q$ estimates

Corollary 2.3.1. *Let $q \in [1, \infty]$ be given and $m \in [1, q]$. Then, the solutions to (2.7) satisfy the $L^m - L^q$ estimates*

$$\big\|\partial_t^j |D|^a u(t,\cdot)\big\|_{L^q} \lesssim t^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{a}{\sigma}-j}\|u_0\|_{L^m} + t^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{a}{\sigma}-j}\|u_1\|_{L^m},$$

where $1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{m}$, $j = 0, 1$, for any $a \geqslant 0$ and for all dimensions $n \geqslant 1$.

2.4. Comparison with known results

First, if we are interested in studying the special case of $\sigma = 1$, in the paper [57] the authors obtained L^1 estimates for oscillating integrals to conclude the following $L^m - L^q$ estimates not necessarily on the conjugate line for Sobolev solutions:

$$\|u(t,\cdot)\|_{L^q} \lesssim t^{-n(1-\frac{1}{r})}\|u_0\|_{L^m} + t^{1-n(1-\frac{1}{r})}\|u_1\|_{L^m},$$

where $1 \leqslant m \leqslant q \leqslant \infty$ and $1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{m}$. The decay rates for solutions produced from the results in [57] are exactly the same as those in Corollary 2.3.1 with $\sigma = 1$ and $a = j = 0$.

In the paper [11], the authors investigated $L^1 \cap L^q - L^q$ estimates for solutions and some of their derivatives as well in the case $\delta = \frac{\sigma}{2}$ with additional L^1 regularity for the data as follows:

$$\big\||D|^a u(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{-\frac{n}{\sigma}(1-\frac{1}{q})-\frac{a}{\sigma}}\|u_0\|_{L^1 \cap H_q^a} + (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{q})-\frac{a}{\sigma}}\|u_1\|_{L^1 \cap H_q^{[a-\sigma]^+}},$$

$$\big\||D|^a u_t(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{-\frac{n}{\sigma}(1-\frac{1}{q})-\frac{a}{\sigma}-1}\|u_0\|_{L^1 \cap H_q^{a+\sigma}} + (1+t)^{-\frac{n}{\sigma}(1-\frac{1}{q})-\frac{a}{\sigma}}\|u_1\|_{L^1 \cap H_q^a},$$

for all non-negative constants a. We see that these results coincide with those in Theorem 2.3.1 if we choose the parameter $m = 1$.

Finally, we want to mention the paper [9] to emphasize some of recent estimates for solutions to structurally damped σ-evolution equations as follows:

$$\big\|\partial_t^j |D|^a u(t,\cdot)\big\|_{L^q} \lesssim t^{-\frac{1}{\sigma}\left(n(\frac{1}{q_0}-\frac{1}{q})+a\right)-j}\|u_0\|_{L^{q_0}} + t^{-\frac{1}{\sigma}\left(n(\frac{1}{q_1}-\frac{1}{q})+a\right)+1-j}\|u_1\|_{L^{q_1}},$$

for any q_0, $q_1 \geqslant 1$, $q \in [\max\{q_0, q_1\}, \infty]$, $a \geqslant 0$ and $j \in \mathbb{N}$. By choosing the values $q_0 = q_1 = m$ satisfying $1 \leqslant m \leqslant q \leqslant \infty$ and $1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{m}$, we may conclude the following estimates:

$$\big\|\partial_t^j |D|^a u(t,\cdot)\big\|_{L^q} \lesssim t^{-\frac{1}{\sigma}\left(n(1-\frac{1}{r})+a\right)-j}\|u_0\|_{L^m} + t^{-\frac{1}{\sigma}\left(n(1-\frac{1}{r})+a\right)+1-j}\|u_1\|_{L^m}.$$

These decay rates from the paper [9] are absolutely the same as those in Corollary 2.3.1.

3. Linear structurally damped σ-evolution models with $\delta \in [0, \frac{\sigma}{2})$

The main purpose of this chapter is to study linear structurally damped σ-evolution models of the form

$$u_{tt} + (-\Delta)^\sigma u + \mu(-\Delta)^\delta u_t = 0, \quad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) \tag{3.1}$$

with $\sigma \geqslant 1$, $\mu > 0$ and $\delta \in [0, \frac{\sigma}{2})$. This is a family of structurally damped σ-evolution models interpolating between models with friction or exterior damping $\delta = 0$ and those with a special damping $\delta = \frac{\sigma}{2}$. Our goal is to obtain $L^q - L^q$ estimates for solutions to (3.1) by assuming additional L^m regularity for the data with $m \in [1, q)$, where $q \in (1, \infty)$ is given.

To do this, let us explain our objectives and strategies as follows:

- By using the partial Fourier transformation we can reduce the partial differential equation to study an ordinary differential equation parameterized by ξ.

- Main difficulties that we will cope within the case $\delta \in (0, \frac{\sigma}{2})$ are to derive $L^1 - L^1$ estimates for oscillating integrals appearing in the representation of solutions. For this reason, we will apply the theory of modified Bessel funtions and Faà di Bruno's formula.

- For the sake of the asymptotic behavior of the characteristic roots, we may obtain L^∞ estimates for oscillating integrals. By an interpolation theorem, we also get L^r estimates with $r \in [1, \infty]$ for oscillating integrals.

- Applying Young's convolution inequality we may conclude $L^m - L^q$ estimates with $q \in [1, \infty]$ and $m \in [1, q]$, $L^m \cap L^q - L^q$ and $L^q - L^q$ estimates with $q \in (1, \infty)$ and $m \in [1, q)$ for solutions to (3.1) in the case $\delta \in (0, \frac{\sigma}{2})$.

- In the case $\delta = 0$ (friction or external damping): An analogous way as we did in the case $\delta \in (0, \frac{\sigma}{2})$ gives $L^m - L^q$ estimates with $q \in [1, \infty]$ and $m \in [1, q]$ for small frequencies. For large frequencies, we will apply the Mikhlin- Hörmander multiplier theorem to get $L^q - L^q$ estimates with $q \in (1, \infty)$. Then, we may conclude $L^m \cap L^q - L^q$ and $L^q - L^q$ estimates with $q \in (1, \infty)$ and $m \in [1, q)$ for solutions to (3.1).

3.1. A first Cauchy problem for linear structurally damped σ-evolution models

Let us consider the following family of parameter-dependent Cauchy problems:

$$u_{tt} + (-\Delta)^\sigma u + \mu(-\Delta)^\delta u_t = 0, \quad u(s,x) = 0, \quad u_t(s,x) = u_1(x), \tag{3.2}$$

where $s \geqslant 0$ is a fixed non-negative real parameter, $\sigma \geqslant 1$, $\mu > 0$ and $\delta \in (0, \frac{\sigma}{2})$. Thanks to the change of variables $t \to t - s$, we have here in mind the following Cauchy problem:

$$v_{tt} + (-\Delta)^\sigma v + \mu(-\Delta)^\delta v_t = 0, \quad v(0,x) = 0, \quad v_t(0,x) = v_1(x). \tag{3.3}$$

$L^m \cap L^q - L^q$ and $L^q - L^q$ estimates

In this section, we want to prove the following result.

Theorem 3.1.1. *Let $\delta \in \big(0, \frac{\sigma}{2}\big)$ in (3.3), $q \in (1, \infty)$ and $m \in [1, q)$. Then, the energy solutions to (3.3) satisfy the $(L^m \cap L^q) - L^q$ estimates*

$$\|v(t,\cdot)\|_{L^q} \lesssim \begin{cases} t^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\|v_1\|_{L^m\cap L^q} & \text{if } t \in (0,1], \\ (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})}\|v_1\|_{L^m\cap L^q} & \text{if } t \in [1,\infty), \end{cases}$$

$$\big\||D|^\sigma v(t,\cdot)\big\|_{L^q} \lesssim \begin{cases} t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\|v_1\|_{L^m\cap L^q} & \text{if } t \in (0,1], \\ (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\sigma}{2(\sigma-\delta)}}\|v_1\|_{L^m\cap L^q} & \text{if } t \in [1,\infty), \end{cases}$$

$$\|v_t(t,\cdot)\|_{L^q} \lesssim \begin{cases} t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\|v_1\|_{L^m\cap L^q} & \text{if } t \in (0,1], \\ (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}}\|v_1\|_{L^m\cap L^q} & \text{if } t \in [1,\infty), \end{cases}$$

and the $L^q - L^q$ estimates

$$\|v(t,\cdot)\|_{L^q} \lesssim \begin{cases} t^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\|v_1\|_{L^q} & \text{if } t \in (0,1], \\ (1+t)\|v_1\|_{L^q} & \text{if } t \in [1,\infty), \end{cases}$$

$$\big\||D|^\sigma v(t,\cdot)\big\|_{L^q} \lesssim \begin{cases} t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\|v_1\|_{L^q} & \text{if } t \in (0,1], \\ (1+t)^{1-\frac{\sigma}{2(\sigma-\delta)}}\|v_1\|_{L^q} & \text{if } t \in [1,\infty), \end{cases}$$

$$\|v_t(t,\cdot)\|_{L^q} \lesssim \begin{cases} t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\|v_1\|_{L^q} & \text{if } t \in (0,1], \\ (1+t)^{1-\frac{\delta}{\sigma-\delta}}\|v_1\|_{L^q} & \text{if } t \in [1,\infty), \end{cases}$$

where $1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{m}$ and for all dimensions $n \geqslant 1$.

As a consequence of Theorem 3.1.1 we conclude the following result.

Theorem 3.1.2. *Let $\delta \in \big(0, \frac{\sigma}{2}\big)$ in (3.2), $q \in (1, \infty)$ and $m \in [1, q)$. Then, the energy solutions to (3.2) satisfy the $(L^m \cap L^q) - L^q$ estimates*

$$\|u(t,\cdot)\|_{L^q} \lesssim \begin{cases} (t-s)^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\|u_1\|_{L^m\cap L^q} & \text{if } t \in (s,s+1], \\ (1+t-s)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})}\|u_1\|_{L^m\cap L^q} & \text{if } t \in [s+1,\infty), \end{cases}$$

$$\big\||D|^\sigma u(t,\cdot)\big\|_{L^q} \lesssim \begin{cases} (t-s)^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\|u_1\|_{L^m\cap L^q} & \text{if } t \in (s,s+1], \\ (1+t-s)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\sigma}{2(\sigma-\delta)}}\|u_1\|_{L^m\cap L^q} & \text{if } t \in [s+1,\infty), \end{cases}$$

$$\|u_t(t,\cdot)\|_{L^q} \lesssim \begin{cases} (t-s)^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\|u_1\|_{L^m\cap L^q} & \text{if } t \in (s,s+1], \\ (1+t-s)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}}\|u_1\|_{L^m\cap L^q} & \text{if } t \in [s+1,\infty), \end{cases}$$

and the $L^q - L^q$ estimates

$$\|u(t,\cdot)\|_{L^q} \lesssim \begin{cases} (t-s)^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\|u_1\|_{L^q} & \text{if } t \in (s,s+1], \\ (1+t-s)\|u_1\|_{L^q} & \text{if } t \in [s+1,\infty), \end{cases}$$

$$\big\||D|^\sigma u(t,\cdot)\big\|_{L^q} \lesssim \begin{cases} (t-s)^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\|u_1\|_{L^q} & \text{if } t \in (s,s+1], \\ (1+t-s)^{1-\frac{\sigma}{2(\sigma-\delta)}}\|u_1\|_{L^q} & \text{if } t \in [s+1,\infty), \end{cases}$$

$$\|u_t(t,\cdot)\|_{L^q} \lesssim \begin{cases} (t-s)^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)-\frac{a}{2\delta}}\|u_1\|_{L^q} & \text{if } t \in (s,s+1], \\ (1+t-s)^{1-\frac{\delta}{\sigma-\delta}}\|u_1\|_{L^q} & \text{if } t \in [s+1,\infty), \end{cases}$$

where $1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{m}$ and for all dimensions $n \geqslant 1$.

Using partial Fourier transformation to (3.3) we obtain the Cauchy problem for $\widehat{v}(t,\xi) := \mathfrak{F}(v(t,x))$ and $\widehat{v_1}(\xi) := \mathfrak{F}(v_1(x))$

$$\widehat{v}_{tt} + \mu|\xi|^{2\delta}\widehat{v}_t + |\xi|^{2\sigma}\widehat{v} = 0, \quad \widehat{v}(0,\xi) = 0, \quad \widehat{v}_t(0,\xi) = \widehat{v_1}(\xi). \tag{3.4}$$

We may choose without loss of generality $\mu = 1$ in (3.3). The characteristic roots are

$$\lambda_{1,2} = \lambda_{1,2}(\xi) = \frac{1}{2}\Big(-|\xi|^{2\delta} \pm \sqrt{|\xi|^{4\delta} - 4|\xi|^{2\sigma}}\Big).$$

The solutions to (3.4) are presented by the following formula (here we assume $\lambda_1 \neq \lambda_2$):

$$\widehat{v}(t,\xi) = \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}\widehat{v_1}(\xi) =: \widehat{K_1}(t,\xi)\widehat{v_1}(\xi).$$

Taking account of the cases of small and large frequencies separately we have

1. $\lambda_1 \sim -|\xi|^{2(\sigma-\delta)}, \quad \lambda_2 \sim -|\xi|^{2\delta}, \quad \lambda_1 - \lambda_2 \sim |\xi|^{2\delta} \quad$ for small $|\xi| \in \big(0, 4^{-\frac{1}{\sigma-2\delta}}\big)$,

2. $\lambda_{1,2} \sim -|\xi|^{2\delta} \pm i|\xi|^{\sigma}, \quad \lambda_1 - \lambda_2 \sim i|\xi|^{\sigma} \quad$ for large $|\xi| \in \big(4^{\frac{1}{\sigma-2\delta}}, \infty\big)$.

Let $\chi_k = \chi_k(|\xi|)$ with $k = 1,2,3$ be smooth cut-off functions having the following properties:

$$\chi_1(|\xi|) = \begin{cases} 1 & \text{if } |\xi| \leqslant 4^{-\frac{1}{\sigma-2\delta}}, \\ 0 & \text{if } |\xi| \geqslant 3^{-\frac{1}{\sigma-2\delta}}, \end{cases} \qquad \chi_3(|\xi|) = \begin{cases} 1 & \text{if } |\xi| \geqslant 4^{\frac{1}{\sigma-2\delta}}, \\ 0 & \text{if } |\xi| \leqslant 3^{\frac{1}{\sigma-2\delta}}, \end{cases}$$

and $\chi_2(|\xi|) = 1 - \chi_1(|\xi|) - \chi_3(|\xi|)$.

We note that $\chi_2(|\xi|) = 1$ if $3^{-\frac{1}{\sigma-2\delta}} \leqslant |\xi| \leqslant 3^{\frac{1}{\sigma-2\delta}}$ and $\chi_2(|\xi|) = 0$ if $|\xi| \leqslant 4^{-\frac{1}{\sigma-2\delta}}$ or $|\xi| \geqslant 4^{\frac{1}{\sigma-2\delta}}$. Let us now decompose the solutions to (3.3) into three parts localized separately to low, middle and high frequencies, that is,

$$v(t,x) = v_{\chi_1}(t,x) + v_{\chi_2}(t,x) + v_{\chi_3}(t,x),$$

where

$$v_{\chi_k}(t,x) = \mathfrak{F}^{-1}\big(\chi_k(|\xi|)\widehat{v}(t,\xi)\big) \quad \text{with } k = 1,2,3.$$

In order to estimate the L^q norms of solutions in (3.3) with additional L^m regularity of the data, we shall estimate the L^r norms of general terms of the form $\mathfrak{F}^{-1}\big(\widehat{K_j}(t,\xi)\chi_k(|\xi|)\big)(t,x)$ with $j = 0,1$ and $k = 1,2,3$, where

$$\widehat{K_0}(t,\xi) = \frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2} \quad \text{and} \quad \widehat{K_1}(t,\xi) = \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}.$$

The proof of Theorem 3.1.1 is divided into several steps as follows:

L^1 estimates for small frequencies

Proposition 3.1.1. *The estimates*

$$\big\|\mathfrak{F}^{-1}\big(\widehat{K_0}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^1} \lesssim 1 \quad \textit{and} \quad \big\|\mathfrak{F}^{-1}\big(\widehat{K_1}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^1} \lesssim t$$

hold for all $t > 0$.

Proof. Our approach is based on the paper [57]. In order to prove the above estimates for small $|\xi|$, we can apply the modified Bessel functions, carry out partial integrations and perform change of variables. According to a modification of Proposition 4 and Proposition 5 in [57], we have to study the three oscillating integrals

$$\mathfrak{F}^{-1}\big(e^{(\lambda_2-\lambda_1)t}\chi_1(|\xi|)\big)(t,x), \quad \mathfrak{F}^{-1}\big(|\xi|^{2\sigma-4\delta}e^{\lambda_2 t}\chi_1(|\xi|)\big)(t,x), \quad \mathfrak{F}^{-1}\big(e^{\lambda_1 t}\chi_1(|\xi|)\big)(t,x).$$

We obtain some auxiliary estimates for small frequencies as follows:

Lemma 3.1.1. *The estimates*

$$\begin{aligned}
&\big\|\mathfrak{F}^{-1}\big(e^{(\lambda_2-\lambda_1)t}\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^1} \lesssim 1, \\
&\big\|\mathfrak{F}^{-1}\big(|\xi|^{2\sigma-4\delta}e^{\lambda_2 t}\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^1} \lesssim 1, \\
&\big\|\mathfrak{F}^{-1}\big(e^{\lambda_1 t}\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^1} \lesssim 1
\end{aligned}$$

hold for all $t > 0$.

Taking account of $\widehat{K_0}(t,\xi)$ by using the asymptotic behavior of the characteristic roots we estimate as follows:

$$\big\|\mathfrak{F}^{-1}\big(\widehat{K_0}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^1} \lesssim \big\|\mathfrak{F}^{-1}\big(|\xi|^{2\sigma-4\delta}e^{\lambda_2 t}\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^1} + \big\|\mathfrak{F}^{-1}\big(e^{\lambda_1 t}\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^1} \lesssim 1.$$

In order to estimate $\widehat{K_1}(t,\xi)$, we may re-write

$$\widehat{K_1}(t,\xi) = t\int_0^1 e^{\theta\lambda_2 t+(1-\theta)\lambda_1 t}d\theta = te^{\lambda_1 t}\int_0^1 e^{\theta(\lambda_2-\lambda_1)t}d\theta.$$

Applying Young's convolution inequality from Proposition B.1.1 allows the following conclusion:

$$\begin{aligned}\big\|\mathfrak{F}^{-1}\big(\widehat{K_1}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^1} &\lesssim t\int_0^1 \Big\|\mathfrak{F}^{-1}\Big(e^{\lambda_1 t}e^{\theta(\lambda_2-\lambda_1)t}\chi_1(|\xi|)\Big)(t,\cdot)\Big\|_{L^1}d\theta \\ &\lesssim t\int_0^1 \Big\|\mathfrak{F}^{-1}\Big(e^{\lambda_1 t}\chi_1(|\xi|)\Big)(t,\cdot)\Big\|_{L^1}\Big\|\mathfrak{F}^{-1}\Big(e^{\theta(\lambda_2-\lambda_1)t}\chi_1(|\xi|)\Big)(t,\cdot)\Big\|_{L^1}d\theta \lesssim t.\end{aligned}$$

Therefore, we may conclude the desired estimates in Proposition 3.1.1. □

Following the approach of the proof of Proposition 3.1.1 we may prove the following statements.

Lemma 3.1.2. *The estimates*

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a e^{(\lambda_2-\lambda_1)t}\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^1} \lesssim \begin{cases} 1 & \text{if } t\in(0,1], \\ t^{-\frac{a}{2\delta}} & \text{if } t\in[1,\infty),\end{cases}$$

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^{a+2\sigma-4\delta} e^{\lambda_2 t}\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^1} \lesssim \begin{cases} 1 & \text{if } t\in(0,1], \\ t^{-\frac{a}{2\delta}} & \text{if } t\in[1,\infty),\end{cases}$$

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a e^{\lambda_1 t}\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^1} \lesssim \begin{cases} 1 & \text{if } t\in(0,1], \\ t^{-\frac{a}{2(\sigma-\delta)}} & \text{if } t\in[1,\infty),\end{cases}$$

hold for any non-negative number a.

Proposition 3.1.2. *The estimates*

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a \widehat{K_0}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^1} \lesssim \begin{cases} 1 & \text{if } t\in(0,1], \\ t^{-\frac{a}{2(\sigma-\delta)}} & \text{if } t\in[1,\infty),\end{cases}$$

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a \widehat{K_1}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^1} \lesssim \begin{cases} t & \text{if } t\in(0,1], \\ t^{1-\frac{a}{2(\sigma-\delta)}} & \text{if } t\in[1,\infty),\end{cases}$$

hold for any non-negative number a.

L^1 estimates for large frequencies

Proposition 3.1.3. *The following estimates hold in $\mathbb{R}^n$:*

$$\big\|\mathfrak{F}^{-1}\big(\widehat{K_0}(t,\xi)\chi_3(|\xi|)\big)(t,\cdot)\big\|_{L^1} \lesssim \begin{cases} t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)} & \text{if } t\in(0,1], \\ e^{-ct} & \text{if } t\in[1,\infty),\end{cases}$$

$$\big\|\mathfrak{F}^{-1}\big(\widehat{K_1}(t,\xi)\chi_3(|\xi|)\big)(t,\cdot)\big\|_{L^1} \lesssim \begin{cases} t^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)} & \text{if } t\in(0,1], \\ e^{-ct} & \text{if } t\in[1,\infty),\end{cases}$$

where c is a suitable positive constant.

In order to obtain the desired estimates for the norm of the Fourier multipliers localized to large frequencies, we re-write

$$\widehat{K_0}(t,\xi) = e^{-\frac{1}{2}|\xi|^{2\delta}t}\cos\Big(|\xi|^\sigma\sqrt{1-\frac{1}{4|\xi|^{2\sigma-4\delta}}}t\Big) + e^{-\frac{1}{2}|\xi|^{2\delta}t}|\xi|^{2\delta}\frac{\sin\big(|\xi|^\sigma\sqrt{1-\frac{1}{4|\xi|^{2\sigma-4\delta}}}t\big)}{2|\xi|^\sigma\sqrt{1-\frac{1}{4|\xi|^{2\sigma-4\delta}}}}$$

and

$$\widehat{K_1}(t,\xi) = e^{-\frac{1}{2}|\xi|^{2\delta}t}\frac{\sin\Big(|\xi|^\sigma\sqrt{1-\frac{1}{4|\xi|^{2\sigma-4\delta}}}t\Big)}{|\xi|^\sigma\sqrt{1-\frac{1}{4|\xi|^{2\sigma-4\delta}}}}.$$

Hence, it seems to be reasonable to divide the proof into two steps. In the first step we derive estimates for the oscillating integrals

$$\mathfrak{F}^{-1}\Big(e^{-c_1|\xi|^{2\delta}t}|\xi|^{2\beta}\frac{\sin(c_2|\xi|^\sigma t)}{|\xi|^\sigma}\chi_3(|\xi|)\Big)(t,x)$$

and

$$\mathfrak{F}^{-1}\Big(e^{-c_1|\xi|^{2\delta}t}\cos(c_2|\xi|^\sigma t)\chi_3(|\xi|)\Big)(t,x),$$

where $\beta \geqslant 0$, c_1 is a positive constant and $c_2 \neq 0$ is a real constant. Then, we estimate the two following oscillating integrals in the second step:

$$\mathfrak{F}^{-1}\Big(e^{-c_1|\xi|^{2\delta}t}|\xi|^{2\beta}\frac{\sin(c_2|\xi|^\sigma f(|\xi|)t)}{|\xi|^\sigma f(|\xi|)}\chi_3(|\xi|)\Big)(t,x)$$

and

$$\mathfrak{F}^{-1}\Big(e^{-c_1|\xi|^{2\delta}t}\cos(c_2|\xi|^\sigma f(|\xi|)t)\chi_3(|\xi|)\Big)(t,x),$$

where

$$f(|\xi|) = \sqrt{1-\frac{1}{4|\xi|^{2\sigma-4\delta}}}.$$

Lemma 3.1.3. *The following estimate holds in $\mathbb{R}^n$:*

$$\Big\|\mathfrak{F}^{-1}\Big(e^{-c_1|\xi|^{2\delta}t}|\xi|^{2\beta}\frac{\sin(c_2|\xi|^\sigma t)}{|\xi|^\sigma}\chi_3(|\xi|)\Big)(t,\cdot)\Big\|_{L^1} \lesssim \begin{cases} t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)+\frac{\sigma-2\beta}{2\delta}} & \text{if } t \in (0,1], \\ e^{-ct} & \text{if } t\in[1,\infty), \end{cases}$$

where $\beta \geqslant 0$ and c is a suitable positive constant. Moreover, c_1 is a positive constant and $c_2 \neq 0$ is a real constant.

Proof. We follow ideas from the proof of Proposition 4 in [57]. Many steps in our proof are similar to those from Proposition 4 devoting to small frequencies, nevertheless we will present the proof in detail to feel changes related to our interest for large frequencies. Let us divide the proof into two cases: $t \in [1,\infty)$ and $t \in (0,1]$. First, in order to treat the first case $t \in [1,\infty)$, we localize to small $|x| \leqslant 1$. Then, we obtain immediately the exponential decay. For this reason, we assume now $|x| \geqslant 1$. We introduce the function

$$I(t,x) := \mathfrak{F}^{-1}\Big(e^{-c_1|\xi|^{2\delta}t}|\xi|^{2\beta-\sigma}\sin(c_2|\xi|^\sigma t)\chi_3(|\xi|)\Big)(t,x).$$

Since the functions in the parenthesis are radially symmetric with respect to ξ, the inverse Fourier transform is radially symmetric with respect to x, too. Using the modified Bessel functions we get

$$I(t,x) = c\int_0^\infty e^{-c_1r^{2\delta}t}r^{2\beta-\sigma}\sin(c_2r^\sigma t)\chi_3(r)r^{n-1}\tilde{\mathcal{J}}_{\frac{n}{2}-1}(r|x|)dr. \tag{3.5}$$

Let us consider odd spatial dimensions $n = 2m+1, m \geqslant 1$. By introducing the vector field $Xf(r) := \frac{d}{dr}\big(\frac{1}{r}f(r)\big)$ as in [57], we carry out $m+1$ steps of partial integration to obtain

$$I(t,x) = -\frac{c}{|x|^n}\int_0^\infty \partial_r\Big(X^m\Big(e^{-c_1r^{2\delta}t}\sin(c_2r^\sigma t)\chi_3(r)r^{2\beta-\sigma+2m}\Big)\Big)\sin(r|x|)dr. \tag{3.6}$$

A standard calculation leads to

$$\begin{aligned} I(t,x) = &\sum_{j=0}^{m}\sum_{k=0}^{j+1}\frac{c_{jk}}{|x|^n}\int_0^\infty \partial_r^{j+1-k}e^{-c_1r^{2\delta}t}\,\partial_r^k\Big(\sin(c_2r^\sigma t)\chi_3(r)\Big)r^{2\beta-\sigma+j}\sin(r|x|)dr \\ &+\sum_{j=0}^{m}\sum_{k=0}^{j}\frac{c_{jk}}{|x|^n}\int_0^\infty \partial_r^{j-k}e^{-c_1r^{2\delta}t}\,\partial_r^{k+1}\Big(\sin(c_2r^\sigma t)\chi_3(r)\Big)r^{2\beta-\sigma+j}\sin(r|x|)dr \\ &+\sum_{j=1}^{m}\sum_{k=0}^{j}\frac{c_{jk}}{|x|^n}\int_0^\infty \partial_r^{j-k}e^{-c_1r^{2\delta}t}\,\partial_r^{k}\Big(\sin(c_2r^\sigma t)\chi_3(r)\Big)r^{2\beta-\sigma+j-1}\sin(r|x|)dr \end{aligned}$$

with some constants c_{jk}. For this reason, we only need to study the integrals

$$I_{j,k}(t,x) := \int_0^\infty \partial_r^{j+1-k} e^{-c_1 r^{2\delta}t}\, \partial_r^k \Big(\sin(c_2 r^\sigma t)\chi_3(r)\Big) r^{2\beta-\sigma+j} \sin(r|x|)dr. \tag{3.7}$$

Due to the large values of r, we can see that on the support of χ_3 and on the support of its derivatives it holds

$$\begin{aligned} \big|\partial_r^l e^{-c_1 r^{2\delta}t}\big| &\lesssim e^{-c_1 r^{2\delta}t} r^{l(2\delta-1)}t^l, \\ \Big|\partial_r^l \Big(\sin(c_2 r^\sigma t)\chi_3(r)\Big)\Big| &\lesssim r^{l(\sigma-1)}t^l \end{aligned}$$

for $l = 0, \cdots, m$. Hence, we imply for large r, $j = 0, \cdots, m$ and $k = 0, \cdots, j$ the estimates

$$\Big|\partial_r^{j+1-k} e^{-c_1 r^{2\delta}t}\, \partial_r^k \Big(\sin(c_2 r^\sigma t)\chi_3(r)\Big) r^{2\beta-\sigma+j}\Big| \lesssim e^{-c_1 r^{2\delta}t} t^{j+1} r^{2\delta(j+1)+k(\sigma-2\delta)+2\beta-1}$$

on the support of χ_3 and on the support of its derivatives. By splitting of the integral (3.7) into two parts, we get on the one hand

$$\Big|\int_0^{\frac{\pi}{2|x|}} \partial_r^{j+1-k} e^{-c_1 r^{2\delta}t}\, \partial_r^k \Big(\sin(c_2 r^\sigma t)\chi_3(r)\Big) r^{2\beta-\sigma+j} \sin(r|x|)dr\Big| \lesssim \frac{1}{|x|^{2\delta}} e^{-ct} \tag{3.8}$$

for some constant $c > 0$. On the other hand, we can carry out one more step of partial integration in the remaining integral as follows:

$$\begin{aligned} &\Big|\int_{\frac{\pi}{2|x|}}^\infty \partial_r^{j+1-k} e^{-c_1 r^{2\delta}t}\, \partial_r^k \Big(\sin(c_2 r^\sigma t)\chi_3(r)\Big) r^{2\beta-\sigma+j} \sin(r|x|)dr\Big| \\ &\quad\lesssim \frac{1}{|x|}\Big|\partial_r^{j+1-k} e^{-c_1 r^{2\delta}t}\, \partial_r^k \Big(\sin(c_2 r^\sigma t)\chi_3(r)\Big) r^{2\beta-\sigma+j} \cos(r|x|)\Big|_{r=\frac{\pi}{2|x|}}^\infty \\ &\qquad + \frac{1}{|x|}\int_{\frac{\pi}{2|x|}}^\infty \Big|\partial_r \Big(\partial_r^{j+1-k} e^{-c_1 r^{2\delta}t}\, \partial_r^k \Big(\sin(c_2 r^\sigma t)\chi_3(r)\Big) r^{2\beta-\sigma+j}\Big) \cos(r|x|)\Big| dr \lesssim \frac{1}{|x|} e^{-ct} \end{aligned} \tag{3.9}$$

for some constant $c > 0$. Here we also note that for all $j = 0, \cdots, m$ and $k = 0, \cdots, j$ we have the estimates

$$\Big|\partial_r \Big(\partial_r^{j+1-k} e^{-c_1 r^{2\delta}t} \partial_r^k \Big(\sin(c_2 r^\sigma t)\chi_3(r)\Big) r^{2\beta-\sigma+j}\Big)\Big| \lesssim e^{-c_1 r^{2\delta}t} t^{j+2} r^{2\delta(j+1)+k(\sigma-2\delta)+2\beta-2}.$$

Therefore, from (3.6) to (3.9) we have produced terms $|x|^{-(n+2\delta)}$ and $|x|^{-(n+1)}$ which guarantee the L^1 property in x. Summarizing, it implies for all $t \in [1,\infty)$ and $n = 2m+1$ the following estimates:

$$\Big\|\mathfrak{F}^{-1}\Big(e^{-c_1|\xi|^{2\delta}t}|\xi|^{2\beta-\sigma} \sin(c_2|\xi|^\sigma t)\chi_3(|\xi|)\Big)(t,\cdot)\Big\|_{L^1(|x|\geqslant 1)} \lesssim e^{-ct} \quad \text{for some } c > 0.$$

Let us consider even spatial dimensions $n = 2m, m \geqslant 1$ in the first case $t \in [1,\infty)$. Carrying out $m-1$ steps of partial integration we obtain

$$\begin{aligned} I(t,x) &= \frac{c}{|x|^{2m-2}} \int_0^\infty X^{m-1}\Big(e^{-c_1 r^{2\delta}t} \sin(c_2 r^\sigma t)\chi_3(r) r^{2\beta-\sigma+2m-1}\Big)\tilde{\mathcal{J}}_0(r|x|)dr \\ &= \sum_{j=0}^{m-1} \frac{c_j}{|x|^{2m-2}} \int_0^\infty \partial_r^j\Big(e^{-c_1 r^{2\delta}t} \sin(c_2 r^\sigma t)\chi_3(r) r^{2\beta-\sigma}\Big) r^{j+1}\tilde{\mathcal{J}}_0(r|x|)dr \\ &=: \sum_{j=0}^{m-1} c_j I_j(t,x). \end{aligned} \tag{3.10}$$

Applying the first rule of the modified Bessel functions for $\mu = 1$ and the fifth rule for $\mu = 0$ from Proposition B.3.2, after two more steps of partial integration we have

$$I_0(t,x) = -\frac{1}{|x|^n} \int_1^\infty \partial_r\Big(\partial_r\Big(e^{-c_1 r^{2\delta}t} \sin(c_2 r^\sigma t)\chi_3(r) r^{2\beta-\sigma}\Big) r\Big)\tilde{\mathcal{J}}_0(r|x|)dr. \tag{3.11}$$

Noting that for large r and all $j=0,\cdots,m$ we have the inequality

$$\Big|\partial_r^j\Big(e^{-c_1r^{2\delta}t}\sin(c_2r^\sigma t)\chi_3(r)r^{2\beta-\sigma}\Big)\Big| \lesssim e^{-c_1r^{2\delta}t}t^jr^{j(\sigma-1)+2\beta-\sigma}.$$

Hence, we get

$$\Big|\partial_r\Big(\partial_r\Big(e^{-c_1r^{2\delta}t}\sin(c_2r^\sigma t)\chi_3(r)r^{2\beta-\sigma}\Big)r\Big)\Big| \lesssim e^{-c_1r^{2\delta}t}t^2r^{\sigma+2\beta-1}$$

on the support of χ_3. Now using the estimate $|\tilde{\mathcal{J}}_0(s)|\leqslant Cs^{-\frac{1}{2}}$ for $s>1$ we conclude

$$\begin{aligned}&\Big|\int_1^\infty\partial_r\Big(\partial_r\Big(e^{-c_1r^{2\delta}t}\sin(c_2r^\sigma t)\chi_3(r)r^{2\beta-\sigma}\Big)r\Big)\tilde{\mathcal{J}}_0(r|x|)dr\Big|\\ &\quad\lesssim\int_1^\infty e^{-c_1r^{2\delta}t}t^2r^{\sigma+2\beta-1}\frac{1}{(r|x|)^{\frac{1}{2}}}dr=\frac{1}{|x|^{\frac{1}{2}}}t^2\int_1^\infty e^{-c_1r^{2\delta}t}r^{\sigma+2\beta-\frac{3}{2}}dr\lesssim\frac{1}{|x|^{\frac{1}{2}}}e^{-ct}\end{aligned}\tag{3.12}$$

for some constant $c>0$. Therefore, from (3.11) and (3.12) we have

$$\|I_0(t,\cdot)\|_{L^1(|x|\geqslant1)}\lesssim e^{-ct}\quad\text{for all } t\in[1,\infty),\ \text{and some constant } c>0.$$

Let $j\in[1,m-1]$ be an integer. By using again the first rule of the modified Bessel functions for $\mu=1$ and the fifth rule for $\mu=0$ from Proposition B.3.2 and carrying out partial integration, we may re-write $I_j(t,x)$ in (3.10) as follows:

$$\begin{aligned}I_j(t,x)=&-\frac{1}{|x|^{2m}}\int_0^\infty\partial_r\Big(\partial_r^{j+1}\Big(e^{-c_1r^{2\delta}t}\sin(c_2r^\sigma t)\chi_3(r)r^{2\beta-\sigma}\Big)r^{j+1}\Big)\tilde{\mathcal{J}}_0(r|x|)dr\\ &-\frac{j}{|x|^{2m}}\int_0^\infty\partial_r\Big(\partial_r^{j}\Big(e^{-c_1r^{2\delta}t}\sin(c_2r^\sigma t)\chi_3(r)r^{2\beta-\sigma}\Big)r^{j}\Big)\tilde{\mathcal{J}}_0(r|x|)dr.\end{aligned}$$

Applying an analogous treatment as we did for $I_0=I_0(t,x)$ implies

$$\|I_j(t,\cdot)\|_{L^1(|x|\geqslant1)}\lesssim e^{-ct}\quad\text{for all } t\in[1,\infty)\text{ and } j=1,\cdots,m-1,$$

where c is a suitable positive constant. Therefore, we have the following desired estimates for all $t\in[1,\infty)$ and $n=2m$:

$$\Big\|\mathfrak{F}^{-1}\Big(e^{-c_1|\xi|^{2\delta}t}|\xi|^{2\beta-\sigma}\sin(c_2|\xi|^\sigma t)\chi_3(|\xi|)\Big)(t,\cdot)\Big\|_{L^1(|x|\geqslant1)}\lesssim e^{-ct}\quad\text{for some constant } c>0.$$

Let us turn to the second case $t\in(0,1]$. By the change of variables $\xi=t^{-\frac{1}{2\delta}}\eta$ we get

$$\begin{aligned}&\mathfrak{F}^{-1}\Big(e^{-c_1|\xi|^{2\delta}t}|\xi|^{2\beta-\sigma}\sin(c_2|\xi|^\sigma t)\chi_3(|\xi|)\Big)(t,x)\\ &\quad=t^{-\frac{n+2\beta-\sigma}{2\delta}}\mathfrak{F}^{-1}\Big(e^{-c_1|\eta|^{2\delta}}|\eta|^{2\beta-\sigma}\sin(c_2|\eta|^\sigma t^{1-\frac{\sigma}{2\delta}})\chi_3(t^{-\frac{1}{2\delta}}|\eta|)\Big)(t,t^{-\frac{1}{2\delta}}x).\end{aligned}$$

Hence, we have

$$\begin{aligned}&\Big\|\mathfrak{F}^{-1}\Big(e^{-c_1|\xi|^{2\delta}t}|\xi|^{2\beta-\sigma}\sin(c_2|\xi|^\sigma t)\chi_3(|\xi|)\Big)(t,\cdot)\Big\|_{L^1}\\ &\quad=t^{\frac{\sigma-2\beta}{2\delta}}\Big\|\mathfrak{F}^{-1}\Big(e^{-c_1|\eta|^{2\delta}}|\eta|^{2\beta-\sigma}\sin(c_2|\eta|^\sigma t^{1-\frac{\sigma}{2\delta}})\chi_3(t^{-\frac{1}{2\delta}}|\eta|)\Big)(t,\cdot)\Big\|_{L^1}.\end{aligned}$$

For this reason, we only need to study the Fourier multipliers in the form

$$H(t,x):=\mathfrak{F}^{-1}\Big(e^{-c_1|\eta|^{2\delta}}|\eta|^{2\beta-\sigma}\sin(c_2|\eta|^\sigma t^{1-\frac{\sigma}{2\delta}})\chi_3(t^{-\frac{1}{2\delta}}|\eta|)\Big)(t,x).$$

First, we localize to small $|x|\leqslant1$. Then, we derive immediately

$$\|H(t,\cdot)\|_{L^1(|x|\leqslant1)}\lesssim t^{1-\frac{\sigma}{2\delta}}.$$

Therefore, we may conclude for $|x|\leqslant1$ the estimates

$$\Big\|\mathfrak{F}^{-1}\Big(e^{-c_1|\xi|^{2\delta}t}|\xi|^{2\beta-\sigma}\sin(c_2|\xi|^\sigma t)\chi_3(|\xi|)\Big)(t,\cdot)\Big\|_{L^1(|x|\leqslant1)}\lesssim t^{1-\frac{\beta}{\delta}}.$$

We assume now $|x| \geqslant 1$. Using the modified Bessel functions we shall estimate

$$H(t,x) = c\int_0^\infty e^{-c_1 r^{2\delta}} r^{2\beta-\sigma} \sin(c_2 r^\sigma t^{1-\frac{\sigma}{2\delta}})\chi_3(t^{-\frac{1}{2\delta}}r) r^{n-1}\tilde{\mathcal{J}}_{\frac{n}{2}-1}(r|x|)dr. \tag{3.13}$$

Let us consider odd spatial dimensions $n = 2m+1, m \geqslant 1$. Then, carrying out $m+1$ steps of partial integration we re-write (3.13) as follows:

$$\begin{aligned} H(t,x) &= \frac{c}{|x|^n}\int_0^\infty \partial_r\Big(X^m\Big(e^{-c_1 r^{2\delta}} \sin(c_2 r^\sigma t^{1-\frac{\sigma}{2\delta}})\chi_3(t^{-\frac{1}{2\delta}}r) r^{2\beta-\sigma+2m}\Big)\Big)\sin(r|x|)dr \\ &=: \sum_{1\leqslant j+k\leqslant m+1,\ j,k\geqslant 0} \frac{c_{jk}}{|x|^n}\int_0^\infty \partial_r^{j+1} e^{-c_1 r^{2\delta}} \partial_r^k\Big(\sin(c_2 r^\sigma t^{1-\frac{\sigma}{2\delta}})\chi_3(t^{-\frac{1}{2\delta}}r)\Big) \\ &\qquad\qquad \times r^{j+k+2\beta-\sigma}\sin(r|x|)dr. \end{aligned} \tag{3.14}$$

In order to estimate the function $H = H(t,x)$, we use the following auxiliary estimates:

$$\big|\partial_r^j e^{-c_1 r^{2\delta}}\big| \lesssim \begin{cases} e^{-c_1 r^{2\delta}} & \text{if } j=0,\\ e^{-c_1 r^{2\delta}}(r^{2\delta-j} + r^{j(2\delta-1)}) \lesssim e^{-c_1 r^{2\delta}} r^{2\delta-j}(1+r^{2\delta})^{j-1} & \text{if } j = 1,\cdots,m, \end{cases}$$

$$\big|\partial_r^j \sin(c_2 r^\sigma t^{1-\frac{\sigma}{2\delta}})\big| \lesssim \begin{cases} r^\sigma t^{1-\frac{\sigma}{2\delta}} & \text{if } j=0,\\ r^{\sigma-j}t^{1-\frac{\sigma}{2\delta}} + (r^{\sigma-1}t^{1-\frac{\sigma}{2\delta}})^j \lesssim r^{\sigma-j}t^{1-\frac{\sigma}{2\delta}}(1+r^\sigma t^{1-\frac{\sigma}{2\delta}})^{j-1} & \text{if } j=1,\cdots,m. \end{cases}$$

From the above estimates we may derive

$$\Big|\partial_r^k\Big(\sin(c_2 r^\sigma t^{1-\frac{\sigma}{2\delta}})\chi_3(t^{-\frac{1}{2\delta}}r)\Big)\Big| \lesssim \begin{cases} r^\sigma t^{1-\frac{\sigma}{2\delta}} & \text{if } k=0,\\ r^{\sigma-k}t^{1-\frac{\sigma}{2\delta}}(1+r^\sigma t^{1-\frac{\sigma}{2\delta}})^{k-1} & \text{if } k=1,\cdots,m. \end{cases}$$

Hence, we have

$$\begin{aligned} &\Big|\partial_r^{j+1} e^{-c_1 r^{2\delta}} \partial_r^k\Big(\sin(c_2 r^\sigma t^{1-\frac{\sigma}{2\delta}})\chi_3(t^{-\frac{1}{2\delta}}r)\Big) r^{j+k+2\beta-\sigma}\Big| \\ &\qquad \lesssim \begin{cases} e^{-c_1 r^{2\delta}} t^{1-\frac{\sigma}{2\delta}} r^{2\delta+2\beta-1}(1+r^{2\delta})^j & \text{if } k=0,\\ e^{-c_1 r^{2\delta}} t^{1-\frac{\sigma}{2\delta}} r^{2\delta+2\beta-1}(1+r^\sigma t^{1-\frac{\sigma}{2\delta}})^{k+j-1} & \text{if } k=1,\cdots,m, \end{cases} \end{aligned}$$

where we also note that $|\xi| \in [1,\infty)$, that is, $r \in [t^{\frac{1}{2\delta}},\infty)$ and $rt^{-\frac{1}{2\delta}} \geqslant 1$. Now, let us devote to $k=0$. By splitting the integral in (3.14) into two parts, on the one hand we obtain the following estimate for $t^{\frac{1}{2\delta}} < \frac{\pi}{2|x|}$:

$$\begin{aligned} &\Big|\int_{t^{\frac{1}{2\delta}}}^{\frac{\pi}{2|x|}} \partial_r^{j+1} e^{-c_1 r^{2\delta}} \sin(c_2 r^\sigma t^{1-\frac{\sigma}{2\delta}})\chi_3(t^{-\frac{1}{2\delta}}r) r^{j+2\beta-\sigma}\sin(r|x|)dr\Big| \\ &\quad \lesssim t^{1-\frac{\sigma}{2\delta}}\int_{t^{\frac{1}{2\delta}}}^{\frac{\pi}{2|x|}} r^{2\delta+2\beta-1}(1+r^{2\delta})^j dr \lesssim t^{1-\frac{\sigma}{2\delta}}\Big(\frac{1}{|x|^{2\delta+2\beta}} + \frac{1}{|x|^{2\delta(j+1)+2\beta}}\Big) \lesssim \frac{t^{1-\frac{\sigma}{2\delta}}}{|x|^{2\delta+2\beta}}. \end{aligned} \tag{3.15}$$

On the other hand, carrying out one more step of partial integration we derive

$$\begin{aligned} &\Big|\int_{\frac{\pi}{2|x|}}^\infty \partial_r^{j+1} e^{-c_1 r^{2\delta}} \sin(c_2 r^\sigma t^{1-\frac{\sigma}{2\delta}})\chi_3(t^{-\frac{1}{2\delta}}r) r^{j+2\beta-\sigma}\sin(r|x|)dr\Big| \\ &\quad \lesssim \frac{1}{|x|}\Big|\partial_r^{j+1} e^{-c_1 r^{2\delta}} \sin(c_2 r^\sigma t^{1-\frac{\sigma}{2\delta}})\chi_3(t^{-\frac{1}{2\delta}}r) r^{j+2\beta-\sigma}\cos(r|x|)\Big|_{r=\frac{\pi}{2|x|}}^\infty \\ &\qquad + \frac{1}{|x|}\int_{\frac{\pi}{2|x|}}^\infty \Big|\partial_r\Big(\partial_r^{j+1} e^{-c_1 r^{2\delta}} \sin(c_2 r^\sigma t^{1-\frac{\sigma}{2\delta}})\chi_3(t^{-\frac{1}{2\delta}}r) r^{j+2\beta-\sigma}\Big)\cos(r|x|)\Big|dr \\ &\quad \lesssim \frac{t^{1-\frac{\sigma}{2\delta}}}{|x|}\int_{\frac{\pi}{2|x|}}^\infty e^{-c_1 r^{2\delta}} r^{2\delta+2\beta-2}(1+r^{2\delta})^{j+1}dr \\ &\quad \lesssim \frac{t^{1-\frac{\sigma}{2\delta}}}{|x|}\Big(\int_{\frac{\pi}{2|x|}}^1 r^{2\delta+2\beta-2}dr + \int_1^\infty e^{-c_1 r^{2\delta}} r^{2\delta(j+2)+2\beta-2}dr\Big) \\ &\quad \lesssim \begin{cases} t^{1-\frac{\sigma}{2\delta}}\Big(\frac{1}{|x|} + \frac{1}{|x|^{2\delta+2\beta}}\Big) & \text{if } 2\delta+2\beta \neq 1\\ \frac{t^{1-\frac{\sigma}{2\delta}}}{|x|}\log(e+|x|) & \text{if } 2\delta+2\beta = 1 \end{cases} \lesssim t^{1-\frac{\sigma}{2\delta}}\Big(\frac{1}{|x|^{\frac{1}{2}}} + \frac{1}{|x|^{2\delta+2\beta}}\Big), \end{aligned} \tag{3.16}$$

where we also note that

$$\Big|\partial_r\Big(\partial_r^{j+1}e^{-c_1r^{2\delta}}\sin(c_2r^\sigma t^{1-\frac{\sigma}{2\delta}})\chi_3(t^{-\frac{1}{2\delta}}r)r^{j+2\beta-\sigma}\Big)\Big| \lesssim e^{-c_1r^{2\delta}}t^{1-\frac{\sigma}{2\delta}}r^{2\delta+2\beta-2}(1+r^{2\delta})^{j+1}.$$

For $k=1,\cdots,m$, after an analogous treatment as we did for $k=0$, we get

$$\Big|\int_{t^{\frac{1}{2\delta}}}^{\frac{\pi}{2|x|}}\partial_r^{j+1}e^{-c_1r^{2\delta}}\partial_r^k\Big(\sin(c_2r^\sigma t^{1-\frac{\sigma}{2\delta}})\chi_3(t^{-\frac{1}{2\delta}}r)\Big)r^{j+k+2\beta-\sigma}\sin(r|x|)dr\Big| \lesssim \frac{t^{(k+j)(1-\frac{\sigma}{2\delta})}}{|x|^{2\delta+2\beta}}, \tag{3.17}$$

and

$$\begin{aligned}\Big|\int_{\frac{\pi}{2|x|}}^{\infty}\partial_r^{j+1}e^{-c_1r^{2\delta}}\partial_r^k\Big(\sin(c_2r^\sigma t^{1-\frac{\sigma}{2\delta}})\chi_3(t^{-\frac{1}{2\delta}}r)\Big)r^{j+k+2\beta-\sigma}\sin(r|x|)dr\Big|\\ \lesssim t^{(k+j+1)(1-\frac{\sigma}{2\delta})}\Big(\frac{1}{|x|^{\frac{1}{2}}}+\frac{1}{|x|^{2\delta+2\beta}}\Big),\end{aligned} \tag{3.18}$$

where we can see that

$$\begin{aligned}\Big|\partial_r\Big(\partial_r^{j+1}e^{-c_1r^{2\delta}}\partial_r^k\Big(\sin(c_2r^\sigma t^{1-\frac{\sigma}{2\delta}})\chi_3(t^{-\frac{1}{2\delta}}r)\Big)r^{j+k+2\beta-\sigma}\Big)\Big|\\ \lesssim e^{-c_1r^{2\delta}}t^{1-\frac{\sigma}{2\delta}}r^{2\delta+2\beta-2}(1+r^\sigma t^{1-\frac{\sigma}{2\delta}})^{k+j}.\end{aligned}$$

Hence, from (3.14) to (3.18) we have produced terms $|x|^{-(n+2\delta+2\beta)}$ and $|x|^{-(n+\frac{1}{2})}$ which guarantee the L^1 property in x. For this reason, we arrive for all $t\in(0,1]$ and $n=2m+1$ at the following estimates:

$$\Big\|\mathfrak{F}^{-1}\Big(e^{-c_1|\xi|^{2\delta}t}|\xi|^{2\beta-\sigma}\sin(c_2|\xi|^\sigma t)\chi_3(|\xi|)\Big)(t,\cdot)\Big\|_{L^1(|x|\geqslant 1)} \lesssim t^{-(m+2)(\frac{\sigma}{2\delta}-1)+\frac{\sigma-2\beta}{2\delta}}.$$

Let us consider even spatial dimensions $n=2m, m\geqslant 1$. Carrying out $m-1$ steps of partial integration we re-write (3.13) as follows:

$$\begin{aligned}H(t,x)&=\frac{c}{|x|^{2m-2}}\int_0^\infty X^{m-1}\Big(e^{-c_1r^{2\delta}}\sin(c_2r^\sigma t^{1-\frac{\sigma}{2\delta}})\chi_3(t^{-\frac{1}{2\delta}}r)r^{2\beta-\sigma+2m-1}\Big)\tilde{\mathcal{J}}_0(r|x|)dr\\ &=\sum_{j=0}^{m-1}\frac{c_j}{|x|^{2m-2}}\int_0^\infty\partial_r^j\Big(e^{-c_1r^{2\delta}}\sin(c_2r^\sigma t^{1-\frac{\sigma}{2\delta}})\chi_3(t^{-\frac{1}{2\delta}}r)r^{2\beta-\sigma}\Big)r^{j+1}\tilde{\mathcal{J}}_0(r|x|)dr\\ &=:\sum_{j=0}^{m-1}c_jI_j(t,x).\end{aligned}$$

Using the first rule of the modified Bessel functions for $\mu=1$ and the fifth rule for $\mu=0$ from Proposition B.3.2 and performing two more steps of partial integration we get

$$|I_0(t,x)|=\frac{1}{|x|^{2m}}\int_0^\infty\Big|\partial_r\Big(\partial_r\Big(e^{-c_1r^{2\delta}}\sin(c_2r^\sigma t^{1-\frac{\sigma}{2\delta}})\chi_3(t^{-\frac{1}{2\delta}}r)r^{2\beta-\sigma}\Big)r\Big)\tilde{\mathcal{J}}_0(r|x|)\Big|dr.$$

We can see that for $j=1,\cdots,m$ we have

$$\Big|\partial_r^j\Big(e^{-c_1r^{2\delta}}\sin(c_2r^\sigma t^{1-\frac{\sigma}{2\delta}})\chi_3(t^{-\frac{1}{2\delta}}r)r^{2\beta-\sigma}\Big)\Big| \lesssim e^{-c_1r^{2\delta}}t^{1-\frac{\sigma}{2\delta}}(1+r^\sigma t^{1-\frac{\sigma}{2\delta}})^{j-1}(r^{2\delta+2\beta-j}+r^{2\beta-j})$$

on the support of $\chi_3(t^{-\frac{1}{2\delta}}r)$ and on the support of its derivatives. Therefore, we may conclude

$$\Big|\partial_r\Big(\partial_r\Big(e^{-c_1r^{2\delta}}\sin(c_2r^\sigma t^{1-\frac{\sigma}{2\delta}})\chi_3(t^{-\frac{1}{2\delta}}r)r^{2\beta-\sigma}\Big)r\Big)\Big| \lesssim e^{-c_1r^{2\delta}}t^{1-\frac{\sigma}{2\delta}}(1+r^\sigma t^{1-\frac{\sigma}{2\delta}})(r^{2\delta+2\beta-1}+r^{2\beta-1}).$$

Since $|\tilde{\mathcal{J}}_0(s)| \leqslant C$ for $s \in [0,1]$ we obtain for $t^{\frac{1}{2\delta}} < \frac{1}{|x|}$ the estimates

$$\begin{aligned}
&\int_{t^{\frac{1}{2\delta}}}^{\frac{1}{|x|}} \Big|\partial_r\Big(\partial_r\Big(e^{-c_1 r^{2\delta}} \sin(c_2 r^\sigma t^{1-\frac{\sigma}{2\delta}})\chi_3(t^{-\frac{1}{2\delta}}r) r^{2\beta-\sigma}\Big) r\Big)\tilde{\mathcal{J}}_0(r|x|)\Big| dr \\
&\quad\lesssim t^{1-\frac{\sigma}{2\delta}} \int_{t^{\frac{1}{2\delta}}}^{\frac{1}{|x|}} e^{-c_1 r^{2\delta}} (1 + r^\sigma t^{1-\frac{\sigma}{2\delta}}) r^{2\beta-1} dr \quad \Big(\text{since } r^{2\delta} \leqslant 1 \text{ for } r \leqslant \frac{1}{|x|}\Big) \\
&\quad\lesssim t^{1-\frac{\sigma}{2\delta}} \int_{t^{\frac{1}{2\delta}}}^{\frac{1}{|x|}} r^{2\beta-1} dr + t^{2(1-\frac{\sigma}{2\delta})} \int_{t^{\frac{1}{2\delta}}}^{\frac{1}{|x|}} r^{\sigma+2\beta-1} dr \\
&\quad\lesssim t^{2(1-\frac{\sigma}{2\delta})} \int_{t^{\frac{1}{2\delta}}}^{\frac{1}{|x|}} r^{2\beta-1+\sigma-2\delta} dr + \frac{t^{2(1-\frac{\sigma}{2\delta})}}{|x|^{\sigma+2\beta}} \quad \Big(\text{since } r^{2\delta-\sigma} \leqslant t^{1-\frac{\sigma}{2\delta}} \text{ for } r \geqslant t^{\frac{1}{2\delta}}\Big) \\
&\quad\lesssim t^{2(1-\frac{\sigma}{2\delta})} \Big(\frac{1}{|x|^{\sigma-2\delta+2\beta}} + \frac{1}{|x|^{\sigma+2\beta}}\Big) \lesssim t^{2(1-\frac{\sigma}{2\delta})} \frac{1}{|x|^{\sigma-2\delta+2\beta}}. \qquad (3.19)
\end{aligned}$$

Moreover, we use $|\tilde{\mathcal{J}}_0(s)| \leqslant C s^{-\frac{1}{2}}$ for $s > 1$ to conclude

$$\begin{aligned}
&\int_{\frac{1}{|x|}}^{\infty} \Big|\partial_r\Big(\partial_r\Big(e^{-c_1 r^{2\delta}} \sin(c_2 r^\sigma t^{1-\frac{\sigma}{2\delta}})\chi_3(t^{-\frac{1}{2\delta}}r) r^{2\beta-\sigma}\Big) r\Big)\tilde{\mathcal{J}}_0(r|x|)\Big| dr \\
&\lesssim \frac{t^{1-\frac{\sigma}{2\delta}}}{|x|^{\frac{1}{2}}} \int_{\frac{1}{|x|}}^{\infty} e^{-c_1 r^{2\delta}} (1 + r^\sigma t^{1-\frac{\sigma}{2\delta}})(r^{2\delta+2\beta-\frac{3}{2}} + r^{2\beta-\frac{3}{2}}) dr \\
&\lesssim \frac{t^{1-\frac{\sigma}{2\delta}}}{|x|^{\frac{1}{2}}} \int_{\frac{1}{|x|}}^{1} r^{2\beta-\frac{3}{2}} dr + \frac{t^{2(1-\frac{\sigma}{2\delta})}}{|x|^{\frac{1}{2}}} \\
&\lesssim \frac{t^{2(1-\frac{\sigma}{2\delta})}}{|x|^{\frac{1}{2}}} \int_{\frac{1}{|x|}}^{1} r^{2\beta-\frac{3}{2}+\sigma-2\delta} dr + \frac{t^{2(1-\frac{\sigma}{2\delta})}}{|x|^{\frac{1}{2}}} \quad \Big(\text{since } r^{2\delta-\sigma} \leqslant t^{1-\frac{\sigma}{2\delta}}\Big) \\
&\lesssim \begin{cases} t^{2(1-\frac{\sigma}{2\delta})}\Big(\frac{1}{|x|^{\frac{1}{2}}} + \frac{1}{|x|^{\sigma-2\delta+2\beta}}\Big) & \text{if } \sigma-2\delta+2\beta \neq \frac{1}{2} \\ \frac{t^{2(1-\frac{\sigma}{2\delta})}}{|x|^{\frac{1}{2}}} \log(e+|x|) & \text{if } \sigma-2\delta+2\beta = \frac{1}{2} \end{cases} \lesssim t^{2(1-\frac{\sigma}{2\delta})}\Big(\frac{1}{|x|^{\frac{1}{4}}} + \frac{1}{|x|^{\sigma-2\delta+2\beta}}\Big). \qquad (3.20)
\end{aligned}$$

Hence, from (3.19) and (3.20) we have produced terms $|x|^{-(n+\frac{1}{4})}$ and $|x|^{-(n+\sigma-2\delta+2\beta)}$ which guarantee the L^1 property in x. Summarizing, we arrive at the estimate

$$\|I_0(t,\cdot)\|_{L^1(|x|\geqslant 1)} \lesssim t^{2(1-\frac{\sigma}{2\delta})} \quad \text{for all} \quad t \in (0,1].$$

Let $j \in [1, m-1]$ be an integer. Then, repeating the above arguments we also derive for $t \in (0,1]$

$$\|I_j(t,\cdot)\|_{L^1(|x|\geqslant 1)} \lesssim t^{(j+2)(1-\frac{\sigma}{2\delta})}.$$

Therefore, we have proved that for all $t \in (0,1]$ and $n = 2m$ the following estimates hold:

$$\Big\|\mathfrak{F}^{-1}\Big(e^{-c_1|\xi|^{2\delta}t}|\xi|^{2\beta-\sigma}\sin(c_2|\xi|^\sigma t)\chi_3(|\xi|)\Big)(t,\cdot)\Big\|_{L^1(|x|\geqslant 1)} \lesssim t^{-(m+1)(\frac{\sigma}{2\delta}-1)+\frac{\sigma-2\beta}{2\delta}}.$$

Summarizing, the proof of Lemma 3.1.3 is completed. □

Remark 3.1.1. In the proof of Lemma 3.1.3 we explained our considerations for $n \geqslant 2$. Nevertheless, repeating the steps of the proof for odd spatial dimensions we conclude that the statements of this lemma also hold for $n = 1$. Here in the latter case we notice that we only carry out partial integration with no necessity to introduce the vector field $Xf(r)$ as we did in (3.6) and (3.14).

Following the proof of Lemma 3.1.3 we may conclude the following L^1 estimates, too.

Lemma 3.1.4. *The following estimates hold in $\mathbb{R}^n$ for any $n \geqslant 1$:*

$$\Big\|\mathfrak{F}^{-1}\Big(e^{-c_1|\xi|^{2\delta}t}|\xi|^{2\beta}\cos(c_2|\xi|^\sigma t)\chi_3(|\xi|)\Big)(t,\cdot)\Big\|_{L^1} \lesssim \begin{cases} t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)-\frac{\beta}{\delta}} & \text{if } t \in (0,1], \\ e^{-ct} & \text{if } t \in [1,\infty), \end{cases}$$

where $\beta \geqslant 0$ and c is a suitable positive constant. Moreover, c_1 is a positive constant and $c_2 \neq 0$ is a real constant.

Remark 3.1.2. Here we want to underline that all the statements in Lemma 3.1.4 remain valid for any $2\beta \geqslant -\sigma$.

Finally, we consider oscillating integrals with a more complicated oscillating integrand. We are going to prove the following result.

Lemma 3.1.5. *The following estimates hold in $\mathbb{R}^n$ for any $n \geqslant 1$:*

$$\Big\|\mathfrak{F}^{-1}\Big(e^{-c_1|\xi|^{2\delta}t}|\xi|^{2\beta}\frac{\sin\big(c_2|\xi|^\sigma f(|\xi|)t\big)}{|\xi|^\sigma f(|\xi|)}\chi_3(|\xi|)\Big)(t,\cdot)\Big\|_{L^1} \lesssim \begin{cases} t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)+\frac{\sigma-2\beta}{2\delta}} & \text{if } t\in(0,1], \\ e^{-ct} & \text{if } t\in[1,\infty), \end{cases}$$

where

$$f(|\xi|)=\sqrt{1-\frac{1}{4|\xi|^{2\sigma-4\delta}}},$$

$\beta \geqslant 0$ and c is a suitable positive constant. Moreover, c_1 is a positive constant and $c_2 \neq 0$ is a real constant.

Proof. The proof of this lemma is similar to that of Lemma 3.1.3. For this reason, we only present the steps which are different. Then, we shall repeat some of the arguments as we did in the proof of Lemma 3.1.3 to conclude the desired estimates.

First, let us consider $|x| \geqslant 1$ and $t \in [1,\infty)$. In order to obtain exponential decay estimates in both cases of odd spatial dimensions $n = 2m+1$ and even spatial dimensions $n = 2m$ with $m \geqslant 1$, we shall prove the following estimates on the support of $\chi_3(r)$ and on the support of its derivatives:

$$\Big|\partial_r^k\Big(\frac{\sin\big(c_2r^\sigma f(r)t\big)}{f(r)}\Big)\Big| \lesssim r^{k(\sigma-1)}t^k \quad \text{for } k=1,\cdots,m,$$

where

$$f(r)=\sqrt{1-\frac{1}{4r^{2\sigma-4\delta}}}.$$

Indeed, we shall apply Faà di Bruno's formula as a main tool. We divide the proof of the above estimates into several sub-steps as follows:

Step 1: Applying Proposition B.4.1 with $h(s)=\sqrt{s}$ and $g(r)=1-\frac{1}{4}r^{-2(\sigma-2\delta)}$ we get

$$\begin{aligned}\big|\partial_r^k f(r)\big| &\lesssim \Big|\sum_{1\cdot m_1+\cdots+k\cdot m_k=k,\, m_i\geqslant 0} g(r)^{\frac{1}{2}-(m_1+\cdots+m_k)}\prod_{j=1}^k\Big(-\frac{1}{4}r^{-2(\sigma-2\delta)-j}\Big)^{m_j}\Big| \\ &\lesssim \sum_{1\cdot m_1+\cdots+k\cdot m_k=k,\, m_i\geqslant 0} r^{-2(\sigma-2\delta)(m_1+\cdots+m_k)-k} \lesssim r^{-k}\ \Big(\text{since } \frac{3}{4}\leqslant g(r)\leqslant 1 \text{ for } r\geqslant 1\Big).\end{aligned}$$

An analogous treatment gives

$$\Big|\partial_r^k\Big(\frac{1}{f(r)}\Big)\Big| \lesssim r^{-k} \quad \text{for } k=1,\cdots,m. \tag{3.21}$$

Step 2: Applying Proposition B.4.1 with $h(s)=\sin(c_2\, s)$ and $g(r)=r^\sigma f(r)t$ we obtain

$$\begin{aligned}\big|\partial_r^k\sin\big(c_2r^\sigma f(r)t\big)\big| &\lesssim \Big|\sum_{1\cdot m_1+\cdots+k\cdot m_k=k,\, m_i\geqslant 0}\sin\big(c_2r^\sigma f(r)t\big)^{(m_1+\cdots+m_k)}\prod_{j=1}^k\Big(\partial_r^j\big(r^\sigma f(r)t\big)\Big)^{m_j}\Big| \\ &\lesssim \Big|\sum_{1\cdot m_1+\cdots+k\cdot m_k=k,\, m_i\geqslant 0}\prod_{j=1}^k\Big(t\sum_{l=0}^j C_j^l r^{\sigma-j+l}f^{(l)}(r)\Big)^{m_j}\Big| \\ &\lesssim \sum_{1\cdot m_1+\cdots+k\cdot m_k=k,\, m_i\geqslant 0}\prod_{j=1}^k(t\,r^{\sigma-j})^{m_j} \\ &\lesssim \sum_{1\cdot m_1+\cdots+k\cdot m_k=k,\, m_i\geqslant 0} r^{-k}(t\,r^\sigma)^{m_1+\cdots+m_k} \lesssim t^kr^{k(\sigma-1)}.\end{aligned} \tag{3.22}$$

Hence, from (3.21) and (3.22) using the product rule for higher derivatives we may conclude

$$\Big|\partial_r^k\Big(\frac{\sin\big(c_2r^\sigma f(r)t\big)}{f(r)}\Big)\Big|\lesssim t^kr^{k(\sigma-1)}\quad\text{for } k=1,\cdots,m.$$

Next, let us turn to the case $|x|\geqslant 1$ and $t\in(0,1]$. In order to prove the desired estimates by using similar ideas as in the proof of Lemma 3.1.3, we need to assert the following auxiliary estimates on the support of $\chi_3(t^{-\frac{1}{2\delta}}r)$ and on the support of its partial derivatives:

$$\Big|\partial_r^k\Big(\frac{\sin\big(c_2r^\sigma f(r)t^{1-\frac{\sigma}{2\delta}}\big)}{f(r)}\Big)\Big|\lesssim t^{1-\frac{\sigma}{2\delta}}r^{\sigma-k}(1+r^\sigma t^{1-\frac{\sigma}{2\delta}})^{k-1}\quad\text{for } k=1,\cdots,m,$$

where

$$f(r)=\sqrt{1-\frac{1}{4}t^{\frac{\sigma-2\delta}{\delta}}r^{-2(\sigma-2\delta)}}.$$

Indeed, we shall divide our proof into several sub-steps as follows:

Step 1: Applying Proposition B.4.1 with $h(s)=\sqrt{s}$ and $g(r)=1-\frac{1}{4}t^{\frac{\sigma-2\delta}{\delta}}r^{-2(\sigma-2\delta)}$ we obtain

$$\begin{aligned}
\big|\partial_r^kf(r)\big|&\lesssim\Big|\sum_{1\cdot m_1+\cdots+k\cdot m_k=k,\,m_i\geqslant0}g(r)^{\frac12-(m_1+\cdots+m_k)}\prod_{j=1}^k\Big(-\frac14t^{\frac{\sigma-2\delta}{\delta}}r^{-2(\sigma-2\delta)-j}\Big)^{m_j}\Big|\\
&\lesssim\sum_{1\cdot m_1+\cdots+k\cdot m_k=k,\,m_i\geqslant0}\big(t^{\frac{\sigma-2\delta}{\delta}}r^{-2(\sigma-2\delta)}\big)^{m_1+\cdots+m_k}r^{-k}\quad\Big(\text{since }\frac34\leqslant g(r)\leqslant1\text{ for }r\geqslant t^{\frac{1}{2\delta}}\Big)\\
&\lesssim r^{-k}\sum_{1\cdot m_1+\cdots+k\cdot m_k=k,\,m_i\geqslant0}(t^{-\frac{1}{2\delta}}r)^{-2(\sigma-2\delta)(m_1+\cdots+m_k)}\\
&\lesssim r^{-k}\quad\Big(\text{since }t^{-\frac{1}{2\delta}}r\geqslant1\text{ for }r\geqslant t^{\frac{1}{2\delta}}\Big).
\end{aligned}$$

In an analogous way we may derive the estimates

$$\Big|\partial_r^k\Big(\frac{1}{f(r)}\Big)\Big|\lesssim r^{-k}\quad\text{for } k=1,\cdots,m. \tag{3.23}$$

Step 2: Applying Proposition B.4.1 with $h(s)=\sin(c_2\,s)$ and $g(r)=r^\sigma f(r)t^{1-\frac{\sigma}{2\delta}}$ we derive

$$\begin{aligned}
&\big|\partial_r^k\sin\big(c_2r^\sigma f(r)t^{1-\frac{\sigma}{2\delta}}\big)\big|\\
&\lesssim\Big|\sum_{1\cdot m_1+\cdots+k\cdot m_k=k,\,m_i\geqslant0}\sin\big(c_2r^\sigma f(r)t^{1-\frac{\sigma}{2\delta}}\big)^{(m_1+\cdots+m_k)}\prod_{j=1}^k\Big(\partial_r^j\big(r^\sigma f(r)t^{1-\frac{\sigma}{2\delta}}\big)\Big)^{m_j}\Big|\\
&\lesssim\Big|\sum_{1\cdot m_1+\cdots+k\cdot m_k=k,\,m_i\geqslant0}\prod_{j=1}^k\Big(t^{1-\frac{\sigma}{2\delta}}\sum_{l=0}^jC_j^lr^{\sigma-j+l}f^{(l)}(r)\Big)^{m_j}\Big|\\
&\lesssim\sum_{1\cdot m_1+\cdots+k\cdot m_k=k,\,m_i\geqslant0}\prod_{j=1}^k(t^{1-\frac{\sigma}{2\delta}}r^{\sigma-j})^{m_j}\lesssim t^{1-\frac{\sigma}{2\delta}}r^{\sigma-k}\sum_{1\cdot m_1+\cdots+k\cdot m_k=k,\,m_i\geqslant0}(t^{1-\frac{\sigma}{2\delta}}r^\sigma)^{m_1+\cdots+m_k-1}\\
&\lesssim t^{1-\frac{\sigma}{2\delta}}r^{\sigma-k}\big(1+(t^{1-\frac{\sigma}{2\delta}}r^\sigma)^{k-1}\big)\lesssim t^{1-\frac{\sigma}{2\delta}}r^{\sigma-k}\big(1+t^{1-\frac{\sigma}{2\delta}}r^\sigma\big)^{k-1}.
\end{aligned}\tag{3.24}$$

Hence, from (3.23) and (3.24) using the product rule for higher derivatives we may conclude

$$\Big|\partial_r^k\Big(\frac{\sin\big(c_2r^\sigma f(r)t^{1-\frac{\sigma}{2\delta}}\big)}{f(r)}\Big)\Big|\lesssim t^{1-\frac{\sigma}{2\delta}}r^{\sigma-k}\big(1+t^{1-\frac{\sigma}{2\delta}}r^\sigma\big)^{k-1}\quad\text{for } k=1,\cdots,m.$$

Summarizing, the proof of Lemma 3.1.5 is completed. □

Following the steps of the proof of Lemma 3.1.5 we may prove the following statement, too.

Lemma 3.1.6. *The following estimates hold in $\mathbb{R}^n$ for any $n\geqslant1$:*

$$\Big\|\mathfrak{F}^{-1}\Big(e^{-c_1|\xi|^{2\delta}t}|\xi|^{2\beta}\cos\big(c_2|\xi|^\sigma f(|\xi|)t\big)\chi_3(|\xi|)\Big)(t,\cdot)\Big\|_{L^1}\lesssim\begin{cases}t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)-\frac{\beta}{\delta}} & \text{if } t\in(0,1],\\ e^{-ct} & \text{if } t\in[1,\infty),\end{cases}$$

where

$$f(|\xi|)=\sqrt{1-\frac{1}{4|\xi|^{2\sigma-4\delta}}},$$

$\beta\geqslant 0$ *and* c *is a suitable positive constant. Moreover,* c_1 *is a positive constant and* $c_2\neq 0$ *is a real constant.*

Remark 3.1.3. Here we want to underline that all the statements in Lemma 3.1.6 remain valid for any $2\beta\geqslant -\sigma$.

Proof of Proposition 3.1.3. In order to prove the first statement, we replace $\beta=\delta$ and $\beta=0$, respectively, in Lemmas 3.1.5 and 3.1.6. Then, plugging $\beta=0$ in Lemma 3.1.5 we may conclude the second statement. Therefore, this completes our proof. □

Following the approach of the proof of Proposition 3.1.3 we may prove the following statements.

Proposition 3.1.4. *The following estimates hold in* $\mathbb{R}^n$ *for any* $n\geqslant 1$*:*

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_0}(t,\xi)\chi_3(|\xi|)\big)(t,\cdot)\big\|_{L^1}\lesssim\begin{cases}t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)-\frac{a}{2\delta}} & \text{if } t\in(0,1],\\ e^{-ct} & \text{if } t\in[1,\infty),\end{cases}$$

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_1}(t,\xi)\chi_3(|\xi|)\big)(t,\cdot)\big\|_{L^1}\lesssim\begin{cases}t^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)-\frac{a}{2\delta}} & \text{if } t\in(0,1],\\ e^{-ct} & \text{if } t\in[1,\infty),\end{cases}$$

where c *is a suitable positive constant and for any non-negative number* a*.*

Proof. In order to obtain estimates for the following norms of Fourier multipliers localized to large frequencies:

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_0}(t,\xi)\chi_3(|\xi|)\big)(t,\cdot)\big\|_{L^1} \text{ and } \big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_1}(t,\xi)\chi_3(|\xi|)\big)(t,\cdot)\big\|_{L^1},$$

we can re-write

$$|\xi|^a\widehat{K_0}(t,\xi)=e^{-\frac12|\xi|^{2\delta}t}|\xi|^a\cos\Big(|\xi|^\sigma\sqrt{1-\frac{1}{4|\xi|^{2\sigma-4\delta}}}\,t\Big)+e^{-\frac12|\xi|^{2\delta}t}|\xi|^{a+2\delta}\frac{\sin\Big(|\xi|^\sigma\sqrt{1-\frac{1}{4|\xi|^{2\sigma-4\delta}}}\,t\Big)}{2|\xi|^\sigma\sqrt{1-\frac{1}{4|\xi|^{2\sigma-4\delta}}}},$$

and

$$|\xi|^a\widehat{K_1}(t,\xi)=e^{-\frac12|\xi|^{2\delta}t}|\xi|^a\frac{\sin\Big(|\xi|^\sigma\sqrt{1-\frac{1}{4|\xi|^{2\sigma-4\delta}}}\,t\Big)}{|\xi|^\sigma\sqrt{1-\frac{1}{4|\xi|^{2\sigma-4\delta}}}}.$$

By choosing the values $2\beta=a+2\delta$ and $2\beta=a$ in Lemma 3.1.5 and 3.1.6, respectively, we may prove the first statement. Moreover, replacing $2\beta=a$ in Lemma 3.1.5 we may conclude the second statement. This completes our proof. □

Remark 3.1.4. Here we want to underline that the first statement in Proposition 3.1.4 remains valid for any $a\geqslant -2\delta$.

Estimates for middle frequencies

Now let us turn to consider some estimates for Fourier multipliers localized to middle frequencies, where $3^{-\frac{1}{\sigma-2\delta}}<|\xi|<3^{\frac{1}{\sigma-2\delta}}$. Our goal is to derive exponential decay estimates for suitable localized Fourier multipliers.

Proposition 3.1.5. *The following estimates hold in* $\mathbb{R}^n$ *for any* $n\geqslant 1$*:*

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_0}(t,\xi)\chi_2(|\xi|)\big)(t,\cdot)\big\|_{L^1}\lesssim e^{-ct},$$
$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_1}(t,\xi)\chi_2(|\xi|)\big)(t,\cdot)\big\|_{L^1}\lesssim e^{-ct},$$
$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_0}(t,\xi)\chi_2(|\xi|)\big)(t,\cdot)\big\|_{L^\infty}\lesssim e^{-ct},$$
$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_1}(t,\xi)\chi_2(|\xi|)\big)(t,\cdot)\big\|_{L^\infty}\lesssim e^{-ct},$$

where c *is a suitable positive constant and for any non-negative number* a*.*

Proof. At first, with $3^{-\frac{1}{\sigma-2\delta}} < |\xi| < 3^{\frac{1}{\sigma-2\delta}}$ we use Cauchy's integral formula to re-write the above Fourier multipliers in the following form:

$$\widehat{K_0}(t,\xi)\chi_2(|\xi|) = \frac{1}{2\pi i}\Big(\int_\Gamma \frac{(z+|\xi|^{2\delta})e^{zt}}{z^2+|\xi|^{2\delta}z+|\xi|^{2\sigma}}dz\Big)\chi_2(|\xi|), \tag{3.25}$$

$$\widehat{K_1}(t,\xi)\chi_2(|\xi|) = \frac{1}{2\pi i}\Big(\int_\Gamma \frac{e^{zt}}{z^2+|\xi|^{2\delta}z+|\xi|^{2\sigma}}dz\Big)\chi_2(|\xi|), \tag{3.26}$$

where Γ is a closed curve containing the two characteristic roots $\lambda_{1,2}$. We can see that $\lambda_1 = \lambda_2$ when $|\xi| = 2^{-\frac{1}{\sigma-2\delta}}$ and $\{\xi \in \mathbb{R}^n \,:\, |\xi| = 2^{-\frac{1}{\sigma-2\delta}}\}$ is not a singular set because we may give equivalent formulas as follows:

$$\widehat{K_0}(t,\xi) = e^{\lambda_2 t} - \lambda_2 e^{\lambda_2 t}\int_0^t e^{(\lambda_1-\lambda_2)s}ds \quad \text{and} \quad \widehat{K_1}(t,\xi) = e^{\lambda_2 t}\int_0^t e^{(\lambda_1-\lambda_2)s}ds.$$

Therefore, it is reasonable to assume $\lambda_1 \neq \lambda_2$. Since $3^{-\frac{1}{\sigma-2\delta}} < |\xi| < 3^{\frac{1}{\sigma-2\delta}}$, this curve additionally is contained in $\{z \in \mathbb{C} \,:\, \operatorname{Re} z \leqslant -c\}$, where c is a positive constant. In order to verify (3.25), we express

$$\frac{(z+|\xi|^{2\delta})e^{zt}}{z^2+|\xi|^{2\delta}z+|\xi|^{2\sigma}} = \frac{(z+|\xi|^{2\delta})e^{zt}}{(z-\lambda_1)(z-\lambda_2)} = -\frac{\lambda_2}{\lambda_1-\lambda_2}\frac{e^{zt}}{z-\lambda_1} + \frac{\lambda_1}{\lambda_1-\lambda_2}\frac{e^{zt}}{z-\lambda_2}.$$

For this reason, applying Cauchy's integral formula we obtain

$$\begin{aligned}
&\frac{1}{2\pi i}\Big(\int_\Gamma \frac{(z+|\xi|^{2\delta})e^{zt}}{z^2+|\xi|^{2\delta}z+|\xi|^{2\sigma}}dz\Big)\chi_2(|\xi|)\\
&\quad = -\frac{\lambda_2}{\lambda_1-\lambda_2}\Big(\frac{1}{2\pi i}\int_{\Gamma_1}\frac{e^{zt}}{z-\lambda_1}dz\Big)\chi_2(|\xi|) + \frac{\lambda_1}{\lambda_1-\lambda_2}\Big(\frac{1}{2\pi i}\int_{\Gamma_2}\frac{e^{zt}}{z-\lambda_2}dz\Big)\chi_2(|\xi|)\\
&\quad = -\frac{\lambda_2}{\lambda_1-\lambda_2}e^{\lambda_1 t}\chi_2(|\xi|) + \frac{\lambda_1}{\lambda_1-\lambda_2}e^{\lambda_2 t}\chi_2(|\xi|) = \widehat{K_0}(t,\xi)\chi_2(|\xi|).
\end{aligned}$$

Here we split the curve Γ into two closed sub-curves separately Γ_1 and Γ_2 containing λ_1 and λ_2, respectively. In the same way we may conclude the relation (3.26). Now, taking account of estimates for $\widehat{K_0}(t,\xi)$ we get

$$\begin{aligned}
\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_0}(t,\xi)\chi_2(|\xi|)\big) &= \int_{\mathbb{R}^n} e^{ix\xi}|\xi|^a\widehat{K_0}(t,\xi)\chi_2(|\xi|)d\xi\\
&= \sum_{k=1}^n \frac{x_k}{i|x|^2}\int_{\mathbb{R}^n}\partial_{\xi_k}\big(e^{ix\xi}\big)|\xi|^a\widehat{K_0}(t,\xi)\chi_2(|\xi|)d\xi,
\end{aligned}$$

where we use the following formula:

$$\sum_{k=1}^n \frac{x_k}{i|x|^2}\partial_{\xi_k}e^{ix\xi} = e^{ix\xi}.$$

By induction argument, carrying out m steps of partial integration we derive

$$\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_0}(t,\xi)\chi_2(|\xi|)\big) = C\sum_{|\alpha|=m}\frac{(ix)^\alpha}{|x|^{2|\alpha|}}\mathfrak{F}^{-1}\Big(\partial_\xi^\alpha\big(|\xi|^a\widehat{K_0}(t,\xi)\chi_2(|\xi|)\big)\Big),$$

for any non-negative integer m and C is a suitable constant. Hence, we arrive at the following estimates:

$$\begin{aligned}
\big|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_0}(t,\xi)\chi_2(|\xi|)\big)\big| &\lesssim |x|^{-m}\Big\|\mathfrak{F}^{-1}\Big(\partial_\xi^\alpha\big(|\xi|^a\widehat{K_0}(t,\xi)\chi_2(|\xi|)\big)\Big)\Big\|_{L^\infty}\\
&\lesssim |x|^{-m}\big\|\partial_\xi^\alpha\big(|\xi|^a\widehat{K_0}(t,\xi)\chi_2(|\xi|)\big)\big\|_{L^1} \lesssim |x|^{-m}e^{-ct},
\end{aligned}$$

since $3^{-\frac{1}{\sigma-2\delta}} < |\xi| < 3^{\frac{1}{\sigma-2\delta}}$. This estimate immediately implies

$$\begin{aligned}
\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_0}(t,\xi)\chi_2(|\xi|)\big)(t,\cdot)\big\|_{L^1} &\lesssim e^{-ct},\\
\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_0}(t,\xi)\chi_2(|\xi|)\big)(t,\cdot)\big\|_{L^\infty} &\lesssim e^{-ct}.
\end{aligned}$$

In an analogous way, we may also conclude

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_1}(t,\xi)\chi_2(|\xi|)\big)(t,\cdot)\big\|_{L^1} \lesssim e^{-ct},$$
$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_1}(t,\xi)\chi_2(|\xi|)\big)(t,\cdot)\big\|_{L^\infty} \lesssim e^{-ct}.$$

Summarizing, the proof of Proposition 3.1.5 is completed. □

From the statements of Proposition 3.1.1, Proposition 3.1.3 and Proposition 3.1.5 we may conclude the following L^1 estimates.

Proposition 3.1.6. *The following estimates hold in $\mathbb{R}^n$:*

$$\big\|\mathfrak{F}^{-1}\big(\widehat{K_0}(t,\xi)\big)(t,\cdot)\big\|_{L^1} \lesssim \begin{cases} t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)} & \text{if } t\in(0,1], \\ 1 & \text{if } t\in[1,\infty), \end{cases}$$
$$\big\|\mathfrak{F}^{-1}\big(\widehat{K_1}(t,\xi)\big)(t,\cdot)\big\|_{L^1} \lesssim \begin{cases} t^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)} & \text{if } t\in(0,1], \\ t & \text{if } t\in[1,\infty). \end{cases}$$

Finally, from the statements of Proposition 3.1.2, Proposition 3.1.4 and Proposition 3.1.5, we may conclude the following L^1 estimates.

Proposition 3.1.7. *The following estimates hold in $\mathbb{R}^n$:*

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_0}(t,\xi)\big)(t,\cdot)\big\|_{L^1} \lesssim \begin{cases} t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)-\frac{a}{2\delta}} & \text{if } t\in(0,1], \\ t^{-\frac{a}{2(\sigma-\delta)}} & \text{if } t\in[1,\infty), \end{cases}$$
$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_1}(t,\xi)\big)(t,\cdot)\big\|_{L^1} \lesssim \begin{cases} t^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)-\frac{a}{2\delta}} & \text{if } t\in(0,1], \\ t^{1-\frac{a}{2(\sigma-\delta)}} & \text{if } t\in[1,\infty), \end{cases}$$

for any non-negative number a.

L^∞ estimates

Proposition 3.1.8. *The following estimates hold in $\mathbb{R}^n$:*

$$\big\|\mathfrak{F}^{-1}\big(\widehat{K_0}\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^\infty} \lesssim \begin{cases} 1 & \text{if } t\in(0,1], \\ t^{-\frac{n}{2(\sigma-\delta)}} & \text{if } t\in[1,\infty), \end{cases}$$
$$\big\|\mathfrak{F}^{-1}\big(\widehat{K_0}\chi_3(|\xi|)\big)(t,\cdot)\big\|_{L^\infty} \lesssim t^{-\frac{n}{2\delta}} \quad \text{for all } t\in(0,\infty),$$
$$\big\|\mathfrak{F}^{-1}\big(\widehat{K_1}\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^\infty} \lesssim \begin{cases} t & \text{if } t\in(0,1], \\ t^{1-\frac{n}{2(\sigma-\delta)}} & \text{if } t\in[1,\infty), \end{cases}$$
$$\big\|\mathfrak{F}^{-1}\big(\widehat{K_1}\chi_3(|\xi|)\big)(t,\cdot)\big\|_{L^\infty} \lesssim t^{1-\frac{n}{2\delta}} \quad \text{for all } t\in(0,\infty).$$

Proof. First let us turn to estimate the above terms for small frequencies. For the sake of the asymptotic behavior of the characteristic roots, we re-write $\widehat{K_1}(t,\xi)$ for small $|\xi|$ as follows:

$$\widehat{K_1}(t,\xi) = e^{\lambda_1 t}\frac{1-e^{(\lambda_2-\lambda_1)t}}{\lambda_1-\lambda_2} = te^{\lambda_1 t}\int_0^1 e^{-\theta\sqrt{|\xi|^{4\delta}-4|\xi|^{2\sigma}}t}d\theta.$$

Hence, we arrive at $|\widehat{K_1}(t,\xi)| \lesssim te^{-|\xi|^{2(\sigma-\delta)}t}$ for small $|\xi|$ to derive

$$\begin{aligned}\big|\mathfrak{F}^{-1}\big(\widehat{K_1}(t,\xi)\chi_1(|\xi|)\big)(t,x)\big| &\lesssim \Big|\int_{\mathbb{R}^n} e^{ix\xi}\widehat{K_1}(t,\xi)\chi_1(|\xi|)d\xi\Big| \\ &\lesssim t\int_0^1 e^{-|\xi|^{2(\sigma-\delta)}t}|\xi|^{n-1}d|\xi| \lesssim \begin{cases} t & \text{if } t\in(0,1], \\ t^{1-\frac{n}{2(\sigma-\delta)}} & \text{if } t\in[1,\infty). \end{cases}\end{aligned}$$

Moreover, for small $|\xi|$ we can see that $\widehat{K_0}(t,\xi) = -\lambda_1 \widehat{K_1}(t,\xi) + e^{\lambda_1 t}$. Since $|\widehat{K_1}(t,\xi)| \lesssim te^{-|\xi|^{2(\sigma-\delta)}t}$, we get for small frequencies the estimate

$$|\widehat{K_0}(t,\xi)| \lesssim (1+t|\xi|^{2(\sigma-\delta)})e^{-|\xi|^{2(\sigma-\delta)}t} \lesssim e^{-c|\xi|^{2(\sigma-\delta)}t}$$

for some positive constant c. Therefore, we conclude

$$\begin{aligned}\big|\mathfrak{F}^{-1}\big(\widehat{K_0}(t,\xi)\chi_1(|\xi|)\big)(t,x)\big| &\lesssim \Big|\int_{\mathbb{R}^n} e^{ix\xi}\widehat{K_0}(t,\xi)\chi_1(|\xi|)d\xi\Big| \\ &\lesssim \int_0^1 e^{-c|\xi|^{2(\sigma-\delta)}t}|\xi|^{n-1}d|\xi| \lesssim \begin{cases} 1 & \text{if } t\in(0,1], \\ t^{-\frac{n}{2(\sigma-\delta)}} & \text{if } t\in[1,\infty). \end{cases}\end{aligned}$$

Let us turn to consider the term $\widehat{K_0}(t,\xi)$ for large $|\xi|$. Thanks to the asymptotic behavior of the characteristic roots, we may estimate

$$\big|\widehat{K_0}(t,\xi)\chi_3(|\xi|)\big| = \Big|\frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1-\lambda_2}\chi_3(|\xi|)\Big| \lesssim \frac{|\xi|^\sigma e^{-|\xi|^{2\delta}t}}{|\xi|^\sigma} \lesssim e^{-|\xi|^{2\delta}t}.$$

Hence, we obtain

$$\big|\mathfrak{F}^{-1}\big(\widehat{K_0}(t,\xi)\chi_3(|\xi|)\big)(t,x)\big| \lesssim \int_1^\infty e^{-|\xi|^{2\delta}t}|\xi|^{n-1}d|\xi| \lesssim t^{-\frac{n}{2\delta}} \quad \text{for all } t\in(0,\infty).$$

Eventually, in order to estimate the term $|\mathfrak{F}^{-1}\big(\widehat{K_1}(t,\xi)\chi_3(|\xi|)\big)(t,x)|$ for all $t\in(0,\infty)$, we recall for large values of $|\xi|$ the relation

$$\widehat{K_1}(t,\xi) = e^{\lambda_1 t}\frac{1-e^{(\lambda_2-\lambda_1)t}}{\lambda_1-\lambda_2} = te^{\lambda_1 t}\int_0^1 e^{-\theta i\sqrt{4|\xi|^{2\sigma}-|\xi|^{4\delta}}t}d\theta.$$

As a result, we may conclude

$$\big|\mathfrak{F}^{-1}\big(\widehat{K_1}(t,\xi)\chi_3(|\xi|)\big)(t,x)\big| \lesssim \Big|\int_{\mathbb{R}^n} e^{ix\xi}\widehat{K_1}(t,\xi)\chi_3(|\xi|)d\xi\Big| \lesssim t\int_1^\infty e^{-|\xi|^{2\delta}t}|\xi|^{n-1}d|\xi| \lesssim t^{1-\frac{n}{2\delta}}.$$

Summarizing, Proposition 3.1.8 is proved. □

From Proposition 3.1.5 and Proposition 3.1.8 the following statement follows immediately.

Proposition 3.1.9. *The following estimates hold in* $\mathbb{R}^n$*:*

$$\big\|\mathfrak{F}^{-1}\big(\widehat{K_0}(t,\xi)\big)(t,\cdot)\big\|_{L^\infty} \lesssim \begin{cases} t^{-\frac{n}{2\delta}} & \text{if } t\in(0,1], \\ t^{-\frac{n}{2(\sigma-\delta)}} & \text{if } t\in[1,\infty), \end{cases}$$

$$\big\|\mathfrak{F}^{-1}\big(\widehat{K_1}(t,\xi)\big)(t,\cdot)\big\|_{L^\infty} \lesssim \begin{cases} t^{1-\frac{n}{2\delta}} & \text{if } t\in(0,1], \\ t^{1-\frac{n}{2(\sigma-\delta)}} & \text{if } t\in[1,\infty). \end{cases}$$

Finally, following the approach of the proof of Proposition 3.1.8 we may prove the following statements.

Proposition 3.1.10. *The following estimates hold in* $\mathbb{R}^n$*:*

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_0}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^\infty} \lesssim \begin{cases} 1 & \text{if } t\in(0,1], \\ t^{-\frac{n+a}{2(\sigma-\delta)}} & \text{if } t\in[1,\infty), \end{cases}$$

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_0}(t,\xi)\chi_3(|\xi|)\big)(t,\cdot)\big\|_{L^\infty} \lesssim t^{-\frac{n+a}{2\delta}} \quad \text{for all } t\in(0,\infty),$$

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_1}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^\infty} \lesssim \begin{cases} t & \text{if } t\in(0,1], \\ t^{1-\frac{n+a}{2(\sigma-\delta)}} & \text{if } t\in[1,\infty), \end{cases}$$

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_1}(t,\xi)\chi_3(|\xi|)\big)(t,\cdot)\big\|_{L^\infty} \lesssim t^{1-\frac{n+a}{2\delta}} \quad \text{for all } t\in(0,\infty),$$

for any non-negative number a*.*

Proposition 3.1.11. *The following estimates hold in $\mathbb{R}^n$:*

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_0}(t,\xi)\big)(t,\cdot)\big\|_{L^\infty} \lesssim \begin{cases} t^{-\frac{n+a}{2\delta}} & \text{if } t\in(0,1], \\ t^{-\frac{n+a}{2(\sigma-\delta)}} & \text{if } t\in[1,\infty), \end{cases}$$

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_1}(t,\xi)\big)(t,\cdot)\big\|_{L^\infty} \lesssim \begin{cases} t^{1-\frac{n+a}{2\delta}} & \text{if } t\in(0,1], \\ t^{1-\frac{n+a}{2(\sigma-\delta)}} & \text{if } t\in[1,\infty), \end{cases}$$

for any non-negative number a.

L^r estimates

By applying an interpolation theorem, from the statements of Propositions 3.1.6 and 3.1.9 we may conclude the following L^r estimates.

Proposition 3.1.12. *The following estimates hold in $\mathbb{R}^n$:*

$$\big\|\mathfrak{F}^{-1}\big(\widehat{K_0}(t,\xi)\big)(t,\cdot)\big\|_{L^r} \lesssim \begin{cases} t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})} & \text{if } t\in(0,1], \\ t^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})} & \text{if } t\in[1,\infty), \end{cases}$$

$$\big\|\mathfrak{F}^{-1}\big(\widehat{K_1}(t,\xi)\big)(t,\cdot)\big\|_{L^r} \lesssim \begin{cases} t^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})} & \text{if } t\in(0,1], \\ t^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})} & \text{if } t\in[1,\infty), \end{cases}$$

for all $r\in[1,\infty]$.

From the statements of Propositions 3.1.7 and 3.1.11, by applying an interpolation theorem we may conclude the following L^r estimates.

Proposition 3.1.13. *The following estimates hold in $\mathbb{R}^n$:*

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_0}(t,\xi)\big)(t,\cdot)\big\|_{L^r} \lesssim \begin{cases} t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{a}{2\delta}} & \text{if } t\in(0,1], \\ t^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{a}{2(\sigma-\delta)}} & \text{if } t\in[1,\infty), \end{cases}$$

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_1}(t,\xi)\big)(t,\cdot)\big\|_{L^r} \lesssim \begin{cases} t^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{a}{2\delta}} & \text{if } t\in(0,1], \\ t^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{a}{2(\sigma-\delta)}} & \text{if } t\in[1,\infty), \end{cases}$$

for all $r\in[1,\infty]$ and any non-negative number a.

Proof of Theorem 3.1.1. In order to obtain $(L^m\cap L^q)-L^q$ estimates, we estimate the L^q norm of the low-frequency part of solutions by the L^m norm of the data, whereas their high-frequency and middle-frequency parts are estimated by using L^q-L^q estimates. Thanks to Proposition 3.1.2 and Proposition 3.1.10, we derive

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_0}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^r} \lesssim \begin{cases} 1 & \text{if } t\in(0,1], \\ t^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{a}{2(\sigma-\delta)}} & \text{if } t\in[1,\infty), \end{cases}$$

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_1}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^r} \lesssim \begin{cases} t & \text{if } t\in(0,1], \\ t^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{a}{2(\sigma-\delta)}} & \text{if } t\in[1,\infty), \end{cases}$$

for all $r\in[1,\infty]$ and any non-negative number a. Applying Young's convolution inequality from Proposition B.1.1 we have

$$\|v_{\chi_1}(t,\cdot)\|_{L^q} \lesssim \big\|\mathfrak{F}^{-1}\big(\widehat{K_1}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^r}\|v_1\|_{L^m} \lesssim \begin{cases} t\,\|v_1\|_{L^m} & \text{if } t\in(0,1], \\ t^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})}\|v_1\|_{L^m} & \text{if } t\in[1,\infty), \end{cases}$$

and

$$\|v_{\chi_2}(t,\cdot)\|_{L^q} \lesssim \big\|\mathfrak{F}^{-1}\big(\widehat{K_1}(t,\xi)\chi_2(|\xi|)\big)(t,\cdot)\big\|_{L^1}\|v_1\|_{L^q} \lesssim e^{-ct}\|v_1\|_{L^q} \quad \text{for all } t\in(0,\infty),$$

$$\|v_{\chi_3}(t,\cdot)\|_{L^q} \lesssim \big\|\mathfrak{F}^{-1}\big(\widehat{K_1}(t,\xi)\chi_3(|\xi|)\big)(t,\cdot)\big\|_{L^1}\|v_1\|_{L^q} \lesssim \begin{cases} t^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\|v_1\|_{L^q} & \text{if } t\in(0,1], \\ e^{-ct}\|v_1\|_{L^q} & \text{if } t\in[1,\infty), \end{cases}$$

where c is a suitable positive constant. Hence, we may conclude

$$\|v(t,\cdot)\|_{L^q}\lesssim\begin{cases}t^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\|v_1\|_{L^m\cap L^q} & \text{if } t\in(0,1],\\ (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})}\|v_1\|_{L^m\cap L^q} & \text{if } t\in[1,\infty).\end{cases}$$

Analogously, we have

$$\begin{aligned}\big\||D|^\sigma v_{\chi_1}(t,\cdot)\big\|_{L^q}&\lesssim\big\|\mathfrak{F}^{-1}\big(|\xi|^\sigma\widehat{K_1}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^r}\|v_1\|_{L^m}\\&\lesssim\begin{cases}t\,\|v_1\|_{L^m} & \text{if } t\in(0,1],\\ t^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\sigma}{2(\sigma-\delta)}}\|v_1\|_{L^m} & \text{if } t\in[1,\infty),\end{cases}\end{aligned}$$

and

$$\begin{aligned}\big\||D|^\sigma v_{\chi_2}(t,\cdot)\big\|_{L^q}&\lesssim\big\|\mathfrak{F}^{-1}\big(|\xi|^\sigma\widehat{K_1}(t,\xi)\chi_2(|\xi|)\big)(t,\cdot)\big\|_{L^1}\|v_1\|_{L^q}\lesssim e^{-ct}\|v_1\|_{L^q}\quad\text{for all } t\in(0,\infty),\\ \big\||D|^\sigma v_{\chi_3}(t,\cdot)\big\|_{L^q}&\lesssim\big\|\mathfrak{F}^{-1}\big(|\xi|^\sigma\widehat{K_1}(t,\xi)\chi_3(|\xi|)\big)(t,\cdot)\big\|_{L^1}\|v_1\|_{L^q}\lesssim\begin{cases}t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\|v_1\|_{L^q} & \text{if } t\in(0,1],\\ e^{-ct}\|v_1\|_{L^q} & \text{if } t\in[1,\infty),\end{cases}\end{aligned}$$

where c is a suitable positive constant. Therefore, we may conclude

$$\big\||D|^\sigma v(t,\cdot)\big\|_{L^q}\lesssim\begin{cases}t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\|v_1\|_{L^m\cap L^q} & \text{if } t\in(0,1],\\ (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\sigma}{2(\sigma-\delta)}}\|v_1\|_{L^m\cap L^q} & \text{if } t\in[1,\infty).\end{cases}$$

Now let us turn to estimate the norm $\|v_t(t,\cdot)\|_{L^q}$. We rewrite

$$\partial_t\widehat{K_1}(t,\xi)=\widehat{K_0}(t,\xi)+(\lambda_1+\lambda_2)\widehat{K_1}(t,\xi)=\widehat{K_0}(t,\xi)-|\xi|^{2\delta}\widehat{K_1}(t,\xi).$$

Applying again Young's convolution inequality from Proposition B.1.1 we get

$$\begin{aligned}\|\partial_t v_{\chi_1}(t,\cdot)\|_{L^q}&=\big\|\mathfrak{F}^{-1}\big(\big(\widehat{K_0}(t,\xi)-|\xi|^{2\delta}\widehat{K_1}(t,\xi)\big)\chi_1(|\xi|)\widehat{v_1}(\xi)\big)(t,\cdot)\big\|_{L^q}\\&\lesssim\Big(\big\|\mathfrak{F}^{-1}\big(\widehat{K_0}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^r}+\big\|\mathfrak{F}^{-1}\big(|\xi|^{2\delta}\widehat{K_1}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^r}\Big)\|v_1\|_{L^m}\\&\lesssim\begin{cases}(1+t)\|v_1\|_{L^m} & \text{if } t\in(0,1],\\ \big(t^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})}+t^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}}\big)\|v_1\|_{L^m} & \text{if } t\in[1,\infty),\end{cases}\\&\lesssim\begin{cases}\|v_1\|_{L^m} & \text{if } t\in(0,1],\\ t^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}}\|v_1\|_{L^m} & \text{if } t\in[1,\infty),\end{cases}\end{aligned}$$

and

$$\begin{aligned}\|\partial_t v_{\chi_2}(t,\cdot)\|_{L^q}&\lesssim\Big(\big\|\mathfrak{F}^{-1}\big(\widehat{K_0}(t,\xi)\chi_2(|\xi|)\big)(t,\cdot)\big\|_{L^1}+\big\|\mathfrak{F}^{-1}\big(|\xi|^{2\delta}\widehat{K_1}(t,\xi)\chi_2(|\xi|)\big)(t,\cdot)\big\|_{L^1}\Big)\|v_1\|_{L^q}\\&\lesssim e^{-ct}\|v_1\|_{L^q}\quad\text{for all } t\in(0,\infty),\\ \|\partial_t v_{\chi_3}(t,\cdot)\|_{L^q}&\lesssim\Big(\big\|\mathfrak{F}^{-1}\big(\widehat{K_0}(t,\xi)\chi_3(|\xi|)\big)(t,\cdot)\big\|_{L^1}+\big\|\mathfrak{F}^{-1}\big(|\xi|^{2\delta}\widehat{K_1}(t,\xi)\chi_3(|\xi|)\big)(t,\cdot)\big\|_{L^1}\Big)\|v_1\|_{L^q}\\&\lesssim\begin{cases}\big(t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}+t^{-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\big)\|v_1\|_{L^q} & \text{if } t\in(0,1],\\ e^{-ct}\|v_1\|_{L^q} & \text{if } t\in[1,\infty),\end{cases}\\&\lesssim\begin{cases}t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\|v_1\|_{L^q} & \text{if } t\in(0,1],\\ e^{-ct}\|v_1\|_{L^q} & \text{if } t\in[1,\infty),\end{cases}\end{aligned}$$

where c is a suitable positive constant. Therefore, we imply

$$\|v_t(t,\cdot)\|_{L^q}\lesssim\begin{cases}t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\|v_1\|_{L^m\cap L^q} & \text{if } t\in(0,1],\\ (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}}\|v_1\|_{L^m\cap L^q} & \text{if } t\in[1,\infty).\end{cases}$$

Summarizing, the proof of Theorem 3.1.1 is completed. □

$L^m - L^q$ estimates

From the proof of Theorem 3.1.1 and the statements of Proposition 3.1.13 we have the following corollary.

Corollary 3.1.1. *Let $\delta \in \big(0, \frac{\sigma}{2}\big)$ in (3.3), $q \in [1,\infty]$ and $m \in [1,q]$. Then, the energy solutions to (3.3) satisfy the $L^m - L^q$ estimates*

$$\|v(t,\cdot)\|_{L^q} \lesssim \begin{cases} t^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|v_1\|_{L^m} & \text{if } t \in (0,1],\\ t^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})}\|v_1\|_{L^m} & \text{if } t \in [1,\infty),\end{cases}$$
$$\big\||D|^\sigma v(t,\cdot)\big\|_{L^q} \lesssim \begin{cases} t^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{\sigma}{2\delta}}\|v_1\|_{L^m} & \text{if } t \in (0,1],\\ t^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\sigma}{2(\sigma-\delta)}}\|v_1\|_{L^m} & \text{if } t \in [1,\infty),\end{cases}$$
$$\|v_t(t,\cdot)\|_{L^q} \lesssim \begin{cases} t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|v_1\|_{L^m} & \text{if } t \in (0,1],\\ t^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}}\|v_1\|_{L^m} & \text{if } t \in [1,\infty),\end{cases}$$

and the energy solutions to (3.2) satisfy the $L^m - L^q$ estimates

$$\|u(t,\cdot)\|_{L^q} \lesssim \begin{cases} (t-s)^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|u_1\|_{L^m} & \text{if } t \in (s,s+1],\\ (t-s)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})}\|u_1\|_{L^m} & \text{if } t \in [s+1,\infty),\end{cases}$$
$$\big\||D|^\sigma u(t,\cdot)\big\|_{L^q} \lesssim \begin{cases} (t-s)^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{\sigma}{2\delta}}\|u_1\|_{L^m} & \text{if } t \in (s,s+1],\\ (t-s)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\sigma}{2(\sigma-\delta)}}\|u_1\|_{L^m} & \text{if } t \in [s+1,\infty),\end{cases}$$
$$\|u_t(t,\cdot)\|_{L^q} \lesssim \begin{cases} (t-s)^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|u_1\|_{L^m} & \text{if } t \in (s,s+1],\\ (t-s)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}}\|u_1\|_{L^m} & \text{if } t \in [s+1,\infty),\end{cases}$$

where $1+\frac{1}{q} = \frac{1}{r}+\frac{1}{m}$ and for all dimensions $n \geqslant 1$.

3.2. A second Cauchy problem for linear structurally damped σ-evolution models

Let us turn to the following Cauchy problem:

$$u_{tt} + (-\Delta)^\sigma u + \mu(-\Delta)^\delta u_t = 0, \quad u(0,x) = u_0(x), \quad u_t(0,x) = 0, \tag{3.27}$$

where $\sigma \geqslant 1$, $\mu > 0$ and $\delta \in (0, \frac{\sigma}{2})$.

$L^m \cap L^q - L^q$ and $L^q - L^q$ estimates

In this section, we are going to prove the following result.

Theorem 3.2.1. *Let $\delta \in \big(0, \frac{\sigma}{2}\big)$ in (3.27), $q \in (1,\infty)$ and $m \in [1,q)$. Then, the energy solutions to (3.27) satisfy the $(L^m \cap L^q) - L^q$ estimates*

$$\|u(t,\cdot)\|_{L^q} \lesssim \begin{cases} t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\|u_0\|_{L^m\cap L^q} & \text{if } t \in (0,1],\\ (1+t)^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})}\|u_0\|_{L^m\cap L^q} & \text{if } t \in [1,\infty),\end{cases}$$
$$\big\||D|^\sigma u(t,\cdot)\big\|_{L^q} \lesssim \begin{cases} t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\|u_0\|_{L^m\cap H^\sigma_q} & \text{if } t \in (0,1],\\ (1+t)^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\sigma}{2(\sigma-\delta)}}\|u_0\|_{L^m\cap H^\sigma_q} & \text{if } t \in [1,\infty),\end{cases}$$
$$\|u_t(t,\cdot)\|_{L^q} \lesssim \begin{cases} t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\|u_0\|_{L^m\cap H^\sigma_q} & \text{if } t \in (0,1],\\ (1+t)^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}}\|u_0\|_{L^m\cap H^\sigma_q} & \text{if } t \in [1,\infty),\end{cases}$$

and the $L^q - L^q$ estimates

$$\|u(t,\cdot)\|_{L^q} \lesssim \begin{cases} t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\|u_0\|_{L^q} & \text{if } t \in (0,1], \\ \|u_0\|_{L^q} & \text{if } t \in [1,\infty), \end{cases}$$

$$\||D|^\sigma u(t,\cdot)\|_{L^q} \lesssim \begin{cases} t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\|u_0\|_{H^\sigma_q} & \text{if } t \in (0,1], \\ (1+t)^{-\frac{\sigma}{2(\sigma-\delta)}}\|u_0\|_{H^\sigma_q} & \text{if } t \in [1,\infty), \end{cases}$$

$$\|u_t(t,\cdot)\|_{L^q} \lesssim \begin{cases} t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\|u_0\|_{H^\sigma_q} & \text{if } t \in (0,1], \\ (1+t)^{-\frac{\delta}{\sigma-\delta}}\|u_0\|_{H^\sigma_q} & \text{if } t \in [1,\infty), \end{cases}$$

where $1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{m}$ and for all dimensions $n \geqslant 1$.

Proof. Applying the partial Fourier transformation to (3.27) we obtain the Cauchy problem for $\widehat{u}(t,\xi) := \mathfrak{F}(u(t,x))$ and $\widehat{u_0}(\xi) := \mathfrak{F}(u_0(x))$ as follows:

$$\widehat{u}_{tt} + \mu|\xi|^{2\delta}\widehat{u}_t + |\xi|^{2\sigma}\widehat{u} = 0, \quad \widehat{u}(0,\xi) = \widehat{u_0}(\xi), \quad \widehat{u}_t(0,\xi) = 0. \tag{3.28}$$

We may choose without loss of generality $\mu = 1$ in (3.27). The characteristic roots are

$$\lambda_{1,2} = \lambda_{1,2}(\xi) = \frac{1}{2}\Big(-|\xi|^{2\delta} \pm \sqrt{|\xi|^{4\delta} - 4|\xi|^{2\sigma}}\Big).$$

The solutions to (3.28) are given by the following formula (here we assume $\lambda_1 \neq \lambda_2$):

$$\widehat{u}(t,\xi) = \frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2}\widehat{u_0}(\xi) =: \widehat{K_0}(t,\xi)\widehat{u_0}(\xi).$$

Taking account of the cases of small and large frequencies separately we have

1. $\lambda_1 \sim -|\xi|^{2(\sigma-\delta)}, \quad \lambda_2 \sim -|\xi|^{2\delta}, \quad \lambda_1 - \lambda_2 \sim |\xi|^{2\delta}$ for small $|\xi| \in (0, 4^{-\frac{1}{\sigma-2\delta}})$,
2. $\lambda_{1,2} \sim -|\xi|^{2\delta} \pm i|\xi|^\sigma, \quad \lambda_1 - \lambda_2 \sim i|\xi|^\sigma$ for large $|\xi| \in (4^{\frac{1}{\sigma-2\delta}}, \infty)$.

As in Section 3.1, we now decompose the solution to (3.27) into three parts localized separately to low, middle and high frequencies, that is,

$$u(t,x) = u_{\chi_1}(t,x) + u_{\chi_2}(t,x) + u_{\chi_3}(t,x),$$

where

$$u_{\chi_k}(t,x) = \mathfrak{F}^{-1}\big(\chi_k(|\xi|)\widehat{u}(t,\xi)\big) \quad \text{with } k = 1,2,3.$$

In order to obtain $(L^m \cap L^q) - L^q$ estimates, we estimate the L^q norm of the low-frequency part of solutions by the L^m norm of the data, whereas their high-frequency and middle-frequency parts are estimated by using $L^q - L^q$ estimates. Thanks to Proposition 3.1.2 and Proposition 3.1.10, we derive

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_0}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^r} \lesssim \begin{cases} 1 & \text{if } t \in (0,1], \\ t^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{a}{2(\sigma-\delta)}} & \text{if } t \in [1,\infty), \end{cases}$$

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_1}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^r} \lesssim \begin{cases} t & \text{if } t \in (0,1], \\ t^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{a}{2(\sigma-\delta)}} & \text{if } t \in [1,\infty), \end{cases}$$

for all $r \in [1,\infty]$ and any non-negative number a. Applying Young's convolution inequality from Proposition B.1.1 we have

$$\|u_{\chi_1}(t,\cdot)\|_{L^q} \lesssim \big\|\mathfrak{F}^{-1}\big(\widehat{K_0}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^r}\|u_0\|_{L^m} \lesssim \begin{cases} \|u_0\|_{L^m} & \text{if } t \in (0,1], \\ t^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})}\|u_0\|_{L^m} & \text{if } t \in [1,\infty), \end{cases}$$

and

$$\|u_{\chi_2}(t,\cdot)\|_{L^q} \lesssim \big\|\mathfrak{F}^{-1}\big(\widehat{K_0}(t,\xi)\chi_2(|\xi|)\big)(t,\cdot)\big\|_{L^1}\|u_0\|_{L^q} \lesssim e^{-ct}\|u_0\|_{L^q} \quad \text{for all } t \in (0,\infty),$$

$$\|u_{\chi_3}(t,\cdot)\|_{L^q} \lesssim \big\|\mathfrak{F}^{-1}\big(\widehat{K_0}(t,\xi)\chi_3(|\xi|)\big)(t,\cdot)\big\|_{L^1}\|u_0\|_{L^q} \lesssim \begin{cases} t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\|u_0\|_{L^q} & \text{if } t \in (0,1], \\ e^{-ct}\|u_0\|_{L^q} & \text{if } t \in [1,\infty), \end{cases}$$

where c is a suitable positive constant. Hence, we may conclude

$$\|u(t,\cdot)\|_{L^q} \lesssim \begin{cases} t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\|u_0\|_{L^m\cap L^q} & \text{if } t\in(0,1],\\ (1+t)^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})}\|u_0\|_{L^m\cap L^q} & \text{if } t\in[1,\infty). \end{cases}$$

In an analogous way we derive

$$\begin{aligned} \big\||D|^\sigma u_{\chi_1}(t,\cdot)\big\|_{L^q} &\lesssim \big\|\mathfrak{F}^{-1}\big(|\xi|^\sigma \widehat{K_0}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^r}\|u_0\|_{L^m} \\ &\lesssim \begin{cases} \|u_0\|_{L^m} & \text{if } t\in(0,1],\\ t^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\sigma}{2(\sigma-\delta)}}\|u_0\|_{L^m} & \text{if } t\in[1,\infty), \end{cases} \end{aligned}$$

and

$$\begin{aligned} \big\||D|^\sigma u_{\chi_2}(t,\cdot)\big\|_{L^q} &\lesssim \big\|\mathfrak{F}^{-1}\big(|\xi|^\sigma \widehat{K_0}(t,\xi)\chi_2(|\xi|)\big)(t,\cdot)\big\|_{L^1}\|u_0\|_{L^q} \lesssim e^{-ct}\|u_0\|_{L^q} \quad \text{for all } t\in(0,\infty),\\ \big\||D|^\sigma u_{\chi_3}(t,\cdot)\big\|_{L^q} &\lesssim \big\|\mathfrak{F}^{-1}\big(\widehat{K_0}(t,\xi)\chi_3(|\xi|)\big)(t,\cdot)\big\|_{L^1}\big\|\mathfrak{F}^{-1}\big(|\xi|^\sigma\widehat{u_0}(\xi)\big)\big\|_{L^q}\\ &\lesssim \begin{cases} t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\|u_0\|_{\dot{H}^\sigma_q} & \text{if } t\in(0,1],\\ e^{-ct}\|u_0\|_{\dot{H}^\sigma_q} & \text{if } t\in[1,\infty), \end{cases}\\ &\lesssim \begin{cases} t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\|u_0\|_{H^\sigma_q} & \text{if } t\in(0,1],\\ e^{-ct}\|u_0\|_{H^\sigma_q} & \text{if } t\in[1,\infty), \end{cases} \end{aligned}$$

where c is a suitable positive constant. Hence, we may conclude

$$\big\||D|^\sigma u(t,\cdot)\big\|_{L^q} \lesssim \begin{cases} t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\|u_0\|_{L^m\cap H^\sigma_q} & \text{if } t\in(0,1],\\ (1+t)^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\sigma}{2(\sigma-\delta)}}\|u_0\|_{L^m\cap H^\sigma_q} & \text{if } t\in[1,\infty). \end{cases}$$

Finally, let us turn to estimate the norm $\|u_t(t,\cdot)\|_{L^q}$. We rewrite

$$\partial_t\widehat{K_0}(t,\xi) = -\lambda_1\lambda_2\widehat{K_1}(t,\xi) = -|\xi|^{2\sigma}\widehat{K_1}(t,\xi).$$

Applying again Young's convolution inequality from Proposition B.1.1 we get

$$\begin{aligned} \|\partial_t u_{\chi_1}(t,\cdot)\|_{L^q} &= \big\|\mathfrak{F}^{-1}\big(|\xi|^{2\sigma}\widehat{K_1}(t,\xi)\chi_1(|\xi|)\widehat{u_0}(\xi)\big)(t,\cdot)\big\|_{L^q} \lesssim \big\|\mathfrak{F}^{-1}\big(|\xi|^{2\sigma}\widehat{K_1}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^r}\|u_0\|_{L^m}\\ &\lesssim \begin{cases} t\,\|u_0\|_{L^m} & \text{if } t\in(0,1],\\ t^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\sigma}{\sigma-\delta}}\|u_0\|_{L^m} & \text{if } t\in[1,\infty), \end{cases}\\ &\lesssim \begin{cases} t\,\|u_0\|_{L^m} & \text{if } t\in(0,1],\\ t^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}}\|u_0\|_{L^m} & \text{if } t\in[1,\infty), \end{cases} \end{aligned}$$

and

$$\begin{aligned} \|\partial_t u_{\chi_2}(t,\cdot)\|_{L^q} &\lesssim \big\|\mathfrak{F}^{-1}\big(|\xi|^{2\sigma}\widehat{K_1}(t,\xi)\chi_2(|\xi|)\big)(t,\cdot)\big\|_{L^1}\|u_0\|_{L^q} \lesssim e^{-ct}\|u_0\|_{L^q} \quad \text{for all } t\in(0,\infty),\\ \|\partial_t u_{\chi_3}(t,\cdot)\|_{L^q} &\lesssim \big\|\mathfrak{F}^{-1}\big(|\xi|^{\sigma}\widehat{K_1}(t,\xi)\chi_3(|\xi|)\big)(t,\cdot)\big\|_{L^1}\big\|\mathfrak{F}^{-1}\big(|\xi|^\sigma\widehat{u_0}(\xi)\big)\big\|_{L^q}\\ &\lesssim \begin{cases} t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\|u_0\|_{\dot{H}^\sigma_q} & \text{if } t\in(0,1],\\ e^{-ct}\|u_0\|_{\dot{H}^\sigma_q} & \text{if } t\in[1,\infty), \end{cases}\\ &\lesssim \begin{cases} t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\|u_0\|_{H^\sigma_q} & \text{if } t\in(0,1],\\ e^{-ct}\|u_0\|_{H^\sigma_q} & \text{if } t\in[1,\infty), \end{cases} \end{aligned}$$

where c is a suitable positive constant. Therefore, we may conclude

$$\|u_t(t,\cdot)\|_{L^q} \lesssim \begin{cases} t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\|u_0\|_{L^m\cap H^\sigma_q} & \text{if } t\in(0,1],\\ (1+t)^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}}\|u_0\|_{L^m\cap H^\sigma_q} & \text{if } t\in[1,\infty). \end{cases}$$

Summarizing, the proof of Theorem 3.2.1 is completed. □

$L^m - L^q$ estimates

From the proof of Theorem 3.2.1 and the statements of Proposition 3.1.13 we have the following corollary.

Corollary 3.2.1. *Let $\delta \in \big(0, \frac{\sigma}{2}\big)$ in (3.27), $q \in [1, \infty]$ and $m \in [1, q]$. Then, the energy solutions to (3.27) satisfy the $L^m - L^q$ estimates*

$$\|u(t,\cdot)\|_{L^q} \lesssim \begin{cases} t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|u_0\|_{L^m} & \text{if } t \in (0,1], \\ t^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})}\|u_0\|_{L^m} & \text{if } t \in [1,\infty), \end{cases}$$

$$\big\||D|^\sigma u(t,\cdot)\big\|_{L^q} \lesssim \begin{cases} t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{\sigma}{2\delta}}\|u_0\|_{L^m} & \text{if } t \in (0,1], \\ t^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\sigma}{2(\sigma-\delta)}}\|u_0\|_{L^m} & \text{if } t \in [1,\infty), \end{cases}$$

$$\|u_t(t,\cdot)\|_{L^q} \lesssim \begin{cases} t^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{\sigma}{\delta}}\|u_0\|_{L^m} & \text{if } t \in (0,1], \\ t^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}}\|u_0\|_{L^m} & \text{if } t \in [1,\infty), \end{cases}$$

where $1+\frac{1}{q}=\frac{1}{r}+\frac{1}{m}$ and for all dimensions $n \geqslant 1$.

3.3. A third Cauchy problem for linear structurally damped σ-evolution models

In this section, let us consider the Cauchy problem for structurally damped σ-evolution models in the form

$$u_{tt}+(-\Delta)^\sigma u+\mu(-\Delta)^\delta u_t=0, \quad u(0,x)=u_0(x), \quad u_t(0,x)=u_1(x), \tag{3.29}$$

with $\sigma \geqslant 1$, $\mu > 0$ and $\delta \in (0, \frac{\sigma}{2})$.

We may summarize the results from Sections 3.1 and 3.2 as follows:

$L^m \cap L^q - L^q$ and $L^q - L^q$ estimates

Theorem 3.3.1. *Let $\delta \in \big(0, \frac{\sigma}{2}\big)$ in (3.29), $q \in (1, \infty)$ and $m \in [1, q)$. Then, the energy solutions to (3.29) satisfy the $(L^m \cap L^q) - L^q$ estimates*

$$\|u(t,\cdot)\|_{L^q} \lesssim \begin{cases} t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\|u_0\|_{L^m\cap L^q}+t^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\|u_1\|_{L^m\cap L^q} & \text{if } t \in (0,1], \\ (1+t)^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})}\|u_0\|_{L^m\cap L^q} & \\ \quad +(1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})}\|u_1\|_{L^m\cap L^q} & \text{if } t \in [1,\infty), \end{cases}$$

$$\big\||D|^\sigma u(t,\cdot)\big\|_{L^q} \lesssim \begin{cases} t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\big(\|u_0\|_{L^m\cap H^\sigma_q}+\|u_1\|_{L^m\cap L^q}\big) & \text{if } t \in (0,1], \\ (1+t)^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\sigma}{2(\sigma-\delta)}}\|u_0\|_{L^m\cap H^\sigma_q} & \\ \quad +(1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\sigma}{2(\sigma-\delta)}}\|u_1\|_{L^m\cap L^q} & \text{if } t \in [1,\infty), \end{cases}$$

$$\|u_t(t,\cdot)\|_{L^q} \lesssim \begin{cases} t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\big(\|u_0\|_{L^m\cap H^\sigma_q}+\|u_1\|_{L^m\cap L^q}\big) & \text{if } t \in (0,1], \\ (1+t)^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}}\|u_0\|_{L^m\cap H^\sigma_q} & \\ \quad +(1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}}\|u_1\|_{L^m\cap L^q} & \text{if } t \in [1,\infty), \end{cases}$$

and the $L^q - L^q$ estimates

$$\|u(t,\cdot)\|_{L^q} \lesssim \begin{cases} t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\|u_0\|_{L^q}+t^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\|u_1\|_{L^q} & \text{if } t \in (0,1], \\ \|u_0\|_{L^q}+(1+t)\|u_1\|_{L^q} & \text{if } t \in [1,\infty), \end{cases}$$

$$\big\||D|^\sigma u(t,\cdot)\big\|_{L^q} \lesssim \begin{cases} t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\big(\|u_0\|_{H^\sigma_q}+\|u_1\|_{L^q}\big) & \text{if } t \in (0,1], \\ (1+t)^{-\frac{\sigma}{2(\sigma-\delta)}}\|u_0\|_{H^\sigma_q}+(1+t)^{1-\frac{\sigma}{2(\sigma-\delta)}}\|u_1\|_{L^q} & \text{if } t \in [1,\infty), \end{cases}$$

$$\|u_t(t,\cdot)\|_{L^q} \lesssim \begin{cases} t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\big(\|u_0\|_{H^\sigma_q}+\|u_1\|_{L^q}\big) & \text{if } t \in (0,1], \\ (1+t)^{-\frac{\delta}{\sigma-\delta}}\|u_0\|_{H^\sigma_q}+(1+t)^{1-\frac{\delta}{\sigma-\delta}}\|u_1\|_{L^q} & \text{if } t \in [1,\infty), \end{cases}$$

where $1+\frac{1}{q}=\frac{1}{r}+\frac{1}{m}$ and for all dimensions $n \geqslant 1$.

We may prove similar estimates to those in Theorem 3.3.1. Namely for any $a \geqslant 0$, we have the following further results.

Theorem 3.3.2. *Let $\delta \in \big(0, \frac{\sigma}{2}\big)$ in (3.29), $q \in (1,\infty)$ and $m \in [1,q)$. Then, the Sobolev solutions to (3.29) satisfy the $(L^m \cap L^q) - L^q$ estimates*

$$\big\||D|^a u(t,\cdot)\big\|_{L^q} \lesssim \begin{cases} t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\|u_0\|_{L^m\cap H^a_q} \\ \qquad +t^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)-\frac{a}{2\delta}}\|u_1\|_{L^m\cap H^{[a-\sigma]^+}_q} & \text{if } t\in(0,1], \\ (1+t)^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{a}{2(\sigma-\delta)}}\|u_0\|_{L^m\cap H^a_q} \\ \qquad +(1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{a}{2(\sigma-\delta)}}\|u_1\|_{L^m\cap H^{[a-\sigma]^+}_q} & \text{if } t\in[1,\infty), \end{cases}$$

$$\big\||D|^a u_t(t,\cdot)\big\|_{L^q} \lesssim \begin{cases} t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\big(\|u_0\|_{L^m\cap H^{a+\sigma}_q}+\|u_1\|_{L^m\cap H^a_q}\big) & \text{if } t\in(0,1], \\ (1+t)^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{a+2\delta}{2(\sigma-\delta)}}\|u_0\|_{L^m\cap H^{a+\sigma}_q} \\ \qquad +(1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{a+2\delta}{2(\sigma-\delta)}}\|u_1\|_{L^m\cap H^a_q} & \text{if } t\in[1,\infty), \end{cases}$$

and the $L^q - L^q$ estimates

$$\big\||D|^a u(t,\cdot)\big\|_{L^q} \lesssim \begin{cases} t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\|u_0\|_{H^\sigma_q}+t^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)-\frac{a}{2\delta}}\|u_1\|_{H^{[a-\sigma]^+}_q} & \text{if } t\in(0,1], \\ (1+t)^{-\frac{a}{2(\sigma-\delta)}}\|u_0\|_{H^a_q}+(1+t)^{1-\frac{a}{2(\sigma-\delta)}}\|u_1\|_{H^{[a-\sigma]^+}_q} & \text{if } t\in[1,\infty), \end{cases}$$

$$\big\||D|^a u_t(t,\cdot)\big\|_{L^q} \lesssim \begin{cases} t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\big(\|u_0\|_{H^{a+\sigma}_q}+\|u_1\|_{H^{a,q}}\big) & \text{if } t\in(0,1], \\ (1+t)^{-\frac{a+2\delta}{2(\sigma-\delta)}}\|u_0\|_{H^{a+\sigma,q}}+(1+t)^{1-\frac{a+2\delta}{2(\sigma-\delta)}}\|u_1\|_{H^a_q} & \text{if } t\in[1,\infty), \end{cases}$$

for any $a \geqslant 0$, where $1+\frac{1}{q}=\frac{1}{r}+\frac{1}{m}$ and all dimensions $n \geqslant 1$.

Proof. In order to estimate some partial derivatives of solutions, we use a suitable regularity of the data u_0 and u_1 depending on the order of a. Then, repeating an analogous treatment as we did in the proofs of Theorems 3.1.1 and 3.2.1 we may conclude all the desired estimates. □

For space dimensions $n > 2\delta$ we obtain the following better estimates for solutions to (3.29).

Theorem 3.3.3. *Let $\delta \in \big(0, \frac{\sigma}{2}\big)$ in (3.29), $q \in (1,\infty)$ and $m \in [1,q)$. Then, the energy solutions to (3.29) satisfy the $(L^m \cap L^q) - L^q$ estimates*

$$\|u(t,\cdot)\|_{L^q} \lesssim \begin{cases} t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\|u_0\|_{L^m\cap L^q}+t^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\|u_1\|_{L^m\cap L^q} & \text{if } t\in(0,1], \\ (1+t)^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})}\|u_0\|_{L^m\cap L^q} \\ \qquad +(1+t)^{1-\frac{n+2(\sigma-2\delta)}{2(\sigma-\delta)}(1-\frac{1}{r})}\|u_1\|_{L^m\cap L^q} & \text{if } t\in[1,\infty), \end{cases}$$

$$\big\||D|^\sigma u(t,\cdot)\big\|_{L^q} \lesssim \begin{cases} t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\big(\|u_0\|_{L^m\cap H^\sigma_q}+\|u_1\|_{L^m\cap L^q}\big) & \text{if } t\in(0,1], \\ (1+t)^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\sigma}{2(\sigma-\delta)}}\|u_0\|_{L^m\cap H^\sigma_q} \\ \qquad +(1+t)^{1-\frac{n+2(\sigma-2\delta)}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\sigma}{2(\sigma-\delta)}}\|u_1\|_{L^m\cap L^q} & \text{if } t\in[1,\infty), \end{cases}$$

$$\|u_t(t,\cdot)\|_{L^q} \lesssim \begin{cases} t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\big(\|u_0\|_{L^m\cap H^\sigma_q}+\|u_1\|_{L^m\cap L^q}\big) & \text{if } t\in(0,1], \\ (1+t)^{-\frac{n+2(\sigma-2\delta)}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}}\|u_0\|_{L^m\cap H^\sigma_q} \\ \qquad +(1+t)^{1-\frac{n+2(\sigma-2\delta)}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}}\|u_1\|_{L^m\cap L^q} & \text{if } t\in[1,\infty), \end{cases}$$

and the $L^q - L^q$ estimates

$$\|u(t,\cdot)\|_{L^q} \lesssim \begin{cases} t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\|u_0\|_{L^q}+t^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\|u_1\|_{L^q} & \text{if } t\in(0,1], \\ \|u_0\|_{L^q}+(1+t)\|u_1\|_{L^q} & \text{if } t\in[1,\infty), \end{cases}$$

$$\big\||D|^\sigma u(t,\cdot)\big\|_{L^q} \lesssim \begin{cases} t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\big(\|u_0\|_{H^\sigma_q}+\|u_1\|_{L^q}\big) & \text{if } t\in(0,1], \\ (1+t)^{-\frac{\sigma}{2(\sigma-\delta)}}\|u_0\|_{H^\sigma_q}+(1+t)^{1-\frac{\sigma}{2(\sigma-\delta)}}\|u_1\|_{L^q} & \text{if } t\in[1,\infty), \end{cases}$$

$$\|u_t(t,\cdot)\|_{L^q} \lesssim \begin{cases} t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\big(\|u_0\|_{H^\sigma_q}+\|u_1\|_{L^q}\big) & \text{if } t\in(0,1], \\ (1+t)^{-\frac{\delta}{\sigma-\delta}}\|u_0\|_{H^\sigma_q}+(1+t)^{1-\frac{\delta}{\sigma-\delta}}\|u_1\|_{L^q} & \text{if } t\in[1,\infty), \end{cases}$$

where $1+\frac{1}{q}=\frac{1}{r}+\frac{1}{m}$ and the constraint condition to the space dimension $n > 2\delta$.

Proof. For space dimensions $n > 2\delta$ we may improve the estimate for the norm

$$\big\|\mathfrak{F}^{-1}\big(\widehat{K_1}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^\infty}$$

in Proposition 3.1.8 to obtain a better estimate. Namely, because of the asymptotic behavior of the characteristic roots, we obtain for small frequencies the following estimate:

$$|\widehat{K_1}(t,\xi)| \lesssim \frac{e^{-|\xi|^{2(\sigma-\delta)}t} + e^{-|\xi|^{2\delta}t}}{|\xi|^{2\delta}} \lesssim |\xi|^{-2\delta} e^{-|\xi|^{2(\sigma-\delta)}t}.$$

Hence, we derive

$$\begin{aligned}\big|\mathfrak{F}^{-1}\big(\widehat{K_1}(t,\xi)\chi_1(|\xi|)\big)(t,x)\big| &\lesssim \Big|\int_{\mathbb{R}^n} e^{ix\xi}\widehat{K_1}(t,\xi)\chi_1(|\xi|)d\xi\Big| \\ &\lesssim \int_0^1 e^{-|\xi|^{2(\sigma-\delta)}t}|\xi|^{n-2\delta-1}d|\xi| \lesssim t^{-\frac{n-2\delta}{2(\sigma-\delta)}} = t^{1-\frac{n+2(\sigma-2\delta)}{2(\sigma-\delta)}}\end{aligned}$$

for t large and under the restriction to the dimension $n > 2\delta$. For this reason, it follows for any $a \geqslant 0$ the estimate

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_1}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^\infty} \lesssim \begin{cases} t & \text{if } t \in (0,1], \\ t^{1-\frac{n+2(\sigma-2\delta)+a}{2(\sigma-\delta)}} & \text{if } t \in [1,\infty). \end{cases}$$

Using again Proposition 3.1.2 and the remaining estimates in Proposition 3.1.8 we get

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_0}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^r} \lesssim \begin{cases} 1 & \text{if } t \in (0,1], \\ t^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{a}{2(\sigma-\delta)}} & \text{if } t \in [1,\infty), \end{cases}$$

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_1}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^r} \lesssim \begin{cases} t & \text{if } t \in (0,1], \\ t^{1-\frac{n+2(\sigma-2\delta)}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{a}{2(\sigma-\delta)}} & \text{if } t \in [1,\infty), \end{cases}$$

for all $r \in [1,\infty]$ and any non-negative number a. Then, repeating an analogous approach to prove the statements of Theorems 3.1.1 and 3.2.1 we may conclude all the desired estimates. □

$L^m - L^q$ estimates

Corollary 3.3.1. *Let $\delta \in \big(0, \frac{\sigma}{2}\big)$ in (3.29), $q \in [1,\infty]$ and $m \in [1,q]$. Then, the energy solutions to (3.29) satisfy the $L^m - L^q$ estimates*

$$\|u(t,\cdot)\|_{L^q} \lesssim \begin{cases} t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|u_0\|_{L^m} \\ \quad +t^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|u_1\|_{L^m} & \text{if } t \in (0,1], \\ t^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})}\|u_0\|_{L^m} \\ \quad +t^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})}\|u_1\|_{L^m} & \text{if } t \in [1,\infty), \end{cases}$$

$$\big\||D|^\sigma u(t,\cdot)\big\|_{L^q} \lesssim \begin{cases} t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{\sigma}{2\delta}}\|u_0\|_{L^m} \\ \quad +t^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{\sigma}{2\delta}}\|u_1\|_{L^m} & \text{if } t \in (0,1], \\ t^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\sigma}{2(\sigma-\delta)}}\|u_0\|_{L^m} \\ \quad +t^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\sigma}{2(\sigma-\delta)}}\|u_1\|_{L^m} & \text{if } t \in [1,\infty), \end{cases}$$

$$\|u_t(t,\cdot)\|_{L^q} \lesssim \begin{cases} t^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{\sigma}{\delta}}\|u_0\|_{L^m} \\ \quad +t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|u_1\|_{L^m} & \text{if } t \in (0,1], \\ t^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}}\|u_0\|_{L^m} \\ \quad +t^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}}\|u_1\|_{L^m} & \text{if } t \in [1,\infty), \end{cases}$$

where $1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{m}$ and for all dimensions $n \geqslant 1$.

3.4. Linear σ-evolution models with friction or external damping

The main purpose of this section is to study the σ-evolution models with friction or external damping in the following form:

$$u_{tt} + (-\Delta)^\sigma u + u_t = 0, \quad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) \tag{3.30}$$

with $\sigma \geqslant 1$. Our goal is to obtain $L^q - L^q$ estimates for solutions to (3.30) assuming additional L^m regularity for the data with $m \in [1,q)$, where $q \in (1,\infty)$ is given.

Using partial Fourier transformation to (3.30) we obtain the Cauchy problem for $\hat{u}(t,\xi) := \mathfrak{F}(u(t,x))$, $\widehat{u_0}(\xi) := \mathfrak{F}(u_0(x))$ and $\widehat{u_1}(\xi) := \mathfrak{F}(u_1(x))$ as follows:

$$\hat{u}_{tt} + \hat{u}_t + |\xi|^{2\sigma}\hat{u} = 0, \quad \hat{u}(0,\xi) = \widehat{u_0}(\xi), \quad \hat{u}_t(0,\xi) = \widehat{u_1}(\xi). \tag{3.31}$$

The characteristic roots are

$$\lambda_{1,2} = \lambda_{1,2}(\xi) = \frac{1}{2}\Big(-1 \pm \sqrt{1-4|\xi|^{2\sigma}}\Big).$$

The solutions to (3.31) are presented by the following formula (here we assume $\lambda_1 \neq \lambda_2$):

$$\hat{u}(t,\xi) = \frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2}\widehat{u_0}(\xi) + \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}\widehat{u_1}(\xi) =: \widehat{K_0}(t,\xi)\widehat{u_0}(\xi) + \widehat{K_1}(t,\xi)\widehat{u_1}(\xi).$$

Taking account of the cases of small and large frequencies separately we have

1. $\lambda_{1,2} = \lambda_{1,2}(\xi) = \frac{1}{2}\big(-1 \pm \sqrt{1-4|\xi|^{2\sigma}}\big)$
   and $\lambda_1 \sim -|\xi|^{2\sigma}, \quad \lambda_2 \sim -1, \quad \lambda_1 - \lambda_2 \sim 1 \quad$ for $|\xi| \in (0, 4^{-\frac{1}{\sigma}})$,
2. $\lambda_{1,2} = \lambda_{1,2}(\xi) = \frac{1}{2}\big(-1 \pm i\sqrt{4|\xi|^{2\sigma}-1}\big)$
   and $\lambda_{1,2} \sim -1 \pm i|\xi|^\sigma, \quad \lambda_1 - \lambda_2 \sim i|\xi|^\sigma \quad$ for $|\xi| \in (4^{\frac{1}{\sigma}}, \infty)$.

Let $\chi_k = \chi_k(|\xi|)$ with $k = 1,2,3$ be smooth cut-off functions having the following properties:

$$\chi_1(|\xi|) = \begin{cases} 1 & \text{if } |\xi| \leqslant 4^{-\frac{1}{\sigma}}, \\ 0 & \text{if } |\xi| \geqslant 3^{-\frac{1}{\sigma}}, \end{cases} \qquad \chi_3(|\xi|) = \begin{cases} 1 & \text{if } |\xi| \geqslant 4^{\frac{1}{\sigma}}, \\ 0 & \text{if } |\xi| \leqslant 3^{\frac{1}{\sigma}}, \end{cases}$$

and $\chi_2(|\xi|) = 1 - \chi_1(|\xi|) - \chi_3(|\xi|)$.

We note that $\chi_2(|\xi|) = 1$ if $3^{-\frac{1}{\sigma}} \leqslant |\xi| \leqslant 3^{\frac{1}{\sigma}}$ and $\chi_2(|\xi|) = 0$ if $|\xi| \leqslant 4^{-\frac{1}{\sigma}}$ or $|\xi| \geqslant 4^{\frac{1}{\sigma}}$. Let us now decompose the solutions to (3.30) into three parts localized separately to low, middle and high frequencies, that is,

$$u(t,x) = u_{\chi_1}(t,x) + u_{\chi_2}(t,x) + u_{\chi_3}(t,x),$$

where

$$u_{\chi_k}(t,x) = \mathfrak{F}^{-1}\big(\chi_k(|\xi|)\hat{u}(t,\xi)\big) \quad \text{with } k = 1,2,3.$$

$L^m - L^q$ estimates for small frequencies

Following the approach of the proof of Proposition 3.1.2 we obtain the following L^1 estimates for small frequencies.

Proposition 3.4.1. *The estimates*

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a \widehat{K_0}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^1} \lesssim \begin{cases} 1 & \text{if } t \in (0,1], \\ t^{-\frac{a}{2\sigma}} & \text{if } t \in [1,\infty), \end{cases}$$

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a \widehat{K_1}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^1} \lesssim \begin{cases} t & \text{if } t \in (0,1], \\ t^{1-\frac{a}{2\sigma}} & \text{if } t \in [1,\infty), \end{cases}$$

hold for any non-negative number a.

Following the approach of the proof of Proposition 3.1.10 we obtain the following L^∞ estimates for small frequencies.

Proposition 3.4.2. *The following estimates hold in* $\mathbb{R}^n$*:*

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a \widehat{K_0}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^\infty} \lesssim \begin{cases} 1 & \text{if } t \in (0,1], \\ t^{-\frac{n+a}{2\sigma}} & \text{if } t \in [1,\infty), \end{cases}$$

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a \widehat{K_1}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^\infty} \lesssim \begin{cases} t & \text{if } t \in (0,1], \\ t^{1-\frac{n+a}{2\sigma}} & \text{if } t \in [1,\infty), \end{cases}$$

for any non-negative number a*.*

By interpolation theorem, from the statements of Propositions 3.4.1 and 3.4.2 we may conclude the following L^r estimates for small frequencies.

Proposition 3.4.3. *The following estimates hold in* $\mathbb{R}^n$*:*

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a \widehat{K_0}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^r} \lesssim (1+t)^{-\frac{n}{2\sigma}(1-\frac{1}{r})-\frac{a}{2\sigma}},$$

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a \widehat{K_1}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^r} \lesssim (1+t)^{1-\frac{n}{2\sigma}(1-\frac{1}{r})-\frac{a}{2\sigma}},$$

for all $r \in [1,\infty]$ *and any non-negative number* a*.*

Finally, we can conclude the following result.

Theorem 3.4.1. *Let* $q \in [1,\infty]$ *and* $m \in [1,q]$*. Then, the Sobolev solutions to (3.30) satisfy the* $L^m - L^q$ *estimates*

$$\big\||D|^a u_{\chi_1}(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{-\frac{n}{2\sigma}(1-\frac{1}{r})-\frac{a}{2\sigma}}\|u_0\|_{L^m} + (1+t)^{1-\frac{n}{2\sigma}(1-\frac{1}{r})-\frac{a}{2\sigma}}\|u_1\|_{L^m},$$

$$\big\|\partial_t |D|^a u_{\chi_1}(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{-\frac{n}{2\sigma}(1-\frac{1}{r})-\frac{a}{2\sigma}}\|u_0\|_{L^m} + (1+t)^{1-\frac{n}{2\sigma}(1-\frac{1}{r})-\frac{a}{2\sigma}}\|u_1\|_{L^m},$$

where $1+\frac{1}{q} = \frac{1}{r}+\frac{1}{m}$ *and for all* $a \geqslant 0$*.*

Proof. In order to prove the first statement, we apply Young's convolution inequality from Proposition B.1.1 as we did in the proofs of Theorems 3.1.1 and 3.2.1 and use the statements in Proposition 3.4.3. Taking account of some estimates related to derivative in time we note that

$$\partial_t \widehat{K_0}(t,\xi) = -|\xi|^{2\sigma}\widehat{K_1}(t,\xi) \quad \text{and} \quad \partial_t \widehat{K_1}(t,\xi) = \widehat{K_0}(t,\xi) - \widehat{K_1}(t,\xi).$$

Then, applying again Young's convolution inequality from Proposition B.1.1 and Proposition 3.4.3, we may conclude the second statement. Hence, the proof of Theorem 3.4.1 is completed. □

$L^q - L^q$ estimates for large frequencies

First we recall that the characteristic roots are

$$\lambda_{1,2} = \lambda_{1,2}(\xi) = \frac{1}{2}\Big(-1 \pm i\sqrt{4|\xi|^{2\sigma}-1}\Big)$$

for large $|\xi|$. We re-write $\widehat{K_0}(t,\xi)$ and $\widehat{K_1}(t,\xi)$ as follows:

$$\widehat{K_0}(t,\xi) = e^{-\frac{t}{2}}\cos\Big(\sqrt{|\xi|^{2\sigma}-\frac{1}{4}}t\Big) + e^{-\frac{t}{2}}\frac{\sin\big(\sqrt{|\xi|^{2\sigma}-\frac{1}{4}}t\big)}{2\sqrt{|\xi|^{2\sigma}-\frac{1}{4}}},$$

and

$$\widehat{K_1}(t,\xi) = e^{-\frac{t}{2}}\frac{\sin\big(\sqrt{|\xi|^{2\sigma}-\frac{1}{4}}t\big)}{\sqrt{|\xi|^{2\sigma}-\frac{1}{4}}}.$$

Now we denote

$$K_0^{cos}(t,x) := \mathfrak{F}^{-1}\Big(e^{-\frac{t}{2}}\cos\Big(\sqrt{|\xi|^{2\sigma}-\frac{1}{4}}t\Big)\widehat{u_0}(\xi)\chi_3(|\xi|)\Big)(t,x),$$

$$K_0^{sin}(t,x) := \mathfrak{F}^{-1}\Big(e^{-\frac{t}{2}}\frac{\sin\big(\sqrt{|\xi|^{2\sigma}-\frac{1}{4}}t\big)}{2\sqrt{|\xi|^{2\sigma}-\frac{1}{4}}}\widehat{u_0}(\xi)\chi_3(|\xi|)\Big)(t,x),$$

$$K_1^{sin}(t,x) := \mathfrak{F}^{-1}\Big(e^{-\frac{t}{2}}\frac{\sin\big(\sqrt{|\xi|^{2\sigma}-\frac{1}{4}}t\big)}{\sqrt{|\xi|^{2\sigma}-\frac{1}{4}}}\widehat{u_1}(\xi)\chi_3(|\xi|)\Big)(t,x).$$

We shall prove the following results.

Proposition 3.4.4. *The following estimates hold in* $\mathbb{R}^n$*:*

$$\big\|\partial_t^j |D|^a K_0^{cos}(t,\cdot)\big\|_{L^q} \lesssim e^{-ct}\|u_0\|_{H_q^{a+j\sigma+s_0}},$$
$$\big\|\partial_t^j |D|^a K_0^{sin}(t,\cdot)\big\|_{L^q} \lesssim e^{-ct}\|u_0\|_{H_q^{[a+(j-1)\sigma+s_0]^+}},$$
$$\big\|\partial_t^j |D|^a K_1^{sin}(t,\cdot)\big\|_{L^q} \lesssim e^{-ct}\|u_0\|_{H_q^{[a+(j-1)\sigma+s_0]^+}},$$

for any $t>0$, $a \geqslant 0$, integer $j \geqslant 0$ and a suitable positive constant c. Here we denote $s_0 := n\sigma\big|\frac{1}{q}-\frac{1}{2}\big|$.

According to view of the Mikhlin- Hörmander multiplier theorem in [9] and [49] and its applications of the Fourier multipliers, in order to Theorem 3.4.2, we need to show the following auxiliary estimates.

Lemma 3.4.1. *The following estimates hold in* $\mathbb{R}^n$ *for large* $|\xi|$*:*

$$\big|\partial_\xi^\alpha |\xi|^{2\sigma}\big| \lesssim |\xi|^{2\sigma-|\alpha|} \text{ for all } \alpha, \tag{3.32}$$
$$\big|\partial_\xi^\alpha |\xi|^{2p\sigma}\big| \lesssim |\xi|^{2p\sigma-|\alpha|} \text{ for all } \alpha \text{ and } p \in \mathbb{R}, \tag{3.33}$$
$$\Big|\partial_\xi^\alpha\Big(\sqrt{|\xi|^{2\sigma}-\frac{1}{4}}\Big)\Big| \lesssim |\xi|^{\sigma-|\alpha|} \text{ for all } \alpha, \tag{3.34}$$
$$\Big|\partial_\xi^\alpha\Big(\sqrt{|\xi|^{2\sigma}-\frac{1}{4}}\Big)^p\Big| \lesssim |\xi|^{p\sigma-|\alpha|} \text{ for all } \alpha \text{ and } p \in \mathbb{R}, \tag{3.35}$$
$$\Big|\partial_\xi^\alpha \cos\Big(\sqrt{|\xi|^{2\sigma}-\frac{1}{4}}t\Big)\Big| \lesssim \big(t+t^{|\alpha|}\big)|\xi|^{\sigma|\alpha|-|\alpha|} \text{ for all } \alpha \text{ and } t>0, \tag{3.36}$$
$$\Big|\partial_\xi^\alpha \sin\Big(\sqrt{|\xi|^{2\sigma}-\frac{1}{4}}t\Big)\Big| \lesssim \big(t+t^{|\alpha|}\big)|\xi|^{\sigma|\alpha|-|\alpha|} \text{ for all } \alpha \text{ and } t>0, \tag{3.37}$$
$$\Big|\partial_\xi^\alpha\Big(\Big(\sqrt{|\xi|^{2\sigma}-\frac{1}{4}}\Big)^j\cos\Big(\sqrt{|\xi|^{2\sigma}-\frac{1}{4}}t\Big)\Big)\Big| \lesssim \big(1+t^{|\alpha|}\big)|\xi|^{j\sigma+\sigma|\alpha|-|\alpha|}, \tag{3.38}$$
for all α, for any $j \geqslant 0$ and $t>0$,
$$\Big|\partial_\xi^\alpha\Big(\Big(\sqrt{|\xi|^{2\sigma}-\frac{1}{4}}\Big)^{j-1}\sin\Big(\sqrt{|\xi|^{2\sigma}-\frac{1}{4}}t\Big)\Big)\Big| \lesssim \big(1+t^{|\alpha|}\big)|\xi|^{(j-1)\sigma+\sigma|\alpha|-|\alpha|}, \tag{3.39}$$
for all α, for any $j \geqslant 0$ and $t>0$,
$$\Big|\partial_\xi^\alpha\Big(|\xi|^b\Big(\sqrt{|\xi|^{2\sigma}-\frac{1}{4}}\Big)^j\cos\Big(\sqrt{|\xi|^{2\sigma}-\frac{1}{4}}t\Big)\Big)\Big| \lesssim \big(1+t^{|\alpha|}\big)|\xi|^{b+j\sigma+\sigma|\alpha|-|\alpha|}, \tag{3.40}$$
for all α, for any $b \in \mathbb{R}$, $j \geqslant 0$ and $t>0$,
$$\Big|\partial_\xi^\alpha\Big(|\xi|^b\Big(\sqrt{|\xi|^{2\sigma}-\frac{1}{4}}\Big)^{j-1}\sin\Big(\sqrt{|\xi|^{2\sigma}-\frac{1}{4}}t\Big)\Big)\Big| \lesssim \big(1+t^{|\alpha|}\big)|\xi|^{b+(j-1)\sigma+\sigma|\alpha|-|\alpha|}, \tag{3.41}$$
for all α, for any $b \in \mathbb{R}$, $j \geqslant 0$ and $t>0$.

Proof. In order to prove all statements in Lemma 3.4.1, we shall apply Lemma B.6.2 and Leibniz rule for multivariable calculus. Indeed, we will indicate the proof of the above estimates as follows:

To (3.32): Applying Lemma B.6.2 with $h(s)=s^\sigma$ and $f(\xi)=|\xi|^2$ we derive

$$\begin{aligned}\big|\partial_\xi^\alpha|\xi|^{2\sigma}\big| &= \Big|\sum_{k=1}^{|\alpha|} h^{(k)}\big(|\xi|^2\big)\Big(\sum_{\substack{\gamma_1+\cdots+\gamma_k\leqslant\alpha\\ |\gamma_1|+\cdots+|\gamma_k|=|\alpha|,\,|\gamma_i|\geqslant 1}} \partial_\xi^{\gamma_1}\big(|\xi|^2\big)\cdots\partial_\xi^{\gamma_k}\big(|\xi|^2\big)\Big)\Big|\\ &\lesssim \sum_{k=1}^{|\alpha|}|\xi|^{2(\sigma-k)}\Big(\sum_{\substack{\gamma_1+\cdots+\gamma_k\leqslant\alpha\\ |\gamma_1|+\cdots+|\gamma_k|=|\alpha|,\,|\gamma_i|\geqslant 1}}|\xi|^{2-|\gamma_1|+\cdots+2-|\gamma_k|}\Big)\\ &\lesssim \sum_{k=1}^{|\alpha|}|\xi|^{2(\sigma-k)+2k-|\alpha|} \lesssim |\xi|^{2\sigma-|\alpha|}.\end{aligned}$$

To (3.33): Applying Lemma B.6.2 with $h(s) = s^p$ and $f(\xi) = |\xi|^{2\sigma}$ we have

$$\begin{aligned}
\big|\partial_\xi^\alpha |\xi|^{2p\sigma}\big| &= \Big|\sum_{k=1}^{|\alpha|} h^{(k)}\big(|\xi|^{2\sigma}\big)\Big(\sum_{\substack{\gamma_1+\cdots+\gamma_k\leqslant\alpha\\ |\gamma_1|+\cdots+|\gamma_k|=|\alpha|,\,|\gamma_i|\geqslant 1}} \partial_\xi^{\gamma_1}\big(|\xi|^{2\sigma}\big)\cdots\partial_\xi^{\gamma_k}\big(|\xi|^{2\sigma}\big)\Big)\Big|\\
&\lesssim \sum_{k=1}^{|\alpha|} |\xi|^{2\sigma(p-k)}\Big(\sum_{\substack{\gamma_1+\cdots+\gamma_k\leqslant\alpha\\ |\gamma_1|+\cdots+|\gamma_k|=|\alpha|,\,|\gamma_i|\geqslant 1}} |\xi|^{2\sigma-|\gamma_1|+\cdots+2\sigma-|\gamma_k|}\Big)\\
&\lesssim \sum_{k=1}^{|\alpha|} |\xi|^{2\sigma(p-k)+2\sigma k-|\alpha|} \lesssim |\xi|^{2p\sigma-|\alpha|}.
\end{aligned}$$

To (3.34): Applying Lemma B.6.2 with $h(s) = s^{\frac{1}{2}}$ and $f(\xi) = |\xi|^{2\sigma} - \frac{1}{4}$ we have

$$\begin{aligned}
\Big|\partial_\xi^\alpha\Big(\sqrt{|\xi|^{2\sigma}-\frac{1}{4}}\Big)\Big| &= \Big|\sum_{k=1}^{|\alpha|} h^{(k)}\Big(|\xi|^{2\sigma}-\frac{1}{4}\Big)\Big(\sum_{\substack{\gamma_1+\cdots+\gamma_k\leqslant\alpha\\ |\gamma_1|+\cdots+|\gamma_k|=|\alpha|,\,|\gamma_i|\geqslant 1}} \partial_\xi^{\gamma_1}\Big(|\xi|^{2\sigma}-\frac{1}{4}\Big)\cdots\partial_\xi^{\gamma_k}\Big(|\xi|^{2\sigma}-\frac{1}{4}\Big)\Big)\Big|\\
&\lesssim \sum_{k=1}^{|\alpha|} \Big(|\xi|^{2\sigma}-\frac{1}{4}\Big)^{\frac{1}{2}-k}\Big(\sum_{\substack{\gamma_1+\cdots+\gamma_k\leqslant\alpha\\ |\gamma_1|+\cdots+|\gamma_k|=|\alpha|,\,|\gamma_i|\geqslant 1}} |\xi|^{2\sigma-|\gamma_1|+\cdots+2\sigma-|\gamma_k|}\Big)\\
&\lesssim \sum_{k=1}^{|\alpha|} \Big(|\xi|^{2\sigma}-\frac{1}{4}\Big)^{\frac{1}{2}-k}|\xi|^{2\sigma k-|\alpha|} \lesssim |\xi|^{\sigma-|\alpha|}.
\end{aligned}$$

To (3.35): Applying Lemma B.6.2 with $h(s) = s^p$ and $f(\xi) = \sqrt{|\xi|^{2\sigma}-\frac{1}{4}}$ we have

$$\begin{aligned}
&\Big|\partial_\xi^\alpha\Big(\sqrt{|\xi|^{2\sigma}-\frac{1}{4}}\Big)^p\Big|\\
&= \Big|\sum_{k=1}^{|\alpha|} h^{(k)}\Big(\sqrt{|\xi|^{2\sigma}-\frac{1}{4}}\Big)\Big(\sum_{\substack{\gamma_1+\cdots+\gamma_k\leqslant\alpha\\ |\gamma_1|+\cdots+|\gamma_k|=|\alpha|,\,|\gamma_i|\geqslant 1}} \partial_\xi^{\gamma_1}\Big(\sqrt{|\xi|^{2\sigma}-\frac{1}{4}}\Big)\cdots\partial_\xi^{\gamma_k}\Big(\sqrt{|\xi|^{2\sigma}-\frac{1}{4}}\Big)\Big)\Big|\\
&\lesssim \sum_{k=1}^{|\alpha|} \Big(\sqrt{|\xi|^{2\sigma}-\frac{1}{4}}\Big)^{p-k}\Big(\sum_{\substack{\gamma_1+\cdots+\gamma_k\leqslant\alpha\\ |\gamma_1|+\cdots+|\gamma_k|=|\alpha|,\,|\gamma_i|\geqslant 1}} |\xi|^{\sigma-|\gamma_1|+\cdots+\sigma-|\gamma_k|}\Big)\\
&\lesssim \sum_{k=1}^{|\alpha|} \Big(\sqrt{|\xi|^{2\sigma}-\frac{1}{4}}\Big)^{p-k}|\xi|^{\sigma k-|\alpha|} \lesssim |\xi|^{p\sigma-|\alpha|}.
\end{aligned}$$

To (3.36): Applying Lemma B.6.2 with $h(s) = \cos(st)$ and $f(\xi) = \sqrt{|\xi|^{2\sigma}-\frac{1}{4}}$ we have

$$\begin{aligned}
&\Big|\partial_\xi^\alpha \cos\Big(\sqrt{|\xi|^{2\sigma}-\frac{1}{4}}t\Big)\Big|\\
&= \Big|\sum_{k=1}^{|\alpha|} h^{(k)}\Big(\sqrt{|\xi|^{2\sigma}-\frac{1}{4}}\Big)\Big(\sum_{\substack{\gamma_1+\cdots+\gamma_k\leqslant\alpha\\ |\gamma_1|+\cdots+|\gamma_k|=|\alpha|,\,|\gamma_i|\geqslant 1}} \partial_\xi^{\gamma_1}\Big(\sqrt{|\xi|^{2\sigma}-\frac{1}{4}}\Big)\cdots\partial_\xi^{\gamma_k}\Big(\sqrt{|\xi|^{2\sigma}-\frac{1}{4}}\Big)\Big)\Big|\\
&\lesssim \sum_{k=1}^{|\alpha|} t^k \cos^{(k)}\Big(\sqrt{|\xi|^{2\sigma}-\frac{1}{4}}t\Big)\Big(\sum_{\substack{\gamma_1+\cdots+\gamma_k\leqslant\alpha\\ |\gamma_1|+\cdots+|\gamma_k|=|\alpha|,\,|\gamma_i|\geqslant 1}} |\xi|^{\sigma-|\gamma_1|+\cdots+\sigma-|\gamma_k|}\Big)\\
&\lesssim \sum_{k=1}^{|\alpha|} t^k|\xi|^{\sigma k-|\alpha|} \lesssim \big(t+t^{|\alpha|}\big)|\xi|^{\sigma|\alpha|-|\alpha|}.
\end{aligned}$$

To (3.37): In the same argument as we did in (3.36), we can conclude (3.37).

To (3.38): Combining (3.35), (3.36) we can conclude (3.38) by using the Leibniz rule.

To (3.39): Combining (3.35), (3.37) we can conclude (3.39) by using the Leibniz rule.

To (3.40): Combining (3.33), (3.38) we can conclude (3.40) by using the Leibniz rule.

To (3.41): Combining (3.33), (3.39) we can conclude (3.41) by using the Leibniz rule. □

Proof of Proposition 3.4.4. Firstly, taking account of estimates for $K_0^{cos}(t,x)$ and some of its partial derivatives we write

$$\partial_t^j |D|^a K_0^{cos}(t,x) = \mathfrak{F}^{-1}\Big(e^{-\frac{t}{2}}|\xi|^{-j\sigma-s_0}\Big(\sqrt{|\xi|^{2\sigma}-\frac{1}{4}}\Big)^j \cos\Big(\sqrt{|\xi|^{2\sigma}-\frac{1}{4}}t\Big)\chi_3(|\xi|)|\xi|^{a+j\sigma+s_0}\widehat{u_0}(\xi)\Big)(t,x).$$

By choosing $b=-j\sigma-s_0$ in (3.40), we get

$$\begin{aligned}&\Big|\partial_\xi^\alpha\Big(e^{-\frac{t}{2}}|\xi|^{-j\sigma-s_0}\Big(\sqrt{|\xi|^{2\sigma}-\frac{1}{4}}\Big)^j \cos\Big(\sqrt{|\xi|^{2\sigma}-\frac{1}{4}}t\Big)\Big)\Big|\\ &\qquad\lesssim e^{-\frac{t}{2}}\big(1+t^{|\alpha|}\big)|\xi|^{-s_0}\big(|\xi|^{\sigma-1}\big)^{-|\alpha|} \lesssim e^{-ct}|\xi|^{-n\sigma|\frac{1}{q}-\frac{1}{2}|}\big(|\xi|^{\sigma-1}\big)^{-|\alpha|},\end{aligned}$$

where c is a suitable positive constant. By Proposition B.5.1, we may arrive at

$$\begin{aligned}\big\|\partial_t^j|D|^aK_0^{cos}(t,\cdot)\big\|_{L^q} &\lesssim e^{-ct}\Big\|\mathfrak{F}^{-1}\Big(|\xi|^{a+j\sigma+s_0}\widehat{u_0}(\xi)\Big)\Big\|_{L^q}\\ &\lesssim e^{-ct}\|u_0\|_{\dot{H}_q^{a+j\sigma+s_0}} \lesssim e^{-ct}\|u_0\|_{H_q^{a+j\sigma+s_0}}.\end{aligned}$$

Now in order to estimate K_0^{sin} and some of its partial derivatives, we will divide our considerations into two cases. In the first case, if $a+(j-1)\sigma+s_0\geqslant 0$, then we write

$$\begin{aligned}&\partial_t^j|D|^aK_0^{sin}(t,x)\\ &\quad=\mathfrak{F}^{-1}\Big(\frac{1}{2}|\xi|^{-(j-1)\sigma-s_0}\Big(\sqrt{|\xi|^{2\sigma}-\frac{1}{4}}\Big)^{j-1}\sin\Big(\sqrt{|\xi|^{2\sigma}-\frac{1}{4}}t\Big)\chi_3(|\xi|)|\xi|^{a+(j-1)\sigma+s_0}\widehat{u_0}(\xi)\Big)(t,x).\end{aligned}$$

By choosing $b=-(j-1)\sigma-s_0$ in (3.41), we get

$$\begin{aligned}&\Big|\partial_\xi^\alpha\Big(|\xi|^{-(j-1)\sigma-s_0}\Big(\sqrt{|\xi|^{2\sigma}-\frac{1}{4}}\Big)^{j-1}\sin\Big(\sqrt{|\xi|^{2\sigma}-\frac{1}{4}}t\Big)\Big)\Big|\\ &\qquad\lesssim e^{-\frac{t}{2}}\big(1+t^{|\alpha|}\big)|\xi|^{-s_0}\big(|\xi|^{\sigma-1}\big)^{-|\alpha|} \lesssim e^{-ct}|\xi|^{-n\sigma|\frac{1}{q}-\frac{1}{2}|}\big(|\xi|^{\sigma-1}\big)^{-|\alpha|},\end{aligned}$$

where c is a suitable positive constant. By Proposition B.5.1, we may conclude

$$\begin{aligned}\big\|\partial_t^j|D|^aK_0^{sin}(t,\cdot)\big\|_{L^q} &\lesssim e^{-ct}\Big\|\mathfrak{F}^{-1}\Big(|\xi|^{a+(j-1)\sigma+s_0}\widehat{u_0}(\xi)\Big)\Big\|_{L^q}\\ &\lesssim e^{-ct}\|u_0\|_{\dot{H}_q^{a+(j-1)\sigma+s_0}} \lesssim e^{-ct}\|u_0\|_{H_q^{a+(j-1)\sigma+s_0}}.\end{aligned}$$

In the second case, if $a+(j-1)\sigma+s_0<0$, then we write

$$\partial_t^j|D|^aK_0^{sin}(t,x)=\mathfrak{F}^{-1}\Big(\frac{1}{2}|\xi|^a\Big(\sqrt{|\xi|^{2\sigma}-\frac{1}{4}}\Big)^{j-1}\sin\Big(\sqrt{|\xi|^{2\sigma}-\frac{1}{4}}t\Big)\chi_3(|\xi|)\widehat{u_0}(\xi)\Big)(t,x).$$

By choosing $b=a$ in (3.41), we get

$$\begin{aligned}&\Big|\partial_\xi^\alpha\Big(|\xi|^a\Big(\sqrt{|\xi|^{2\sigma}-\frac{1}{4}}\Big)^{j-1}\sin\Big(\sqrt{|\xi|^{2\sigma}-\frac{1}{4}}t\Big)\Big)\Big|\\ &\qquad\lesssim e^{-\frac{t}{2}}\big(1+t^{|\alpha|}\big)|\xi|^{a+(j-1)\sigma}\big(|\xi|^{\sigma-1}\big)^{-|\alpha|} \lesssim e^{-ct}|\xi|^{-s_0}\big(|\xi|^{\sigma-1}\big)^{-|\alpha|}\\ &\qquad\lesssim e^{-ct}|\xi|^{-n\sigma|\frac{1}{q}-\frac{1}{2}|}\big(|\xi|^{\sigma-1}\big)^{-|\alpha|},\end{aligned}$$

where c is a suitable positive constant. By Proposition B.5.1, we may arrive at

$$\big\|\partial_t^j|D|^aK_0^{sin}(t,\cdot)\big\|_{L^q}\lesssim e^{-ct}\|u_0\|_{L^q}.$$

Therefore, we have shown the estimates

$$\big\|\partial_t^j |D|^a K_0^{sin}(t,\cdot)\big\|_{L^q} \lesssim e^{-ct}\|u_0\|_{H_q^{[a+(j-1)\sigma+s_0]^+}}.$$

In the analogous argument, we may also conclude the following estimates

$$\big\|\partial_t^j |D|^a K_1^{sin}(t,\cdot)\big\|_{L^q} \lesssim e^{-ct}\|u_0\|_{H_q^{[a+(j-1)\sigma+s_0]^+}}.$$

Summarizing, the proof to Proposition 3.4.4 is completed. □

From the statements in Proposition 3.4.4 we obtain immediately the following result.

Theorem 3.4.2. *Let $q \in (1,\infty)$. Then, the Sobolev solutions to (3.30) satisfy the $L^q - L^q$ estimates*

$$\big\|\partial_t^j |D|^a u_{\chi_3}(t,\cdot)\big\|_{L^q} \lesssim e^{-ct}\Big(\|u_0\|_{H_q^{a+j\sigma+s_0}} + \|u_1\|_{H_q^{[a+(j-1)\sigma+s_0]^+}}\Big),$$

where $s_0 = n\sigma\big|\frac{1}{q}-\frac{1}{2}\big|$, for any $t>0$, $a \geqslant 0$, integer $j \geqslant 0$ and a suitable positive constant c.

$L^q - L^q$ estimates for middle frequencies

Now let us turn to consider some estimates for middle frequencies, where $3^{-\frac{1}{\sigma}} \leqslant |\xi| \leqslant 3^{\frac{1}{\sigma}}$. Our goal is to derive the exponential decay for solutions and some of their derivatives to (3.30).

Theorem 3.4.3. *Let $q \in [1,\infty]$. Then, the Sobolev solutions to (3.30) satisfy the $L^q - L^q$ estimates*

$$\big\|\partial_t^j |D|^a u_{\chi_2}(t,\cdot)\big\|_{L^q} \lesssim e^{-ct}\|(u_0,u_1)\|_{L^q}$$

for any $t>0$, $a \geqslant 0$, integer $j \geqslant 0$ and a suitable positive constant c.

Proof. Indeed, following the proof of Proposition 3.1.5 we may arrive at the exponential decay for the following estimates:

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a \partial_t^j \widehat{K_0}(t,\xi)\chi_2(|\xi|)\big)(t,\cdot)\big\|_{L^1} \lesssim e^{-ct},$$
$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a \partial_t^j \widehat{K_1}(t,\xi)\chi_2(|\xi|)\big)(t,\cdot)\big\|_{L^1} \lesssim e^{-ct}.$$

Therefore, applying Young's convolution inequality from Proposition B.1.1 we get

$$\begin{aligned}\big\|\partial_t^j |D|^a u_{\chi_2}(t,\cdot)\big\|_{L^q} &\lesssim \big\|\mathfrak{F}^{-1}\big(|\xi|^a \partial_t^j \widehat{K_0}(t,\xi)\chi_2(|\xi|)\big)(t,\cdot)\big\|_{L^1}\|u_0\|_{L^q}\\ &\quad + \big\|\mathfrak{F}^{-1}\big(|\xi|^a \partial_t^j \widehat{K_1}(t,\xi)\chi_2(|\xi|)\big)(t,\cdot)\big\|_{L^1}\|u_1\|_{L^q}\\ &\lesssim e^{-ct}\|(u_0,u_1)\|_{L^q}.\end{aligned}$$

Summarizing, the proof to Theorem 3.4.3 is completed. □

Finally, from the statements in Theorems 3.4.1, 3.4.2 and 3.4.3 we conclude the following result.

Theorem 3.4.4. *Let $q \in (1,\infty)$ and $m \in [1,q)$. Then, the Sobolev solutions to (3.30) satisfy the $(L^m \cap L^q) - L^q$ estimates*

$$\big\||D|^a u(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{-\frac{n}{2\sigma}(1-\frac{1}{r})-\frac{a}{2\sigma}}\|u_0\|_{L^m\cap H_q^{s_0+a}} + (1+t)^{1-\frac{n}{2\sigma}(1-\frac{1}{r})-\frac{a}{2\sigma}}\|u_1\|_{L^m\cap H_q^{[s_0+a-\sigma]^+}},$$
$$\big\||D|^a u_t(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{-\frac{n}{2\sigma}(1-\frac{1}{r})-\frac{a}{2\sigma}}\|u_0\|_{L^m\cap H_q^{s_0+\sigma+a}} + (1+t)^{1-\frac{n}{2\sigma}(1-\frac{1}{r})-\frac{a}{2\sigma}}\|u_1\|_{L^m\cap H_q^{s_0+a}},$$

and the $L^q - L^q$ estimates

$$\big\||D|^a u(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{-\frac{a}{2\sigma}}\|u_0\|_{H_q^{s_0+a}} + (1+t)^{1-\frac{a}{2\sigma}}\|u_1\|_{H_q^{[s_0+a-\sigma]^+}},$$
$$\big\||D|^a u_t(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{-\frac{a}{2\sigma}}\|u_0\|_{H_q^{s_0+\sigma+a}} + (1+t)^{1-\frac{a}{2\sigma}}\|u_1\|_{H_q^{s_0+a}},$$

for any $a \geqslant 0$, where $1+\frac{1}{q} = \frac{1}{r}+\frac{1}{m}$ and $s_0 = n\sigma\big|\frac{1}{q}-\frac{1}{2}\big|$.

3.5. Comparison with known results

In this section, we explain some comparisons between the above obtained estimates and known results.

3.5.1. The case $\delta \in (0, \frac{\sigma}{2})$

First if we are interested in studying the special case of $\sigma = 1$ and $\delta \in (0, \frac{1}{2})$, in the paper [57] the authors obtained L^1 estimates for oscillating integrals to conclude $L^m - L^q$ estimates not necessarily on the conjugate line for solution as follows:

$$\|u(t,\cdot)\|_{L^q} \lesssim \begin{cases} t^{-([\frac{n}{2}])(\frac{1}{2\delta}-1)\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|u_0\|_{L^m} + t^{1-[\frac{n-2}{2}](\frac{1}{2\delta}-1)\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|u_1\|_{L^m} & \text{if } t \in (0,1], \\ t^{-\frac{n}{2(1-\delta)}(1-\frac{1}{r})}\|u_0\|_{L^m} + t^{1-\frac{n+2-4\delta}{2(1-\delta)}(1-\frac{1}{r})}\|u_1\|_{L^m} & \text{if } t \in [1,\infty), \end{cases}$$

where $1 \leqslant m \leqslant q \leqslant \infty$ and $1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{m}$. Here the authors took into considerations the connection to Fourier multipliers appearing for wave models. The decay rates for solutions produced from the results of [57] are somehow better than those in Corollary 3.3.1 with $\sigma = 1$ and $\delta \in (0, \frac{1}{2})$. However, these decay rates are almost the same if we consider the case of sufficiently large space dimensions n.

In the paper [11], the authors investigated $L^1 \cap L^2 - L^2$ estimates for solutions and some of their derivatives as well in the case $\delta \in (0, \frac{\sigma}{2})$ with additional L^1 regularity for the data as follows:

$$\begin{aligned} \|u(t,\cdot)\|_{L^2} &\lesssim (1+t)^{-\frac{n}{4(\sigma-\delta)}}\|u_0\|_{L^1\cap L^2} + (1+t)^{1-\frac{n}{4(\sigma-\delta)}}\|u_1\|_{L^1\cap L^2}, \\ \big\||D|^\sigma u(t,\cdot)\big\|_{L^2} &\lesssim (1+t)^{-\frac{n+2\sigma}{4(\sigma-\delta)}}\|u_0\|_{L^1\cap H^\sigma} + (1+t)^{1-\frac{n+2\sigma}{4(\sigma-\delta)}}\|u_1\|_{L^1\cap L^2}, \\ \|u_t(t,\cdot)\|_{L^2} &\lesssim (1+t)^{-\frac{n+4\delta}{4(\sigma-\delta)}}\|u_0\|_{L^1\cap H^\sigma} + (1+t)^{1-\frac{n+4\delta}{4(\sigma-\delta)}}\|u_1\|_{L^1\cap L^2}, \end{aligned}$$

for all space dimensions n. Moreover, in [11] there are other sharper results under a restriction to the space dimension. Namely, if $n \geqslant 4\delta$, then the following $L^1 \cap L^2 - L^2$ estimates hold:

$$\begin{aligned} \|u(t,\cdot)\|_{L^2} &\lesssim \begin{cases} (1+t)^{-\frac{n}{4(\sigma-\delta)}}\|u_0\|_{L^1\cap L^2} + (1+t)^{-\frac{n-4\delta}{4(\sigma-\delta)}}\|u_1\|_{L^1\cap L^2} & \text{if } n > 4\delta, \\ (1+t)^{-\frac{n}{4(\sigma-\delta)}}\|u_0\|_{L^1\cap L^2} + \log(e+t)\|u_1\|_{L^1\cap L^2} & \text{if } n = 4\delta, \end{cases} \\ \big\||D|^\sigma u(t,\cdot)\big\|_{L^2} &\lesssim (1+t)^{-\frac{n+2\sigma}{4(\sigma-\delta)}}\|u_0\|_{L^1\cap H^\sigma} + (1+t)^{-\frac{n+2\sigma-4\delta}{4(\sigma-\delta)}}\|u_1\|_{L^1\cap L^2} \quad \text{if } n \geqslant 4\delta, \\ \|u_t(t,\cdot)\|_{L^2} &\lesssim (1+t)^{-\frac{n}{4(\sigma-\delta)}-1}\|u_0\|_{L^1\cap H^\sigma} + (1+t)^{-\frac{n-4\delta}{4(\sigma-\delta)}-1}\|u_1\|_{L^1\cap L^2} \quad \text{if } n \geqslant 4\delta. \end{aligned}$$

We see that these results coincide with those in Theorem 3.3.1 if we only consider large t under the choice of parameters $m = 1$ and $q = 2$ for all space dimensions n. Nevertheless, the authors considered $L^1 \cap L^2 - L^2$ estimates by using Parseval's formula. For this reason, some results in the paper [11] with a restriction to the space dimensions $n \geqslant 4\delta$ are obviously somehow better than those in Theorem 3.3.3 under the assumption of space dimensions $n > 2\delta$.

Finally, we want to mention the paper [9] to emphasize some of recent estimates for solutions to structurally damped σ-evolution equations as follows:

$$\begin{aligned} \big\|\partial_t^k |D|^a u_{\chi_1}(t,\cdot)\big\|_{L^q} &\lesssim (1+t)^{-\frac{1}{2(\sigma-\delta)}\left(n(\frac{1}{q_0}-\frac{1}{q})+a\right)-k}\|u_0\|_{L^{q_0}} \\ &\quad + (1+t)^{-\frac{1}{2(\sigma-\delta)}\left(n(\frac{1}{q_1}-\frac{1}{q})+a-2\delta\right)-k}\|u_1\|_{L^{q_1}} \end{aligned} \tag{3.42}$$

for any $q_0,\, q_1 \geqslant 1$, $q \in [\max\{q_0, q_1\}, \infty]$, $a \geqslant 0$, $k \in \mathbb{N}$ provided that $\frac{1}{q_1} - \frac{1}{q} + \frac{a-2\delta}{n} \geqslant 0$, and

$$\begin{aligned} \big\|\partial_t^k |D|^a u_{\chi_3}(t,\cdot)\big\|_{L^q} &\lesssim e^{-ct} t^{-n\theta(q_0,q)\frac{\sigma-2\delta}{2\delta}-\frac{1}{2\delta}\left(n(\frac{1}{q_0}-\frac{1}{q})+a+k\sigma\right)}\|u_0\|_{L^{q_0}} \\ &\quad + e^{-ct} t^{-n\theta(q_1,q)\frac{\sigma-2\delta}{2\delta}-\frac{1}{2\delta}\left(n(\frac{1}{q_1}-\frac{1}{q})+a+(k-1)\sigma\right)}\|u_1\|_{L^{q_1}} \end{aligned} \tag{3.43}$$

for $1 < q_j \leqslant q < \infty$, $j = 0, 1$ with some positive constant c, where

$$\theta(q_j, q) = \begin{cases} \frac{1}{2} - \frac{1}{q_j} & \text{if } 2 \leqslant q_j, \\ 0 & \text{if } q_j \leqslant 2 \leqslant q, \\ \frac{1}{q} - \frac{1}{2} & \text{if } q \leqslant 2. \end{cases}$$

By choosing the values $q_0 = q_1 = m$ in (3.42) and $q_0 = q_1 = q$ in (3.43), respectively, we may conclude the following estimates:

$$\|u_{\chi_1}(t,\cdot)\|_{L^q} \lesssim (1+t)^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})}\|u_0\|_{L^m} + (1+t)^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})+\frac{\delta}{\sigma-\delta}}\|u_1\|_{L^m},$$
$$\big\||D|^\sigma u_{\chi_1}(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\sigma}{2(\sigma-\delta)}}\|u_0\|_{L^m} + (1+t)^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\sigma-2\delta}{2(\sigma-\delta)}}\|u_1\|_{L^m},$$
$$\|\partial_t u_{\chi_1}(t,\cdot)\|_{L^q} \lesssim (1+t)^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-1}\|u_0\|_{L^m} + (1+t)^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\sigma-2\delta}{\sigma-\delta}}\|u_1\|_{L^m},$$

and

$$\|u_{\chi_3}(t,\cdot)\|_{L^q} \lesssim e^{-ct}t^{-n|\frac{1}{q}-\frac{1}{2}|(\frac{\sigma}{2\delta}-1)}\|u_0\|_{L^q} + e^{-ct}t^{-n|\frac{1}{q}-\frac{1}{2}|(\frac{\sigma}{2\delta}-1)+\frac{\sigma}{2\delta}}\|u_1\|_{L^q},$$
$$\big\||D|^\sigma u_{\chi_3}(t,\cdot)\big\|_{L^q} \lesssim e^{-ct}t^{-n|\frac{1}{q}-\frac{1}{2}|(\frac{\sigma}{2\delta}-1)-\frac{\sigma}{2\delta}}\|u_0\|_{L^q} + e^{-ct}t^{-n|\frac{1}{q}-\frac{1}{2}|(\frac{\sigma}{2\delta}-1)}\|u_1\|_{L^q},$$
$$\|\partial_t u_{\chi_3}(t,\cdot)\|_{L^q} \lesssim e^{-ct}t^{-n|\frac{1}{q}-\frac{1}{2}|(\frac{\sigma}{2\delta}-1)-\frac{\sigma}{2\delta}}\|u_0\|_{L^q} + e^{-ct}t^{-n|\frac{1}{q}-\frac{1}{2}|(\frac{\sigma}{2\delta}-1)}\|u_1\|_{L^q}.$$

The authors of [9] found an explicit way to obtain $L^p - L^q$ estimates for solutions and some of their partial derivatives by using the Mikhlin-Hörmander multiplier theorem for kernels localized to high frequencies. Moreover, the choice of two entire numbers $\sigma \in \mathbb{N}\backslash\{0\}$ and $\delta \in \mathbb{N}$ in [9] is important to prove blow-up results. The decay rates from the paper [9] are almost the same as those in Theorem 3.3.1 if we consider the case of sufficiently large space dimensions n.

3.5.2. The case $\delta = 0$

We can see that the decay rates from Theorem 3.4.4 are exactly the same as those from Propositions 4.1 and 4.2 in the paper [9].

4. Linear structurally damped σ-evolution models with $\delta \in (\frac{\sigma}{2}, \sigma]$

The main purpose of this chapter is to study the linear structurally damped σ-evolution models of the form

$$u_{tt} + (-\Delta)^\sigma u + \mu(-\Delta)^\delta u_t = 0, \quad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) \tag{4.1}$$

with $\sigma \geqslant 1$, $\mu > 0$ and $\delta \in (\frac{\sigma}{2}, \sigma]$. This is a family of structurally damped σ-evolution models interpolating between models with a special damping $\delta = \frac{\sigma}{2}$ and those with visco-elastic type damping $\delta = \sigma$. Our goal is to obtain $L^q - L^q$ estimates for solutions to (4.1) by assuming additional L^m regularity for the data with $m \in [1, q)$, where $q \in (1, \infty)$ is given.

To do this, let us explain our objectives and strategies as follows:

- By using the partial Fourier transformation we can reduce the partial differential equation to study an ordinary differential equation parameterized by ξ.

- Main difficulties that we will cope within the case $\delta \in (\frac{\sigma}{2}, \sigma)$ are to derive $L^1 - L^1$ estimates for oscillating integrals appearing in the representation of solutions. For this reason, we will apply the theory of modified Bessel funtions and Faà di Bruno's formula.

- For the sake of the asymptotic behavior of the characteristic roots, we may obtain L^∞ estimates for oscillating integrals. By an interpolation theorem, we also get L^r estimates with $r \in [1, \infty]$ for oscillating integrals.

- Applying Young's convolution inequality we may conclude $L^m - L^q$ estimates with $q \in [1, \infty]$ and $m \in [1, q]$, $L^m \cap L^q - L^q$ and $L^q - L^q$ estimates with $q \in (1, \infty)$ and $m \in [1, q)$ for solutions to (4.1) in the case $\delta \in (\frac{\sigma}{2}, \sigma)$.

- In the case $\delta = \sigma$ (visco-elastic damping): Following an analogous way as we did in the case $\delta \in (\frac{\sigma}{2}, \sigma)$ implies $L^m - L^q$ estimates with $q \in [1, \infty]$ and $m \in [1, q]$ for small frequencies. For large frequencies, we will apply the Mikhlin- Hömander multiplier theorem to get $L^q - L^q$ estimates with $q \in (1, \infty)$. Then, we may conclude $L^m \cap L^q - L^q$ and $L^q - L^q$ estimates with $q \in (1, \infty)$ and $m \in [1, q)$ for solutions to (4.1).

4.1. A first Cauchy problem for linear structurally damped σ-evolution models

Let us consider the following family of parameter-dependent Cauchy problems:

$$u_{tt} + (-\Delta)^\sigma u + \mu(-\Delta)^\delta u_t = 0, \quad u(s,x) = 0, \quad u_t(s,x) = u_1(x), \tag{4.2}$$

where $s \geqslant 0$ is a fixed non-negative real parameter, $\sigma \geqslant 1$, $\mu > 0$ and $\delta \in (\frac{\sigma}{2}, \sigma)$. Thanks to the change of variables $t \to t - s$, we have here in mind the following Cauchy problem:

$$v_{tt} + (-\Delta)^\sigma v + \mu(-\Delta)^\delta v_t = 0, \quad v(0,x) = 0, \quad v_t(0,x) = v_1(x). \tag{4.3}$$

$L^m \cap L^q - L^q$ and $L^q - L^q$ estimates

In this section, we want to prove the following result.

Theorem 4.1.1. *Let $\delta \in \big(\frac{\sigma}{2}, \sigma\big)$ in (4.3), $q \in (1, \infty)$ and $m \in [1, q)$. Then, the energy solutions to (4.3) satisfy the $(L^m \cap L^q) - L^q$ estimates*

$$\begin{aligned}
\|v(t,\cdot)\|_{L^q} &\lesssim (1+t)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|v_1\|_{L^m\cap L^q},\\
\big\||D|^\sigma v(t,\cdot)\big\|_{L^q} &\lesssim (1+t)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{\sigma}{2\delta}}\|v_1\|_{L^m\cap L^q},\\
\|v_t(t,\cdot)\|_{L^q} &\lesssim (1+t)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|v_1\|_{L^m\cap L^q},\\
\big\||D|^{2\delta} v(t,\cdot)\big\|_{L^q} &\lesssim (1+t)^{(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|v_1\|_{L^m\cap L^q},
\end{aligned}$$

and the $L^q - L^q$ estimates

$$\begin{aligned}
\|v(t,\cdot)\|_{L^q} &\lesssim (1+t)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})}\|v_1\|_{L^q},\\
\big\||D|^\sigma v(t,\cdot)\big\|_{L^q} &\lesssim (1+t)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})}\|v_1\|_{L^q},\\
\|v_t(t,\cdot)\|_{L^q} &\lesssim (1+t)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})}\|v_1\|_{L^q},\\
\big\||D|^{2\delta} v(t,\cdot)\big\|_{L^q} &\lesssim (1+t)^{(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})}\|v_1\|_{L^q},
\end{aligned}$$

where $1+\frac{1}{q} = \frac{1}{r} + \frac{1}{m}$ and for all dimensions $n \geqslant 1$.

As a consequence of Theorem 4.1.1 we may conclude the following theorem.

Theorem 4.1.2. *Let $\delta \in \big(\frac{\sigma}{2}, \sigma\big)$ in (4.2), $q \in (1, \infty)$ and $m \in [1, q)$. Then, the energy solutions to (4.2) satisfy the $(L^m \cap L^q) - L^q$ estimates*

$$\begin{aligned}
\|u(t,\cdot)\|_{L^q} &\lesssim (1+t-s)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|u_1\|_{L^m\cap L^q},\\
\big\||D|^\sigma u(t,\cdot)\big\|_{L^q} &\lesssim (1+t-s)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{\sigma}{2\delta}}\|u_1\|_{L^m\cap L^q},\\
\|u_t(t,\cdot)\|_{L^q} &\lesssim (1+t-s)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|u_1\|_{L^m\cap L^q},\\
\big\||D|^{2\delta} u(t,\cdot)\big\|_{L^q} &\lesssim (1+t-s)^{(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|u_1\|_{L^m\cap L^q},
\end{aligned}$$

and the $L^q - L^q$ estimates

$$\begin{aligned}
\|u(t,\cdot)\|_{L^q} &\lesssim (1+t-s)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})}\|u_1\|_{L^q},\\
\big\||D|^\sigma u(t,\cdot)\big\|_{L^q} &\lesssim (1+t-s)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})}\|u_1\|_{L^q},\\
\|u_t(t,\cdot)\|_{L^q} &\lesssim (1+t-s)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})}\|u_1\|_{L^q},\\
\big\||D|^{2\delta} u(t,\cdot)\big\|_{L^q} &\lesssim (1+t-s)^{(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})}\|u_1\|_{L^q},
\end{aligned}$$

where $1 < r \leqslant \infty$ satisfying $1+\frac{1}{q} = \frac{1}{r} + \frac{1}{m}$ and for all dimensions $n \geqslant 1$.

Using partial Fourier transformation to (4.3) we obtain the Cauchy problem for $\widehat{v}(t,\xi) := \mathfrak{F}(v(t,x))$ and $\widehat{v_1}(\xi) := \mathfrak{F}(v_1(x))$

$$\widehat{v}_{tt} + \mu|\xi|^{2\delta}\widehat{v}_t + |\xi|^{2\sigma}\widehat{v} = 0, \quad \widehat{v}(0,\xi) = 0, \quad \widehat{v}_t(0,\xi) = \widehat{v_1}(\xi). \tag{4.4}$$

We may choose without loss of generality $\mu = 1$ in (4.3). The characteristic roots are

$$\lambda_{1,2} = \lambda_{1,2}(\xi) = \frac{1}{2}\Big(-|\xi|^{2\delta} \pm \sqrt{|\xi|^{4\delta} - 4|\xi|^{2\sigma}}\Big).$$

The solutions to (4.4) are presented by the following formula (here we assume $\lambda_1 \neq \lambda_2$):

$$\widehat{v}(t,\xi) = \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}\widehat{v_1}(\xi) =: \widehat{K_1}(t,\xi)\widehat{v_1}(\xi).$$

Taking account of the cases of small and large frequencies separately we have

1. $\lambda_{1,2} \sim -|\xi|^{2\delta} \pm i|\xi|^\sigma, \quad \lambda_1 - \lambda_2 \sim i|\xi|^\sigma$ for small $|\xi| \in (0, 4^{-\frac{1}{2\delta-\sigma}})$,

2. $\lambda_1 \sim -|\xi|^{2(\sigma-\delta)}, \quad \lambda_2 \sim -|\xi|^{2\delta}, \quad \lambda_1-\lambda_2 \sim |\xi|^{2\delta}$ for large $|\xi| \in (4^{\frac{1}{2\delta-\sigma}}, \infty)$.

Let $\chi_k = \chi_k(|\xi|)$ with $k=1,2,3$ be smooth cut-off functions having the following properties:

$$\chi_1(|\xi|) = \begin{cases} 1 & \text{if } |\xi| \leqslant 4^{-\frac{1}{2\delta-\sigma}}, \\ 0 & \text{if } |\xi| \geqslant 3^{-\frac{1}{2\delta-\sigma}}, \end{cases} \qquad \chi_3(|\xi|) = \begin{cases} 1 & \text{if } |\xi| \geqslant 4^{\frac{1}{2\delta-\sigma}}, \\ 0 & \text{if } |\xi| \leqslant 3^{\frac{1}{2\delta-\sigma}}, \end{cases}$$

and $\chi_2 = 1-\chi_1(|\xi|)-\chi_3(|\xi|)$.

We note that $\chi_2(|\xi|)=1$ if $3^{-\frac{1}{2\delta-\sigma}} \leqslant |\xi| \leqslant 3^{\frac{1}{2\delta-\sigma}}$ and $\chi_2(|\xi|)=0$ if $|\xi| \leqslant 4^{-\frac{1}{2\delta-\sigma}}$ or $|\xi| \geqslant 4^{\frac{1}{2\delta-\sigma}}$.

Let us now decompose the solutions to (4.3) into three parts localized separately to low, middle and high frequencies, that is,

$$v(t,x) = v_{\chi_1}(t,x) + v_{\chi_2}(t,x) + v_{\chi_3}(t,x),$$

where

$$v_{\chi_k}(t,x) = \mathfrak{F}^{-1}\big(\chi_k(|\xi|)\widehat{v}(t,\xi)\big) \quad \text{with } k=1,2,3.$$

In order to estimate the L^q norms of solutions in (4.3) with additional L^m regularity of the data, we shall estimate the L^r norms of general terms of the form $\mathfrak{F}^{-1}\big(\widehat{K_j}(t,\xi)\chi_k(|\xi|)\big)(t,x)$ with $j=0,1$ and $k=1,2,3$, where

$$\widehat{K_0}(t,\xi) = \frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1-\lambda_2} \quad \text{and} \quad \widehat{K_1}(t,\xi) = \frac{e^{\lambda_1 t}-e^{\lambda_2 t}}{\lambda_1-\lambda_2}.$$

The proof of Theorem 4.1.1 is divided into several steps as follows:

L^1 estimates for small frequencies

Proposition 4.1.1. *The following estimates hold in* $\mathbb{R}^n$*:*

$$\big\|\mathfrak{F}^{-1}\big(\widehat{K_0}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^1} \lesssim \begin{cases} 1 & \text{if } t \in (0,1], \\ t^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})} & \text{if } t \in [1,\infty), \end{cases}$$

$$\big\|\mathfrak{F}^{-1}\big(\widehat{K_1}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^1} \lesssim \begin{cases} t & \text{if } t \in (0,1], \\ t^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})} & \text{if } t \in [1,\infty). \end{cases}$$

To derive the desired estimates for the norm of the Fourier multipliers localized to small frequencies, we write

$$\widehat{K_0}(t,\xi) = e^{-\frac{1}{2}|\xi|^{2\delta}t}\cos\Big(|\xi|^\sigma\sqrt{1-\frac{1}{4}|\xi|^{4\delta-2\sigma}}t\Big) + e^{-\frac{1}{2}|\xi|^{2\delta}t}|\xi|^{2\delta}\frac{\sin\big(|\xi|^\sigma\sqrt{1-\frac{1}{4}|\xi|^{4\delta-2\sigma}}t\big)}{2|\xi|^\sigma\sqrt{1-\frac{1}{4}|\xi|^{4\delta-2\sigma}}},$$

and

$$\widehat{K_1}(t,\xi) = e^{-\frac{1}{2}|\xi|^{2\delta}t}\frac{\sin\big(|\xi|^\sigma\sqrt{1-\frac{1}{4}|\xi|^{4\delta-2\sigma}}t\big)}{|\xi|^\sigma\sqrt{1-\frac{1}{4}|\xi|^{4\delta-2\sigma}}}.$$

For this reason, we will split our proof into two steps. In the first step we derive L^1 estimates for the following oscillating integrals:

$$\mathfrak{F}^{-1}\Big(e^{-c_1|\xi|^{2\delta}t}|\xi|^{2\beta}\frac{\sin(c_2|\xi|^\sigma t)}{|\xi|^\sigma}\chi_1(|\xi|)\Big)(t,x),$$

and

$$\mathfrak{F}^{-1}\Big(e^{-c_1|\xi|^{2\delta}t}|\xi|^{2\beta}\cos(c_2|\xi|^\sigma t)\chi_1(|\xi|)\Big)(t,x),$$

where $\beta \geqslant 0$, c_1 is a positive constant and $c_2 \neq 0$ is a real constant. Then, in the second step we estimate the following more structured oscillating integrals:

$$\mathfrak{F}^{-1}\Big(e^{-c_1|\xi|^{2\delta}t}|\xi|^{2\beta}\frac{\sin\big(c_2|\xi|^\sigma f(|\xi|)t\big)}{|\xi|^\sigma f(|\xi|)}\chi_1(|\xi|)\Big)(t,x),$$

and

$$\mathfrak{F}^{-1}\Big(e^{-c_1|\xi|^{2\delta}t}|\xi|^{2\beta}\cos\big(c_2|\xi|^\sigma f(|\xi|)t\big)\chi_1(|\xi|)\Big)(t,x),$$

where

$$f(|\xi|)=\sqrt{1-\frac{1}{4}|\xi|^{4\delta-2\sigma}}.$$

Lemma 4.1.1. *The following estimates hold in $\mathbb{R}^n$ for any $n \geqslant 1$:*

$$\Big\|\mathfrak{F}^{-1}\Big(e^{-c_1|\xi|^{2\delta}t}|\xi|^{2\beta}\frac{\sin(c_2|\xi|^\sigma t)}{|\xi|^\sigma}\chi_1(|\xi|)\Big)(t,\cdot)\Big\|_{L^1} \lesssim \begin{cases} t & \text{if } t\in(0,1],\\ t^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})+\frac{\sigma-2\beta}{2\delta}} & \text{if } t\in[1,\infty),\end{cases}$$

with $\beta \geqslant 0$. Here c_1 is a positive and $c_2 \neq 0$ is a real constant.

Proof. We follow ideas from the proofs of Proposition 4 in [57] and Lemma 3.1.3. Many steps in our proof are similar to the proofs of these results. Hence, it is reasonable to present only the steps which are different. Let us divide the proof into two cases: $t \in (0,1]$ and $t \in [1,\infty)$. First, in order to treat the case $t \in (0,1]$, we localize to small $|x| \leqslant 1$. Then, we derive immediately for small values of $|\xi|$ the estimate

$$\Big\|\mathfrak{F}^{-1}\Big(e^{-c_1|\xi|^{2\delta}t}|\xi|^{2\beta-\sigma}\sin(c_2|\xi|^\sigma t)\chi_1(|\xi|)\Big)(t,\cdot)\Big\|_{L^1(|x|\leqslant 1)} \lesssim t. \tag{4.5}$$

For this reason, we assume now $|x| \geqslant 1$. We introduce the function

$$I(t,x):=\mathfrak{F}^{-1}\Big(e^{-c_1|\xi|^{2\delta}t}|\xi|^{2\beta-\sigma}\sin(c_2|\xi|^\sigma t)\chi_1(|\xi|)\Big)(t,x).$$

Because the functions in the parenthesis are radial symmetric with respect to ξ, the inverse Fourier transform is radial symmetric with respect to x, too. Applying the modified Bessel functions leads to

$$I(t,x)=c\int_0^\infty e^{-c_1r^{2\delta}t}r^{2\beta-\sigma}\sin(c_2r^\sigma t)\chi_1(r)r^{n-1}\tilde{\mathcal{J}}_{\frac{n}{2}-1}(r|x|)\,dr. \tag{4.6}$$

Let us consider odd spatial dimensions $n=2m+1, m\geqslant 1$. We introduce the vector field $Xf(r):=\frac{d}{dr}\big(\frac{1}{r}f(r)\big)$ as in the proof of Proposition 4 in [57]. Then, carrying out $m+1$ steps of partial integration we have

$$I(t,x)=-\frac{c}{|x|^n}\int_0^\infty \partial_r\big(X^m\big(e^{-c_1r^{2\delta}t}\sin(c_2r^\sigma t)\chi_1(r)r^{2\beta-\sigma+2m}\big)\big)\sin(r|x|)\,dr. \tag{4.7}$$

A standard calculation leads to the following presentation of the right-hand side of (4.7):

$$\begin{aligned} I(t,x) &= \sum_{j=0}^m\sum_{k=0}^{j+1}\frac{c_{jk}}{|x|^n}\int_0^\infty \partial_r^{j+1-k}e^{-c_1r^{2\delta}t}\partial_r^k\big(\sin(c_2r^\sigma t)\chi_1(r)\big)r^{2\beta-\sigma+j}\sin(r|x|)\,dr\\ &\quad+\sum_{j=0}^m\sum_{k=0}^{j}\frac{c_{jk}}{|x|^n}\int_0^\infty \partial_r^{j-k}e^{-c_1r^{2\delta}t}\partial_r^{k+1}\big(\sin(c_2r^\sigma t)\chi_1(r)\big)r^{2\beta-\sigma+j}\sin(r|x|)\,dr\\ &\quad+\sum_{j=1}^m\sum_{k=0}^{j}\frac{c_{jk}}{|x|^n}\int_0^\infty \partial_r^{j-k}e^{-c_1r^{2\delta}t}\partial_r^{k}\big(\sin(c_2r^\sigma t)\chi_1(r)\big)r^{2\beta-\sigma+j-1}\sin(r|x|)\,dr \end{aligned}$$

with some constants c_{jk}. Now, we estimate the integrals

$$I_{j,k}(t,x):=\int_0^\infty \partial_r^{j+1-k}e^{-c_1r^{2\delta}t}\partial_r^k\big(\sin(c_2r^\sigma t)\chi_1(r)\big)r^{2\beta-\sigma+j}\sin(r|x|)\,dr. \tag{4.8}$$

Because of small values of r, we notice that the following estimates hold on the support of χ_1 and on the support of its derivatives:

$$\big|\partial_r^l e^{-c_1r^{2\delta}t}\big| \lesssim \begin{cases} 1 & \text{if } l=0,\\ r^{2\delta-l}t & \text{if } l=1,\cdots,m,\end{cases}$$

$$\big|\partial_r^l\big(\sin(c_2r^\sigma t)\chi_1(r)\big)\big| \lesssim r^{\sigma-l}t \quad \text{for all } l=0,\cdots,m.$$

As a result, we obtain for small r, $j = 0, \cdots, m$ and $k = 0, \cdots, j$ the estimates

$$\big|\partial_r^{j+1-k} e^{-c_1 r^{2\delta} t} \partial_r^k \big(\sin(c_2 r^\sigma t)\chi_1(r)\big) r^{2\beta-\sigma+j}\big| \lesssim r^{2\delta+2\beta-1} t^2$$

on the support of χ_1 and on the support of its derivatives. We divide the integral (4.8) into two parts to derive on the one hand

$$\Big|\int_0^{\frac{\pi}{2|x|}} \partial_r^{j+1-k} e^{-c_1 r^{2\delta} t} \partial_r^k \big(\sin(c_2 r^\sigma t)\chi_1(r)\big) r^{2\beta-\sigma+j} \sin(r|x|)\, dr\Big| \lesssim \frac{t^2}{|x|^{2\delta}}. \tag{4.9}$$

On the other hand, we can carry out one more step of partial integration in estimating the remaining integral as follows:

$$\begin{aligned} &\Big|\int_{\frac{\pi}{2|x|}}^\infty \partial_r^{j+1-k} e^{-c_1 r^{2\delta} t} \partial_r^k \big(\sin(c_2 r^\sigma t)\chi_1(r)\big) r^{2\beta-\sigma+j} \sin(r|x|)\, dr\Big| \\ &\quad\lesssim \frac{1}{|x|}\Big|\partial_r^{j+1-k} e^{-c_1 r^{2\delta} t} \partial_r^k \big(\sin(c_2 r^\sigma t)\chi_1(r)\big) r^{2\beta-\sigma+j} \cos(r|x|)\Big|_{r=\frac{\pi}{2|x|}}^\infty \\ &\qquad + \frac{1}{|x|}\int_{\frac{\pi}{2|x|}}^\infty \Big|\partial_r\Big(\partial_r^{j+1-k} e^{-c_1 r^{2\delta} t} \partial_r^k \big(\sin(c_2 r^\sigma t)\chi_1(r)\big) r^{2\beta-\sigma+j}\Big) \cos(r|x|)\Big|\, dr \lesssim \frac{t^2}{|x|}, \end{aligned} \tag{4.10}$$

since $2\delta + 2\beta > \sigma \geqslant 1$. Here we also note that for all $j = 0, \cdots, m$ and $k = 0, \cdots, j$ we have

$$\Big|\partial_r\Big(\partial_r^{j+1-k} e^{-c_1 r^{2\delta} t} \partial_r^k \big(\sin(c_2 r^\sigma t)\chi_1(r)\big) r^{2\beta-\sigma+j}\Big)\Big| \lesssim r^{2\delta+2\beta-2} t^2.$$

Hence, from (4.7) to (4.10) we have produced terms $|x|^{-(n+2\delta)}$ and $|x|^{-(n+1)}$ which guarantee the L^1 property in x to prove that for all $t \in (0, 1]$ and $n = 2m + 1$ the following estimates hold:

$$\big\|\mathfrak{F}^{-1}\big(e^{-c_1|\xi|^{2\delta} t}|\xi|^{2\beta-\sigma} \sin(c_2|\xi|^\sigma t)\chi_1(|\xi|)\big)(t, \cdot)\big\|_{L^1(|x|\geqslant 1)} \lesssim t^2. \tag{4.11}$$

Let us consider even spatial dimensions $n = 2m$, $m \geqslant 1$, in the first case $t \in (0, 1]$. Then, applying the first rule for the modified Bessel functions for $\mu = 1$ and the fifth rule for $\mu = 0$ from Proposition B.3.2, and repeating the above calculations as we did to get (4.11) we may conclude the following estimates for $n = 2m$, $m \geqslant 1$:

$$\big\|\mathfrak{F}^{-1}\big(e^{-c_1|\xi|^{2\delta} t}|\xi|^{2\beta-\sigma} \sin(c_2|\xi|^\sigma t)\chi_1(|\xi|)\big)(t, \cdot)\big\|_{L^1(|x|\geqslant 1)} \lesssim t^2. \tag{4.12}$$

Let us turn to the second case $t \in [1, \infty)$. Then, by the change of variables $\xi = t^{-\frac{1}{2\delta}}\eta$ as we did in the proof of the case $t \in (0, 1]$ to Lemma 3.1.3 we will follow the steps of the proof of this lemma to conclude the following estimates:

$$\big\|\mathfrak{F}^{-1}\big(e^{-c_1|\xi|^{2\delta} t}|\xi|^{2\beta-\sigma} \sin(c_2|\xi|^\sigma t)\chi_1(|\xi|)\big)(t, \cdot)\big\|_{L^1(|x|\leqslant 1)} \lesssim t^{1-\frac{\beta}{\delta}}, \tag{4.13}$$

and

$$\begin{aligned} &\big\|\mathfrak{F}^{-1}\big(e^{-c_1|\xi|^{2\delta} t}|\xi|^{2\beta-\sigma} \sin(c_2|\xi|^\sigma t)\chi_1(|\xi|)\big)(t, \cdot)\big\|_{L^1(|x|\geqslant 1)} \\ &\quad\lesssim \begin{cases} t^{(m+2)(1-\frac{\sigma}{2\delta})+\frac{\sigma-2\beta}{2\delta}} & \text{if } n = 2m+1, \\ t^{(m+1)(1-\frac{\sigma}{2\delta})+\frac{\sigma-2\beta}{2\delta}} & \text{if } n = 2m. \end{cases} \end{aligned} \tag{4.14}$$

Here we also note that $|\xi| \in (0, 1]$, that is, $r \in (0, t^{\frac{1}{2\delta}}]$ and $rt^{-\frac{1}{2\delta}} \leqslant 1$ which are useful in our proof. Summarizing, from (4.5) and (4.11) to (4.14) the statements of Lemma 4.1.1 are proved. □

Remark 4.1.1. Let us explain the results for the case $n = 1$. We explained the proofs to Lemma 4.1.1 for $n \geqslant 2$ only. However, in the case $n = 1$ we only carry out partial integration with no need of the support of the vector field $Xf(r)$ as we did in (4.7). Then, following the steps of our considerations for odd spatial dimensions we may conclude that the statements of this lemma also hold for $n = 1$.

Following the proof of Lemma 4.1.1 we may prove the following L^1 estimate, too.

Lemma 4.1.2. *The following estimates hold in $\mathbb{R}^n$ for any $n \geqslant 1$:*

$$\Big\|\mathfrak{F}^{-1}\Big(e^{-c_1|\xi|^{2\delta}t}|\xi|^{2\beta}\cos(c_2|\xi|^\sigma t)\chi_1(|\xi|)\Big)(t,\cdot)\Big\|_{L^1} \lesssim \begin{cases} 1 & \text{if } t \in (0,1], \\ t^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})-\frac{\beta}{\delta}} & \text{if } t \in [1,\infty), \end{cases}$$

with $\beta \geqslant 0$. Here c_1 is a positive and $c_2 \neq 0$ is a real constant.

Finally, we consider oscillating integrals with more complicated oscillations in the integrand. We are going to prove the following result.

Lemma 4.1.3. *The following estimates hold in $\mathbb{R}^n$ for any $n \geqslant 1$:*

$$\Big\|\mathfrak{F}^{-1}\Big(e^{-c_1|\xi|^{2\delta}t}|\xi|^{2\beta}\frac{\sin(c_2|\xi|^\sigma f(|\xi|)t)}{|\xi|^\sigma f(|\xi|)}\chi_1(|\xi|)\Big)(t,\cdot)\Big\|_{L^1} \lesssim \begin{cases} t & \text{if } t \in (0,1], \\ t^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})+\frac{\sigma-2\beta}{2\delta}} & \text{if } t \in [1,\infty), \end{cases}$$

where

$$f(|\xi|) = \sqrt{1-\frac{1}{4}|\xi|^{4\delta-2\sigma}}$$

and $\beta \geqslant 0$. Here c_1 is a positive and $c_2 \neq 0$ is a real constant.

Proof. We will follow the proof of Lemma 3.1.5. Hence, it is reasonable to present only the steps which are different. Then, we shall repeat some of the arguments as we did in the proof of Lemma 4.1.1 to conclude the desired estimates.

First, let us consider $|x| \geqslant 1$ and $t \in (0,1]$. To obtain the first desired estimate in both cases of odd spatial dimensions $n = 2m+1$ and even spatial dimensions $n = 2m$ with $m \geqslant 1$, we assert the following estimates on the support of $\chi_1(r)$ and on the support of its derivatives:

$$\Big|\partial_t^k\Big(\frac{\sin\big(c_2 r^\sigma f(r)t\big)}{f(r)}\Big)\Big| \lesssim r^{\sigma-k}t \quad \text{for all } k = 1,\cdots,m,$$

where

$$f(r) = \sqrt{1-\frac{1}{4}r^{4\delta-2\sigma}}.$$

Here Faà di Bruno's formula comes into play for verifying all our estimates. We split the proof of the above estimates into several sub-steps as follows:

Step 1: Applying Proposition B.4.1 with $h(s) = \sqrt{s}$ and $g(r) = 1-\frac{1}{4}r^{2(2\delta-\sigma)}$ we have

$$\begin{aligned}\big|\partial_r^k f(r)\big| &\lesssim \Big|\sum_{1\cdot m_1+\cdots+k\cdot m_k=k,\, m_i\geqslant 0} g(r)^{\frac{1}{2}-(m_1+\cdots+m_k)}\prod_{j=1}^k\Big(-\frac{1}{4}r^{2(2\delta-\sigma)-j}\Big)^{m_j}\Big| \\ &\lesssim \sum_{1\cdot m_1+\cdots+k\cdot m_k=k,\, m_i\geqslant 0} r^{2(2\delta-\sigma)(m_1+\cdots+m_k)-k} \lesssim r^{-k}.\end{aligned}$$

Here we used $\frac{3}{4} \leqslant g(r) \leqslant 1$ for $r \leqslant 1$. In the same way we derive

$$\Big|\partial_r^k\Big(\frac{1}{f(r)}\Big)\Big| \lesssim r^{-k} \quad \text{for } k = 1,\cdots,m. \tag{4.15}$$

Step 2: Applying Proposition B.4.1 with $h(s) = \sin(c_2\, s)$ and $g(r) = r^\sigma f(r)t$ we get

$$\begin{aligned}\big|\partial_r^k \sin\big(c_2 r^\sigma f(r)t\big)\big| &\lesssim \Big|\sum_{1\cdot m_1+\cdots+k\cdot m_k=k,\, m_i\geqslant 0} \sin\big(c_2 r^\sigma f(r)t\big)^{(m_1+\cdots+m_k)}\prod_{j=1}^k\Big(\partial_r^j\big(r^\sigma f(r)t\big)\Big)^{m_j}\Big| \\ &\lesssim \Big|\sum_{1\cdot m_1+\cdots+k\cdot m_k=k,\, m_i\geqslant 0}\prod_{j=1}^k\Big(t\sum_{l=0}^j C_j^l r^{\sigma-j+l}f^{(l)}(r)\Big)^{m_j}\Big| \\ &\lesssim \sum_{1\cdot m_1+\cdots+k\cdot m_k=k,\, m_i\geqslant 0}\prod_{j=1}^k(t\,r^{\sigma-j})^{m_j} \\ &\lesssim \sum_{1\cdot m_1+\cdots+k\cdot m_k=k,\, m_i\geqslant 0} r^{-k}(t\,r^\sigma)^{m_1+\cdots+m_k} \lesssim r^{\sigma-k}t.\end{aligned} \tag{4.16}$$

Therefore, from (4.15) and (4.16) using the product rule for higher derivatives we may conclude

$$\Big|\partial_r^k\Big(\frac{\sin\big(c_2 r^\sigma f(r)t\big)}{f(r)}\Big)\Big| \lesssim r^{\sigma-k}t \quad \text{for } k=1,\cdots,m.$$

Next, let us turn to consider $|x|\geqslant 1$ *and* $t\in[1,\infty)$. To derive the desired estimates by using similar ideas as in the proof of Lemma 4.1.1, we shall prove the following auxiliary estimates on the support of $\chi_1(t^{-\frac{1}{2\delta}}r)$ and on the support of its partial derivatives:

$$\Big|\partial_r^k\Big(\frac{\sin\big(c_2 r^\sigma f(r)t^{1-\frac{\sigma}{2\delta}}\big)}{f(r)}\Big)\Big| \lesssim t^{1-\frac{\sigma}{2\delta}}r^{\sigma-k}(1+r^\sigma t^{1-\frac{\sigma}{2\delta}})^{k-1} \quad \text{for } k=1,\cdots,m,$$

where

$$f(r)=\sqrt{1-\frac{1}{4}t^{\frac{\sigma-2\delta}{\delta}}r^{2(2\delta-\sigma)}}.$$

Step 1: Applying Proposition B.4.1 with $h(s)=\sqrt{s}$ and $g(r)=1-\frac{1}{4}t^{\frac{\sigma-2\delta}{\delta}}r^{2(2\delta-\sigma)}$ we get

$$\begin{aligned}
\big|\partial_r^k f(r)\big| &\lesssim \Big|\sum_{1\cdot m_1+\cdots+k\cdot m_k=k,\, m_i\geqslant 0} g(r)^{\frac{1}{2}-(m_1+\cdots+m_k)}\prod_{j=1}^k\Big(-\frac{1}{4}t^{\frac{\sigma-2\delta}{\delta}}r^{2(2\delta-\sigma)-j}\Big)^{m_j}\Big|\\
&\lesssim \sum_{1\cdot m_1+\cdots+k\cdot m_k=k,\, m_i\geqslant 0}\big(t^{\frac{\sigma-2\delta}{\delta}}r^{2(2\delta-\sigma)}\big)^{m_1+\cdots+m_k}r^{-k}\\
&\lesssim r^{-k}\sum_{1\cdot m_1+\cdots+k\cdot m_k=k,\, m_i\geqslant 0}(t^{-\frac{1}{2\delta}}r)^{2(2\delta-\sigma)(m_1+\cdots+m_k)}\lesssim r^{-k}.
\end{aligned}$$

Here we used $\frac{3}{4}\leqslant g(r)\leqslant 1$ and $t^{-\frac{1}{2\delta}}r\leqslant 1$ for $r\leqslant t^{\frac{1}{2\delta}}$. An analogous treatment leads to

$$\Big|\partial_r^k\Big(\frac{1}{f(r)}\Big)\Big| \lesssim r^{-k} \quad \text{for } k=1,\cdots,m. \tag{4.17}$$

Step 2: Repeating the proof as we did in Lemma 3.1.5 we have the following estimates:

$$\big|\partial_r^k \sin\big(c_2 r^\sigma f(r)t^{1-\frac{\sigma}{2\delta}}\big)\big| \lesssim t^{1-\frac{\sigma}{2\delta}}r^{\sigma-k}\big(1+t^{1-\frac{\sigma}{2\delta}}r^\sigma\big)^{k-1}. \tag{4.18}$$

Therefore, from (4.17) and (4.18) using the product rule for higher derivatives we may conclude

$$\Big|\partial_r^k\Big(\frac{\sin\big(c_2 r^\sigma f(r)t^{1-\frac{\sigma}{2\delta}}\big)}{f(r)}\Big)\Big| \lesssim t^{1-\frac{\sigma}{2\delta}}r^{\sigma-k}\big(1+t^{1-\frac{\sigma}{2\delta}}r^\sigma\big)^{k-1} \quad \text{for } k=1,\cdots,m.$$

Summarizing, Lemma 4.1.3 is proved. □

Following the steps of the proof of Lemma 4.1.3 we may conclude the following statement, too.

Lemma 4.1.4. *The following estimates hold in* $\mathbb{R}^n$ *for any* $n\geqslant 1$*:*

$$\Big\|\mathfrak{F}^{-1}\Big(e^{-c_1|\xi|^{2\delta}t}|\xi|^{2\beta}\cos\big(c_2|\xi|^\sigma f(|\xi|)t\big)\chi_1(|\xi|)\Big)(t,\cdot)\Big\|_{L^1}\lesssim\begin{cases}1 & \text{if } t\in(0,1],\\ t^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})-\frac{\beta}{\delta}} & \text{if } t\in[1,\infty),\end{cases}$$

where

$$f(|\xi|)=\sqrt{1-\frac{1}{4}|\xi|^{4\delta-2\sigma}}$$

and $\beta\geqslant 0$. *Here* c_1 *is a positive and* $c_2\neq 0$ *is a real constant.*

Proof of Proposition 4.1.1. In order to prove the first statement, we replace $\beta=\delta$ and $\beta=0$, respectively, in Lemmas 4.1.3 and 4.1.4. Then, plugging $\beta=0$ in Lemma 4.1.3 we may conclude the second statement. Therefore, this completes our proof. □

Following the approach of the proof of Proposition 4.1.1 we may prove the following statements.

Proposition 4.1.2. *The following estimates hold in $\mathbb{R}^n$:*

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a \widehat{K_0}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^1} \lesssim \begin{cases} 1 & \text{if } t \in (0,1], \\ t^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})-\frac{a}{2\delta}} & \text{if } t \in [1,\infty), \end{cases}$$

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a \widehat{K_1}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^1} \lesssim \begin{cases} t & \text{if } t \in (0,1], \\ t^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})-\frac{a}{2\delta}} & \text{if } t \in [1,\infty), \end{cases}$$

for any non-negative number a.

Proof. To derive the desired estimates for the norm of the Fourier multipliers localized to small frequencies, we write

$$|\xi|^a \widehat{K_0}(t,\xi) = e^{-\frac{1}{2}|\xi|^{2\delta}t}|\xi|^a \cos\Big(|\xi|^\sigma\sqrt{1-\frac{1}{4}|\xi|^{4\delta-2\sigma}}t\Big) + e^{-\frac{1}{2}|\xi|^{2\delta}t}|\xi|^{a+2\delta}\frac{\sin\big(|\xi|^\sigma\sqrt{1-\frac{1}{4}|\xi|^{4\delta-2\sigma}}t\big)}{2|\xi|^\sigma\sqrt{1-\frac{1}{4}|\xi|^{4\delta-2\sigma}}},$$

and

$$|\xi|^a \widehat{K_1}(t,\xi) = e^{-\frac{1}{2}|\xi|^{2\delta}t}|\xi|^a\frac{\sin\big(|\xi|^\sigma\sqrt{1-\frac{1}{4}|\xi|^{4\delta-2\sigma}}t\big)}{|\xi|^\sigma\sqrt{1-\frac{1}{4}|\xi|^{4\delta-2\sigma}}}.$$

By choosing the values $2\beta = a + 2\delta$ and $2\beta = a$ in Lemma 4.1.3 and 4.1.4, respectively, we may conclude the first statement. Then, plugging $2\beta = a$ in Lemma 4.1.3 we may conclude the second statement. Therefore, this completes our proof. □

L^1 estimates for large frequencies

Proposition 4.1.3. *The estimates*

$$\big\|\mathfrak{F}^{-1}\big(\widehat{K_0}(t,\xi)\chi_3(|\xi|)\big)(t,\cdot)\big\|_{L^1} \lesssim 1 \quad \text{and} \quad \big\|\mathfrak{F}^{-1}\big(\widehat{K_1}(t,\xi)\chi_3(|\xi|)\big)(t,\cdot)\big\|_{L^1} \lesssim t$$

hold for all $t > 0$.

Proof. Our approach is based on the paper [57]. In order to prove the above estimates for large $|\xi|$, we can apply the modified Bessel functions, carry out partial integrations and perform change of variables. According to a modification of the proof from Lemma 16 to Lemma 20 in [57], we have to study the three oscillating integrals

$$\mathfrak{F}^{-1}\big(e^{(\lambda_2-\lambda_1)t}\chi_3(|\xi|)\big)(t,x), \quad \mathfrak{F}^{-1}\big(|\xi|^{2\sigma-4\delta}e^{\lambda_2 t}\chi_3(|\xi|)\big)(t,x), \quad \mathfrak{F}^{-1}\big(e^{\lambda_1 t}\chi_3(|\xi|)\big)(t,x).$$

Hence, we obtain the following results.

Lemma 4.1.5. *The estimates*

$$\begin{aligned} &\big\|\mathfrak{F}^{-1}\big(e^{(\lambda_2-\lambda_1)t}\chi_3(|\xi|)\big)(t,\cdot)\big\|_{L^1} \lesssim 1, \\ &\big\|\mathfrak{F}^{-1}\big(|\xi|^{2\sigma-4\delta}e^{\lambda_2 t}\chi_3(|\xi|)\big)(t,\cdot)\big\|_{L^1} \lesssim 1, \\ &\big\|\mathfrak{F}^{-1}\big(e^{\lambda_1 t}\chi_3(|\xi|)\big)(t,\cdot)\big\|_{L^1} \lesssim 1 \end{aligned}$$

hold for all $t > 0$.

Then, repeating the proof of Proposition 3.1.1 we may conclude the desired estimates in Proposition 4.1.3. □

Following the approach of the proof of Proposition 4.1.3 we may prove the following statements.

Lemma 4.1.6. *The estimates*

$$\begin{aligned} &\big\|\mathfrak{F}^{-1}\big(|\xi|^a e^{(\lambda_2-\lambda_1)t}\chi_3(|\xi|)\big)(t,\cdot)\big\|_{L^1} \lesssim t^{-\frac{a}{2\delta}}, \\ &\big\|\mathfrak{F}^{-1}\big(|\xi|^{a+2\sigma-4\delta}e^{\lambda_2 t}\chi_3(|\xi|)\big)(t,\cdot)\big\|_{L^1} \lesssim t^{-\frac{a}{2\delta}}, \\ &\big\|\mathfrak{F}^{-1}\big(|\xi|^a e^{\lambda_1 t}\chi_3(|\xi|)\big)(t,\cdot)\big\|_{L^1} \lesssim t^{-\frac{a}{2(\sigma-\delta)}} \end{aligned}$$

hold for all $t > 0$ and any non-negative number a.

Proposition 4.1.4. *The estimates*

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_0}(t,\xi)\chi_3(|\xi|)\big)(t,\cdot)\big\|_{L^1}\lesssim\begin{cases}t^{-\frac{a}{2(\sigma-\delta)}} & \text{if } t\in(0,1],\\ t^{-\frac{a}{2\delta}} & \text{if } t\in[1,\infty),\end{cases}$$

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_1}(t,\xi)\chi_3(|\xi|)\big)(t,\cdot)\big\|_{L^1}\lesssim\begin{cases}t^{1-\frac{a}{2\delta}} & \text{if } t\in(0,1],\\ t^{1-\frac{a}{2(\sigma-\delta)}} & \text{if } t\in[1,\infty),\end{cases}$$

hold for any non-negative number a.

Estimates for middle frequencies

Now let us turn to consider some estimates for Fourier multipliers localized to middle frequencies, where $3^{-\frac{1}{2\delta-\sigma}}\leqslant|\xi|\leqslant 3^{\frac{1}{2\delta-\sigma}}$. Then, following the proof of Proposition 3.1.5 we may arrive at the exponential decay for the following norms.

Proposition 4.1.5. *The following estimates hold in $\mathbb{R}^n$ for any $n\geqslant 1$:*

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_0}(t,\xi)\chi_2(|\xi|)\big)(t,\cdot)\big\|_{L^1}\lesssim e^{-ct},$$
$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_1}(t,\xi)\chi_2(|\xi|)\big)(t,\cdot)\big\|_{L^1}\lesssim e^{-ct},$$
$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_0}(t,\xi)\chi_2(|\xi|)\big)(t,\cdot)\big\|_{L^\infty}\lesssim e^{-ct},$$
$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_1}(t,\xi)\chi_2(|\xi|)\big)(t,\cdot)\big\|_{L^\infty}\lesssim e^{-ct},$$

where c is a suitable positive constant and for any non-negative number a.

From the statements of Proposition 4.1.1, Proposition 4.1.3 and Proposition 4.1.5 we may conclude the following L^1 estimates.

Proposition 4.1.6. *The following estimates hold in $\mathbb{R}^n$:*

$$\big\|\mathfrak{F}^{-1}\big(\widehat{K_0}(t,\xi)\big)(t,\cdot)\big\|_{L^1}\lesssim\begin{cases}1 & \text{if } t\in(0,1],\\ t^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})} & \text{if } t\in[1,\infty),\end{cases}$$

$$\big\|\mathfrak{F}^{-1}\big(\widehat{K_1}(t,\xi)\big)(t,\cdot)\big\|_{L^1}\lesssim\begin{cases}t & \text{if } t\in(0,1],\\ t^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})} & \text{if } t\in[1,\infty).\end{cases}$$

Finally, from the statements of Proposition 4.1.2, Proposition 4.1.4 and Proposition 4.1.5 we may conclude the following L^1 estimates.

Proposition 4.1.7. *The following estimates hold in $\mathbb{R}^n$:*

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_0}(t,\xi)\big)(t,\cdot)\big\|_{L^1}\lesssim\begin{cases}t^{-\frac{a}{2(\sigma-\delta)}} & \text{if } t\in(0,1],\\ t^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})-\frac{a}{2\delta}} & \text{if } t\in[1,\infty),\end{cases}$$

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_1}(t,\xi)\big)(t,\cdot)\big\|_{L^1}\lesssim\begin{cases}t^{1-\frac{a}{2\delta}} & \text{if } t\in(0,1],\\ t^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})-\frac{a}{2\delta}} & \text{if } t\in[1,\infty),\end{cases}$$

for any non-negative number a.

L^∞ estimates

Proposition 4.1.8. *The following estimates hold in $\mathbb{R}^n$:*

$$\big\|\mathfrak{F}^{-1}\big(\widehat{K_0}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^\infty}\lesssim\begin{cases}1 & \text{if } t\in(0,1],\\ t^{-\frac{n}{2\delta}} & \text{if } t\in[1,\infty),\end{cases}$$

$$\big\|\mathfrak{F}^{-1}\big(\widehat{K_0}(t,\xi)\chi_3(|\xi|)\big)(t,\cdot)\big\|_{L^\infty}\lesssim t^{-\frac{n}{2(\sigma-\delta)}}\quad \text{for all } t\in(0,\infty),$$

$$\big\|\mathfrak{F}^{-1}\big(\widehat{K_1}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^\infty}\lesssim\begin{cases}t & \text{if } t\in(0,1],\\ t^{1-\frac{n}{2\delta}} & \text{if } t\in[1,\infty),\end{cases}$$

$$\big\|\mathfrak{F}^{-1}\big(\widehat{K_1}(t,\xi)\chi_3(|\xi|)\big)(t,\cdot)\big\|_{L^\infty}\lesssim t^{1-\frac{n}{2(\sigma-\delta)}}\quad \text{for all } t\in(0,\infty).$$

Proof. First let us consider estimates for small frequencies. For the sake of the asymptotic behavior of the characteristic roots, we re-write $\widehat{K_1}(t,\xi)$ as follows:

$$\widehat{K_1}(t,\xi) = \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} = e^{\lambda_1 t}\frac{1 - e^{(\lambda_2-\lambda_1)t}}{\lambda_1 - \lambda_2} = te^{\lambda_1 t}\int_0^1 e^{-i\theta\sqrt{4|\xi|^{2\sigma}-|\xi|^{4\delta}}t}d\theta.$$

Hence, we arrive at $|\widehat{K_1}(t,\xi)| \lesssim te^{-|\xi|^{2\delta}t}$ for small $|\xi|$ to derive

$$\begin{aligned}\big|\mathfrak{F}^{-1}\big(\widehat{K_1}(t,\xi)\chi_1(|\xi|)\big)(t,x)\big| &\lesssim \Big|\int_{\mathbb{R}^n} e^{ix\xi}\widehat{K_1}(t,\xi)\chi_1(|\xi|)d\xi\Big| \\ &\lesssim t\int_0^1 e^{-|\xi|^{2\delta}t}|\xi|^{n-1}d|\xi| \lesssim \begin{cases} t & \text{if } t\in(0,1], \\ t^{1-\frac{n}{2\delta}} & \text{if } t\in[1,\infty). \end{cases}\end{aligned}$$

Taking account of $\widehat{K_0}(t,\xi)$ for small frequencies, thanks to the asymptotic behavior of the characteristic roots we estimate

$$|\widehat{K_0}(t,\xi)| = \Big|\frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2}\Big| \lesssim e^{-|\xi|^{2\delta}t}.$$

Therefore, we conclude

$$\begin{aligned}\big|\mathfrak{F}^{-1}\big(\widehat{K_0}(t,\xi)\chi_1(|\xi|)\big)(t,x)\big| &\lesssim \Big|\int_{\mathbb{R}^n} e^{ix\xi}\widehat{K_0}(t,\xi)\chi_1(|\xi|)d\xi\Big| \\ &\lesssim \int_0^1 e^{-|\xi|^{2\delta}t}|\xi|^{n-1}d|\xi| \lesssim \begin{cases} 1 & \text{if } t\in(0,1], \\ t^{-\frac{n}{2\delta}} & \text{if } t\in[1,\infty). \end{cases}\end{aligned}$$

Let us turn to consider the term $\widehat{K_1}(t,\xi)$ for large $|\xi|$. We get

$$\widehat{K_1}(t,\xi) = e^{\lambda_1 t}\frac{1 - e^{(\lambda_2-\lambda_1)t}}{\lambda_1 - \lambda_2} = te^{\lambda_1 t}\int_0^1 e^{-\theta\sqrt{|\xi|^{4\delta}-4|\xi|^{2\sigma}}t}d\theta.$$

As a result, we estimate $|\widehat{K_1}(t,\xi)| \lesssim te^{-|\xi|^{2(\sigma-\delta)}t}$ for large $|\xi|$ to conclude

$$\big|\mathfrak{F}^{-1}\big(\widehat{K_1}(t,\xi)\chi_3(|\xi|)\big)\big| \lesssim t\int_1^\infty e^{-|\xi|^{2(\sigma-\delta)}t}|\xi|^{n-1}d|\xi| \lesssim t^{1-\frac{n}{2(\sigma-\delta)}} \quad \text{for all } t\in(0,\infty).$$

Finally, in order to estimate the term $\big|\mathfrak{F}^{-1}\big(\widehat{K_0}(t,\xi)\chi_3(|\xi|)\big)(t,x)\big|$, we can see that

$$\widehat{K_0}(t,\xi) = -\lambda_1\widehat{K_1}(t,\xi) + e^{\lambda_1 t}.$$

Since $|\widehat{K_1}(t,\xi)| \lesssim te^{-|\xi|^{2(\sigma-\delta)}t}$, we find for large $|\xi|$

$$|\widehat{K_0}(t,\xi)| \lesssim (1 + t|\xi|^{2(\sigma-\delta)})e^{-|\xi|^{2(\sigma-\delta)}t} \lesssim e^{-c|\xi|^{2(\sigma-\delta)}t},$$

for some positive constants c. Hence, we obtain

$$\big|\mathfrak{F}^{-1}\big(\widehat{K_0}(t,\xi)\chi_3(|\xi|)\big)(t,x)\big| \lesssim \int_1^\infty e^{-c|\xi|^{2(\sigma-\delta)}t}|\xi|^{n-1}d|\xi| \lesssim t^{-\frac{n}{2(\sigma-\delta)}} \quad \text{for all } t\in(0,\infty).$$

Summarizing, Proposition 4.1.8 is proved. □

From Proposition 4.1.5 and Propostition 4.1.8 we may conclude the following statement.

Proposition 4.1.9. *The following estimates hold in $\mathbb{R}^n$:*

$$\big\|\mathfrak{F}^{-1}\big(\widehat{K_0}(t,\xi)\big)(t,\cdot)\big\|_{L^\infty} \lesssim \begin{cases} t^{-\frac{n}{2(\sigma-\delta)}} & \text{if } t\in(0,1], \\ t^{-\frac{n}{2\delta}} & \text{if } t\in[1,\infty), \end{cases}$$

$$\big\|\mathfrak{F}^{-1}\big(\widehat{K_1}(t,\xi)\big)(t,\cdot)\big\|_{L^\infty} \lesssim \begin{cases} t^{1-\frac{n}{2(\sigma-\delta)}} & \text{if } t\in(0,1], \\ t^{1-\frac{n}{2\delta}} & \text{if } t\in[1,\infty). \end{cases}$$

Finally, following the approach of the proof of Proposition 4.1.8 we may prove the following statements.

Proposition 4.1.10. *The following estimates hold in* $\mathbb{R}^n$*:*

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_0}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^\infty} \lesssim \begin{cases} 1 & \text{if } t\in(0,1],\\ t^{-\frac{n+a}{2\delta}} & \text{if } t\in[1,\infty),\end{cases}$$

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_0}(t,\xi)\chi_3(|\xi|)\big)(t,\cdot)\big\|_{L^\infty} \lesssim t^{-\frac{n+a}{2(\sigma-\delta)}} \quad \text{for all } t\in(0,\infty),$$

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_1}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^\infty} \lesssim \begin{cases} t & \text{if } t\in(0,1],\\ t^{1-\frac{n+a}{2\delta}} & \text{if } t\in[1,\infty),\end{cases}$$

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_1}(t,\xi)\chi_3(|\xi|)\big)(t,\cdot)\big\|_{L^\infty} \lesssim t^{1-\frac{n+a}{2(\sigma-\delta)}} \quad \text{for all } t\in(0,\infty),$$

for any non-negative number a*.*

Proposition 4.1.11. *The following estimates hold in* $\mathbb{R}^n$*:*

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_0}(t,\xi)\big)(t,\cdot)\big\|_{L^\infty} \lesssim \begin{cases} t^{-\frac{n+a}{2(\sigma-\delta)}} & \text{if } t\in(0,1],\\ t^{-\frac{n+a}{2\delta}} & \text{if } t\in[1,\infty),\end{cases}$$

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_1}(t,\xi)\big)(t,\cdot)\big\|_{L^\infty} \lesssim \begin{cases} t^{1-\frac{n+a}{2(\sigma-\delta)}} & \text{if } t\in(0,1],\\ t^{1-\frac{n+a}{2\delta}} & \text{if } t\in[1,\infty),\end{cases}$$

for any non-negative number a*.*

L^r estimates

By an interpolation theorem, from the statements of Propositions 4.1.6 and 4.1.9 we may conclude the following statement.

Proposition 4.1.12. *The following estimates hold in* $\mathbb{R}^n$*:*

$$\big\|\mathfrak{F}^{-1}\big(\widehat{K_0}(t,\xi)\big)(t,\cdot)\big\|_{L^r} \lesssim \begin{cases} t^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})} & \text{if } t\in(0,1],\\ t^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})} & \text{if } t\in[1,\infty),\end{cases}$$

$$\big\|\mathfrak{F}^{-1}\big(\widehat{K_1}(t,\xi)\big)(t,\cdot)\big\|_{L^r} \lesssim \begin{cases} t^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})} & \text{if } t\in(0,1],\\ t^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})} & \text{if } t\in[1,\infty),\end{cases}$$

for all $r\in[1,\infty]$*.*

From the statements of Propositions 4.1.7 and 4.1.11, by applying an interpolation theorem we may conclude the following statement.

Proposition 4.1.13. *The following estimates hold in* $\mathbb{R}^n$*:*

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_0}(t,\xi)\big)(t,\cdot)\big\|_{L^r} \lesssim \begin{cases} t^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{a}{2(\sigma-\delta)}} & \text{if } t\in(0,1],\\ t^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{a}{2\delta}} & \text{if } t\in[1,\infty),\end{cases}$$

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_1}(t,\xi)\big)(t,\cdot)\big\|_{L^r} \lesssim \begin{cases} t^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{a}{2(\sigma-\delta)}} & \text{if } t\in(0,1],\\ t^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{a}{2\delta}} & \text{if } t\in[1,\infty),\end{cases}$$

for all $r\in[1,\infty]$ *and any non-negative number* a*.*

Proof of Theorem 4.1.1. In order to obtain $(L^m\cap L^q)-L^q$ estimates, we estimate the L^q norm of the low-frequency part of solutions by the L^m norm of the data, whereas their high-frequency part are estimated on L^q-L^q basis. Thanks to Proposition 4.1.2 and Proposition 4.1.10, we derive

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_0}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^r} \lesssim \begin{cases} 1 & \text{if } t\in(0,1],\\ t^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{a}{2\delta}} & \text{if } t\in[1,\infty),\end{cases}$$

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_1}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^r} \lesssim \begin{cases} t & \text{if } t\in(0,1],\\ t^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{a}{2\delta}} & \text{if } t\in[1,\infty),\end{cases}$$

for all $r \in [1,\infty]$ and any non-negative number a. Applying Young's convolution inequality from Proposition B.1.1 we derive

$$\|v_{\chi_1}(t,\cdot)\|_{L^q} \lesssim \big\|\mathfrak{F}^{-1}\big(\widehat{K_1}(t,\xi)\chi_1(\xi)\big)(t,\cdot)\big\|_{L^r}\|v_1\|_{L^m} \lesssim \begin{cases} t\,\|v_1\|_{L^m} & \text{if } t \in (0,1],\\ t^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|v_1\|_{L^m} & \text{if } t \in [1,\infty), \end{cases}$$

and

$$\|v_{\chi_2}(t,\cdot)\|_{L^q} \lesssim \big\|\mathfrak{F}^{-1}\big(\widehat{K_1}(t,\xi)\chi_2(|\xi|)\big)(t,\cdot)\big\|_{L^1}\|v_1\|_{L^q} \lesssim e^{-ct}\|v_1\|_{L^q} \quad \text{for all } t \in (0,\infty),$$
$$\|v_{\chi_3}(t,\cdot)\|_{L^q} \lesssim \big\|\mathfrak{F}^{-1}\big(\widehat{K_1}(t,\xi)\chi_3(|\xi|)\big)(t,\cdot)\big\|_{L^1}\|v_1\|_{L^q} \lesssim t\,\|v_1\|_{L^q} \quad \text{for all } t \in (0,\infty),$$

where c is a suitable positive constant. Hence, we may conclude

$$\|v(t,\cdot)\|_{L^q} \lesssim (1+t)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|v_1\|_{L^m\cap L^q} \quad \text{for all } t \in (0,\infty).$$

Analogously, we derive

$$\begin{aligned}\big\||D|^\sigma v_{\chi_1}(t,\cdot)\big\|_{L^q} &\lesssim \big\|\mathfrak{F}^{-1}\big(|\xi|^\sigma\widehat{K_1}(t,\xi)\chi_1(\xi)\big)(t,\cdot)\big\|_{L^r}\|v_1\|_{L^m}\\ &\lesssim \begin{cases} t\,\|v_1\|_{L^m} & \text{if } t \in (0,1],\\ t^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{\sigma}{2\delta}}\|v_1\|_{L^m} & \text{if } t \in [1,\infty), \end{cases}\end{aligned}$$

and

$$\big\||D|^\sigma v_{\chi_2}(t,\cdot)\big\|_{L^q} \lesssim \big\|\mathfrak{F}^{-1}\big(|\xi|^\sigma\widehat{K_1}(t,\xi)\chi_2(|\xi|)\big)(t,\cdot)\big\|_{L^1}\|v_1\|_{L^q} \lesssim e^{-ct}\|v_1\|_{L^q} \quad \text{for all } t \in (0,\infty),$$
$$\begin{aligned}\big\||D|^\sigma v_{\chi_3}(t,\cdot)\big\|_{L^q} &\lesssim \big\|\mathfrak{F}^{-1}\big(|\xi|^\sigma\widehat{K_1}(t,\xi)\chi_3(|\xi|)\big)(t,\cdot)\big\|_{L^1}\|v_1\|_{L^q}\\ &\lesssim \begin{cases} t^{1-\frac{\sigma}{2\delta}}\|v_1\|_{L^q} & \text{if } t \in (0,1]\\ t^{1-\frac{\sigma}{2(\sigma-\delta)}}\|v_1\|_{L^q} & \text{if } t \in [1,\infty) \end{cases} \lesssim \|v_1\|_{L^q} \quad \text{for all } t \in (0,\infty),\end{aligned}$$

where c is a suitable positive constant. Hence, we may conclude

$$\big\||D|^\sigma v(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{\sigma}{2\delta}}\|v_1\|_{L^m\cap L^q} \quad \text{for all } t \in (0,\infty).$$

In the same way we also obtain

$$\big\||D|^{2\delta} v(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|v_1\|_{L^m\cap L^q} \quad \text{for all } t \in (0,\infty).$$

Now let us turn to estimate for the term $\|v_t(t,\cdot)\|_{L^q}$. We rewrite

$$\partial_t\widehat{K_1}(t,\xi) = \widehat{K_0}(t,\xi) + (\lambda_1+\lambda_2)\widehat{K_1}(t,\xi) = \widehat{K_0}(t,\xi) - |\xi|^{2\delta}\widehat{K_1}(t,\xi).$$

Applying again Young's convolution inequality from Proposition B.1.1 we get

$$\begin{aligned}\|\partial_t v_{\chi_1}(t,\cdot)\|_{L^q} &= \big\|\mathfrak{F}^{-1}\big(\big(\widehat{K_0}(t,\xi) - |\xi|^{2\delta}\widehat{K_1}(t,\xi)\big)\chi_1(|\xi|)\widehat{v_1}(\xi)\big)(t,\cdot)\big\|_{L^q}\\ &\lesssim \Big(\big\|\mathfrak{F}^{-1}\big(\widehat{K_0}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^r} + \big\|\mathfrak{F}^{-1}\big(|\xi|^{2\delta}\widehat{K_1}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^r}\Big)\|v_1\|_{L^m}\\ &\lesssim \begin{cases} (1+t)\|v_1\|_{L^m} & \text{if } t \in (0,1],\\ \big(t^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})} + t^{(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\big)\|v_1\|_{L^m} & \text{if } t \in [1,\infty), \end{cases}\\ &\lesssim \begin{cases} \|v_1\|_{L^m} & \text{if } t \in (0,1],\\ t^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|v_1\|_{L^m} & \text{if } t \in [1,\infty), \end{cases}\end{aligned}$$

and

$$\begin{aligned}\|\partial_t v_{\chi_2}(t,\cdot)\|_{L^q} &\lesssim \Big(\big\|\mathfrak{F}^{-1}\big(\widehat{K_0}(t,\xi)\chi_2(|\xi|)\big)(t,\cdot)\big\|_{L^1} + \big\|\mathfrak{F}^{-1}\big(|\xi|^{2\delta}\widehat{K_1}(t,\xi)\chi_2(|\xi|)\big)(t,\cdot)\big\|_{L^1}\Big)\|v_1\|_{L^q}\\ &\lesssim e^{-ct}\|v_1\|_{L^q} \quad \text{for all } t \in (0,\infty),\\ \|\partial_t v_{\chi_3}(t,\cdot)\|_{L^q} &\lesssim \Big(\big\|\mathfrak{F}^{-1}\big(\widehat{K_0}(t,\xi)\chi_3(|\xi|)\big)(t,\cdot)\big\|_{L^1} + \big\|\mathfrak{F}^{-1}\big(|\xi|^{2\delta}\widehat{K_1}(t,\xi)\chi_3(|\xi|)\big)(t,\cdot)\big\|_{L^1}\Big)\|v_1\|_{L^q}\\ &\lesssim \begin{cases} \|v_1\|_{L^q} & \text{if } t \in (0,1]\\ \big(1+t^{1-\frac{\delta}{\sigma-\delta}}\big)\|v_1\|_{L^q} & \text{if } t \in [1,\infty) \end{cases} \lesssim \|v_1\|_{L^q} \quad \text{for all } t \in (0,\infty),\end{aligned}$$

where c is a suitable positive constant. Therefore, we may imply

$$\|v_t(t,\cdot)\|_{L^q} \lesssim (1+t)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|v_1\|_{L^m\cap L^q} \quad \text{for all } t \in (0,\infty).$$

Summarizing, the proof of Theorem 4.1.1 is completed. □

$L^m - L^q$ estimates

From the proof of Theorem 4.1.1 and the statements of Proposition 4.1.13 we have the following corollary.

Corollary 4.1.1. *Let $\delta \in \big(\frac{\sigma}{2},\sigma\big)$ in (4.3), $q \in [1,\infty]$ and $m \in [1,q]$. Then, the energy solutions to (4.3) satisfy the $L^m - L^q$ estimates*

$$\|v(t,\cdot)\|_{L^q} \lesssim \begin{cases} t^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})}\|v_1\|_{L^m} & \text{if } t \in (0,1], \\ t^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|v_1\|_{L^m} & \text{if } t \in [1,\infty), \end{cases}$$

$$\big\||D|^\sigma v(t,\cdot)\big\|_{L^q} \lesssim \begin{cases} t^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\sigma}{2(\sigma-\delta)}}\|v_1\|_{L^m} & \text{if } t \in (0,1], \\ t^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{\sigma}{2\delta}}\|v_1\|_{L^m} & \text{if } t \in [1,\infty), \end{cases}$$

$$\|v_t(t,\cdot)\|_{L^q} \lesssim \begin{cases} t^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}}\|v_1\|_{L^m} & \text{if } t \in (0,1], \\ t^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|v_1\|_{L^m} & \text{if } t \in [1,\infty), \end{cases}$$

$$\big\||D|^{2\delta} v(t,\cdot)\big\|_{L^q} \lesssim \begin{cases} t^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}}\|v_1\|_{L^m} & \text{if } t \in (0,1], \\ t^{(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|v_1\|_{L^m} & \text{if } t \in [1,\infty), \end{cases}$$

and the energy solutions to (4.2) satisfy the $L^m - L^q$ estimates

$$\|u(t,\cdot)\|_{L^q} \lesssim \begin{cases} (t-s)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})}\|u_1\|_{L^m} & \text{if } t \in (s,s+1], \\ (t-s)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|u_1\|_{L^m} & \text{if } t \in [s+1,\infty), \end{cases}$$

$$\big\||D|^\sigma u(t,\cdot)\big\|_{L^q} \lesssim \begin{cases} (t-s)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\sigma}{2(\sigma-\delta)}}\|u_1\|_{L^m} & \text{if } t \in (s,s+1], \\ (t-s)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{\sigma}{2\delta}}\|u_1\|_{L^m} & \text{if } t \in [s+1,\infty), \end{cases}$$

$$\|u_t(t,\cdot)\|_{L^q} \lesssim \begin{cases} (t-s)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}}\|u_1\|_{L^m} & \text{if } t \in (s,s+1], \\ (t-s)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|u_1\|_{L^m} & \text{if } t \in [s+1,\infty), \end{cases}$$

$$\big\||D|^{2\delta} u(t,\cdot)\big\|_{L^q} \lesssim \begin{cases} (t-s)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}}\|u_1\|_{L^m} & \text{if } t \in (s,s+1], \\ (t-s)^{(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|u_1\|_{L^m} & \text{if } t \in [s+1,\infty), \end{cases}$$

where $1+\frac{1}{q}=\frac{1}{r}+\frac{1}{m}$ and for all dimensions $n \geqslant 1$.

4.2. A second Cauchy problem for linear structurally damped σ-evolution models

Let us turn to the following Cauchy problem:

$$u_{tt}+(-\Delta)^\sigma u+\mu(-\Delta)^\delta u_t=0, \quad u(0,x)=u_0(x), \quad u_t(0,x)=0, \tag{4.19}$$

where $\sigma \geqslant 1$, $\mu>0$ and $\delta \in (\frac{\sigma}{2},\sigma)$.

$L^m \cap L^q - L^q$ and $L^q - L^q$ estimates

In this section, we want to prove the following result.

Theorem 4.2.1. *Let $\delta \in \big(\frac{\sigma}{2},\sigma\big)$ in (4.19), $q \in (1,\infty)$ be given and $m \in [1,q)$. Then, the energy solutions to (4.19) satisfy the $(L^m \cap L^q) - L^q$ estimates*

$$\|u(t,\cdot)\|_{L^q} \lesssim (1+t)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|u_0\|_{L^m\cap L^q},$$

$$\big\||D|^\sigma u(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{\sigma}{2\delta}}\|u_0\|_{L^m\cap H^\sigma_q},$$

$$\|u_t(t,\cdot)\|_{L^q} \lesssim (1+t)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{\sigma}{\delta}}\|u_0\|_{L^m\cap H^{2(\sigma-\delta)}_q},$$

$$\big\||D|^{2\delta} u(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-1}\|u_0\|_{L^m\cap H^{2\delta}_q},$$

and the $L^q - L^q$ estimates

$$\|u(t,\cdot)\|_{L^q} \lesssim (1+t)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})}\|u_0\|_{L^q},$$
$$\big\||D|^\sigma u(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})-\frac{\sigma}{2\delta}}\|u_0\|_{H^\sigma_q},$$
$$\|u_t(t,\cdot)\|_{L^q} \lesssim (1+t)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})-\frac{\sigma}{2\delta}}\|u_0\|_{H^{2(\sigma-\delta)}_q},$$
$$\big\||D|^{2\delta} u(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})-1}\|u_0\|_{H^{2\delta}_q},$$

where $1+\frac{1}{q} = \frac{1}{r}+\frac{1}{m}$ and for all dimensions n.

Proof. Applying the partial Fourier transformation to (4.19) we obtain the Cauchy problem for $\widehat{u}(t,\xi) := \mathfrak{F}(u(t,x))$ and $\widehat{u_0}(\xi) := \mathfrak{F}(u_0(x))$ as follows:

$$\widehat{u}_{tt} + \mu|\xi|^{2\delta}\widehat{u}_t + |\xi|^{2\sigma}\widehat{u} = 0, \quad \widehat{u}(0,\xi) = \widehat{u_0}(\xi), \quad \widehat{u}_t(0,\xi) = 0. \tag{4.20}$$

We may choose without loss of generality $\mu = 1$ in (4.19). The characteristic roots are

$$\lambda_{1,2} = \lambda_{1,2}(\xi) = \frac{1}{2}\Big(-|\xi|^{2\delta} \pm \sqrt{|\xi|^{4\delta} - 4|\xi|^{2\sigma}}\Big).$$

The solutions to (4.20) are presented by the following formula (here we assume $\lambda_1 \neq \lambda_2$):

$$\widehat{u}(t,\xi) = \frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2}\widehat{u_0}(\xi) =: \widehat{K_0}(t,\xi)\widehat{u_0}(\xi).$$

Taking account of the cases of small and large frequencies separately we have

1. $\lambda_{1,2} \sim -|\xi|^{2\delta} \pm i|\xi|^\sigma, \quad \lambda_1 - \lambda_2 \sim i|\xi|^\sigma$ for small $|\xi| \in (0, 3^{-\frac{1}{2\delta-\sigma}})$,
2. $\lambda_1 \sim -|\xi|^{2(\sigma-\delta)}, \quad \lambda_2 \sim -|\xi|^{2\delta}, \quad \lambda_1 - \lambda_2 \sim |\xi|^{2\delta}$ for large $|\xi| \in (3^{\frac{1}{2\delta-\sigma}}, \infty)$.

As in Section 4.1, we now decompose the solution to (4.19) into three parts localized separately to low, middle and high frequencies, that is,

$$u(t,x) = u_{\chi_1}(t,x) + u_{\chi_2}(t,x) + u_{\chi_3}(t,x),$$

where

$$u_{\chi_k}(t,x) = \mathfrak{F}^{-1}\big(\chi_k(|\xi|)\widehat{u}(t,\xi)\big) \quad \text{with } k = 1,2,3.$$

In order to obtain the $(L^m \cap L^q) - L^q$ estimates, we estimate the L^q norm of the low-frequency part of solutions by the L^m norm of the data, whereas their high-frequency and middle-frequency parts are estimated on $L^q - L^q$ basis. Thanks to Proposition 4.1.2 and Proposition 4.1.10, we derive

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_0}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^r} \lesssim \begin{cases} 1 & \text{if } t \in (0,1], \\ t^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r} - \frac{n}{2\delta}(1-\frac{1}{r}) - \frac{a}{2\delta}} & \text{if } t \in [1,\infty), \end{cases}$$
$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_1}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^r} \lesssim \begin{cases} t & \text{if } t \in (0,1], \\ t^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r} - \frac{n}{2\delta}(1-\frac{1}{r}) - \frac{a}{2\delta}} & \text{if } t \in [1,\infty), \end{cases}$$

for all $r \in [1,\infty]$ and any non-negative number a. Applying Young's convolution inequality from Proposition B.1.1 we derive

$$\|u_{\chi_1}(t,\cdot)\|_{L^q} \lesssim \big\|\mathfrak{F}^{-1}\big(\widehat{K_0}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^r}\|u_0\|_{L^m} \lesssim \begin{cases} \|u_0\|_{L^m} & \text{if } t \in (0,1], \\ t^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r} - \frac{n}{2\delta}(1-\frac{1}{r})}\|u_0\|_{L^m} & \text{if } t \in [1,\infty), \end{cases}$$

and

$$\|u_{\chi_2}(t,\cdot)\|_{L^q} \lesssim \big\|\mathfrak{F}^{-1}\big(\widehat{K_0}(t,\xi)\chi_2(|\xi|)\big)(t,\cdot)\big\|_{L^1}\|u_0\|_{L^q} \lesssim \|u_0\|_{L^q} \lesssim e^{-ct}\|u_0\|_{L^q} \quad \text{for all } t \in (0,\infty),$$
$$\|u_{\chi_3}(t,\cdot)\|_{L^q} \lesssim \big\|\mathfrak{F}^{-1}\big(\widehat{K_0}(t,\xi)\chi_3(|\xi|)\big)(t,\cdot)\big\|_{L^1}\|u_0\|_{L^q} \lesssim \|u_0\|_{L^q} \quad \text{for all } t \in (0,\infty),$$

where c is a suitable positive constant. Hence, we may conclude

$$\|u(t,\cdot)\|_{L^q} \lesssim (1+t)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r} - \frac{n}{2\delta}(1-\frac{1}{r})}\|u_0\|_{L^m \cap L^q} \quad \text{for all } t \in (0,\infty).$$

In an analogous way we derive

$$\big\||D|^\sigma u_{\chi_1}(t,\cdot)\big\|_{L^q} \lesssim \big\|\mathfrak{F}^{-1}\big(|\xi|^\sigma \widehat{K_0}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^r}\|u_0\|_{L^m}$$
$$\lesssim \begin{cases} \|u_0\|_{L^m} & \text{if } t\in(0,1],\\ t^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{\sigma}{2\delta}}\|u_0\|_{L^m} & \text{if } t\in[1,\infty),\end{cases}$$

and

$$\big\||D|^\sigma u_{\chi_2}(t,\cdot)\big\|_{L^q} \lesssim \big\|\mathfrak{F}^{-1}\big(|\xi|^\sigma \widehat{K_0}(t,\xi)\chi_2(|\xi|)\big)(t,\cdot)\big\|_{L^1}\|u_0\|_{L^q} \lesssim e^{-ct}\|u_0\|_{L^q} \quad \text{for all } t\in(0,\infty),$$
$$\big\||D|^\sigma u_{\chi_3}(t,\cdot)\big\|_{L^q} \lesssim \big\|\mathfrak{F}^{-1}\big(\widehat{K_0}(t,\xi)\chi_3(|\xi|)\big)(t,\cdot)\big\|_{L^1}\big\|\mathfrak{F}^{-1}\big(|\xi|^\sigma\widehat{u_0}(\xi)\big)(t,\cdot)\big\|_{L^q}$$
$$\lesssim \|u_0\|_{\dot{H}^\sigma_q} \lesssim \|u_0\|_{H^\sigma_q} \quad \text{for all } t\in(0,\infty),$$

where c is a suitable positive constant. Hence, we may conclude

$$\big\||D|^\sigma u(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{\sigma}{2\delta}}\|u_0\|_{L^m\cap H^\sigma_q} \quad \text{for all } t\in(0,\infty).$$

After the same treatment we also obtain

$$\big\||D|^{2\delta} u(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-1}\|u_0\|_{L^m\cap H^{2\delta}_q} \quad \text{for all } t\in(0,\infty).$$

Now let us turn to estimate for the norm $\|u_t(t,\cdot)\|_{L^q}$. We rewrite

$$\partial_t\widehat{K_0}(t,\xi) = -\lambda_1\lambda_2\widehat{K_1}(t,\xi) = -|\xi|^{2\sigma}\widehat{K_1}(t,\xi).$$

Applying again Young's convolution inequality from Proposition B.1.1 we get

$$\|\partial_t u_{\chi_1}(t,\cdot)\|_{L^q} = \big\|\mathfrak{F}^{-1}\big(|\xi|^{2\sigma}\widehat{K_1}(t,\xi)\chi_1(|\xi|)\widehat{u_0}(\xi)\big)(t,\cdot)\big\|_{L^q} \lesssim \big\|\mathfrak{F}^{-1}\big(|\xi|^{2\sigma}\widehat{K_1}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^r}\|u_0\|_{L^m}$$
$$\lesssim \begin{cases} t\,\|u_0\|_{L^m} & \text{if } t\in(0,1],\\ t^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{\sigma}{\delta}}\|u_0\|_{L^m} & \text{if } t\in[1,\infty),\end{cases}$$

and

$$\|\partial_t u_{\chi_2}(t,\cdot)\|_{L^q} \lesssim \big\|\mathfrak{F}^{-1}\big(|\xi|^{2\sigma}\widehat{K_1}(t,\xi)\chi_2(|\xi|)\big)(t,\cdot)\big\|_{L^1}\|u_0\|_{L^q} \lesssim e^{-ct}\|u_0\|_{L^q} \quad \text{for all } t\in(0,\infty),$$
$$\|\partial_t u_{\chi_3}(t,\cdot)\|_{L^q} \lesssim \big\|\mathfrak{F}^{-1}\big(|\xi|^{2\delta}\widehat{K_1}(t,\xi)\chi_3(|\xi|)\big)(t,\cdot)\big\|_{L^1}\big\|\mathfrak{F}^{-1}\big(|\xi|^{2(\sigma-\delta)}\widehat{u_0}(\xi)\big)\big\|_{L^q}$$
$$\lesssim \begin{cases} \|u_0\|_{\dot{H}^{2(\sigma-\delta)}_q} & \text{if } t\in(0,1],\\ t^{1-\frac{\delta}{\sigma-\delta}}\|u_0\|_{\dot{H}^{2(\sigma-\delta)}_q} & \text{if } t\in[1,\infty),\end{cases}$$
$$\lesssim \|u_0\|_{\dot{H}^{2(\sigma-\delta)}_q} \lesssim \|u_0\|_{H^{2(\sigma-\delta)}_q} \quad \text{for all } t\in(0,\infty),$$

where c is a suitable positive constant. Therefore, we may conclude

$$\|u_t(t,\cdot)\|_{L^q} \lesssim (1+t)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{\sigma}{\delta}}\|u_0\|_{L^m\cap H^{2(\sigma-\delta)}_q} \quad \text{for all } t\in(0,\infty).$$

Summarizing, the proof of Theorem 4.2.1 is completed. □

$L^m - L^q$ estimates

From the proof of Theorem 4.2.1 and the statements of Proposition 4.1.13 we have the following corollary.

Corollary 4.2.1. *Let $\delta\in\big(\frac{\sigma}{2},\sigma\big)$ in (4.19), $q\in[1,\infty]$ and $m\in[1,q]$. Then, the energy solutions to (4.19) satisfy the L^m-L^q estimates*

$$\|u(t,\cdot)\|_{L^q} \lesssim \begin{cases} t^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})}\|u_0\|_{L^m} & \textit{if } t\in(0,1],\\ t^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|u_0\|_{L^m} & \textit{if } t\in[1,\infty),\end{cases}$$
$$\big\||D|^\sigma u(t,\cdot)\big\|_{L^q} \lesssim \begin{cases} t^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\sigma}{2(\sigma-\delta)}}\|u_0\|_{L^m} & \textit{if } t\in(0,1],\\ t^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{\sigma}{2\delta}}\|u_0\|_{L^m} & \textit{if } t\in[1,\infty),\end{cases}$$
$$\|u_t(t,\cdot)\|_{L^q} \lesssim \begin{cases} t^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\sigma}{\sigma-\delta}}\|u_0\|_{L^m} & \textit{if } t\in(0,1],\\ t^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{\sigma}{\delta}}\|u_0\|_{L^m} & \textit{if } t\in[1,\infty),\end{cases}$$
$$\big\||D|^{2\delta} u(t,\cdot)\big\|_{L^q} \lesssim \begin{cases} t^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}}\|u_0\|_{L^m} & \textit{if } t\in(0,1],\\ t^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-1}\|u_0\|_{L^m} & \textit{if } t\in[1,\infty),\end{cases}$$

where $1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{m}$ *and for all dimensions* $n \geq 1$.

4.3. A third Cauchy problem for linear structurally damped σ-evolution models

In this section, let us consider the Cauchy problem for structurally damped σ-evolution models in the form

$$u_{tt} + (-\Delta)^\sigma u + \mu(-\Delta)^\delta u_t = 0, \quad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \tag{4.21}$$

with $\sigma \geq 1$, $\mu > 0$ and $\delta \in (\frac{\sigma}{2}, \sigma)$.

We may summarize the results from Sections 4.1 and 4.2 as follows:

$L^m \cap L^q - L^q$ **and** $L^q - L^q$ **estimates**

Theorem 4.3.1. *Let* $\delta \in \big(\frac{\sigma}{2}, \sigma\big)$ *in (4.21),* $q \in (1,\infty)$ *and* $m \in [1,q)$. *Then, the energy solutions to (4.21) satisfy the* $(L^m \cap L^q) - L^q$ *estimates*

$$\begin{aligned}
\|u(t,\cdot)\|_{L^q} &\lesssim (1+t)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|u_0\|_{L^m\cap L^q} \\
&\quad + (1+t)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|u_1\|_{L^m\cap L^q}, \\
\big\||D|^\sigma u(t,\cdot)\big\|_{L^q} &\lesssim (1+t)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{\sigma}{2\delta}}\|u_0\|_{L^m\cap H^\sigma_q} \\
&\quad + (1+t)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{\sigma}{2\delta}}\|u_1\|_{L^m\cap L^q}, \\
\|u_t(t,\cdot)\|_{L^q} &\lesssim (1+t)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{\sigma}{\delta}}\|u_0\|_{L^m\cap H^{2(\sigma-\delta)}_q} \\
&\quad + (1+t)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|u_1\|_{L^m\cap L^q}, \\
\big\||D|^{2\delta} u(t,\cdot)\big\|_{L^q} &\lesssim (1+t)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-1}\|u_0\|_{L^m\cap H^{2\delta}_q} \\
&\quad + (1+t)^{(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|u_1\|_{L^m\cap L^q},
\end{aligned}$$

and the $L^q - L^q$ *estimates*

$$\begin{aligned}
\|u(t,\cdot)\|_{L^q} &\lesssim (1+t)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})}\|u_0\|_{L^q} + (1+t)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})}\|u_1\|_{L^q}, \\
\big\||D|^\sigma u(t,\cdot)\big\|_{L^q} &\lesssim (1+t)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})-\frac{\sigma}{2\delta}}\|u_0\|_{H^\sigma_q} + (1+t)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})}\|u_1\|_{L^q}, \\
\|u_t(t,\cdot)\|_{L^q} &\lesssim (1+t)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})-\frac{\sigma}{2\delta}}\|u_0\|_{H^{2(\sigma-\delta)}_q} + (1+t)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})}\|u_1\|_{L^q}, \\
\big\||D|^{2\delta} u(t,\cdot)\big\|_{L^q} &\lesssim (1+t)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})-1}\|u_0\|_{H^{2\delta}_q} + (1+t)^{(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})}\|u_1\|_{L^q},
\end{aligned}$$

where $1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{m}$ *and for all dimensions* $n \geq 1$.

For space dimensions $n > \sigma$, we obtain the following better estimates for solutions to (4.21):

Theorem 4.3.2. *Let* $\delta \in \big(\frac{\sigma}{2}, \sigma\big)$ *in (4.21),* $q \in (1,\infty)$ *and* $m \in [1,q)$. *Then, the energy solutions to (4.21) satisfy the* $(L^m \cap L^q) - L^q$ *estimates*

$$\begin{aligned}
\|u(t,\cdot)\|_{L^q} &\lesssim (1+t)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|u_0\|_{L^m\cap L^q} \\
&\quad + (1+t)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})+\frac{\sigma}{2\delta}}\|u_1\|_{L^m\cap L^q}, \\
\big\||D|^\sigma u(t,\cdot)\big\|_{L^q} &\lesssim (1+t)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{\sigma}{2\delta}}\|u_0\|_{L^m\cap H^\sigma_q} \\
&\quad + (1+t)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|u_1\|_{L^m\cap L^q}, \\
\|u_t(t,\cdot)\|_{L^q} &\lesssim (1+t)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{\sigma}{2\delta}}\|u_0\|_{L^m\cap H^{2(\sigma-\delta)}_q} \\
&\quad + (1+t)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|u_1\|_{L^m\cap L^q}, \\
\big\||D|^{2\delta} u(t,\cdot)\big\|_{L^q} &\lesssim (1+t)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-1}\|u_0\|_{L^m\cap H^{2\delta}_q} \\
&\quad + (1+t)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-1+\frac{\sigma}{2\delta}}\|u_1\|_{L^m\cap L^q},
\end{aligned}$$

and the $L^q - L^q$ estimates

$$\begin{aligned}
\|u(t,\cdot)\|_{L^q} &\lesssim (1+t)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})}\|u_0\|_{L^q} + (1+t)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})}\|u_1\|_{L^q},\\
\big\||D|^\sigma u(t,\cdot)\big\|_{L^q} &\lesssim (1+t)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})-\frac{\sigma}{2\delta}}\|u_0\|_{H^\sigma_q} + (1+t)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})}\|u_1\|_{L^q},\\
\|u_t(t,\cdot)\|_{L^q} &\lesssim (1+t)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})-\frac{\sigma}{2\delta}}\|u_0\|_{H^{2(\sigma-\delta)}_q} + (1+t)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})}\|u_1\|_{L^q},\\
\big\||D|^{2\delta} u(t,\cdot)\big\|_{L^q} &\lesssim (1+t)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})-1}\|u_0\|_{H^{2\delta}_q} + (1+t)^{(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})}\|u_1\|_{L^q},
\end{aligned}$$

where $1+\frac{1}{q}=\frac{1}{r}+\frac{1}{m}$ and the constraint condition to the space dimension $n>\sigma$.

Proof. With space dimensions $n>\sigma$ we may improve the estimate for

$$\big\|\mathfrak{F}^{-1}\big(\widehat{K_1}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^\infty}$$

in Proposition 4.1.8 to obtain a better estimate. Namely, because of the asymptotic behaviour of the characteristic roots, we obtain for small frequencies the following estimate:

$$|\widehat{K_1}(t,\xi)| \lesssim |\xi|^{-\sigma}e^{-|\xi|^{2\delta}t}.$$

Hence, we derive

$$\big|\mathfrak{F}^{-1}\big(\widehat{K_1}(t,\xi)\chi_1(|\xi|)\big)(t,x)\big| \lesssim \Big|\int_{\mathbb{R}^n} e^{ix\xi}\widehat{K_1}(t,\xi)\chi_1(|\xi|)d\xi\Big| \lesssim \int_0^1 e^{-|\xi|^{2\delta}t}|\xi|^{n-\sigma-1}d|\xi| \lesssim t^{-\frac{n-\sigma}{2\delta}}$$

for t large and under the restriction to the dimension $n>\sigma$. For this reason, it follows for any $a\geqslant 0$ the estimate

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_1}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^\infty} \lesssim \begin{cases} t & \text{if } t\in(0,1],\\ t^{-\frac{n+a-\sigma}{2\delta}} & \text{if } t\in[1,\infty).\end{cases}$$

Using again Proposition 4.1.2 and the remaining estimates in Proposition 4.1.8 we get

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_0}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^r} \lesssim \begin{cases} 1 & \text{if } t\in(0,1],\\ t^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{a}{2\delta}} & \text{if } t\in[1,\infty),\end{cases}$$

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{K_1}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^r} \lesssim \begin{cases} t & \text{if } t\in(0,1],\\ t^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{a-\sigma}{2\delta}} & \text{if } t\in[1,\infty),\end{cases}$$

for all $r\in[1,\infty]$ and any non-negative number a. Then, repeating an analogous approach to prove Theorems 4.1.1 and 4.2.1 we may conlude all the statements in Theorem 4.3.2. ☐

We may prove similar estimates to those in Theorem 4.3.2. Namely for any $a\geqslant 0$, we have the following further results.

Theorem 4.3.3. *Let $\delta\in\big(\frac{\sigma}{2},\sigma\big)$ in (4.21), $q\in(1,\infty)$ and $m\in[1,q)$. Then, the Sobolev solutions to (4.21) satisfy the $(L^m\cap L^q)-L^q$ estimates*

$$\begin{aligned}
\big\||D|^a u(t,\cdot)\big\|_{L^q} &\lesssim (1+t)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{a}{2\delta}}\|u_0\|_{L^m\cap H^a_q}\\
&\quad + (1+t)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{a}{2\delta}}\|u_1\|_{L^m\cap H^{[a-2\delta]^+}_q},\\
\big\||D|^a u_t(t,\cdot)\big\|_{L^q} &\lesssim (1+t)^{(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{a+2(\sigma-\delta)}{2\delta}}\|u_0\|_{L^m\cap H^{a+2(\sigma-\delta)}_q}\\
&\quad + (1+t)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{a}{2\delta}}\|u_1\|_{L^m\cap H^a_q},
\end{aligned}$$

and the L^q-L^q estimates

$$\begin{aligned}
\big\||D|^a u(t,\cdot)\big\|_{L^q} &\lesssim (1+t)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})-\frac{a}{2\delta}}\|u_0\|_{H^a_q} + (1+t)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})-\frac{a}{2\delta}}\|u_1\|_{H^{[a-2\delta]^+}_q},\\
\big\||D|^a u_t(t,\cdot)\big\|_{L^q} &\lesssim (1+t)^{(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})-\frac{a+2(\sigma-\delta)}{2\delta}}\|u_0\|_{H^{a+2(\sigma-\delta)}_q} + (1+t)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})-\frac{a}{2\delta}}\|u_1\|_{H^a_q},
\end{aligned}$$

for any $a\geqslant 0$, where $1+\frac{1}{q}=\frac{1}{r}+\frac{1}{m}$ and all dimensions $n\geqslant 1$.

Proof. In order to estimate some of derivatives of solutions we use a suitable regularity of data u_0 and u_1 depending on the order of a. Then, repeating an analogous treatment as we did in the proofs of Theorem 4.1.1 and Theorem 4.2.1 we may conlude all the statements in Theorem 4.3.3. ☐

$L^m - L^q$ estimates

Corollary 4.3.1. *Let $\delta \in \big(\frac{\sigma}{2}, \sigma\big)$ in (4.21), $q \in [1,\infty]$ and $m \in [1,q]$. Then, the energy solutions to (4.21) satisfy the $L^m - L^q$ estimates*

$$\|u(t,\cdot)\|_{L^q} \lesssim \begin{cases} t^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})}\|u_0\|_{L^m} + t^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})}\|u_1\|_{L^m} & \text{if } t \in (0,1], \\ t^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|u_0\|_{L^m} \\ \quad + t^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|u_1\|_{L^m} & \text{if } t \in [1,\infty), \end{cases}$$

$$\big\||D|^\sigma u(t,\cdot)\big\|_{L^q} \lesssim \begin{cases} t^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\sigma}{2(\sigma-\delta)}}\|u_0\|_{L^m} + t^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\sigma}{2(\sigma-\delta)}}\|u_1\|_{L^m} & \text{if } t \in (0,1], \\ t^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{\sigma}{2\delta}}\|u_0\|_{L^m} \\ \quad + t^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{\sigma}{2\delta}}\|u_1\|_{L^m} & \text{if } t \in [1,\infty), \end{cases}$$

$$\|u_t(t,\cdot)\|_{L^q} \lesssim \begin{cases} t^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\sigma}{\sigma-\delta}}\|u_0\|_{L^m} + t^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}}\|u_1\|_{L^m} & \text{if } t \in (0,1], \\ t^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{\sigma}{\delta}}\|u_0\|_{L^m} \\ \quad + t^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|u_1\|_{L^m} & \text{if } t \in [1,\infty), \end{cases}$$

$$\big\||D|^{2\delta} u(t,\cdot)\big\|_{L^q} \lesssim \begin{cases} t^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}}\|u_0\|_{L^m} + t^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}}\|u_1\|_{L^m} & \text{if } t \in (0,1], \\ t^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-1}\|u_0\|_{L^m} \\ \quad + t^{(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|u_1\|_{L^m} & \text{if } t \in [1,\infty), \end{cases}$$

where $1+\frac{1}{q} = \frac{1}{r}+\frac{1}{m}$ and for all dimensions $n \geqslant 1$.

4.4. Linear visco-elastic damped σ-evolution models

The main purpose of this section is to study visco-elastic damped σ-evolution models in the following form:

$$u_{tt} + (-\Delta)^\sigma u + (-\Delta)^\sigma u_t = 0, \quad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) \tag{4.22}$$

with $\sigma \geqslant 1$. Our goal is to obtain $L^q - L^q$ estimates for solutions to (4.22) assuming additional L^m regularity for the data with $m \in [1,q)$, where $q \in (1,\infty)$ is given.

Using partial Fourier transformation to (4.22) we obtain the Cauchy problem for $\hat{u}(t,\xi) := \mathfrak{F}(u(t,x))$, $\widehat{u_0}(\xi) := \mathfrak{F}(u_0(x))$ and $\widehat{u_1}(\xi) := \mathfrak{F}(u_1(x))$ as follows:

$$\hat{u}_{tt} + |\xi|^{2\sigma}\hat{u}_t + |\xi|^{2\sigma}\hat{u} = 0, \quad \hat{u}(0,\xi) = \widehat{u_0}(\xi), \quad \hat{u}_t(0,\xi) = \widehat{u_1}(\xi). \tag{4.23}$$

The characteristic roots are

$$\lambda_{1,2} = \lambda_{1,2}(\xi) = \frac{1}{2}\Big(-|\xi|^{2\sigma} \pm \sqrt{|\xi|^{4\sigma} - 4|\xi|^{2\sigma}}\Big).$$

The solutions to (4.23) are presented by the following formula (here we assume $\lambda_1 \neq \lambda_2$):

$$\hat{u}(t,\xi) = \frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2}\widehat{u_0}(\xi) + \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}\widehat{u_1}(\xi) =: \widehat{K_0}(t,\xi)\widehat{u_0}(\xi) + \widehat{K_1}(t,\xi)\widehat{u_1}(\xi).$$

Taking account of the cases of small and large frequencies separately we have

1. $\lambda_{1,2} = \lambda_{1,2}(\xi) = -\frac{1}{2}\big(|\xi|^{2\sigma} \mp i\sqrt{4|\xi|^{2\sigma} - |\xi|^{4\sigma}}\big)$
   and $\lambda_{1,2} \sim -|\xi|^{2\sigma} \pm i|\xi|^\sigma$, $\quad \lambda_1 - \lambda_2 \sim i|\xi|^\sigma \quad$ for $|\xi| \in (0, 4^{-\frac{1}{\sigma}})$,

2. $\lambda_{1,2} = \lambda_{1,2}(\xi) = -\frac{1}{2}\big(|\xi|^{2\sigma} \mp \sqrt{|\xi|^{4\sigma} - 4|\xi|^{2\sigma}}\big)$
   and $\lambda_1 \sim -1$, $\quad \lambda_2 \sim -|\xi|^{2\sigma}$, $\quad \lambda_1 - \lambda_2 \sim |\xi|^{2\sigma} \quad$ for $|\xi| \in (4^{\frac{1}{\sigma}}, \infty)$.

Let $\chi_k = \chi_k(|\xi|)$ with $k = 1,2,3$ be smooth cut-off functions having the following properties:

$$\chi_1(|\xi|) = \begin{cases} 1 & \text{if } |\xi| \leqslant 4^{-\frac{1}{\sigma}}, \\ 0 & \text{if } |\xi| \geqslant 3^{-\frac{1}{\sigma}}, \end{cases} \qquad \chi_3(|\xi|) = \begin{cases} 1 & \text{if } |\xi| \geqslant 4^{\frac{1}{\sigma}}, \\ 0 & \text{if } |\xi| \leqslant 3^{\frac{1}{\sigma}}, \end{cases}$$

and $\chi_2(|\xi|) = 1 - \chi_1(|\xi|) - \chi_3(|\xi|)$.

We note that $\chi_2(|\xi|) = 1$ if $3^{-\frac{1}{\sigma}} \leqslant |\xi| \leqslant 3^{\frac{1}{\sigma}}$ and $\chi_2(|\xi|) = 0$ if $|\xi| \leqslant 4^{-\frac{1}{\sigma}}$ or $|\xi| \geqslant 4^{\frac{1}{\sigma}}$. Let us now decompose the solutions to (4.22) into three parts localized separately to low, middle and high frequencies, that is,

$$u(t,x) = u_{\chi_1}(t,x) + u_{\chi_2}(t,x) + u_{\chi_3}(t,x),$$

where

$$u_{\chi_k}(t,x) = \mathfrak{F}^{-1}\big(\chi_k(|\xi|)\hat{u}(t,\xi)\big) \quad \text{with } k = 1,2,3.$$

$L^m - L^q$ estimates for small frequencies

Following the approach of the proof of Lemma 4.1.3 we can see that the result in Lemma 4.1.3 still holds in the case $\delta = \sigma$. Therefore, we obtain the following L^1 estimates for small frequencies.

Proposition 4.4.1. *The following estimates hold in $\mathbb{R}^n$:*

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a \widehat{K_0}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^1} \lesssim \begin{cases} 1 & \text{if } t \in (0,1], \\ t^{\frac{1}{2}(2+[\frac{n}{2}])-\frac{a}{2\sigma}} & \text{if } t \in [1,\infty), \end{cases}$$

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a \widehat{K_1}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^1} \lesssim \begin{cases} t & \text{if } t \in (0,1], \\ t^{\frac{1}{2}(3+[\frac{n}{2}])-\frac{a}{2\sigma}} & \text{if } t \in [1,\infty), \end{cases}$$

for any non-negative number a.

Following the approach of the proof of Proposition 4.1.8 we obtain the following L^∞ estimates for small frequencies.

Proposition 4.4.2. *The following estimates hold in $\mathbb{R}^n$:*

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a \widehat{K_0}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^\infty} \lesssim \begin{cases} 1 & \text{if } t \in (0,1], \\ t^{-\frac{n+a}{2\sigma}} & \text{if } t \in [1,\infty), \end{cases}$$

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a \widehat{K_1}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^\infty} \lesssim \begin{cases} t & \text{if } t \in (0,1], \\ t^{1-\frac{n+a}{2\sigma}} & \text{if } t \in [1,\infty), \end{cases}$$

for any non-negative number a.

By an interpolation theorem, from the statements of Propositions 4.4.1 and 4.4.2 we may conclude the following L^r estimates for small frequencies.

Proposition 4.4.3. *The following estimates hold in $\mathbb{R}^n$:*

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a \widehat{K_0}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^r} \lesssim \begin{cases} 1 & \text{if } t \in (0,1], \\ t^{\frac{1}{2}(2+[\frac{n}{2}])\frac{1}{r}-\frac{n}{2\sigma}(1-\frac{1}{r})-\frac{a}{2\sigma}} & \text{if } t \in [1,\infty), \end{cases}$$

$$\big\|\mathfrak{F}^{-1}\big(|\xi|^a \widehat{K_1}(t,\xi)\chi_1(|\xi|)\big)(t,\cdot)\big\|_{L^r} \lesssim \begin{cases} t & \text{if } t \in (0,1], \\ t^{1+\frac{1}{2}(1+[\frac{n}{2}])\frac{1}{r}-\frac{n}{2\sigma}(1-\frac{1}{r})-\frac{a}{2\sigma}} & \text{if } t \in [1,\infty), \end{cases}$$

for all $r \in [1,\infty]$ and any non-negative number a.

Finally, we may conclude the following result.

Theorem 4.4.1. *Let $q \in [1,\infty]$ and $m \in [1,q]$. Then, the Sobolev solutions to (4.22) satisfy the $L^m - L^q$ estimates*

$$\big\||D|^a u_{\chi_1}(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{\frac{1}{2}(2+[\frac{n}{2}])\frac{1}{r}-\frac{n}{2\sigma}(1-\frac{1}{r})-\frac{a}{2\sigma}}\|u_0\|_{L^m} + (1+t)^{1+\frac{1}{2}(1+[\frac{n}{2}])\frac{1}{r}-\frac{n}{2\sigma}(1-\frac{1}{r})-\frac{a}{2\sigma}}\|u_1\|_{L^m},$$

$$\big\|\partial_t|D|^a u_{\chi_1}(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{\frac{1}{2}(1+[\frac{n}{2}])\frac{1}{r}-\frac{n}{2\sigma}(1-\frac{1}{r})-\frac{a}{2\sigma}}\|u_0\|_{L^m} + (1+t)^{\frac{1}{2}(2+[\frac{n}{2}])\frac{1}{r}-\frac{n}{2\sigma}(1-\frac{1}{r})-\frac{a}{2\sigma}}\|u_1\|_{L^m},$$

where $1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{m}$ and for all $a \geqslant 0$.

Proof. In order to prove the first statement, we apply Young's convolution inequality from Proposition B.1.1 as we did in the proof of Theorems 4.1.1 and 4.2.1 and use the statements in Proposition 4.4.3. Taking account of some estimates related to the partial derivative in time of solutions we note that

$$\partial_t\widehat{K_0}(t,\xi) = -|\xi|^{2\sigma}\widehat{K_1}(t,\xi) \quad \text{and} \quad \partial_t\widehat{K_1}(t,\xi) = \widehat{K_0}(t,\xi) - |\xi|^{2\sigma}\widehat{K_1}(t,\xi).$$

Then, applying again Young's convolution inequality from Proposition B.1.1 and Proposition 4.4.3, we may conclude the second statement. Hence, the proof of Theorem 4.4.1 is completed. □

$L^q - L^q$ estimates for large frequencies

First, we can re-write the characteristic roots as follows:

$$\lambda_1(\xi) = -1 - \gamma(\xi) \text{ and } \lambda_2(\xi) = -|\xi|^{2\sigma} + 1 + \gamma(\xi), \tag{4.24}$$

where

$$\gamma(\xi) = -1 + g\Big(\frac{4}{|\xi|^{2\sigma}}\Big) \text{ and } g(s) = \int_0^1 (1-\theta s)^{-\frac{1}{2}} d\theta. \tag{4.25}$$

Now, we introduce the following abbreviations:

$$K_0^1(t,x) := \mathfrak{F}^{-1}\Big(\frac{\lambda_2(\xi)e^{\lambda_1(\xi)t}}{\lambda_1(\xi)-\lambda_2(\xi)}\widehat{u_0}(\xi)\chi_3(|\xi|)\Big)(t,x), \quad K_0^2(t,x) := \mathfrak{F}^{-1}\Big(\frac{\lambda_1(\xi)e^{\lambda_2(\xi)t}}{\lambda_1(\xi)-\lambda_2(\xi)}\widehat{u_0}(\xi)\chi_3(|\xi|)\Big)(t,x),$$
$$K_1^1(t,x) := \mathfrak{F}^{-1}\Big(\frac{e^{\lambda_1(\xi)t}}{\lambda_1(\xi)-\lambda_2(\xi)}\widehat{u_1}(\xi)\chi_3(|\xi|)\Big)(t,x), \quad K_1^2(t,x) := \mathfrak{F}^{-1}\Big(\frac{e^{\lambda_2(\xi)t}}{\lambda_1(\xi)-\lambda_2(\xi)}\widehat{u_1}(\xi)\chi_3(|\xi|)\Big)(t,x).$$

We shall prove the following results.

Proposition 4.4.4. *Let $q \in (1,\infty)$. Then, the following estimates hold:*

$$\big\|\partial_t^j |D|^a K_0^1(t,\cdot)\big\|_{L^q} \lesssim e^{-ct}\|u_0\|_{H^a_q},$$
$$\big\|\partial_t^j |D|^a K_0^2(t,\cdot)\big\|_{L^q} \lesssim e^{-ct}\|u_0\|_{H_q^{[2\sigma j-2\sigma+a]^+}},$$
$$\big\|\partial_t^j |D|^a K_1^1(t,\cdot)\big\|_{L^q} \lesssim e^{-ct}\|u_1\|_{H_q^{[a-2\sigma]^+}},$$
$$\big\|\partial_t^j |D|^a K_1^2(t,\cdot)\big\|_{L^q} \lesssim e^{-ct}\|u_1\|_{H_q^{[2\sigma j-2\sigma+a]^+}},$$

for any $t>0$, $a \geqslant 0$, integer $j \geqslant 0$ and a suitable positive constant c.

Proposition 4.4.5. *Let $q = 1$ or ∞. Let us assume $\sigma > 1$. Then, the following estimates hold:*

$$\big\|\partial_t^j |D|^a K_0^1(t,\cdot)\big\|_{L^q} \lesssim e^{-ct}\|u_0\|_{H^a_q},$$
$$\big\|\partial_t^j |D|^a K_0^2(t,\cdot)\big\|_{L^q} \lesssim e^{-ct}\|u_0\|_{H_q^{2\sigma j+[a-\sigma]^+}},$$
$$\big\|\partial_t^j |D|^a K_1^1(t,\cdot)\big\|_{L^q} \lesssim e^{-ct}\|u_1\|_{H_q^{[a-\sigma]^+}},$$
$$\big\|\partial_t^j |D|^a K_1^2(t,\cdot)\big\|_{L^q} \lesssim e^{-ct}\|u_1\|_{H_q^{2\sigma j+[a-\sigma]^+}},$$

for any $t>0$, $a \geqslant 0$, integer $j \geqslant 0$ and a suitable positive constant c.

According to the application of the Mikhlin-Hörmander multiplier theorem (see also [9, 49]) for Fourier multipliers from Proposition B.5.1, in order to prove Proposition 4.4.4, we shall show the following auxiliary estimates.

Lemma 4.4.1. *The following estimates hold in $\mathbb{R}^n$ for sufficiently large $|\xi|$:*

$$\big|\partial_\xi^\alpha |\xi|^{-2\sigma}\big| \lesssim |\xi|^{-2\sigma-|\alpha|} \text{ for all } \alpha, \tag{4.26}$$
$$\big|\partial_\xi^\alpha |\xi|^{2p\sigma}\big| \lesssim |\xi|^{2p\sigma-|\alpha|} \text{ for all } \alpha \text{ and } p \in \mathbb{R}, \tag{4.27}$$
$$\Big|\partial_\xi^\alpha g\Big(\frac{4}{|\xi|^{2\sigma}}\Big)\Big| \lesssim |\xi|^{-2\sigma-|\alpha|} \text{ for all } |\alpha| \geqslant 1, \text{ and } \Big|g\Big(\frac{4}{|\xi|^{2\sigma}}\Big)\Big| \lesssim 1, \tag{4.28}$$
$$\big|\partial_\xi^\alpha \gamma(\xi)\big| \lesssim |\xi|^{-2\sigma-|\alpha|} \text{ for all } \alpha, \tag{4.29}$$
$$\big|\partial_\xi^\alpha \lambda_2(\xi)\big| \lesssim |\xi|^{2\sigma-|\alpha|} \text{ for all } \alpha, \tag{4.30}$$
$$\big|\partial_\xi^\alpha \lambda_1(\xi)\big| \lesssim |\xi|^{-2\sigma-|\alpha|} \text{ for all } |\alpha| \geqslant 1, \text{ and } \big|\lambda_1(\xi)\big| \lesssim 1, \tag{4.31}$$
$$\Big|\partial_\xi^\alpha \big(\lambda_1(\xi)-\lambda_2(\xi)\big)^{-1}\Big| \lesssim |\xi|^{-2\sigma-|\alpha|} \text{ for all } \alpha, \tag{4.32}$$

$$\big|\partial_\xi^\alpha \lambda_2^j(\xi)\big| \lesssim |\xi|^{2\sigma j-|\alpha|} \textit{ for all } \alpha \textit{ and } j \geqslant 0, \tag{4.33}$$

$$\big|\partial_\xi^\alpha \lambda_1^j(\xi)\big| \lesssim |\xi|^{-|\alpha|} \textit{ for all } \alpha \textit{ and } j \geqslant 0, \tag{4.34}$$

$$\big|\partial_\xi^\alpha \big(|\xi|^b \lambda_2^j(\xi)\big)\big| \lesssim |\xi|^{2\sigma j+b-|\alpha|} \textit{ for all } \alpha, \textit{ for any } b \in \mathbb{R} \textit{ and } j \geqslant 0, \tag{4.35}$$

$$\big|\partial_\xi^\alpha \big(|\xi|^b \lambda_1^j(\xi)\big)\big| \lesssim |\xi|^{b-|\alpha|} \textit{ for all } \alpha, \textit{ for any } b \in \mathbb{R} \textit{ and } j \geqslant 0, \tag{4.36}$$

$$\big|\partial_\xi^\alpha \big(e^{\lambda_2(\xi)t}\big)\big| \lesssim e^{-ct}|\xi|^{-|\alpha|} \tag{4.37}$$

for all α and $t > 0$, where c is a suitable positive constant,

$$\big|\partial_\xi^\alpha \big(e^{\lambda_1(\xi)t}\big)\big| \lesssim e^{-ct}|\xi|^{-|\alpha|} \tag{4.38}$$

for all α and $t > 0$, where c is a suitable positive constant,

$$\Big|\partial_\xi^\alpha \Big(\frac{\lambda_1(\xi)e^{\lambda_2(\xi)t}\lambda_2^j(\xi)|\xi|^b}{\lambda_1(\xi)-\lambda_2(\xi)}\Big)\Big| \lesssim e^{-ct}|\xi|^{2\sigma j+b-2\sigma-|\alpha|} \tag{4.39}$$

for all α, for any $b \in \mathbb{R}$, $j \geqslant 0$ and $t > 0$, where c is a suitable positive constant,

$$\Big|\partial_\xi^\alpha \Big(\frac{e^{\lambda_2(\xi)t}\lambda_2^j(\xi)|\xi|^b}{\lambda_1(\xi)-\lambda_2(\xi)}\Big)\Big| \lesssim e^{-ct}|\xi|^{2\sigma j+b-2\sigma-|\alpha|} \tag{4.40}$$

for all α, for any $b \in \mathbb{R}$, $j \geqslant 0$ and $t > 0$, where c is a suitable positive constant,

$$\Big|\partial_\xi^\alpha \Big(\frac{\lambda_2(\xi)e^{\lambda_1(\xi)t}\lambda_1^j(\xi)|\xi|^b}{\lambda_1(\xi)-\lambda_2(\xi)}\Big)\Big| \lesssim e^{-ct}|\xi|^{b-|\alpha|} \tag{4.41}$$

for all α, for any $b \in \mathbb{R}$, $j \geqslant 0$ and $t > 0$, where c is a suitable positive constant,

$$\Big|\partial_\xi^\alpha \Big(\frac{e^{\lambda_1(\xi)t}\lambda_1^j(\xi)|\xi|^b}{\lambda_1(\xi)-\lambda_2(\xi)}\Big)\Big| \lesssim e^{-ct}|\xi|^{b-2\sigma-|\alpha|} \tag{4.42}$$

for all α, for any $b \in \mathbb{R}$, $j \geqslant 0$ and $t > 0$, where c is a suitable positive constant.

Proof. In order to prove all statements in Lemma 4.4.1, we shall apply Lemma B.6.2 and Leibniz rule of the multivariable calculus. Indeed, we will indicate the proof of the above estimates as follows:

To (4.26): Applying Lemma B.6.2 with $h(s) = s^{-\sigma}$ and $f(\xi) = |\xi|^2$ we derive

$$\begin{aligned}
\big|\partial_\xi^\alpha |\xi|^{-2\sigma}\big| &= \Big|\sum_{k=1}^{|\alpha|} h^{(k)}\big(|\xi|^2\big)\Big(\sum_{\substack{\gamma_1+\cdots+\gamma_k\leqslant\alpha\\ |\gamma_1|+\cdots+|\gamma_k|=|\alpha|,\, |\gamma_i|\geqslant 1}} \partial_\xi^{\gamma_1}\big(|\xi|^2\big)\cdots\partial_\xi^{\gamma_k}\big(|\xi|^2\big)\Big)\Big| \\
&\lesssim \sum_{k=1}^{|\alpha|} |\xi|^{2(-\sigma-k)}\Big(\sum_{\substack{\gamma_1+\cdots+\gamma_k\leqslant\alpha\\ |\gamma_1|+\cdots+|\gamma_k|=|\alpha|,\, |\gamma_i|\geqslant 1}} |\xi|^{2-|\gamma_1|+\cdots+2-|\gamma_k|}\Big) \\
&\lesssim \sum_{k=1}^{|\alpha|} |\xi|^{2(-\sigma-k)+2k-|\alpha|} \lesssim |\xi|^{-2\sigma-|\alpha|}.
\end{aligned}$$

In an analogous way, we may conclude $\big|\partial_\xi^\alpha |\xi|^{2\sigma}\big| \lesssim |\xi|^{2\sigma-|\alpha|}$.

To (4.27): Applying Lemma B.6.2 with $h(s) = s^p$ and $f(\xi) = |\xi|^{2\sigma}$ we have

$$\begin{aligned}
\big|\partial_\xi^\alpha |\xi|^{2p\sigma}\big| &= \Big|\sum_{k=1}^{|\alpha|} h^{(k)}\big(|\xi|^{2\sigma}\big)\Big(\sum_{\substack{\gamma_1+\cdots+\gamma_k\leqslant\alpha\\ |\gamma_1|+\cdots+|\gamma_k|=|\alpha|,\, |\gamma_i|\geqslant 1}} \partial_\xi^{\gamma_1}\big(|\xi|^{2\sigma}\big)\cdots\partial_\xi^{\gamma_k}\big(|\xi|^{2\sigma}\big)\Big)\Big| \\
&\lesssim \sum_{k=1}^{|\alpha|} |\xi|^{2\sigma(p-k)}\Big(\sum_{\substack{\gamma_1+\cdots+\gamma_k\leqslant\alpha\\ |\gamma_1|+\cdots+|\gamma_k|=|\alpha|,\, |\gamma_i|\geqslant 1}} |\xi|^{2\sigma-|\gamma_1|+\cdots+2\sigma-|\gamma_k|}\Big) \\
&\lesssim \sum_{k=1}^{|\alpha|} |\xi|^{2\sigma(p-k)+2\sigma k-|\alpha|} \lesssim |\xi|^{2p\sigma-|\alpha|}.
\end{aligned}$$

To (4.28): Applying Lemma B.6.2 with $h(s) = g(s)$ and $f(\xi) = 4|\xi|^{-2\sigma}$ we have

$$\begin{aligned}
\Big|\partial_\xi^\alpha g\Big(\frac{4}{|\xi|^{2\sigma}}\Big)\Big| &= \Big|\sum_{k=1}^{|\alpha|} g^{(k)}\Big(\frac{4}{|\xi|^{2\sigma}}\Big)\Big(\sum_{\substack{\gamma_1+\cdots+\gamma_k\leqslant\alpha\\ |\gamma_1|+\cdots+|\gamma_k|=|\alpha|,\,|\gamma_i|\geqslant 1}} \partial_\xi^{\gamma_1}\big(4|\xi|^{-2\sigma}\big)\cdots\partial_\xi^{\gamma_k}\big(4|\xi|^{-2\sigma}\big)\Big)\Big| \\
&\lesssim \sum_{k=1}^{|\alpha|}\Big(1-\frac{4}{|\xi|^{2\sigma}}\Big)^{-k-\frac12}\Big(\sum_{\substack{\gamma_1+\cdots+\gamma_k\leqslant\alpha\\ |\gamma_1|+\cdots+|\gamma_k|=|\alpha|,\,|\gamma_i|\geqslant 1}} |\xi|^{-2\sigma-|\gamma_1|-\cdots-2\sigma-|\gamma_k|}\Big) \\
&\lesssim \sum_{k=1}^{|\alpha|}\Big(\frac{|\xi|^{2\sigma}-4}{|\xi|^{2\sigma}}\Big)^{-k-\frac12}|\xi|^{-2\sigma k-|\alpha|} \lesssim \sum_{k=1}^{|\alpha|}\big(|\xi|^{2\sigma}-4\big)^{-k-\frac12}|\xi|^{\sigma-|\alpha|} \\
&\lesssim \sum_{k=1}^{|\alpha|}\big(|\xi|^{2\sigma}-4\big)^{-\frac32}|\xi|^{\sigma-|\alpha|} \lesssim |\xi|^{-2\sigma-|\alpha|}.
\end{aligned}$$

To (4.29), (4.30) and (4.31): These statements are immediately followed by (4.28) and the expression of the characteristic roots by the function $\gamma = \gamma(\xi)$.

To (4.32): Applying Lemma B.6.2 with $h(s) = \frac{1}{s}$ and $f(\xi) = \lambda_1(\xi) - \lambda_2(\xi)$ we get

$$\begin{aligned}
&\Big|\partial_\xi^\alpha\big(\lambda_1(\xi)-\lambda_2(\xi)\big)^{-1}\Big| \\
&= \Big|\sum_{k=1}^{|\alpha|} h^{(k)}\big(\lambda_1(\xi)-\lambda_2(\xi)\big)\Big(\sum_{\substack{\gamma_1+\cdots+\gamma_k\leqslant\alpha\\ |\gamma_1|+\cdots+|\gamma_k|=|\alpha|,\,|\gamma_i|\geqslant 1}} \partial_\xi^{\gamma_1}\big(\lambda_1(\xi)-\lambda_2(\xi)\big)\cdots\partial_\xi^{\gamma_k}\big(\lambda_1(\xi)-\lambda_2(\xi)\big)\Big)\Big| \\
&\lesssim \Big|\sum_{k=1}^{|\alpha|}\big(\lambda_1(\xi)-\lambda_2(\xi)\big)^{-k-1}\Big(\sum_{\substack{\gamma_1+\cdots+\gamma_k\leqslant\alpha\\ |\gamma_1|+\cdots+|\gamma_k|=|\alpha|,\,|\gamma_i|\geqslant 1}} |\xi|^{2\sigma-|\gamma_1|+\cdots+2\sigma-|\gamma_k|}\Big)\Big| \\
&\lesssim \sum_{k=1}^{|\alpha|}|\xi|^{2\sigma(-k-1)+2\sigma k-|\alpha|} \lesssim |\xi|^{-2\sigma-|\alpha|}.
\end{aligned}$$

To (4.33): Applying Lemma B.6.2 with $h(s) = s^j$ and $f(\xi) = \lambda_2(\xi)$ we obtain

$$\begin{aligned}
\big|\partial_\xi^\alpha\lambda_2^j(\xi)\big| &= \Big|\sum_{k=1}^{|\alpha|} h^{(k)}\big(\lambda_2(\xi)\big)\Big(\sum_{\substack{\gamma_1+\cdots+\gamma_k\leqslant\alpha\\ |\gamma_1|+\cdots+|\gamma_k|=|\alpha|,\,|\gamma_i|\geqslant 1}} \partial_\xi^{\gamma_1}\big(\lambda_2(\xi)\big)\cdots\partial_\xi^{\gamma_k}\big(\lambda_2(\xi)\big)\Big)\Big| \\
&\lesssim \Big|\sum_{k=1}^{|\alpha|}\big(\lambda_2(\xi)\big)^{j-k}\Big(\sum_{\substack{\gamma_1+\cdots+\gamma_k\leqslant\alpha\\ |\gamma_1|+\cdots+|\gamma_k|=|\alpha|,\,|\gamma_i|\geqslant 1}} |\xi|^{2\sigma-|\gamma_1|+\cdots+2\sigma-|\gamma_k|}\Big)\Big| \\
&\lesssim \sum_{k=1}^{|\alpha|}|\xi|^{2\sigma(j-k)+2\sigma k-|\alpha|} \lesssim |\xi|^{2\sigma j-|\alpha|}.
\end{aligned}$$

To (4.34): By the same arguments as we did in (4.33), we may conclude (4.34) by using (4.31).

To (4.35): Using the Leibniz rule we obtain

$$\big|\partial_\xi^\alpha\big(|\xi|^b\lambda_2^j(\xi)\big)\big| = \Big|\sum_{\gamma\leqslant\alpha}\binom{\alpha}{\gamma}\partial_\xi^\gamma\big(|\xi|^b\big)\partial_\xi^{\alpha-\gamma}\big(\lambda_2^j(\xi)\big)\Big| \lesssim \sum_{\gamma\leqslant\alpha}|\xi|^{b-|\gamma|}|\xi|^{2\sigma j-|\alpha|+|\gamma|} \lesssim |\xi|^{2\sigma j+b-|\alpha|}.$$

To (4.36): In the same way to verify (4.35), we may conclude (4.36) by using (4.34).

To (4.37): Applying Lemma B.6.2 with $h(s) = e^{st}$ and $f(\xi) = \lambda_2(\xi)$ we have

$$\begin{aligned}
\big|\partial_\xi^\alpha e^{\lambda_2(\xi)t}\big| &= \Big|\sum_{k=1}^{|\alpha|} h^{(k)}\big(\lambda_2(\xi)\big)\Big(\sum_{\substack{\gamma_1+\cdots+\gamma_k\leqslant\alpha\\ |\gamma_1|+\cdots+|\gamma_k|=|\alpha|,\,|\gamma_i|\geqslant 1}} \partial_\xi^{\gamma_1}\big(\lambda_2(\xi)\big)\cdots\partial_\xi^{\gamma_k}\big(\lambda_2(\xi)\big)\Big)\Big| \\
&\lesssim \sum_{k=1}^{|\alpha|} t^k e^{\lambda_2(\xi)t}\Big(\sum_{\substack{\gamma_1+\cdots+\gamma_k\leqslant\alpha\\ |\gamma_1|+\cdots+|\gamma_k|=|\alpha|,\,|\gamma_i|\geqslant 1}} |\xi|^{2\sigma-|\gamma_1|+\cdots+2\sigma-|\gamma_k|}\Big) \\
&\lesssim \sum_{k=1}^{|\alpha|} \big(|\xi|^{2\sigma}t\big)^k e^{-\frac{|\xi|^{2\sigma}}{2}t}|\xi|^{-|\alpha|}\ \Big(\text{since } \lambda_2(\xi)\leqslant -\frac{|\xi|^{2\sigma}}{2}\Big) \\
&\lesssim e^{-ct}|\xi|^{-|\alpha|},\ \text{ where } c \text{ is a suitable positive constant.}
\end{aligned}$$

To (4.38): Applying Lemma B.6.2 with $h(s) = e^{st}$ and $f(\xi) = \lambda_1(\xi)$ we have

$$\begin{aligned}
\big|\partial_\xi^\alpha e^{\lambda_1(\xi)t}\big| &= \Big|\sum_{k=1}^{|\alpha|} h^{(k)}\big(\lambda_1(\xi)\big)\Big(\sum_{\substack{\gamma_1+\cdots+\gamma_k\leqslant\alpha\\ |\gamma_1|+\cdots+|\gamma_k|=|\beta|,\,|\gamma_i|\geqslant 1}} \partial_\xi^{\gamma_1}\big(\lambda_1(\xi)\big)\cdots\partial_\xi^{\gamma_k}\big(\lambda_1(\xi)\big)\Big)\Big| \\
&\lesssim \sum_{k=1}^{|\alpha|} t^k e^{\lambda_1(\xi)t}\Big(\sum_{\substack{\gamma_1+\cdots+\gamma_k\leqslant\alpha\\ |\gamma_1|+\cdots+|\gamma_k|=|\alpha|,\,|\gamma_i|\geqslant 1}} |\xi|^{-2\sigma-|\gamma_1|-\cdots-2\sigma-|\gamma_k|}\Big) \\
&\lesssim \sum_{k=1}^{|\alpha|} t^k e^{-t}|\xi|^{-2\sigma k-|\alpha|}\ (\text{since } \lambda_1(\xi)\leqslant -1) \\
&\lesssim (t+t^{|\alpha|})e^{-t}|\xi|^{-|\alpha|} \lesssim t(1+t)^{|\alpha|-1}e^{-t}|\xi|^{-|\alpha|} \\
&\lesssim e^{-ct}|\xi|^{-|\alpha|},\ \text{ where } c \text{ is a suitable positive constant.}
\end{aligned}$$

To (4.39) and (4.40): Combining (4.27), (4.31) to (4.33) and (4.35) to (4.37) we may conclude (4.39) and (4.40) by using the Leibniz rule.

To (4.41) and (4.42): Combining (4.27), (4.30), (4.32), (4.36) and (4.38) we may conclude (4.41) and (4.42) by using the Leibniz rule. □

Proof of Proposition 4.4.4. First, taking account of the estimates for $K_0^2(t,x)$ and some of its partial derivatives we will divide our considerations into two cases. In the first case, if $2\sigma j - 2\sigma + a \geqslant 0$, then we write

$$\partial_t^j|D|^a K_0^2(t,x) = \mathfrak{F}^{-1}\Big(\frac{\lambda_1(\xi)e^{\lambda_2(\xi)t}\lambda_2^j(\xi)|\xi|^{2\sigma-2\sigma j}}{\lambda_1(\xi)-\lambda_2(\xi)}\chi_3(|\xi|)|\xi|^{2\sigma j-2\sigma+a}\widehat{u_0}(\xi)\Big)(t,x).$$

By choosing $b = 2\sigma - 2\sigma j$ in (4.39), we get for all α the estimates

$$\Big|\partial_\xi^\alpha\Big(\frac{\lambda_1(\xi)e^{\lambda_2(\xi)t}\lambda_2^j(\xi)|\xi|^{2\sigma-2\sigma j}}{\lambda_1(\xi)-\lambda_2(\xi)}\Big)\Big| \lesssim e^{-ct}|\xi|^{-|\alpha|},$$

where c is a suitable positive constant. By Proposition B.5.1, we may conclude

$$\big\|\partial_t^j|D|^a K_0^2(t,\cdot)\big\|_{L^q} \lesssim e^{-ct}\|u_0\|_{H_q^{2\sigma j-2\sigma+a}}. \tag{4.43}$$

In the second case, if $2\sigma j - 2\sigma + a < 0$, then we write

$$\partial_t^j|D|^a K_0^2(t,x) = \mathfrak{F}^{-1}\Big(\frac{\lambda_1(\xi)e^{\lambda_2(\xi)t}\lambda_2^j(\xi)|\xi|^a}{\lambda_1(\xi)-\lambda_2(\xi)}\chi_3(|\xi|)\widehat{u_0}(\xi)\Big)(t,x).$$

By choosing $b = a$ in (4.39), we derive for all α the estimates

$$\Big|\partial_\xi^\alpha\Big(\frac{\lambda_1(\xi)e^{\lambda_2(\xi)t}\lambda_2^j(\xi)|\xi|^a}{\lambda_1(\xi)-\lambda_2(\xi)}\Big)\Big| \lesssim e^{-ct}|\xi|^{2\sigma j+a-2\sigma-|\alpha|} \lesssim e^{-ct}|\xi|^{-|\alpha|},$$

where c is a suitable positive constant. By Proposition B.5.1, we arrive at

$$\big\|\partial_t^j |D|^a K_0^2(t,\cdot)\big\|_{L^q} \lesssim e^{-ct}\|u_0\|_{L^q}. \tag{4.44}$$

Hence, from (4.43) and (4.44) we have proved the second statement in Proposition 4.4.4. By the same arguments we may also conclude the last statement in Proposition 4.4.4 by using (4.40). Let us turn to estimate the two remaining statements. Indeed, we write

$$\partial_t^j |D|^a K_0^1(t,x) = \mathfrak{F}^{-1}\Big(\frac{\lambda_2(\xi)e^{\lambda_1(\xi)t}\lambda_1^j(\xi)}{\lambda_1(\xi)-\lambda_2(\xi)}\chi_3(|\xi|)|\xi|^a\widehat{u_0}(\xi)\Big)(t,x).$$

By choosing $b=0$ in (4.41) we obtain

$$\Big|\partial_\xi^\alpha\Big(\frac{\lambda_2(\xi)e^{\lambda_1(\xi)t}\lambda_1^j(\xi)}{\lambda_1(\xi)-\lambda_2(\xi)}\Big)\Big| \lesssim e^{-ct}|\xi|^{-|\alpha|},$$

where c is a suitable positive constant. By Proposition B.5.1, we arrive at

$$\big\|\partial_t^j |D|^a K_0^1(t,\cdot)\big\|_{L^q} \lesssim e^{-ct}\big\|\mathfrak{F}^{-1}\big(|\xi|^a\widehat{u_0}(\xi)\big)\big\|_{L^q} \lesssim e^{-ct}\|u_0\|_{\dot{H}^a_q} \lesssim e^{-ct}\|u_0\|_{H^a_q}.$$

Finally, taking account of estimates for $K_1^1(t,x)$ and some of its partial derivatives we will divide our consideration into two cases. In the first case, if $a \geqslant 2\sigma$, then we write

$$\partial_t^j |D|^a K_1^1(t,x) = \mathfrak{F}^{-1}\Big(\frac{e^{\lambda_1(\xi)t}\lambda_1^j(\xi)|\xi|^{2\sigma}}{\lambda_1(\xi)-\lambda_2(\xi)}\chi_3(|\xi|)|\xi|^{a-2\sigma}\widehat{u_0}(\xi)\Big)(t,x).$$

By choosing $b=2\sigma$ in (4.42), we obtain

$$\Big|\partial_\xi^\alpha\Big(\frac{e^{\lambda_1(\xi)t}\lambda_1^j(\xi)|\xi|^{2\sigma}}{\lambda_1(\xi)-\lambda_2(\xi)}\Big)\Big| \lesssim e^{-ct}|\xi|^{-|\alpha|},$$

where c is a suitable positive constant. By Proposition B.5.1, we derive

$$\big\|\partial_t^j |D|^a K_1^1(t,\cdot)\big\|_{L^q} \lesssim e^{-ct}\|u_1\|_{\dot{H}^{a-2\sigma}_q} \lesssim e^{-ct}\|u_1\|_{H^{a-2\sigma}_q}.$$

In the second case, if $a < 2\sigma$, then we write

$$\partial_t^j |D|^a K_1^1(t,x) = \mathfrak{F}^{-1}\Big(\frac{e^{\lambda_1(\xi)t}\lambda_1^j(\xi)|\xi|^{a}}{\lambda_1(\xi)-\lambda_2(\xi)}\chi_3(|\xi|)\widehat{u_0}(\xi)\Big)(t,x).$$

By choosing $b=a$ in (4.42), we have

$$\Big|\partial_\xi^\alpha\Big(\frac{e^{\lambda_1(\xi)t}\lambda_1^j(\xi)|\xi|^{a}}{\lambda_1(\xi)-\lambda_2(\xi)}\Big)\Big| \lesssim e^{-ct}|\xi|^{a-2\sigma-|\alpha|} \lesssim e^{-ct}|\xi|^{-|\alpha|},$$

where c is a suitable positive constant. By Proposition B.5.1, we may conclude

$$\big\|\partial_t^j |D|^a K_1^1(t,\cdot)\big\|_{L^q} \lesssim e^{-ct}\|u_1\|_{L^q}.$$

Summarizing, the proof of Proposition 4.4.4 is completed. □

Proof of Proposition 4.4.5. Following the expression as we did in Proposition 4.4.4 we write

$$\begin{aligned}
\partial_t^j |D|^a K_0^2(t,x) &= \mathfrak{F}^{-1}\Big(\frac{\lambda_1(\xi)e^{\lambda_2(\xi)t}\lambda_2^j(\xi)|\xi|^{\min\{a,\sigma\}-2\sigma j}}{\lambda_1(\xi)-\lambda_2(\xi)}\chi_3(|\xi|)|\xi|^{2\sigma j+[a-\sigma]^+}\widehat{u_0}(\xi)\Big)(t,x)\\
&= \mathfrak{F}^{-1}\Big(\frac{\lambda_1(\xi)e^{\lambda_2(\xi)t}\lambda_2^j(\xi)|\xi|^{\min\{a,\sigma\}-2\sigma j}}{\lambda_1(\xi)-\lambda_2(\xi)}\chi_3(|\xi|)\Big)(t,x) \ast |D|^{2\sigma j+[a-\sigma]^+}u_0(x)\\
&=: \mathfrak{F}^{-1}\big(\widehat{L_0^2}(t,\xi)\big)(t,x) \ast |D|^{2\sigma j+[a-\sigma]^+}u_0(x).
\end{aligned}$$

By choosing $b=\min\{a,\sigma\}-2\sigma j$ in (4.39), we get

$$\Big|\partial_\xi^\alpha\big(\widehat{L_0^2}(t,\xi)\big)\Big| \lesssim e^{-ct}|\xi|^{\min\{a,\sigma\}-2\sigma-|\alpha|} \lesssim e^{-ct}|\xi|^{-\sigma-|\alpha|},$$

where c is a suitable positive constant. Since

$$e^{ix\xi} = \sum_{k=1}^{n} \frac{x_k}{i|x|^2} \partial_{\xi_k} e^{ix\xi}, \tag{4.45}$$

carrying out m steps of partial integration we derive

$$\mathfrak{F}^{-1}\big(\widehat{L_0^2}(t,\xi)\big)(t,x) = C \sum_{|\alpha|=m} \frac{(ix)^\alpha}{|x|^{2|\alpha|}} \mathfrak{F}^{-1}\Big(\partial_\xi^\alpha \big(\widehat{L_0^2}(t,\xi)\big)\Big)(t,x).$$

For this reason, we obtain the following estimates:

$$\begin{aligned}
\big|\mathfrak{F}^{-1}\big(\widehat{L_0^2}(t,\xi)\big)(t,x)\big| &\lesssim |x|^{-m}\Big\|\mathfrak{F}^{-1}\Big(\partial_\xi^\alpha \big(\widehat{L_0^2}(t,\xi)\big)\Big)(t,\cdot)\Big\|_{L^\infty} \lesssim |x|^{-m}\big\|\partial_\xi^\alpha \big(L_0^2(t,\xi)\big)(t,\cdot)\big\|_{L^1} \\
&\lesssim |x|^{-m} e^{-ct} \int_1^\infty |\xi|^{-\sigma-m+n-1} d|\xi| \\
&\lesssim e^{-ct} \begin{cases} |x|^{-(n-1)} & \text{if } 0<|x|\leqslant 1 \text{ and } m=n-1, \\ |x|^{-(n+1)} & \text{if } |x|\geqslant 1 \text{ and } m=n+1, \end{cases}
\end{aligned}$$

where the assumption $\sigma > 1$ comes into play. Hence, we arrive at

$$\begin{aligned}
\big\|\mathfrak{F}^{-1}\big(\widehat{L_0^2}(t,\xi)\big)(t,\cdot)\big\|_{L^1} &\lesssim \int_{|x|\leqslant 1} \big|\mathfrak{F}^{-1}\big(\widehat{L_0^2}(t,\xi)\big)(t,x)\big| dx + \int_{|x|\geqslant 1} \big|\mathfrak{F}^{-1}\big(\widehat{L_0^2}(t,\xi)\big)(t,x)\big| dx \\
&\lesssim e^{-ct} \int_0^1 d|x| + e^{-ct} \int_1^\infty |x|^{-2} d|x| \lesssim e^{-ct}.
\end{aligned}$$

Then, employing Young's convolution inequality from Proposition B.1.1 we have proved the second statement in Proposition 4.4.5. In the same way, we may also conclude the last statement and the third statement in Proposition 4.4.5, respectively, by using (4.40) and (4.42). Let us turn to estimate the first statement. Indeed, we can see that

$$\begin{aligned}
\partial_t^j |D|^a K_0^1(t,x) &= \partial_t^j |D|^a \mathfrak{F}^{-1}\Big(e^{\lambda_1(\xi)t} \chi_3(|\xi|)\widehat{u_0}(\xi)\Big)(t,x) \\
&\quad + \partial_t^{j+1} |D|^a \mathfrak{F}^{-1}\Big(\frac{e^{\lambda_1(\xi)t}}{\lambda_2(\xi)-\lambda_1(\xi)} \chi_3(|\xi|)\widehat{u_0}(\xi)\Big)(t,x),
\end{aligned} \tag{4.46}$$

by using the relation

$$\frac{\lambda_2(\xi)e^{\lambda_1(\xi)t}}{\lambda_2(\xi)-\lambda_1(\xi)} = e^{\lambda_1(\xi)t} + \partial_t\Big(\frac{e^{\lambda_1(\xi)t}}{\lambda_2(\xi)-\lambda_1(\xi)}\Big).$$

In an analogous treatment to get the third statement, we derive the following estimate for the second term:

$$\Big\|\partial_t^{j+1} |D|^a \mathfrak{F}^{-1}\Big(\frac{e^{\lambda_1(\xi)t}}{\lambda_2(\xi)-\lambda_1(\xi)} \chi_3(|\xi|)\widehat{u_0}(\xi)\Big)(t,\cdot)\Big\|_{L^q} \lesssim e^{-ct}\|u_0\|_{H_q^{[a-\sigma]^+}}. \tag{4.47}$$

In order to control the first term, using the relation $\lambda_1(\xi) = -1-\gamma(\xi)$ we write

$$e^{\lambda_1(\xi)t} = e^{-t}e^{-\gamma(\xi)t} = e^{-t} - te^{-t}\gamma(\xi)\int_0^1 e^{-\gamma(\xi)tr}dr.$$

Hence, we obtain

$$\begin{aligned}
\mathfrak{F}^{-1}\Big(e^{\lambda_1(\xi)t}\chi_3(|\xi|)\widehat{u_0}(\xi)\Big)(t,x) &= e^{-t}\mathfrak{F}^{-1}\big(\widehat{u_0}(\xi)\big)(x) - e^{-t}\mathfrak{F}^{-1}\big((1-\chi_3(|\xi|))\widehat{u_0}(\xi)\big)(x) \\
&\quad - te^{-t}\mathfrak{F}^{-1}\Big(\mu(\xi)\chi_3(|\xi|)\widehat{u_0}(\xi)\int_0^1 e^{-\gamma(\xi)tr}dr\Big)(t,x).
\end{aligned} \tag{4.48}$$

Obviously, we have

$$\Big\|\partial_t^j |D|^a \Big(e^{-t}\mathfrak{F}^{-1}\big(\widehat{u_0}(\xi)\big)\Big)(t,\cdot)\Big\|_{L^q} = e^{-t}\big\||D|^a u_0\big\|_{L^q} \lesssim e^{-t}\|u_0\|_{H_q^a}. \tag{4.49}$$

Now, we re-write

$$\begin{aligned}
&\partial_t^j |D|^a \Big(te^{-t}\mathfrak{F}^{-1}\Big(\gamma(\xi)\chi_3(|\xi|)\widehat{u_0}(\xi) \int_0^1 e^{-\gamma(\xi)tr} dr\Big)\Big)(t,x) \\
&\quad = \sum_{\ell=0}^{j} \partial_t^{j-\ell}(te^{-t})\, \partial_t^{\ell}|D|^a \mathfrak{F}^{-1}\Big(\gamma(\xi)\chi_3(|\xi|)\widehat{u_0}(\xi) \int_0^1 e^{-\gamma(\xi)tr} dr\Big)(t,x) \\
&\quad = \sum_{\ell=0}^{j} \partial_t^{j-\ell}(te^{-t})\, \mathfrak{F}^{-1}\Big(\gamma(\xi)^{\ell+1}|\xi|^{\min\{a,\sigma\}}\chi_3(|\xi|) \int_0^1 e^{-\gamma(\xi)tr}(-r)^{\ell}\, dr\Big)(t,x) * |D|^{[a-\sigma]^+} u_0(x) \\
&\quad =: \sum_{\ell=0}^{j} \partial_t^{j-\ell}(te^{-t})\, \mathfrak{F}^{-1}\big(\widehat{L_0^1}(t,\xi)\big)(t,x) * |D|^{[a-\sigma]^+} u_0(x).
\end{aligned}$$

Thanks to (4.27) and (4.29), by using the Leibniz rule we have

$$\big|\partial_\xi^\alpha\big(\widehat{L_0^1}(t,\xi)\big)\big| \lesssim e^{\frac{t}{2}}|\xi|^{-2\sigma\ell-2\sigma+\min\{a,\sigma\}-|\alpha|} \lesssim e^{\frac{t}{2}}|\xi|^{-\sigma-|\alpha|}.$$

Using again (4.45), and carrying out $n-1$ and $n+1$ steps of partial integration imply

$$\big|\mathfrak{F}^{-1}\big(\widehat{L_0^1}(t,\xi)\big)(t,x)\big| \lesssim e^{\frac{t}{2}} \begin{cases} |x|^{-(n-1)} & \text{if } 0<|x|\leqslant 1, \\ |x|^{-(n+1)} & \text{if } |x|\geqslant 1. \end{cases}$$

It follows

$$\Big|\sum_{\ell=0}^{j} \partial_t^{j-\ell}(te^{-t})\, \mathfrak{F}^{-1}\big(\widehat{L_0^1}(t,\xi)\big)(t,x)\Big| \lesssim e^{-ct} \begin{cases} |x|^{-(n-1)} & \text{if } 0<|x|\leqslant 1, \\ |x|^{-(n+1)} & \text{if } |x|\geqslant 1, \end{cases}$$

where c is a sufficiently positive constant. Therefore, we derive

$$\Big\|\sum_{\ell=0}^{j} \partial_t^{j-\ell}(te^{-t})\, \mathfrak{F}^{-1}\big(\widehat{L_0^1}(t,\xi)\big)(t,\cdot)\Big\|_{L^1} \lesssim e^{-ct}.$$

Applying Young's convolution inequality from Proposition B.1.1 gives

$$\Big\|\partial_t^j |D|^a \Big(te^{-t}\mathfrak{F}^{-1}\Big(\gamma(\xi)\chi_3(|\xi|)\widehat{u_0}(\xi) \int_0^1 e^{-\gamma(\xi)tr} dr\Big)\Big)(t,\cdot)\Big\|_{L^q} \lesssim e^{-ct}\|u_0\|_{H_q^{[a-\sigma]^+}}. \tag{4.50}$$

Moreover, due to $1-\chi_3(|\xi|) \in C_0^\infty$, we derive

$$\Big\|\partial_t^j |D|^a \Big(e^{-t}\mathfrak{F}^{-1}\big(1-\chi_3(|\xi|)\big)\Big)(t,\cdot)\Big\|_{L^1} \lesssim e^{-t}.$$

By using again Young's convolution inequality from Proposition B.1.1 we obtain

$$\Big\|\partial_t^j |D|^a \Big(e^{-t}\mathfrak{F}^{-1}\big((1-\chi_3(|\xi|))\widehat{u_0}(\xi)\big)\Big)(t,\cdot)\Big\|_{L^q} \lesssim e^{-t}\|u_0\|_{L^q}. \tag{4.51}$$

Combining from (4.46) to (4.51) we may conclude the first statement in Proposition 4.4.5. Summarizing, the proof of Proposition 4.4.5 is completed. □

From the statements in Propositions 4.4.4 and 4.4.5 we obtain immediately the following result.

Theorem 4.4.2. *The Sobolev solutions to (4.22) satisfy the $L^q - L^q$ estimates*

$$\big\|\partial_t^j |D|^a u_{\chi_3}(t,\cdot)\big\|_{L^q} \lesssim e^{-ct}\Big(\|(u_0,u_1)\|_{H_q^{[2\sigma j-2\sigma+a]^+}} + \|u_0\|_{H_q^a} + \|u_1\|_{H_q^{[a-2\sigma]^+}}\Big) \text{ if } q\in(1,\infty) \text{ and } \sigma\geqslant 1,$$

$$\big\|\partial_t^j |D|^a u_{\chi_3}(t,\cdot)\big\|_{L^q} \lesssim e^{-ct}\Big(\|(u_0,u_1)\|_{H_q^{2\sigma j+[a-\sigma]^+}} + \|u_0\|_{H_q^a} + \|u_1\|_{H_q^{[a-\sigma]^+}}\Big) \text{ if } q=1 \text{ or } \infty \text{ and } \sigma>1,$$

for any $t>0$, $a\geqslant 0$, integer $j\geqslant 0$ and a suitable positive constant c.

$L^q - L^q$ estimates for middle frequencies

Now let us turn to consider some estimates for middle frequencies, where $3^{-\frac{1}{\sigma}} \leqslant |\xi| \leqslant 3^{\frac{1}{\sigma}}$. Our goal is to derive the exponential decay for solutions and some of their derivatives to (4.22).

Theorem 4.4.3. *Let $q \in [1, \infty]$. The Sobolev solutions to (4.22) satisfy the $L^q - L^q$ estimates*

$$\big\|\partial_t^j |D|^a u_{\chi_2}(t,\cdot)\big\|_{L^q} \lesssim e^{-ct}\|(u_0,u_1)\|_{L^q}$$

for any $t > 0$, $a \geqslant 0$, integer $j \geqslant 0$ and a suitable positive constant c.

Proof. Indeed, following the proof of Proposition 3.1.5 we may arrive at the exponential decay for the following estimates:

$$\begin{aligned}
&\big\|\mathfrak{F}^{-1}\big(|\xi|^a \partial_t^j \widehat{K_0}(t,\xi)\chi_2(|\xi|)\big)(t,\cdot)\big\|_{L^1} \lesssim e^{-ct},\\
&\big\|\mathfrak{F}^{-1}\big(|\xi|^a \partial_t^j \widehat{K_1}(t,\xi)\chi_2(|\xi|)\big)(t,\cdot)\big\|_{L^1} \lesssim e^{-ct}.
\end{aligned}$$

Therefore, applying Young's convolution inequality from Proposition B.1.1 we get

$$\begin{aligned}
\big\|\partial_t^j |D|^a u_{\chi_2}(t,\cdot)\big\|_{L^q} &\lesssim \big\|\mathfrak{F}^{-1}\big(|\xi|^a \partial_t^j \widehat{K_0}(t,\xi)\chi_2(|\xi|)\big)(t,\cdot)\big\|_{L^1}\|u_0\|_{L^q}\\
&\quad + \big\|\mathfrak{F}^{-1}\big(|\xi|^a \partial_t^j \widehat{K_1}(t,\xi)\chi_2(|\xi|)\big)(t,\cdot)\big\|_{L^1}\|u_1\|_{L^q}\\
&\lesssim e^{-ct}\|(u_0,u_1)\|_{L^q}.
\end{aligned}$$

Summarizing, the proof to Theorem 4.4.3 is completed. □

From the statements in Theorems 4.4.1, 4.4.2 and 4.4.3, we obtain immediately the following result.

Theorem 4.4.4. *Let $q \in (1,\infty)$ and $m \in [1,q)$. Then, the Sobolev solutions to (4.22) satisfy the $(L^m \cap L^q) - L^q$ estimates*

$$\begin{aligned}
\big\||D|^a u(t,\cdot)\big\|_{L^q} &\lesssim (1+t)^{\frac{1}{2}(2+[\frac{n}{2}])\frac{1}{r}-\frac{n}{2\sigma}(1-\frac{1}{r})-\frac{a}{2\sigma}}\|u_0\|_{L^m\cap H^a_q}\\
&\quad + (1+t)^{1+\frac{1}{2}(1+[\frac{n}{2}])\frac{1}{r}-\frac{n}{2\sigma}(1-\frac{1}{r})-\frac{a}{2\sigma}}\|u_1\|_{L^m\cap H_q^{[a-2\sigma]^+}},\\
\big\||D|^a u_t(t,\cdot)\big\|_{L^q} &\lesssim (1+t)^{\frac{1}{2}(1+[\frac{n}{2}])\frac{1}{r}-\frac{n}{2\sigma}(1-\frac{1}{r})-\frac{a}{2\sigma}}\|u_0\|_{L^m\cap H^a_q}\\
&\quad + (1+t)^{\frac{1}{2}(2+[\frac{n}{2}])\frac{1}{r}-\frac{n}{2\sigma}(1-\frac{1}{r})-\frac{a}{2\sigma}}\|u_1\|_{L^m\cap H^a_q},
\end{aligned}$$

and the $L^q - L^q$ estimates

$$\begin{aligned}
\big\||D|^a u(t,\cdot)\big\|_{L^q} &\lesssim (1+t)^{\frac{1}{2}(2+[\frac{n}{2}])-\frac{a}{2\sigma}}\|u_0\|_{H^a_q} + (1+t)^{1+\frac{1}{2}(1+[\frac{n}{2}])-\frac{a}{2\sigma}}\|u_1\|_{H_q^{[a-2\sigma]^+}},\\
\big\||D|^a u_t(t,\cdot)\big\|_{L^q} &\lesssim (1+t)^{\frac{1}{2}(1+[\frac{n}{2}])-\frac{a}{2\sigma}}\|u_0\|_{H^a_q} + (1+t)^{\frac{1}{2}(2+[\frac{n}{2}])-\frac{a}{2\sigma}}\|u_1\|_{H^a_q},
\end{aligned}$$

where $1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{m}$ and for all $a \geqslant 0$.

4.5. Comparison with known results

In this section, we explain some comparisons between the above obtained estimates and known results.

4.5.1. The case $\delta \in (\frac{\sigma}{2}, \sigma)$

First if we are interested in studying the special case of $\sigma = 1$ and $\delta \in (\frac{1}{2}, 1)$, in the paper [57] the authors obtained L^1 estimates for oscillating integrals to conclude the following $L^m - L^q$ estimates not necessarily on the conjugate line for solutions:

$$\|u(t,\cdot)\|_{L^q} \lesssim \begin{cases} t^{-\frac{n+2-4\delta}{2\delta}(1-\frac{1}{r})}\|u_0\|_{L^m} + t^{1-\frac{n}{2\delta}(1-\frac{1}{r})}\|u_1\|_{L^m} & \text{if } t \in (0,1],\\ t^{-\frac{n+2-4\delta}{2\delta}(1-\frac{1}{r})+[\frac{n}{2}](1-\frac{1}{2\delta})\frac{1}{r}}\|u_0\|_{L^m} + t^{1-\frac{n}{2\delta}(1-\frac{1}{r})+[\frac{n}{2}-1](1-\frac{1}{2\delta})\frac{1}{r}}\|u_1\|_{L^m} & \text{if } t \in [1,\infty),\end{cases}$$

where $1 \leqslant m \leqslant q \leqslant \infty$ and $1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{m}$. Here the authors took into considerations the connection to Fourier multipliers appearing for wave models. It is reasonable to see that the decay rates for solution produced from the results in the paper [57] are somehow better than those in Theorem 4.3.1 with $\sigma = 1$ and $\delta \in (\frac{1}{2}, 1)$. However, these decay rates are almost the same if we consider the case of sufficiently large space dimensions n.

In the paper [11], the authors investigated L^2 estimates for solutions and some of their derivatives as well in the case $\delta \in (\frac{\sigma}{2}, \sigma)$ with additional L^1 regularity for the data as follows:

$$\begin{aligned}
\|u(t,\cdot)\|_{L^2} &\lesssim (1+t)^{-\frac{n}{4\delta}}\|u_0\|_{L^1\cap L^2} + (1+t)^{1-\frac{n}{4\delta}}\|u_1\|_{L^1\cap L^2},\\
\big\||D|^\sigma u(t,\cdot)\big\|_{L^2} &\lesssim (1+t)^{-\frac{n+2\sigma}{4\delta}}\|u_0\|_{L^1\cap H^\sigma} + (1+t)^{1-\frac{n+2\sigma}{4\delta}}\|u_1\|_{L^1\cap L^2},\\
\|u_t(t,\cdot)\|_{L^2} &\lesssim (1+t)^{-\frac{n+2\sigma}{4\delta}}\|u_0\|_{L^1\cap H^{2(\sigma-\delta)}} + (1+t)^{1-\frac{n+2\sigma}{4\delta}}\|u_1\|_{L^1\cap L^2},\\
\big\||D|^{2\delta} u(t,\cdot)\big\|_{L^2} &\lesssim (1+t)^{-\frac{n}{4\delta}-1}\|u_0\|_{L^1\cap H^{2\delta}} + (1+t)^{-\frac{n}{4\delta}}\|u_1\|_{L^1\cap L^2},
\end{aligned}$$

for all space dimensions n. Moreover, there are other sharper results under a restriction to the space dimensions $n \geqslant 2\sigma$ in the paper [11] which is stated as follows:

$$\begin{aligned}
\|u(t,\cdot)\|_{L^2} &\lesssim \begin{cases} (1+t)^{-\frac{n}{4\delta}}\|u_0\|_{L^1\cap L^2} + (1+t)^{-\frac{n-2\sigma}{4\delta}}\|u_1\|_{L^1\cap L^2} & \text{if } n > 2\sigma,\\ (1+t)^{-\frac{\sigma}{2\delta}}\|u_0\|_{L^1\cap L^2} + \log(e+t)\|u_1\|_{L^1\cap L^2} & \text{if } n = 2\sigma, \end{cases}\\
\big\||D|^\sigma u(t,\cdot)\big\|_{L^2} &\lesssim (1+t)^{-\frac{n+2\sigma}{4\delta}}\|u_0\|_{L^1\cap H^\sigma} + (1+t)^{-\frac{n}{4\delta}}\|u_1\|_{L^1\cap L^2},\\
\|u_t(t,\cdot)\|_{L^2} &\lesssim (1+t)^{-\frac{n+2\sigma}{4\delta}}\|u_0\|_{L^1\cap H^{2(\sigma-\delta)}} + (1+t)^{-\frac{n}{4\delta}}\|u_1\|_{L^1\cap L^2},\\
\big\||D|^{2\delta} u(t,\cdot)\big\|_{L^2} &\lesssim (1+t)^{-\frac{n}{4\delta}-1}\|u_0\|_{L^1\cap H^{2\delta}} + (1+t)^{-\frac{n-2\sigma}{4\delta}-1}\|u_1\|_{L^1\cap L^2},
\end{aligned}$$

We can see that the authors in [11] considered L^2 estimates by using Parseval's formula under the choice of parameters $m = 1$ and $q = 2$. For this reason, some results in the paper [11] are obviously somehow better than those in Theorem 4.3.1 and Theorem 4.3.2, where we want to derive estimates in gerenal cases $q \in (1, \infty)$ and $m \in [1, q)$. Nevertheless, if we are interested in a restriction to sufficiently large space dimensions n, the results in Theorem 4.3.1 and Theorem 4.3.2 almost coincide with those in [11].

4.5.2. The case $\delta = \sigma$

First if we are interested in studying low frequencies, the author in [69] in the special case of $\sigma = \delta = 1$ obtained L^1 estimates for oscillating integrals to get the following $L^1 - L^1$ estimates:

$$\|u_{\chi_1}(t,\cdot)\|_{L^1} \lesssim (1+t)^{\frac{1}{2}[\frac{n}{2}]}\|u_0\|_{L^1} + (1+t)^{\frac{1}{2}(1+[\frac{n}{2}])}\|u_1\|_{L^1}.$$

From Theorem 4.4.1 we may conclude for $L^1 - L^1$ estimates as follows:

$$\|u_{\chi_1}(t,\cdot)\|_{L^1} \lesssim (1+t)^{\frac{1}{2}(2+[\frac{n}{2}])}\|u_0\|_{L^1} + (1+t)^{\frac{1}{2}(3+[\frac{n}{2}])}\|u_1\|_{L^1}.$$

Here the author in [69] took into considerations the connection to Fourier multipliers appearing for wave models. It is reasonable to see that the decay rates for solutions produced from the results in [69] are somehow better than those in Theorem 4.4.1 with $\sigma = \delta = 1$. However, these decay rates are almost the same if we consider the case of sufficiently large space dimensions n.

In order to compare the estimates for solutions and some of their partial derivatives in the second case of high frequencies, we recall the result from Theorem 4.4.2 as follows:

$$\big\|\partial_t^j |D|^a u_{\chi_3}(t,\cdot)\big\|_{L^q} \lesssim e^{-ct}\Big(\|(u_0,u_1)\|_{H_q^{[2\sigma j-2\sigma+a]^+}} + \|u_0\|_{H_q^a} + \|u_1\|_{H_q^{[a-2\sigma]^+}}\Big).$$

The author in [69] proved the following result:

$$\big\|\partial_t^j \partial_x^\alpha u_{\chi_3}(t,\cdot)\big\|_{L^q} \lesssim e^{-ct}\Big(\|(u_0,u_1)\|_{H_q^{[2j-2+|\alpha|]^+}} + \|u_0\|_{H_q^{|\alpha|}} + \|u_1\|_{H_q^{[|\alpha|-2]^+}}\Big).$$

Finally, we can see that if we choose $\sigma = 1$ in Theorem 4.4.2, these two results are exactly the same.

5. Semi-linear structurally damped σ-evolution models in the case $\delta = \frac{\sigma}{2}$

The main purpose of this chapter is to study the global (in time) existence of small data Sobolev solutions for semi-linear structurally damped σ-evolution models. We will use the decay estimates for solutions and some of their partial derivatives to the previous linear Cauchy problems to treat a family of semi-linear models.

Let us consider the following two Cauchy problems:

$$u_{tt} + (-\Delta)^\sigma u + \mu(-\Delta)^{\frac{\sigma}{2}} u_t = \big||D|^a u\big|^p, \quad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) \tag{5.1}$$

and

$$u_{tt} + (-\Delta)^\sigma u + \mu(-\Delta)^{\frac{\sigma}{2}} u_t = |u_t|^p, \quad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) \tag{5.2}$$

in space dimensions $n \geqslant 2$ with $\sigma \geqslant 1$, $\mu > 0$, $a \in [0,\sigma)$ and a given number $p > 1$.

Let us explain our objectives and strategies as follows:

- The estimates for solutions to the linear Cauchy problems (2.1) are a key tool to deal with the semi-linear Cauchy problems (5.1) and (5.2).
- By using the fractional Gagliardo-Nirenberg inequality, the fractional chain rule, the fractional powers rule and the fractional Sobolev embedding, we obtain global (in time) existence of small data solutions in the energy space, in the solution space below energy space, in the energy space with a suitable higher regularity and in the large regular space.
- Some examples are presented at the end of each theorem to compare with known results.

In the following statements we introduce the data spaces $\mathcal{A}^s_{m,q} := \big(L^m \cap H^s_q\big) \times \big(L^m \cap H^{[s-\sigma]^+}_q\big)$ with the norm

$$\|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}} := \|u_0\|_{L^m} + \|u_0\|_{H^s_q} + \|u_1\|_{L^m} + \|u_1\|_{H^{[s-\sigma]^+}_q},$$

where $s \geqslant 0$, $q \in (1,\infty)$ and $m \in [1,q)$.

5.1. Global (in time) existence of small data solutions to the model (5.1)

5.1.1. Data from the energy space

Let us assume data to belong to the energy space on the base of L^q.

Theorem 5.1.1. *Let $q \in (1,\infty)$ be a fixed constant and $m \in [1,q)$. Let $a \in [0,\sigma)$ and $n \geqslant 1$. We assume the condition*

$$p > 1 + \frac{m(2\sigma - a)}{n - m(\sigma - a)}.$$

Moreover, we suppose the following conditions:

$$p \in \Big[\frac{q}{m}, \infty\Big) \qquad \text{if } n \in \big(m(\sigma-a), q(\sigma-a)\big],$$

$$p \in \Big[\frac{q}{m}, \frac{n}{n-q(\sigma-a)}\Big] \qquad \text{if } n \in \Big(q(\sigma-a), \frac{q^2(\sigma-a)}{q-m}\Big].$$

Then, there exists a constant $\varepsilon > 0$ such that for any small data

$$(u_0,u_1) \in \mathcal{A}^\sigma_{m,q} \text{ satisfying the assumption } \|(u_0,u_1)\|_{\mathcal{A}^\sigma_{m,q}} \leqslant \varepsilon,$$

we have a uniquely determined global (in time) small data energy solution (on the base of L^q)

$$u \in C\big([0,\infty), H^\sigma_q\big) \cap C^1\big([0,\infty), L^q\big)$$

to (5.1). The following estimates hold:

$$\|u(t,\cdot)\|_{L^q} \lesssim (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})}\|(u_0,u_1)\|_{\mathcal{A}^\sigma_{m,q}},$$
$$\big\|\big(|D|^\sigma u(t,\cdot), u_t(t,\cdot)\big)\big\|_{L^q} \lesssim (1+t)^{-\frac{n}{\sigma}(1-\frac{1}{r})}\|(u_0,u_1)\|_{\mathcal{A}^\sigma_{m,q}},$$

where $1+\frac{1}{q} = \frac{1}{r}+\frac{1}{m}$.

Remark 5.1.1. Let us explain the conditions for p and n in Theorem 5.1.1. The conditions $p > 1+\frac{m(2\sigma-a)}{n-m(\sigma-a)}$ and $n > m(\sigma-a)$ imply the same decay estimates for solutions to (5.1) as for solutions to the corresponding linear model with vanishing right-hand side. Hence, we can say that the non-linearity is interpreted as a small perturbation. The other conditions for p come into play after we apply the fractional Gargliardo-Nirenberg inequality. In addtion, the condition for n compared to the term $q(\sigma-a)$ results from the same tool. Eventually, the upper bound for $n \leqslant \frac{q^2(\sigma-a)}{q-m}$ arises from the set of admissible range for p to guarantee it as non-empty set.

Proof. Our approach bases on the paper [12]. By using fundamental solutions we write the solutions to (5.1) with vanishing right-hand side as follows:

$$u^{ln}(t,x) = K_0(t,x) *_x u_0(x) + K_1(t,x) *_x u_1(x),$$

where $K_j(t,x)$ with $j=0,1$ are defined as in Chapter 2. Applying Duhamel's principle leads to the following representation of solutions to (5.1):

$$u(t,x) = K_0(t,x) *_x u_0(x) + K_1(t,x) *_x u_1(x) + \int_0^t K_1(t-\tau,x) *_x \big||D|^a u(\tau,x)\big|^p d\tau.$$

We introduce the data space $\mathcal{A}^\sigma_{m,q} := (L^m \cap H^\sigma_q) \times (L^m \cap L^q)$. Moreover, we introduce for any $t>0$ the function space $X(t) := C\big([0,t], H^\sigma_q\big) \cap C^1\big([0,t], L^q\big)$. For the sake of brevity, we also define the norm

$$\|u\|_{X(t)} := \sup_{0\leqslant\tau\leqslant t}\Big(f_0(\tau)^{-1}\|u(\tau,\cdot)\|_{L^q} + f_\sigma(\tau)^{-1}\big\||D|^\sigma u(\tau,\cdot)\big\|_{L^q} + g(\tau)^{-1}\|u_t(\tau,\cdot)\|_{L^q}\Big)$$

and the space $X_0(t) := C\big([0,t], H^\sigma_q\big)$ with the norm

$$\|w\|_{X_0(t)} := \sup_{0\leqslant\tau\leqslant t}\Big(f_0(\tau)^{-1}\|w(\tau,\cdot)\|_{L^q} + f_\sigma(\tau)^{-1}\big\||D|^\sigma w(\tau,\cdot)\big\|_{L^q}\Big),$$

where from the estimates for solutions and some of their derivatives to the linear Cauchy problems given in Theorem 2.3.1 we choose

$$f_0(\tau) = (1+\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})}, \quad f_\sigma(\tau) = g(\tau) = (1+\tau)^{-\frac{n}{\sigma}(1-\frac{1}{r})}.$$

We define the operator $N:\ X(t) \longrightarrow X(t)$ by the formula

$$Nu(t,x) = K_0(t,x) *_x u_0(x) + K_1(t,x) *_x u_1(x) + \int_0^t K_1(t-\tau,x) *_x \big||D|^a u(\tau,x)\big|^p d\tau.$$

We will prove that the operator N satisfies the following two estimates:

$$\|Nu\|_{X(t)} \lesssim \|(u_0,u_1)\|_{\mathcal{A}^\sigma_{m,q}} + \|u\|^p_{X_0(t)}, \tag{5.3}$$
$$\|Nu-Nv\|_{X(t)} \lesssim \|u-v\|_{X_0(t)}\big(\|u\|^{p-1}_{X_0(t)} + \|v\|^{p-1}_{X_0(t)}\big). \tag{5.4}$$

First let us prove the estimate (5.3). Taking into consideration the estimates for solutions and some of their partial derivatives to the linear Cauchy problems from Chapter 2 for $j,k=0,1$ and $(j,k)\neq(1,1)$ we get the following estimates:

$$\begin{aligned}
&\big\|\partial_t^j|D|^{k\sigma}\big(K_0(t,x) *_x u_0(x) + K_1(t,x) *_x u_1(x)\big)\big\|_{L^q}\\
&\quad\lesssim (1+t)^{-\frac{n}{\sigma}(1-\frac{1}{r})-(k+j)}\|u_0\|_{L^m\cap H^\sigma_q} + (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-(k+j)}\|u_1\|_{L^m\cap L^q}\\
&\quad\lesssim (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-(k+j)}\|(u_0,u_1)\|_{\mathcal{A}^\sigma_{m,q}}.
\end{aligned}$$

In order to control the integral term in the representation of solutions, we use two different strategies for $\tau \in [0, t/2]$ and $\tau \in [t/2, t]$. In particular, we use the $(L^m \cap L^q) - L^q$ estimates if $\tau \in [0, t/2]$ and the $L^q - L^q$ estimates if $\tau \in [t/2, t]$ from Theorem 2.3.1. Therefore, we obtain

$$\begin{aligned}\big\|\partial_t^j |D|^{k\sigma} Nu(t,\cdot)\big\|_{L^q} &\lesssim (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-(k+j)}\|(u_0,u_1)\|_{\mathcal{A}^\sigma_{m,q}} \\ &\quad + \int_0^{t/2} (1+t-\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-(k+j)} \big\| \big||D|^a u(\tau,\cdot)\big|^p \big\|_{L^m \cap L^q} d\tau \\ &\quad + \int_{t/2}^t (1+t-\tau)^{1-(k+j)} \big\| \big||D|^a u(\tau,\cdot)\big|^p \big\|_{L^q} d\tau.\end{aligned}$$

Hence, it is necessary to require the estimates for $\big||D|^a u(\tau,x)\big|^p$ in $L^m \cap L^q$ and L^q as follows:

$$\big\| \big||D|^a u(\tau,\cdot)\big|^p \big\|_{L^m \cap L^q} \lesssim \big\||D|^a u(\tau,\cdot)\big\|^p_{L^{mp}} + \big\||D|^a u(\tau,\cdot)\big\|^p_{L^{qp}}, \quad \big\| \big||D|^a u(\tau,\cdot)\big|^p \big\|_{L^q} = \big\||D^a| u(\tau,\cdot)\big\|^p_{L^{qp}}.$$

Applying the fractional Gagliardo-Nirenberg inequality from Proposition C.1.1 we may conclude

$$\begin{aligned}\big\||D|^a u(\tau,\cdot)\big\|_{L^{qp}} &\lesssim \big\||D|^\sigma u(\tau,\cdot)\big\|^{\theta_{qp}}_{L^q} \|u(\tau,\cdot)\|^{1-\theta_{qp}}_{L^q} \\ &\lesssim \big(f_\sigma(\tau)\|u\|_{X_0(\tau)}\big)^{\theta_{qp}} \big(f_0(\tau)\|u\|_{X_0(\tau)}\big)^{1-\theta_{qp}} \lesssim (1+\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\theta_{qp}}\|u\|_{X_0(\tau)},\end{aligned}$$

and

$$\begin{aligned}\big\||D|^a u(\tau,\cdot)\big\|_{L^{mp}} &\lesssim \big\||D|^\sigma u(\tau,\cdot)\big\|^{\theta_{mp}}_{L^q} \|u(\tau,\cdot)\|^{1-\theta_{mp}}_{L^q} \\ &\lesssim \big(f_\sigma(\tau)\|u(\tau,\cdot)\|_{X_0(\tau)}\big)^{\theta_{mp}} \big(f_0(\tau)\|u\|_{X_0(\tau)}\big)^{1-\theta_{mp}} \lesssim (1+\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\theta_{mp}}\|u\|_{X_0(\tau)},\end{aligned}$$

where

$$\theta_{qp} := \theta_{a,\sigma}(qp,q) = \frac{n}{\sigma}\Big(\frac{1}{q} - \frac{1}{qp} + \frac{a}{n}\Big) \text{ and } \theta_{mp} := \theta_{a,\sigma}(mp,q) = \frac{n}{\sigma}\Big(\frac{1}{q} - \frac{1}{mp} + \frac{a}{n}\Big).$$

As in Corollary C.1.1 we have to guarantee that θ_{qp} and θ_{mp} belong to $[\frac{a}{\sigma}, 1]$. Both conditions imply the restrictions

$$p \in \Big[\frac{q}{m}, \infty\Big) \text{ if } n \leqslant q(\sigma - a), \quad \text{ or } \quad p \in \Big[\frac{q}{m}, \frac{n}{n - q(\sigma - a)}\Big] \text{ if } n > q(\sigma - a).$$

By virtue of $\theta_{mp} < \theta_{qp}$ and the relation $1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{m}$, we derive

$$\begin{aligned}\big\| \big||D|^a u(\tau,\cdot)\big|^p \big\|_{L^m \cap L^q} &\lesssim (1+\tau)^{p\big(1-\frac{n}{\sigma}(1-\frac{1}{r})-\theta_{mp}\big)}\|u\|^p_{X_0(\tau)} \lesssim (1+\tau)^{p-\frac{np}{\sigma}(\frac{1}{m}-\frac{1}{mp}+\frac{a}{n})}\|u\|^p_{X_0(\tau)}, \\ \big\| \big||D|^a u(\tau,\cdot)\big|^p \big\|_{L^q} &\lesssim (1+\tau)^{p\big(1-\frac{n}{\sigma}(1-\frac{1}{r})-\theta_{qp}\big)}\|u\|^p_{X_0(\tau)} \lesssim (1+\tau)^{p-\frac{np}{\sigma}(\frac{1}{m}-\frac{1}{qp}+\frac{a}{n})}\|u\|^p_{X_0(\tau)}.\end{aligned}$$

Summarizing, from both estimates we may conclude

$$\begin{aligned}\big\|\partial_t^j |D|^{k\sigma} Nu(t,\cdot)\big\|_{L^q} &\lesssim (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-(k+j)}\|(u_0,u_1)\|_{\mathcal{A}^\sigma_{m,q}} \\ &\quad + \|u\|^p_{X_0(t)} \int_0^{t/2} (1+t-\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-(k+j)}(1+\tau)^{p-\frac{np}{\sigma}(\frac{1}{m}-\frac{1}{mp}+\frac{a}{n})} d\tau \\ &\quad + \|u\|^p_{X_0(t)} \int_{t/2}^t (1+t-\tau)^{1-(k+j)}(1+\tau)^{p-\frac{np}{\sigma}(\frac{1}{m}-\frac{1}{qp}+\frac{a}{n})} d\tau.\end{aligned}$$

Using $(1+t-\tau) \approx (1+t)$ if $\tau \in [0, t/2]$ and $(1+\tau) \approx (1+t)$ if $\tau \in [t/2, t]$ we get

$$\begin{aligned}&\int_0^{t/2} (1+t-\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-(k+j)}(1+\tau)^{p-\frac{np}{\sigma}(\frac{1}{m}-\frac{1}{mp}+\frac{a}{n})} d\tau \\ &\qquad \lesssim (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-(k+j)} \int_0^{t/2} (1+\tau)^{p-\frac{n}{\sigma}(\frac{p-1}{m}+\frac{ap}{n})} d\tau,\end{aligned}$$

and

$$\int_{t/2}^{t}(1+t-\tau)^{1-(k+j)}(1+\tau)^{p-\frac{np}{\sigma}(\frac{1}{m}-\frac{1}{qp}+\frac{a}{n})}d\tau \lesssim (1+t)^{p-\frac{np}{\sigma}(\frac{1}{m}-\frac{1}{qp}+\frac{a}{n})}\int_{t/2}^{t}(1+t-\tau)^{1-(k+j)}d\tau$$
$$\lesssim (1+t)^{p-\frac{np}{\sigma}(\frac{1}{m}-\frac{1}{qp}+\frac{a}{n})+2-(k+j)} \lesssim (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-(k+j)+1+p-\frac{n}{\sigma}(\frac{p-1}{m}+\frac{ap}{n})}.$$

Because of $p > 1 + \frac{m(2\sigma-a)}{n-m(\sigma-a)}$, it follows immediately

$$p - \frac{n}{\sigma}\Big(\frac{p-1}{m} + \frac{ap}{n}\Big) < -1.$$

Consequently, the function $(1+\tau)^{p-\frac{n}{\sigma}\left(\frac{p-1}{m}+\frac{ap}{n}\right)}$ is integrable over $(0,\infty)$. Hence, we have

$$\int_0^{t/2}(1+t-\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-(k+j)}(1+\tau)^{p-\frac{np}{\sigma}(\frac{1}{m}-\frac{1}{mp}+\frac{a}{n})}d\tau \lesssim (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-(k+j)},$$

and

$$\int_{t/2}^{t}(1+t-\tau)^{1-(k+j)}(1+\tau)^{p-\frac{np}{\sigma}(\frac{1}{m}-\frac{1}{qp}+\frac{a}{n})}d\tau \lesssim (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-(k+j)}.$$

Finally, we conclude for $j,k = 0,1$ and $(j,k) \neq (1,1)$ the estimates

$$\big\|\partial_t^j|D|^{k\sigma}Nu(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-(k+j)}\big(\|(u_0,u_1)\|_{\mathcal{A}^{\sigma}_{m,q}} + \|u\|^p_{X_0(t)}\big).$$

From the definition of the norm in $X(t)$, we obtain immediately the inequality (5.3).

Next let us prove the estimate (5.4). Using again the estimates for solutions and some of their partial derivatives to the linear Cauchy problems from Chapter 2, that is, the $(L^m \cap L^q) - L^q$ estimates if $\tau \in [0,t/2]$ and the $L^q - L^q$ estimates if $\tau \in [t/2,t]$ from Theorem 2.3.1, then we may derive for two functions u and v from $X(t)$ the estimates

$$\begin{aligned}&\big\|\partial_t^j|D|^{k\sigma}(Nu(t,\cdot)-Nv(t,\cdot))\big\|_{L^q}\\ &\quad\lesssim \int_0^{t/2}(1+t-\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-(k+j)}\big\|\,\big||D|^a u(\tau,\cdot)\big|^p - \big||D|^a v(\tau,\cdot)\big|^p\big\|_{L^m\cap L^q}d\tau\\ &\qquad+\int_{t/2}^{t}(1+t-\tau)^{1-(k+j)}\big\|\,\big||D|^a u(\tau,\cdot)\big|^p - \big||D|^a v(\tau,\cdot)\big|^p\big\|_{L^q}d\tau.\end{aligned}$$

By using Hölder's inequality, we get

$$\begin{aligned}&\big\|\,\big||D|^a u(\tau,\cdot)\big|^p - \big||D|^a v(\tau,\cdot)\big|^p\big\|_{L^q}\\ &\quad\lesssim \big\||D|^a u(\tau,\cdot)-|D|^a v(\tau,\cdot)\big\|_{L^{qp}}\big(\big\||D|^a u(\tau,\cdot)\big\|^{p-1}_{L^{qp}} + \big\||D|^a v(\tau,\cdot)\big\|^{p-1}_{L^{qp}}\big),\end{aligned}$$

and

$$\begin{aligned}&\big\|\,\big||D|^a u(\tau,\cdot)\big|^p - \big||D|^a v(\tau,\cdot)\big|^p\big\|_{L^m}\\ &\quad\lesssim \big\||D|^a u(\tau,\cdot)-|D|^a v(\tau,\cdot)\big\|_{L^{mp}}\big(\big\||D|^a u(\tau,\cdot)\big\|^{p-1}_{L^{mp}} + \big\||D|^a v(\tau,\cdot)\big\|^{p-1}_{L^{mp}}\big).\end{aligned}$$

Analogously to the proof of (5.3), applying the fractional Gagliardo-Nirenberg inequality from Proposition C.1.1 to the norms

$$\big\||D|^a u(\tau,\cdot)-|D|^a v(\tau,\cdot)\big\|_{L^\eta},\quad \big\||D|^a u(\tau,\cdot)\big\|_{L^\eta},\quad \big\||D|^a v(\tau,\cdot)\big\|_{L^\eta}$$

with $\eta = qp$ and $\eta = mp$ we derive the following estimates:

$$\begin{aligned}\big\||D|^a u(\tau,\cdot)-|D|^a v(\tau,\cdot)\big\|_{L^{qp}} &\lesssim (1+\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\theta_{qp}}\|u-v\|_{X_0(\tau)},\\ \big\||D|^a u(\tau,\cdot)\big\|_{L^{qp}} &\lesssim (1+\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\theta_{qp}}\|u\|_{X_0(\tau)},\\ \big\||D|^a v(\tau,\cdot)\big\|_{L^{qp}} &\lesssim (1+\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\theta_{qp}}\|v\|_{X_0(\tau)},\end{aligned}$$

and

$$\begin{aligned} \big\||D|^a u(\tau,\cdot)-|D|^a v(\tau,\cdot)\big\|_{L^{mp}} &\lesssim (1+\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\theta_{mp}}\|u-v\|_{X_0(\tau)},\\ \big\||D|^a u(\tau,\cdot)\big\|_{L^{mp}} &\lesssim (1+\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\theta_{mp}}\|u\|_{X_0(\tau)},\\ \big\||D|^a v(\tau,\cdot)\big\|_{L^{mp}} &\lesssim (1+\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\theta_{mp}}\|v\|_{X_0(\tau)}. \end{aligned}$$

Therefore, thanks to $\theta_{mp}<\theta_{qp}$ and the relation $1+\frac{1}{q}=\frac{1}{r}+\frac{1}{m}$, we obtain

$$\begin{aligned} &\big\|\,\big||D|^a u(\tau,\cdot)\big|^p-\big||D|^a v(\tau,\cdot)\big|^p\big\|_{L^m\cap L^q}\\ &\quad\lesssim (1+\tau)^{p-\frac{np}{\sigma}(\frac{1}{m}-\frac{1}{mp}+\frac{a}{n})}\|u-v\|_{X_0(\tau)}\big(\|u\|^{p-1}_{X_0(\tau)}+\|v\|^{p-1}_{X_0(\tau)}\big), \end{aligned}$$

and

$$\begin{aligned} &\big\|\,\big||D|^a u(\tau,\cdot)\big|^p-\big||D|^a v(\tau,\cdot)\big|^p\big\|_{L^q}\\ &\quad\lesssim (1+\tau)^{p-\frac{np}{\sigma}(\frac{1}{m}-\frac{1}{qp}+\frac{a}{n})}\|u-v\|_{X_0(\tau)}\big(\|u\|^{p-1}_{X_0(\tau)}+\|v\|^{p-1}_{X_0(\tau)}\big). \end{aligned}$$

Applying again an analogous treatment as we did in the proof of (5.3) we may conclude for $j,k=0,1$ and $(j,k)\neq(1,1)$ the estimates

$$\big\|\partial_t^j|D|^{k\sigma}\big(Nu(t,\cdot)-Nv(t,\cdot)\big)\big\|_{L^q}\lesssim(1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-(k+j)}\|u-v\|_{X_0(t)}\big(\|u\|^{p-1}_{X_0(t)}+\|v\|^{p-1}_{X_0(t)}\big).$$

From the definition of the norm in $X(t)$, we obtain immediately the inequality (5.4).
Summarizing, the proof of Theorem 5.1.1 is completed. □

Example 5.1.1. If we choose $m=1$ and $q\in(1,2]$ in Theorem 5.1.1, then it becomes Theorem 1 in the paper [11]. For this reason, Theorem 5.1.1 is a generalization of Theorem 1 in the paper [11].

Example 5.1.2. If we choose $m=1$, $\sigma=1$, $a=0$ and the space dimensions $n=2,3,4$ in Theorem 5.1.1, then it becomes Theorem 1 in the paper [12] with $q\in(1,2]$. Moreover, by choosing some other suitable parameters for $q\in(1,\infty)$, we can see that the results of Theorem 5.1.1 bring some flexibility in comparison with those from Theorem 1 in the paper [12] (see the following table):

	Theorem 5.1.1	Theorem 1 in [12]
$q\in(1,2]$ and $n=2,3,4$	$p\in[q,\frac{n}{n-q}]$ and $p>1+\frac{2}{n-1}$	$p\in[q,\frac{n}{n-q}]$ and $p>1+\frac{2}{n-1}$
$q=4$ and $n=2,3,4$	$p\in[4,\infty)$	no result for p
$q=4$ and $n=5$	$p\in[4,5]$	no result for p

Tab. 5.1.: Comparison between the obtained results.

Remark 5.1.2. In this remark, we allow a loss of decay in estimates for solutions in comparison with the corresponding decay estimates for solutions to the linear Cauchy problem with vanishing right-hand side. We follow the proof of Theorem 5.1.1. Having this in mind we fix the data space and the solution space as in Theorem 5.1.1, but we use the norm

$$\|u\|_{X(t)}:=\sup_{0\leqslant\tau\leqslant t}\Big(f_{\varepsilon_1}(\tau)^{-1}\|u(\tau,\cdot)\|_{L^q}+f_{\varepsilon_2}(\tau)^{-1}\big\||D|^\sigma u(\tau,\cdot)\big\|_{L^q}+f_{\varepsilon_3}(\tau)^{-1}\|u_t(\tau,\cdot)\|_{L^q}\Big),$$

and the space $X_0(t):=C\big([0,t],H^\sigma_q\big)$ with the norm

$$\|w\|_{X_0(t)}:=\sup_{0\leqslant\tau\leqslant t}\Big(f_{\varepsilon_1}(\tau)^{-1}\|w(\tau,\cdot)\|_{L^q}+f_{\varepsilon_2}(\tau)^{-1}\big\||D|^\sigma w(\tau,\cdot)\big\|_{L^q}\Big),$$

where

$$f_{\varepsilon_1}(\tau)=(1+\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})+\varepsilon_1},\quad f_{\varepsilon_2}(\tau)=(1+\tau)^{-\frac{n}{\sigma}(1-\frac{1}{r})+\varepsilon_2}\text{ and }f_{\varepsilon_3}(\tau)=(1+\tau)^{-\frac{n}{\sigma}(1-\frac{1}{r})+\varepsilon_3},$$

for some positive constants ε_1, ε_2 and ε_3.

Case 1: If we assume that $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_0$ are sufficiently small positive constants, then the condition for exponents p is the same as in Theorem 5.1.1, in particular,

$$p > 1 + \frac{m(2\sigma - a)}{n - m(\sigma - a)}.$$

Case 2: Now we fix the parameter $\varepsilon_1 := (1 - \frac{1}{p})\big(-1 + \frac{n}{\sigma}(1 - \frac{1}{r})\big)$. Next we choose $\varepsilon_2 = 1 + \varepsilon_1$ and $\varepsilon_3 = 1$. Then, we have to guarantee another condition for the space dimension n as follows:

$$n > \frac{2mq\sigma}{q - m}$$

which we may avoid the condition for the exponent $p > 1 + \frac{m(2\sigma - a)}{n - m(\sigma - a)}$ in Theorem 5.1.1.

5.1.2. Data below the energy space

In the second case we assume that the data belong to the Sobolev space on the base of L^q only. For this reason, we get Sobolev solutions instead of energy solutions.

Theorem 5.1.2. *Let $s \in (0, \sigma)$ and $n \geqslant 1$. Let $q \in (1, \infty)$ be a fixed constant, $m \in [1, q)$ and $a \in [0, s)$. We assume the condition*

$$p > 1 + \frac{m(2\sigma - a)}{n - m(\sigma - a)}.$$

Moreover, we suppose the following conditions:

$$p \in \Big[\frac{q}{m}, \infty\Big) \qquad \text{if } n \in \big(m(\sigma - a), q(s - a)\big],$$
$$p \in \Big[\frac{q}{m}, \frac{n}{n - q(s - a)}\Big] \qquad \text{if } n \in \Big(q(s - a), \frac{q^2(s - a)}{q - m}\Big].$$

Then, there exists a constant $\varepsilon > 0$ such that for any small data

$$(u_0, u_1) \in \mathcal{A}^s_{m,q} \text{ satisfying the assumption } \|(u_0, u_1)\|_{\mathcal{A}^s_{m,q}} \leqslant \varepsilon,$$

we have a uniquely determined global (in time) small data Sobolev solution

$$u \in C\big([0, \infty), H^s_q\big)$$

to (5.1). The following estimates hold:

$$\|u(t, \cdot)\|_{L^q} \lesssim (1 + t)^{1 - \frac{n}{\sigma}(1 - \frac{1}{r})} \|(u_0, u_1)\|_{\mathcal{A}^s_{m,q}},$$
$$\big\||D|^s u(t, \cdot)\big\|_{L^q} \lesssim (1 + t)^{1 - \frac{n}{\sigma}(1 - \frac{1}{r}) - \frac{s}{\sigma}} \|(u_0, u_1)\|_{\mathcal{A}^s_{m,q}},$$

where $1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{m}$.

Proof. We introduce the data space $\mathcal{A}^s_{m,q} := (L^m \cap H^s_q) \times (L^m \cap L^q)$. Moreover, we introduce for any $t > 0$ the function space $X(t) := C\big([0, t], H^s_q\big)$. For the sake of brevity, we also define the norm

$$\|u\|_{X(t)} := \sup_{0 \leqslant \tau \leqslant t} \Big((1 + \tau)^{-1 + \frac{n}{\sigma}(1 - \frac{1}{r})} \|u(\tau, \cdot)\|_{L^q} + (1 + \tau)^{-1 + \frac{n}{\sigma}(1 - \frac{1}{r}) + \frac{s}{\sigma}} \big\||D|^s u(\tau, \cdot)\big\|_{L^q}\Big).$$

We define the operator $N :\ X(t) \longrightarrow X(t)$ by the formula

$$Nu(t, x) = K_0(t, x) \ast_x u_0(x) + K_1(t, x) \ast_x u_1(x) + \int_0^t K_1(t - \tau, x) \ast_x \big||D|^a u(\tau, x)\big|^p d\tau.$$

We will prove that the operator N satisfies the following two estimates:

$$\|Nu\|_{X(t)} \lesssim \|(u_0, u_1)\|_{\mathcal{A}^s_{m,q}} + \|u\|^p_{X(t)}, \tag{5.5}$$
$$\|Nu - Nv\|_{X(t)} \lesssim \|u - v\|_{X(t)} \big(\|u\|^{p-1}_{X(t)} + \|v\|^{p-1}_{X(t)}\big). \tag{5.6}$$

First let us prove the estimate (5.5). Taking into consideration the estimates for solutions and some of their partial derivatives to the linear Cauchy problems from Chapter 2 for $k = 0, 1$ we get the following estimates:

$$\begin{aligned}\big\||D|^{ks}\big(K_0(t,x) *_x u_0(x) + K_1(t,x) *_x u_1(x)\big)\big\|_{L^q} \\ &\lesssim (1+t)^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{ks}{\sigma}}\|u_0\|_{L^m\cap H^s_q} + (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{ks}{\sigma}}\|u_1\|_{L^m\cap L^q} \\ &\lesssim (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{ks}{\sigma}}\|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}}.\end{aligned}$$

In order to control the integral term in the representation of solutions, we use the $(L^m \cap L^q) - L^q$ estimates if $\tau \in [0, t/2]$ and the $L^q - L^q$ estimates if $\tau \in [t/2, t]$ from Theorem 2.3.1. Therefore, we obtain

$$\begin{aligned}\big\||D|^{ks}Nu(t,\cdot)\big\|_{L^q} &\lesssim (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{ks}{\sigma}}\|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}} \\ &\quad + \int_0^{t/2}(1+t-\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{ks}{\sigma}}\big\|\big||D|^a u(\tau,\cdot)\big|^p\big\|_{L^m\cap L^q}d\tau \\ &\quad + \int_{t/2}^t (1+t-\tau)^{1-\frac{ks}{\sigma}}\big\|\big||D|^a u(\tau,\cdot)\big|^p\big\|_{L^q}d\tau.\end{aligned}$$

Hence, it is necessary to require the estimates for $\big||D|^a u(\tau,x)\big|^p$ in $L^m \cap L^q$ and L^q as follows:

$$\big\|\big||D|^a u(\tau,\cdot)\big|^p\big\|_{L^m\cap L^q} \lesssim \big\||D|^a u(\tau,\cdot)\big\|^p_{L^{mp}} + \big\||D|^a u(\tau,\cdot)\big\|^p_{L^{qp}}, \quad \big\|\big||D|^a u(\tau,\cdot)\big|^p\big\|_{L^q} = \big\||D^a|u(\tau,\cdot)\big\|^p_{L^{qp}}.$$

Applying the fractional Gagliardo-Nirenberg inequality from Proposition C.1.1 we may conclude

$$\big\||D|^a u(\tau,\cdot)\big\|_{L^{qp}} \lesssim \big\||D|^s u(\tau,\cdot)\big\|^{\theta_{qp}}_{L^q}\|u(\tau,\cdot)\|^{1-\theta_{qp}}_{L^q} \lesssim (1+\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s}{\sigma}\theta_{qp}}\|u\|_{X(\tau)},$$

and

$$\big\||D|^a u(\tau,\cdot)\big\|_{L^{mp}} \lesssim \big\||D|^s u(\tau,\cdot)\big\|^{\theta_{mp}}_{L^q}\|u(\tau,\cdot)\|^{1-\theta_{mp}}_{L^q} \lesssim (1+\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s}{\sigma}\theta_{mp}}\|u\|_{X(\tau)},$$

where

$$\theta_{qp} := \theta_{a,s}(qp,q) = \frac{n}{s}\Big(\frac{1}{q}-\frac{1}{qp}+\frac{a}{n}\Big) \text{ and } \theta_{mp} := \theta_{a,s}(mp,q) = \frac{n}{s}\Big(\frac{1}{q}-\frac{1}{mp}+\frac{a}{n}\Big).$$

As in Corollary C.1.1 we have to guarantee that θ_{qp} and θ_{mp} belong to $[\frac{a}{s}, 1]$. Both conditions imply the restrictions

$$p \in \Big[\frac{q}{m}, \infty\Big) \text{ if } n \leqslant q(s-a), \quad \text{or} \quad p \in \Big[\frac{q}{m}, \frac{n}{n-q(s-a)}\Big] \text{ if } n > q(s-a).$$

By virtue of $\theta_{mp} < \theta_{qp}$ and the relation $1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{m}$, we derive

$$\begin{aligned}\big\|\big||D|^a u(\tau,\cdot)\big|^p\big\|_{L^m\cap L^q} &\lesssim (1+\tau)^{p\big(1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s}{\sigma}\theta_{mp}\big)}\|u\|^p_{X(\tau)} \lesssim (1+\tau)^{p-\frac{np}{\sigma}(\frac{1}{m}-\frac{1}{mp}+\frac{a}{n})}\|u\|^p_{X(\tau)}, \\ \big\|\big||D|^a u(\tau,\cdot)\big|^p\big\|_{L^q} &\lesssim (1+\tau)^{p\big(1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s}{\sigma}\theta_{qp}\big)}\|u\|^p_{X(\tau)} \lesssim (1+\tau)^{p-\frac{np}{\sigma}(\frac{1}{m}-\frac{1}{qp}+\frac{a}{n})}\|u\|^p_{X(\tau)}.\end{aligned}$$

Summarizing, from both estimates we may conclude

$$\begin{aligned}\big\||D|^{ks}Nu(t,\cdot)\big\|_{L^q} &\lesssim (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{ks}{\sigma}}\|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}} \\ &\quad + \|u\|^p_{X(t)}\int_0^{t/2}(1+t-\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{ks}{\sigma}}(1+\tau)^{p-\frac{np}{\sigma}(\frac{1}{m}-\frac{1}{mp}+\frac{a}{n})}d\tau \\ &\quad + \|u\|^p_{X(t)}\int_{t/2}^t (1+t-\tau)^{1-\frac{s}{\sigma}}(1+\tau)^{p-\frac{np}{\sigma}(\frac{1}{m}-\frac{1}{qp}+\frac{a}{n})}d\tau.\end{aligned}$$

Using $(1+t-\tau) \approx (1+t)$ if $\tau \in [0, t/2]$ and $(1+\tau) \approx (1+t)$ if $\tau \in [t/2, t]$ we get

$$\begin{aligned}&\int_0^{t/2}(1+t-\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{ks}{\sigma}}(1+\tau)^{p-\frac{np}{\sigma}(\frac{1}{m}-\frac{1}{mp}+\frac{a}{n})}d\tau \\ &\qquad \lesssim (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{ks}{\sigma}}\int_0^{t/2}(1+\tau)^{p-\frac{n}{\sigma}(\frac{p-1}{m}+\frac{ap}{n})}d\tau,\end{aligned}$$

and

$$\int_{t/2}^{t}(1+t-\tau)^{1-\frac{ks}{\sigma}}(1+\tau)^{p-\frac{np}{\sigma}(\frac{1}{m}-\frac{1}{qp}+\frac{a}{n})}d\tau \lesssim (1+t)^{p-\frac{np}{\sigma}(\frac{1}{m}-\frac{1}{qp}+\frac{a}{n})}\int_{t/2}^{t}(1+t-\tau)^{1-\frac{ks}{\sigma}}d\tau$$
$$\lesssim (1+t)^{p-\frac{np}{\sigma}(\frac{1}{m}-\frac{1}{qp}+\frac{a}{n})+2-\frac{ks}{\sigma}} \lesssim (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{ks}{\sigma}+1+p-\frac{n}{\sigma}(\frac{p-1}{m}+\frac{ap}{n})}.$$

Because of $p > 1+\frac{m(2\sigma-a)}{n-m(\sigma-a)}$, it follows immediately

$$p-\frac{n}{\sigma}\Big(\frac{p-1}{m}+\frac{ap}{n}\Big) < -1.$$

Consequently, the function $(1+\tau)^{p-\frac{n}{\sigma}\left(\frac{p-1}{m}+\frac{ap}{n}\right)}$ is integrable over $(0,\infty)$. Hence, we have

$$\int_{0}^{t/2}(1+t-\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{ks}{\sigma}}(1+\tau)^{p-\frac{np}{\sigma}(\frac{1}{m}-\frac{1}{mp}+\frac{a}{n})}d\tau \lesssim (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{ks}{\sigma}},$$

and

$$\int_{t/2}^{t}(1+t-\tau)^{1-\frac{ks}{\sigma}}(1+\tau)^{p-\frac{np}{\sigma}(\frac{1}{m}-\frac{1}{qp}+\frac{a}{n})}d\tau \lesssim (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{ks}{\sigma}}.$$

Finally, we conclude for $k=0,1$ the estimates

$$\big\||D|^{ks}Nu(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{ks}{\sigma}}\big(\|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}}+\|u\|^p_{X(t)}\big).$$

From the definition of the norm in $X(t)$ we obtain immediately the inequality (5.5).

Next let us prove the estimate (5.6). Using again the $(L^m\cap L^q)-L^q$ estimates if $\tau\in[0,t/2]$ and the L^q-L^q estimates if $\tau\in[t/2,t]$ from Theorem 2.3.1 we derive for two functions u and v from $X(t)$ the estimates

$$\begin{aligned}&\big\||D|^{ks}(Nu(t,\cdot)-Nv(t,\cdot))\big\|_{L^q}\\ &\quad\lesssim \int_0^{t/2}(1+t-\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{ks}{\sigma}}\big\|\,\big||D|^a u(\tau,\cdot)\big|^p-\big||D|^a v(\tau,\cdot)\big|^p\big\|_{L^m\cap L^q}d\tau\\ &\qquad+\int_{t/2}^{t}(1+t-\tau)^{1-\frac{ks}{\sigma}}\big\|\,\big||D|^a u(\tau,\cdot)\big|^p-\big||D|^a v(\tau,\cdot)\big|^p\big\|_{L^q}d\tau.\end{aligned}$$

By using Hölder's inequality and applying again an analogous treatment as we did in the proof of (5.4) in Theorem 5.1.1, we may conclude for $k=0,1$ the estimates

$$\big\||D|^{ks}(Nu(t,\cdot)-Nv(t,\cdot))\big\|_{L^q} \lesssim (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{ks}{\sigma}}\|u-v\|_{X(t)}\big(\|u\|^{p-1}_{X(t)}+\|v\|^{p-1}_{X(t)}\big).$$

From the definition of the norm in $X(t)$ it follows immediately the inequality (5.6).
Summarizing, the proof of Theorem 5.1.2 is completed. □

Example 5.1.3. By choosing $m=1$, $q=1.1$, $\sigma=1$, $s=0.99$ and $a=0.1$ we obtain the following admissible range of the exponents p in Theorem 5.1.2:

$$p\in(20,\infty) \quad \text{and} \quad n=1.$$

5.1.3. Data from the energy space with suitable higher regularity

Now we assume that the data in (5.1) belong to the energy space with a suitable higher regularity. We have the following result.

Theorem 5.1.3. *Let $\sigma < s \leqslant \sigma+\frac{n}{q}$ and $n\geqslant 1$. Let $q\in(1,\infty)$ be a fixed constant, $m\in[1,q)$ and $a\in[0,\sigma)$. We assume the condition*

$$p > 1+\max\Big\{\frac{m(2\sigma-a)}{n-m(\sigma-a)}, \lceil s-\sigma\rceil\Big\}.$$

Moreover, we suppose the following conditions:

$$p \in \Big[\frac{q}{m}, \infty\Big) \qquad \text{if } n \in \big(m(\sigma-a), q(s-a)\big],$$
$$p \in \Big[\frac{q}{m}, 1+\frac{q(\sigma-a)}{n-q(s-a)}\Big] \qquad \text{if } n \in \Big(q(s-a), q(s-a)+\frac{mq(\sigma-a)}{q-m}\Big].$$

Then, there exists a constant $\varepsilon > 0$ such that for any small data

$$(u_0,u_1) \in \mathcal{A}^s_{m,q} \text{ satisfying the assumption } \|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}} \leqslant \varepsilon,$$

we have a uniquely determined global (in time) small data energy solution

$$u \in C\big([0,\infty), H^s_q\big) \cap C^1\big([0,\infty), H^{s-\sigma}_q\big)$$

to (5.1). The following estimates hold:

$$\|u(t,\cdot)\|_{L^q} \lesssim (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})}\|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}}, \tag{5.7}$$
$$\|u_t(t,\cdot)\|_{L^q} \lesssim (1+t)^{-\frac{n}{\sigma}(1-\frac{1}{r})}\|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}}, \tag{5.8}$$
$$\big\|\big(|D|^s u(t,\cdot), |D|^{s-\sigma}u_t(t,\cdot)\big)\big\|_{L^q} \lesssim (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s}{\sigma}}\|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}}, \tag{5.9}$$

where $1+\frac{1}{q} = \frac{1}{r}+\frac{1}{m}$.

Remark 5.1.3. Let us explain the conditions for p and n in Theorem 5.1.3. Because we want to use the fractional chain rule, the condition $p > 1+\lceil s-\sigma\rceil$ is necessary to assume. The upper bound for $p \leqslant 1+\frac{q(\sigma-a)}{n-q(s-a)}$ appears to guarantee the possibility for choosing the parameters satisfying the fractional chain rule (the existence of these paremeters is presented more in detail in another remark at the end of the proof). Moreover, the conditions $p > 1+\frac{m(2\sigma-a)}{n-m(\sigma-a)}$ and $n > m(\sigma-a)$ imply the same decay estimates for solutions to (5.1) as for solutions to the corresponding linear model with vanishing right-hand side. Hence, we can say that the non-linearity is interpreted as a small perturbation. The other conditions for p come into play after we apply the fractional Gargliardo-Nirenberg inequality. In addtion, the condition for n compared to the term $q(\sigma-a)$ results from the same tool. Eventually, the upper bound for $n \leqslant q(s-a)+\frac{mq(\sigma-a)}{q-m}$ arises from the set of admissible range for p to guarantee that this set is non-empty.

Proof. We introduce the data space $\mathcal{A}^s_{m,q} := (L^m \cap H^s_q) \times (L^m \cap H^{s-\sigma}_q)$, the function space $X(t) := C\big([0,t], H^s_q\big) \cap C^1\big([0,t], H^{s-\sigma}_q\big)$ with the norm

$$\begin{aligned}\|u\|_{X(t)} := \sup_{0\leqslant\tau\leqslant t}\Big(&(1+\tau)^{\frac{n}{\sigma}(1-\frac{1}{r})-1}\|u(\tau,\cdot)\|_{L^q} + (1+\tau)^{-1+\frac{n}{\sigma}(1-\frac{1}{r})+\frac{s}{\sigma}}\big\||D|^s u(\tau,\cdot)\big\|_{L^q}\\ &+ (1+\tau)^{\frac{n}{\sigma}(1-\frac{1}{r})}\|u_t(\tau,\cdot)\|_{L^q} + (1+\tau)^{-1+\frac{n}{\sigma}(1-\frac{1}{r})+\frac{s}{\sigma}}\big\||D|^{s-\sigma} u_t(\tau,\cdot)\big\|_{L^q}\Big),\end{aligned}$$

and the space $X_0(t) := C\big([0,t], H^s_q\big)$ with the norm

$$\|w\|_{X_0(t)} := \sup_{0\leqslant\tau\leqslant t}\Big((1+\tau)^{\frac{n}{\sigma}(1-\frac{1}{r})-1}\|w(\tau,\cdot)\|_{L^q} + (1+\tau)^{-1+\frac{n}{\sigma}(1-\frac{1}{r})+\frac{s}{\sigma}}\big\||D|^s w(\tau,\cdot)\big\|_{L^q}\Big).$$

We define a mapping $N: \ X(t) \longrightarrow X(t)$ in the following way:

$$Nu(t,x) = K_0(t,x) \ast_x u_0(x) + K_1(t,x) \ast_x u_1(x) + \int_0^t K_1(t-\tau,x) \ast_x \big||D|^a u(\tau,x)\big|^p d\tau.$$

In order to conclude the uniqueness and the global (in time) existence of small data solutions to (5.1), we have to prove the following pair of inequalities:

$$\|Nu\|_{X(t)} \lesssim \|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}} + \|u\|^p_{X_0(t)}, \tag{5.10}$$
$$\|Nu-Nv\|_{X(t)} \lesssim \|u-v\|_{X_0(t)}\big(\|u\|^{p-1}_{X_0(t)} + \|v\|^{p-1}_{X_0(t)}\big). \tag{5.11}$$

First let us prove the inequality (5.10). Our proof is divided into four steps.

Step 1: We need to estimate the norm $\|Nu(t,\cdot)\|_{L^q}$. We apply the $L^m \cap L^q - L^q$ estimates if $\tau \in [0,t/2]$ and $L^q - L^q$ estimates if $\tau \in [t/2,t]$ from Theorem 2.3.1 to conclude

$$\|Nu(t,\cdot)\|_{L^q} \lesssim (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})}\|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}} + \int_0^{t/2}(1+t-\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})}\big\|\,\big||D|^a u(\tau,\cdot)\big|^p\big\|_{L^m\cap L^q}d\tau$$
$$+ \int_{t/2}^t (1+t-\tau)\big\|\,\big||D|^a u(\tau,\cdot)\big|^p\big\|_{L^q}d\tau.$$

We have
$$\big\|\,\big||D|^a u(\tau,\cdot)\big|^p\big\|_{L^m\cap L^q} \lesssim \big\||D|^a u(\tau,\cdot)\big\|^p_{L^{mp}} + \big\||D|^a u(\tau,\cdot)\big\|^p_{L^{qp}}.$$

To estimate the norm $\big\||D|^a u(\tau,\cdot)\big\|^p_{L^{kp}}$ with $k=q,m$, we apply the fractional Gagliardo-Nirenberg inequality from Proposition C.1.1 to conclude

$$\big\||D|^a u(\tau,\cdot)\big\|_{L^{qp}} \lesssim \big\||D|^s u(\tau,\cdot)\big\|^{\theta_{qp}}_{L^q}\,\|u(\tau,\cdot)\|^{1-\theta_{qp}}_{L^q} \lesssim (1+\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s}{\sigma}\theta_{qp}}\|u\|_{X_0(\tau)},$$

and
$$\big\||D|^a u(\tau,\cdot)\big\|_{L^{mp}} \lesssim \big\||D|^s u(\tau,\cdot)\big\|^{\theta_{mp}}_{L^q}\,\|u(\tau,\cdot)\|^{1-\theta_{mp}}_{L^q} \lesssim (1+\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s}{\sigma}\theta_{mp}}\|u\|_{X_0(\tau)},$$

where

$$\theta_{qp} := \theta_{a,s}(qp,q) = \frac{n}{s}\Big(\frac{1}{q}-\frac{1}{qp}+\frac{a}{n}\Big) \text{ and } \theta_{mp} := \theta_{a,s}(mp,q) = \frac{n}{s}\Big(\frac{1}{q}-\frac{1}{mp}+\frac{a}{n}\Big).$$

Due to Corollary C.1.1 we have to guarantee that θ_{qp} and θ_{mp} belong to $[\frac{a}{s},1]$. Since $\theta_{mp} < \theta_{qp}$, we obtain

$$p \in \Big[\frac{q}{m},\infty\Big) \text{ if } n \leqslant q(s-a), \quad \text{or} \quad p \in \Big[\frac{q}{m},\frac{n}{n-q(s-a)}\Big] \text{ if } n > q(s-a).$$

By virtue of the relation $1+\frac{1}{q} = \frac{1}{r}+\frac{1}{m}$, we derive

$$\big\|\,\big||D|^a u(\tau,\cdot)\big|^p\big\|_{L^m\cap L^q} \lesssim (1+\tau)^{p\left(1-\frac{n}{\sigma}(\frac{1}{m}-\frac{1}{mp}+\frac{a}{n})\right)}\|u\|^p_{X_0(\tau)},$$
$$\big\|\,\big||D|^a u(\tau,\cdot)\big|^p\big\|_{L^q} \lesssim (1+\tau)^{p\left(1-\frac{n}{\sigma}(\frac{1}{m}-\frac{1}{qp}+\frac{a}{n})\right)}\|u\|^p_{X_0(\tau)}.$$

Hence, we get

$$\int_0^{t/2}(1+t-\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})}\big\|\,\big||D|^a u(\tau,\cdot)\big|^p\big\|_{L^m\cap L^q}d\tau$$
$$\lesssim (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})}\|u\|^p_{X_0(t)}\int_0^{t/2}(1+\tau)^{p-\frac{n}{\sigma}(\frac{p-1}{m}+\frac{ap}{n})}d\tau,$$

and

$$\int_{t/2}^t (1+t-\tau)\big\|\,\big||D|^a u(\tau,\cdot)\big|^p\big\|_{L^q}d\tau \lesssim (1+t)^{p\left(1-\frac{n}{\sigma}(\frac{1}{m}-\frac{1}{qp}+\frac{a}{n})\right)}\|u\|^p_{X_0(t)}\int_{t/2}^t(1+t-\tau)d\tau$$
$$\lesssim (1+t)^{2+p\left(1-\frac{n}{\sigma}(\frac{1}{m}-\frac{1}{qp}+\frac{a}{n})\right)}\|u\|^p_{X_0(t)} \lesssim (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})+1+p-\frac{n}{\sigma}(\frac{p-1}{m}+\frac{ap}{n})}\|u\|^p_{X_0(t)},$$

where we use the estimates $(1+t-\tau) \approx (1+t)$ if $\tau \in [0,t/2]$ and $(1+\tau) \approx (1+t)$ if $\tau \in [t/2,t]$. Because of $p > 1+\frac{m(2\sigma-a)}{n-m(\sigma-a)}$, we have immediately

$$p - \frac{n}{\sigma}\Big(\frac{p-1}{m}+\frac{ap}{n}\Big) < -1.$$

Consequently, the term $(1+\tau)^{p-\frac{n}{\sigma}\left(\frac{p-1}{m}+\frac{ap}{n}\right)}$ is integrable over $(0,\infty)$. Hence, we get

$$\int_0^{t/2}(1+t-\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})}\big\|\,\big||D|^a u(\tau,\cdot)\big|^p\big\|_{L^m\cap L^q}d\tau \lesssim (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})}\|u\|^p_{X_0(t)},$$

and

$$\int_{t/2}^{t}(1+t-\tau)\big\|\,\big||D|^a u(\tau,\cdot)\big|^p\big\|_{L^q}d\tau \lesssim (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})}\|u\|^p_{X_0(t)}.$$

Therefore, we arrive at the following desired estimate:

$$\|Nu(t,\cdot)\|_{L^q} \lesssim (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})}\big(\|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}}+\|u\|^p_{X_0(t)}\big). \tag{5.12}$$

Step 2: We need to estimate the norm $\|\partial_t Nu(t,\cdot)\|_{L^q}$. Differentiating $Nu(t,x)$ with respect to t we obtain

$$\partial_t Nu(t,x)=\partial_t\big(K_0(t,x)\ast_x u_0(x)+K_1(t,x)\ast_x u_1(x)\big)+\int_0^t \partial_t\big(K_1(t-\tau,x)\ast_x \big||D|^a u(\tau,x)\big|^p\big)d\tau.$$

We apply the $L^m\cap L^q-L^q$ estimates if $\tau\in[0,t/2]$ and the L^q-L^q estimates if $\tau\in[t/2,t]$ from Theorem 2.3.1 to conclude

$$\begin{aligned}\|\partial_t Nu(t,\cdot)\|_{L^q} &\lesssim (1+t)^{-\frac{n}{\sigma}(1-\frac{1}{r})}\|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}}\\ &\quad+\int_0^{t/2}(1+t-\tau)^{-\frac{n}{\sigma}(1-\frac{1}{r})}\big\|\,\big||D|^a u(\tau,\cdot)\big|^p\big\|_{L^m\cap L^q}d\tau+\int_{t/2}^{t}\big\|\,\big||D|^a u(\tau,\cdot)\big|^p\big\|_{L^q}d\tau.\end{aligned}$$

Following the same ideas for deriving (5.12) we may conclude

$$\|\partial_t Nu(t,\cdot)\|_{L^q}\lesssim (1+\tau)^{-\frac{n}{\sigma}(1-\frac{1}{r})}\big(\|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}}+\|u\|^p_{X_0(t)}\big),$$

under the same assumptions for p, that is,

$$p\in\Big[\frac{q}{m},\infty\Big)\ \text{if}\ n\leqslant q(s-a),\qquad\text{or}\qquad p\in\Big[\frac{q}{m},\frac{n}{n-q(s-a)}\Big]\ \text{if}\ n>q(s-a),$$

and

$$p>1+\frac{m(2\sigma-a)}{n-m(\sigma-a)}.$$

Step 3: Let us estimate the norm $\big\|\partial_t|D|^{s-\sigma}Nu(t,\cdot)\big\|_{L^q}$. We use

$$\begin{aligned}\partial_t|D|^{s-\sigma}Nu(t,x)&=\partial_t|D|^{s-\sigma}\big(K_0(t,x)\ast_x u_0(x)+K_1(t,x)\ast_x u_1(x)\big)\\ &\quad+\int_0^t\partial_t|D|^{s-\sigma}\big(K_1(t-\tau,x)\ast_x\big||D|^a u(\tau,x)\big|^p\big)d\tau.\end{aligned}$$

We apply the $(L^m\cap L^q)-L^q$ estimates if $\tau\in[0,t/2]$ and the L^q-L^q estimates if $\tau\in[t/2,t]$ from Theorem 2.3.1 to derive

$$\begin{aligned}\big\|\partial_t|D|^{s-\sigma}Nu(t,\cdot)\big\|_{L^q}&\lesssim(1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s}{\sigma}}\|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}}\\ &\quad+\int_0^{t/2}(1+t-\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s}{\sigma}}\big\|\,\big||D|^a u(\tau,\cdot)\big|^p\big\|_{L^m\cap H^{s-\sigma}_q}d\tau\\ &\quad+\int_{t/2}^{t}(1+t-\tau)^{1-\frac{s}{\sigma}}\big\|\,\big||D|^a u(\tau,\cdot)\big|^p\big\|_{H^{s-\sigma}_q}d\tau\\ &=(1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s}{\sigma}}\|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}}\\ &\quad+\int_0^{t/2}(1+t-\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s}{\sigma}}\big\|\,\big||D|^a u(\tau,\cdot)\big|^p\big\|_{L^m\cap L^q\cap\dot H^{s-\sigma}_q}\\ &\quad+\int_{t/2}^{t}(1+t-\tau)^{1-\frac{s}{\sigma}}\big\|\,\big||D|^a u(\tau,\cdot)\big|^p\big\|_{L^q\cap\dot H^{s-\sigma}_q}d\tau.\end{aligned}$$

The integrals with $\big\|\,\big||D|^a u(\tau,\cdot)\big|^p\big\|_{L^m\cap L^q}$ and $\big\|\,\big||D|^a u(\tau,\cdot)\big|^p\big\|_{L^q}$ will be handled as before if we apply the conditions for p, that is,

$$p\in\Big[\frac{q}{m},\infty\Big)\ \text{if}\ n\leqslant q(s-a),\qquad\text{or}\qquad p\in\Big[\frac{q}{m},\frac{n}{n-q(s-a)}\Big]\ \text{if}\ n>q(s-a),$$

and

$$p > 1 + \frac{m(2\sigma - a)}{n - m(\sigma - a)}.$$

Hence, we get

$$\int_0^{t/2} (1+t-\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s}{\sigma}} \big\| \big||D|^a u(\tau,\cdot)\big|^p \big\|_{L^m \cap L^q} d\tau \lesssim (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s}{\sigma}} \|u\|^p_{X_0(t)},$$

and

$$\int_{t/2}^t (1+t-\tau)^{1-\frac{s}{\sigma}} \big\| \big||D|^a u(\tau,\cdot)\big|^p \big\|_{L^q} d\tau \lesssim (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s}{\sigma}} \|u\|^p_{X_0(t)}.$$

To estimate the integrals with the norm $\big\| \big||D|^a u(\tau,\cdot)\big|^p \big\|_{\dot{H}^{s-\sigma}_q}$, we shall apply Proposition C.3.2 for the fractional chain rule with $p > \lceil s-\sigma \rceil$. Therefore, we obtain

$$\big\| \big||D|^a u(\tau,\cdot)\big|^p \big\|_{\dot{H}^{s-\sigma}_q} \lesssim \big\||D|^a u(\tau,\cdot)\big\|^{p-1}_{L^{q_1}} \big\||D|^{s-\sigma+a} u(\tau,\cdot)\big\|_{L^{q_2}}, \text{ where } \frac{1}{q} = \frac{p-1}{q_1} + \frac{1}{q_2}.$$

Applying the fractional Gagliardo-Nirenberg inequality from Proposition C.1.1 we have

$$\big\||D|^a u(\tau,\cdot)\big\|_{L^{q_1}} \lesssim \|u(\tau,\cdot)\|^{1-\theta_{q_1}}_{L^q} \big\||D|^s u(\tau,\cdot)\big\|^{\theta_{q_1}}_{L^q} \lesssim (1+\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s}{\sigma}\theta_{q_1}} \|u\|_{X_0(\tau)},$$

and

$$\big\||D|^{s-\sigma+a} u(\tau,\cdot)\big\|_{L^{q_2}} \lesssim \|u(\tau,\cdot)\|^{1-\theta_{q_2}}_{L^q} \big\||D|^s u(\tau,\cdot)\big\|^{\theta_{q_2}}_{L^q} \lesssim (1+\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s}{\sigma}\theta_{q_2}} \|u\|_{X_0(\tau)},$$

where

$$\theta_{q_1} := \theta_{a,s}(q_1,q) = \frac{n}{s}\Big(\frac{1}{q} - \frac{1}{q_1} + \frac{a}{n}\Big) \text{ and } \theta_{q_2} := \theta_{s-\sigma+a,s}(q_2,q) = \frac{n}{s}\Big(\frac{1}{q} - \frac{1}{q_2} + \frac{s-\sigma+a}{n}\Big).$$

Hence, we may conclude

$$\begin{aligned}\big\| \big||D|^a u(\tau,\cdot)\big|^p \big\|_{\dot{H}^{s-\sigma}_q} &\lesssim (1+\tau)^{p-\frac{np}{\sigma}(1-\frac{1}{r})-\frac{s}{\sigma}\big((p-1)\theta_{q_1}+\theta_{q_2}\big)} \|u\|^p_{X_0(\tau)} \\ &\lesssim (1+\tau)^{p-\frac{np}{\sigma}(\frac{1}{m}-\frac{1}{qp}+\frac{a}{n})-\frac{s-\sigma}{\sigma}} \|u\|^p_{X_0(\tau)},\end{aligned}$$

where we can see that $(p-1)\theta_{q_1} + \theta_{q_2} = \frac{n}{s}\big(\frac{p-1}{q} + \frac{s-\sigma+ap}{n}\big)$. Here we have to guarantee that $\theta_{q_1} \in [\frac{a}{s}, 1]$ and $\theta_{q_2} \in [\frac{s-\sigma+a}{s}, 1]$. Both conditions imply the restrictions

$$1 < p \leqslant 1 + \frac{q(\sigma-a)}{n-q(s-a)} \text{ if } n > q(s-a), \quad \text{ or } \quad p > 1 \text{ if } n \leqslant q(s-a).$$

Therefore, we have shown the estimates

$$\big\|\partial_t |D|^{s-\sigma} Nu(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s}{\sigma}} \big(\|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}} + \|u\|^p_{X_0(t)}\big). \tag{5.13}$$

Step 4: Let us estimate the norm $\big\||D|^s Nu(t,\cdot)\big\|_{L^q}$. We use

$$\begin{aligned}|D|^s Nu(t,x) &= |D|^s\big(K_0(t,x) \ast_x u_0(x) + K_1(t,x) \ast_x u_1(x)\big) \\ &\quad + \int_0^t |D|^s\big(K_1(t-\tau,x) \ast_x \big||D|^a u(\tau,x)\big|^p\big) d\tau.\end{aligned}$$

By applying again the $(L^m \cap L^q) - L^q$ estimates if $\tau \in [0,t/2]$ and the $L^q - L^q$ estimates if $\tau \in [t/2,t]$ from Theorem 2.3.1, we derive

$$\begin{aligned}\big\||D|^s Nu(t,\cdot)\big\|_{L^q} &\lesssim (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s}{\sigma}} \|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}} \\ &\quad + \int_0^{t/2} (1+t-\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s}{\sigma}} \big\| \big||D|^a u(\tau,\cdot)\big|^p \big\|_{L^m \cap H^{s-\sigma}_q} d\tau \\ &\quad + \int_{t/2}^t (1+t-\tau)^{1-\frac{s}{\sigma}} \big\| \big||D|^a u(\tau,\cdot)\big|^p \big\|_{H^{s-\sigma}_q} d\tau.\end{aligned}$$

Following the approach to show (5.13) we may conclude

$$\big\||D|^s Nu(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s}{\sigma}}\big(\|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}} + \|u\|^p_{X_0(t)}\big).$$

Summarizing, from the definition of the norm in $X(t)$ we obtain immediately the inequality (5.10).

Next let us prove the inequality (5.11). Our proof is also divided into four steps.

Step 1: We need to estimate the norm $\|Nu(t,\cdot)-Nv(t,\cdot)\|_{L^q}$. Using the $(L^m\cap L^q)-L^q$ estimates if $\tau\in[0,t/2]$ and the L^q-L^q estimates if $\tau\in[t/2,t]$ from Theorem 2.3.1 we derive for two functions u and v from $X(t)$ the estimate

$$\begin{aligned}\|Nu(t,\cdot)-Nv(t,\cdot)\|_{L^q} &\lesssim \int_0^{t/2}(1+t-\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})}\big\|\,\big||D|^a u(\tau,\cdot)\big|^p-\big||D|^a v(\tau,\cdot)\big|^p\big\|_{L^m\cap L^q}d\tau\\ &\quad+\int_{t/2}^t(1+t-\tau)\big\|\,\big||D|^a u(\tau,\cdot)\big|^p-\big||D|^a v(\tau,\cdot)\big|^p\big\|_{L^q}d\tau.\end{aligned}$$

By using Hölder's inequality and applying again the same ideas as we did in the proof of (5.4) and Step 1 to prove (5.10) we may conclude

$$\|Nu(t,\cdot)-Nv(t,\cdot)\|_{L^q}\lesssim(1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})}\|u-v\|_{X_0(t)}\big(\|u\|^{p-1}_{X_0(t)}+\|v\|^{p-1}_{X_0(t)}\big).$$

Step 2: We need to estimate the norm $\|\partial_t Nu(t,\cdot)-\partial_t Nv(t,\cdot)\|_{L^q}$. We use

$$\begin{aligned}\|\partial_t Nu(t,\cdot)-\partial_t Nv(t,\cdot)\|_{L^q} &\lesssim \int_0^{t/2}(1+t-\tau)^{-\frac{n}{\sigma}(1-\frac{1}{r})}\big\|\,\big||D|^a u(\tau,\cdot)\big|^p-\big||D|^a v(\tau,\cdot)\big|^p\big\|_{L^m\cap L^q}d\tau\\ &\quad+\int_{t/2}^t\big\|\,\big||D|^a u(\tau,\cdot)\big|^p-\big||D|^a v(\tau,\cdot)\big|^p\big\|_{L^q}d\tau.\end{aligned}$$

Using the same approach as we did in the proof of (5.4) and Step 2 to prove (5.10) we conclude

$$\|\partial_t Nu(t,\cdot)-\partial_t Nv(t,\cdot)\|_{L^q}\lesssim(1+t)^{-\frac{n}{\sigma}(1-\frac{1}{r})}\|u-v\|_{X_0(t)}\big(\|u\|^{p-1}_{X_0(t)}+\|v\|^{p-1}_{X_0(t)}\big).$$

Step 3: Let us estimate the norm $\big\|\partial_t|D|^{s-\sigma}Nu(t,\cdot)-\partial_t|D|^{s-\sigma}Nv(t,\cdot)\big\|_{L^q}$. We use

$$\begin{aligned}&\big\|\partial_t|D|^{s-\sigma}Nu(t,\cdot)-\partial_t|D|^{s-\sigma}Nv(t,\cdot)\big\|_{L^q}\\ &\quad\lesssim\int_0^{t/2}(1+t-\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s}{\sigma}}\big\|\,\big||D|^a u(\tau,\cdot)\big|^p-\big||D|^a v(\tau,\cdot)\big|^p\big\|_{L^m\cap H^{s-\sigma}_q}d\tau\\ &\qquad+\int_{t/2}^t(1+t-\tau)^{1-\frac{s}{\sigma}}\big\|\,\big||D|^a u(\tau,\cdot)\big|^p-\big||D|^a v(\tau,\cdot)\big|^p\big\|_{H^{s-\sigma}_q}d\tau\\ &\quad=\int_0^{t/2}(1+t-\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s}{\sigma}}\big\|\,\big||D|^a u(\tau,\cdot)\big|^p-\big||D|^a v(\tau,\cdot)\big|^p\big\|_{L^m\cap L^q\cap\dot{H}^{s-\sigma}_q}d\tau\\ &\qquad+\int_{t/2}^t(1+t-\tau)^{1-\frac{s}{\sigma}}\big\|\,\big||D|^a u(\tau,\cdot)\big|^p-\big||D|^a v(\tau,\cdot)\big|^p\big\|_{L^q\cap\dot{H}^{s-\sigma}_q}d\tau.\end{aligned}$$

The integrals with $\big\|\,\big||D|^a u(\tau,\cdot)\big|^p-\big||D|^a v(\tau,\cdot)\big|^p\big\|_{L^m\cap L^q}$ and $\big\|\,\big||D|^a u(\tau,\cdot)\big|^p-\big||D|^a v(\tau,\cdot)\big|^p\big\|_{L^q}$ will be handled as we did in Step 1 to prove (5.11). Hence, we get

$$\begin{aligned}&\int_0^{t/2}(1+t-\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s}{\sigma}}\big\|\,\big||D|^a u(\tau,\cdot)\big|^p-\big||D|^a v(\tau,\cdot)\big|^p\big\|_{L^m\cap L^q}d\tau\\ &\qquad\lesssim(1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s}{\sigma}}\|u-v\|_{X_0(t)}\big(\|u\|^{p-1}_{X_0(t)}+\|v\|^{p-1}_{X_0(t)}\big),\end{aligned}$$

and

$$\begin{aligned}&\int_{t/2}^t(1+t-\tau)^{1-\frac{s}{\sigma}}\big\|\,\big||D|^a u(\tau,\cdot)\big|^p-\big||D|^a v(\tau,\cdot)\big|^p\big\|_{L^q}d\tau\\ &\qquad\lesssim(1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s}{\sigma}}\|u-v\|_{X_0(t)}\big(\|u\|^{p-1}_{X_0(t)}+\|v\|^{p-1}_{X_0(t)}\big).\end{aligned}$$

Let us now turn to estimate the norms $\big\| \big||D|^a u(\tau,\cdot)\big|^p - \big||D|^a v(\tau,\cdot)\big|^p \big\|_{\dot{H}^{s-\sigma}_q}$. By using the integral representation

$$\big||D|^a u(\tau,x)\big|^p - \big||D|^a v(\tau,x)\big|^p = p\int_0^1 \big(|D|^a u(\tau,x) - |D|^a v(\tau,x)\big) G\big(\omega |D|^a u(\tau,x) + (1-\omega)|D|^a v(\tau,x)\big)\, d\omega,$$

where $G(u) = u|u|^{p-2}$, we obtain

$$\begin{aligned}
&\big\| \big||D|^a u(\tau,\cdot)\big|^p - \big||D|^a v(\tau,\cdot)\big|^p \big\|_{\dot{H}^{s-\sigma}_q} \\
&\quad \lesssim \int_0^1 \Big\| |D|^{s-\sigma}\Big(|D|^a \big(u(\tau,\cdot) - v(\tau,\cdot)\big) G\big(\omega|D|^a u(\tau,\cdot) + (1-\omega)|D|^a v(\tau,\cdot)\big)\Big)\Big\|_{L^q}\, d\omega.
\end{aligned}$$

Thanks to the fractional Leibniz formula from Proposition C.2.1, we may proceed as follows:

$$\begin{aligned}
&\big\| \big||D|^a u(\tau,\cdot)\big|^p - \big||D|^a v(\tau,\cdot)\big|^p \big\|_{\dot{H}^{s-\sigma}_q} \\
&\quad \lesssim \int_0^1 \big\| |D|^{s-\sigma+a}\big(u(\tau,\cdot) - v(\tau,\cdot)\big)\big\|_{L^{r_1}} \big\| G\big(\omega|D|^a u(\tau,\cdot) + (1-\omega)|D|^a v(\tau,\cdot)\big)\big\|_{L^{r_2}} d\omega \\
&\qquad + \int_0^1 \big\| |D|^a\big(u(\tau,\cdot) - v(\tau,\cdot)\big)\big\|_{L^{r_3}} \big\| |D|^{s-\sigma} G\big(\omega|D|^a u(\tau,\cdot) + (1-\omega)|D|^a v(\tau,\cdot)\big)\big\|_{L^{r_4}} d\omega \\
&\quad \lesssim \big\| |D|^{s-\sigma+a}\big(u(\tau,\cdot) - v(\tau,\cdot)\big)\big\|_{L^{r_1}} \int_0^1 \big\| G\big(\omega|D|^a u(\tau,\cdot) + (1-\omega)|D|^a v(\tau,\cdot)\big)\big\|_{L^{r_2}} d\omega \\
&\qquad + \big\| |D|^a\big(u(\tau,\cdot) - v(\tau,\cdot)\big)\big\|_{L^{r_3}} \int_0^1 \big\| |D|^{s-\sigma} G\big(\omega|D|^a u(\tau,\cdot) + (1-\omega)|D|^a v(\tau,\cdot)\big)\big\|_{L^{r_4}} d\omega \\
&\quad \lesssim \big\| |D|^{s-\sigma+a}\big(u(\tau,\cdot) - v(\tau,\cdot)\big)\big\|_{L^{r_1}} \Big(\big\| |D|^a u(\tau,\cdot)\big\|^{p-1}_{L^{r_2(p-1)}} + \big\| |D|^a v(\tau,\cdot)\big\|^{p-1}_{L^{r_2(p-1)}} \Big) \\
&\qquad + \big\| |D|^a\big(u(\tau,\cdot) - v(\tau,\cdot)\big)\big\|_{L^{r_3}} \int_0^1 \big\| |D|^{s-\sigma} G\big(\omega|D|^a u(\tau,\cdot) + (1-\omega)|D|^a v(\tau,\cdot)\big)\big\|_{L^{r_4}} d\omega,
\end{aligned}$$

where

$$\frac{1}{q} = \frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r_3} + \frac{1}{r_4}.$$

Employing the fractional Gargliardo-Nirenberg inequality from Proposition C.1.1 implies

$$\begin{aligned}
\big\| |D|^{s-\sigma+a}\big(u(\tau,\cdot) - v(\tau,\cdot)\big)\big\|_{L^{r_1}} &\lesssim \|u(\tau,\cdot) - v(\tau,\cdot)\|^{\theta_1}_{\dot{H}^s_q} \|u(\tau,\cdot) - v(\tau,\cdot)\|^{1-\theta_1}_{L^q} \\
\big\| |D|^a u(\tau,\cdot)\big\|_{L^{r_2(p-1)}} &\lesssim \|u(\tau,\cdot)\|^{\theta_2}_{\dot{H}^s_q} \|u(\tau,\cdot)\|^{1-\theta_2}_{L^q} \\
\big\| |D|^a v(\tau,\cdot)\big\|_{L^{r_2(p-1)}} &\lesssim \|v(\tau,\cdot)\|^{\theta_2}_{\dot{H}^s_q} \|v(\tau,\cdot)\|^{1-\theta_2}_{L^q} \\
\big\| |D|^a\big(u(\tau,\cdot) - v(\tau,\cdot)\big)\big\|_{L^{r_3}} &\lesssim \|u(\tau,\cdot) - v(\tau,\cdot)\|^{\theta_3}_{\dot{H}^s_q} \|u(\tau,\cdot) - v(\tau,\cdot)\|^{1-\theta_3}_{L^q},
\end{aligned}$$

where

$$\begin{aligned}
\theta_1 &:= \theta_{s-\sigma+a,s}(r_1,q) = \frac{n}{s}\Big(\frac{1}{q} - \frac{1}{r_1} + \frac{s-\sigma+a}{n}\Big) \in \Big[\frac{s-\sigma+a}{s}, 1\Big], \\
\theta_2 &:= \theta_{a,s}(r_2(p-1),q) = \frac{n}{s}\Big(\frac{1}{q} - \frac{1}{r_2(p-1)} + \frac{a}{n}\Big) \in \Big[\frac{a}{n}, 1\Big], \\
\text{and }\ \theta_3 &:= \theta_{a,s}(r_3,q) = \frac{n}{s}\Big(\frac{1}{q} - \frac{1}{r_3} + \frac{a}{n}\Big) \in \Big[\frac{a}{n}, 1\Big].
\end{aligned}$$

Moreover, since $\omega \in [0,1]$ is a parameter, we may apply again the fractional chain rule with $p > 1 + \lceil s-\sigma \rceil$ from Proposition C.3.2 and the fractional Gagliardo-Nirenberg inequality from Proposition C.1.1 to conclude

$$\begin{aligned}
&\big\| |D|^{s-\sigma} G\big(\omega|D|^a u(\tau,\cdot) + (1-\omega)|D|^a v(\tau,\cdot)\big)\big\|_{L^{r_4}} \\
&\quad \lesssim \big\| |D|^a\big(\omega u(\tau,\cdot) + (1-\omega)v(\tau,\cdot)\big)\big\|^{p-2}_{L^{r_5}} \big\| |D|^{s-\sigma+a}\big(\omega u(\tau,\cdot) + (1-\omega)v(\tau,\cdot)\big)\big\|_{L^{r_6}} \\
&\quad \lesssim \big\| \omega u(\tau,\cdot) + (1-\omega)v(\tau,\cdot)\big\|^{(p-2)\theta_5+\theta_6}_{\dot{H}^s_q} \big\| \omega u(\tau,\cdot) + (1-\omega)v(\tau,\cdot)\big\|^{(p-2)(1-\theta_5)+1-\theta_6}_{L^q},
\end{aligned}$$

where

$$\frac{1}{r_4}=\frac{p-2}{r_5}+\frac{1}{r_6},\ \ \theta_5:=\theta_{a,s}(r_5,q)=\frac{n}{s}\Big(\frac{1}{q}-\frac{1}{r_5}+\frac{a}{n}\Big)\in\Big[\frac{a}{s},1\Big],$$
$$\text{and }\ \theta_6:=\theta_{s-\sigma+a,s}(r_6,q)=\frac{n}{s}\Big(\frac{1}{q}-\frac{1}{r_6}+\frac{s-\sigma+a}{n}\Big)\in\Big[\frac{s-\sigma+a}{s},1\Big].$$

Hence, we derive

$$\int_0^1 \big\||D|^{s-\sigma}G\big(\omega|D|^a u(\tau,\cdot)+(1-\omega)|D|^a v(\tau,\cdot)\big)\big\|_{L^{r_4}}d\omega$$
$$\lesssim \big(\|u(\tau,\cdot)\|_{\dot{H}^s_q}+\|v(\tau,\cdot)\|_{\dot{H}^s_q}\big)^{(p-2)\theta_5+\theta_6}\big(\|u(\tau,\cdot)\|_{L^q}+\|v(\tau,\cdot)\|_{L^q}\big)^{(p-2)(1-\theta_5)+1-\theta_6}.$$

Combining all previous estimates we get

$$\big\|\,\big||D|^a u(\tau,\cdot)\big|^p-\big||D|^a v(\tau,\cdot)\big|^p\big\|_{\dot{H}^{s-\sigma}_q}$$
$$\lesssim (1+\tau)^{p-\frac{np}{\sigma}(1-\frac{1}{r})-\frac{n}{\sigma}(\frac{p-1}{q}+\frac{s-\sigma+ap}{n})}\|u-v\|_{X_0(\tau)}\big(\|u\|^{p-1}_{X_0(\tau)}+\|v\|^{p-1}_{X_0(\tau)}\big)$$
$$\lesssim (1+\tau)^{p-\frac{np}{\sigma}(\frac{1}{m}-\frac{1}{qp}+\frac{a}{n})-\frac{s-\sigma}{\sigma}}\|u-v\|_{X_0(\tau)}\big(\|u\|^{p-1}_{X_0(\tau)}+\|v\|^{p-1}_{X_0(\tau)}\big),$$

where we note that

$$\theta_1+(p-1)\theta_2=\theta_3+(p-2)\theta_5+\theta_6=\frac{n}{s}\Big(\frac{p-1}{q}+\frac{s-\sigma+ap}{n}\Big).$$

Therefore, we may conclude

$$\int_0^{t/2}(1+t-\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s}{\sigma}}\big\|\,\big||D|^a u(\tau,\cdot)\big|^p-\big||D|^a v(\tau,\cdot)\big|^p\big\|_{\dot{H}^{s-\sigma}_q}d\tau$$
$$\lesssim (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s}{\sigma}}\|u-v\|_{X_0(t)}\big(\|u\|^{p-1}_{X_0(t)}+\|v\|^{p-1}_{X_0(t)}\big),$$

and

$$\int_{t/2}^{t}(1+t-\tau)^{1-\frac{s}{\sigma}}\big\|\,\big||D|^a u(\tau,\cdot)\big|^p-\big||D|^a v(\tau,\cdot)\big|^p\big\|_{\dot{H}^{s-\sigma}_q}d\tau$$
$$\lesssim (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s}{\sigma}}\|u-v\|_{X_0(t)}\big(\|u\|^{p-1}_{X_0(t)}+\|v\|^{p-1}_{X_0(t)}\big).$$

Summarizing, we have proved the estimates

$$\big\|\partial_t|D|^{s-\sigma}Nu(t,\cdot)-\partial_t|D|^{s-\sigma}Nv(t,\cdot)\big\|_{L^q}\lesssim (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s}{\sigma}}\|u-v\|_{X_0(t)}\big(\|u\|^{p-1}_{X_0(t)}+\|v\|^{p-1}_{X_0(t)}\big).$$

Step 4: Let us estimate the norm $\big\||D|^s Nu(t,\cdot)-|D|^s Nv(t,\cdot)\big\|_{L^q}$. We use

$$\big\||D|^s Nu(t,\cdot)-|D|^s Nv(t,\cdot)\big\|_{L^q}$$
$$\lesssim \int_0^{t/2}(1+t-\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s}{\sigma}}\big\|\,\big||D|^a u(\tau,\cdot)\big|^p-\big|D|^a v(\tau,\cdot)\big|^p\big\|_{L^m\cap H^{s-\sigma}_q}d\tau$$
$$+\int_{t/2}^{t}(1+t-\tau)^{1-\frac{s}{\sigma}}\big\|\,\big||D|^a u(\tau,\cdot)\big|^p-\big||D|^a v(\tau,\cdot)\big|^p\big\|_{H^{s-\sigma}_q}d\tau.$$

By the same treatment as in Step 3 to prove (5.11) we may conclude

$$\big\||D|^s Nu(t,\cdot)-|D|^s Nv(t,\cdot)\big\|_{L^q}\lesssim (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s}{\sigma}}\|u-v\|_{X_0(t)}\big(\|u\|^{p-1}_{X_0(t)}+\|v\|^{p-1}_{X_0(t)}\big).$$

Summarizing, from the definition of the norm in $X(t)$ and all the previous estimates we have completed the proof of (5.11). □

Remark 5.1.4. In this remark, we want to clarify the possibility to choose actually the parameters q_1, q_2, $r_1, \cdots, r_6$ and $\theta_1, \cdots, \theta_6$ as required in the proof to Theorem 5.1.3.

Firstly, let us see that we may choose q_1, q_2 such that

$$\frac{1}{q} = \frac{p-1}{q_1} + \frac{1}{q_2}, \quad \theta_{q_1} = \frac{n}{s}\Big(\frac{1}{q} - \frac{1}{q_1} + \frac{a}{s}\Big) \in \Big[\frac{a}{n}, 1\Big]$$
$$\text{and } \theta_{q_2} = \frac{n}{s}\Big(\frac{1}{q} - \frac{1}{q_2} + \frac{s-\sigma+a}{n}\Big) \in \Big[\frac{s-\sigma+a}{s}, 1\Big]$$

thanks to the following condition:

$$2 \leqslant p \leqslant 1 + \frac{q(\sigma-a)}{n-q(s-a)} \text{ if } n > q(s-a), \quad \text{or } p \geqslant 2 \text{ if } n \leqslant q(s-a). \tag{5.14}$$

Namely, we may describe the requirements on θ_{q_1} and θ_{q_2} in terms of conditions on q_1 and q_2 as follows:

$$\frac{1}{q_1} \in \Big[\frac{1}{q} - \frac{s-a}{n}, \frac{1}{q}\Big] \quad \text{and} \quad \frac{1}{q_2} \in \Big[\frac{1}{q} - \frac{\sigma-a}{n}, \frac{1}{q}\Big].$$

Combining the second condition on q_2 and using the expression $\frac{1}{q_2} = \frac{1}{q} - \frac{p-1}{q_1}$ we may obtain the following condition on q_1:

$$\frac{1}{q_1} \leqslant \frac{\sigma-a}{n(p-1)} \quad \text{since } p \geqslant 2.$$

Hence, in order to guarantee the existence of two parameters q_1 and q_2 it is sufficient to intersect two condition intervals for q_1 to become a non-empty intersection. This is possible by the following condition:

$$\frac{1}{q} - \frac{s-a}{n} \leqslant \frac{\sigma-a}{n(p-1)},$$

which implies immediately (5.14). For the choice of parameters r_1, r_2 such that

$$\theta_1 = \frac{n}{s}\Big(\frac{1}{q} - \frac{1}{r_1} + \frac{s-\sigma+a}{n}\Big) \in \Big[\frac{s-\sigma+a}{s}, 1\Big],$$
$$\theta_2 = \frac{n}{s}\Big(\frac{1}{q} - \frac{1}{r_2(p-1)} + \frac{a}{s}\Big) \in \Big[\frac{a}{n}, 1\Big] \quad \text{and} \quad \frac{1}{q} = \frac{1}{r_1} + \frac{1}{r_2},$$

we repeat exactly the same arguments to find (5.14) with r_1 in place of q_2 and r_2 in place of $\frac{q_1}{p-1}$.

Let us turn now to explain the existence of suitable parameters $r_3, \cdots, r_6$ and $\theta_3, \cdots, \theta_6$. In the first step, our goal is to clarify parameters r_3, r_4 such that

$$\frac{1}{q} = \frac{1}{r_3} + \frac{1}{r_4} \quad \text{and} \quad \theta_3 = \frac{n}{s}\Big(\frac{1}{q} - \frac{1}{r_3} + \frac{a}{n}\Big) \in \Big[\frac{a}{s}, 1\Big].$$

By re-writing $\frac{1}{r_4} = \frac{1}{q} - \frac{1}{r_3}$ we express the condition on θ_3 equivalent to the condition on r_4 as follows:

$$\frac{1}{r_4} \in \Big[0, \frac{s-a}{n}\Big].$$

Therefore, choosing r_4 in the above admissible range we take r_3 to guarantee $\theta_3 \in [0, 1]$.

In the second step, taking account of conditions on

$$\theta_5 = \frac{n}{s}\Big(\frac{1}{q} - \frac{1}{r_5} + \frac{a}{s}\Big) \in \Big[\frac{a}{n}, 1\Big] \quad \text{and} \quad \theta_6 = \frac{n}{s}\Big(\frac{1}{q} - \frac{1}{r_6} + \frac{s-\sigma+a}{n}\Big) \in \Big[\frac{s-\sigma+a}{s}, 1\Big]$$

we re-write as conditions on parameters r_5 and r_6 as follows:

$$\frac{1}{r_5} \in \Big[\frac{1}{q} - \frac{s-a}{n}, \frac{1}{q}\Big] \quad \text{and} \quad \frac{1}{r_6} \in \Big[\frac{1}{q} - \frac{\sigma-a}{n}, \frac{1}{q}\Big].$$

Moreover, using the sum $\frac{1}{r_4} = \frac{p-2}{r_5} + \frac{1}{r_6}$ and the above obtained condition $\frac{1}{r_4} \in [0, \frac{s-a}{n}]$ we express the condition on r_6 in an equivalent way as a condition on r_5, in particular,

$$\frac{1}{r_6} \leqslant \frac{s-a}{n} - (p-2)\frac{1}{r_5} \quad \text{since } p \geqslant 2.$$

Hence, in order to ensure that we get a non-empty range for the parameter r_6 we need to have the following second condition for r_5 as follows:

$$(p-2)\frac{1}{r_5} \leqslant \frac{s+\sigma-2a}{n} - \frac{1}{q}.$$

Finally, we check the conditions on p and n coming from the requirement that the admissible range of r_5 becomes non-empty, that is,

$$(p-2)\Big(\frac{1}{q} - \frac{s-a}{n}\Big) \leqslant \frac{s+\sigma-2a}{n} - \frac{1}{q},$$

which leads again to (5.14).

Summarizing, we have shown that (5.14) is sufficient to guarantee the possibility to choose suitable parameters q_1, q_2, $r_1, \cdots, r_6$ and $\theta_1, \cdots, \theta_6$ in the proof to Theorem 5.1.3.

Example 5.1.4. By choosing $m = 1$, $q = 1.1$, $\sigma = 1$, $s = 2$ and $a = 0$ we obtain the following admissible range of the exponents p in Theorem 5.1.4:

$$p \in (3, \infty) \text{ if } n = 2, \qquad \text{or} \qquad p \in (2, 2.375) \text{ if } n = 3.$$

5.1.4. Large regular data

Next, we obtain the following result for large regular data by using the fractional powers rule and the fractional Sobolev embedding.

Theorem 5.1.4. *Let $s > \sigma + \frac{n}{q}$ and $n \geqslant 1$. Let $q \in (1, \infty)$ be a fixed constant, $m \in [1, q)$ and $a \in [0, \sigma)$. We assume the condition*

$$p > 1 + \max\Big\{\frac{m(2\sigma-a)}{n-m(\sigma-a)}, s-\sigma, 1\Big\}.$$

Moreover, we suppose the following conditions:

$$p \in \Big[\frac{q}{m}, \infty\Big) \text{ and } n > m(\sigma-a).$$

Then, there exists a constant $\varepsilon > 0$ such that for any small data

$$(u_0, u_1) \in \mathcal{A}^s_{m,q} \text{ satisfying the assumption } \|(u_0, u_1)\|_{\mathcal{A}^s_{m,q}} \leqslant \varepsilon,$$

we have a uniquely determined global (in time) small data energy solution

$$u \in C\big([0,\infty), H^s_q\big) \cap C^1\big([0,\infty), H^{s-\sigma}_q\big)$$

to (5.1). Moreover, the estimates (5.7) to (5.9) hold.

Remark 5.1.5. Let us explain the conditions for p and n in Theorem 5.1.4. Because we want to use the fractional powers rule, the conditions $p > 1 + s - \sigma$ and $p > 2$ are necessary to assume. Moreover, the conditions $p > 1 + \frac{m(2\sigma-a)}{n-m(\sigma-a)}$ and $n > m(\sigma - a)$ imply the same decay estimates for solutions to (5.1) as for solutions to the corresponding linear model with vanishing right-hand side. Hence, we can say that the non-linearity is interpreted as a small perturbation. Finally, the remaining conditions for p come into play after we apply the fractional Gargliardo-Nirenberg inequality.

Proof. We introduce the definitions of spaces $\mathcal{A}^s_{m,q}$, $X(t)$ and $X_0(t)$ in the same way as in the proof of Theorem 5.1.3. We repeat exactly on the one hand the same estimates for the terms $\big||D|^a u(\tau,\cdot)\big|^p$ and $\big||D|^a u(\tau,\cdot)\big|^p - \big||D|^a v(\tau,\cdot)\big|^p$ in L^m and L^q. On the other hand, we estimate the above terms in $\dot{H}^{s-\sigma}_q$ by using the fractional powers rule and the fractional Sobolev embedding.

In the first step, let us begin with $\big\| \big||D|^a u(\tau,\cdot)\big|^p \big\|_{\dot{H}^{s-\sigma}_q}$. We shall apply Corollary C.4.1 for the fractional powers rule with $s - \sigma \in (\frac{n}{q}, p)$. Therefore, we obtain

$$\begin{aligned}
\big\| \big||D|^a u(\tau,\cdot)\big|^p \big\|_{\dot{H}^{s-\sigma}_q} &\lesssim \big\||D|^a u(\tau,\cdot)\big\|_{\dot{H}^{s-\sigma}_q} \big\||D|^a u(\tau,\cdot)\big\|^{p-1}_{L^\infty} \\
&\lesssim \big\||D|^a u(\tau,\cdot)\big\|_{\dot{H}^{s-\sigma}_q} \Big(\big\||D|^a u(\tau,\cdot)\big\|_{\dot{H}^{s^*}_q} + \big\||D|^a u(\tau,\cdot)\big\|_{\dot{H}^{s-\sigma}_q}\Big)^{p-1}.
\end{aligned}$$

Here we used Corollary C.5.1 with a suitable $s^* < \frac{n}{q}$. Applying the fractional Gagliardo-Nirenberg inequality from Proposition C.1.1 we have

$$\big\||D|^{s-\sigma+a}u(\tau,\cdot)\big\|_{L^q} \lesssim \|u(\tau,\cdot)\|_{L^q}^{1-\theta_1}\,\big\||D|^s u(\tau,\cdot)\big\|_{L^q}^{\theta_1}$$
$$\lesssim (1+\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s}{\sigma}\theta_1}\|u\|_{X_0(\tau)} \lesssim (1+\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s-\sigma+a}{\sigma}}\|u\|_{X_0(\tau)},$$

and

$$\big\||D|^{s^*+a}u(\tau,\cdot)\big\|_{L^q} \lesssim \|u(\tau,\cdot)\|_{L^q}^{1-\theta_2}\,\big\||D|^s u(\tau,\cdot)\big\|_{L^q}^{\theta_2}$$
$$\lesssim (1+\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s}{\sigma}\theta_2}\|u\|_{X_0(\tau)} \lesssim (1+\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s^*+a}{\sigma}}\|u\|_{X_0(\tau)},$$

where $\theta_1 = \frac{s-\sigma+a}{s}$ and $\theta_2 = \frac{s^*+a}{s}$. Hence, we derive

$$\big\|\,\big||D|^a u(\tau,\cdot)\big|^p\big\|_{\dot{H}^{s-\sigma}_q} \lesssim (1+\tau)^{p\left(1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{a}{\sigma}\right)-\frac{s-\sigma}{\sigma}-(p-1)\frac{s^*}{\sigma}}\|u\|^p_{X_0(\tau)}$$
$$\lesssim (1+\tau)^{p-\frac{n}{\sigma}(\frac{p-1}{m}+\frac{ap}{n})}\|u\|^p_{X_0(\tau)},$$

if we choose $s^* = \frac{n}{q} - \varepsilon_0$ where $\varepsilon_0 > 0$ is a sufficiently small. Therefore, we may conclude

$$\int_0^{t/2}(1+t-\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s}{\sigma}}\big\|\,\big||D|^a u(\tau,\cdot)\big|^p\big\|_{\dot{H}^{s-\sigma}_q}d\tau \lesssim (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s}{\sigma}}\|u\|^p_{X_0(t)}.$$

Moreover, in an analogous way we can also derive

$$\int_{t/2}^{t}(1+t-\tau)^{-\frac{s-\sigma}{\sigma}}\big\|\,\big||D|^a u(\tau,\cdot)\big|^p\big\|_{\dot{H}^{s-\sigma}_q}d\tau \lesssim (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s}{\sigma}}\|u\|^p_{X_0(t)}.$$

Finally, let us turn to estimate the norms $\big\|\,\big||D|^a u(\tau,\cdot)\big|^p - \big||D|^a v(\tau,\cdot)\big|^p\big\|_{\dot{H}^{s-\sigma}_q}$. By using the integral representation

$$\big||D|^a u(\tau,x)\big|^p - \big||D|^a v(\tau,x)\big|^p = p\int_0^1 |D|^a\big(u(\tau,x)-v(\tau,x)\big)G\big(\omega|D|^a u(\tau,x)+(1-\omega)|D|^a v(\tau,x)\big)d\omega,$$

where $G(u) = u|u|^{p-2}$, we obtain

$$\big\|\,\big||D|^a u(\tau,\cdot)\big|^p - \big||D|^a v(\tau,\cdot)\big|^p\big\|_{\dot{H}^{s-\sigma}_q}$$
$$\lesssim \int_0^1 \big\||D|^a\big(u(\tau,\cdot)-v(\tau,\cdot)\big)G\big(\omega|D|^a u(\tau,\cdot)+(1-\omega)|D|^a v(\tau,\cdot)\big)\big\|_{\dot{H}^{s-\sigma}_q}d\omega.$$

Thanks to the fractional powers rule from Corollary C.4.2, we can proceed as follows:

$$\big\|\,\big||D|^a u(\tau,\cdot)\big|^p - \big||D|^a v(\tau,\cdot)\big|^p\big\|_{\dot{H}^{s-\sigma}_q}$$
$$\lesssim \int_0^1 \big\||D|^a\big(u(\tau,\cdot)-v(\tau,\cdot)\big)\big\|_{\dot{H}^{s-\sigma}_q}\big\|G\big(\omega|D|^a u(\tau,\cdot)+(1-\omega)|D|^a v(\tau,\cdot)\big)\big\|_{L^\infty}d\omega$$
$$+\int_0^1 \big\||D|^a\big(u(\tau,\cdot)-v(\tau,\cdot)\big)\big\|_{L^\infty}\big\|G\big(\omega|D|^a u(\tau,\cdot)+(1-\omega)|D|^a v(\tau,\cdot)\big)\big\|_{\dot{H}^{s-\sigma}_q}d\omega$$
$$\lesssim \int_0^1 \big\||D|^a\big(u(\tau,\cdot)-v(\tau,\cdot)\big)\big\|_{\dot{H}^{s-\sigma}_q}\big\|\omega|D|^a u(\tau,\cdot)+(1-\omega)|D|^a v(\tau,\cdot)\big\|^{p-1}_{L^\infty}d\omega$$
$$+\int_0^1 \big\||D|^a\big(u(\tau,\cdot)-v(\tau,\cdot)\big)\big\|_{L^\infty}\big\|G\big(\omega|D|^a u(\tau,\cdot)+(1-\omega)|D|^a v(\tau,\cdot)\big)\big\|_{\dot{H}^{s-\sigma}_q}d\omega.$$

Applying Corollary C.4.1 with $p > 2$ and $s-\sigma \in (\frac{n}{q}, p-1)$ we get

$$\big\|G\big(\omega|D|^a u(\tau,\cdot)+(1-\omega)|D|^a v(\tau,\cdot)\big)\big\|_{\dot{H}^{s-\sigma}_q}$$
$$\lesssim \big\|\omega|D|^a u(\tau,\cdot)+(1-\omega)|D|^a v(\tau,\cdot)\big\|_{\dot{H}^{s-\sigma}_q}\big\|\omega|D|^a u(\tau,\cdot)+(1-\omega)|D|^a v(\tau,\cdot)\big\|^{p-2}_{L^\infty}.$$

Using again Corollary C.5.1 with a suitable $s^* < \frac{n}{q}$ in order to estimate all the L^∞ norms we get

$$\big\||D|^a\big(u(\tau,\cdot)-v(\tau,\cdot)\big)\big\|_{L^\infty} \lesssim \big\||D|^a\big(u(\tau,\cdot)-v(\tau,\cdot)\big)\big\|_{\dot{H}^{s^*}_q} + \big\||D|^a\big(u(\tau,\cdot)-v(\tau,\cdot)\big)\big\|_{\dot{H}^{s-\sigma}_q},$$

and

$$\begin{aligned}&\big\|\omega|D|^a u(\tau,\cdot)+(1-\omega)|D|^a v(\tau,\cdot)\big\|_{L^\infty}\\ &\quad\lesssim \big\|\omega|D|^a u(\tau,\cdot)+(1-\omega)|D|^a v(\tau,\cdot)\big\|_{\dot{H}^{s^*}_q} + \big\|\omega|D|^a u(\tau,\cdot)+(1-\omega)|D|^a v(\tau,\cdot)\big\|_{\dot{H}^{s-\sigma}_q}.\end{aligned}$$

Applying again the fractional Gagliardo-Nirenberg inequality from Proposition C.1.1 we have

$$\begin{aligned}\big\||D|^a\big(u(\tau,\cdot)-v(\tau,\cdot)\big)\big\|_{\dot{H}^{s-\sigma}_q} &\lesssim (1+\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s-\sigma+a}{\sigma}}\|u-v\|_{X_0(\tau)},\\ \big\|\omega|D|^a u(\tau,\cdot)+(1-\omega)|D|^a v(\tau,\cdot)\big\|_{\dot{H}^{s-\sigma}_q} &\lesssim (1+\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s-\sigma+a}{\sigma}}\|\omega u+(1-\omega)v\|_{X_0(\tau)},\end{aligned}$$

and

$$\begin{aligned}\big\||D|^a\big(u(\tau,\cdot)-v(\tau,\cdot)\big)\big\|_{\dot{H}^{s^*}_q} &\lesssim (1+\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s^*+a}{\sigma}}\|u-v\|_{X_0(\tau)},\\ \big\|\omega|D|^a u(\tau,\cdot)+(1-\omega)|D|^a v(\tau,\cdot)\big\|_{\dot{H}^{s^*}_q} &\lesssim (1+\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s^*+a}{\sigma}}\|\omega u+(1-\omega)v\|_{X_0(\tau)}.\end{aligned}$$

Therefore, from the definition of the norm in $X_0(t)$ we may conclude

$$\begin{aligned}&\big\|\,\big||D|^a u(\tau,\cdot)\big|^p-\big||D|^a v(\tau,\cdot)\big|^p\big\|_{\dot{H}^{s-\sigma}_q}\\ &\quad\lesssim \int_0^1 (1+\tau)^{p\left(1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{a}{\sigma}\right)-\frac{s-\sigma}{\sigma}-(p-1)\frac{s^*}{\sigma}}\|u-v\|_{X_0(\tau)}\|\omega u+(1-\omega)v\|^{p-1}_{X_0(\tau)}d\omega\\ &\quad\lesssim (1+\tau)^{p\left(1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{a}{\sigma}\right)-\frac{s-\sigma}{\sigma}-(p-1)\frac{s^*}{\sigma}}\|u-v\|_{X_0(\tau)}\big(\|u\|^{p-1}_{X_0(\tau)}+\|v\|^{p-1}_{X_0(\tau)}\big).\end{aligned}$$

By an analogous argument as we applied in the first step we obtain

$$\begin{aligned}&\int_0^{t/2}(1+t-\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s}{\sigma}}\big\|\,\big||D|^a u(\tau,\cdot)\big|^p-\big||D|^a v(\tau,\cdot)\big|^p\big\|_{\dot{H}^{s-\sigma}_q}d\tau\\ &\qquad\lesssim (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s}{\sigma}}\|u-v\|_{X_0(t)}\big(\|u\|^{p-1}_{X_0(t)}+\|v\|^{p-1}_{X_0(t)}\big),\end{aligned}$$

and

$$\begin{aligned}&\int_{t/2}^{t}(1+t-\tau)^{-\frac{s-\sigma}{\sigma}}\big\|\,\big||D|^a u(\tau,\cdot)\big|^p-\big||D|^a v(\tau,\cdot)\big|^p\big\|_{\dot{H}^{s-\sigma}_q}d\tau\\ &\qquad\lesssim (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s}{\sigma}}\|u-v\|_{X_0(t)}\big(\|u\|^{p-1}_{X_0(t)}+\|v\|^{p-1}_{X_0(t)}\big).\end{aligned}$$

Summarizing, the proof of Theorem 5.1.4 is completed. □

Example 5.1.5. By choosing $m=1$, $q=5$, $\sigma=1$, $s=5$ and $a=0$ we obtain the following admissible range of exponents p:

$$p\in(5,\infty)\quad\text{for all } n\in[2,19].$$

5.2. Global (in time) existence of small data solutions to the model (5.2)

Now, we assume that the data belong to a function space with large regularity. We obtain the following result.

Theorem 5.2.1. *Let $q \in (1,\infty)$ be a fixed constant, $m \in [1,q)$ and $n \geqslant 1$. We assume the regularity $s > \sigma + \frac{n}{q}$, and the exponent p satisfies the following conditions:*

$$p \in \Big[\frac{q}{m},\infty\Big) \text{ and } p > 1 + \max\Big\{\frac{m\sigma}{n}, s-\sigma, 1\Big\}.$$

Then, there exists a constant $\varepsilon > 0$ such that for any small data

$$(u_0,u_1) \in \mathcal{A}^s_{m,q} \text{ satisfying the assumption } \|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}} \leqslant \varepsilon,$$

we have a uniquely determined global (in time) small data energy solution

$$u \in C\big([0,\infty), H^s_q\big) \cap C^1\big([0,\infty), H^{s-\sigma}_q\big)$$

to (5.2). Moreover, the estimates (5.7) to (5.9) hold.

Remark 5.2.1. Let us explain the conditions for p and n in Theorem 5.2.1. Because we want to use the fractional powers rule, the conditions $p > s$ and $p > 2$ are necessary to assume. We are not interested to have a restriction to the upper bound for p. For this reason, we suppose the condition $s > \sigma + \frac{n}{q}$. Moreover, the interval of admissible exponents $p \in [\frac{q}{m},\infty)$ comes from applying the fractional Gargliardo-Nirenberg inequality. The remaining conditions for p and s imply the same decay estimates for solutions to (5.2) as for solutions to the corresponding linear model with vanishing right-hand side. Hence, we can say that the non-linearity is interpreted as a small perturbation.

Proof. We introduce the definitions of spaces $\mathcal{A}^s_{m,q}$ and $X(t)$ in the same way as in the proof of Theorem 5.1.3. We define a mapping $N:\ X(t) \longrightarrow X(t)$ in the following way:

$$Nu(t,x) = K_0(t,x) \ast_x u_0(x) + K_1(t,x) \ast_x u_1(x) + \int_0^t K_1(t-\tau,x) \ast_x |u_t(\tau,x)|^p d\tau.$$

In order to conclude the uniqueness and the global (in time) existence of small data solutions to (5.2) as well, we have to prove the following pair of inequalities:

$$\|Nu\|_{X(t)} \lesssim \|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}} + \|u\|^p_{X(t)}, \tag{5.15}$$

$$\|Nu - Nv\|_{X(t)} \lesssim \|u-v\|_{X(t)}\big(\|u\|^{p-1}_{X(t)} + \|v\|^{p-1}_{X(t)}\big). \tag{5.16}$$

First let us prove the inequality (5.15). Our proof is divided into four steps.

Step 1: We need to estimate the norm $\|Nu(t,\cdot)\|_{L^q}$. We apply the $L^m \cap L^q - L^q$ estimates if $\tau \in [0,t/2]$ and $L^q - L^q$ estimates if $\tau \in [t/2,t]$ from Theorem 2.3.1 to conclude

$$\begin{aligned}\|Nu(t,\cdot)\|_{L^q} \lesssim (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})}\|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}} &+ \int_0^{t/2}(1+t-\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})}\big\||u_t(\tau,\cdot)|^p\big\|_{L^m\cap L^q}d\tau \\ &+ \int_{t/2}^t (1+t-\tau)\big\||u_t(\tau,\cdot)|^p\big\|_{L^q}d\tau.\end{aligned}$$

We have

$$\big\||u_t(\tau,\cdot)|^p\big\|_{L^m\cap L^q} = \big\||u_t(\tau,\cdot)|^p\big\|_{L^m} + \big\||u_t(\tau,\cdot)|^p\big\|_{L^q} \lesssim \|u_t(\tau,\cdot)\|^p_{L^{mp}} + \|u_t(\tau,\cdot)\|^p_{L^{qp}}.$$

To estimate the norm $\|u_t(\tau,\cdot)\|^p_{L^{kp}}$ with $k = q,m$, we apply the fractional Gagliardo-Nirenberg inequality from Proposition C.1.1 to get

$$\|u_t(\tau,\cdot)\|_{L^{qp}} \lesssim \big\||D|^{s-\sigma}u_t(\tau,\cdot)\big\|^{\theta_{qp}}_{L^q}\,\|u_t(\tau,\cdot)\|^{1-\theta_{qp}}_{L^q} \lesssim (1+\tau)^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s-\sigma}{\sigma}\theta_{qp}}\|u\|_{X(\tau)},$$

and

$$\|u_t(\tau,\cdot)\|_{L^{mp}} \lesssim \big\||D|^{s-\sigma}u_t(\tau,\cdot)\big\|^{\theta_{mp}}_{L^q}\,\|u_t(\tau,\cdot)\|^{1-\theta_{mp}}_{L^q} \lesssim (1+\tau)^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s-\sigma}{\sigma}\theta_{mp}}\|u\|_{X(\tau)},$$

where

$$\theta_{qp} := \theta_{0,s-\sigma}(qp,q) = \frac{n}{s-\sigma}\Big(\frac{1}{q}-\frac{1}{qp}\Big) \text{ and } \theta_{mp} := \theta_{0,s-\sigma}(mp,q) = \frac{n}{s-\sigma}\Big(\frac{1}{q}-\frac{1}{mp}\Big).$$

As in Corollary C.1.1 we have to guarantee that θ_{qp} and θ_{mp} belong to $[0,1]$. Since $\theta_{mp} < \theta_{qp}$, we obtain

$$p \in \Big[\frac{q}{m}, \frac{n}{n-q(s-\sigma)}\Big] \text{ if } s < \sigma + \frac{n}{q}, \quad \text{ or } \quad p \in \Big[\frac{q}{m}, \infty\Big) \text{ if } s \geqslant \sigma + \frac{n}{q}.$$

By virtue of the relation $1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{m}$, we derive

$$\big\||u_t(\tau,\cdot)|^p\big\|_{L^m \cap L^q} \lesssim (1+\tau)^{-\frac{n}{m\sigma}(p-1)} \|u\|^p_{X(\tau)},$$
$$\big\||u_t(\tau,\cdot)|^p\big\|_{L^q} \lesssim (1+\tau)^{-\frac{np}{\sigma}\left(\frac{1}{m}-\frac{1}{qp}\right)} \|u\|^p_{X(\tau)}.$$

Hence, we may conclude

$$\int_0^{t/2} (1+t-\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})} \big\||u_t(\tau,\cdot)|^p\big\|_{L^m\cap L^q} d\tau \lesssim (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})} \|u\|^p_{X(t)} \int_0^{t/2} (1+\tau)^{-\frac{n}{m\sigma}(p-1)} d\tau,$$

and

$$\int_{t/2}^t (1+t-\tau) \big\||u_t(\tau,\cdot)|^p\big\|_{L^q} d\tau \lesssim (1+t)^{-\frac{np}{\sigma}\left(\frac{1}{m}-\frac{1}{qp}\right)} \|u\|^p_{X(t)} \int_{t/2}^t (1+t-\tau) d\tau$$
$$\lesssim (1+t)^{2-\frac{np}{\sigma}\left(\frac{1}{m}-\frac{1}{qp}\right)} \|u\|^p_{X(t)} \lesssim (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})+1-\frac{n}{m\sigma}(p-1)} \|u\|^p_{X(t)},$$

where we use the estimates $(1+t-\tau) \approx (1+t)$ if $\tau \in [0,t/2]$ and $(1+\tau) \approx (1+t)$ if $\tau \in [t/2,t]$. Because of $p > 1 + \frac{m\sigma}{n}$, it follows immediately

$$-\frac{n}{m\sigma}(p-1) < -1.$$

Consequently, the term $(1+\tau)^{-\frac{n}{m\sigma}(p-1)}$ is integrable over $(0,\infty)$. Hence, we have

$$\int_0^{t/2} (1+t-\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})} \big\||u_t(\tau,\cdot)|^p\big\|_{L^m\cap L^q} d\tau \lesssim (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})} \|u\|^p_{X(t)},$$

and

$$\int_{t/2}^t (1+t-\tau) \big\||u_t(\tau,\cdot)|^p\big\|_{L^q} d\tau \lesssim (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})} \|u\|^p_{X(t)}.$$

Therefore, we arrive at the following estimate:

$$\|Nu(t,\cdot)\|_{L^q} \lesssim (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})} \Big(\|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}} + \|u\|^p_{X(t)} \Big). \tag{5.17}$$

Step 2: We need to estimate the norm $\|\partial_t Nu(t,\cdot)\|_{L^q}$. Differentiating $Nu = Nu(t,x)$ with respect to t we obtain

$$\partial_t Nu(t,x) = \partial_t\big(K_0(t,x) *_x u_0(x) + K_1(t,x) *_x u_1(x)\big) + \int_0^t \partial_t\big(K_1(t-\tau,x) *_x |u_t(\tau,x)|^p\big) d\tau.$$

We apply the $L^m \cap L^q - L^q$ estimates if $\tau \in [0,t/2]$ and the $L^q - L^q$ estimates if $\tau \in [t/2,t]$ from Theorem 2.3.1 to conclude

$$\|\partial_t Nu(t,\cdot)\|_{L^q} \lesssim (1+t)^{-\frac{n}{\sigma}(1-\frac{1}{r})} \|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}}$$
$$+ \int_0^{t/2} (1+t-\tau)^{-\frac{n}{\sigma}(1-\frac{1}{r})} \big\||u_t(\tau,\cdot)|^p\big\|_{L^m\cap L^q} d\tau + \int_{t/2}^t \big\||u_t(\tau,\cdot)|^p\big\|_{L^q} d\tau.$$

Using the same ideas for deriving (5.17) we may conclude

$$\|\partial_t Nu(t,\cdot)\|_{L^q} \lesssim (1+\tau)^{-\frac{n}{\sigma}(1-\frac{1}{r})} \Big(\|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}} + \|u\|^p_{X(t)} \Big),$$

under the same assumptions for p, that is,

$$p \in \Big[\frac{q}{m}, \frac{n}{n-q(s-\sigma)}\Big] \text{ if } s < \sigma + \frac{n}{q}, \quad \text{ or } \quad p \in \Big[\frac{q}{m}, \infty\Big) \text{ if } s \geqslant \sigma + \frac{n}{q},$$

and

$$p > 1 + \frac{m\sigma}{n}.$$

Step 3: Let us estimate the norm $\big\|\partial_t |D|^{s-\sigma} Nu(t,\cdot)\big\|_{L^q}$. We use

$$\begin{aligned}\partial_t |D|^{s-\sigma} Nu(t,x) &= \partial_t |D|^{s-\sigma}\big(K_0(t,x) *_x u_0(x) + K_1(t,x) *_x u_1(x)\big)\\ &\quad + \int_0^t \partial_t |D|^{s-\sigma}\big(K_1(t-\tau,x) *_x |u_t(\tau,x)|^p\big)d\tau.\end{aligned}$$

We apply the $(L^m \cap L^q) - L^q$ estimates if $\tau \in [0,t/2]$ and the $L^q - L^q$ estimates if $\tau \in [t/2,t]$ from Theorem 2.3.1 to derive

$$\begin{aligned}\big\|\partial_t |D|^{s-\sigma} Nu(t,\cdot)\big\|_{L^q} &\lesssim (1+t)^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s-\sigma}{\sigma}}\|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}}\\ &\quad + \int_0^{t/2}(1+t-\tau)^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s-\sigma}{\sigma}}\big\||u_t(\tau,\cdot)|^p\big\|_{L^m\cap H^{s-\sigma}_q}d\tau\\ &\quad + \int_{t/2}^t (1+t-\tau)^{-\frac{s-\sigma}{\sigma}}\big\||u_t(\tau,\cdot)|^p\big\|_{H^{s-\sigma}_q}d\tau\\ &= (1+t)^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s-\sigma}{\sigma}}\|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}}\\ &\quad + \int_0^{t/2}(1+t-\tau)^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s-\sigma}{\sigma}}\big\||u_t(\tau,\cdot)|^p\big\|_{L^m\cap L^q\cap \dot{H}^{s-\sigma}_q}d\tau\\ &\quad + \int_{t/2}^t (1+t-\tau)^{-\frac{s-\sigma}{\sigma}}\big\||u_t(\tau,\cdot)|^p\big\|_{L^q\cap \dot{H}^{s-\sigma}_q}d\tau.\end{aligned}$$

The integrals with $\big\||u_t(\tau,\cdot)|^p\big\|_{L^m\cap L^q}$ and $\big\||u_t(\tau,\cdot)|^p\big\|_{L^q}$ will be handled as before if we apply the conditions for p, that is,

$$p \in \Big[\frac{q}{m}, \frac{n}{n-q(s-\sigma)}\Big] \text{ if } s < \sigma + \frac{n}{q}, \quad \text{or} \quad p \in \Big[\frac{q}{m}, \infty\Big) \text{ if } s \geqslant \sigma + \frac{n}{q},$$

and

$$p > 1 + \frac{m\sigma}{n}.$$

Hence, we get

$$\int_0^{t/2}(1+t-\tau)^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s-\sigma}{\sigma}}\big\||u_t(\tau,\cdot)|^p\big\|_{L^m\cap L^q}d\tau \lesssim (1+t)^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s-\sigma}{\sigma}}\|u\|^p_{X(t)},$$

and

$$\int_{t/2}^t (1+t-\tau)^{-\frac{s-\sigma}{\sigma}}\big\||u_t(\tau,\cdot)|^p\big\|_{L^q}d\tau \lesssim (1+t)^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s-\sigma}{\sigma}}\|u\|^p_{X(t)}.$$

To estimate the integrals with the norm $\big\||u_t(\tau,\cdot)|^p\big\|_{\dot{H}^{s-\sigma}_q}$, we shall apply Corollary C.4.1 for the fractional powers rule with $s-\sigma \in (\frac{n}{q}, p)$. Therefore, we obtain

$$\begin{aligned}\big\||u_t(\tau,\cdot)|^p\big\|_{\dot{H}^{s-\sigma}_q} &\lesssim \|u_t(\tau,\cdot)\|_{\dot{H}^{s-\sigma}_q}\|u_t(\tau,\cdot)\|^{p-1}_{L^\infty}\\ &\lesssim \|u_t(\tau,\cdot)\|_{\dot{H}^{s-\sigma}_q}\big(\|u_t(\tau,\cdot)\|_{\dot{H}^{s^*}_q} + \|u_t(\tau,\cdot)\|_{\dot{H}^{s-\sigma}_q}\big)^{p-1}.\end{aligned}$$

Here we used Corollary C.5.1 with a suitable $s^* < \frac{n}{q}$. Applying the fractional Gagliardo-Nirenberg inequality from Proposition C.1.1 we have

$$\|u_t(\tau,\cdot)\|_{\dot{H}^{s^*}_q} \lesssim \|u_t(\tau,\cdot)\|^{1-\theta}_{L^q}\big\||D|^{s-\sigma}u_t(\tau,\cdot)\big\|^{\theta}_{L^q} \lesssim (1+\tau)^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s^*}{\sigma}}\|u\|_{X(\tau)},$$

where $\theta = \frac{s^*}{s-\sigma}$. Hence, we derive

$$\big\||u_t(\tau,\cdot)|^p\big\|_{\dot{H}^{s-\sigma}_q} \lesssim (1+\tau)^{-\frac{np}{\sigma}(1-\frac{1}{r})-\frac{s-\sigma}{\sigma}-(p-1)\frac{s^*}{\sigma}}\|u\|^p_{X(\tau)} \lesssim (1+\tau)^{-\frac{n}{m\sigma}(p-1)}\|u\|^p_{X(\tau)},$$

if we choose $s^* = \frac{n}{q} - \varepsilon_0$ where $\varepsilon_0 > 0$ is a sufficiently small. Consequently, we may conclude

$$\int_0^{t/2} (1+t-\tau)^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s-\sigma}{\sigma}} \big\||u_t(\tau,\cdot)|^p\big\|_{\dot{H}^{s-\sigma}_q} d\tau \lesssim (1+t)^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s-\sigma}{\sigma}} \|u\|^p_{X(t)}.$$

Moreover, in an analogous way we can also derive

$$\int_{t/2}^{t} (1+t-\tau)^{-\frac{s-\sigma}{\sigma}} \big\||u_t(\tau,\cdot)|^p\big\|_{\dot{H}^{s-\sigma}_q} d\tau \lesssim (1+t)^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s-\sigma}{\sigma}} \|u\|^p_{X(t)}.$$

Therefore, we have shown the estimates

$$\big\|\partial_t |D|^{s-\sigma} Nu(\tau,\cdot)\big\|_{L^q} \lesssim (1+t)^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s-\sigma}{\sigma}} \big(\|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}} + \|u\|^p_{X(t)}\big). \tag{5.18}$$

Step 4: Let us estimate the norm $\big\||D|^s Nu(t,\cdot)\big\|_{L^q}$. We use

$$|D|^s Nu(t,x) = |D|^s\big(K_0(t,x) *_x u_0(x) + K_1(t,x) *_x u_1(x)\big) + \int_0^t |D|^s\big(K_1(t-\tau,x) *_x |u_t(\tau,x)|^p\big) d\tau.$$

By applying again the $(L^m \cap L^q) - L^q$ estimates if $\tau \in [0,t/2]$ and the $L^q - L^q$ estimates if $\tau \in [t/2,t]$ from Theorem 2.3.1, we derive

$$\begin{aligned}\big\||D|^s Nu(t,\cdot)\big\|_{L^q} &\lesssim (1+t)^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s-\sigma}{\sigma}} \|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}} \\ &\quad + \int_0^{t/2} (1+t-\tau)^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s-\sigma}{\sigma}} \big\||u_t(\tau,\cdot)|^p\big\|_{L^m \cap H^{s-\sigma}_q} d\tau \\ &\quad + \int_{t/2}^{t} (1+t-\tau)^{-\frac{s-\sigma}{\sigma}} \big\||u_t(\tau,\cdot)|^p\big\|_{H^{s-\sigma}_q} d\tau.\end{aligned}$$

Following the approach to show (5.18) we may conclude

$$\big\||D|^s Nu(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s-\sigma}{\sigma}} \big(\|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}} + \|u\|^p_{X(t)}\big).$$

Summarizing, from the definition of the norm in $X(t)$ we obtain immediately the inequality (5.15).

Next let us prove the inequality (5.16). Our proof is divided into four steps.

Step 1: We need to estimate the norm $\|Nu(t,\cdot) - Nv(t,\cdot)\|_{L^q}$. Using the $(L^m \cap L^q) - L^q$ estimates if $\tau \in [0,t/2]$ and the $L^q - L^q$ estimates if $\tau \in [t/2,t]$ from Theorem 2.3.1 we derive for two functions u and v from $X(t)$ the estimate

$$\begin{aligned}\|Nu(t,\cdot) - Nv(t,\cdot)\|_{L^q} &\lesssim \int_0^{t/2} (1+t-\tau)^{1-\frac{n}{\sigma}(1-\frac{1}{r})} \big\||u_t(\tau,\cdot)|^p - |v_t(\tau,\cdot)|^p\big\|_{L^m \cap L^q} d\tau \\ &\quad + \int_{t/2}^{t} (1+t-\tau) \big\||u_t(\tau,\cdot)|^p - |v_t(\tau,\cdot)|^p\big\|_{L^q} d\tau.\end{aligned}$$

By using Hölder's inequality, we may show the estimates

$$\big\||u_t(\tau,\cdot)|^p - |v_t(\tau,\cdot)|^p\big\|_{L^q} \lesssim \|u_t(\tau,\cdot) - v_t(\tau,\cdot)\|_{L^{qp}} \big(\|u_t(\tau,\cdot)\|^{p-1}_{L^{qp}} + \|v_t(\tau,\cdot)\|^{p-1}_{L^{qp}}\big),$$

and

$$\big\||u_t(\tau,\cdot)|^p - |v_t(\tau,\cdot)|^p\big\|_{L^m} \lesssim \|u_t(\tau,\cdot) - v_t(\tau,\cdot)\|_{L^{mp}} \big(\|u_t(\tau,\cdot)\|^{p-1}_{L^{mp}} + \|v_t(\tau,\cdot)\|^{p-1}_{L^{mp}}\big).$$

Analogously to the proof of (5.15), applying the fractional Gagliardo-Nirenberg inequality from Proposition C.1.1 to the norms

$$\|u_t(\tau,\cdot) - v_t(\tau,\cdot)\|_{L^\eta}, \quad \|u_t(\tau,\cdot)\|_{L^\eta}, \quad \|v_t(\tau,\cdot)\|_{L^\eta}$$

with $\eta = qp$ and $\eta = mp$ we derive the following estimates:

$$\begin{aligned}\|u_t(\tau,\cdot) - v_t(\tau,\cdot)\|_{L^{qp}} &\lesssim (1+\tau)^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s-\sigma}{\sigma}\theta_{qp}} \|u-v\|_{X(\tau)}, \\ \|u_t(\tau,\cdot)\|_{L^{qp}} &\lesssim (1+\tau)^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s-\sigma}{\sigma}\theta_{qp}} \|u\|_{X(\tau)}, \\ \|v_t(\tau,\cdot)\|_{L^{qp}} &\lesssim (1+\tau)^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s-\sigma}{\sigma}\theta_{qp}} \|v\|_{X(\tau)},\end{aligned}$$

and

$$\begin{aligned}
\|u_t(\tau,\cdot)-v_t(\tau,\cdot)\|_{L^{mp}} &\lesssim (1+\tau)^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s-\sigma}{\sigma}\theta_{mp}}\|u-v\|_{X(\tau)},\\
\|u_t(\tau,\cdot)\|_{L^{mp}} &\lesssim (1+\tau)^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s-\sigma}{\sigma}\theta_{mp}}\|u\|_{X(\tau)},\\
\|v_t(\tau,\cdot)\|_{L^{mp}} &\lesssim (1+\tau)^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s-\sigma}{\sigma}\theta_{mp}}\|v\|_{X(\tau)}.
\end{aligned}$$

Therefore, thanks to $\theta_{mp} < \theta_{qp}$ and the relation $1+\frac{1}{q}=\frac{1}{r}+\frac{1}{m}$, we obtain

$$\begin{aligned}
\big\||u_t(\tau,\cdot)|^p-|v_t(\tau,\cdot)|^p\big\|_{L^m\cap L^q} &\lesssim (1+\tau)^{-\frac{n}{m\sigma}(p-1)}\|u-v\|_{X(\tau)}\big(\|u\|^{p-1}_{X(\tau)}+\|v\|^{p-1}_{X(\tau)}\big),\\
\big\||u_t(\tau,\cdot)|^p-|v_t(\tau,\cdot)|^p\big\|_{L^q} &\lesssim (1+\tau)^{-\frac{np}{\sigma}(\frac{1}{m}-\frac{1}{qp})}\|u-v\|_{X(\tau)}\big(\|u\|^{p-1}_{X(\tau)}+\|v\|^{p-1}_{X(\tau)}\big).
\end{aligned}$$

Applying again the same ideas as we did in Step 1 to prove (5.15) we may conclude

$$\|Nu(t,\cdot)-Nv(t,\cdot)\|_{L^q} \lesssim (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})}\|u-v\|_{X(t)}\big(\|u\|^{p-1}_{X(t)}+\|v\|^{p-1}_{X(t)}\big).$$

Step 2: We need to estimate the norm $\|\partial_t Nu(t,\cdot)-\partial_t Nv(t,\cdot)\|_{L^q}$. We use

$$\begin{aligned}
\|\partial_t Nu(t,\cdot)-\partial_t Nv(t,\cdot)\|_{L^q} \lesssim& \int_0^{t/2}(1+t-\tau)^{-\frac{n}{\sigma}(1-\frac{1}{r})}\big\||u_t(\tau,\cdot)|^p-|v_t(\tau,\cdot)|^p\big\|_{L^m\cap L^q}d\tau\\
&+\int_{t/2}^t \big\||u_t(\tau,\cdot)|^p-|v_t(\tau,\cdot)|^p\big\|_{L^q}d\tau.
\end{aligned}$$

Using the same approach of Step 2 to prove (5.15) and Step 1 to prove (5.16) we may conclude

$$\|\partial_t Nu(t,\cdot)-\partial_t Nv(t,\cdot)\|_{L^q} \lesssim (1+t)^{-\frac{n}{\sigma}(1-\frac{1}{r})}\|u-v\|_{X(t)}\big(\|u\|^{p-1}_{X(t)}+\|v\|^{p-1}_{X(t)}\big).$$

Step 3: Let us estimate the norm $\big\|\partial_t|D|^{s-\sigma}Nu(t,\cdot)-\partial_t|D|^{s-\sigma}Nv(t,\cdot)\big\|_{L^q}$. We use

$$\begin{aligned}
&\big\|\partial_t|D|^{s-\sigma}Nu(t,\cdot)-\partial_t|D|^{s-\sigma}Nv(t,\cdot)\big\|_{L^q}\\
&\lesssim \int_0^{t/2}(1+t-\tau)^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s-\sigma}{\sigma}}\big\||u_t(\tau,\cdot)|^p-|v_t(\tau,\cdot)|^p\big\|_{L^m\cap H^{s-\sigma}_q}d\tau\\
&\quad+\int_{t/2}^t(1+t-\tau)^{-\frac{s-\sigma}{\sigma}}\big\||u_t(\tau,\cdot)|^p-|v_t(\tau,\cdot)|^p\big\|_{H^{s-\sigma}_q}d\tau\\
&=\int_0^{t/2}(1+t-\tau)^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s-\sigma}{\sigma}}\big\||u_t(\tau,\cdot)|^p-|v_t(\tau,\cdot)|^p\big\|_{L^m\cap L^q\cap \dot{H}^{s-\sigma}_q}d\tau\\
&\quad+\int_{t/2}^t(1+t-\tau)^{-\frac{s-\sigma}{\sigma}}\big\||u_t(\tau,\cdot)|^p-|v_t(\tau,\cdot)|^p\big\|_{L^q\cap\dot{H}^{s-\sigma}_q}d\tau.
\end{aligned}$$

The integrals with $\big\||u_t(\tau,\cdot)|^p-|v_t(\tau,\cdot)|^p\big\|_{L^m\cap L^q}$ and $\big\||u_t(\tau,\cdot)|^p-|v_t(\tau,\cdot)|^p\big\|_{L^q}$ will be handled as we did in Step 1 to prove (5.16). Hence, we get

$$\begin{aligned}
&\int_0^{t/2}(1+t-\tau)^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s-\sigma}{\sigma}}\big\||u_t(\tau,\cdot)|^p-|v_t(\tau,\cdot)|^p\big\|_{L^m\cap L^q}d\tau\\
&\qquad\lesssim (1+t)^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s-\sigma}{\sigma}}\|u-v\|_{X(\tau)}\big(\|u\|^{p-1}_{X(\tau)}+\|v\|^{p-1}_{X(\tau)}\big),
\end{aligned}$$

and

$$\begin{aligned}
&\int_{t/2}^t(1+t-\tau)^{-\frac{s-\sigma}{\sigma}}\big\||u_t(\tau,\cdot)|^p-|v_t(\tau,\cdot)|^p\big\|_{L^q}d\tau\\
&\qquad\lesssim (1+t)^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s-\sigma}{\sigma}}\|u-v\|_{X(\tau)}\big(\|u\|^{p-1}_{X(\tau)}+\|v\|^{p-1}_{X(\tau)}\big).
\end{aligned}$$

Let us now turn to estimate the norm $\big\||u_t(\tau,\cdot)|^p-|v_t(\tau,\cdot)|^p\big\|_{\dot{H}^{s-\sigma}_q}$. By using the integral representation

$$|u_t(\tau,x)|^p-|v_t(\tau,x)|^p = p\int_0^1\big(u_t(\tau,x)-v_t(\tau,x)\big)G\big(\omega u_t(\tau,x)+(1-\omega)v_t(\tau,x)\big)d\omega,$$

where $G(u) = u|u|^{p-2}$, we obtain

$$\big\||u_t(\tau,\cdot)|^p - |v_t(\tau,\cdot)|^p\big\|_{\dot H^{s-\sigma}_q} \lesssim \int_0^1 \big\|\big(u_t(\tau,\cdot)-v_t(\tau,\cdot)\big)G\big(\omega u_t(\tau,\cdot)+(1-\omega)v_t(\tau,\cdot)\big)\big\|_{\dot H^{s-\sigma}_q}d\omega.$$

Thanks to the fractional powers rule from Corollary C.4.2, we can proceed as follows:

$$\begin{aligned}\big\||u_t(\tau,\cdot)|^p - |v_t(\tau,\cdot)|^p\big\|_{\dot H^{s-\sigma}_q} &\lesssim \int_0^1 \|u_t(\tau,\cdot)-v_t(\tau,\cdot)\|_{\dot H^{s-\sigma}_q}\big\|G\big(\omega u_t(\tau,\cdot)+(1-\omega)v_t(\tau,\cdot)\big)\big\|_{L^\infty}d\omega\\ &\quad+\int_0^1 \|u_t(\tau,\cdot)-v_t(\tau,\cdot)\|_{L^\infty}\big\|G\big(\omega u_t(\tau,\cdot)+(1-\omega)v_t(\tau,\cdot)\big)\big\|_{\dot H^{s-\sigma}_q}d\omega\\ &\lesssim \int_0^1 \|u_t(\tau,\cdot)-v_t(\tau,\cdot)\|_{\dot H^{s-\sigma}_q}\|\omega u_t(\tau,\cdot)+(1-\omega)v_t(\tau,\cdot)\|^{p-1}_{L^\infty}d\omega\\ &\quad+\int_0^1 \|u_t(\tau,\cdot)-v_t(\tau,\cdot)\|_{L^\infty}\big\|G\big(\omega u_t(\tau,\cdot)+(1-\omega)v_t(\tau,\cdot)\big)\big\|_{\dot H^{s-\sigma}_q}d\omega.\end{aligned}$$

Applying Corollary C.4.1 with $p>2$ and $s-\sigma\in(\frac{n}{q},p-1)$ we get

$$\begin{aligned}&\big\|G\big(\omega u_t(\tau,\cdot)+(1-\omega)v_t(\tau,\cdot)\big)\big\|_{\dot H^{s-\sigma}_q}\\ &\qquad\lesssim \|\omega u_t(\tau,\cdot)+(1-\omega)v_t(\tau,\cdot)\|_{\dot H^{s-\sigma}_q}\|\omega u_t(\tau,\cdot)+(1-\omega)v_t(\tau,\cdot)\|^{p-2}_{L^\infty}.\end{aligned}$$

Using Corollary C.5.1 with a suitable $s^* < \frac{n}{q}$, we obtain

$$\|u_t(\tau,\cdot)-v_t(\tau,\cdot)\|_{L^\infty} \lesssim \|u_t(\tau,\cdot)-v_t(\tau,\cdot)\|_{\dot H^{s^*}_q} + \|u_t(\tau,\cdot)-v_t(\tau,\cdot)\|_{\dot H^{s-\sigma}_q}$$

and

$$\begin{aligned}&\|\omega u_t(\tau,\cdot)+(1-\omega)v_t(\tau,\cdot)\|_{L^\infty}\\ &\qquad\lesssim \|\omega u_t(\tau,\cdot)+(1-\omega)v_t(\tau,\cdot)\|_{\dot H^{s^*}_q} + \|\omega u_t(\tau,\cdot)+(1-\omega)v_t(\tau,\cdot)\|_{\dot H^{s-\sigma}_q}.\end{aligned}$$

Applying the fractional Gagliardo-Nirenberg inequality from Proposition C.1.1 we have

$$\begin{aligned}\|u_t(\tau,\cdot)-v_t(\tau,\cdot)\|_{\dot H^{s^*}_q} &\lesssim \|u_t(\tau,\cdot)-v_t(\tau,\cdot)\|^{1-\theta}_{L^q}\,\big\||D|^{s-\sigma}\big(u_t(\tau,\cdot)-v_t(\tau,\cdot)\big)\big\|^{\theta}_{L^q}\\ &\lesssim (1+\tau)^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s^*}{\sigma}}\|u-v\|_{X(\tau)},\end{aligned}$$

where $\theta = \frac{s^*}{s-\sigma}$. In the same way, we get

$$\|\omega u_t(\tau,\cdot)+(1-\omega)v_t(\tau,\cdot)\|_{\dot H^{s^*}_q} \lesssim (1+\tau)^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s^*}{\sigma}}\|\omega u_t(\tau,\cdot)+(1-\omega)v_t(\tau,\cdot)\|_{X(\tau)}.$$

Therefore, we may conclude

$$\begin{aligned}&\big\||u_t(\tau,\cdot)|^p-|v_t(\tau,\cdot)|^p\big\|_{\dot H^{s-\sigma}_q}\\ &\quad\lesssim \int_0^1 (1+\tau)^{-\frac{np}{\sigma}(1-\frac{1}{r})-\frac{s-\sigma}{\sigma}-(p-1)\frac{s^*}{\sigma}}\|u-v\|_{X(\tau)}\|\omega u+(1-\omega)v\|^{p-1}_{X(\tau)}d\omega\\ &\quad\lesssim (1+\tau)^{-\frac{np}{\sigma}(1-\frac{1}{r})-\frac{s-\sigma}{\sigma}-(p-1)\frac{s^*}{\sigma}}\|u-v\|_{X(\tau)}\big(\|u\|^{p-1}_{X(\tau)}+\|v\|^{p-1}_{X(\tau)}\big).\end{aligned}$$

By an analogous argument as we did in Step 3 to prove (5.15) we obtain

$$\begin{aligned}&\int_0^{t/2}(1+t-\tau)^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s-\sigma}{\sigma}}\big\||u_t(\tau,\cdot)|^p-|v_t(\tau,\cdot)|^p\big\|_{\dot H^{s-\sigma}_q}d\tau\\ &\qquad\lesssim (1+t)^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s-\sigma}{\sigma}}\|u-v\|_{X(t)}\big(\|u\|^{p-1}_{X(t)}+\|v\|^{p-1}_{X(t)}\big),\end{aligned}$$

and

$$\begin{aligned}&\int_{t/2}^{t}(1+t-\tau)^{-\frac{s-\sigma}{\sigma}}\big\||u_t(\tau,\cdot)|^p-|v_t(\tau,\cdot)|^p\big\|_{\dot H^{s-\sigma}_q}d\tau\\ &\qquad\lesssim (1+t)^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s-\sigma}{\sigma}}\|u-v\|_{X(t)}\big(\|u\|^{p-1}_{X(t)}+\|v\|^{p-1}_{X(t)}\big).\end{aligned}$$

Summarizing, we have proved the estimates

$$\begin{aligned}&\big\|\partial_t|D|^{s-\sigma}Nu(t,\cdot)-\partial_t|D|^{s-\sigma}Nv(t,\cdot)\big\|_{L^q}\\&\qquad\lesssim(1+t)^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s-\sigma}{\sigma}}\|u-v\|_{X(t)}\big(\|u\|^{p-1}_{X(t)}+\|v\|^{p-1}_{X(t)}\big).\end{aligned}$$

Step 4: Let us estimate the norm $\big\||D|^sNu(t,\cdot)-|D|^sNv(t,\cdot)\big\|_{L^q}$. We use

$$\begin{aligned}&\big\||D|^sNu(t,\cdot)-|D|^sNv(t,\cdot)\big\|_{L^q}\\&\quad\lesssim\int_0^{t/2}(1+t-\tau)^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s-\sigma}{\sigma}}\big\||u_t(\tau,\cdot)|^p-v_t(\tau,\cdot)|^p\big\|_{L^m\cap H^{s-\sigma}_q}d\tau\\&\qquad+\int_{t/2}^t(1+t-\tau)^{-\frac{s-\sigma}{\sigma}}\big\||u_t(\tau,\cdot)|^p-|v_t(\tau,\cdot)|^p\big\|_{H^{s-\sigma}_q}d\tau.\end{aligned}$$

Following the treatment as in Step 3 to prove (5.16) we may conclude

$$\big\||D|^sNu(t,\cdot)-|D|^sNv(t,\cdot)\big\|_{L^q}\lesssim(1+t)^{-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s-\sigma}{\sigma}}\|u-v\|_{X(t)}\big(\|u\|^{p-1}_{X(t)}+\|v\|^{p-1}_{X(t)}\big).$$

Summarizing, from the definition of the norm in $X(t)$ and all the previous estimates we have completed the proof of (5.16). □

Example 5.2.1. If we replace $m = 1$ in Theorem 5.2.1, then it becomes Theorem 3 in [11]. Hence, we want to underline that Theorem 5.2.1 is a generalization of the result from Theorem 3 in [11].

Example 5.2.2. By choosing $m = 1$, $q = 5$, $\sigma = 1$ and $s = 4$ we obtain the following admissible range of the exponents p in Theorem 5.2.1:

$$p \in [5,\infty) \quad \text{for all } n \in [1,14].$$

6. Semi-linear structurally damped σ-evolution models in the case $\delta \in \left(0, \frac{\sigma}{2}\right)$

Let us consider the following two Cauchy problems:

$$u_{tt} + (-\Delta)^\sigma u + \mu(-\Delta)^\delta u_t = |u|^p, \quad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) \tag{6.1}$$

and

$$u_{tt} + (-\Delta)^\sigma u + \mu(-\Delta)^\delta u_t = |u_t|^p, \quad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) \tag{6.2}$$

in space dimensions $n \geqslant 2$ with $\sigma \geqslant 1$, $\delta \in (0, \frac{\sigma}{2})$, $\mu > 0$ and a given number $p > 1$.

Let us explain our objectives and strategies as follows:

- The estimates for solutions to the linear Cauchy problems (3.1) are a key tool to deal with the semi-linear Cauchy problems (6.1) and (6.2).
- Because the oscillations in the representation of solutions for the linear Cauchy problems (3.1) produce singular behavior of coefficients as $t \to +0$ in $L^p - L^q$ estimates, we can compensate this singular behavior by assuming higher regularity for the data. Diffierent strategies appear to deal with the semi-linear Cauchy problems (6.1) and (6.2) in the following two considerations: No loss of decay but loss of regularity, loss of decay and loss of regularity.

  Loss of regularity (see, for example, [4, 9, 50, 59]) is a well-known phenomenon describing the effect that the regularity of the obtained solutions to semi-linear models is less than those of the initial data. This phenomenon appearing in our global (in time) existence results is due to the singular behavior of time-dependent coefficients in the estimates for solutions to the linear models localized to high frequencies as $t \to +0$. However, we can compensate this difficulty by assuming higher regularity for the data.

  Loss of decay is understood when the decay rates in the estimates for solutions to semi-linear models are worse than those given for solutions to the linear models with vanishing right-hand side. Additional benefits of allowing loss of decay (see [7]) are to show how the restrictions to the admissible exponents p could be relaxed.
- By using the fractional Gagliardo-Nirenberg inequality, the fractional chain rule, the fractional powers rule, the fractional Sobolev embedding and some auxiliary lemmas, we obtain global (in time) existence of small data solutions in the energy space, in the solution space below energy space, in the energy space with a suitable higher regularity and in the large regular space.
- Some examples are presented at the end of each theorem to compare with known results.

In the following statements we introduce the data spaces $\mathcal{A}^s_{m,q} := \left(L^m \cap H^s_q\right) \times \left(L^m \cap H^{[s-\sigma]^+}_q\right)$ with the norm

$$\|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}} := \|u_0\|_{L^m} + \|u_0\|_{H^s_q} + \|u_1\|_{L^m} + \|u_1\|_{H^{[s-\sigma]^+}_q},$$

where $s \geqslant 0$, $q \in (1,\infty)$ and $m \in [1,q)$. Moreover, we fix the following constants:

$$s_0 := \left(2 + \left[\frac{n}{2}\right]\right)(\sigma - 2\delta), \quad n_0 := \frac{6\delta - 2\sigma}{\sigma - 2\delta} \quad \text{and } n_1 := \frac{4mq(\sigma-\delta)}{q-m}.$$

6.1. No loss of decay but loss of regularity

6.1.1. Solutions in the energy space to the model (6.1)

In the first case, we obtain solutions to (6.1) from energy space on the base of L^q.

Theorem 6.1.1. *Let $q \in (1,\infty)$ be a fixed constant and $m \in [1,q)$. We assume the conditions $\left[\frac{n}{2}\right] < n_0$ and*

$$p > 1 + \frac{\max\left\{n - \frac{m}{q}n + m\sigma,\, 4m(\sigma-\delta)\right\}}{n - 2m(\sigma-\delta)}. \tag{6.3}$$

Moreover, we suppose the following conditions:

$$p \in \Big[\frac{q}{m}, \infty\Big) \text{ if } n \leqslant q\sigma, \quad \text{or} \quad p \in \Big[\frac{q}{m}, \frac{n}{n-q\sigma}\Big] \text{ if } n \in \Big(q\sigma, \frac{q^2\sigma}{q-m}\Big]. \tag{6.4}$$

Then, there exists a constant $\varepsilon > 0$ such that for any small data

$$(u_0,u_1) \in \mathcal{A}^{\sigma+s_0}_{m,q} \text{ satisfying the assumption } \|(u_0,u_1)\|_{\mathcal{A}^{\sigma+s_0}_{m,q}} \leqslant \varepsilon,$$

we have a uniquely determined global (in time) small data energy solution (on the base of L^q)

$$u \in C\big([0,\infty), H^\sigma_q\big) \cap C^1\big([0,\infty), L^q\big)$$

to (6.1). The following estimates hold:

$$\|u(t,\cdot)\|_{L^q} \lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})}\|(u_0,u_1)\|_{\mathcal{A}^{\sigma+s_0}_{m,q}}, \tag{6.5}$$

$$\big\||D|^\sigma u(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\sigma}{2(\sigma-\delta)}}\|(u_0,u_1)\|_{\mathcal{A}^{\sigma+s_0}_{m,q}}, \tag{6.6}$$

$$\|u_t(t,\cdot)\|_{L^q} \lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}}\|(u_0,u_1)\|_{\mathcal{A}^{\sigma+s_0}_{m,q}}, \tag{6.7}$$

where $1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{m}$.

Remark 6.1.1. From the proof of Theorems 3.1.1 and 3.2.1, the solutions u^{ln} to (6.1) with vanishing right-hand side satisfy the following $L^q - L^q$ estimates for $t \in (0,1]$:

$$\|u^{ln}(t,\cdot)\|_{L^q} \lesssim t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\|u_0\|_{L^q} + t^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\|u_1\|_{L^q},$$
$$\big\||D|^\sigma u^{ln}(t,\cdot)\big\|_{L^q} \lesssim t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)-\frac{\sigma}{2\delta}}\|u_0\|_{L^q} + t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\|u_1\|_{L^q},$$
$$\|u^{ln}_t(t,\cdot)\|_{L^q} \lesssim t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)-\frac{\sigma}{2\delta}}\|u_0\|_{L^q} + t^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\|u_1\|_{L^q}.$$

The singular behavior of coefficients as $t \to +0$ brings some problems in the treatment of the semi-linear Cauchy problem (6.1). But we can compensate this singular behavior by assuming higher regularity for the data u_0 and u_1. We are going to prove the following lemma.

Lemma 6.1.1. *The solutions u^{ln} to (6.1) with vanishing right-hand side satisfy the $(L^m \cap L^q) - L^q$ estimates*

$$\|u^{ln}(t,\cdot)\|_{L^q} \lesssim (1+t)^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})}\|u_0\|_{L^m\cap H^{s_0}_q} + (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})}\|u_1\|_{L^m\cap L^q},$$
$$\big\||D|^\sigma u^{ln}(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\sigma}{2(\sigma-\delta)}}\|u_0\|_{L^m\cap H^{\sigma+s_0}_q} + (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\sigma}{2(\sigma-\delta)}}\|u_1\|_{L^m\cap H^{s_0}_q},$$
$$\|u^{ln}_t(t,\cdot)\|_{L^q} \lesssim (1+t)^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}}\|u_0\|_{L^m\cap H^{\sigma+s_0}_q} + (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}}\|u_1\|_{L^m\cap H^{s_0}_q}$$

for any $t > 0$.

Proof. First, we can see that from the condition $[\frac{n}{2}] < n_0$ in Theorem 6.1.1 it follows immediately $s_0 < 2\delta$. Following the proof of Theorems 3.1.1 and 3.2.1 we get

$$\begin{aligned}
\|u^{ln}_{\chi_3}(t,\cdot)\|_{L^q} &\lesssim \big\|\mathfrak{F}^{-1}\big(\widehat{K_0}(t,\xi)\chi_3(|\xi|)\widehat{u_0}(\xi)\big)(t,\cdot)\big\|_{L^q} + \big\|\mathfrak{F}^{-1}\big(\widehat{K_1}(t,\xi)\chi_3(|\xi|)\widehat{u_1}(\xi)\big)(t,\cdot)\big\|_{L^q} \\
&\lesssim \big\|\mathfrak{F}^{-1}\big(|\xi|^{-s_0}\widehat{K_0}(t,\xi)\chi_3(|\xi|)\big)(t,\cdot)\big\|_{L^1}\big\|\mathfrak{F}^{-1}\big(|\xi|^{s_0}\widehat{u_0}(\xi)\big)\big\|_{L^q} \\
&\quad + \big\|\mathfrak{F}^{-1}\big(\widehat{K_1}(t,\xi)\chi_3(|\xi|)\big)(t,\cdot)\big\|_{L^1}\big\|\mathfrak{F}^{-1}\big(\widehat{u_1}(\xi)\big)\big\|_{L^q} \\
&\lesssim \begin{cases} \|u_0\|_{\dot{H}^{s_0}_q} + t^{1-\frac{s_0}{2\delta}}\|u_1\|_{L^q} & \text{if } t \in (0,1], \\ e^{-ct}\big(\|u_0\|_{\dot{H}^{s_0}_q} + \|u_1\|_{L^q}\big) & \text{if } t \in [1,\infty), \end{cases} \\
&\lesssim \begin{cases} \|u_0\|_{H^{s_0}_q} + \|u_1\|_{L^q} & \text{if } t \in (0,1], \\ e^{-ct}\big(\|u_0\|_{H^{s_0}_q} + \|u_1\|_{L^q}\big) & \text{if } t \in [1,\infty), \end{cases}
\end{aligned}$$

$$\begin{aligned}
\big\||D|^\sigma u^{ln}_{\chi_3}(t,\cdot)\big\|_{L^q} &\lesssim \big\|\mathfrak{F}^{-1}\big(|\xi|^\sigma \widehat{K_0}(t,\xi)\chi_3(|\xi|)\widehat{u_0}(\xi)\big)(t,\cdot)\big\|_{L^q} + \big\|\mathfrak{F}^{-1}\big(|\xi|^\sigma \widehat{K_1}(t,\xi)\chi_3(|\xi|)\widehat{u_1}(\xi)\big)(t,\cdot)\big\|_{L^q}\\
&\lesssim \big\|\mathfrak{F}^{-1}\big(|\xi|^{-s_0} \widehat{K_0}(t,\xi)\chi_3(|\xi|)\big)(t,\cdot)\big\|_{L^1}\big\|\mathfrak{F}^{-1}\big(|\xi|^{s_0+\sigma}\widehat{u_0}(\xi)\big)\big\|_{L^q}\\
&\quad + \big\|\mathfrak{F}^{-1}\big(|\xi|^{\sigma-s_0} \widehat{K_1}(t,\xi)\chi_3(|\xi|)\big)(t,\cdot)\big\|_{L^1}\big\|\mathfrak{F}^{-1}\big(|\xi|^{s_0}\widehat{u_1}(\xi)\big)\big\|_{L^q}\\
&\lesssim \begin{cases} \|u_0\|_{\dot{H}^{\sigma+s_0}_q} + \|u_1\|_{\dot{H}^{s_0}_q} & \text{if } t\in(0,1],\\ e^{-ct}\big(\|u_0\|_{\dot{H}^{\sigma+s_0}_q} + \|u_1\|_{\dot{H}^{s_0}_q}\big) & \text{if } t\in[1,\infty), \end{cases}\\
&\lesssim \begin{cases} \|u_0\|_{H^{\sigma+s_0}_q} + \|u_1\|_{H^{s_0}_q} & \text{if } t\in(0,1],\\ e^{-ct}\big(\|u_0\|_{H^{\sigma+s_0}_q} + \|u_1\|_{H^{s_0}_q}\big) & \text{if } t\in[1,\infty), \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
\|\partial_t u^{ln}_{\chi_3}(t,\cdot)\|_{L^q} &\lesssim \big\|\mathfrak{F}^{-1}\big(|\xi|^{2\sigma} \widehat{K_1}(t,\xi)\chi_3(|\xi|)\widehat{u_0}(\xi)\big)(t,\cdot)\big\|_{L^q}\\
&\quad + \big\|\mathfrak{F}^{-1}\big(\big(\widehat{K_0}(t,\xi) - |\xi|^{2\delta}\widehat{K_1}(t,\xi)\big)\chi_3(|\xi|)\widehat{u_1}(\xi)\big)(t,\cdot)\big\|_{L^q}\\
&\lesssim \big\|\mathfrak{F}^{-1}\big(|\xi|^{\sigma-s_0} \widehat{K_1}(t,\xi)\chi_3(|\xi|)\big)(t,\cdot)\big\|_{L^1}\big\|\mathfrak{F}^{-1}\big(|\xi|^{s_0+\sigma}\widehat{u_0}(\xi)\big)\big\|_{L^q}\\
&\quad + \big\|\mathfrak{F}^{-1}\big(|\xi|^{-s_0}\big(\widehat{K_0}(t,\xi) - |\xi|^{2\delta}\widehat{K_1}(t,\xi)\big)\chi_3(|\xi|)\big)(t,\cdot)\big\|_{L^1}\big\|\mathfrak{F}^{-1}\big(|\xi|^{s_0}\widehat{u_1}(\xi)\big)\big\|_{L^q}\\
&\lesssim \begin{cases} \|u_0\|_{\dot{H}^{\sigma+s_0}_q} + \|u_1\|_{\dot{H}^{s_0}_q} & \text{if } t\in(0,1],\\ e^{-ct}\big(\|u_0\|_{\dot{H}^{\sigma+s_0}_q} + \|u_1\|_{\dot{H}^{s_0}_q}\big) & \text{if } t\in[1,\infty), \end{cases}\\
&\lesssim \begin{cases} \|u_0\|_{H^{\sigma+s_0}_q} + \|u_1\|_{H^{s_0}_q} & \text{if } t\in(0,1],\\ e^{-ct}\big(\|u_0\|_{H^{\sigma+s_0}_q} + \|u_1\|_{H^{s_0}_q}\big) & \text{if } t\in[1,\infty), \end{cases}
\end{aligned}$$

where c is a suitable positive constant. Moreover, we have shown the following decay estimates for low frequencies and middle frequencies:

$$\begin{aligned}
\|u^{ln}_{\chi_1}(t,\cdot)\|_{L^q} &\lesssim \begin{cases} \|u_0\|_{L^m} + t\|u_1\|_{L^m} & \text{if } t\in(0,1],\\ t^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})}\|u_0\|_{L^m} + t^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})}\|u_1\|_{L^m} & \text{if } t\in[1,\infty), \end{cases}\\
\big\||D|^\sigma u^{ln}_{\chi_1}(t,\cdot)\big\|_{L^q} &\lesssim \begin{cases} \|u_0\|_{L^m} + t\|u_1\|_{L^m} & \text{if } t\in(0,1],\\ t^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\sigma}{2(\sigma-\delta)}}\|u_0\|_{L^m} + t^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\sigma}{2(\sigma-\delta)}}\|u_1\|_{L^m} & \text{if } t\in[1,\infty), \end{cases}\\
\|\partial_t u^{ln}_{\chi_1}(t,\cdot)\|_{L^q} &\lesssim \begin{cases} t\|u_0\|_{L^m} + \|u_1\|_{L^m} & \text{if } t\in(0,1]\\ t^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}}\|u_0\|_{L^m} + t^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}}\|u_1\|_{L^m} & \text{if } t\in[1,\infty). \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
\|u^{ln}_{\chi_2}(t,\cdot)\|_{L^q} &\lesssim e^{-ct}\big(\|u_0\|_{L^q} + \|u_1\|_{L^q}\big),\\
\big\||D|^\sigma u^{ln}_{\chi_2}(t,\cdot)\big\|_{L^q} &\lesssim e^{-ct}\big(\|u_0\|_{L^q} + \|u_1\|_{L^q}\big),\\
\|\partial_t u^{ln}_{\chi_2}(t,\cdot)\|_{L^q} &\lesssim e^{-ct}\big(\|u_0\|_{L^q} + \|u_1\|_{L^q}\big),
\end{aligned}$$

for all $t\in(0,\infty)$, where c is a suitable positive constant. Summarizing, the proof of Lemma 6.1.1 is completed. □

Remark 6.1.2. In the proof of Lemma 6.1.1 we apply Proposition 3.1.4 with $a = -s_0$ for $\widehat{K_0}(t,\xi)$ and $a = 0$, $a = \sigma - s_0$, $a = 2\delta - s_0$ for $\widehat{K_1}(t,\xi)$. To apply Proposition 3.1.4 with these mentioned parameters, from Remark 3.1.4 we have to guarantee that $s_0 \leqslant 2\delta$. Because of the condition $[\frac{n}{2}] < n_0$ in Theorem 6.1.1, it follows immediately this condition.

Proof of Theorem 6.1.1. By using fundamental solutions we write the solution to (6.1) with vanishing right-hand side as follows:

$$u^{ln}(t,x) = K_0(t,x) *_x u_0(x) + K_1(t,x) *_x u_1(x),$$

where $K_j(t,x)$ with $j = 0,1$ are defined as in Chapter 3. Applying Duhamel's principle leads to the following representation of solutions to (6.1):

$$u(t,x) = K_0(t,x) *_x u_0(x) + K_1(t,x) *_x u_1(x) + \int_0^t K_1(t-\tau,x) *_x |u(\tau,x)|^p d\tau.$$

We introduce the data space $\mathcal{A}^{\sigma+s_0}_{m,q} := (L^m \cap H^{\sigma+s_0}_q) \times (L^m \cap H^{s_0}_q)$. Moreover, we introduce for any $t > 0$ the function space $X(t) := C\big([0,t], H^\sigma_q\big) \cap C^1\big([0,t], L^q\big)$. For the sake of brevity, we also define the norm

$$\|u\|_{X(t)} := \sup_{0\leqslant\tau\leqslant t}\Big(f_0(\tau)^{-1}\|u(\tau,\cdot)\|_{L^q} + f_\sigma(\tau)^{-1}\big\||D|^\sigma u(\tau,\cdot)\big\|_{L^q} + f_\delta(\tau)^{-1}\|u_t(\tau,\cdot)\|_{L^q}\Big)$$

and the space $X_0(t) := C\big([0,t], H^\sigma_q\big)$ with the norm

$$\|w\|_{X_0(t)} := \sup_{0\leqslant\tau\leqslant t}\Big(f_0(\tau)^{-1}\|w(\tau,\cdot)\|_{L^q} + f_\sigma(\tau)^{-1}\big\||D|^\sigma w(\tau,\cdot)\big\|_{L^q}\Big),$$

where from the estimates for solutions and some of their derivatives to the linear Cauchy problems given in Theorem 3.3.1 we choose

$$f_0(\tau) = (1+\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})}, \quad f_\sigma(\tau) = (1+\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\sigma}{2(\sigma-\delta)}},$$

and

$$f_\delta(\tau) = (1+\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}}.$$

We define the operator $N:\ X(t) \longrightarrow X(t)$ by the formula

$$Nu(t,x) = K_0(t,x) *_x u_0(x) + K_1(t,x) *_x u_1(x) + \int_0^t K_1(t-\tau,x) *_x |u(\tau,x)|^p d\tau.$$

We will prove that the operator N satisfies the following two estimates:

$$\|Nu\|_{X(t)} \lesssim \|(u_0,u_1)\|_{\mathcal{A}^{\sigma+s_0}_{m,q}} + \|u\|^p_{X_0(t)}, \tag{6.8}$$

$$\|Nu - Nv\|_{X(t)} \lesssim \|u-v\|_{X_0(t)}\big(\|u\|^{p-1}_{X_0(t)} + \|v\|^{p-1}_{X_0(t)}\big). \tag{6.9}$$

First let us prove the estimate (6.8). Taking into consideration the estimates for solutions and some of their partial derivatives to the linear Cauchy problems in Lemma 6.1.1 we get the following estimates for $j,k = 0,1$ and $(j,k) \neq (1,1)$:

$$\begin{aligned}&\big\|\partial_t^j|D|^{k\sigma}\big(K_0(t,x) *_x u_0(x) + K_1(t,x) *_x u_1(x)\big)\big\|_{L^q}\\ &\quad\lesssim (1+t)^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{k\sigma+2j\delta}{2(\sigma-\delta)}}\|u_0\|_{L^m\cap H^{\sigma+s_0}_q} + (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{k\sigma+2j\delta}{2(\sigma-\delta)}}\|u_1\|_{L^m\cap H^{s_0}_q}\\ &\quad\lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{k\sigma+2j\delta}{2(\sigma-\delta)}}\|(u_0,u_1)\|_{\mathcal{A}^{\sigma+s_0}_{m,q}}.\end{aligned}$$

In order to control the integral term in the representation of solutions, we use two different strategies for $\tau \in [0,[t-1]^+]$ and $\tau \in [[t-1]^+,t]$. In particular, we use the $(L^m \cap L^q) - L^q$ estimates if $\tau \in [0,[t-1]^+]$ and the $L^q - L^q$ estimates if $\tau \in [[t-1]^+,t]$ from Theorem 3.3.1. Therefore, we obtain

$$\begin{aligned}\big\|\partial_t^j|D|^{k\sigma}Nu(t,\cdot)\big\|_{L^q} &\lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{k\sigma+2j\delta}{2(\sigma-\delta)}}\|(u_0,u_1)\|_{\mathcal{A}^{\sigma+s_0}_{m,q}}\\ &\quad+\int_0^{[t-1]^+}(1+t-\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{k\sigma+2j\delta}{2(\sigma-\delta)}}\big\||u(\tau,\cdot)|^p\big\|_{L^m\cap L^q}d\tau\\ &\quad+\int_{[t-1]^+}^t(t-\tau)^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)-(k+j)\frac{\sigma}{2\delta}}\big\||u(\tau,\cdot)|^p\big\|_{L^q}d\tau.\end{aligned}$$

Hence, it is necessary to require the estimates for $|u(\tau,x)|^p$ in $L^m \cap L^q$ and L^q as follows:

$$\big\||u(\tau,\cdot)|^p\big\|_{L^m\cap L^q} \lesssim \|u(\tau,\cdot)\|^p_{L^{mp}} + \|u(\tau,\cdot)\|^p_{L^{qp}}, \quad \big\||u(\tau,\cdot)|^p\big\|_{L^q} = \|u(\tau,\cdot)\|^p_{L^{qp}}.$$

Applying the fractional Gagliardo-Nirenberg inequality from Proposition C.1.1 we may conclude

$$\begin{aligned}\|u(\tau,\cdot)\|_{L^{qp}} &\lesssim \big\||D|^\sigma u(\tau,\cdot)\big\|^{\theta_{qp}}_{L^q}\|u(\tau,\cdot)\|^{1-\theta_{qp}}_{L^q} \lesssim \big(f_\sigma(\tau)\|u\|_{X_0(\tau)}\big)^{\theta_{qp}}\big(f_0(\tau)\|u\|_{X_0(\tau)}\big)^{1-\theta_{qp}}\\ &\lesssim (1+\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\sigma}{2(\sigma-\delta)}\theta_{qp}}\|u\|_{X_0(\tau)},\end{aligned}$$

and

$$\begin{aligned}\|u(\tau,\cdot)\|_{L^{mp}} &\lesssim \big\||D|^\sigma u(\tau,\cdot)\big\|_{L^q}^{\theta_{mp}} \|u(\tau,\cdot)\|_{L^q}^{1-\theta_{mp}} \lesssim \big(f_\sigma(\tau)\|u\|_{X_0(\tau)}\big)^{\theta_{mp}} \big(f_0(\tau)\|u\|_{X_0(\tau)}\big)^{1-\theta_{mp}}\\ &\lesssim (1+\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\sigma}{2(\sigma-\delta)}\theta_{mp}}\|u\|_{X_0(\tau)},\end{aligned}$$

where

$$\theta_{qp} := \theta_{0,\sigma}(qp,q) = \frac{n}{\sigma}\Big(\frac{1}{q}-\frac{1}{qp}\Big) \text{ and } \theta_{mp} := \theta_{0,\sigma}(mp,q) = \frac{n}{\sigma}\Big(\frac{1}{q}-\frac{1}{mp}\Big).$$

As in Corollary C.1.1 we have to guarantee that θ_{qp} and θ_{mp} belong to $[0,1]$. Both conditions imply the restriction

$$p \in \Big[\frac{q}{m},\infty\Big) \text{ if } n \leqslant q\sigma, \qquad \text{or} \qquad p \in \Big[\frac{q}{m},\frac{n}{n-q\sigma}\Big] \text{ if } n > q\sigma.$$

By virtue of $\theta_{mp} < \theta_{qp}$ and the relation $1+\frac{1}{q} = \frac{1}{r}+\frac{1}{m}$, we derive

$$\begin{aligned}\big\||u(\tau,\cdot)|^p\big\|_{L^m\cap L^q} &\lesssim (1+\tau)^{p\big(1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\sigma}{2(\sigma-\delta)}\theta_{mp}\big)}\|u\|^p_{X_0(\tau)}\\ &\lesssim (1+\tau)^{p-\frac{n}{2m(\sigma-\delta)}(p-1)}\|u\|^p_{X_0(\tau)},\\ \big\||u(\tau,\cdot)|^p\big\|_{L^q} &\lesssim (1+\tau)^{p\big(1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\sigma}{2(\sigma-\delta)}\theta_{qp}\big)}\|u\|^p_{X_0(\tau)}\\ &\lesssim (1+\tau)^{p-\frac{np}{2(\sigma-\delta)}(\frac{1}{m}-\frac{1}{qp})}\|u\|^p_{X_0(\tau)}.\end{aligned}$$

Summarizing, from both estimates we may conclude

$$\begin{aligned}\big\|\partial_t^j|D|^{k\sigma}Nu(t,\cdot)\big\|_{L^q} &\lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{k\sigma+2j\delta}{2(\sigma-\delta)}}\|(u_0,u_1)\|_{\mathcal{A}^{\sigma+s_0}_{m,q}}\\ &\quad+\|u\|^p_{X_0(t)}\int_0^{[t-1]^+}(1+t-\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{k\sigma+2j\delta}{2(\sigma-\delta)}}(1+\tau)^{p-\frac{n}{2m(\sigma-\delta)}(p-1)}d\tau\\ &\quad+\|u\|^p_{X_0(t)}\int_{[t-1]^+}^t(t-\tau)^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)-(k+j)\frac{\sigma}{2\delta}}(1+\tau)^{p-\frac{np}{2(\sigma-\delta)}(\frac{1}{m}-\frac{1}{qp})}d\tau.\end{aligned}$$

The key tool relies now in the application of Lemma B.6.1. Because of

$$p > 1+\frac{\max\big\{n-\frac{m}{q}n+m\sigma,\, 4m(\sigma-\delta)\big\}}{n-2m(\sigma-\delta)},$$

we obtain

$$p-\frac{n}{2m(\sigma-\delta)}(p-1) < -1,$$

and

$$1-\frac{n}{2(\sigma-\delta)}\Big(1-\frac{1}{r}\Big)-\frac{k\sigma+2j\delta}{2(\sigma-\delta)} \geqslant p-\frac{n}{2m(\sigma-\delta)}(p-1).$$

Hence, after applying Lemma B.6.1 by choosing

$$\alpha = -1+\frac{n}{2(\sigma-\delta)}\Big(1-\frac{1}{r}\Big)+\frac{k\sigma+2j\delta}{2(\sigma-\delta)} \quad \text{and } \beta = -p+\frac{n}{2m(\sigma-\delta)}(p-1)$$

we get

$$\begin{aligned}&\int_0^{[t-1]^+}(1+t-\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{k\sigma+2j\delta}{2(\sigma-\delta)}}(1+\tau)^{p-\frac{n}{2m(\sigma-\delta)}(p-1)}d\tau\\ &\quad\lesssim \int_0^t(1+t-\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{k\sigma+2j\delta}{2(\sigma-\delta)}}(1+\tau)^{p-\frac{n}{2m(\sigma-\delta)}(p-1)}d\tau \lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{k\sigma+2j\delta}{2(\sigma-\delta)}}.\end{aligned}$$

Moreover, since the condition $\big[\frac{n}{2}\big] < n_0$ holds, it follows

$$1-\Big(1+\Big[\frac{n}{2}\Big]\Big)\Big(\frac{\sigma}{2\delta}-1\Big)-(k+j)\frac{\sigma}{2\delta} > -1.$$

Therefore, we estimate

$$\begin{aligned}
&\int_{[t-1]^+}^t (t-\tau)^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)-(k+j)\frac{\sigma}{2\delta}}(1+\tau)^{p-\frac{np}{2(\sigma-\delta)}(\frac{1}{m}-\frac{1}{qp})}d\tau \\
&\quad\lesssim (1+t)^{p-\frac{np}{2(\sigma-\delta)}(\frac{1}{m}-\frac{1}{qp})}\int_{[t-1]^+}^t (t-\tau)^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)-(k+j)\frac{\sigma}{2\delta}}d\tau \\
&\quad\lesssim (1+t)^{p-\frac{np}{2(\sigma-\delta)}(\frac{1}{m}-\frac{1}{qp})}\int_0^1 r^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)-(k+j)\frac{\sigma}{2\delta}}dr \\
&\quad\lesssim (1+t)^{p-\frac{np}{2(\sigma-\delta)}(\frac{1}{m}-\frac{1}{qp})} \lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{k\sigma+2j\delta}{2(\sigma-\delta)}},
\end{aligned}$$

since

$$p-\frac{np}{2(\sigma-\delta)}\Big(\frac{1}{m}-\frac{1}{qp}\Big) < p-\frac{n}{2m(\sigma-\delta)}(p-1) \leqslant 1-\frac{n}{2(\sigma-\delta)}\Big(1-\frac{1}{r}\Big)-\frac{k\sigma+2j\delta}{2(\sigma-\delta)}.$$

Finally, we conclude for $j,k=0,1$ and $(j,k)\neq(1,1)$ the estimates

$$\big\|\partial_t^j|D|^{k\sigma}Nu(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{k\sigma+2j\delta}{2(\sigma-\delta)}}\Big(\|(u_0,u_1)\|_{\mathcal{A}^{\sigma+s_0}_{m,q}}+\|u\|^p_{X_0(t)}\Big).$$

From the definition of the norm in $X(t)$, we obtain immediately the inequality (6.8).

Next let us prove the estimate (6.9). Using again the estimates for solutions and some of their partial derivatives to the linear Cauchy problems from Theorem 3.3.1, that is, the $(L^m\cap L^q)-L^q$ estimates if $\tau\in[0,[t-1]^+]$ and the L^q-L^q estimates if $\tau\in[[t-1]^+,t]$ we derive for two functions u and v from $X(t)$ the estimate

$$\begin{aligned}
&\big\|\partial_t^j|D|^{k\sigma}(Nu(t,\cdot)-Nv(t,\cdot))\big\|_{L^q} \\
&\quad\lesssim \int_0^{[t-1]^+}(1+t-\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{k\sigma+2j\delta}{2(\sigma-\delta)}}\big\||u(\tau,\cdot)|^p-|v(\tau,\cdot)|^p\big\|_{L^m\cap L^q}d\tau \\
&\qquad+\int_{[t-1]^+}^t (t-\tau)^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)-(k+j)\frac{\sigma}{2\delta}}\big\||u(\tau,\cdot)|^p-|v(\tau,\cdot)|^p\big\|_{L^q}d\tau.
\end{aligned}$$

By using Hölder's inequality, we get

$$\begin{aligned}
\big\||u(\tau,\cdot)|^p-|v(\tau,\cdot)|^p\big\|_{L^q} &\lesssim \|u(\tau,\cdot)-v(\tau,\cdot)\|_{L^{qp}}\big(\|u(\tau,\cdot)\|^{p-1}_{L^{qp}}+\|v(\tau,\cdot)\|^{p-1}_{L^{qp}}\big),\\
\big\||u(\tau,\cdot)|^p-|v(\tau,\cdot)|^p\big\|_{L^m} &\lesssim \|u(\tau,\cdot)-v(\tau,\cdot)\|_{L^{mp}}\big(\|u(\tau,\cdot)\|^{p-1}_{L^{mp}}+\|v(\tau,\cdot)\|^{p-1}_{L^{mp}}\big).
\end{aligned}$$

Analogously to the proof of (6.8), applying the fractional Gagliardo-Nirenberg inequality from Proposition C.1.1 to the norms

$$\|u(\tau,\cdot)-v(\tau,\cdot)\|_{L^\eta},\ \|u(\tau,\cdot)\|_{L^\eta},\ \|v(\tau,\cdot)\|_{L^\eta}$$

with $\eta=qp$ and $\eta=mp$ we derive the following estimates:

$$\begin{aligned}
\|u(\tau,\cdot)-v(\tau,\cdot)\|_{L^{qp}} &\lesssim (1+\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\sigma}{2(\sigma-\delta)}\theta_{qp}}\|u-v\|_{X_0(\tau)},\\
\|u(\tau,\cdot)\|_{L^{qp}} &\lesssim (1+\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\sigma}{2(\sigma-\delta)}\theta_{qp}}\|u\|_{X_0(\tau)},\\
\|v(\tau,\cdot)\|_{L^{qp}} &\lesssim (1+\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\sigma}{2(\sigma-\delta)}\theta_{qp}}\|v\|_{X_0(\tau)},
\end{aligned}$$

and

$$\begin{aligned}
\|u(\tau,\cdot)-v(\tau,\cdot)\|_{L^{mp}} &\lesssim (1+\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\sigma}{2(\sigma-\delta)}\theta_{mp}}\|u-v\|_{X_0(\tau)},\\
\|u(\tau,\cdot)\|_{L^{mp}} &\lesssim (1+\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\sigma}{2(\sigma-\delta)}\theta_{mp}}\|u\|_{X_0(\tau)},\\
\|v(\tau,\cdot)\|_{L^{mp}} &\lesssim (1+\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\sigma}{2(\sigma-\delta)}\theta_{mp}}\|v\|_{X_0(\tau)}.
\end{aligned}$$

Therefore, thanks to $\theta_{mp}<\theta_{qp}$ and the relation $1+\frac{1}{q}=\frac{1}{r}+\frac{1}{m}$, we obtain

$$\begin{aligned}
\big\||u(\tau,\cdot)|^p-|v(\tau,\cdot)|^p\big\|_{L^m\cap L^q} &\lesssim (1+\tau)^{p-\frac{n}{2m(\sigma-\delta)}(p-1)}\|u-v\|_{X_0(\tau)}\big(\|u\|^{p-1}_{X_0(\tau)}+\|v\|^{p-1}_{X_0(\tau)}\big),\\
\big\||u(\tau,\cdot)|^p-|v(\tau,\cdot)|^p\big\|_{L^q} &\lesssim (1+\tau)^{p-\frac{np}{2(\sigma-\delta)}(\frac{1}{m}-\frac{1}{qp})}\|u-v\|_{X_0(\tau)}\big(\|u\|^{p-1}_{X_0(\tau)}+\|v\|^{p-1}_{X_0(\tau)}\big).
\end{aligned}$$

Applying again an analogous treatment as we did in the proof of (6.8) we may conclude for $j,k=0,1$ and $(j,k)\neq(1,1)$ the estimates

$$\big\|\partial_t^j|D|^{k\sigma}(Nu(t,\cdot)-Nv(t,\cdot))\big\|_{L^q}\lesssim(1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{k\sigma+2j\delta}{2(\sigma-\delta)}}\|u-v\|_{X_0(t)}\big(\|u\|^{p-1}_{X_0(t)}+\|v\|^{p-1}_{X_0(t)}\big).$$

From the definition of the norm in $X(t)$, we obtain immediately the inequality (6.9).
Summarizing, the proof of Theorem 6.1.1 is completed. □

Remark 6.1.3. In Theorem 6.1.1 we may simplify the restriction of admissible exponent p if we assume another condition for the space dimension n instead. In other word, if we have the following condition

$$n\leqslant n_2:=\frac{mq(3\sigma-4\delta)}{q-m},$$

then it follows $4m(\sigma-\delta)\geqslant n-\frac{m}{q}n+m\sigma$. This inequality allows us to remove the restriction of $p>1+\frac{n-\frac{m}{q}n+m\sigma}{n-2m(\sigma-\delta)}$, which appears in Theorem 6.1.1 to derive

$$p>1+\frac{4m(\sigma-\delta)}{n-2m(\sigma-\delta)}.$$

Example 6.1.1. Here we want to make a comparison between Theorem 6.1.1 and Theorem 3 in the paper [12] by choosing $m=1$, $q=2$, $\sigma=1$ and $\delta=0.45$. In general, with these selected parameters the admissible range of the exponents p in Theorem 3 of [12] is more flexible than the result in Theorem 6.1.1 for the space dimensions $n=2,3,4$ (see the following table):

	Theorem 6.1.1	Theorem 3 in [12]
$n=2$	$p\in(3.44,\infty)$	$p\in(2.82,\infty)$
$n=3$	$p\in(2.32,3]$	$p\in[2,3]$
$n=4$	empty	$p=2$

Tab. 6.1.: The first comparison between the obtained results.

Example 6.1.2. In this example, we want to emphasize that the results in Theorem 6.1.1 allow some flexibility in comparison with those from Theorem 3 in the paper [12] if we choose $m=1$, $q=3$, $\sigma=1$ and $\delta=0.45$ (see the following table):

	Theorem 6.1.1	Theorem 3 in [12]
$n=2$	$p\in(3.59,\infty)$	empty
$n=3$	$p\in[3,\infty)$	empty
$n=4$	$p\in[3,4]$	empty

Tab. 6.2.: The second comparison between the obtained results.

6.1.2. Solutions below the energy space to the model (6.1)

In the second case we obtain solutions from Sobolev space on the base of L^q.

Theorem 6.1.2. *Let $q\in(1,\infty)$ be a fixed constant, $m\in[1,q)$ and $0<s<\sigma$. We assume the conditions $\big[\frac{n}{2}\big]<n_0$ and*

$$p>1+\frac{\max\big\{n-\frac{m}{q}n+ms,\,4m(\sigma-\delta)\big\}}{n-2m(\sigma-\delta)}. \tag{6.10}$$

Moreover, we suppose the following conditions:

$$p\in\Big[\frac{q}{m},\infty\Big)\text{ if }n\leqslant qs,\qquad\text{or}\qquad p\in\Big[\frac{q}{m},\frac{n}{n-qs}\Big]\text{ if }n\in\Big(qs,\frac{q^2s}{q-m}\Big]. \tag{6.11}$$

Then, there exists a constant $\varepsilon > 0$ such that for any small data

$$(u_0, u_1) \in \mathcal{A}^{s+s_0}_{m,q} \text{ satisfying the assumption } \|(u_0, u_1)\|_{\mathcal{A}^{s+s_0}_{m,q}} \leqslant \varepsilon,$$

we have a uniquely determined global (in time) small data Sobolev solution

$$u \in C\big([0,\infty), H^s_q\big)$$

to (6.1). The following estimates hold:

$$\|u(t,\cdot)\|_{L^q} \lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})} \|(u_0,u_1)\|_{\mathcal{A}^{s+s_0}_{m,q}}, \tag{6.12}$$

$$\big\||D|^s u(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}} \|(u_0,u_1)\|_{\mathcal{A}^{s+s_0}_{m,q}}, \tag{6.13}$$

where $1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{m}$.

Following the proof of Lemma 6.1.1 we may prove the following lemma.

Lemma 6.1.2. *The solutions u^{ln} to (6.1) with vanishing right-hand side satisfy the $(L^m \cap L^q) - L^q$ estimates*

$$\|u^{ln}(t,\cdot)\|_{L^q} \lesssim (1+t)^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})} \|u_0\|_{L^m \cap H^{s_0}_q} + (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})} \|u_1\|_{L^m \cap L^q},$$

$$\begin{aligned} \big\||D|^s u^{ln}(t,\cdot)\big\|_{L^q} &\lesssim (1+t)^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}} \|u_0\|_{L^m \cap H^{s+s_0}_q} \\ &\quad + (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}} \|u_1\|_{L^m \cap H^{[s-\sigma+s_0]^+}_q}, \end{aligned}$$

for any $t > 0$.

Proof of Theorem 6.1.2. We introduce the data space $\mathcal{A}^{s+s_0}_{m,q} := (L^m \cap H^{s+s_0}_q) \times (L^m \cap H^{[s-\sigma+s_0]^+}_q)$. Moreover, we introduce for any $t > 0$ the function space $X(t) := C\big([0,t], H^s_q\big)$. For the sake of brevity, we also define the norm

$$\|u\|_{X(t)} := \sup_{0 \leqslant \tau \leqslant t} \Big((1+\tau)^{-1+\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})} \|u(\tau,\cdot)\|_{L^q} + (1+\tau)^{-1+\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})+\frac{s}{2(\sigma-\delta)}} \big\||D|^s u(\tau,\cdot)\big\|_{L^q} \Big).$$

We define the operator $N:\ X(t) \longrightarrow X(t)$ by the formula

$$Nu(t,x) = K_0(t,x) \ast_x u_0(x) + K_1(t,x) \ast_x u_1(x) + \int_0^t K_1(t-\tau,x) \ast_x |u(\tau,x)|^p d\tau.$$

We will prove that the operator N satisfies the following two estimates:

$$\|Nu\|_{X(t)} \lesssim \|(u_0,u_1)\|_{\mathcal{A}^{s+s_0}_{m,q}} + \|u\|^p_{X(t)}, \tag{6.14}$$

$$\|Nu - Nv\|_{X(t)} \lesssim \|u-v\|_{X(t)} \big(\|u\|^{p-1}_{X(t)} + \|v\|^{p-1}_{X(t)}\big). \tag{6.15}$$

First let us prove the estimate (6.14). Taking into consideration the estimates for solutions to the linear Cauchy problems in Lemma 6.1.2 we get the following estimates for $k = 0, 1$:

$$\begin{aligned} &\big\||D|^{ks}\big(K_0(t,x) \ast_x u_0(x) + K_1(t,x) \ast_x u_1(x)\big)\big\|_{L^q} \\ &\quad \lesssim (1+t)^{-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{ks}{2(\sigma-\delta)}} \|u_0\|_{L^m \cap H^{s+s_0}_q} + (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{ks}{2(\sigma-\delta)}} \|u_1\|_{L^m \cap H^{[s-\sigma+s_0]^+}_q} \\ &\quad \lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{ks}{2(\sigma-\delta)}} \|(u_0,u_1)\|_{\mathcal{A}^{s+s_0}_{m,q}}. \end{aligned}$$

In order to control the integral term in the representation of solutions, we use the $(L^m \cap L^q) - L^q$ estimates if $\tau \in [0, [t-1]^+]$ and the $L^q - L^q$ estimates if $\tau \in [[t-1]^+, t]$ from Theorem 3.3.2. Therefore, we obtain

$$\begin{aligned} \big\||D|^{ks} Nu(t,\cdot)\big\|_{L^q} &\lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{ks}{2(\sigma-\delta)}} \|(u_0,u_1)\|_{\mathcal{A}^{s+s_0}_{m,q}} \\ &\quad + \int_0^{[t-1]^+} (1+t-\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{ks}{2(\sigma-\delta)}} \big\||u(\tau,\cdot)|^p\big\|_{L^m \cap L^q} d\tau \\ &\quad + \int_{[t-1]^+}^t (t-\tau)^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)-\frac{ks}{2\delta}} \big\||u(\tau,\cdot)|^p\big\|_{L^q} d\tau. \end{aligned}$$

Hence, it is necessary to require the estimates for $|u(\tau,x)|^p$ in $L^m\cap L^q$ and L^q as follows:

$$\big\||u(\tau,\cdot)|^p\big\|_{L^m\cap L^q}\lesssim\|u(\tau,\cdot)\|^p_{L^{mp}}+\|u(\tau,\cdot)\|^p_{L^{qp}},\quad \big\||u(\tau,\cdot)|^p\big\|_{L^q}=\|u(\tau,\cdot)\|^p_{L^{qp}}.$$

Applying the fractional Gagliardo-Nirenberg inequality from Proposition C.1.1 we may conclude

$$\|u(\tau,\cdot)\|_{L^{qp}}\lesssim\big\||D|^s u(\tau,\cdot)\big\|^{\theta_{qp}}_{L^q}\|u(\tau,\cdot)\|^{1-\theta_{qp}}_{L^q}\lesssim(1+\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}\theta_{qp}}\|u\|_{X(\tau)},$$
$$\|u(\tau,\cdot)\|_{L^{mp}}\lesssim\big\||D|^s u(\tau,\cdot)\big\|^{\theta_{mp}}_{L^q}\|u(\tau,\cdot)\|^{1-\theta_{mp}}_{L^q}\lesssim(1+\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}\theta_{mp}}\|u\|_{X(\tau)},$$

where

$$\theta_{qp}:=\theta_{0,s}(qp,q)=\frac{n}{s}\Big(\frac{1}{q}-\frac{1}{qp}\Big)\text{ and }\theta_{mp}:=\theta_{0,s}(mp,q)=\frac{n}{s}\Big(\frac{1}{q}-\frac{1}{mp}\Big).$$

As in Corollary C.1.1 we have to guarantee that θ_{qp} and θ_{mp} belong to $[0,1]$. Both conditions imply the restriction

$$p\in\Big[\frac{q}{m},\infty\Big)\text{ if }n\leqslant qs,\qquad\text{or}\qquad p\in\Big[\frac{q}{m},\frac{n}{n-qs}\Big]\text{ if }n>qs.$$

By virtue of $\theta_{mp}<\theta_{qp}$ and the relation $1+\frac{1}{q}=\frac{1}{r}+\frac{1}{m}$, we derive

$$\big\||u(\tau,\cdot)|^p\big\|_{L^m\cap L^q}\lesssim(1+\tau)^{p\left(1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}\theta_{mp}\right)}\|u\|^p_{X(\tau)}\lesssim(1+\tau)^{p-\frac{n}{2m(\sigma-\delta)}(p-1)}\|u\|^p_{X(\tau)},$$

and

$$\big\||u(\tau,\cdot)|^p\big\|_{L^q}\lesssim(1+\tau)^{p\left(1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}\theta_{qp}\right)}\|u\|^p_{X(\tau)}\lesssim(1+\tau)^{p-\frac{np}{2(\sigma-\delta)}(\frac{1}{m}-\frac{1}{qp})}\|u\|^p_{X(\tau)}.$$

Summarizing, from both estimates we may conclude

$$\begin{aligned}\big\||D|^{ks}Nu(t,\cdot)\big\|_{L^q}&\lesssim(1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{ks}{2(\sigma-\delta)}}\|(u_0,u_1)\|_{\mathcal{A}^{s+s_0}_{m,q}}\\&\quad+\|u\|^p_{X(t)}\int_0^{[t-1]^+}(1+t-\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{ks}{2(\sigma-\delta)}}(1+\tau)^{p-\frac{n}{2m(\sigma-\delta)}(p-1)}d\tau\\&\quad+\|u\|^p_{X(t)}\int_{[t-1]^+}^t(t-\tau)^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)-\frac{ks}{2\delta}}(1+\tau)^{p-\frac{np}{2(\sigma-\delta)}(\frac{1}{m}-\frac{1}{qp})}d\tau.\end{aligned}$$

The key tool relies now in the application of Lemma B.6.1. Because of

$$p>1+\frac{\max\big\{n-\frac{m}{q}n+ms,\,4m(\sigma-\delta)\big\}}{n-2m(\sigma-\delta)},$$

we obtain

$$p-\frac{n}{2m(\sigma-\delta)}(p-1)<-1,$$

and

$$1-\frac{n}{2(\sigma-\delta)}\Big(1-\frac{1}{r}\Big)-\frac{ks}{2(\sigma-\delta)}\geqslant p-\frac{n}{2m(\sigma-\delta)}(p-1).$$

Hence, applying Lemma B.6.1 by choosing

$$\alpha=-1+\frac{n}{2(\sigma-\delta)}\Big(1-\frac{1}{r}\Big)+\frac{ks}{2(\sigma-\delta)}\text{ and }\beta=-p+\frac{n}{2m(\sigma-\delta)}(p-1)$$

we get

$$\begin{aligned}&\int_0^{[t-1]^+}(1+t-\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{ks}{2(\sigma-\delta)}}(1+\tau)^{p-\frac{n}{2m(\sigma-\delta)}(p-1)}d\tau\\&\quad\lesssim\int_0^t(1+t-\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{ks}{2(\sigma-\delta)}}(1+\tau)^{p-\frac{n}{2m(\sigma-\delta)}(p-1)}d\tau\lesssim(1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{ks}{2(\sigma-\delta)}}.\end{aligned}$$

Moreover, since the condition $\big[\frac{n}{2}\big]<n_0$ holds, it follows

$$1-\Big(1+\Big[\frac{n}{2}\Big]\Big)\Big(\frac{\sigma}{2\delta}-1\Big)-\frac{ks}{2\delta}>-1.$$

Therefore, we estimate

$$\begin{aligned}
&\int_{[t-1]^+}^{t} (t-\tau)^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)-\frac{ks}{2\delta}} (1+\tau)^{p-\frac{np}{2(\sigma-\delta)}(\frac{1}{m}-\frac{1}{qp})} d\tau \\
&\qquad \lesssim (1+t)^{p-\frac{np}{2(\sigma-\delta)}(\frac{1}{m}-\frac{1}{qp})} \int_{[t-1]^+}^{t} (t-\tau)^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)-\frac{ks}{2\delta}} d\tau \\
&\qquad \lesssim (1+t)^{p-\frac{np}{2(\sigma-\delta)}(\frac{1}{m}-\frac{1}{qp})} \int_0^1 r^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)-\frac{ks}{2\delta}} dr \\
&\qquad \lesssim (1+t)^{p-\frac{np}{2(\sigma-\delta)}(\frac{1}{m}-\frac{1}{qp})} \lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{ks}{2(\sigma-\delta)}},
\end{aligned}$$

since

$$p-\frac{np}{2(\sigma-\delta)}\Big(\frac{1}{m}-\frac{1}{qp}\Big) < p-\frac{n}{2m(\sigma-\delta)}(p-1) \leqslant 1-\frac{n}{2(\sigma-\delta)}\Big(1-\frac{1}{r}\Big)-\frac{ks}{2(\sigma-\delta)}.$$

Finally, we conclude for $k=0,1$ the estimates

$$\big\||D|^{ks}Nu(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{ks}{2(\sigma-\delta)}} \Big(\|(u_0,u_1)\|_{\mathcal{A}^{s+s_0}_{m,q}} + \|u\|^p_{X(t)}\Big).$$

From the definition of the norm in $X(t)$ we obtain immediately the inequality (6.14).

Next let us prove the estimate (6.15). Using again the $(L^m \cap L^q) - L^q$ estimates if $\tau \in [0,[t-1]^+]$ and the $L^q - L^q$ estimates if $\tau \in [[t-1]^+,t]$ from Theorem 3.3.2 we derive for two functions u and v from $X(t)$ the estimates

$$\begin{aligned}
&\big\||D|^{ks}(Nu(t,\cdot)-Nv(t,\cdot))\big\|_{L^q} \\
&\quad \lesssim \int_0^{[t-1]^+} (1+t-\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{ks}{2(\sigma-\delta)}} \big\||u(\tau,\cdot)|^p-|v(\tau,\cdot)|^p\big\|_{L^m\cap L^q} d\tau \\
&\qquad + \int_{[t-1]^+}^{t} (t-\tau)^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)-\frac{ks}{2\delta}} \big\||u(\tau,\cdot)|^p-|v(\tau,\cdot)|^p\big\|_{L^q} d\tau.
\end{aligned}$$

By using Hölder's inequality and applying again an analogous treatment as we did in the proof of (6.9) in Theorem 6.1.1, we may conclude for $k=0,1$ the estimates

$$\big\||D|^{ks}(Nu(t,\cdot)-Nv(t,\cdot))\big\|_{L^q} \lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{ks}{2(\sigma-\delta)}} \|u-v\|_{X(t)} \big(\|u\|^{p-1}_{X(t)} + \|v\|^{p-1}_{X(t)}\big).$$

From the definition of the norm in $X(t)$ it follows immediately the inequality (6.15).
Summarizing, the proof of Theorem 6.1.2 is completed. □

Remark 6.1.4. In Theorem 6.1.2, if we assume the following condition for the space dimension:

$$n \leqslant n_2 := \frac{mq(4\sigma-4\delta-s)}{q-m},$$

then the relation $4m(\sigma-\delta) \geqslant n-\frac{m}{q}n+ms$ holds. This inequality allows us to simplify the restriction of admissible exponents p in Theorem 6.1.2 as follows:

$$p > 1+\frac{4m(\sigma-\delta)}{n-2m(\sigma-\delta)}.$$

Example 6.1.3. By choosing $m=1$, $q=2$, $\sigma=2$, $\delta=0.9$ and $s=1.5$ we obtain the admissible range of the exponents p as follows:

$$p \in (6.5,\infty) \text{ if } n=3, \qquad \text{or} \qquad p \in (3.44,4] \text{ if } n=4.$$

6.1.3. Solutions in the energy space with suitable higher regularity to the model (6.1)

In the third case, we obtain solutions to (6.1) belonging to the energy space (on the base of L^q) with a suitable higher regularity.

Theorem 6.1.3. *Let $q \in (1, \infty)$ be a fixed constant, $m \in [1, q)$ and $\sigma < s \leqslant \sigma + \frac{n}{q}$. We assume that the exponent p satisfies the conditions $p > 1 + \lceil s - \sigma \rceil$ and*

$$p > 1 + \frac{\max\left\{n - \frac{m}{q}n + ms,\, 4m(\sigma - \delta)\right\}}{n - 2m(\sigma - \delta)}, \tag{6.16}$$

where $\left[\frac{n}{2}\right] < n_0$. Moreover, we suppose the following conditions:

$$p \in \left[\frac{q}{m}, \infty\right) \text{ if } n \leqslant qs, \quad \text{or} \quad p \in \left[\frac{q}{m}, 1 + \frac{q\sigma}{n - qs}\right] \text{ if } n \in \left(qs, qs + \frac{qm\sigma}{q - m}\right]. \tag{6.17}$$

Then, there exists a constant $\varepsilon > 0$ such that for any small data

$$(u_0, u_1) \in \mathcal{A}^{s+s_0}_{m,q} \text{ satisfying the assumption } \|(u_0, u_1)\|_{\mathcal{A}^{s+s_0}_{m,q}} \leqslant \varepsilon,$$

we have a uniquely determined global (in time) small data energy solution

$$u \in C\big([0, \infty), H^s_q\big) \cap C^1\big([0, \infty), H^{s-\sigma}_q\big)$$

to (6.1). The following estimates hold:

$$\|u(t, \cdot)\|_{L^q} \lesssim (1 + t)^{1 - \frac{n}{2(\sigma-\delta)}(1 - \frac{1}{r})} \|(u_0, u_1)\|_{\mathcal{A}^{s+s_0}_{m,q}}, \tag{6.18}$$

$$\big\||D|^s u(t, \cdot)\big\|_{L^q} \lesssim (1 + t)^{1 - \frac{n}{2(\sigma-\delta)}(1 - \frac{1}{r}) - \frac{s}{2(\sigma-\delta)}} \|(u_0, u_1)\|_{\mathcal{A}^{s+s_0}_{m,q}}, \tag{6.19}$$

$$\|u_t(t, \cdot)\|_{L^q} \lesssim (1 + t)^{1 - \frac{n}{2(\sigma-\delta)}(1 - \frac{1}{r}) - \frac{\delta}{\sigma-\delta}} \|(u_0, u_1)\|_{\mathcal{A}^{s+s_0}_{m,q}}, \tag{6.20}$$

$$\big\||D|^{s-\sigma} u_t(t, \cdot)\big\|_{L^q} \lesssim (1 + t)^{1 - \frac{n}{2(\sigma-\delta)}(1 - \frac{1}{r}) - \frac{s-\sigma+2\delta}{2(\sigma-\delta)}} \|(u_0, u_1)\|_{\mathcal{A}^{s+s_0}_{m,q}}, \tag{6.21}$$

where $1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{m}$.

Following the proof of Lemma 6.1.1 we may prove the following lemma.

Lemma 6.1.3. *The solutions u^{ln} to (6.1) with vanishing right-hand side satisfy the $(L^m \cap L^q) - L^q$ estimates*

$$\begin{aligned}
\|u^{ln}(t, \cdot)\|_{L^q} &\lesssim (1 + t)^{-\frac{n}{2(\sigma-\delta)}(1 - \frac{1}{r})} \|u_0\|_{L^m \cap H^{s_0}_q} + (1 + t)^{1 - \frac{n}{2(\sigma-\delta)}(1 - \frac{1}{r})} \|u_1\|_{L^m \cap L^q}, \\
\big\||D|^s u^{ln}(t, \cdot)\big\|_{L^q} &\lesssim (1 + t)^{-\frac{n}{2(\sigma-\delta)}(1 - \frac{1}{r}) - \frac{s}{2(\sigma-\delta)}} \|u_0\|_{L^m \cap H^{s+s_0}_q} \\
&\quad + (1 + t)^{1 - \frac{n}{2(\sigma-\delta)}(1 - \frac{1}{r}) - \frac{s}{2(\sigma-\delta)}} \|u_1\|_{L^m \cap H^{s-\sigma+s_0}_q}, \\
\|u^{ln}_t(t, \cdot)\|_{L^q} &\lesssim (1 + t)^{-\frac{n}{2(\sigma-\delta)}(-\frac{1}{r}) - \frac{\delta}{\sigma-\delta}} \|u_0\|_{L^m \cap H^{\sigma+s_0}_q} \\
&\quad + (1 + t)^{1 - \frac{n}{2(\sigma-\delta)}(1 - \frac{1}{r}) - \frac{\delta}{\sigma-\delta}} \|u_1\|_{L^m \cap H^{s_0}_q}, \\
\big\||D|^{s-\sigma} u^{ln}_t(t, \cdot)\big\|_{L^q} &\lesssim (1 + t)^{-\frac{n}{2(\sigma-\delta)}(1 - \frac{1}{r}) - \frac{s-\sigma+2\delta}{2(\sigma-\delta)}} \|u_0\|_{L^m \cap H^{s+s_0}_q} \\
&\quad + (1 + t)^{1 - \frac{n}{2(\sigma-\delta)}(1 - \frac{1}{r}) - \frac{s-\sigma+2\delta}{2(\sigma-\delta)}} \|u_1\|_{L^m \cap H^{s-\sigma+s_0}_q},
\end{aligned}$$

for any $t > 0$.

Proof of Theorem 6.1.3. We introduce the data space $\mathcal{A}^{s+s_0}_{m,q} := (L^m \cap H^{s+s_0}_q) \times (L^m \cap H^{s-\sigma+s_0}_q)$. Moreover, we introduce for any $t > 0$ the function space $X(t) := C\big([0, t], H^s_q\big) \cap C^1\big([0, t], H^{s-\sigma}_q\big)$. For the sake of brevity, we also define the norm

$$\begin{aligned}
\|u\|_{X(t)} := \sup_{0 \leqslant \tau \leqslant t} \Big(&(1 + \tau)^{-1 + \frac{n}{2(\sigma-\delta)}(1 - \frac{1}{r})} \|u(\tau, \cdot)\|_{L^q} + (1 + \tau)^{-1 + \frac{n}{2(\sigma-\delta)}(1 - \frac{1}{r}) + \frac{s}{2(\sigma-\delta)}} \big\||D|^s u(\tau, \cdot)\big\|_{L^q} \\
&+ (1 + \tau)^{-1 + \frac{n}{2(\sigma-\delta)}(1 - \frac{1}{r}) + \frac{\delta}{\sigma-\delta}} \|u_t(\tau, \cdot)\|_{L^q} \\
&+ (1 + \tau)^{-1 + \frac{n}{2(\sigma-\delta)}(1 - \frac{1}{r}) + \frac{s-\sigma+2\delta}{2(\sigma-\delta)}} \big\||D|^{s-\sigma} u_t(\tau, \cdot)\big\|_{L^q} \Big),
\end{aligned}$$

and the space $X_0(t) := C\big([0,t], H^s_q\big)$ with the norm

$$\|w\|_{X_0(t)} := \sup_{0\leqslant\tau\leqslant t} \Big((1+\tau)^{-1+\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})} \|w(\tau,\cdot)\|_{L^q} + (1+\tau)^{-1+\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})+\frac{s}{2(\sigma-\delta)}} \big\||D|^s w(\tau,\cdot)\big\|_{L^q} \Big).$$

We define the operator $N: \quad u \in X(t) \longrightarrow Nu \in X(t)$ by the formula

$$Nu(t,x) = K_0(t,x) *_x u_0(x) + K_1(t,x) *_x u_1(x) + \int_0^t K_1(t-\tau,x) *_x |u(\tau,x)|^p d\tau.$$

We will prove that the operator N satisfies the following two estimates:

$$\|Nu\|_{X(t)} \lesssim \|(u_0,u_1)\|_{\mathcal{A}^{s+s_0}_{m,q}} + \|u\|^p_{X_0(t)}, \tag{6.22}$$

$$\|Nu - Nv\|_{X(t)} \lesssim \|u-v\|_{X_0(t)} \big(\|u\|^{p-1}_{X_0(t)} + \|v\|^{p-1}_{X_0(t)} \big). \tag{6.23}$$

First let us prove the inequality (6.22). Our proof is divided into four steps.

Step 1: We need to estimate the norm $\|Nu(t,\cdot)\|_{L^q}$. We apply the $L^m \cap L^q - L^q$ estimates if $\tau \in [0,[t-1]^+]$ and the $L^q - L^q$ estimates if $\tau \in [[t-1]^+, t]$ from Theorem 3.3.1 to conclude

$$\begin{aligned}\|Nu(t,\cdot)\|_{L^q} &\lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})} \|(u_0,u_1)\|_{\mathcal{A}^{s+s_0}_{m,q}} \\ &\quad + \int_0^{[t-1]^+} (1+t-\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})} \big\||u(\tau,\cdot)|^p\big\|_{L^m\cap L^q} d\tau \\ &\quad + \int_{[t-1]^+}^t (t-\tau)^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)} \big\||u(\tau,\cdot)|^p\big\|_{L^q} d\tau.\end{aligned}$$

We have

$$\big\||u(\tau,\cdot)|^p\big\|_{L^m\cap L^q} = \big\||u(\tau,\cdot)|^p\big\|_{L^m} + \big\||u(\tau,\cdot)|^p\big\|_{L^q} \lesssim \|u(\tau,\cdot)\|^p_{L^{mp}} + \|u(\tau,\cdot)\|^p_{L^{qp}}.$$

To estimate the norm $\|u(\tau,\cdot)\|^p_{L^{kp}}$ with $k = q,m$, we apply the fractional Gagliardo-Nirenberg inequality from Proposition C.1.1 to obtain

$$\|u(\tau,\cdot)\|_{L^{qp}} \lesssim \big\||D|^s u(\tau,\cdot)\big\|^{\theta_{qp}}_{L^q} \|u(\tau,\cdot)\|^{1-\theta_{qp}}_{L^q} \lesssim (1+\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}\theta_{qp}} \|u\|_{X_0(\tau)},$$

and

$$\|u(\tau,\cdot)\|_{L^{mp}} \lesssim \big\||D|^s u(\tau,\cdot)\big\|^{\theta_{mp}}_{L^q} \|u(\tau,\cdot)\|^{1-\theta_{mp}}_{L^q} \lesssim (1+\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}\theta_{mp}} \|u\|_{X_0(\tau)},$$

where

$$\theta_{qp} := \theta_{0,s}(qp,q) = \frac{n}{s}\Big(\frac{1}{q} - \frac{1}{qp}\Big) \text{ and } \theta_{mp} := \theta_{0,s}(mp,q) = \frac{n}{s}\Big(\frac{1}{q} - \frac{1}{mp}\Big).$$

As in Corollary C.1.1 we have to guarantee that θ_{qp} and θ_{mp} belong to $[0,1]$. Both conditions follow the restrictions

$$p \in \Big[\frac{q}{m}, \frac{n}{n-qs}\Big] \text{ if } n > qs, \qquad \text{or} \qquad p \in \Big[\frac{q}{m}, \infty\Big) \text{ if } n \leqslant qs.$$

By virtue of the relation $1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{m}$, we derive

$$\begin{aligned}\big\||u(\tau,\cdot)|^p\big\|_{L^m\cap L^q} &\lesssim (1+\tau)^{p\big(1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}\theta_{mp}\big)} \|u\|^p_{X_0(\tau)} \lesssim (1+\tau)^{p-\frac{n}{2m(\sigma-\delta)}(p-1)} \|u\|^p_{X_0(\tau)}, \\ \big\||u(\tau,\cdot)|^p\big\|_{L^q} &\lesssim (1+\tau)^{p\big(1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}\theta_{qp}\big)} \|u\|^p_{X_0(\tau)} \lesssim (1+\tau)^{p-\frac{np}{2(\sigma-\delta)}(\frac{1}{m}-\frac{1}{qp})} \|u\|^p_{X_0(\tau)}.\end{aligned}$$

From both estimates we may conclude

$$\begin{aligned}\|Nu(t,\cdot)\|_{L^q} &\lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})} \|(u_0,u_1)\|_{\mathcal{A}^{s+s_0}_{m,q}} \\ &\quad + \|u\|^p_{X_0(t)} \int_0^{[t-1]^+} (1+t-\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})} (1+\tau)^{p-\frac{n}{2m(\sigma-\delta)}(p-1)} d\tau \\ &\quad + \|u\|^p_{X_0(t)} \int_{[t-1]^+}^t (t-\tau)^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)} (1+\tau)^{p-\frac{np}{2(\sigma-\delta)}(\frac{1}{m}-\frac{1}{qp})} d\tau.\end{aligned}$$

The key tool relies now in the application of Lemma B.6.1. Because of

$$p > 1 + \frac{\max\big\{n - \frac{m}{q}n + ms,\, 4m(\sigma-\delta)\big\}}{n - 2m(\sigma-\delta)},$$

we obtain

$$p - \frac{n}{2m(\sigma-\delta)}(p-1) < -1,$$

and

$$1 - \frac{n}{2(\sigma-\delta)}\Big(1-\frac{1}{r}\Big) \geqslant p - \frac{n}{2m(\sigma-\delta)}(p-1).$$

Hence, after applying Lemma B.6.1 by choosing

$$\alpha = -1 + \frac{n}{2(\sigma-\delta)}\Big(1-\frac{1}{r}\Big) \text{ and } \beta = -p + \frac{n}{2m(\sigma-\delta)}(p-1)$$

we get

$$\begin{aligned}\int_0^{[t-1]^+} & (1+t-\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})}(1+\tau)^{p-\frac{n}{2m(\sigma-\delta)}(p-1)}d\tau \\ &\lesssim \int_0^t (1+t-\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})}(1+\tau)^{p-\frac{n}{2m(\sigma-\delta)}(p-1)}d\tau \lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})}.\end{aligned}$$

Moreover, since $\big[\frac{n}{2}\big] < n_0$ holds, it follows

$$1 - \Big(1+\Big[\frac{n}{2}\Big]\Big)\Big(\frac{\sigma}{2\delta}-1\Big) > -1.$$

Therefore, we estimate

$$\begin{aligned}\int_{[t-1]^+}^t & (t-\tau)^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}(1+\tau)^{p-\frac{np}{2(\sigma-\delta)}(\frac{1}{m}-\frac{1}{qp})}d\tau \\ &\lesssim (1+t)^{p-\frac{np}{2(\sigma-\delta)}(\frac{1}{m}-\frac{1}{qp})}\int_{[t-1]^+}^t (t-\tau)^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}d\tau \\ &\lesssim (1+t)^{p-\frac{np}{2(\sigma-\delta)}(\frac{1}{m}-\frac{1}{qp})}\int_0^1 r^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}dr \\ &\lesssim (1+t)^{p-\frac{np}{2(\sigma-\delta)}(\frac{1}{m}-\frac{1}{qp})} \lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})},\end{aligned}$$

since

$$p - \frac{np}{2(\sigma-\delta)}\Big(\frac{1}{m}-\frac{1}{qp}\Big) < p - \frac{n}{2m(\sigma-\delta)}(p-1) \leqslant 1 - \frac{n}{2(\sigma-\delta)}\Big(1-\frac{1}{r}\Big).$$

Hence, we arrive at the following estimate:

$$\|Nu(t,\cdot)\|_{L^q} \lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})}\Big(\|(u_0,u_1)\|_{\mathcal{A}^{s+s_0}_{m,q}} + \|u\|^p_{X_0(t)}\Big). \tag{6.24}$$

Step 2: We need to estimate the norm $\|\partial_t Nu(t,\cdot)\|_{L^q}$. Differentiating $Nu(t,x)$ with respect to t we obtain

$$\partial_t Nu(t,x) = \partial_t\big(K_0(t,x)*_x u_0(x) + K_1(t,x)*_x u_1(x)\big) + \int_0^t \partial_t\big(K_1(t-\tau,x)*_x |u(\tau,x)|^p\big)d\tau.$$

We apply the $L^m\cap L^q - L^q$ estimates if $\tau\in[0,[t-1]^+]$ and the L^q-L^q estimates if $\tau\in[[t-1]^+,t]$ from Theorem 3.3.1 to conclude

$$\begin{aligned}\|\partial_t Nu(t,\cdot)\|_{L^q} \lesssim{}& (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}}\|(u_0,u_1)\|_{\mathcal{A}^{s+s_0}_{m,q}} \\ &+ \int_0^{[t-1]^+}(1+t-\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}}\big\||u(\tau,\cdot)|^p\big\|_{L^m\cap L^q}d\tau \\ &+ \int_{[t-1]^+}^t (t-\tau)^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\big\||u(\tau,\cdot)|^p\big\|_{L^q}d\tau.\end{aligned}$$

Using the same ideas for deriving (6.24) we may conclude

$$\|\partial_t Nu(t,\cdot)\|_{L^q} \lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}} \Big(\|(u_0,u_1)\|_{\mathcal{A}^{s+s_0}_{m,q}} + \|u\|^p_{X_0(t)} \Big).$$

Step 3: Let us estimate the norm $\big\||D|^s Nu(t,\cdot)\big\|_{L^q}$. We use

$$|D|^s Nu(t,x) = |D|^s \big(K_0(t,x) *_x u_0(x) + K_1(t,x) *_x u_1(x) \big) + \int_0^t |D|^s \big(K_1(t-\tau,x) *_x |u(\tau,x)|^p \big) d\tau.$$

We apply the $(L^m \cap L^q) - L^q$ estimates if $\tau \in [0,[t-1]^+]$ and the $L^q - L^q$ estimates if $\tau \in [[t-1]^+, t]$ from Theorem 3.3.2 to derive

$$\begin{aligned}
\big\||D|^s Nu(t,\cdot)\big\|_{L^q} &\lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}} \|(u_0,u_1)\|_{\mathcal{A}^{s+s_0}_{m,q}} \\
&\quad + \int_0^{[t-1]^+} (1+t-\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}} \big\||u(\tau,\cdot)|^p\big\|_{L^m \cap H^{s-\sigma}_q} d\tau \\
&\quad + \int_{[t-1]^+}^t (t-\tau)^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)} \big\||u(\tau,\cdot)|^p\big\|_{H^{s-\sigma}_q} d\tau \\
&= (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}} \|(u_0,u_1)\|_{\mathcal{A}^{s+s_0}_{m,q}} \\
&\quad + \int_0^{[t-1]^+} (1+t-\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}} \big\||u(\tau,\cdot)|^p\big\|_{L^m \cap L^q \cap \dot{H}^{s-\sigma}_q} d\tau \\
&\quad + \int_{[t-1]^+}^t (t-\tau)^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)} \big\||u(\tau,\cdot)|^p\big\|_{L^q \cap \dot{H}^{s-\sigma}_q} d\tau.
\end{aligned}$$

The integrals with $\big\||u(\tau,\cdot)|^p\big\|_{L^m \cap L^q}$ and $\big\||u(\tau,\cdot)|^p\big\|_{L^q}$ will be handled as before if we apply the conditions for p and n, that is,

$$p \in \Big[\frac{q}{m}, \frac{n}{n-qs}\Big] \text{ if } n > qs, \quad \text{ or } \quad p \in \Big[\frac{q}{m}, \infty\Big) \text{ if } n \leqslant qs,$$

and

$$p > 1 + \frac{\max\big\{n - \frac{m}{q}n + ms,\, 4m(\sigma-\delta)\big\}}{n - 2m(\sigma-\delta)}, \quad \Big[\frac{n}{2}\Big] < n_0.$$

Hence, we get

$$\begin{aligned}
&\int_0^{[t-1]^+} (1+t-\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}} \big\||u(\tau,\cdot)|^p\big\|_{L^m \cap L^q} d\tau \\
&\qquad \lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}} \|u\|^p_{X_0(t)},
\end{aligned}$$

and

$$\int_{[t-1]^+}^t (t-\tau)^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)} \big\||u(\tau,\cdot)|^p\big\|_{L^q} d\tau \lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}} \|u\|^p_{X_0(t)}.$$

To estimate the integrals with the norm $\big\||u(\tau,\cdot)|^p\big\|_{\dot{H}^{s-\sigma}_q}$, we shall apply Proposition C.3.2 for the fractional chain rule with $p > \lceil s-\sigma \rceil$. Therefore, we obtain

$$\big\||u(\tau,\cdot)|^p\big\|_{\dot{H}^{s-\sigma}_q} \lesssim \|u(\tau,\cdot)\|^{p-1}_{L^{q_1}} \big\||D|^{s-\sigma} u(\tau,\cdot)\big\|_{L^{q_2}}, \text{ where } \frac{1}{q} = \frac{p-1}{q_1} + \frac{1}{q_2}.$$

Applying the fractional Gagliardo-Nirenberg inequality from Proposition C.1.1 we have

$$\|u(\tau,\cdot)\|_{L^{q_1}} \lesssim \|u(\tau,\cdot)\|^{1-\theta_{q_1}}_{L^q} \big\||D|^s u(\tau,\cdot)\big\|^{\theta_{q_1}}_{L^q} \lesssim (1+\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}\theta_{q_1}} \|u\|_{X_0(\tau)},$$

and

$$\big\||D|^{s-\sigma} u(\tau,\cdot)\big\|_{L^{q_2}} \lesssim \|u(\tau,\cdot)\|^{1-\theta_{q_2}}_{L^q} \big\||D|^s u(\tau,\cdot)\big\|^{\theta_{q_2}}_{L^q} \lesssim (1+\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}\theta_{q_2}} \|u\|_{X_0(\tau)},$$

where

$$\theta_{q_1} := \theta_{0,s}(q_1,q) = \frac{n}{s}\Big(\frac{1}{q} - \frac{1}{q_1}\Big) \text{ and } \theta_{q_2} := \theta_{s-\sigma,s}(q_2,q) = \frac{n}{s}\Big(\frac{1}{q} - \frac{1}{q_2} + \frac{s-\sigma}{n}\Big).$$

Hence, we may conclude

$$\begin{aligned}\big\||u(\tau,\cdot)|^p\big\|_{\dot H^{s-\sigma}_q} &\lesssim (1+\tau)^{p-\frac{np}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}\big((p-1)\theta_{q_1}+\theta_{q_2}\big)}\|u\|^p_{X_0(\tau)}\\ &\lesssim (1+\tau)^{p-\frac{np}{2(\sigma-\delta)}(\frac{1}{m}-\frac{1}{qp})-\frac{s-\sigma}{2(\sigma-\delta)}}\|u\|^p_{X_0(\tau)},\end{aligned}$$

where we can see that $(p-1)\theta_{q_1}+\theta_{q_2}=\frac{n}{s}\big(\frac{p-1}{q}+\frac{s-\sigma}{n}\big)$. Here we have to guarantee that $\theta_{q_1}\in[0,1]$ and $\theta_{q_2}\in[\frac{s-\sigma}{s},1]$. Both conditions imply the restrictions

$$1<p\leqslant 1+\frac{q\sigma}{n-qs}\text{ if }n>qs,\quad\text{ or }\quad p>1\text{ if }n\leqslant qs.$$

Therefore, we have shown the estimates

$$\big\||D|^sNu(t,\cdot)\big\|_{L^q}\lesssim(1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}}\Big(\|(u_0,u_1)\|_{\mathcal{A}^{s+s_0}_{m,q}}+\|u\|^p_{X_0(t)}\Big).\tag{6.25}$$

Step 4: Let us estimate the norm $\big\|\partial_t|D|^{s-\sigma}Nu(t,\cdot)\big\|_{L^q}$. We use

$$\begin{aligned}\partial_t|D|^{s-\sigma}Nu(t,x)&=\partial_t|D|^{s-\sigma}\big(K_0(t,x)*_x u_0(x)+K_1(t,x)*_x u_1(x)\big)\\&\quad+\int_0^t\partial_t|D|^{s-\sigma}\big(K_1(t-\tau,x)*_x|u(\tau,x)|^p\big)d\tau.\end{aligned}$$

By applying again the $(L^m\cap L^q)-L^q$ estimates if $\tau\in[0,[t-1]^+]$ and the L^q-L^q estimates if $\tau\in[[t-1]^+,t]$ from Theorem 3.3.2, we derive

$$\begin{aligned}\big\|\partial_t|D|^{s-\sigma}Nu(t,\cdot)\big\|_{L^q}&\lesssim(1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s-\sigma+2\delta}{2(\sigma-\delta)}}\|(u_0,u_1)\|_{\mathcal{A}^{s+s_0}_{m,q}}\\&\quad+\int_0^{[t-1]^+}(1+t-\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s-\sigma+2\delta}{2(\sigma-\delta)}}\big\||u(\tau,\cdot)|^p\big\|_{L^m\cap H^{s-\sigma}_q}d\tau\\&\quad+\int_{[t-1]^+}^t(t-\tau)^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\big\||u(\tau,\cdot)|^p\big\|_{H^{s-\sigma}_q}d\tau.\end{aligned}$$

Following the approach to show (6.25) we may conclude

$$\big\|\partial_t|D|^{s-\sigma}Nu(t,\cdot)\big\|_{L^q}\lesssim(1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s-\sigma+2\delta}{2(\sigma-\delta)}}\Big(\|(u_0,u_1)\|_{\mathcal{A}^{s+s_0}_{m,q}}+\|u\|^p_{X_0(t)}\Big).$$

Summarizing, from the definition of the norm in $X(t)$ we obtain immediately the inequality (6.22).

Next let us prove the inequality (6.23). Our proof is also divided into four steps.

Step 1: We need to estimate the norm $\|Nu(t,\cdot)-Nv(t,\cdot)\|_{L^q}$. We use the $(L^m\cap L^q)-L^q$ estimates if $\tau\in[0,[t-1]^+]$ and the L^q-L^q estimates if $\tau\in[t-1]^+,t]$ from Theorem 3.3.1 to derive for two functions u and v from $X(t)$ the estimate

$$\begin{aligned}\|Nu(t,\cdot)-Nv(t,\cdot)\|_{L^q}&\lesssim\int_0^{[t-1]^+}(1+t-\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})}\big\||u(\tau,\cdot)|^p-|v(\tau,\cdot)|^p\big\|_{L^m\cap L^q}d\tau\\&\quad+\int_{[t-1]^+}^t(t-\tau)^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\big\||u(\tau,\cdot)|^p-|v(\tau,\cdot)|^p\big\|_{L^q}d\tau.\end{aligned}$$

By using Hölder's inequality and applying again the same ideas as we did in the proof of (6.9) and Step 1 to prove (6.22) we may conclude

$$\|Nu(t,\cdot)-Nv(t,\cdot)\|_{L^q}\lesssim(1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})}\|u-v\|_{X_0(t)}\big(\|u\|^{p-1}_{X_0(t)}+\|v\|^{p-1}_{X_0(t)}\big).$$

Step 2: We need to estimate the norm $\|\partial_tNu(t,\cdot)-\partial_tNv(t,\cdot)\|_{L^q}$. We use

$$\begin{aligned}\|\partial_tNu(t,\cdot)-\partial_tNv(t,\cdot)\|_{L^q}&\lesssim\int_0^{[t-1]^+}(1+t-\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}}\big\||u(\tau,\cdot)|^p-|v(\tau,\cdot)|^p\big\|_{L^m\cap L^q}d\tau\\&\quad+\int_{[t-1]^+}^t(t-\tau)^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\big\||u(\tau,\cdot)|^p-|v(\tau,\cdot)|^p\big\|_{L^q}d\tau.\end{aligned}$$

Using the same approach of (6.9) and Step 2 to prove (6.22) we conclude

$$\|\partial_t Nu(t,\cdot) - \partial_t Nv(t,\cdot)\|_{L^q} \lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}} \|u-v\|_{X_0(t)} \big(\|u\|^{p-1}_{X_0(t)} + \|v\|^{p-1}_{X_0(t)}\big).$$

Step 3: Let us estimate the norm $\big\||D|^s Nu(t,\cdot) - |D|^s Nv(t,\cdot)\big\|_{L^q}$. We use

$$\begin{aligned}
&\big\||D|^s Nu(t,\cdot) - |D|^s Nv(t,\cdot)\big\|_{L^q} \\
&\lesssim \int_0^{[t-1]^+} (1+t-\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}} \big\||u(\tau,\cdot)|^p - |v(\tau,\cdot)|^p\big\|_{L^m \cap H^{s-\sigma}_q} d\tau \\
&\quad + \int_{[t-1]^+}^t (t-\tau)^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)} \big\||u(\tau,\cdot)|^p - |v(\tau,\cdot)|^p\big\|_{H^{s-\sigma}_q} d\tau \\
&= \int_0^{[t-1]^+} (1+t-\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}} \big\||u(\tau,\cdot)|^p - |v(\tau,\cdot)|^p\big\|_{L^m \cap L^q \cap \dot{H}^{s-\sigma}_q} d\tau \\
&\quad + \int_{[t-1]^+}^t (t-\tau)^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)} \big\||u(\tau,\cdot)|^p - |v(\tau,\cdot)|^p\big\|_{L^q \cap \dot{H}^{s-\sigma}_q} d\tau.
\end{aligned}$$

The integrals with $\big\||u(\tau,\cdot)|^p - |v(\tau,\cdot)|^p\big\|_{L^m \cap L^q}$ and $\big\||u(\tau,\cdot)|^p - |v(\tau,\cdot)|^p\big\|_{L^q}$ will be handled as we did in the proof of (6.9) and Step 3 to prove (6.22). Hence, we get

$$\begin{aligned}
&\int_0^{[t-1]^+} (1+t-\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}} \big\||u(\tau,\cdot)|^p - |v(\tau,\cdot)|^p\big\|_{L^m \cap L^q} d\tau \\
&\qquad \lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}} \|u-v\|_{X_0(t)} \big(\|u\|^{p-1}_{X_0(t)} + \|v\|^{p-1}_{X_0(t)}\big),
\end{aligned}$$

and

$$\begin{aligned}
&\int_{[t-1]^+}^t (t-\tau)^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)} \big\||u(\tau,\cdot)|^p - |v(\tau,\cdot)|^p\big\|_{L^q} d\tau \\
&\qquad \lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}} \|u-v\|_{X_0(t)} \big(\|u\|^{p-1}_{X_0(t)} + \|v\|^{p-1}_{X_0(t)}\big).
\end{aligned}$$

Let us now turn to estimate the norm $\big\||u(\tau,\cdot)|^p - |v(\tau,\cdot)|^p\big\|_{\dot{H}^{s-\sigma}_q}$. By using the integral representation

$$|u(\tau,x)|^p - |v(\tau,x)|^p = p\int_0^1 \big(u(\tau,x) - v(\tau,x)\big) G\big(\omega u(\tau,x) + (1-\omega)v(\tau,x)\big) d\omega,$$

where $G(u) = u|u|^{p-2}$, we obtain

$$\big\||u(\tau,\cdot)|^p - |v(\tau,\cdot)|^p\big\|_{\dot{H}^{s-\sigma}_q} \lesssim \int_0^1 \Big\||D|^{s-\sigma}\Big(\big(u(\tau,\cdot) - v(\tau,\cdot)\big) G\big(\omega u(\tau,\cdot) + (1-\omega)v(\tau,\cdot)\big)\Big)\Big\|_{L^q} d\omega.$$

Thanks to the fractional Leibniz formula from Proposition C.2.1, we may estimate a product in $\dot{H}^{s-\sigma}_q$ as follows:

$$\begin{aligned}
\big\||u(\tau,\cdot)|^p - |v(\tau,\cdot)|^p\big\|_{\dot{H}^{s-\sigma}_q} &\lesssim \int_0^1 \big\||D|^{s-\sigma}\big(u(\tau,\cdot) - v(\tau,\cdot)\big)\big\|_{L^{r_1}} \big\|G\big(\omega u(\tau,\cdot) + (1-\omega)v(\tau,\cdot)\big)\big\|_{L^{r_2}} d\omega \\
&\quad + \int_0^1 \|u(\tau,\cdot) - v(\tau,\cdot)\|_{L^{r_3}} \big\||D|^{s-\sigma} G\big(\omega u(\tau,\cdot) + (1-\omega)v(\tau,\cdot)\big)\big\|_{L^{r_4}} d\omega \\
&\lesssim \big\||D|^{s-\sigma}\big(u(\tau,\cdot) - v(\tau,\cdot)\big)\big\|_{L^{r_1}} \int_0^1 \big\|G\big(\omega u(\tau,\cdot) + (1-\omega)v(\tau,\cdot)\big)\big\|_{L^{r_2}} d\omega \\
&\quad + \|u(\tau,\cdot) - v(\tau,\cdot)\|_{L^{r_3}} \int_0^1 \big\||D|^{s-\sigma} G\big(\omega u(\tau,\cdot) + (1-\omega)v(\tau,\cdot)\big)\big\|_{L^{r_4}} d\omega \\
&\lesssim \big\||D|^{s-\sigma}\big(u(\tau,\cdot) - v(\tau,\cdot)\big)\big\|_{L^{r_1}} \big(\|u(\tau,\cdot)\|^{p-1}_{L^{r_2(p-1)}} + \|v(\tau,\cdot)\|^{p-1}_{L^{r_2(p-1)}}\big) \\
&\quad + \|u(\tau,\cdot) - v(\tau,\cdot)\|_{L^{r_3}} \int_0^1 \big\||D|^{s-\sigma} G\big(\omega u(\tau,\cdot) + (1-\omega)v(\tau,\cdot)\big)\big\|_{L^{r_4}} d\omega,
\end{aligned}$$

where
$$\frac{1}{q}=\frac{1}{r_1}+\frac{1}{r_2}=\frac{1}{r_3}+\frac{1}{r_4}.$$
Employing the fractional Gargliardo-Nirenberg inequality from Proposition C.1.1 implies
$$\big\||D|^{s-\sigma}\big(u(\tau,\cdot)-v(\tau,\cdot)\big)\big\|_{L^{r_1}}\lesssim\|u(\tau,\cdot)-v(\tau,\cdot)\|_{\dot H^s_q}^{\theta_1}\|u(\tau,\cdot)-v(\tau,\cdot)\|_{L^q}^{1-\theta_1},$$
$$\|u(\tau,\cdot)\|_{L^{r_2(p-1)}}\lesssim\|u(\tau,\cdot)\|_{\dot H^s_q}^{\theta_2}\|u(\tau,\cdot)\|_{L^q}^{1-\theta_2},$$
$$\|v(\tau,\cdot)\|_{L^{r_2(p-1)}}\lesssim\|v(\tau,\cdot)\|_{\dot H^s_q}^{\theta_2}\|v(\tau,\cdot)\|_{L^q}^{1-\theta_2},$$
$$\|u(\tau,\cdot)-v(\tau,\cdot)\|_{L^{r_3}}\lesssim\|u(\tau,\cdot)-v(\tau,\cdot)\|_{\dot H^s_q}^{\theta_3}\|u(\tau,\cdot)-v(\tau,\cdot)\|_{L^q}^{1-\theta_3},$$
where
$$\theta_1:=\theta_{s-\sigma,s}(r_1,q)=\frac{n}{s}\Big(\frac{1}{q}-\frac{1}{r_1}+\frac{s-\sigma}{n}\Big)\in\Big[\frac{s-\sigma}{s},1\Big],$$
$$\theta_2:=\theta_{0,s}(r_2(p-1),q)=\frac{n}{s}\Big(\frac{1}{q}-\frac{1}{r_2(p-1)}\Big)\in[0,1]\ \text{ and }\ \theta_3:=\theta_{0,s}(r_3,q)=\frac{n}{s}\Big(\frac{1}{q}-\frac{1}{r_3}\Big)\in[0,1].$$
Moreover, since $\omega\in[0,1]$ is a parameter, we may apply again the fractional chain rule with $p>1+\lceil s-\sigma\rceil$ from Proposition C.3.2 and the fractional Gagliardo-Nirenberg inequality from Proposition C.1.1 to conclude
$$\begin{aligned}&\big\||D|^{s-\sigma}G\big(\omega u(\tau,\cdot)+(1-\omega)v(\tau,\cdot)\big)\big\|_{L^{r_4}}\\&\quad\lesssim\|\omega u(\tau,\cdot)+(1-\omega)v(\tau,\cdot)\|_{L^{r_5}}^{p-2}\big\||D|^{s-\sigma}(\omega u(\tau,\cdot)+(1-\omega)v(\tau,\cdot))\big\|_{L^{r_6}}\\&\quad\lesssim\|\omega u(\tau,\cdot)+(1-\omega)v(\tau,\cdot)\|_{\dot H^s_q}^{(p-2)\theta_5+\theta_6}\|\omega u(\tau,\cdot)+(1-\omega)v(\tau,\cdot)\|_{L^q}^{(p-2)(1-\theta_5)+1-\theta_6},\end{aligned}$$
where
$$\frac{1}{r_4}=\frac{p-2}{r_5}+\frac{1}{r_6},\quad\theta_5:=\theta_{0,s}(r_5,q)=\frac{n}{s}\Big(\frac{1}{q}-\frac{1}{r_5}\Big)\in[0,1],$$
$$\text{and }\theta_6:=\theta_{s-\sigma,s}(r_1,q)=\frac{n}{s}\Big(\frac{1}{q}-\frac{1}{r_6}+\frac{s-\sigma}{n}\Big)\in\Big[\frac{s-\sigma}{s},1\Big].$$
Hence, we derive
$$\begin{aligned}&\int_0^1\big\||D|^{s-\sigma}G\big(\omega u(\tau,\cdot)+(1-\omega)v(\tau,\cdot)\big)\big\|_{L^{r_4}}d\omega\\&\quad\lesssim\big(\|u(\tau,\cdot)\|_{\dot H^{s,q}}+\|v(\tau,\cdot)\|_{\dot H^{s,q}}\big)^{(p-2)\theta_5+\theta_6}\big(\|u(\tau,\cdot)\|_{L^q}+\|v(\tau,\cdot)\|_{L^q}\big)^{(p-2)(1-\theta_5)+1-\theta_6}.\end{aligned}$$
Therefore, we conclude
$$\begin{aligned}&\big\||u(\tau,\cdot)|^p-|v(\tau,\cdot)|^p\big\|_{\dot H^{s-\sigma}_q}\\&\quad\lesssim(1+\tau)^{p-\frac{np}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{n}{2(\sigma-\delta)}(\frac{p-1}{q}+\frac{s-\sigma}{n})}\|u-v\|_{X_0(\tau)}\big(\|u\|_{X_0(\tau)}^{p-1}+\|v\|_{X_0(\tau)}^{p-1}\big)\\&\quad\lesssim(1+\tau)^{p-\frac{np}{2(\sigma-\delta)}(\frac{1}{m}-\frac{1}{qp})-\frac{s-\sigma}{2(\sigma-\delta)}}\|u-v\|_{X_0(\tau)}\big(\|u\|_{X_0(\tau)}^{p-1}+\|v\|_{X_0(\tau)}^{p-1}\big),\end{aligned}$$
where we note that
$$\theta_1+(p-1)\theta_2=\theta_3+(p-2)\theta_5+\theta_6=\frac{n}{s}\Big(\frac{p-1}{q}+\frac{s-\sigma}{n}\Big).$$
Summarizing, we have proved the estimates
$$\big\||D|^sNu(t,\cdot)-|D|^sNv(t,\cdot)\big\|_{L^q}\lesssim(1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}}\|u-v\|_{X_0(t)}\big(\|u\|_{X_0(t)}^{p-1}+\|v\|_{X_0(t)}^{p-1}\big).$$
Step 4: Let us estimate the norm $\big\|\partial_t|D|^{s-\sigma}Nu(t,\cdot)-\partial_t|D|^{s-\sigma}Nv(t,\cdot)\big\|_{L^q}$. We use
$$\begin{aligned}&\big\|\partial_t|D|^{s-\sigma}Nu(t,\cdot)-\partial_t|D|^{s-\sigma}Nv(t,\cdot)\big\|_{L^q}\\&\quad\lesssim\int_0^{[t-1]^+}(1+t-\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s-\sigma+2\delta}{2(\sigma-\delta)}}\big\||u(\tau,\cdot)|^p-v(\tau,\cdot)|^p\big\|_{L^m\cap H^{s-\sigma}_q}d\tau\\&\qquad+\int_{[t-1]^+}^t(t-\tau)^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\big\||u(\tau,\cdot)|^p-|v(\tau,\cdot)|^p\big\|_{H^{s-\sigma}_q}d\tau.\end{aligned}$$

By the same treatment as in Step 3 to prove (6.23) we may conclude

$$\begin{aligned}&\big\|\partial_t|D|^{s-\sigma}Nu(t,\cdot)-\partial_t|D|^{s-\sigma}Nv(t,\cdot)\big\|_{L^q}\\ &\lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s-\sigma+2\delta}{2(\sigma-\delta)}}\|u-v\|_{X_0(t)}\big(\|u\|^{p-1}_{X_0(t)}+\|v\|^{p-1}_{X_0(t)}\big).\end{aligned}$$

Summarizing, from the definition of the norm in $X(t)$ and all the previous estimates we have completed the proof of (6.23). □

Remark 6.1.5. Explaining the possibility to choose the suitable parameters q_1, q_2, $r_1, \cdots, r_6$ and $\theta_1, \cdots, \theta_6$ appearing in the proof to Theorem 6.1.3 is the same as that in Remark 5.1.4. Following the explanations as we did in Remark 5.1.4 we may conclude the following conditions:

$$2 \leqslant p \leqslant 1+\frac{q\sigma}{n-qs} \text{ if } n > qs, \quad \text{ or } \quad p \geqslant 2 \text{ if } n \leqslant qs,$$

which are sufficient to guarantee the existence of all these parameters satisfying the required conditions.

Remark 6.1.6. If we assume the following condition for the space dimension:

$$n \leqslant n_2 := \frac{mq(4\sigma-4\delta-s)}{q-m},$$

then it follows $4m(\sigma-\delta) \geqslant n-\frac{m}{q}n+ms$. This inequality allows us to avoid the restriction of $p > 1+\frac{n-\frac{m}{q}n+ms}{n-2m(\sigma-\delta)}$ in Theorem 6.1.3. Hence, we only need to guarantee the following restriction:

$$p > 1+\frac{4m(\sigma-\delta)}{n-2m(\sigma-\delta)}.$$

Example 6.1.4. By choosing $m=1$, $q=2$, $\sigma=2$, $\delta=0.9$ and $s=2.5$ we obtain the following admissible range of exponents p:

n	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
p	$p \in (6.5,\infty)$	$p \in (3.5,\infty)$	$p \in (2.79,\infty)$	$p \in (2.45,5]$	$p \in (2.25,3]$	$p \in (2.12,2.33]$

Tab. 6.3.: The admissible range of exponents p depends on the space dimension n.

6.1.4. Large regular solutions to the model (6.1)

Finally, we obtain large regular solutions to (6.1) by using the fractional powers rule and the fractional Sobolev embedding.

Theorem 6.1.4. *Let $s > \sigma+\frac{n}{q}$. Let $q \in (1,\infty)$ be a fixed constant and $m \in [1,q)$. We assume that the exponent p satisfies the conditions $p > 1+s-\sigma$ and*

$$p > 1+\frac{\max\big\{n-\frac{m}{q}n+ms,\, 4m(\sigma-\delta)\big\}}{n-2m(\sigma-\delta)}, \tag{6.26}$$

where $\big[\frac{n}{2}\big] < n_0$. Moreover, we suppose the following conditions:

$$p \in \Big[\frac{q}{m},\infty\Big) \text{ and } n > 2m(\sigma-\delta). \tag{6.27}$$

Then, there exists a constant $\varepsilon > 0$ such that for any small data

$$(u_0,u_1) \in \mathcal{A}^{s+s_0}_{m,q} \text{ satisfying the assumption } \|(u_0,u_1)\|_{\mathcal{A}^{s+s_0}_{m,q}} \leqslant \varepsilon,$$

we have a uniquely determined global (in time) small data energy solution

$$u \in C\big([0,\infty),H^s_q\big) \cap C^1\big([0,\infty),H^{s-\sigma}_q\big)$$

to (6.1). Moreover, the estimates (6.18) to (6.21) hold.

Proof. We introduce the definitions of spaces $\mathcal{A}^{s+s_0}_{m,q}$, $X(t)$ and $X_0(t)$ as in the proof of Theorem 6.1.3. We repeat exactly on the one hand the same estimates for the terms $|u(\tau,\cdot)|^p$ and $|u(\tau,\cdot)|^p-|v(\tau,\cdot)|^p$ in L^m and L^q. On the other hand, we estimate the above terms in $\dot{H}^{s-\sigma}_q$ by using the fractional powers rule and the fractional Sobolev embedding.

In the first step, let us begin with $\big\||u(\tau,\cdot)|^p\big\|_{\dot{H}^{s-\sigma}_q}$. We shall apply Corollary C.4.1 for the fractional powers rule with $s-\sigma\in(\frac{n}{q},p)$. Therefore, we obtain

$$\begin{aligned}\big\||u(\tau,\cdot)|^p\big\|_{\dot{H}^{s-\sigma}_q} &\lesssim \|u(\tau,\cdot)\|_{\dot{H}^{s-\sigma}_q}\|u(\tau,\cdot)\|^{p-1}_{L^\infty}\\ &\lesssim \|u(\tau,\cdot)\|_{\dot{H}^{s-\sigma}_q}\big(\|u(\tau,\cdot)\|_{\dot{H}^{s^*}_q}+\|u(\tau,\cdot)\|_{\dot{H}^{s-\sigma}_q}\big)^{p-1}.\end{aligned}$$

Here we used Corollary C.5.1 with a suitable $s^*<\frac{n}{q}$. Applying the fractional Gagliardo-Nirenberg inequality from Proposition C.1.1 we have

$$\begin{aligned}\|u(\tau,\cdot)\|_{\dot{H}^{s-\sigma}_q} &\lesssim \|u(\tau,\cdot)\|^{1-\theta_1}_{L^q}\,\||D|^s u(\tau,\cdot)\|^{\theta_1}_{L^q}\lesssim (1+\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s-\sigma}{2(\sigma-\delta)}}\|u\|_{X_0(\tau)},\\ \|u(\tau,\cdot)\|_{\dot{H}^{s^*}_q} &\lesssim \|u(\tau,\cdot)\|^{1-\theta_2}_{L^q}\,\||D|^s u(\tau,\cdot)\|^{\theta_2}_{L^q}\lesssim (1+\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s^*}{2(\sigma-\delta)}}\|u\|_{X_0(\tau)},\end{aligned}$$

where $\theta_1=1-\frac{\sigma}{s}$ and $\theta_2=\frac{s^*}{s}$. Hence, we derive

$$\begin{aligned}\big\||u(\tau,\cdot)|^p\big\|_{\dot{H}^{s-\sigma}_q} &\lesssim (1+\tau)^{p\left(1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})\right)-\frac{s-\sigma}{2(\sigma-\delta)}-(p-1)\frac{s^*}{2(\sigma-\delta)}}\|u\|_{X_0(\tau)}\\ &\lesssim (1+\tau)^{p-\frac{n}{2m(\sigma-\delta)}(p-1)}\|u\|^p_{X_0(\tau)},\end{aligned}$$

if we choose $s^*=\frac{n}{q}-\varepsilon_0$ with a sufficiently small $\varepsilon_0>0$. Therefore, by an analogous argument as we did in the proof of Theorem 6.1.3 we may conclude

$$\int_0^{[t-1]^+}(1+t-\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}}\big\||u(\tau,\cdot)|^p\big\|_{\dot{H}^{s-\sigma}_q}d\tau\lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}}\|u\|^p_{X_0(t)},$$

and

$$\int_{[t-1]^+}^t (t-\tau)^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\big\||u(\tau,\cdot)|^p\big\|_{\dot{H}^{s-\sigma}_q}d\tau\lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}}\|u\|^p_{X_0(t)}.$$

Finally, let us turn to estimate the norm $\big\||u(\tau,\cdot)|^p-|v(\tau,\cdot)|^p\big\|_{\dot{H}^{s-\sigma}_q}$. Then, repeating the proof of the second step of Theorem 5.1.4 and using the same treatment as in the proof of the above first step we get

$$\big\||u(\tau,\cdot)|^p-|v(\tau,\cdot)|^p\big\|_{\dot{H}^{s-\sigma}_q}\lesssim (1+\tau)^{p-\frac{n}{2m(\sigma-\delta)}(p-1)}\|u-v\|_{X_0(\tau)}\big(\|u\|^{p-1}_{X_0(\tau)}+\|v\|^{p-1}_{X_0(\tau)}\big).$$

Hence, we arrive at

$$\begin{aligned}&\int_0^{[t-1]^+}(1+t-\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}}\big\||u(\tau,\cdot)|^p-|v(\tau,\cdot)|^p\big\|_{\dot{H}^{s-\sigma,q}}d\tau\\ &\qquad\lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}}\|u-v\|_{X_0(t)}\big(\|u\|^{p-1}_{X_0(t)}+\|v\|^{p-1}_{X_0(t)}\big),\end{aligned}$$

and

$$\begin{aligned}&\int_{[t-1]^+}^t (t-\tau)^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\big\||u(\tau,\cdot)|^p-|v(\tau,\cdot)|^p\big\|_{\dot{H}^{s-\sigma,q}}d\tau\\ &\qquad\lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}}\|u-v\|_{X_0(t)}\big(\|u\|^{p-1}_{X_0(t)}+\|v\|^{p-1}_{X_0(t)}\big).\end{aligned}$$

Summarizing, the proof of Theorem 6.1.4 is completed. □

Example 6.1.5. By choosing $m=1$, $q=2$, $\sigma=2$, $\delta=0.9$ and $s=5$ we obtain the following admissible range of exponents p:

n	$n = 3$	$n = 4$	$n = 5$
p	$p \in (9.125, \infty)$	$p \in (4.889, \infty)$	$p \in (4, \infty)$

Tab. 6.4.: The admissible range of exponents p depends on the space dimension n.

6.1.5. Large regular solution to the model (6.2)

In this section, we obtain large regular solutions to (6.2) by using the fractional powers rule and the fractional Sobolev embedding.

Theorem 6.1.5. *Let $s > \sigma + \frac{n}{q}$. Let $q \in (1, \infty)$ be a fixed constant and $m \in [1, q)$. We assume that the exponent p satisfies the conditions $p > 1 + s - \sigma$ and*

$$p > 1 + \frac{\max\big\{n - \frac{m}{q}n + m(s - 2\delta),\, 2m(2\sigma - 3\delta)\big\}}{n - 2m(\sigma - 2\delta)}, \tag{6.28}$$

where $\big[\frac{n}{2}\big] < n_0$. Moreover, we suppose the following conditions:

$$p \in \Big[\frac{q}{m}, \infty\Big) \text{ and } n > 2m(\sigma - 2\delta). \tag{6.29}$$

Then, there exists a constant $\varepsilon > 0$ such that for any small data

$$(u_0, u_1) \in \mathcal{A}^{s+s_0}_{m,q} \text{ satisfying the assumption } \|(u_0, u_1)\|_{\mathcal{A}^{s+s_0}_{m,q}} \leqslant \varepsilon,$$

we have a uniquely determined global (in time) small data energy solution

$$u \in C\big([0, \infty), H^s_q\big) \cap C^1\big([0, \infty), H^{s-\sigma}_q\big)$$

to (6.2). Moreover, the estimates (6.18) to (6.21) hold.

Proof. We introduce the definitions of spaces $\mathcal{A}^{s+s_0}_{m,q}$ and $X(t)$ as in the proof of Theorem 6.1.3. We define the operator $N : \; X(t) \longrightarrow X(t)$ by the formula

$$Nu(t, x) = K_0(t, x) \ast_x u_0(x) + K_1(t, x) \ast_x u_1(x) + \int_0^t K_1(t - \tau, x) \ast_x |u_t(\tau, x)|^p d\tau.$$

We will prove that the operator N satisfies the following two estimates:

$$\|Nu\|_{X(t)} \lesssim \|(u_0, u_1)\|_{\mathcal{A}^{s+s_0}_{m,q}} + \|u\|^p_{X(t)}, \tag{6.30}$$

$$\|Nu - Nv\|_{X(t)} \lesssim \|u - v\|_{X(t)}\big(\|u\|^{p-1}_{X(t)} + \|v\|^{p-1}_{X(t)}\big). \tag{6.31}$$

First let us prove the inequality (6.30). Our proof is divided into four steps.

Step 1: We need to estimate the norm $\|Nu(t, \cdot)\|_{L^q}$. We apply the $L^m \cap L^q - L^q$ estimates if $\tau \in [0, [t-1]^+]$ and the $L^q - L^q$ estimates if $\tau \in [[t-1]^+, t]$ from Theorem 3.3.1 to conclude

$$\begin{aligned}\|Nu(t, \cdot)\|_{L^q} &\lesssim (1 + t)^{1 - \frac{n}{2(\sigma - \delta)}(1 - \frac{1}{r})}\|(u_0, u_1)\|_{\mathcal{A}^{s+s_0}_{m,q}} \\ &\quad + \int_0^{[t-1]^+} (1 + t - \tau)^{1 - \frac{n}{2(\sigma - \delta)}(1 - \frac{1}{r})}\big\||u_t(\tau, \cdot)|^p\big\|_{L^m \cap L^q} d\tau \\ &\quad + \int_{[t-1]^+}^t (t - \tau)^{1 - (1 + [\frac{n}{2}])(\frac{\sigma}{2\delta} - 1)}\big\||u_t(\tau, \cdot)|^p\big\|_{L^q} d\tau.\end{aligned}$$

We have

$$\big\||u_t(\tau, \cdot)|^p\big\|_{L^m \cap L^q} = \big\||u_t(\tau, \cdot)|^p\big\|_{L^m} + \big\||u_t(\tau, \cdot)|^p\big\|_{L^q} \lesssim \|u_t(\tau, \cdot)\|^p_{L^{mp}} + \|u_t(\tau, \cdot)\|^p_{L^{qp}}.$$

To estimate the norm $\|u_t(\tau, \cdot)\|^p_{L^{kp}}$ with $k = q, m$, we apply the fractional Gagliardo-Nirenberg inequality from Proposition C.1.1 to obtain

$$\|u_t(\tau, \cdot)\|_{L^{qp}} \lesssim \big\||D|^{s-\sigma}u_t(\tau, \cdot)\big\|^{\theta_{qp}}_{L^q} \|u_t(\tau, \cdot)\|^{1-\theta_{qp}}_{L^q} \lesssim (1 + \tau)^{1 - \frac{n}{2(\sigma - \delta)}(1 - \frac{1}{r}) - \frac{\delta}{\sigma - \delta} - \frac{s - \sigma}{2(\sigma - \delta)}\theta_{qp}}\|u\|_{X(\tau)},$$

and

$$\|u_t(\tau,\cdot)\|_{L^{mp}} \lesssim \big\||D|^{s-\sigma}u_t(\tau,\cdot)\big\|_{L^q}^{\theta_{mp}} \|u_t(\tau,\cdot)\|_{L^q}^{1-\theta_{mp}} \lesssim (1+\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}-\frac{s-\sigma}{2(\sigma-\delta)}\theta_{mp}}\|u\|_{X(\tau)},$$

where

$$\theta_{qp} := \theta_{0,s-\sigma}(qp,q) = \frac{n}{s-\sigma}\Big(\frac{1}{q}-\frac{1}{qp}\Big) \text{ and } \theta_{mp} := \theta_{0,s-\sigma}(mp,q) = \frac{n}{s-\sigma}\Big(\frac{1}{q}-\frac{1}{mp}\Big).$$

As in Corollary C.1.1 we have to guarantee that θ_{qp} and θ_{mp} belong to $[0,1]$. Both conditions imply the restrictions

$$p \in \Big[\frac{q}{m}, \frac{n}{n-q(s-\sigma)}\Big] \text{ if } s<\sigma+\frac{n}{q}, \qquad \text{or} \qquad p\in\Big[\frac{q}{m},\infty\Big) \text{ if } s\geqslant \sigma+\frac{n}{q}.$$

By virtue of the relation $1+\frac{1}{q}=\frac{1}{r}+\frac{1}{m}$, we derive

$$\begin{aligned}
\big\||u_t(\tau,\cdot)|^p\big\|_{L^m\cap L^q} &\lesssim (1+\tau)^{p\left(1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}-\frac{s-\sigma}{2(\sigma-\delta)}\theta_{mp}\right)}\|u\|^p_{X(\tau)}\\
&\lesssim (1+\tau)^{p\left(1-\frac{n}{2(\sigma-\delta)}(\frac{1}{m}-\frac{1}{mp})-\frac{\delta}{\sigma-\delta}\right)}\|u\|^p_{X(\tau)},\\
\big\||u_t(\tau,\cdot)|^p\big\|_{L^q} &\lesssim (1+\tau)^{p\left(1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}-\frac{s-\sigma}{2(\sigma-\delta)}\theta_{qp}\right)}\|u\|^p_{X(\tau)}\\
&\lesssim (1+\tau)^{p\left(1-\frac{n}{2(\sigma-\delta)}(\frac{1}{m}-\frac{1}{qp})-\frac{\delta}{\sigma-\delta}\right)}\|u\|^p_{X(\tau)}.
\end{aligned}$$

From both estimates we may conclude

$$\begin{aligned}
\|Nu(t,\cdot)\|_{L^q} &\lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})}\|(u_0,u_1)\|_{\mathcal{A}^{s+s_0}_{m,q}}\\
&\quad+\|u\|^p_{X(t)}\int_0^{[t-1]^+}(1+t-\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})}(1+\tau)^{p\left(1-\frac{n}{2(\sigma-\delta)}(\frac{1}{m}-\frac{1}{mp})-\frac{\delta}{\sigma-\delta}\right)}d\tau\\
&\quad+\|u\|^p_{X(t)}\int_{[t-1]^+}^t(t-\tau)^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}(1+\tau)^{p\left(1-\frac{n}{2(\sigma-\delta)}(\frac{1}{m}-\frac{1}{qp})-\frac{\delta}{\sigma-\delta}\right)}d\tau.
\end{aligned}$$

The key tool relies now in the application of Lemma B.6.1. Because of

$$p>1+\frac{\max\big\{n-\frac{m}{q}n+m(s-2\delta),\,2m(2\sigma-3\delta)\big\}}{n-2m(\sigma-2\delta)},$$

we obtain

$$p\Big(1-\frac{n}{2(\sigma-\delta)}\Big(\frac{1}{m}-\frac{1}{mp}\Big)-\frac{\delta}{\sigma-\delta}\Big)<-1,$$

and

$$1-\frac{n}{2(\sigma-\delta)}\Big(1-\frac{1}{r}\Big)\geqslant p\Big(1-\frac{n}{2(\sigma-\delta)}\Big(\frac{1}{m}-\frac{1}{mp}\Big)-\frac{\delta}{\sigma-\delta}\Big).$$

Hence, after applying Lemma B.6.1 by choosing

$$\alpha=-1+\frac{n}{2(\sigma-\delta)}\Big(1-\frac{1}{r}\Big) \text{ and } \beta=p\Big(-1+\frac{n}{2(\sigma-\delta)}\Big(\frac{1}{m}-\frac{1}{mp}\Big)+\frac{\delta}{\sigma-\delta}\Big)$$

we get

$$\begin{aligned}
&\int_0^{[t-1]^+}(1+t-\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})}(1+\tau)^{p\left(1-\frac{n}{2(\sigma-\delta)}(\frac{1}{m}-\frac{1}{mp})-\frac{\delta}{\sigma-\delta}\right)}d\tau\\
&\quad\lesssim\int_0^t(1+t-\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})}(1+\tau)^{p\left(1-\frac{n}{2(\sigma-\delta)}(\frac{1}{m}-\frac{1}{mp})-\frac{\delta}{\sigma-\delta}\right)}d\tau\lesssim(1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})}.
\end{aligned}$$

Moreover, since $\big[\frac{n}{2}\big]<n_0$ holds, it follows

$$1-\Big(1+\Big[\frac{n}{2}\Big]\Big)\Big(\frac{\sigma}{2\delta}-1\Big)>-1.$$

Therefore, we can estimate

$$\int_{[t-1]^+}^t (t-\tau)^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}(1+\tau)^{p\left(1-\frac{n}{2(\sigma-\delta)}(\frac{1}{m}-\frac{1}{qp})-\frac{\delta}{\sigma-\delta}\right)}d\tau$$
$$\lesssim (1+t)^{p\left(1-\frac{n}{2(\sigma-\delta)}(\frac{1}{m}-\frac{1}{qp})-\frac{\delta}{\sigma-\delta}\right)}\int_{[t-1]^+}^t (t-\tau)^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}d\tau$$
$$\lesssim (1+t)^{p\left(1-\frac{n}{2(\sigma-\delta)}(\frac{1}{m}-\frac{1}{qp})-\frac{\delta}{\sigma-\delta}\right)}\int_0^1 r^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}dr$$
$$\lesssim (1+t)^{p\left(1-\frac{n}{2(\sigma-\delta)}(\frac{1}{m}-\frac{1}{qp})-\frac{\delta}{\sigma-\delta}\right)} \lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})},$$

since

$$p\Big(1-\frac{n}{2(\sigma-\delta)}\Big(\frac{1}{m}-\frac{1}{qp}\Big)-\frac{\delta}{\sigma-\delta}\Big) < 1-\frac{n}{2(\sigma-\delta)}\Big(1-\frac{1}{r}\Big).$$

Hence, we arrive at the following estimate:

$$\|Nu(t,\cdot)\|_{L^q} \lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})}\Big(\|(u_0,u_1)\|_{\mathcal{A}^{s+s_0}_{m,q}} + \|u\|^p_{X(t)}\Big). \tag{6.32}$$

Step 2: We need to estimate the norm $\|\partial_t Nu(t,\cdot)\|_{L^q}$. Differentiating $Nu(t,x)$ with respect to t we obtain

$$\partial_t Nu(t,x) = \partial_t\big(K_0(t,x) *_x u_0(x) + K_1(t,x) *_x u_1(x)\big) + \int_0^t \partial_t\big(K_1(t-\tau,x) *_x |u_t(\tau,x)|^p\big)d\tau.$$

We apply the $L^m \cap L^q - L^q$ estimates if $\tau \in [0,[t-1]^+]$ and the $L^q - L^q$ estimates if $\tau \in [[t-1]^+, t]$ from Theorem 3.3.1 to conclude

$$\|\partial_t Nu(t,\cdot)\|_{L^q} \lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}}\|(u_0,u_1)\|_{\mathcal{A}^{s+s_0}_{m,q}}$$
$$+\int_0^{[t-1]^+}(1+t-\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}}\big\||u_t(\tau,\cdot)|^p\big\|_{L^m\cap L^q}d\tau$$
$$+\int_{[t-1]^+}^t (t-\tau)^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\big\||u_t(\tau,\cdot)|^p\big\|_{L^q}d\tau.$$

Using the same way for deriving (6.32) we may conclude

$$\|\partial_t Nu(t,\cdot)\|_{L^q} \lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}}\Big(\|(u_0,u_1)\|_{\mathcal{A}^{s+s_0}_{m,q}} + \|u\|^p_{X(t)}\Big).$$

Step 3: Let us estimate the norm $\big\||D|^s Nu(t,\cdot)\big\|_{L^q}$. We use

$$|D|^s Nu(t,x) = |D|^s\big(K_0(t,x) *_x u_0(x) + K_1(t,x) *_x u_1(x)\big) + \int_0^t |D|^s\big(K_1(t-\tau,x) *_x |u_t(\tau,x)|^p\big)d\tau.$$

We apply the $(L^m \cap L^q) - L^q$ estimates if $\tau \in [0,[t-1]^+]$ and the $L^q - L^q$ estimates if $\tau \in [[t-1]^+, t]$ from Theorem 3.3.2 to derive

$$\big\||D|^s Nu(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}}\|(u_0,u_1)\|_{\mathcal{A}^{s+s_0}_{m,q}}$$
$$+\int_0^{[t-1]^+}(1+t-\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}}\big\||u_t(\tau,\cdot)|^p\big\|_{L^m\cap H^{s-\sigma}_q}d\tau$$
$$+\int_{[t-1]^+}^t (t-\tau)^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\big\||u_t(\tau,\cdot)|^p\big\|_{H^{s-\sigma}_q}d\tau$$
$$= (1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}}\|(u_0,u_1)\|_{\mathcal{A}^{s+s_0}_{m,q}}$$
$$+\int_0^{[t-1]^+}(1+t-\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}}\big\||u_t(\tau,\cdot)|^p\big\|_{L^m\cap L^q\cap \dot{H}^{s-\sigma}_q}d\tau$$
$$+\int_{[t-1]^+}^t (t-\tau)^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\big\||u_t(\tau,\cdot)|^p\big\|_{L^q\cap \dot{H}^{s-\sigma}_q}d\tau.$$

The integrals with $\big\||u_t(\tau,\cdot)|^p\big\|_{L^m\cap L^q}$ and $\big\||u_t(\tau,\cdot)|^p\big\|_{L^q}$ will be handled as before if we apply the conditions for p and n, that is,

$$p\in\Big[\frac{q}{m},\frac{n}{n-q(s-\sigma)}\Big] \text{ if } s<\sigma+\frac{n}{q}, \quad \text{or} \quad p\in\Big[\frac{q}{m},\infty\Big) \text{ if } s\geqslant\sigma+\frac{n}{q},$$

and

$$p>1+\frac{\max\big\{n-\frac{m}{q}n+m(s-2\delta),\,2m(2\sigma-3\delta)\big\}}{n-2m(\sigma-2\delta)}, \quad \Big[\frac{n}{2}\Big]<n_0.$$

Hence, we get

$$\int_0^{[t-1]^+}(1+t-\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}}\big\||u_t(\tau,\cdot)|^p\big\|_{L^m\cap L^q}d\tau\lesssim(1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}}\|u\|^p_{X(t)}$$

and

$$\int_{[t-1]^+}^t(t-\tau)^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\big\||u_t(\tau,\cdot)|^p\big\|_{L^q}d\tau\lesssim(1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}}\|u\|^p_{X(t)}.$$

To estimate the integrals with the norm $\big\||u_t(\tau,\cdot)|^p\big\|_{\dot H^{s-\sigma}_q}$, we shall apply Corollary C.4.1 for the fractional powers rule with $s-\sigma\in(\frac{n}{q},p)$. Therefore, we obtain

$$\begin{aligned}\big\||u_t(\tau,\cdot)|^p\big\|_{\dot H^{s-\sigma}_q}&\lesssim\|u_t(\tau,\cdot)\|_{\dot H^{s-\sigma}_q}\|u_t(\tau,\cdot)\|^{p-1}_{L^\infty}\\&\lesssim\|u_t(\tau,\cdot)\|_{\dot H^{s-\sigma}_q}\big(\|u_t(\tau,\cdot)\|_{\dot H^{s^*}_q}+\|u_t(\tau,\cdot)\|_{\dot H^{s-\sigma}_q}\big)^{p-1}.\end{aligned}$$

Here we used Corollary C.5.1 with a suitable $s^*<\frac{n}{q}$. Applying the fractional Gagliardo-Nirenberg inequality from Proposition C.1.1 we have

$$\|u_t(\tau,\cdot)\|_{\dot H^{s^*}_q}\lesssim\|u_t(\tau,\cdot)\|^{1-\theta}_{L^q}\,\big\||D|^{s-\sigma}u_t(\tau,\cdot)\big\|^\theta_{L^q}\lesssim(1+\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}-\frac{s^*}{2(\sigma-\delta)}}\|u\|_{X(\tau)},$$

where $\theta=\frac{s^*}{s-\sigma}$. Hence, we derive

$$\begin{aligned}\big\||u_t(\tau,\cdot)|^p\big\|_{\dot H^{s-\sigma}_q}&\lesssim(1+\tau)^{p\big(1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}\big)-\frac{s-\sigma}{2(\sigma-\delta)}-\frac{s^*}{2(\sigma-\delta)}}\|u\|^p_{X(\tau)}\\&\lesssim(1+\tau)^{p\big(1-\frac{n}{2(\sigma-\delta)}(\frac{1}{m}-\frac{1}{mp})-\frac{\delta}{\sigma-\delta}\big)}\|u\|^p_{X(\tau)},\end{aligned}$$

if we choose $s^*=\frac{n}{q}-\varepsilon_0$ where $\varepsilon_0>0$ is sufficiently small. By an analogous argument as we did in Step 1 we obtain

$$\int_0^{[t-1]^+}(1+t-\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}}\big\||u_t(\tau,\cdot)|^p\big\|_{\dot H^{s-\sigma}_q}d\tau\lesssim(1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}}\|u\|^p_{X(t)},$$

$$\int_{[t-1]^+}^t(t-\tau)^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\big\||u_t(\tau,\cdot)|^p\big\|_{\dot H^{s-\sigma}_q}d\tau\lesssim(1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}}\|u\|^p_{X(t)}.$$

Therefore, we have shown the estimates

$$\big\||D|^sNu(t,\cdot)\big\|_{L^q}\lesssim(1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}}\Big(\|(u_0,u_1)\|_{\mathcal{A}^{s+s_0}_{m,q}}+\|u\|^p_{X(t)}\Big). \tag{6.33}$$

Step 4: Let us estimate the norm $\big\|\partial_t|D|^{s-\sigma}Nu(t,\cdot)\big\|_{L^q}$. We use

$$\begin{aligned}\partial_t|D|^{s-\sigma}Nu(t,x)&=\partial_t|D|^{s-\sigma}\big(K_0(t,x)*_x u_0(x)+K_1(t,x)*_x u_1(x)\big)\\&\quad+\int_0^t\partial_t|D|^{s-\sigma}\big(K_1(t-\tau,x)*_x|u_t(\tau,x)|^p\big)d\tau.\end{aligned}$$

By applying again the $(L^m\cap L^q)-L^q$ estimates if $\tau\in[0,[t-1]^+]$ and the L^q-L^q estimates if $\tau\in[[t-1]^+,t]$ from Theorem 3.3.2, we derive

$$\begin{aligned}\big\|\partial_t|D|^{s-\sigma}Nu(t,\cdot)\big\|_{L^q}&\lesssim(1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s-\sigma+2\delta}{2(\sigma-\delta)}}\|(u_0,u_1)\|_{\mathcal{A}^{s+s_0}_{m,q}}\\&\quad+\int_0^{[t-1]^+}(1+t-\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s-\sigma+2\delta}{2(\sigma-\delta)}}\big\||u_t(\tau,\cdot)|^p\big\|_{L^m\cap H^{s-\sigma}_q}d\tau\\&\quad+\int_{[t-1]^+}^t(t-\tau)^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)}\big\||u_t(\tau,\cdot)|^p\big\|_{H^{s-\sigma}_q}d\tau.\end{aligned}$$

Following the approach to show (6.33) we may conclude

$$\big\|\partial_t |D|^{s-\sigma} Nu(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s-\sigma+2\delta}{2(\sigma-\delta)}} \Big(\|(u_0,u_1)\|_{\mathcal{A}^{s+s_0}_{m,q}} + \|u\|^p_{X(t)}\Big).$$

Summarizing, from the definition of the norm in $X(t)$ we obtain immediately the inequality (6.30).

Next let us prove the inequality (6.31). Our proof is also divided into four steps.

Step 1: We need to estimate the norm $\|Nu(t,\cdot)-Nv(t,\cdot)\|_{L^q}$. We use the $(L^m \cap L^q) - L^q$ estimates if $\tau \in [0,[t-1]^+]$ and the $L^q - L^q$ estimates if $\tau \in [t-1]^+, t]$ from Theorem 3.3.1 to derive for two functions u and v from $X(t)$ the estimate

$$\begin{aligned}\|Nu(t,\cdot)-Nv(t,\cdot)\|_{L^q} \lesssim &\int_0^{[t-1]^+} (1+t-\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})} \big\||u_t(\tau,\cdot)|^p - |v_t(\tau,\cdot)|^p\big\|_{L^m\cap L^q} d\tau \\ &+ \int_{[t-1]^+}^t (t-\tau)^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)} \big\||u_t(\tau,\cdot)|^p - |v_t(\tau,\cdot)|^p\big\|_{L^q} d\tau.\end{aligned}$$

By using Hölder's inequality and applying again the same ideas as we did in the proof of (6.9) and Step 1 to prove (6.30) we may conclude

$$\|Nu(t,\cdot)-Nv(t,\cdot)\|_{L^q} \lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})} \|u-v\|_{X(t)} \big(\|u\|^{p-1}_{X(t)} + \|v\|^{p-1}_{X(t)}\big).$$

Step 2: We need to estimate the norm $\|\partial_t Nu(t,\cdot)-\partial_t Nv(t,\cdot)\|_{L^q}$. We use

$$\begin{aligned}\|\partial_t Nu(t,\cdot)-\partial_t Nv(t,\cdot)\|_{L^q} \lesssim &\int_0^{[t-1]^+} (1+t-\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}} \big\||u_t(\tau,\cdot)|^p - |v_t(\tau,\cdot)|^p\big\|_{L^m\cap L^q} d\tau \\ &+ \int_{[t-1]^+}^t (t-\tau)^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)} \big\||u_t(\tau,\cdot)|^p - |v_t(\tau,\cdot)|^p\big\|_{L^q} d\tau.\end{aligned}$$

Using the same approach to derive (6.9) and Step 2 to prove (6.30) we conclude

$$\|\partial_t Nu(t,\cdot)-\partial_t Nv(t,\cdot)\|_{L^q} \lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}} \|u-v\|_{X(t)} \big(\|u\|^{p-1}_{X(t)} + \|v\|^{p-1}_{X(t)}\big).$$

Step 3: Let us estimate the norm $\big\||D|^s Nu(t,\cdot)-|D|^s Nv(t,\cdot)\big\|_{L^q}$. We use

$$\begin{aligned}&\big\||D|^s Nu(t,\cdot)-|D|^s Nv(t,\cdot)\big\|_{L^q} \\ &\lesssim \int_0^{[t-1]^+} (1+t-\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}} \big\||u_t(\tau,\cdot)|^p - |v_t(\tau,\cdot)|^p\big\|_{L^m\cap H^{s-\sigma}_q} d\tau \\ &\quad + \int_{[t-1]^+}^t (t-\tau)^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)} \big\||u_t(\tau,\cdot)|^p - |v_t(\tau,\cdot)|^p\big\|_{H^{s-\sigma}_q} d\tau \\ &= \int_0^{[t-1]^+} (1+t-\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}} \big\||u_t(\tau,\cdot)|^p - |v_t(\tau,\cdot)|^p\big\|_{L^m\cap L^q\cap \dot{H}^{s-\sigma}_q} d\tau \\ &\quad + \int_{[t-1]^+}^t (t-\tau)^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)} \big\||u_t(\tau,\cdot)|^p - |v_t(\tau,\cdot)|^p\big\|_{L^q\cap \dot{H}^{s-\sigma}_q} d\tau.\end{aligned}$$

The integrals with $\big\||u_t(\tau,\cdot)|^p - |v_t(\tau,\cdot)|^p\big\|_{L^m\cap L^q}$ and $\big\||u_t(\tau,\cdot)|^p - |v_t(\tau,\cdot)|^p\big\|_{L^q}$ will be handled as we did in the proof of (6.9) and Step 3 to prove (6.30). Hence, we get

$$\begin{aligned}&\int_0^{[t-1]^+} (1+t-\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}} \big\||u_t(\tau,\cdot)|^p - |v_t(\tau,\cdot)|^p\big\|_{L^m\cap L^q} d\tau \\ &\qquad \lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}} \|u-v\|_{X(t)} \big(\|u\|^{p-1}_{X(t)} + \|v\|^{p-1}_{X(t)}\big),\end{aligned}$$

and

$$\begin{aligned}&\int_{[t-1]^+}^t (t-\tau)^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)} \big\||u_t(\tau,\cdot)|^p - |v_t(\tau,\cdot)|^p\big\|_{L^q} d\tau \\ &\qquad \lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}} \|u-v\|_{X(t)} \big(\|u\|^{p-1}_{X(t)} + \|v\|^{p-1}_{X(t)}\big).\end{aligned}$$

Let us now turn to estimate the norm $\big\||u_t(\tau,\cdot)|^p-|v_t(\tau,\cdot)|^p\big\|_{\dot H^{s-\sigma}_q}$. By using the integral representation

$$|u_t(\tau,x)|^p-|v_t(\tau,x)|^p=p\int_0^1\big(u_t(\tau,x)-v_t(\tau,x)\big)G\big(\omega u_t(\tau,x)+(1-\omega)v_t(\tau,x)\big)d\omega,$$

where $G(u)=u|u|^{p-2}$, we obtain

$$\big\||u_t(\tau,\cdot)|^p-|v_t(\tau,\cdot)|^p\big\|_{\dot H^{s-\sigma}_q}\lesssim\int_0^1\Big\||D|^{s-\sigma}\Big(\big(u_t(\tau,\cdot)-v_t(\tau,\cdot)\big)G\big(\omega u_t(\tau,\cdot)+(1-\omega)v_t(\tau,\cdot)\big)\Big)\Big\|_{L^q}d\omega.$$

Thanks to the fractional powers rule from Corollary C.4.2, we can proceed as follows:

$$\begin{aligned}\big\||u_t(\tau,\cdot)|^p-|v_t(\tau,\cdot)|^p\big\|_{\dot H^{s-\sigma}_q}&\lesssim\int_0^1\|u_t(\tau,\cdot)-v_t(\tau,\cdot)\|_{\dot H^{s-\sigma}_q}\big\|G\big(\omega u_t(\tau,\cdot)+(1-\omega)v_t(\tau,\cdot)\big)\big\|_{L^\infty}d\omega\\&\quad+\int_0^1\|u_t(\tau,\cdot)-v_t(\tau,\cdot)\|_{L^\infty}\big\|G\big(\omega u_t(\tau,\cdot)+(1-\omega)v_t(\tau,\cdot)\big)\big\|_{\dot H^{s-\sigma}_q}d\omega\\&\lesssim\int_0^1\|u_t(\tau,\cdot)-v_t(\tau,\cdot)\|_{\dot H^{s-\sigma}_q}\|\omega u_t(\tau,\cdot)+(1-\omega)v_t(\tau,\cdot)\|^{p-1}_{L^\infty}d\omega\\&\quad+\int_0^1\|u_t(\tau,\cdot)-v_t(\tau,\cdot)\|_{L^\infty}\big\|G\big(\omega u_t(\tau,\cdot)+(1-\omega)v_t(\tau,\cdot)\big)\big\|_{\dot H^{s-\sigma}_q}d\omega.\end{aligned}$$

Applying Corollary C.4.1 with $p>2$ and $s-\sigma\in(\frac{n}{q},p-1)$ we get

$$\begin{aligned}&\big\|G\big(\omega u_t(\tau,\cdot)+(1-\omega)v_t(\tau,\cdot)\big)\big\|_{\dot H^{s-\sigma}_q}\\&\quad\lesssim\|\omega u_t(\tau,\cdot)+(1-\omega)v_t(\tau,\cdot)\|_{\dot H^{s-\sigma}_q}\|\omega u_t(\tau,\cdot)+(1-\omega)v_t(\tau,\cdot)\|^{p-2}_{L^\infty}.\end{aligned}$$

Using Corollary C.5.1 with a suitable $s^*<\frac{n}{q}$ we get

$$\|u_t(\tau,\cdot)-v_t(\tau,\cdot)\|_{L^\infty}\lesssim\|u_t(\tau,\cdot)-v_t(\tau,\cdot)\|_{\dot H^{s^*}_q}+\|u_t(\tau,\cdot)-v_t(\tau,\cdot)\|_{\dot H^{s-\sigma}_q},$$

and

$$\begin{aligned}&\|\omega u_t(\tau,\cdot)+(1-\omega)v_t(\tau,\cdot)\|_{L^\infty}\\&\quad\lesssim\|\omega u_t(\tau,\cdot)+(1-\omega)v_t(\tau,\cdot)\|_{\dot H^{s^*}_q}+\|\omega u_t(\tau,\cdot)+(1-\omega)v_t(\tau,\cdot)\|_{\dot H^{s-\sigma}_q}.\end{aligned}$$

Applying the fractional Gagliardo-Nirenberg inequality from Proposition C.1.1 we have

$$\begin{aligned}\|u_t(\tau,\cdot)-v_t(\tau,\cdot)\|_{\dot H^{s^*}_q}&\lesssim\|u_t(\tau,\cdot)-v_t(\tau,\cdot)\|^{1-\theta}_{L^q}\big\||D|^{s-\sigma}\big(u_t(\tau,\cdot)-v_t(\tau,\cdot)\big)\big\|^{\theta}_{L^q}\\&\lesssim(1+\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}-\frac{s^*}{2(\sigma-\delta)}}\|u-v\|_{X(\tau)},\end{aligned}$$

where $\theta=\frac{s^*}{s-\sigma}$. In the same way, we get

$$\|\omega u_t(\tau,\cdot)+(1-\omega)v_t(\tau,\cdot)\|_{\dot H^{s^*}_q}\lesssim(1+\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}-\frac{s^*}{2(\sigma-\delta)}}\|\omega u_t(\tau,\cdot)+(1-\omega)v_t(\tau,\cdot)\|_{X(\tau)}.$$

Therefore, we may conclude

$$\begin{aligned}&\big\||u_t(\tau,\cdot)|^p-|v_t(\tau,\cdot)|^p\big\|_{\dot H^{s-\sigma}_q}\\&\quad\lesssim\int_0^1(1+\tau)^{p\left(1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}\right)-\frac{s-\sigma}{2(\sigma-\delta)}-\frac{s^*}{2(\sigma-\delta)}}\|u-v\|_{X(\tau)}\|\omega u+(1-\omega)v\|^{p-1}_{X(\tau)}d\omega\\&\quad\lesssim(1+\tau)^{p\left(1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}\right)-\frac{s-\sigma}{2(\sigma-\delta)}-\frac{s^*}{2(\sigma-\delta)}}\|u-v\|_{X(\tau)}\big(\|u\|^{p-1}_{X(\tau)}+\|v\|^{p-1}_{X(\tau)}\big).\end{aligned}$$

By an analogous argument as we did in Step 1 to prove (6.30) we obtain

$$\begin{aligned}&\int_0^{[t-1]^+}(1+t-\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}}\big\||u_t(\tau,\cdot)|^p-|v_t(\tau,\cdot)|^p\big\|_{\dot H^{s-\sigma}_q}d\tau\\&\qquad\lesssim(1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}}\|u-v\|_{X(t)}\big(\|u\|^{p-1}_{X(t)}+\|v\|^{p-1}_{X(t)}\big),\end{aligned}$$

and

$$\begin{aligned}
&\int_{[t-1]^+}^t (t-\tau)^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)} \big\| |u_t(\tau,\cdot)|^p - |v_t(\tau,\cdot)|^p \big\|_{\dot{H}^{s-\sigma}_q} d\tau \\
&\qquad \lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}} \|u-v\|_{X(t)} \big(\|u\|^{p-1}_{X(t)} + \|v\|^{p-1}_{X(t)} \big).
\end{aligned}$$

Summarizing, we have proved the estimates

$$\big\| |D|^s Nu(t,\cdot) - |D|^s Nv(t,\cdot) \big\|_{L^q} \lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}} \|u-v\|_{X(t)} \big(\|u\|^{p-1}_{X(t)} + \|v\|^{p-1}_{X(t)} \big).$$

Step 4: Let us estimate the norm $\big\|\partial_t |D|^{s-\sigma} Nu(t,\cdot) - \partial_t |D|^{s-\sigma} Nv(t,\cdot)\big\|_{L^q}$. We use

$$\begin{aligned}
&\big\|\partial_t |D|^{s-\sigma} Nu(t,\cdot) - \partial_t |D|^{s-\sigma} Nv(t,\cdot)\big\|_{L^q} \\
&\quad \lesssim \int_0^{[t-1]^+} (1+t-\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s-\sigma+2\delta}{2(\sigma-\delta)}} \big\| |u_t(\tau,\cdot)|^p - v_t(\tau,\cdot)|^p \big\|_{L^m \cap H^{s-\sigma}_q} d\tau \\
&\qquad + \int_{[t-1]^+}^t (t-\tau)^{-(2+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)} \big\| |u_t(\tau,\cdot)|^p - |v_t(\tau,\cdot)|^p \big\|_{H^{s-\sigma}_q} d\tau.
\end{aligned}$$

By the same treatment as in Step 3 to prove (6.31) we may conclude

$$\begin{aligned}
&\big\|\partial_t |D|^{s-\sigma} Nu(t,\cdot) - \partial_t |D|^{s-\sigma} Nv(t,\cdot)\big\|_{L^q} \\
&\qquad \lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s-\sigma+2\delta}{2(\sigma-\delta)}} \|u-v\|_{X(t)} \big(\|u\|^{p-1}_{X(t)} + \|v\|^{p-1}_{X(t)} \big).
\end{aligned}$$

Summarizing, from the definition of the norm in $X(t)$ and all the previous estimates we have completed the proof of (6.31). □

Remark 6.1.7. From the condition $s > \sigma + \frac{n}{q}$, we can see that

$$1 + \frac{n - \frac{m}{q}n + m(s-2\delta)}{n - 2m(\sigma-2\delta)} > 2.$$

Hence, the condition $p > 2$ in the proof of Theorem 6.1.5 can be omitted. Moreover, if we introduce the another condition for the space dimension n, namely

$$n \leqslant n_2 := \frac{mq(4\sigma - 4\delta - s)}{q-m},$$

then it follows $2m(2\sigma - 3\delta) \geqslant n - \frac{m}{q}n + m(s-2\delta)$. This inequality allows us to avoid the restriction of $p > 1 + \frac{n-\frac{m}{q}n+m(s-2\delta)}{n-2m(\sigma-2\delta)}$ in Theorem 6.1.5. Hence, we only need to guarantee the following restriction:

$$p > 1 + \frac{2m(2\sigma-3\delta)}{n-2m(\sigma-2\delta)}.$$

Example 6.1.6. By choosing $m = 1$, $q = 2$, $\sigma = 2$, $\delta = 0.9$ and $s = 5$ we obtain the following admissible range of exponents p:

$$p \in (7.17, \infty) \text{ if } n = 1, \quad \text{or} \quad p \in (4, \infty) \text{ if } n = 2, 3, 4, 5.$$

6.2. Loss of decay and loss of regularity

In this section, we show how the restrictions to the admissible exponents p appearing in all the theorems of Section 6.1 can be relaxed. We will use some decay rates for solutions or some of their partial derivatives to the semi-linear models which are worse than those given for solutions to the corresponding linear models with vanishing right-hand side to treat the semi-linear models (6.1) and (6.2). Consequently, we allow loss of decay. This strategy comes into play to bring some advantage to weaken the restrictions to the admissible exponents p in comparison with those in the previous section.

6.2.1. Solutions in the energy space to the model (6.1)

In the first case we obtain solutions from energy space on the base of L^q.

Theorem 6.2.1. *Under the assumptions of Theorem 6.1.1, if condition (6.3) is replaced by $n > n_1$, then we have the same conclusions of Theorem 6.1.1. But the estimates (6.5) to (6.7) are modified in the following way:*

$$\big\|\big(|D|^\sigma u(t,\cdot), u(t,\cdot)\big)\big\|_{L^q} \lesssim (1+t)^{\frac{1}{p}\big(1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})\big)}\|(u_0,u_1)\|_{\mathcal{A}^{\sigma+s_0}_{m,q}}, \tag{6.34}$$

$$\|u_t(t,\cdot)\|_{L^q} \lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})}\|(u_0,u_1)\|_{\mathcal{A}^{\sigma+s_0}_{m,q}}. \tag{6.35}$$

Proof. We follow the proof of Theorem 6.1.1. Having this in mind we fix the data space and the solution space as in Theorem 6.1.1, but we use the norm

$$\|u\|_{X(t)} := \sup_{0\leqslant\tau\leqslant t}\Big(f_{\varepsilon_1}(\tau)^{-1}\|u(\tau,\cdot)\|_{L^q} + f_{\varepsilon_2}(\tau)^{-1}\big\||D|^\sigma u(\tau,\cdot)\big\|_{L^q} + f_{\varepsilon_3}(\tau)^{-1}\|u_t(\tau,\cdot)\|_{L^q}\Big),$$

and the space $X_0(t) := C([0,t], H^{\sigma,q})$ with the norm

$$\|w\|_{X_0(t)} := \sup_{0\leqslant\tau\leqslant t}\Big(f_{\varepsilon_1}(\tau)^{-1}\|w(\tau,\cdot)\|_{L^q} + f_{\varepsilon_2}(\tau)^{-1}\big\||D|^\sigma w(\tau,\cdot)\big\|_{L^q}\Big),$$

where

$$f_{\varepsilon_1}(\tau) = (1+\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})+\varepsilon_1}, \quad f_{\varepsilon_2}(\tau) = (1+\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\sigma}{2(\sigma-\delta)}+\varepsilon_2},$$

and

$$f_{\varepsilon_3}(\tau) = (1+\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}+\varepsilon_3}$$

for some positive constants ε_1, ε_2 and ε_3. Here these constants stand for the loss of decay in comparison with the corresponding decay estimates for solutions to the linear Cauchy problem with vanishing right-hand side.

First let us prove the estimate (6.8). Repeating the proof of Theorem 6.1.1 we derive the following estimates:

$$\begin{aligned}
\big\|\partial_t^j|D|^{k\sigma}Nu(t,\cdot)\big\|_{L^q} &\lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{k\sigma+2j\delta}{2(\sigma-\delta)}}\|(u_0,u_1)\|_{\mathcal{A}^{\sigma+s_0}_{m,q}}\\
&+\|u\|^p_{X_0(t)}\int_0^{[t-1]^+}(1+t-\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{k\sigma+2j\delta}{2(\sigma-\delta)}}(1+\tau)^{p-\frac{n}{2m(\sigma-\delta)}(p-1)+p(\varepsilon_2\theta_{mp}+\varepsilon_1(1-\theta_{mp}))}d\tau\\
&+\|u\|^p_{X_0(t)}\int_{[t-1]^+}^t(t-\tau)^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)-(k+j)\frac{\sigma}{2\delta}}(1+\tau)^{p-\frac{np}{2(\sigma-\delta)}(\frac{1}{m}-\frac{1}{qp})+p(\varepsilon_2\theta_{qp}+\varepsilon_1(1-\theta_{qp}))}d\tau.
\end{aligned}$$

Now we fix the constant $\varepsilon_1 := \big(1-\frac{1}{p}\big)\big(-1+\frac{n}{2(\sigma-\delta)}\big(1-\frac{1}{r}\big)\big)$.

Due to $n > n_1$, it follows $-1+\frac{n}{2(\sigma-\delta)}\big(1-\frac{1}{r}\big) > 1$ and ε_1 is positive. Next we choose $\varepsilon_2 = \frac{\sigma}{2(\sigma-\delta)}+\varepsilon_1$ and $\varepsilon_3 = \frac{\delta}{\sigma-\delta}$. Then, we have

$$\begin{aligned}
\big\|\partial_t^j|D|^{k\sigma}Nu(t,\cdot)\big\|_{L^q} &\lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{k\sigma+2j\delta}{2(\sigma-\delta)}}\|(u_0,u_1)\|_{\mathcal{A}^{\sigma+s_0}_{m,q}}\\
&+\|u\|^p_{X_0(t)}\int_0^{[t-1]^+}(1+t-\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{k\sigma+2j\delta}{2(\sigma-\delta)}}(1+\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})}d\tau\\
&+\|u\|^p_{X_0(t)}\int_{[t-1]^+}^t(t-\tau)^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)-(k+j)\frac{\sigma}{2\delta}}(1+\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})}d\tau.
\end{aligned}$$

After applying Lemma B.6.1 by choosing $\alpha = -1+\frac{n}{2(\sigma-\delta)}\big(1-\frac{1}{r}\big)+\frac{k\sigma+2j\delta}{2(\sigma-\delta)}$ and $\beta = -1+\frac{n}{2(\sigma-\delta)}\big(1-\frac{1}{r}\big)$ we get

$$\int_0^{[t-1]^+}(1+t-\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{k\sigma+2j\delta}{2(\sigma-\delta)}}(1+\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})}d\tau \lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})}.$$

Following the same arguments we used in the proof of Theorem 6.1.1, the condition $\big[\frac{n}{2}\big] < n_0$ implies

$$\int_{[t-1]^+}^t(t-\tau)^{1-(1+[\frac{n}{2}])(\frac{\sigma}{2\delta}-1)-(k+j)\frac{\sigma}{2\delta}}(1+\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})}d\tau \lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})}.$$

Finally, we conclude the following estimates:

$$\begin{aligned}
\|Nu(t,\cdot)\|_{L^q} &\lesssim f_{\varepsilon_1}(t)\,\big(\|(u_0,u_1)\|_{\mathcal{A}^{\sigma+s_0}_{m,q}} + \|u\|^p_{X_0(t)}\big),\\
\big\||D|^\sigma Nu(t,\cdot)\big\|_{L^q} &\lesssim f_{\varepsilon_2}(t)\,\big(\|(u_0,u_1)\|_{\mathcal{A}^{\sigma+s_0}_{m,q}} + \|u\|^p_{X_0(t)}\big),\\
\|\partial_t Nu(t,\cdot)\|_{L^q} &\lesssim f_{\varepsilon_3}(t)\,\big(\|(u_0,u_1)\|_{\mathcal{A}^{\sigma+s_0}_{m,q}} + \|u\|^p_{X_0(t)}\big).
\end{aligned}$$

From the definition of the norm in $X(t)$, we obtain immediately the inequality (6.8).

Next let us prove the inequality (6.9). An analogous treatment as we did in the proof of Theorem 6.1.1 and the above arguments give the following estimates:

$$\begin{aligned}
\|Nu(t,\cdot)-Nv(t,\cdot)\|_{L^q} &\lesssim f_{\varepsilon_1}(t)\,\|u-v\|_{X_0(t)}\big(\|u\|^{p-1}_{X_0(t)} + \|v\|^{p-1}_{X_0(t)}\big),\\
\big\||D|^\sigma Nu(t,\cdot)-|D|^\sigma Nv(t,\cdot)\big\|_{L^q} &\lesssim f_{\varepsilon_2}(t)\,\|u-v\|_{X_0(t)}\big(\|u\|^{p-1}_{X_0(t)} + \|v\|^{p-1}_{X_0(t)}\big),\\
\big\|\partial_t Nu(t,\cdot)-\partial_t Nv(t,\cdot)\big\|_{L^q} &\lesssim f_{\varepsilon_3}(t)\,\|u-v\|_{X_0(t)}\big(\|u\|^{p-1}_{X_0(t)} + \|v\|^{p-1}_{X_0(t)}\big).
\end{aligned}$$

From the definition of the norm in $X(t)$, we obtain immediately the inequality (6.9).
Summarizing, the proof of Theorem 6.2.1 is completed. □

Remark 6.2.1. Here we can see that some loss of decay appears in Theorem 6.2.1. Although the loss of decay is not arbitrarily small, for example, $\varepsilon_2 > \frac{\sigma}{2(\sigma-\delta)}$, we can reduce the restrictions of admissible exponents p in comparison with those of Theorem 6.1.1. In other words, the admissible interval of exponents p is more relaxed because we only need to guarantee the conditions coming from Gagliardo-Nirenberg inequality. Moreover, if we assume $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_0$, where

$$\varepsilon_0 := \Big(1-\frac{1}{p}\Big)\Big(\frac{n}{2m(\sigma-\delta)}-1\Big) - \frac{n}{2p(\sigma-\delta)}\Big(1-\frac{1}{r}\Big),$$

then we have to guarantee another condition for the exponent p in Theorem 6.2.1 as follows:

$$p \geqslant \frac{2n-\frac{mn}{q}-2m(\sigma-\delta)}{n-m(3\sigma-2\delta)}.$$

Additionally, we want to emphasize that the results in Theorem 6.2.1 bring some flexibility in comparison with those in Theorem 6.1.1. For the sake of brevity, let us denote and recall two parameters as follows:

$$p_1 := 1 + \frac{\max\big\{n-\frac{m}{q}n+m\sigma,\, 4m(\sigma-\delta)\big\}}{n-2m(\sigma-\delta)} \quad \text{and} \quad n_1 := \frac{4mq(\sigma-\delta)}{q-m}$$

appearing in Theorem 6.1.1 and Theorem 6.2.1, respectively. In order to observe the whole picture for the admissible interval of exponents p, for example, we may summarize our results depending on using no loss of decay or loss of decay in Theorem 6.1.1 and Theorem 6.2.1 in the following picture:

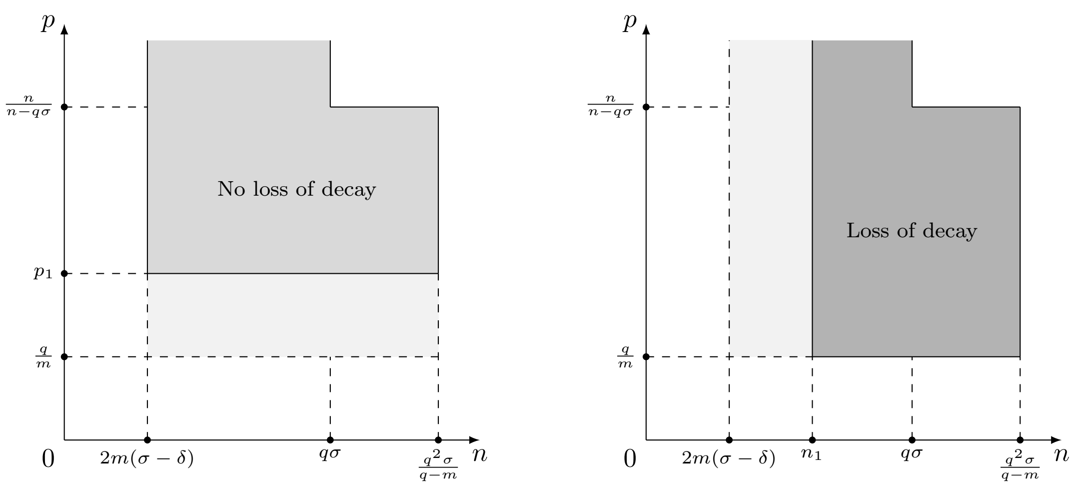

Fig. 6.1.: The admissible set of exponents p and dimensions n without loss of decay and with loss of decay.

Example 6.2.1. In the first example, we choose $m = 1$, $q = 2$, $\sigma = 2$ and $\delta = 0.875$ in Theorem 6.1.1 and Theorem 6.2.1. Here we can see that the idea of allowing a loss of decay cannot be applied to Theorem 6.2.1 because there is no value of n satisfying the condition $n > n_1$. Hence, we only obtain the results of Theorem 6.1.1 without a loss of decay (see the following table):

n	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$
p	$p \in (7, \infty)$	$p \in (5, \infty)$	$p \in (2.63, 5]$	$p \in (2.5, 3]$	$p \in (2.27, 2.33]$

Tab. 6.5.: The admissible interval of exponents p depends on the space dimension n without loss of decay.

Example 6.2.2. In the second example, we choose $m = 1$, $q = 4$, $\sigma = 2$ and $\delta = 0.875$ in Theorem 6.1.1 and Theorem 6.2.1. Then, we can see that the loss of decay works in Theorem 6.2.1 because there exist values of n satisfying the condition $n > n_1$. Hence, using loss of decay in Theorem 6.2.1 brings some flexibility for the following admissible intervals of exponents p:

n	$n = 7$	$n = 8$	$n = 9$
p	$p \in [4, \infty)$	$p \in [4, \infty)$	$p \in [4, 9]$

Tab. 6.6.: The admissible interval of exponents p depends on the space dimension n with loss of decay.

6.2.2. Solutions below the energy space to the model (6.1)

In the second case we obtain solutions from Sobolev space on the base of L^q.

Theorem 6.2.2. *Under the assumptions of Theorem 6.1.2, if the condition (6.10) is replaced by $n > n_1$, then we have the same conclusions of Theorem 6.1.2. But the estimates (6.12) to (6.13) are modified in the following way:*

$$\big\|\big(|D|^s u(t,\cdot), u(t,\cdot)\big)\big\|_{L^q} \lesssim (1+t)^{\frac{1}{p}(1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r}))} \|(u_0, u_1)\|_{\mathcal{A}^{s+s_0}_{m,q}}. \tag{6.36}$$

Proof. We follow the proof of Theorems 6.1.2 and 6.2.1. Having this in mind we fix the data space and the solution space as in the proof of Theorem 6.1.2, but we use the norm

$$\|u\|_{X(t)} := \sup_{0\leqslant\tau\leqslant t} \Big(f_{\varepsilon_1}(\tau)^{-1} \|u(\tau,\cdot)\|_{L^q} + f_{\varepsilon_2}(\tau)^{-1} \big\||D|^s u(\tau,\cdot)\big\|_{L^q} \Big),$$

where

$$f_{\varepsilon_1}(\tau) = (1+\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})+\varepsilon_1} \text{ and } f_{\varepsilon_2}(\tau) = (1+\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}+\varepsilon_2},$$

for some positive constants ε_1 and ε_2. Here these constants stand for the loss of decay in comparison with the corresponding decay estimates for solutions to the linear Cauchy problem with vanishing right-hand side.

Now we fix the constant $\varepsilon_1 := \big(1 - \frac{1}{p}\big)\big(-1 + \frac{n}{2(\sigma-\delta)}\big(1 - \frac{1}{r}\big)\big)$. Next we choose $\varepsilon_2 = \frac{s}{2(\sigma-\delta)} + \varepsilon_1$. Then, following the proofs of Theorems 6.1.2 and 6.2.1 we may prove Theorem 6.2.2. □

Remark 6.2.2. Here we can see that some loss of decay appears in Theorem 6.2.2. This brings some benefits to reduce the restrictions of admissible exponent p in comparison with those from Theorem 6.1.2. In other words, the admissible interval of exponents p is more relaxed because we only need to guarantee the conditions coming from Gagliardo-Nirenberg inequality. Moreover, if we assume $\varepsilon_1 = \varepsilon_2 = \varepsilon_0$, where

$$\varepsilon_0 := \Big(1 - \frac{1}{p}\Big)\Big(\frac{n}{2m(\sigma-\delta)} - 1\Big) - \frac{n}{2p(\sigma-\delta)}\Big(1 - \frac{1}{r}\Big),$$

then we have to guarantee another condition for exponents p in Theorem 6.2.2 as follows:

$$p \geqslant \frac{2n - \frac{mn}{q} - 2m(\sigma-\delta)}{n - 2m(\sigma-2\delta) - ms}.$$

Example 6.2.3. If we choose $m = 1$, $q = 4$, $\sigma = 2$, $\delta = 0.875$ and $s = 1.8$ in Theorem 6.2.2, then we can see that the loss of decay works in Theorem 6.2.2. Hence, we obtain the following admissible intervals of exponents p:

n	$n = 7$	$n = 8$	$n = 9$
p	$p \in [4, \infty)$	$p \in [4, 10]$	$p \in [4, 5]$

Tab. 6.7.: The admissible interval of exponents p depends on the space dimension n with loss of decay.

6.2.3. Solutions in the energy space with suitable higher regularity to the model (6.1)

Now we obtain solutions belonging to the energy space with a suitable higher regularity.

Theorem 6.2.3. *Under the assumptions of Theorem 6.1.3, if condition (6.16) is replaced by $n > n_1$, then we have the same conclusions of Theorem 6.1.3. But the estimates (6.18) to (6.21) are modified in the following way:*

$$\big\|\big(|D|^s u(t,\cdot), u(t,\cdot)\big)\big\|_{L^q} \lesssim (1+t)^{\frac{1}{p}(1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r}))}\|(u_0,u_1)\|_{\mathcal{A}^{s+s_0}_{m,q}}, \tag{6.37}$$

$$\big\|\big(|D|^{s-\sigma} u_t(t,\cdot), u_t(t,\cdot)\big)\big\|_{L^q} \lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})}\|(u_0,u_1)\|_{\mathcal{A}^{s+s_0}_{m,q}}. \tag{6.38}$$

Proof. We follow the proof of Theorems 6.1.3 and 6.2.1. Having this in mind we fix the data space and the solution space as in Theorem 6.1.3, but we use the norm

$$\|u\|_{X(t)} := \sup_{0\leqslant\tau\leqslant t}\Big(f_{\varepsilon_1}(\tau)^{-1}\|u(\tau,\cdot)\|_{L^q} + f_{\varepsilon_2}(\tau)^{-1}\big\||D|^s u(\tau,\cdot)\big\|_{L^q}$$
$$+ f_{\varepsilon_3}(\tau)^{-1}\|u_t(\tau,\cdot)\|_{L^q} + f_{\varepsilon_4}(\tau)^{-1}\big\||D|^{s-\sigma} u_t(\tau,\cdot)\big\|_{L^q}\Big),$$

and the space $X_0(t) := C([0,t], H^s_q)$ with the norm

$$\|w\|_{X_0(t)} := \sup_{0\leqslant\tau\leqslant t}\Big(f_{\varepsilon_1}(\tau)^{-1}\|u(\tau,\cdot)\|_{L^q} + f_{\varepsilon_2}(\tau)^{-1}\big\||D|^s u(\tau,\cdot)\big\|_{L^q}\Big),$$

where

$$f_{\varepsilon_1}(\tau) = (1+\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})+\varepsilon_1}, \quad f_{\varepsilon_2}(\tau) = (1+\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s}{2(\sigma-\delta)}+\varepsilon_2},$$

and

$$f_{\varepsilon_3}(\tau) = (1+\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{\delta}{\sigma-\delta}+\varepsilon_3}, \quad f_{\varepsilon_4}(\tau) = (1+\tau)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})-\frac{s-\sigma+2\delta}{2(\sigma-\delta)}+\varepsilon_4}$$

for some positive constant ε_j with $j = 1,2,3,4$. Here these constants stand for the loss of decay in comparison with the corresponding decay estimates for solutions to the linear Cauchy problem with vanishing right-hand side.

Now we fix the constant $\varepsilon_1 := \big(1-\frac{1}{p}\big)\big(-1+\frac{n}{2(\sigma-\delta)}\big(1-\frac{1}{r}\big)\big)$. Next we choose $\varepsilon_2 = \frac{s}{2(\sigma-\delta)} + \varepsilon_1$, $\varepsilon_3 = \frac{\delta}{\sigma-\delta}$ and $\varepsilon_4 = \frac{s-\sigma+2\delta}{2(\sigma-\delta)}$. Then, following the proofs of Theorems 6.1.3 and 6.2.1 we may prove Theorem 6.2.3. □

Remark 6.2.3. Here we can see that some loss of decay appears in Theorem 6.2.3. This brings some benefits to reduce the restrictions of admissible exponents p in comparison with those from Theorem 6.1.3. In other words, the admissible interval of exponents p is more relaxed because we only need to guarantee the conditions coming from the fractional Gagliardo-Nirenberg inequality and the fractional chain rule.

Example 6.2.4. If we choose $m = 1$, $q = 4$, $\sigma = 2$, $\delta = 0.875$ and $s = 2.5$ in Theorem 6.2.3, then we can see that the idea to allow loss of decay works in Theorem 6.2.3. Hence, we obtain the following admissible interval of exponents p:

$$p \in [4,\infty) \quad \text{for all } n = 7,8,9.$$

6.2.4. Large regular solutions to the model (6.1)

Now we obtain large regular solutions to (6.1).

Theorem 6.2.4. *Under the assumptions of Theorem 6.1.4, if the condition (6.26) is replaced by $n > n_1$, then we have the same conclusions of Theorem 6.1.4. But the estimates (6.18) to (6.21) are modified in the following way:*

$$\big\|\big(|D|^s u(t,\cdot), u(t,\cdot)\big)\big\|_{L^q} \lesssim (1+t)^{\frac{1}{p}(1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r}))}\|(u_0,u_1)\|_{\mathcal{A}^{s+s_0}_{m,q}}, \tag{6.39}$$

$$\big\|\big(|D|^{s-\sigma} u_t(t,\cdot), u_t(t,\cdot)\big)\big\|_{L^q} \lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})}\|(u_0,u_1)\|_{\mathcal{A}^{s+s_0}_{m,q}}. \tag{6.40}$$

Proof. We follow the proof of Theorems 6.1.4 and 6.2.1. Having this in mind we fix the data space and the solution space as in the proof of Theorem 6.1.4, but we use the norm as in the proof of Theorem 6.2.3.

Now we fix the constant ε_1, and choose ε_j with $j = 2, 3, 4$ as in the proof of Theorem 6.2.3. Then, following the proofs of Theorems 6.1.4 and 6.2.1 we may prove Theorem 6.2.4. □

Example 6.2.5. If we choose $m = 1$, $q = 10$, $\sigma = 2$, $\delta = 0.875$ and $s = 10.5$ in Theorem 6.2.4, then we can see that the loss of decay works in Theorem 6.2.4. Hence, we obtain the following admissible interval of exponents p:

n	$n=6$	$n=7$	$n=8$	$n=9$
p	$p \in [10, 15.17]$	$p \in [10, 13.14]$	$p \in [10, 11.625]$	$p \in [10, 10.44]$

Tab. 6.8.: The admissible range of exponents p depends on the space dimension n with loss of decay.

6.2.5. Large regular solutions to the model (6.2)

Finally, we obtain large regular solutions to (6.2).

Theorem 6.2.5. *Under the assumptions of Theorem 6.1.5, if the condition (6.28) is replaced by $n > n_1$, then we have the same conclusions of Theorem 6.1.5. But the estimates (6.18) to (6.21) are modified in the following way:*

$$\big\|\big(|D|^s u(t,\cdot), u(t,\cdot)\big)\big\|_{L^q} \lesssim (1+t)^{1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r})}\|(u_0,u_1)\|_{\mathcal{A}^{s+s_0}_{m,q}}, \tag{6.41}$$

$$\big\|\big(|D|^{s-\sigma} u_t(t,\cdot), u_t(t,\cdot)\big)\big\|_{L^q} \lesssim (1+t)^{\frac{1}{p}(1-\frac{n}{2(\sigma-\delta)}(1-\frac{1}{r}))}\|(u_0,u_1)\|_{\mathcal{A}^{s+s_0}_{m,q}}. \tag{6.42}$$

Proof. We follow the proof of Theorems 6.1.5 and 6.2.1. Having this in mind we fix the data space and the solution space as in the proof of Theorem 6.1.5, but we use the norm as in the proof of Theorem 6.2.3.

Now we fix the constant $\varepsilon := \big(1-\frac{1}{p}\big)\big(-1+\frac{n}{2(\sigma-\delta)}\big(1-\frac{1}{r}\big)\big)$. Next we choose $\varepsilon_1 = 0$, $\varepsilon_2 = \frac{s}{2(\sigma-\delta)}$, $\varepsilon_3 = \frac{\delta}{\sigma-\delta} + \varepsilon$ and $\varepsilon_4 = \frac{s-\sigma+2\delta}{2(\sigma-\delta)} + \varepsilon$. Then, following the proofs of Theorems 6.1.5 and 6.2.1 we may prove Theorem 6.2.5. □

Example 6.2.6. If we choose $m = 1$, $q = 10$, $\sigma = 2$, $\delta = 0.875$ and $s = 10.5$ in Theorem 6.2.5, then we can see that the loss of decay works in Theorem 6.2.5. Hence, we obtain the following admissible interval of exponents p:

n	$n=6$	$n=7$	$n=8$	$n=9$
p	$p \in [10, 15.17]$	$p \in [10, 13.14]$	$p \in [10, 11.625]$	$p \in [10, 10.44]$

Tab. 6.9.: The admissible range of exponents p depends on the space dimension n with loss of decay.

7. Semi-linear structurally damped σ-evolution models in the case $\delta \in \left(\frac{\sigma}{2}, \sigma\right]$

Let us consider the following two Cauchy problems:

$$u_{tt} + (-\Delta)^\sigma u + \mu(-\Delta)^\delta u_t = |u|^p, \quad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) \tag{7.1}$$

and

$$u_{tt} + (-\Delta)^\sigma u + \mu(-\Delta)^\delta u_t = |u_t|^p, \quad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) \tag{7.2}$$

in space dimensions $n \geqslant 2$ with $\sigma \geqslant 1$, $\delta \in (\frac{\sigma}{2}, \sigma]$, $\mu > 0$ and a given number $p > 1$.

Let us explain our objectives and strategies as follows:

- The estimates for solutions to the linear Cauchy problems (4.1) are a key tool to deal with the semi-linear Cauchy problems (7.1) and (7.2).
- By using the fractional Gagliardo-Nirenberg inequality, the fractional chain rule, the fractional powers rule, the fractional Sobolev embedding and some auxiliary inequalities, we obtain global (in time) existence of small data solutions in the energy space, in the solution space below energy space, in the energy space with a suitable higher regularity and in the large regular space.
- Some examples are presented at the end of each theorem to compare with known results.

In the following statements we introduce the data spaces $\mathcal{A}^s_{m,q} := \left(L^m \cap H^s_q\right) \times \left(L^m \cap H^{[s-2\delta]^+}_q\right)$ with the norm

$$\|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}} := \|u_0\|_{L^m} + \|u_0\|_{H^s_q} + \|u_1\|_{L^m} + \|u_1\|_{H^{[s-2\delta]^+}_q},$$

where $s \geqslant 0$, $q \in (1,\infty)$ and $m \in [1,q)$. Moreover, we fix the following constants:

$$\kappa_1 := 1 + \Big(1 + \Big[\frac{n}{2}\Big]\Big)\Big(1 - \frac{\sigma}{2\delta}\Big)\Big(1 + \frac{1}{q} - \frac{1}{m}\Big) \quad \text{and} \quad \kappa_2 := \Big(2 + \Big[\frac{n}{2}\Big]\Big)\Big(1 - \frac{\sigma}{2\delta}\Big)\Big(1 + \frac{1}{q} - \frac{1}{m}\Big).$$

7.1. Global (in time) existence of small data solutions to the model (7.1)

7.1.1. Data from the energy space

In the first result we assume data from energy space on the base of L^q.

Theorem 7.1.1. *Let $q \in (1,\infty)$ be a fixed constant, $m \in [1,q)$ and $n \geqslant 1$. We assume the condition*

$$p > 1 + \frac{\max\left\{2m\delta(1+\kappa_1),\, n - \frac{m}{q}n + 2m\delta\right\}}{n - 2m\delta\kappa_1}. \tag{7.3}$$

Moreover, we suppose the following conditions:

$$p \in \Big[\frac{q}{m}, \infty\Big) \text{ if } n \leqslant 2q\delta, \quad \text{or} \quad p \in \Big[\frac{q}{m}, \frac{n}{n-2q\delta}\Big] \text{ if } n \in \Big(2q\delta, \frac{2q^2\delta}{q-m}\Big]. \tag{7.4}$$

Then, there exists a constant $\varepsilon > 0$ such that for any small data

$$(u_0,u_1) \in \mathcal{A}^{2\delta}_{m,q} \text{ satisfying the assumption } \|(u_0,u_1)\|_{\mathcal{A}^{2\delta}_{m,q}} \leqslant \varepsilon,$$

we have a uniquely determined global (in time) small data energy solution (on the base of L^q)

$$u \in C\big([0,\infty), H^{2\delta}_q\big) \cap C^1\big([0,\infty), L^q\big)$$

to (7.1). The following estimates hold:

$$\big\|u(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|(u_0,u_1)\|_{\mathcal{A}^0_{m,q}}, \tag{7.5}$$

$$\big\||D|^\sigma u(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{\sigma}{2\delta}}\|(u_0,u_1)\|_{\mathcal{A}^\sigma_{m,q}}, \tag{7.6}$$

$$\big\|u_t(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|(u_0,u_1)\|_{\mathcal{A}^{2(\sigma-\delta)}_{m,q}}, \tag{7.7}$$

$$\big\||D|^{2\delta} u(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|(u_0,u_1)\|_{\mathcal{A}^{2\delta}_{m,q}}, \tag{7.8}$$

where $1+\frac{1}{q}=\frac{1}{r}+\frac{1}{m}$.

Proof. We introduce the data space $\mathcal{A}^{2\delta}_{m,q} := (L^m \cap H^{2\delta}_q)\times(L^m\cap L^q)$. Moreover, we introduce for any $t>0$ the function space $X(t) := C\big([0,t],H^{2\delta}_q\big)\cap C^1\big([0,t],L^q\big)$. For the sake of brevity, we also define the norm

$$\|u\|_{X(t)} := \sup_{0\leqslant\tau\leqslant t}\Big(f_0(\tau)^{-1}\|u(\tau,\cdot)\|_{L^q}+f_\sigma(\tau)^{-1}\big\||D|^\sigma u(\tau,\cdot)\big\|_{L^q}+g(\tau)^{-1}\|u_t(\tau,\cdot)\|_{L^q}$$
$$+f_\delta(\tau)^{-1}\big\||D|^{2\delta}u(\tau,\cdot)\big\|_{L^q}\Big),$$

and the space $X_0(t) := C\big([0,t],H^{2\delta}_q\big)$ with the norm

$$\|w\|_{X_0(t)} := \sup_{0\leqslant\tau\leqslant t}\Big(f_0(\tau)^{-1}\|w(\tau,\cdot)\|_{L^q}+f_\delta(\tau)^{-1}\big\||D|^{2\delta}w(\tau,\cdot)\big\|_{L^q}\Big),$$

where

$$f_0(\tau)=(1+\tau)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})},\quad f_\sigma(\tau)=(1+\tau)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{\sigma}{2\delta}},$$

and

$$g(\tau)=(1+\tau)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})},\quad f_\delta(\tau)=(1+\tau)^{(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}.$$

We define the operator $N:\ X(t)\longrightarrow X(t)$ by the formula

$$Nu(t,x)=K_0(t,x)\ast_x u_0(x)+K_1(t,x)\ast_x u_1(x)+\int_0^t K_1(t-\tau,x)\ast_x\big|u(\tau,x)\big|^p d\tau,$$

where $K_j(t,x)$ with $j=0,1$ are defined as in Chapter 4. We will prove that the operator N satisfies the following two estimates:

$$\|Nu\|_{X(t)} \lesssim \|(u_0,u_1)\|_{\mathcal{A}^{2\delta}_{m,q}}+\|u\|^p_{X_0(t)}, \tag{7.9}$$

$$\|Nu-Nv\|_{X(t)} \lesssim \|u-v\|_{X_0(t)}\big(\|u\|^{p-1}_{X_0(t)}+\|v\|^{p-1}_{X_0(t)}\big). \tag{7.10}$$

First let us prove the estimate (7.9). Taking into consideration the estimates for solutions and some of their partial derivatives to the linear Cauchy problems from Theorems 4.3.1 and 4.4.4 we get the following estimates with $s=0,\ \sigma,\ 2\delta$:

$$\big\||D|^s\big(K_0(t,x)\ast_x u_0(x)+K_1(t,x)\ast_x u_1(x)\big)\big\|_{L^q} \lesssim (1+t)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{s}{2\delta}}\|(u_0,u_1)\|_{\mathcal{A}^{2\delta}_{m,q}},$$

$$\big\|\partial_t\big(K_0(t,x)\ast_x u_0(x)+K_1(t,x)\ast_x u_1(x)\big)\big\|_{L^q} \lesssim (1+t)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|(u_0,u_1)\|_{\mathcal{A}^{2\delta}_{m,q}}.$$

In order to control the integral term in the representation of solutions, we use the $(L^m\cap L^q)-L^q$ estimates from Theorems 4.3.1 and 4.4.4. Therefore, we obtain

$$\big\|Nu(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|(u_0,u_1)\|_{\mathcal{A}^{2\delta}_{m,q}}$$
$$+\int_0^t(1+t-\tau)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\big\||u(\tau,\cdot)|^p\big\|_{L^m\cap L^q}d\tau.$$

Hence, it is necessary to require the estimates for $\big|u(\tau,x)\big|^p$ in $L^m\cap L^q$ as follows:

$$\big\||u(\tau,\cdot)|^p\big\|_{L^m\cap L^q} \lesssim \|u(\tau,\cdot)\|^p_{L^{mp}}+\|u(\tau,\cdot)\|^p_{L^{qp}}.$$

Applying the fractional Gagliardo-Nirenberg inequality from Proposition C.1.1 we may conclude

$$\begin{aligned}\|u(\tau,\cdot)\|_{L^{qp}} &\lesssim \big\||D|^{2\delta}u(\tau,\cdot)\big\|_{L^q}^{\theta_{qp}}\,\|u(\tau,\cdot)\|_{L^q}^{1-\theta_{qp}} \lesssim \big(f_\delta(\tau)\|u\|_{X_0(\tau)}\big)^{\theta_{qp}}\big(f_0(\tau)\|u\|_{X_0(\tau)}\big)^{1-\theta_{qp}}\\ &\lesssim (1+\tau)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\theta_{qp}}\|u\|_{X_0(\tau)},\end{aligned}$$

and

$$\begin{aligned}\|u(\tau,\cdot)\|_{L^{mp}} &\lesssim \big\||D|^{2\delta}u(\tau,\cdot)\big\|_{L^q}^{\theta_{mp}}\,\|u(\tau,\cdot)\|_{L^q}^{1-\theta_{mp}} \lesssim \big(f_\delta(\tau)\|u\|_{X_0(\tau)}\big)^{\theta_{mp}}\big(f_0(\tau)\|u\|_{X_0(\tau)}\big)^{1-\theta_{mp}}\\ &\lesssim (1+\tau)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\theta_{mp}}\|u\|_{X_0(\tau)},\end{aligned}$$

where

$$\theta_{qp}:=\theta_{0,2\delta}(qp,q)=\frac{n}{2\delta}\Big(\frac{1}{q}-\frac{1}{qp}\Big)\text{ and }\theta_{mp}:=\theta_{0,2\delta}(mp,q)=\frac{n}{2\delta}\Big(\frac{1}{q}-\frac{1}{mp}\Big).$$

As in Corollary C.1.1 we have to guarantee that θ_{qp} and θ_{mp} belong to $[0,1]$. Both conditions imply the restrictions

$$p\in\Big[\frac{q}{m},\infty\Big)\text{ if }n\leqslant 2q\delta,\qquad\text{or}\qquad p\in\Big[\frac{q}{m},\frac{n}{n-2q\delta}\Big]\text{ if }n>2q\delta.$$

By virtue of $\theta_{mp}<\theta_{qp}$ and the relation $1+\frac{1}{q}=\frac{1}{r}+\frac{1}{m}$, we derive

$$\big\||u(\tau,\cdot)|^p\big\|_{L^m\cap L^q}\lesssim(1+\tau)^{p\left(1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(\frac{1}{m}-\frac{1}{mp})\right)}\|u\|_{X_0(\tau)}^p.$$

Summarizing, from both estimates we may conclude

$$\begin{aligned}\big\|Nu(t,\cdot)\big\|_{L^q}&\lesssim(1+t)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|(u_0,u_1)\|_{\mathcal{A}^{2\delta}_{m,q}}\\ &+\|u\|_{X_0(t)}^p\int_0^t(1+t-\tau)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}(1+\tau)^{p\left(1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(\frac{1}{m}-\frac{1}{mp})\right)}d\tau.\end{aligned}$$

The key tool relies now in the application of Lemma B.6.1. Because of $p>1+\frac{2m\delta(1+\kappa_1)}{n-2m\delta\kappa_1}$, it follows

$$p\Big(1+\Big(1+\Big[\frac{n}{2}\Big]\Big)\Big(1-\frac{\sigma}{2\delta}\Big)\frac{1}{r}-\frac{n}{2\delta}\Big(\frac{1}{m}-\frac{1}{mp}\Big)\Big)<-1.$$

After applying Lemma B.6.1 with the condition $p>1+\frac{n-\frac{m}{q}n+2m\delta}{n-2m\delta\kappa_1}$, we get

$$\begin{aligned}&\int_0^t(1+t-\tau)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}(1+\tau)^{p\left(1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(\frac{1}{m}-\frac{1}{mp})\right)}d\tau\\ &\qquad\lesssim(1+t)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}.\end{aligned}$$

Finally, we conclude the following estimate:

$$\big\|Nu(t,\cdot)\big\|_{L^q}\lesssim(1+t)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\Big(\|(u_0,u_1)\|_{\mathcal{A}^{2\delta}_{m,q}}+\|u\|_{X_0(t)}^p\Big).$$

Analogously, we arrive at

$$\begin{aligned}\big\||D|^sNu(t,\cdot)\big\|_{L^q}&\lesssim(1+t)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{\sigma}{2\delta}}\Big(\|(u_0,u_1)\|_{\mathcal{A}^{2\delta}_{m,q}}+\|u\|_{X_0(t)}^p\Big),\\ \big\|\partial_tNu(t,\cdot)\big\|_{L^q}&\lesssim(1+t)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\Big(\|(u_0,u_1)\|_{\mathcal{A}^{2\delta}_{m,q}}+\|u\|_{X_0(t)}^p\Big),\end{aligned}$$

for $s=\sigma,\,2\delta$. From the definition of the norm in $X(t)$, we obtain immediately the inequality (7.9).

Next let us prove the estimate (7.10). Using again the $(L^m\cap L^q)-L^q$ estimates from Theorems 4.3.1 and 4.4.4 we have for two functions u and v from $X(t)$ the following estimates for $k=0,1$:

$$\begin{aligned}&\big\||D|^{2k\delta}\big(Nu(t,\cdot)-Nv(t,\cdot)\big)\big\|_{L^q}\\ &\qquad\lesssim\int_0^t(1+t-\tau)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-k}\big\||u(\tau,\cdot)|^p-|v(\tau,\cdot)|^p\big\|_{L^m\cap L^q}\,d\tau.\end{aligned}$$

Applying Hölder's inequality leads to

$$\big\||u(\tau,\cdot)|^p - |v(\tau,\cdot)|^p\big\|_{L^q} \lesssim \|u(\tau,\cdot) - v(\tau,\cdot)\|_{L^{qp}}\big(\|u(\tau,\cdot)\|^{p-1}_{L^{qp}} + \|v(\tau,\cdot)\|^{p-1}_{L^{qp}}\big),$$
$$\big\||u(\tau,\cdot)|^p - |v(\tau,\cdot)|^p\big\|_{L^m} \lesssim \|u(\tau,\cdot) - v(\tau,\cdot)\|_{L^{mp}}\big(\|u(\tau,\cdot)\|^{p-1}_{L^{mp}} + \|v(\tau,\cdot)\|^{p-1}_{L^{mp}}\big).$$

In the same way as in the proof of (7.9), after employing the fractional Gagliardo-Nirenberg inequality from Proposition C.1.1 to the norms

$$\|u(\tau,\cdot) - v(\tau,\cdot)\|_{L^\eta},\ \|u(\tau,\cdot)\|_{L^\eta},\ \|v(\tau,\cdot)\|_{L^\eta}$$

with $\eta = qp$ and $\eta = mp$ we have for $k = 0, 1$ the estimates

$$\begin{aligned}&\big\||D|^{2\delta k}\big(Nu(t,\cdot) - Nv(t,\cdot)\big)\big\|_{L^q}\\ &\qquad \lesssim (1+t)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r} - \frac{n}{2\delta}(1-\frac{1}{r}) - k}\|u - v\|_{X_0(t)}\big(\|u\|^{p-1}_{X_0(t)} + \|v\|^{p-1}_{X_0(t)}\big).\end{aligned}$$

Analogously, we also derive

$$\begin{aligned}&\big\||D|^{\sigma}\big(Nu(t,\cdot) - Nv(t,\cdot)\big)\big\|_{L^q}\\ &\qquad \lesssim (1+t)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r} - \frac{n}{2\delta}(1-\frac{1}{r}) - \frac{\sigma}{2\delta}}\|u - v\|_{X_0(t)}\big(\|u\|^{p-1}_{X_0(t)} + \|v\|^{p-1}_{X_0(t)}\big),\\ &\big\|\partial_t\big(Nu(t,\cdot) - Nv(t,\cdot)\big)\big\|_{L^q}\\ &\qquad \lesssim (1+t)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r} - \frac{n}{2\delta}(1-\frac{1}{r})}\|u - v\|_{X_0(t)}\big(\|u\|^{p-1}_{X_0(t)} + \|v\|^{p-1}_{X_0(t)}\big).\end{aligned}$$

From the definition of the norm in $X(t)$, we may conclude the inequality (7.10).
Summarizing, the proof of Theorem 7.1.1 is completed. □

Example 7.1.1. In the first example, we want to make a comparison between the statements from Theorem 7.1.1 and those from Theorem 4 in the paper [12] by choosing $m = 1$, $q = 2$, $\sigma = 1$ and $\delta = 0.8$. In general, with these selected parameters the admissible range of exponents p in Theorem 4 of [12] is more flexible than the result in Theorem 7.1.1 for the space dimensions $n = 4, 5, 6$. Nevertheless, for $n = 3$ the result in Theorem 4 of [12] is empty while the admissible range of exponents p in Theorem 7.1.1 is available (see the following table):

	Theorem 7.1.1	Theorem 4 in [12]
$n = 3$	$p \in (5.75, \infty)$	empty
$n = 4$	$p \in (3.73, 5]$	$p \in [2, 5]$
$n = 5$	$p \in (2.64, 2.78]$	$p \in [2, 2.78]$
$n = 6$	empty	$p \in [2, 2.14]$

Tab. 7.1.: The first comparison between the obtained results.

Example 7.1.2. In the second example, we want to emphasize that the results from Theorem 7.1.1 allow some flexibility in comparison with those from Theorem 4 in the paper [12] if we choose $m = 1$, $q = 3$, $\sigma = 1.4$ and $\delta = 1$ (see the following table):

	Theorem 7.1.1	Theorem 4 in [12]
$n = 3$	$p \in (8.33, \infty)$	empty
$n = 4$	$p \in (4.33, \infty)$	empty
$n = 5$	$p \in (3.22, \infty)$	empty
$n = 6$	$p \in [3, \infty)$	empty
$n = 7$	$p \in [3, 7]$	empty
$n = 8$	$p \in [3, 4]$	empty

Tab. 7.2.: The second comparison between the obtained results.

Example 7.1.3. In the third example, we want to make a comparison between the statements from Theorem 7.1.1 and those from Theorem 2 in the paper [12] in the viscoelastic damping case by choosing $m = 1$, $q = 2$ and $\sigma = \delta = 1$. In general, with these selected parameters the admissible range of exponents p in Theorem 2 of [12] is larger than that in Theorem 7.1.1 for the space dimensions $n = 4, 5$ (see the following table):

	Theorem 7.1.1	Theorem 2 in [12]
$n = 4$	$p \in (12, \infty)$	$p \in [2, \infty)$
$n = 5$	$p \in (4.67, 5]$	$p \in [2, 5]$

Tab. 7.3.: The third comparison between the obtained results.

Example 7.1.4. In the fourth example, we want to emphasize that in the viscoelastic damping case the results from Theorem 7.1.1 allow some flexibility in comparison with those from Theorem 2 in the paper [12] if we choose $m = 1$, $q = 3$ and $\sigma = \delta = 1$ (see the following table):

	Theorem 7.1.1	Theorem 2 in [12]
$n = 3$	$p \in (15, \infty)$	empty
$n = 4$	$p \in (6, \infty)$	empty
$n = 5$	$p \in (3.67, \infty)$	empty
$n = 6$	$p \in (3.25, \infty)$	empty
$n = 7$	$p \in [3, 7]$	empty
$n = 8$	$p \in [3, 4]$	empty

Tab. 7.4.: The fourth comparison between the obtained results.

Remark 7.1.1. In this remark, we allow a loss of decay in estimates for solutions to semi-linear models in comparison with the corresponding decay estimates for solutions of the linear Cauchy problem with vanishing right-hand side. We follow the proof of Theorem 7.1.1. Having this in mind we fix the data space and the solution space as in Theorem 7.1.1, but we use the norm

$$\|u\|_{X(t)} := \sup_{0\leqslant\tau\leqslant t}\Big(f_{\varepsilon_1}(\tau)^{-1}\|u(\tau,\cdot)\|_{L^q} + f_{\varepsilon_2}(\tau)^{-1}\big\||D|^\sigma u(\tau,\cdot)\big\|_{L^q} + f_{\varepsilon_3}(\tau)^{-1}\|u_t(\tau,\cdot)\|_{L^q}$$
$$+ f_{\varepsilon_4}(\tau)^{-1}\big\||D|^{2\delta} u(\tau,\cdot)\big\|_{L^q}\Big),$$

and the space $X_0(t) := C\big([0,t], H^{2\delta}_q\big)$ with the norm

$$\|w\|_{X_0(t)} := \sup_{0\leqslant\tau\leqslant t}\Big(f_{\varepsilon_1}(\tau)^{-1}\|w(\tau,\cdot)\|_{L^q} + f_{\varepsilon_4}(\tau)^{-1}\big\||D|^{2\delta} w(\tau,\cdot)\big\|_{L^q}\Big),$$

where

$$f_{\varepsilon_1}(\tau) = (1+\tau)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})+\varepsilon_1}, \quad f_{\varepsilon_2}(\tau) = (1+\tau)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{\sigma}{2\delta}+\varepsilon_2},$$

and

$$f_{\varepsilon_3}(\tau) = (1+\tau)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})+\varepsilon_3}, \quad f_{\varepsilon_4}(\tau) = (1+\tau)^{(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})+\varepsilon_4}.$$

for some positive constant ε_j with $j = 1, \cdots, 4$.

Now we fix the constant $\varepsilon_1 := (1-\frac{1}{p})\big(\frac{n}{2\delta}(1-\frac{1}{r})-\kappa_1\big)$. Next we choose $\varepsilon_2 = \frac{\sigma}{2\delta}$, $\varepsilon_3 = 1-\frac{\sigma}{2\delta}\big(\frac{1}{m}-\frac{1}{q}\big)$ and $\varepsilon_4 = 1+\varepsilon_1$. Then, we have to guarantee the following condition for the space dimension:

$$n > \frac{2mq\delta(1+\kappa_1)}{q-m},$$

which allows us to omit the condition (7.3).

7.1.2. Data below the energy space

In the second result we assume data from Sobolev space on the base of L^q.

Theorem 7.1.2. *Let $q \in (1, \infty)$ be a fixed constant and $m \in [1, q)$. Let $0 < s < 2\delta$ and $n \geqslant 1$. We assume the condition*

$$p > 1 + \frac{\max\big\{2m\delta(1+\kappa_1),\, n - \frac{m}{q}n + ms\big\}}{n - 2m\delta\kappa_1}. \tag{7.11}$$

Moreover, we suppose the following conditions:

$$p \in \Big[\frac{q}{m}, \infty\Big) \text{ if } n \leqslant qs, \quad \text{or} \quad p \in \Big[\frac{q}{m}, \frac{n}{n-qs}\Big] \text{ if } n \in \Big(qs, \frac{q^2 s}{q-m}\Big]. \tag{7.12}$$

Then, there exists a constant $\varepsilon > 0$ such that for any small data

$$(u_0, u_1) \in \mathcal{A}^s_{m,q} \text{ satisfying the assumption } \|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}} \leqslant \varepsilon,$$

we have a uniquely determined global (in time) small data Sobolev solution

$$u \in C\big([0,\infty), H^s_q\big)$$

to (7.1). The following estimates hold:

$$\big\|u(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r} - \frac{n}{2\delta}(1-\frac{1}{r})} \|(u_0,u_1)\|_{\mathcal{A}^0_{m,q}}, \tag{7.13}$$

$$\big\||D|^s u(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r} - \frac{n}{2\delta}(1-\frac{1}{r}) - \frac{s}{2\delta}} \|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}}, \tag{7.14}$$

where $1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{m}$.

Proof. We introduce the data space $\mathcal{A}^s_{m,q} := (L^m \cap H^s_q) \times (L^m \cap L^q)$. Moreover, we introduce for any $t > 0$ the function space $X(t) := C\big([0,t], H^s_q\big)$. For the sake of brevity, we also define the norm

$$\|u\|_{X(t)} := \sup_{0 \leqslant \tau \leqslant t} \Big(f_0(\tau)^{-1} \|u(\tau,\cdot)\|_{L^q} + f_s(\tau)^{-1} \big\||D|^s u(\tau,\cdot)\big\|_{L^q} \Big),$$

where

$$f_0(\tau) = (1+\tau)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r} - \frac{n}{2\delta}(1-\frac{1}{r})}, \quad f_s(\tau) = (1+\tau)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r} - \frac{n}{2\delta}(1-\frac{1}{r}) - \frac{s}{2\delta}}.$$

We define the operator $N:\ X(t) \longrightarrow X(t)$ by the formula

$$Nu(t,x) = K_0(t,x) \ast_x u_0(x) + K_1(t,x) \ast_x u_1(x) + \int_0^t K_1(t-\tau,x) \ast_x \big|u(\tau,x)\big|^p d\tau.$$

We will prove that the operator N satisfies the following two estimates:

$$\|Nu\|_{X(t)} \lesssim \|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}} + \|u\|^p_{X(t)}, \tag{7.15}$$

$$\|Nu - Nv\|_{X(t)} \lesssim \|u - v\|_{X(t)} \big(\|u\|^{p-1}_{X(t)} + \|v\|^{p-1}_{X(t)}\big). \tag{7.16}$$

First let us prove the estimate (7.15). Taking into consideration the estimates for solutions and some of their partial derivatives to the linear Cauchy problems in Theorems 4.3.3 and 4.4.4 we get the following estimates for $k = 0, 1$:

$$\big\||D|^{ks}\big(K_0(t,x) \ast_x u_0(x) + K_1(t,x) \ast_x u_1(x)\big)\big\|_{L^q} \lesssim (1+t)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r} - \frac{n}{2\delta}(1-\frac{1}{r}) - \frac{ks}{2\delta}} \|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}}.$$

In order to control the integral term in the representation of solutions, we use the $(L^m \cap L^q) - L^q$ estimates from Theorems 4.3.3 and 4.4.4. Therefore, we obtain

$$\begin{aligned} \big\||D|^{ks} Nu(t,\cdot)\big\|_{L^q} &\lesssim (1+t)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r} - \frac{n}{2\delta}(1-\frac{1}{r}) - \frac{ks}{2\delta}} \|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}} \\ &\quad + \int_0^t (1+t-\tau)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r} - \frac{n}{2\delta}(1-\frac{1}{r}) - \frac{ks}{2\delta}} \big\||u(\tau,\cdot)|^p\big\|_{L^m \cap L^q} d\tau. \end{aligned}$$

Hence, it is necessary to require the estimates for $|u(\tau,x)|^p$ in $L^m \cap L^q$ as follows:

$$\big\||u(\tau,\cdot)|^p\big\|_{L^m\cap L^q} \lesssim \|u(\tau,\cdot)\|^p_{L^{mp}} + \|u(\tau,\cdot)\|^p_{L^{qp}}.$$

Applying the fractional Gagliardo-Nirenberg inequality from Proposition C.1.1 we may conclude

$$\begin{aligned}\|u(\tau,\cdot)\|_{L^{qp}} &\lesssim \big\||D|^s u(\tau,\cdot)\big\|^{\theta_{qp}}_{L^q}\, \|u(\tau,\cdot)\|^{1-\theta_{qp}}_{L^q} \lesssim \big(f_s(\tau)\|u\|_{X(\tau)}\big)^{\theta_{qp}}\big(f_0(\tau)\|u\|_{X(\tau)}\big)^{1-\theta_{qp}}\\ &\lesssim (1+\tau)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{s}{2\delta}\theta_{qp}}\|u\|_{X(\tau)},\end{aligned}$$

and

$$\begin{aligned}\|u(\tau,\cdot)\|_{L^{mp}} &\lesssim \big\||D|^s u(\tau,\cdot)\big\|^{\theta_{mp}}_{L^q}\, \|u(\tau,\cdot)\|^{1-\theta_{mp}}_{L^q} \lesssim \big(f_s(\tau)\|u\|_{X(\tau)}\big)^{\theta_{mp}}\big(f_0(\tau)\|u\|_{X(\tau)}\big)^{1-\theta_{mp}}\\ &\lesssim (1+\tau)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{s}{2\delta}\theta_{mp}}\|u\|_{X(\tau)},\end{aligned}$$

where

$$\theta_{qp} := \theta_{0,s}(qp,q) = \frac{n}{s}\Big(\frac{1}{q}-\frac{1}{qp}\Big) \text{ and } \theta_{mp} := \theta_{0,s}(mp,q) = \frac{n}{s}\Big(\frac{1}{q}-\frac{1}{mp}\Big).$$

As in Corollary C.1.1 we have to guarantee that θ_{qp} and θ_{mp} belong to $[0,1]$. Both conditions imply the restrictions

$$p \in \Big[\frac{q}{m},\infty\Big) \text{ if } n \leqslant qs, \quad \text{ or } \quad p \in \Big[\frac{q}{m},\frac{n}{n-qs}\Big] \text{ if } n > qs.$$

By virtue of $\theta_{mp} < \theta_{qp}$ and the relation $1+\frac{1}{q} = \frac{1}{r}+\frac{1}{m}$, we derive

$$\big\||u(\tau,\cdot)|^p\big\|_{L^m\cap L^q} \lesssim (1+\tau)^{p\big(1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(\frac{1}{m}-\frac{1}{mp})\big)}\|u\|^p_{X(\tau)}.$$

Summarizing, from both estimates we may conclude

$$\begin{aligned}\big\||D|^{ks}Nu(t,\cdot)\big\|_{L^q} &\lesssim (1+t)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{ks}{2\delta}}\|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}}\\ &+\|u\|^p_{X(t)}\int_0^t (1+t-\tau)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{ks}{2\delta}}(1+\tau)^{p\big(1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(\frac{1}{m}-\frac{1}{mp})\big)}d\tau.\end{aligned}$$

The key tool relies now in the application of Lemma B.6.1. Because of $p > 1+\frac{2m\delta(1+\kappa_1)}{n-2m\delta\kappa_1}$, it follows

$$p\Big(1+\Big(1+\Big[\frac{n}{2}\Big]\Big)\Big(1-\frac{\sigma}{2\delta}\Big)\frac{1}{r}-\frac{n}{2\delta}\Big(\frac{1}{m}-\frac{1}{mp}\Big)\Big) < -1.$$

After applying Lemma B.6.1 with the condition $p > 1+\frac{n-\frac{m}{q}n+ms}{n-2m\delta\kappa_1}$, we get

$$\begin{aligned}&\int_0^t (1+t-\tau)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{ks}{2\delta}}(1+\tau)^{p\big(1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(\frac{1}{m}-\frac{1}{mp})\big)}d\tau\\ &\quad\lesssim (1+t)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{ks}{2\delta}}.\end{aligned}$$

Finally, we conclude the following estimates for $k=0,1$:

$$\big\||D|^{ks}Nu(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{ks}{2\delta}}\Big(\|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}}+\|u\|^p_{X(t)}\Big).$$

From the definition of the norm in $X(t)$, we obtain immediately the inequality (7.15).

Next let us prove the estimate (7.16). By applying again an analogous treatment as we did in the proof of (7.10) in Theorem 7.1.1, we may conclude the following estimates for $k=0,1$:

$$\big\||D|^{ks}\big(Nu(t,\cdot)-Nv(t,\cdot)\big)\big\|_{L^q} \lesssim (1+t)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{ks}{2\delta}}\|u-v\|_{X(t)}\big(\|u\|^{p-1}_{X(t)}+\|v\|^{p-1}_{X(t)}\big).$$

From the definition of the norm in $X(t)$, we obtain immediately the inequality (7.16).
Summarizing, the proof of Theorem 7.1.2 is completed. □

Remark 7.1.2. We want to underline that due to the flexibility of the choice of parameter $q \in (1,\infty)$, we really get a result for arbitrarily small positive s in Theorem 7.1.2. In particular, if we take any small positive $s=\epsilon$, then we may also choose, for example, a sufficiently large $q=\frac{1}{\epsilon^2}$ and $m=1$ in order to guarantee the existence of both an admissible space dimension n and admissible exponents p satisfying the required conditions in Theorem 7.1.2.

Example 7.1.5. In the first example, by choosing $m = 1$, $q = 2$, $\sigma = 1.8$, $\delta = 1$ and $s = 1.5$ we obtain the following admissible range of exponents p in the structural damping case:

$$p \in (6.25, \infty) \text{ if } n = 3, \quad \text{or} \quad p \in (3.53, 4] \text{ if } n = 4.$$

Example 7.1.6. By choosing $m = 1$, $q = 3$, $\sigma = \delta = 1$ and $s = 1.5$ we obtain the following admissible range of exponents p in the viscoelastic damping case:

n	$n = 3$	$n = 4$	$n = 5$	$n = 6$
p	$p \in (15, \infty)$	$p \in (6, \infty)$	$p \in (3.5, 10]$	$p \in (3.06, 4]$

Tab. 7.5.: The admissible range of exponents p depends on the space dimension n.

7.1.3. Data from the energy space with suitable higher regularity

The third result contains Sobolev solutions to (7.1) belonging to the energy space (on the base of L^q) with a suitable higher regularity.

Theorem 7.1.3. *Let $q \in (1, \infty)$ be a fixed constant and $m \in [1, q)$. Let $2\delta < s \leqslant 2\delta + \frac{n}{q}$ and $n \geqslant 1$. We assume that the exponent $p > 1 + \lceil s - 2\delta \rceil$ satisfies the condition*

$$p > 1 + \frac{\max\big\{2m\delta(1+\kappa_1),\, n - \frac{m}{q}n + ms\big\}}{n - 2m\delta\kappa_1}. \tag{7.17}$$

Moreover, we suppose the following conditions:

$$p \in \Big[\frac{q}{m}, \infty\Big) \text{ if } n \leqslant qs, \quad \text{or} \quad p \in \Big[\frac{q}{m}, 1 + \frac{2q\delta}{n - qs}\Big] \text{ if } n \in \Big(qs, qs + \frac{2mq\delta}{q - m}\Big]. \tag{7.18}$$

Then, there exists a constant $\varepsilon > 0$ such that for any small data

$$(u_0, u_1) \in \mathcal{A}^s_{m,q} \text{ satisfying the assumption } \|(u_0, u_1)\|_{\mathcal{A}^s_{m,q}} \leqslant \varepsilon,$$

we have a uniquely determined global (in time) small data energy solution

$$u \in C\big([0, \infty), H^s_q\big) \cap C^1\big([0, \infty), H^{s-2\delta}_q\big)$$

to (7.1). The following estimates hold:

$$\|u(t, \cdot)\|_{L^q} \lesssim (1+t)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r} - \frac{n}{2\delta}(1-\frac{1}{r})} \|(u_0, u_1)\|_{\mathcal{A}^0_{m,q}}, \tag{7.19}$$

$$\big\||D|^s u(t, \cdot)\big\|_{L^q} \lesssim (1+t)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r} - \frac{n}{2\delta}(1-\frac{1}{r}) - \frac{s}{2\delta}} \|(u_0, u_1)\|_{\mathcal{A}^s_{m,q}}, \tag{7.20}$$

$$\|u_t(t, \cdot)\|_{L^q} \lesssim (1+t)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r} - \frac{n}{2\delta}(1-\frac{1}{r})} \|(u_0, u_1)\|_{\mathcal{A}^{2(\sigma-\delta)}_{m,q}}, \tag{7.21}$$

$$\big\||D|^{s-2\delta} u_t(t, \cdot)\big\|_{L^q} \lesssim (1+t)^{1+(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r} - \frac{n}{2\delta}(1-\frac{1}{r}) - \frac{s}{2\delta}} \|(u_0, u_1)\|_{\mathcal{A}^s_{m,q}}, \tag{7.22}$$

where $1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{m}$.

Proof. We introduce the data space $\mathcal{A}^s_{m,q} := (L^m \cap H^s_q) \times (L^m \cap H^{s-2\delta}_q)$, the function space $X(t) := C\big([0, t], H^s_q\big) \cap C^1\big([0, t], H^{s-2\delta}_q\big)$ with the norm

$$\|u\|_{X(t)} := \sup_{0 \leqslant \tau \leqslant t} \Big(f_0(\tau)^{-1} \|u(\tau, \cdot)\|_{L^q} + f_s(\tau)^{-1} \big\||D|^s u(\tau, \cdot)\big\|_{L^q} + g_0(\tau)^{-1} \|u_t(\tau, \cdot)\|_{L^q} + g_s(\tau)^{-1} \big\||D|^{s-2\delta} u_t(\tau, \cdot)\big\|_{L^q} \Big),$$

and the space $X_0(t) := C\big([0, t], H^s_q\big)$ with the norm

$$\|w\|_{X_0(t)} := \sup_{0 \leqslant \tau \leqslant t} \Big(f_0(\tau)^{-1} \|w(\tau, \cdot)\|_{L^q} + f_s(\tau)^{-1} \big\||D|^s w(\tau, \cdot)\big\|_{L^q} \Big),$$

where

$$f_0(\tau)=(1+\tau)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})},\quad f_s(\tau)=(1+\tau)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{s}{2\delta}},$$

and

$$g_0(\tau)=(1+\tau)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})},\quad g_s(\tau)=(1+\tau)^{1+(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{s}{2\delta}}.$$

We define a mapping $N:\ X(t)\longrightarrow X(t)$ in the following way:

$$Nu(t,x)=K_0(t,x)*_x u_0(x)+K_1(t,x)*_x u_1(x)+\int_0^t K_1(t-\tau,x)*_x |u(\tau,x)|^p d\tau.$$

In order to conclude the uniqueness and the global (in time) existence of small data solutions to (7.1), we have to prove the following pair of inequalities:

$$\|Nu\|_{X(t)}\lesssim \|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}}+\|u\|^p_{X_0(t)}, \tag{7.23}$$

$$\|Nu-Nv\|_{X(t)}\lesssim \|u-v\|_{X_0(t)}\big(\|u\|^{p-1}_{X_0(t)}+\|v\|^{p-1}_{X_0(t)}\big). \tag{7.24}$$

First let us prove the inequality (7.23). Our proof is divided into four steps.

Step 1: We need to estimate the norm $\|Nu(t,\cdot)\|_{L^q}$. We use the $(L^m\cap L^q)-L^q$ estimates from Theorems 4.3.1 and 4.4.4 to obtain

$$\begin{aligned}\big\|Nu(t,\cdot)\big\|_{L^q}&\lesssim (1+t)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r}}\|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}}\\&\quad+\int_0^t(1+t-\tau)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\big\||u(\tau,\cdot)|^p\big\|_{L^m\cap L^q}d\tau.\end{aligned}$$

Hence, it is necessary to require the estimates for $\big|u(\tau,x)\big|^p$ in $L^m\cap L^q$ as follows:

$$\big\||u(\tau,\cdot)|^p\big\|_{L^m\cap L^q}\lesssim \|u(\tau,\cdot)\|^p_{L^{mp}}+\|u(\tau,\cdot)\|^p_{L^{qp}}.$$

Applying the fractional Gagliardo-Nirenberg inequality from Proposition C.1.1 we may conclude

$$\begin{aligned}\|u(\tau,\cdot)\|_{L^{qp}}&\lesssim \big\||D|^s u(\tau,\cdot)\big\|^{\theta_{qp}}_{L^q}\|u(\tau,\cdot)\|^{1-\theta_{qp}}_{L^q}\lesssim \big(f_s(\tau)\|u\|_{X_0(\tau)}\big)^{\theta_{qp}}\big(f_0(\tau)\|u\|_{X_0(\tau)}\big)^{1-\theta_{qp}}\\&\lesssim (1+\tau)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{s}{2\delta}\theta_{qp}}\|u\|_{X_0(\tau)},\end{aligned}$$

and

$$\begin{aligned}\|u(\tau,\cdot)\|_{L^{mp}}&\lesssim \big\||D|^s u(\tau,\cdot)\big\|^{\theta_{mp}}_{L^q}\|u(\tau,\cdot)\|^{1-\theta_{mp}}_{L^q}\lesssim \big(f_s(\tau)\|u\|_{X_0(\tau)}\big)^{\theta_{mp}}\big(f_0(\tau)\|u\|_{X_0(\tau)}\big)^{1-\theta_{mp}}\\&\lesssim (1+\tau)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{s}{2\delta}\theta_{mp}}\|u\|_{X_0(\tau)},\end{aligned}$$

where

$$\theta_{qp}:=\theta_{0,s}(qp,q)=\frac{n}{s}\Big(\frac{1}{q}-\frac{1}{qp}\Big)\text{ and }\theta_{mp}:=\theta_{0,s}(mp,q)=\frac{n}{s}\Big(\frac{1}{q}-\frac{1}{mp}\Big).$$

As in Corollary C.1.1 we have to guarantee that θ_{qp} and θ_{mp} belong to $[0,1]$. Both conditions imply the restrictions

$$p\in\Big[\frac{q}{m},\infty\Big)\text{ if }n\leqslant qs,\qquad\text{or}\qquad p\in\Big[\frac{q}{m},\frac{n}{n-qs}\Big]\text{ if }n>qs.$$

By virtue of $\theta_{mp}<\theta_{qp}$ and the relation $1+\frac{1}{q}=\frac{1}{r}+\frac{1}{m}$, we derive

$$\big\||u(\tau,\cdot)|^p\big\|_{L^m\cap L^q}\lesssim (1+\tau)^{p\big(1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(\frac{1}{m}-\frac{1}{mp})\big)}\|u\|^p_{X_0(\tau)}.$$

Summarizing, from both estimates we may conclude

$$\begin{aligned}\|Nu(t,\cdot)\|_{L^q}&\lesssim (1+t)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}}\\&\quad+\|u\|^p_{X_0(t)}\int_0^t(1+t-\tau)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}(1+\tau)^{p\big(1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(\frac{1}{m}-\frac{1}{mp})\big)}d\tau.\end{aligned}$$

The key tool relies now in the application of Lemma B.6.1. Because of $p > 1 + \frac{2m\delta(1+\kappa_1)}{n-2m\delta\kappa_1}$, it follows

$$p\Big(1+\Big(1+\Big[\frac{n}{2}\Big]\Big)\Big(1-\frac{\sigma}{2\delta}\Big)\frac{1}{r}-\frac{n}{2\delta}\Big(\frac{1}{m}-\frac{1}{mp}\Big)\Big) < -1.$$

After applying Lemma B.6.1 with the condition $p > 1 + \frac{n-\frac{m}{q}n+ms}{n-2m\delta\kappa_1}$, we get

$$\begin{aligned}\int_0^t &(1+t-\tau)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}(1+\tau)^{p\big(1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(\frac{1}{m}-\frac{1}{mp})\big)}d\tau\\ &\lesssim (1+t)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}.\end{aligned}$$

Therefore, we arrive at the following desired estimate:

$$\|Nu(t,\cdot)\|_{L^q} \lesssim (1+t)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\big(\|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}}+\|u\|^p_{X_0(t)}\big). \tag{7.25}$$

Step 2: We need to estimate the norm $\|\partial_t Nu(t,\cdot)\|_{L^q}$. Differentiating $Nu(t,x)$ with respect to t we obtain

$$\partial_t Nu(t,x) = \partial_t\big(K_0(t,x)*_x u_0(x)+K_1(t,x)*_x u_1(x)\big)+\int_0^t \partial_t\big(K_1(t-\tau,x)*_x|u(\tau,x)|^p\big)d\tau.$$

We apply the $L^m\cap L^q - L^q$ estimates from Theorems 4.3.1 and 4.4.4 to conclude

$$\begin{aligned}\|\partial_t Nu(t,\cdot)\|_{L^q} &\lesssim (1+t)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}}\\ &\quad+\int_0^t (1+t-\tau)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\big\||u(\tau,\cdot)|^p\big\|_{L^m\cap L^q}d\tau.\end{aligned}$$

Using the same ideas for deriving (7.25) we may conclude

$$\|\partial_t Nu(t,\cdot)\|_{L^q} \lesssim (1+t)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\big(\|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}}+\|u\|^p_{X_0(t)}\big),$$

under the same assumptions for p, that is,

$$p \in \Big[\frac{q}{m},\infty\Big) \text{ if } n \leqslant qs, \quad \text{ or } \quad p \in \Big[\frac{q}{m},\frac{n}{n-qs}\Big] \text{ if } n > qs,$$

and

$$p > 1+\frac{\max\{2m\delta(1+\kappa_1),\, n-\frac{m}{q}n+ms\}}{n-2m\delta\kappa_1}.$$

Step 3: Let us estimate the norm $\big\|\partial_t|D|^{s-2\delta}Nu(t,\cdot)\big\|_{L^q}$. We use

$$\begin{aligned}\partial_t|D|^{s-2\delta}Nu(t,x) &= \partial_t|D|^{s-2\delta}\big(K_0(t,x)*_x u_0(x)+K_1(t,x)*_x u_1(x)\big)\\ &\quad+\int_0^t \partial_t|D|^{s-2\delta}\big(K_1(t-\tau,x)*_x|u(\tau,x)|^p\big)d\tau.\end{aligned}$$

We apply the $(L^m\cap L^q)-L^q$ estimates from Theorems 4.3.1 and 4.4.4 to derive

$$\begin{aligned}\big\|\partial_t|D|^{s-2\delta}Nu(t,\cdot)\big\|_{L^q} &\lesssim (1+t)^{1+(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{s}{2\delta}}\|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}}\\ &\quad+\int_0^t (1+t-\tau)^{1+(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{s}{2\delta}}\big\||u(\tau,\cdot)|^p\big\|_{L^m\cap H^{s-2\delta}_q}d\tau\\ &= (1+t)^{1+(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{s}{2\delta}}\|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}}\\ &\quad+\int_0^t (1+t-\tau)^{1+(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{s}{2\delta}}\big\||u(\tau,\cdot)|^p\big\|_{L^m\cap L^q\cap \dot{H}^{s-2\delta}_q}d\tau.\end{aligned}$$

The integrals with $\big\||u(\tau,\cdot)|^p\big\|_{L^m\cap L^q}$ will be handled as before to get

$$\begin{aligned}\int_0^t &(1+t-\tau)^{1+(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{s}{2\delta}}\big\||u(\tau,\cdot)|^p\big\|_{L^m\cap L^q}d\tau\\ &\lesssim (1+t)^{1+(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{s}{2\delta}}\|u\|^p_{X_0(t)}.\end{aligned}$$

To estimate the integral with the norm $\big\||u(\tau,\cdot)|^p\big\|_{\dot H^{s-2\delta}_q}$, we shall apply Proposition C.3.2 for the fractional chain rule with $p > \lceil s-2\delta \rceil$. Therefore, we obtain

$$\big\||u(\tau,\cdot)|^p\big\|_{\dot H^{s-2\delta}_q} \lesssim \|u(\tau,\cdot)\|^{p-1}_{L^{q_1}}\,\big\||D|^{s-2\delta}u(\tau,\cdot)\big\|_{L^{q_2}}, \text{ where } \frac{1}{q}=\frac{p-1}{q_1}+\frac{1}{q_2}.$$

Applying the fractional Gagliardo-Nirenberg inequality from Proposition C.1.1 we have

$$\begin{aligned}\|u(\tau,\cdot)\|_{L^{q_1}} &\lesssim \|u(\tau,\cdot)\|^{1-\theta_{q_1}}_{L^q}\,\big\||D|^s u(\tau,\cdot)\big\|^{\theta_{q_1}}_{L^q}\\ &\lesssim (1+\tau)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{s}{2\delta}\theta_{q_1}}\|u\|_{X_0(\tau)},\end{aligned}$$

and

$$\begin{aligned}\big\||D|^{s-2\delta}u(\tau,\cdot)\big\|_{L^{q_2}} &\lesssim \|u(\tau,\cdot)\|^{1-\theta_{q_2}}_{L^q}\,\big\||D|^s u(\tau,\cdot)\big\|^{\theta_{q_2}}_{L^q}\\ &\lesssim (1+\tau)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{s}{2\delta}\theta_{q_2}}\|u\|_{X_0(\tau)},\end{aligned}$$

where

$$\theta_{q_1} := \theta_{0,s}(q_1,q) = \frac{n}{s}\Big(\frac{1}{q}-\frac{1}{q_1}\Big) \text{ and } \theta_{q_2} := \theta_{s-2\delta,s}(q_2,q) = \frac{n}{s}\Big(\frac{1}{q}-\frac{1}{q_2}+\frac{s-2\delta}{n}\Big).$$

Hence, we may conclude

$$\begin{aligned}\big\||u(\tau,\cdot)|^p\big\|_{\dot H^{s-2\delta}_q} &\lesssim (1+\tau)^{p\big(1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})\big)-\frac{s}{2\delta}\big((p-1)\theta_{q_1}+\theta_{q_2}\big)}\|u\|^p_{X_0(\tau)}\\ &\lesssim (1+\tau)^{p\big(1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(\frac{1}{m}-\frac{1}{qp})\big)-\frac{s-2\delta}{2\delta}}\|u\|^p_{X_0(\tau)},\end{aligned}$$

where we can see that $(p-1)\theta_{q_1}+\theta_{q_2} = \frac{n}{s}\big(\frac{p-1}{q}+\frac{s-2\delta}{n}\big)$. Here we have to guarantee that $\theta_{q_1}\in[0,1]$ and $\theta_{q_2}\in[\frac{s-2\delta}{s},1]$. Both conditions imply the restriction

$$1<p\leqslant 1+\frac{2q\delta}{n-qs} \text{ if } n>qs, \quad \text{ or } \quad p>1 \text{ if } n\leqslant qs.$$

Therefore, we have shown the estimates

$$\big\|\partial_t|D|^{s-2\delta}Nu(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{1+(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{s}{2\delta}}\big(\|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}}+\|u\|^p_{X_0(t)}\big). \tag{7.26}$$

Step 4: Let us estimate the norm $\big\||D|^sNu(t,\cdot)\big\|_{L^q}$. We use

$$\begin{aligned}|D|^sNu(t,x) &= |D|^s\big(K_0(t,x)*_x u_0(x)+K_1(t,x)*_x u_1(x)\big)\\ &\quad+\int_0^t |D|^s\big(K_1(t-\tau,x)*_x|u(\tau,x)|^p\big)d\tau.\end{aligned}$$

By applying again the $(L^m\cap L^q)-L^q$ estimates on the interval $\tau\in[0,t]$ from Theorems 4.3.1 and 4.4.4, we derive

$$\begin{aligned}\big\||D|^sNu(t,\cdot)\big\|_{L^q} &\lesssim (1+t)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{s}{2\delta}}\|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}}\\ &\quad+\int_0^t(1+t-\tau)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{s}{2\delta}}\big\||u(\tau,\cdot)|^p\big\|_{L^m\cap H^{s-2\delta}_q}d\tau.\end{aligned}$$

Following the approach to show (7.26) we may conclude

$$\big\||D|^sNu(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{s}{2\delta}}\big(\|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}}+\|u\|^p_{X_0(t)}\big).$$

Summarizing, from the definition of the norm in $X(t)$ we obtain immediately the inequality (7.23).

Next let us prove the inequality (7.24). Following the proof of Theorem 7.1.1, the new difficulty is to estimate the norm

$$\big\||u(\tau,\cdot)|^p-|v(\tau,\cdot)|^p\big\|_{\dot H^{s-2\delta}_q}.$$

The integral representation

$$|u(\tau,x)|^p - |v(\tau,x)|^p = p\int_0^1 \big(u(\tau,x)-v(\tau,x)\big)G\big(\omega u(\tau,x)+(1-\omega)v(\tau,x)\big)\,d\omega,$$

where $G(u) = u|u|^{p-2}$, leads to

$$\big\||u(\tau,\cdot)|^p - |v(\tau,\cdot)|^p\big\|_{\dot{H}^{s-2\delta}_q} \lesssim \int_0^1 \Big\||D|^{s-2\delta}\Big(\big(u(\tau,\cdot)-v(\tau,\cdot)\big)G\big(\omega u(\tau,\cdot)+(1-\omega)v(\tau,\cdot)\big)\Big)\Big\|_{L^q}\,d\omega.$$

Applying the fractional Leibniz formula from Proposition C.2.1 we derive the following estimate:

$$\begin{aligned}\big\||u(\tau,\cdot)|^p - |v(\tau,\cdot)|^p\big\|_{\dot{H}^{s-2\delta}_q} &\lesssim \big\||D|^{s-2\delta}\big(u(\tau,\cdot)-v(\tau,\cdot)\big)\big\|_{L^{r_1}}\int_0^1 \big\|G\big(\omega u(\tau,\cdot)+(1-\omega)v(\tau,\cdot)\big)\big\|_{L^{r_2}}\,d\omega\\ &\quad+\|u(\tau,\cdot)-v(\tau,\cdot)\|_{L^{r_3}}\int_0^1 \big\||D|^{s-2\delta}G\big(\omega u(\tau,\cdot)+(1-\omega)v(\tau,\cdot)\big)\big\|_{L^{r_4}}\,d\omega\\ &\lesssim \big\||D|^{s-2\delta}\big(u(\tau,\cdot)-v(\tau,\cdot)\big)\big\|_{L^{r_1}}\Big(\|u(\tau,\cdot)\|^{p-1}_{L^{r_2(p-1)}}+\|v(\tau,\cdot)\|^{p-1}_{L^{r_2(p-1)}}\Big)\\ &\quad+\|u(\tau,\cdot)-v(\tau,\cdot)\|_{L^{r_3}}\int_0^1 \big\||D|^{s-2\delta}G\big(\omega u(\tau,\cdot)+(1-\omega)v(\tau,\cdot)\big)\big\|_{L^{r_4}}\,d\omega,\end{aligned}$$

where

$$\frac{1}{r_1}+\frac{1}{r_2}=\frac{1}{r_3}+\frac{1}{r_4}=\frac{1}{q}.$$

Taking into consideration the fractional Gagliardo-Nirenberg inequality from Proposition C.1.1 we obtain

$$\begin{aligned}\big\||D|^{s-2\delta}\big(u(\tau,\cdot)-v(\tau,\cdot)\big)\big\|_{L^{r_1}} &\lesssim \|u(\tau,\cdot)-v(\tau,\cdot)\|^{\theta_1}_{\dot{H}^s_q}\,\|u(\tau,\cdot)-v(\tau,\cdot)\|^{1-\theta_1}_{L^q},\\ \|u(\tau,\cdot)\|_{L^{r_2(p-1)}} &\lesssim \|u(\tau,\cdot)\|^{\theta_2}_{\dot{H}^s_q}\,\|u(\tau,\cdot)\|^{1-\theta_2}_{L^q},\\ \|u(\tau,\cdot)-v(\tau,\cdot)\|_{L^{r_3}} &\lesssim \|u(\tau,\cdot)-v(\tau,\cdot)\|^{\theta_3}_{\dot{H}^s_q}\,\|u(\tau,\cdot)-v(\tau,\cdot)\|^{1-\theta_3}_{L^q},\end{aligned}$$

where

$$\theta_1=\frac{n}{s}\Big(\frac{1}{q}-\frac{1}{r_1}+\frac{s-2\delta}{n}\Big)\in\Big[\frac{s-2\delta}{s},1\Big],\ \theta_2=\frac{n}{s}\Big(\frac{1}{q}-\frac{1}{r_2(p-1)}\Big)\in[0,1],\ \text{and } \theta_3=\frac{n}{s}\Big(\frac{1}{q}-\frac{1}{r_3}\Big)\in[0,1].$$

Because $\omega\in[0,1]$ is a parameter, employing again the fractional chain rule with $p>1+\lceil s-2\delta\rceil$ from Proposition C.3.2 and the fractional Gagliardo-Nirenberg inequality we get

$$\begin{aligned}&\big\||D|^{s-2\delta}G\big(\omega u(\tau,\cdot)+(1-\omega)v(\tau,\cdot)\big)\big\|_{L^{r_4}}\\ &\quad\lesssim \|\omega u(\tau,\cdot)+(1-\omega)v(\tau,\cdot)\|^{p-2}_{L^{r_5}}\,\big\||D|^{s-2\delta}\big(\omega u(\tau,\cdot)+(1-\omega)v(\tau,\cdot)\big)\big\|_{L^{r_6}}\\ &\quad\lesssim \|\omega u(\tau,\cdot)+(1-\omega)v(\tau,\cdot)\|^{(p-2)\theta_5+\theta_6}_{\dot{H}^s_q}\,\|\omega u(\tau,\cdot)+(1-\omega)v(\tau,\cdot)\|^{(p-2)(1-\theta_5)+1-\theta_6}_{L^q},\end{aligned}$$

where

$$\frac{p-2}{r_5}+\frac{1}{r_6}=\frac{1}{r_4},\quad \theta_5=\frac{n}{s}\Big(\frac{1}{q}-\frac{1}{r_5}\Big)\in[0,1]\ \text{ and }\ \theta_6=\frac{n}{s}\Big(\frac{1}{q}-\frac{1}{r_6}+\frac{s-2\delta}{n}\Big)\in\Big[\frac{s-2\delta}{s},1\Big].$$

All together it follows

$$\begin{aligned}&\int_0^1 \big\||D|^{s-2\delta}G\big(\omega u(\tau,\cdot)+(1-\omega)v(\tau,\cdot)\big)\big\|_{L^{r_4}}d\omega\\ &\quad\lesssim \big(\|u(\tau,\cdot)\|_{\dot{H}^s_q}+\|v(\tau,\cdot)\|_{\dot{H}^s_q}\big)^{(p-2)\theta_5+\theta_6}\big(\|u(\tau,\cdot)\|_{L^q}+\|v(\tau,\cdot)\|_{L^q}\big)^{(p-2)(1-\theta_5)+1-\theta_6}.\end{aligned}$$

Hence, we derived the following estimate:

$$\begin{aligned}&\big\||u(\tau,\cdot)|^p-|v(\tau,\cdot)|^p\big\|_{\dot{H}^{s-2\delta}_q}\\ &\quad\lesssim (1+\tau)^{p\left(1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(\frac{1}{m}-\frac{1}{qp})\right)-\frac{s-2\delta}{2\delta}}\|u-v\|_{X_0(\tau)}\big(\|u\|^{p-1}_{X_0(\tau)}+\|v\|^{p-1}_{X_0(\tau)}\big),\end{aligned}$$

where we note that

$$\theta_1 + (p-1)\theta_2 = \theta_3 + (p-2)\theta_5 + \theta_6 = \frac{n}{s}\Big(\frac{p-1}{q} + \frac{s-2\delta}{n}\Big).$$

Therefore, we have proved the estimates

$$\begin{aligned}
&\big\||D|^{s-2\delta}\partial_t\big(Nu(t,\cdot) - Nv(t,\cdot)\big)\big\|_{L^q} \\
&\qquad \lesssim (1+t)^{1+(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{s}{2\delta}}\|u-v\|_{X_0(t)}\big(\|u\|^{p-1}_{X_0(t)} + \|v\|^{p-1}_{X_0(t)}\big), \\
&\big\||D|^{s}\big(Nu(t,\cdot) - Nv(t,\cdot)\big)\big\|_{L^q} \\
&\qquad \lesssim (1+t)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{s}{2\delta}}\|u-v\|_{X_0(t)}\big(\|u\|^{p-1}_{X_0(t)} + \|v\|^{p-1}_{X_0(t)}\big).
\end{aligned}$$

From the definition of the norm in $X(t)$ the inequality (7.24) follows.
Summarizing, the proof of Theorem 7.1.3 is complete. □

Remark 7.1.3. One should explain if one can really choose the parameters q_1, q_2, $r_1, \cdots, r_6$ and $\theta_1, \cdots, \theta_6$ as required in the proof of Theorem 7.1.3. Following the explanations as we did in Remark 5.1.4 we may conclude the following conditions:

$$2 \leqslant p \leqslant 1 + \frac{q2\delta}{n-qs} \text{ if } n > qs, \quad \text{ or } \quad p \geqslant 2 \text{ if } n \leqslant qs.$$

These conditions are sufficient to guarantee the existence of all these parameters satisfying the required conditions.

Example 7.1.7. In the first example, by choosing $m = 1$, $q = 2$, $\sigma = 1.8$, $\delta = 1$ and $s = 2.5$ we obtain the following admissible range of exponents p in the structural damping case:

n	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
p	$p \in (6.25, \infty)$	$p \in (3.65, \infty)$	$p \in (2.85, \infty)$	$p \in (2.53, 5]$	$p \in (2.37, 3]$	$p \in (2.18, 2.33]$

Tab. 7.6.: The admissible range of exponents p depends on the space dimension n.

Example 7.1.8. In the second example, by choosing $m = 1$, $q = 2$, $\sigma = \delta = 1$ and $s = 2.5$ we obtain the following admissible range of exponents p in the viscoelastic damping case:

n	$n=4$	$n=5$	$n=6$
p	$p \in (12, \infty)$	$p \in (4.67, \infty)$	$p \in (4, 5]$

Tab. 7.7.: The admissible range of exponents p depends on the space dimension n.

7.1.4. Large regular data

Next, we obtain large regular solutions to (7.1) by using the fractional powers rule and the fractional Sobolev embedding.

Theorem 7.1.4. *Let $q \in (1,\infty)$ be a fixed constant and $m \in [1,q)$. Let $s > 2\delta + \frac{n}{q}$ and $n \geqslant 1$. We assume that the exponent $p > 1 + s - 2\delta$ satisfies the condition*

$$p > 1 + \frac{\max\big\{2m\delta(1+\kappa_1),\; n - \frac{m}{q}n + ms\big\}}{n - 2m\delta\kappa_1}. \tag{7.27}$$

Moreover, we suppose the following conditions:

$$p \in \Big[\frac{q}{m}, \infty\Big) \text{ and } n > 2m\delta\kappa_1. \tag{7.28}$$

Then, there exists a constant $\varepsilon > 0$ such that for any small data

$$(u_0, u_1) \in \mathcal{A}^s_{m,q} \text{ satisfying the assumption } \|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}} \leqslant \varepsilon,$$

we have a uniquely determined global (in time) small data energy solution

$$u \in C\big([0,\infty), H^s_q\big) \cap C^1\big([0,\infty), H^{s-2\delta}_q\big)$$

to (7.1). Moreover, the estimates (7.19) to (7.22) hold.

Proof. We introduce the definitions of spaces $\mathcal{A}^s_{m,q}$, $X(t)$ and $X_0(t)$ as in the proof of Theorem 7.1.3. We repeat exactly on the one hand the same estimates for the terms $|u(\tau,\cdot)|^p$ and $|u(\tau,\cdot)|^p - |v(\tau,\cdot)|^p$ in L^m and L^q. On the other hand, we estimate the above terms in $\dot{H}^{s-2\delta}_q$ by using the fractional powers rule and the fractional Sobolev embedding.

In the first step, let us begin with $\big\||u(\tau,\cdot)|^p\big\|_{\dot{H}^{s-2\delta}_q}$. We shall apply Corollary C.4.1 for the fractional powers rule with $s-2\delta \in (\frac{n}{q}, p)$. Therefore, we obtain

$$\begin{aligned}\big\||u(\tau,\cdot)|^p\big\|_{\dot{H}^{s-2\delta}_q} &\lesssim \|u(\tau,\cdot)\|_{\dot{H}^{s-2\delta}_q}\|u(\tau,\cdot)\|^{p-1}_{L^\infty}\\ &\lesssim \|u(\tau,\cdot)\|_{\dot{H}^{s-2\delta}_q}\big(\|u(\tau,\cdot)\|_{\dot{H}^{s^*}_q} + \|u(\tau,\cdot)\|_{\dot{H}^{s-2\delta}_q}\big)^{p-1}.\end{aligned}$$

Here we used Corollary C.5.1 with a suitable $s^* < \frac{n}{q}$. Applying the fractional Gagliardo-Nirenberg inequality from Proposition C.1.1 we have

$$\|u(\tau,\cdot)\|_{\dot{H}^{s-2\delta}_q} \lesssim \|u(\tau,\cdot)\|^{1-\theta_1}_{L^q}\big\||D|^s u(\tau,\cdot)\big\|^{\theta_1}_{L^q} \lesssim (1+\tau)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{s-2\delta}{2\delta}}\|u\|_{X_0(\tau)},$$

$$\|u(\tau,\cdot)\|_{\dot{H}^{s^*}_q} \lesssim \|u(\tau,\cdot)\|^{1-\theta_2}_{L^q}\big\||D|^s u(\tau,\cdot)\big\|^{\theta_2}_{L^q} \lesssim (1+\tau)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{s^*}{2\delta}}\|u\|_{X_0(\tau)},$$

where $\theta_1 = 1-\frac{2\delta}{s}$ and $\theta_2 = \frac{s^*}{s}$. Hence, we derive

$$\begin{aligned}\big\||u(\tau,\cdot)|^p\big\|_{\dot{H}^{s-2\delta}_q} &\lesssim (1+\tau)^{p\big(1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})\big)-\frac{s-2\delta}{2\delta}-(p-1)\frac{s^*}{2\delta}}\|u\|_{X_0(\tau)}\\ &\lesssim (1+\tau)^{p\big(1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(\frac{1}{m}-\frac{1}{mp})\big)}\|u\|^p_{X_0(\tau)},\end{aligned}$$

if we choose $s^* = \frac{n}{q} - \varepsilon_0$ with a sufficiently small $\varepsilon_0 > 0$. Therefore, by an analogous argument as we did in the proof of Theorem 7.1.3 we may conclude

$$\begin{aligned}&\int_0^t (1+t-\tau)^{1+(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{s}{2\delta}}\big\||u(\tau,\cdot)|^p\big\|_{\dot{H}^{s-2\delta}_q}d\tau\\ &\qquad\lesssim (1+t)^{1+(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{s}{2\delta}}\|u\|^p_{X_0(t)}.\end{aligned}$$

Finally, let us turn to estimate the norm $\big\||u(\tau,\cdot)|^p - |v(\tau,\cdot)|^p\big\|_{\dot{H}^{s-2\delta}_q}$. Then, repeating the proof of the second step of Theorem 5.1.4 and using the same treatment as in the proof of the above first step we get

$$\begin{aligned}&\big\||u(\tau,\cdot)|^p - |v(\tau,\cdot)|^p\big\|_{\dot{H}^{s-2\delta}_q}\\ &\qquad\lesssim (1+\tau)^{p\big(1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(\frac{1}{m}-\frac{1}{mp})\big)}\|u-v\|_{X_0(\tau)}\big(\|u\|^{p-1}_{X_0(\tau)} + \|v\|^{p-1}_{X_0(\tau)}\big).\end{aligned}$$

Hence, we arrive at

$$\begin{aligned}&\int_0^t (1+t-\tau)^{1+(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{s}{2\delta}}\big\||u(\tau,\cdot)|^p - |v(\tau,\cdot)|^p\big\|_{\dot{H}^{s-2\delta}_q}d\tau\\ &\qquad\lesssim (1+t)^{1+(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{s}{2\delta}}\|u-v\|_{X_0(t)}\big(\|u\|^{p-1}_{X_0(t)} + \|v\|^{p-1}_{X_0(t)}\big).\end{aligned}$$

Summarizing, the proof of Theorem 7.1.4 is completed. □

Example 7.1.9. In the first example, by choosing $m = 1$, $q = 4$, $\sigma = 1.8$, $\delta = 1$ and $s = 3.5$ we obtain the following admissible range of exponents p in the structural damping case (see Tab.7.8):

Example 7.1.10. In the second example, by choosing $m = 1$, $q = 4$, $\sigma = \delta = 1.1$ and $s = 3.5$ we obtain the following admissible range of exponents p in the viscoelastic damping case (see Tab.7.9):

n	$n=3$	$n=4$	$n=5$
p	$p \in (7.39, \infty)$	$p \in (4.51, \infty)$	$p \in [4, \infty)$

Tab. 7.8.: The admissible range of exponents p depends on the space dimension n.

n	$n=3$	$n=4$	$n=5$
p	$p \in (24, \infty)$	$p \in (7.67, \infty)$	$p \in (4.67, \infty)$

Tab. 7.9.: The admissible range of exponents p depends on the space dimension n.

7.2. Global (in time) existence of small data solutions to the model (7.2)

Finally, we obtain large regular solutions to (7.2) by using the fractional powers rule and the fractional Sobolev embedding.

Theorem 7.2.1. *Let $q \in (1, \infty)$ be a fixed constant and $m \in [1, q)$. Let $s > 2\delta + \frac{n}{q}$ and $n \geqslant 1$. We assume that the exponent $p > 1 + s - 2\delta$ satisfies the condition*

$$p > 1 + \frac{\max\big\{2m\delta(1+\kappa_2),\, n - \frac{m}{q}n + m(s-\sigma)\big\}}{n - 2m\delta\kappa_2}. \tag{7.29}$$

Moreover, we suppose the following conditions:

$$p \in \Big[\frac{q}{m}, \infty\Big) \text{ and } n > 2m\delta\kappa_2. \tag{7.30}$$

Then, there exists a constant $\varepsilon > 0$ such that for any small data

$$(u_0, u_1) \in \mathcal{A}^s_{m,q} \text{ satisfying the assumption } \|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}} \leqslant \varepsilon,$$

we have a uniquely determined global (in time) small data energy solution

$$u \in C\big([0,\infty), H^s_q\big) \cap C^1\big([0,\infty), H^{s-2\delta}_q\big)$$

to (7.2). Moreover, the estimates (7.19) to (7.22) hold.

Proof. We introduce the definitions of spaces $\mathcal{A}^s_{m,q}$ and $X(t)$ as in the proof of Theorem 7.1.3. We define a mapping $N: \ X(t) \longrightarrow X(t)$ in the following way:

$$Nu(t,x) = K_0(t,x) \ast_x u_0(x) + K_1(t,x) \ast_x u_1(x) + \int_0^t K_1(t-\tau,x) \ast_x |u_t(\tau,x)|^p d\tau.$$

In order to conclude the uniqueness and the global (in time) existence of small data solutions to (7.2), we have to prove the following pair of inequalities:

$$\|Nu\|_{X(t)} \lesssim \|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}} + \|u\|^p_{X(t)}, \tag{7.31}$$

$$\|Nu - Nv\|_{X(t)} \lesssim \|u-v\|_{X(t)}\big(\|u\|^{p-1}_{X(t)} + \|v\|^{p-1}_{X(t)}\big). \tag{7.32}$$

First let us prove the inequality (7.31). Our proof is divided into four steps.

Step 1: We need to estimate the norm $\|Nu(t,\cdot)\|_{L^q}$. We use the $(L^m \cap L^q) - L^q$ estimates from Theorems 4.3.1 and 4.4.4 to obtain

$$\begin{aligned}\big\|Nu(t,\cdot)\big\|_{L^q} &\lesssim (1+t)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r}}\|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}}\\ &\quad + \int_0^t (1+t-\tau)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\big\||u_t(\tau,\cdot)|^p\big\|_{L^m\cap L^q}d\tau.\end{aligned}$$

Hence, it is necessary to require the estimates for $\big|u_t(\tau,x)\big|^p$ in $L^m \cap L^q$ as follows:

$$\big\||u_t(\tau,\cdot)|^p\big\|_{L^m\cap L^q} \lesssim \|u_t(\tau,\cdot)\|^p_{L^{mp}} + \|u_t(\tau,\cdot)\|^p_{L^{qp}}.$$

Applying the fractional Gagliardo-Nirenberg inequality from Proposition C.1.1 we may conclude

$$\begin{aligned}\|u_t(\tau,\cdot)\|_{L^{qp}} &\lesssim \big\||D|^{s-2\delta}u_t(\tau,\cdot)\big\|_{L^q}^{\theta_{qp}}\,\|u_t(\tau,\cdot)\|_{L^q}^{1-\theta_{qp}} \lesssim \big(g_s(\tau)\|u\|_{X(\tau)}\big)^{\theta_{qp}}\big(g_0(\tau)\|u\|_{X(\tau)}\big)^{1-\theta_{qp}}\\ &\lesssim (1+\tau)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{s-2\delta}{2\delta}\theta_{qp}}\|u\|_{X(\tau)},\end{aligned}$$

and

$$\begin{aligned}\|u_t(\tau,\cdot)\|_{L^{mp}} &\lesssim \big\||D|^{s-2\delta}u_t(\tau,\cdot)\big\|_{L^q}^{\theta_{mp}}\,\|u_t(\tau,\cdot)\|_{L^q}^{1-\theta_{mp}} \lesssim \big(g_s(\tau)\|u\|_{X(\tau)}\big)^{\theta_{mp}}\big(g_0(\tau)\|u\|_{X(\tau)}\big)^{1-\theta_{mp}}\\ &\lesssim (1+\tau)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{s-2\delta}{2\delta}\theta_{mp}}\|u\|_{X(\tau)},\end{aligned}$$

where

$$\theta_{qp} := \theta_{0,s-2\delta}(qp,q) = \frac{n}{s-2\delta}\Big(\frac{1}{q}-\frac{1}{qp}\Big) \text{ and } \theta_{mp} := \theta_{0,s-2\delta}(mp,q) = \frac{n}{s-2\delta}\Big(\frac{1}{q}-\frac{1}{mp}\Big).$$

As in Corollary C.1.1 we have to guarantee that θ_{qp} and θ_{mp} belong to $[0,1]$. Both conditions imply the restrictions

$$p \in \Big[\frac{q}{m},\infty\Big) \text{ if } s \geqslant 2\delta+\frac{n}{q}, \qquad \text{or} \qquad p \in \Big[\frac{q}{m},\frac{n}{n-q(s-2\delta)}\Big] \text{ if } s < 2\delta+\frac{n}{q}.$$

By virtue of $\theta_{mp} < \theta_{qp}$ and the relation $1+\frac{1}{q} = \frac{1}{r}+\frac{1}{m}$, we derive

$$\big\||u_t(\tau,\cdot)|^p\big\|_{L^m\cap L^q} \lesssim (1+\tau)^{p\big((2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(\frac{1}{m}-\frac{1}{mp})\big)}\|u\|_{X(\tau)}^p.$$

Summarizing, from both estimates we may conclude

$$\begin{aligned}\|Nu(t,\cdot)\|_{L^q} &\lesssim (1+t)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}}\\ &\quad+\|u\|_{X(t)}^p\int_0^t(1+t-\tau)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}(1+\tau)^{p\big((2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(\frac{1}{m}-\frac{1}{mp})\big)}d\tau.\end{aligned}$$

The key tool relies now in the application of Lemma B.6.1. Because of $p > 1+\frac{2m\delta(1+\kappa_2)}{n-2m\delta\kappa_2}$, it follows

$$p\Big(\Big(2+\Big[\frac{n}{2}\Big]\Big)\Big(1-\frac{\sigma}{2\delta}\Big)\frac{1}{r}-\frac{n}{2\delta}\Big(\frac{1}{m}-\frac{1}{mp}\Big)\Big) < -1.$$

After applying Lemma B.6.1 with the condition $p > 1+\frac{n-\frac{m}{q}n+m(s-\sigma)}{n-2m\delta\kappa_2}$, we get

$$\begin{aligned}&\int_0^t(1+t-\tau)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}(1+\tau)^{p\big((2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(\frac{1}{m}-\frac{1}{mp})\big)}d\tau\\ &\qquad\lesssim (1+t)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}.\end{aligned}$$

Therefore, we arrive at the following desired estimate:

$$\|Nu(t,\cdot)\|_{L^q} \lesssim (1+t)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\big(\|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}}+\|u\|_{X(t)}^p\big). \tag{7.33}$$

Step 2: We need to estimate the norm $\|\partial_t Nu(t,\cdot)\|_{L^q}$. Differentiating $Nu(t,x)$ with respect to t we obtain

$$\partial_t Nu(t,x) = \partial_t\big(K_0(t,x)*_x u_0(x)+K_1(t,x)*_x u_1(x)\big)+\int_0^t\partial_t\big(K_1(t-\tau,x)*_x|u_t(\tau,x)|^p\big)d\tau.$$

We apply the $L^m\cap L^q-L^q$ estimates from Theorems 4.3.1 and 4.4.4 to conclude

$$\begin{aligned}\|\partial_t Nu(t,\cdot)\|_{L^q} &\lesssim (1+t)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}}\\ &\qquad+\int_0^t(1+t-\tau)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\big\||u_t(\tau,\cdot)|^p\big\|_{L^m\cap L^q}d\tau.\end{aligned}$$

Using the same ideas for deriving (7.33) we may conclude

$$\|\partial_t Nu(t,\cdot)\|_{L^q} \lesssim (1+\tau)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})}\big(\|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}}+\|u\|_{X(t)}^p\big),$$

under the same assumptions for p, that is,

$$p \in \Big[\frac{q}{m}, \infty\Big) \text{ if } s \geqslant 2\delta + \frac{n}{q}, \quad \text{ or } \quad p \in \Big[\frac{q}{m}, \frac{n}{n-q(s-2\delta)}\Big] \text{ if } s < 2\delta + \frac{n}{q},$$

and

$$p > 1 + \frac{\max\{2m\delta(1+\kappa_2),\, n - \frac{m}{q}n + m(s-\sigma)\}}{n - 2m\delta\kappa_2}.$$

Step 3: Let us estimate the norm $\big\|\partial_t |D|^{s-2\delta} Nu(t,\cdot)\big\|_{L^q}$. We use

$$\begin{aligned}\partial_t |D|^{s-2\delta} Nu(t,x) &= \partial_t |D|^{s-2\delta}\big(K_0(t,x) *_x u_0(x) + K_1(t,x) *_x u_1(x)\big)\\ &\quad + \int_0^t \partial_t |D|^{s-2\delta}\big(K_1(t-\tau,x) *_x |u_t(\tau,x)|^p\big) d\tau.\end{aligned}$$

We apply the $(L^m \cap L^q) - L^q$ estimates from Theorems 4.3.1 and 4.4.4 to derive

$$\begin{aligned}\big\|\partial_t |D|^{s-2\delta} Nu(t,\cdot)\big\|_{L^q} &\lesssim (1+t)^{1+(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r} - \frac{n}{2\delta}(1-\frac{1}{r}) - \frac{s}{2\delta}} \|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}}\\ &\quad + \int_0^t (1+t-\tau)^{1+(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r} - \frac{n}{2\delta}(1-\frac{1}{r}) - \frac{s}{2\delta}} \big\||u_t(\tau,\cdot)|^p\big\|_{L^m \cap H^{s-2\delta}_q} d\tau\\ &= (1+t)^{1+(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r} - \frac{n}{2\delta}(1-\frac{1}{r}) - \frac{s}{2\delta}} \|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}}\\ &\quad + \int_0^t (1+t-\tau)^{1+(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r} - \frac{n}{2\delta}(1-\frac{1}{r}) - \frac{s}{2\delta}} \big\||u_t(\tau,\cdot)|^p\big\|_{L^m \cap L^q \cap \dot{H}^{s-2\delta}_q} d\tau.\end{aligned}$$

The integrals with $\big\||u_t(\tau,\cdot)|^p\big\|_{L^m \cap L^q}$ will be handled as before to get

$$\begin{aligned}&\int_0^t (1+t-\tau)^{1+(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r} - \frac{n}{2\delta}(1-\frac{1}{r}) - \frac{s}{2\delta}} \big\||u_t(\tau,\cdot)|^p\big\|_{L^m \cap L^q} d\tau\\ &\qquad \lesssim (1+t)^{1+(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r} - \frac{n}{2\delta}(1-\frac{1}{r}) - \frac{s}{2\delta}} \|u\|^p_{X(t)}.\end{aligned}$$

To estimate the integral with the norm $\big\||u_t(\tau,\cdot)|^p\big\|_{\dot{H}^{s-2\delta}_q}$, we shall apply Corollary C.4.1 for the fractional powers rule with $s - 2\delta \in (\frac{n}{q}, p)$. Therefore, we obtain

$$\begin{aligned}\big\||u_t(\tau,\cdot)|^p\big\|_{\dot{H}^{s-2\delta}_q} &\lesssim \|u_t(\tau,\cdot)\|_{\dot{H}^{s-2\delta}_q} \|u_t(\tau,\cdot)\|^{p-1}_{L^\infty}\\ &\lesssim \|u_t(\tau,\cdot)\|_{\dot{H}^{s-2\delta}_q} \big(\|u_t(\tau,\cdot)\|_{\dot{H}^{s^*}_q} + \|u_t(\tau,\cdot)\|_{\dot{H}^{s-2\delta}_q}\big)^{p-1}.\end{aligned}$$

Here we used Corollary C.5.1 with a suitable $s^* < \frac{n}{q}$. Applying the fractional Gagliardo-Nirenberg inequality from Proposition C.1.1 we have

$$\|u_t(\tau,\cdot)\|_{\dot{H}^{s^*}_q} \lesssim \|u_t(\tau,\cdot)\|^{1-\theta}_{L^q} \big\||D|^{s-2\delta} u_t(\tau,\cdot)\big\|^\theta_{L^q} \lesssim (1+\tau)^{(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r} - \frac{n}{2\delta}(1-\frac{1}{r}) - \frac{s^*}{2\delta}} \|u\|_{X(\tau)},$$

where $\theta = \frac{s^*}{s-2\delta}$. Hence, we derive

$$\begin{aligned}\big\||u_t(\tau,\cdot)|^p\big\|_{\dot{H}^{s-2\delta}_q} &\lesssim (1+\tau)^{p\left((2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r} - \frac{n}{2\delta}(1-\frac{1}{r})\right) - \frac{s-2\delta}{2\delta} - (p-1)\frac{s^*}{2\delta}} \|u\|^p_{X(\tau)}\\ &\lesssim (1+\tau)^{p\left((2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r} - \frac{n}{2\delta}(\frac{1}{m} - \frac{1}{mp})\right)} \|u\|^p_{X(\tau)},\end{aligned}$$

if we choose $s^* = \frac{n}{q} - \varepsilon_0$ where $\varepsilon_0 > 0$ is sufficiently small. Therefore, by an analogous argument as we did in the proof of Theorem 7.1.3 we may conclude

$$\begin{aligned}&\int_0^t (1+t-\tau)^{1+(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r} - \frac{n}{2\delta}(1-\frac{1}{r}) - \frac{s}{2\delta}} \big\||u_t(\tau,\cdot)|^p\big\|_{\dot{H}^{s-2\delta}_q} d\tau\\ &\qquad \lesssim (1+t)^{1+(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r} - \frac{n}{2\delta}(1-\frac{1}{r}) - \frac{s}{2\delta}} \|u\|^p_{X(t)}.\end{aligned}$$

Therefore, we have shown the estimates

$$\big\|\partial_t |D|^{s-2\delta} Nu(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{1+(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r} - \frac{n}{2\delta}(1-\frac{1}{r}) - \frac{s}{2\delta}} \big(\|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}} + \|u\|^p_{X(t)}\big). \tag{7.34}$$

Step 4: Let us estimate the norm $\big\||D|^s Nu(t,\cdot)\big\|_{L^q}$. We use

$$|D|^s Nu(t,x) = |D|^s\big(K_0(t,x) \ast_x u_0(x) + K_1(t,x) \ast_x u_1(x)\big) + \int_0^t |D|^s\big(K_1(t-\tau,x) \ast_x |u_t(\tau,x)|^p\big)d\tau.$$

By applying again the $(L^m \cap L^q) - L^q$ estimates from Theorems 4.3.1 and 4.4.4, we derive

$$\big\||D|^s Nu(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{s}{2\delta}}\|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}} + \int_0^t (1+t-\tau)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{s}{2\delta}}\big\||u_t(\tau,\cdot)|^p\big\|_{L^m \cap H^{s-2\delta}_q}d\tau.$$

Following the approach to show (7.34) we arrive at

$$\big\||D|^s Nu(t,\cdot)\big\|_{L^q} \lesssim (1+t)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{s}{2\delta}}\big(\|(u_0,u_1)\|_{\mathcal{A}^s_{m,q}} + \|u\|^p_{X(t)}\big).$$

Summarizing, from the definition of the norm in $X(t)$ we obtain immediately the inequality (7.31).

Next let us prove the inequality (7.32). The new difficulty is to estimate the norm

$$\big\||u_t(\tau,\cdot)|^p - |v_t(\tau,\cdot)|^p\big\|_{\dot{H}^{s-2\delta}_q}.$$

Then, repeating the proof of Theorem 7.1.3 and using an analogous treatment as in the first step we may derive

$$\begin{aligned}
&\big\||D|^{s-2\delta}\partial_t\big(Nu(t,\cdot)-Nv(t,\cdot)\big)\big\|_{L^q} \\
&\qquad \lesssim (1+t)^{1+(2+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{s}{2\delta}}\|u-v\|_{X(t)}\big(\|u\|^{p-1}_{X(t)} + \|v\|^{p-1}_{X(t)}\big), \\
&\big\||D|^s\big(Nu(t,\cdot)-Nv(t,\cdot)\big)\big\|_{L^q} \\
&\qquad \lesssim (1+t)^{1+(1+[\frac{n}{2}])(1-\frac{\sigma}{2\delta})\frac{1}{r}-\frac{n}{2\delta}(1-\frac{1}{r})-\frac{s}{2\delta}}\|u-v\|_{X(t)}\big(\|u\|^{p-1}_{X(t)} + \|v\|^{p-1}_{X(t)}\big).
\end{aligned}$$

From the definition of the norm in $X(t)$ we conclude immediately the inequality (7.32).
This completes the proof of Theorem 7.2.1. □

Example 7.2.1. In the first example, by choosing $m=1$, $q=4$, $\sigma=1.8$, $\delta=1$ and $s=3$ we obtain the following admissible range of exponents p in the structural damping case:

$$p \in [4,\infty) \quad \text{for all } n=1,2,3.$$

Example 7.2.2. In the second example, by choosing $m=1$, $q=4$, $\sigma=\delta=1.1$ and $s=3$ we obtain the following admissible range of exponents p in the viscoelastic damping case:

$$p \in (6.89,\infty) \text{ if } n=1, \quad \text{or} \quad p \in [4,\infty) \text{ if } n=2,3.$$

8. Other qualitative properties of solutions to linear models

Up to now, we have presented $(L^m \cap L^q) - L^q$ and $L^q - L^q$ estimates, with $q \in (1, \infty)$ and $m \in [1, q)$, for solutions and some of their partial derivatives to the Cauchy problem for linear damped σ-evolution models in Chapters 2, 3 and 4. Then, a direct application of these estimates is to prove the global (in time) existence of small data Sobolev solutions to the corresponding semi-linear models from suitable function spaces basing on L^q spaces in Chapters 5, 6 and 7. In this chapter, we explain some other qualitative properties of solutions to the linear models as Gevrey smoothing, propagation of singularities and loss of regularity in the cases of structural damping $\delta \in (0, \sigma)$, external damping $\delta = 0$ and visco-elastic damping $\delta = \sigma$, respectively.

The following linear Cauchy problem is of our interest:

$$u_{tt} + (-\Delta)^\sigma u + \mu(-\Delta)^\delta u_t = 0, \quad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) \tag{8.1}$$

with $\mu > 0$, $\sigma \geqslant 1$ and $\delta \in [0, \sigma]$.

8.1. Gevrey smoothing

We are interested to understand which Gevrey space $\Gamma^{a,s}$ the solution to (8.1) belongs to. For this reason, we will consider our estimates with the L^2 norm and assume for the Cauchy data $u_0 \in \dot{H}^\sigma$ and $u_1 \in L^2$. The study of regularity properties for solutions allows to restrict our considerations for large frequencies in the extended phase space. In order to state our main results, at first we recall the following definitions of the Gevrey space regularity.

Definition 8.1.1. A given function $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ belongs to the Gevrey space $\Gamma^{a,s}$ if and only if there exist positive real constants a and s such that

$$\exp\big(a\langle\xi\rangle^{\frac{1}{s}}\big)\mathfrak{F}(u)(\xi) \in L^2.$$

We write $u \in \Gamma^{a,s}$. By Γ^s we denote the inductive limit of all spaces $\Gamma^{a,s}$, that is, $\Gamma^s := \bigcup_{a>0} \Gamma^{a,s}$.

More precisely, we may define the regularity of Gevrey-Sobolev spaces as follows.

Definition 8.1.2. A given function $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ belongs to the Gevrey-Sobolev space $\Gamma^{a,s,\rho}$ if and only if there exist positive constants a, s and a real constant $\rho \in \mathbb{R}$ such that

$$\exp\big(a\langle\xi\rangle^{\frac{1}{s}}\big)\langle\xi\rangle^\rho\mathfrak{F}(u)(\xi) \in L^2.$$

We write $u \in \Gamma^{a,s,\rho}$. By $\Gamma^{s,\rho}$ we denote the inductive limit of all spaces $\Gamma^{a,s,\rho}$, that is, $\Gamma^{s,\rho} := \bigcup_{a>0} \Gamma^{a,s,\rho}$.

8.1.1. The case $\delta \in \big(0, \frac{\sigma}{2}\big)$

Theorem 8.1.1. *Let us consider the Cauchy problem (8.1) with $\delta \in (0, \frac{\sigma}{2})$. The data (u_0, u_1) are supposed to belong to the space $\in \dot{H}^\sigma \times L^2$. Then, there is a smoothing effect in the sense, that the solutions belong to the Gevrey-Sobolev space and the Gevrey space, respectively, as follows:*

$$u(t,\cdot) \in \Gamma^{\frac{1}{2\delta},\sigma} \text{ and } |D|^\sigma u(t,\cdot),\ u_t(t,\cdot) \in \Gamma^{\frac{1}{2\delta},0} \quad \text{for all } t > 0.$$

Proof. Applying partial Fourier transformation to (8.1) we obtain the Cauchy problem as follows:

$$\widehat{u}_{tt} + \mu|\xi|^{2\delta}\widehat{u}_t + |\xi|^{2\sigma}\widehat{u} = 0, \quad \widehat{u}(0,\xi) = \widehat{u_0}(\xi), \quad \widehat{u}_t(0,\xi) = \widehat{u_1}(\xi). \tag{8.2}$$

We may choose without loss of generality $\mu = 1$ in (8.1). We recall the characteristic roots are

$$\lambda_{1,2} = \lambda_{1,2}(\xi) = \frac{1}{2}\Big(- |\xi|^{2\delta} \pm \sqrt{|\xi|^{4\delta} - 4|\xi|^{2\sigma}}\Big).$$

The solutions to (8.2) are presented by the following formula (here we assume $\lambda_1 \neq \lambda_2$):

$$\widehat{u}(t,\xi) = \frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2}\widehat{u_0}(\xi) + \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}\widehat{u_1}(\xi) =: \widehat{K_0}(t,\xi)\widehat{u_0}(\xi) + \widehat{K_1}(t,\xi)\widehat{u_1}(\xi).$$

Taking account of the cases of large frequencies we have

$$\lambda_{1,2} \sim -|\xi|^{2\delta} \pm i|\xi|^{\sigma} \quad \text{and} \quad \lambda_1 - \lambda_2 \sim i|\xi|^{\sigma}.$$

We introduce the smooth cut-off functions $\chi_k = \chi_k(|\xi|)$ with $k = 1, 2, 3$ as in Section 3.1. Using the asymptotic behavior of the characteristic roots we find the estimates for large frequencies

$$\big|\widehat{K_0}(t,\xi)\big| \lesssim e^{-c|\xi|^{2\delta}t}, \qquad \big|\widehat{K_1}(t,\xi)\big| \lesssim |\xi|^{-\sigma}e^{-c|\xi|^{2\delta}t},$$
$$\big|\partial_t\widehat{K_0}(t,\xi)\big| \lesssim |\xi|^{\sigma}e^{-c|\xi|^{2\delta}t}, \qquad \big|\partial_t\widehat{K_1}(t,\xi)\big| \lesssim e^{-c|\xi|^{2\delta}t},$$

for some positive constant c. Hence, we derive the following estimate for solutions $\widehat{u}(t,\xi) = \mathfrak{F}(u(t,x))$ localized to large frequencies:

$$\begin{aligned}
&\int_{\mathbb{R}^n} \exp\big(2c|\xi|^{2\delta}t\big)(1+|\xi|^2)^{\sigma}|\widehat{u}(t,\xi)|^2\chi_3(|\xi|)d\xi \\
&\quad\lesssim \int_{\mathbb{R}^n} \exp\big(2c|\xi|^{2\delta}t\big)(1+|\xi|^2)^{\sigma}\big|\widehat{K_0}(t,\xi)\widehat{u_0}(\xi) + \widehat{K_1}(t,\xi)\widehat{u_1}(\xi)\big|^2\chi_3(|\xi|)d\xi \\
&\quad\lesssim \int_{\mathbb{R}^n} \exp\big(2c|\xi|^{2\delta}t\big)(1+|\xi|^2)^{\sigma}|\widehat{K_0}(t,\xi)|^2|\widehat{u_0}(\xi)|^2\chi_3(|\xi|)d\xi \\
&\qquad + \int_{\mathbb{R}^n} \exp\big(2c|\xi|^{2\delta}t\big)(1+|\xi|^2)^{\sigma}|\widehat{K_1}(t,\xi)|^2|\widehat{u_1}(\xi)|^2\chi_3(|\xi|)d\xi \\
&\quad\lesssim \int_{\mathbb{R}^n} |\xi|^{2\sigma}|\widehat{u_0}(\xi)|^2\chi_3(|\xi|)d\xi + \int_{\mathbb{R}^n} |\widehat{u_1}(\xi)|^2\chi_3(|\xi|)d\xi.
\end{aligned}$$

Moreover, we have for $|\xi|^{\sigma}\widehat{u}(t,\xi) = \mathfrak{F}_{x\to\xi}(|D|^{\sigma}u(t,x))$ the following estimate:

$$\begin{aligned}
&\int_{\mathbb{R}^n} \exp\big(2c|\xi|^{2\delta}t\big)|\xi|^{2\sigma}|\widehat{u}(t,\xi)|^2\chi_3(|\xi|)d\xi \\
&\quad\lesssim \int_{\mathbb{R}^n} \exp\big(2c|\xi|^{2\delta}t\big)|\xi|^{2\sigma}\big|\widehat{K_0}(t,\xi)\widehat{u_0}(\xi) + \widehat{K_1}(t,\xi)\widehat{u_1}(\xi)\big|^2\chi_3(|\xi|)d\xi \\
&\quad\lesssim \int_{\mathbb{R}^n} \exp\big(2c|\xi|^{2\delta}t\big)|\xi|^{2\sigma}|\widehat{K_0}(t,\xi)|^2|\widehat{u_0}(\xi)|^2\chi_3(|\xi|)d\xi \\
&\qquad + \int_{\mathbb{R}^n} \exp\big(2c|\xi|^{2\delta}t\big)|\xi|^{2\sigma}|\widehat{K_1}(t,\xi)|^2|\widehat{u_1}(\xi)|^2\chi_3(|\xi|)d\xi \\
&\quad\lesssim \int_{\mathbb{R}^n} |\xi|^{2\sigma}|\widehat{u_0}(\xi)|^2\chi_3(|\xi|)d\xi + \int_{\mathbb{R}^n} |\widehat{u_1}(\xi)|^2\chi_3(|\xi|)d\xi.
\end{aligned}$$

For the partial derivative in time $\widehat{u}_t(t,\xi) = \mathfrak{F}(u_t(t,x))$, we may conclude

$$\begin{aligned}
&\int_{\mathbb{R}^n} \exp\big(2c|\xi|^{2\delta}t\big)|\widehat{u}_t(t,\xi)|^2\chi_3(|\xi|)d\xi \\
&\quad\lesssim \int_{\mathbb{R}^n} \exp\big(2c|\xi|^{2\delta}t\big)\big|\partial_t\widehat{K_0}(t,\xi)\widehat{u_0}(\xi) + \partial_t\widehat{K_1}(t,\xi)\widehat{u_1}(\xi)\big|^2\chi_3(|\xi|)d\xi \\
&\quad\lesssim \int_{\mathbb{R}^n} \exp\big(2c|\xi|^{2\delta}t\big)|\partial_t\widehat{K_0}(t,\xi)|^2|\widehat{u_0}(\xi)|^2\chi_3(|\xi|)d\xi \\
&\qquad + \int_{\mathbb{R}^n} \exp\big(2c|\xi|^{2\delta}t\big)|\partial_t\widehat{K_1}(t,\xi)|^2|\widehat{u_1}(\xi)|^2\chi_3(|\xi|)d\xi \\
&\quad\lesssim \int_{\mathbb{R}^n} |\xi|^{2\sigma}|\widehat{u_0}(\xi)|^2\chi_3(|\xi|)d\xi + \int_{\mathbb{R}^n} |\widehat{u_1}(\xi)|^2\chi_3(|\xi|)d\xi.
\end{aligned}$$

Therefore, by Definitions 8.1.1 and 8.1.2 we may conclude immediately $u(t,\cdot) \in \Gamma^{\frac{1}{2\delta},\sigma}$ and $|D|^{\sigma}u(t,\cdot)$, $u_t(t,\cdot) \in \Gamma^{\frac{1}{2\delta},0}$ for all $t > 0$ what we wanted to prove. $\square$

8.1.2. The case $\delta = \frac{\sigma}{2}$

Theorem 8.1.2. *Let us consider the Cauchy problem (8.1) with $\delta = \frac{\sigma}{2}$. The data (u_0, u_1) are supposed to belong to the space $\dot{H}^\sigma \times L^2$. Then, there is a smoothing effect in the sense, that the solutions belong to the Gevrey-Sobolev space and the Gevrey space, respectively, as follows:*

$$u(t,\cdot) \in \Gamma^{\frac{1}{\sigma},\sigma} \text{ and } |D|^\sigma u(t,\cdot),\ u_t(t,\cdot) \in \Gamma^{\frac{1}{\sigma},0} \quad \text{for all } t > 0.$$

Proof. Here we divide our considerations into two cases: $\mu \neq 2$ and $\mu = 2$.

Case 1: $\mu \neq 2$. We have the characteristic roots

$$\lambda_{1,2} = \begin{cases} \frac{1}{2}|\xi|^\sigma\big(-\mu \pm \sqrt{\mu^2-4}\big) & \text{if } \mu \in (2,\infty), \\ \frac{1}{2}|\xi|^\sigma\big(-\mu \pm i\sqrt{4-\mu^2}\big) & \text{if } \mu \in (0,2). \end{cases}$$

Taking account of the asymptotic behavior of the characteristic roots we obtain the following estimates:

$$\big|\widehat{K_0}(t,\xi)\big| \lesssim e^{-c|\xi|^\sigma t}, \qquad \big|\widehat{K_1}(t,\xi)\big| \lesssim |\xi|^{-\sigma} e^{-c|\xi|^\sigma t},$$
$$\big|\partial_t\widehat{K_0}(t,\xi)\big| \lesssim |\xi|^\sigma e^{-c|\xi|^\sigma t}, \qquad \big|\partial_t\widehat{K_1}(t,\xi)\big| \lesssim e^{-c|\xi|^\sigma t},$$

for some positive constant c. Hence, in an analogous way as in the proof of Theorem 8.1.1 we may prove the statements in Theorem 8.1.2 with $\mu \neq 2$.

Case 2: $\mu = 2$. We have a double root $\lambda_{1,2}(\xi) = -|\xi|^\sigma$. The solutions and their derivative in time to (8.2) are, respectively,

$$\widehat{u}(t,\xi) = (1 + t\,|\xi|^\sigma)e^{-|\xi|^\sigma t}\widehat{u_0}(\xi) + t\,e^{-|\xi|^\sigma t}\widehat{u_1}(\xi),$$

and

$$\widehat{u}_t(t,\xi) = -t\,|\xi|^{2\sigma}e^{-|\xi|^\sigma t}\widehat{u_0}(\xi) + (1 - t\,|\xi|^\sigma)e^{-|\xi|^\sigma t}\widehat{u_1}(\xi).$$

Now let us turn to estimate for high frequencies. First we get the following estimate for solutions:

$$\begin{aligned} \big\|e^{\frac{1}{2}|\xi|^\sigma t}\langle\xi\rangle^\sigma \widehat{u}(t,\xi)\chi_3(|\xi|)\big\|_{L^2} &\lesssim \big\|e^{-\frac{1}{2}|\xi|^\sigma t}|\xi|^\sigma\widehat{u_0}(\xi)\chi_3(|\xi|)\big\|_{L^2} + t\,\big\|e^{-\frac{1}{2}|\xi|^\sigma t}|\xi|^{2\sigma}\widehat{u_0}(\xi)\chi_3(|\xi|)\big\|_{L^2} \\ &\quad + t\,\big\|e^{-\frac{1}{2}|\xi|^\sigma t}|\xi|^\sigma\widehat{u_1}(\xi)\chi_3(|\xi|)\big\|_{L^2}. \end{aligned}$$

Thanks to Parseval's formula, we derive

$$\begin{aligned} \big\|e^{-\frac{1}{2}|\xi|^\sigma t}|\xi|^\sigma\widehat{u_0}(\xi)\chi_3(|\xi|)\big\|_{L^2} &\lesssim \big\|e^{-\frac{1}{2}|\xi|^\sigma t}\chi_3(|\xi|)\big\|_{L^\infty}\big\||\xi|^\sigma\widehat{u_0}(\xi)\big\|_{L^2} \\ &\lesssim \big\||\xi|^\sigma\widehat{u_0}(\xi)\big\|_{L^2} = \|u_0\|_{\dot{H}^\sigma}, \\ t\,\big\|e^{-\frac{1}{2}|\xi|^\sigma t}|\xi|^{2\sigma}\widehat{u_0}(\xi)\chi_3(|\xi|)\big\|_{L^2} &\lesssim \big\|t\,|\xi|^\sigma e^{-\frac{1}{2}|\xi|^\sigma t}\chi_3(|\xi|)\big\|_{L^\infty}\big\||\xi|^\sigma\widehat{u_0}(\xi)\big\|_{L^2} \\ &\lesssim \big\||\xi|^\sigma\widehat{u_0}(\xi)\big\|_{L^2} = \|u_0\|_{\dot{H}^\sigma}, \end{aligned}$$

and

$$\begin{aligned} t\,\big\|e^{-\frac{1}{2}|\xi|^\sigma t}|\xi|^\sigma\widehat{u_1}(\xi)\chi_3(|\xi|)\big\|_{L^2} &\lesssim \big\|t\,|\xi|^\sigma e^{-\frac{1}{2}|\xi|^\sigma t}\chi_3(|\xi|)\big\|_{L^\infty}\big\|\widehat{u_1}(\xi)\big\|_{L^2} \\ &\lesssim \big\|\widehat{u_1}(\xi)\big\|_{L^2} = \|u_1\|_{L^2}. \end{aligned}$$

Hence, we arrive at

$$\big\|e^{\frac{1}{2}|\xi|^\sigma t}\langle\xi\rangle^\sigma \widehat{u}(t,\xi)\chi_3(|\xi|)\big\|_{L^2} \lesssim \|u_0\|_{\dot{H}^\sigma} + \|u_1\|_{L^2}.$$

Analogously, we also obtain the following estimates:

$$\big\|e^{\frac{1}{2}|\xi|^\sigma t}|\xi|^\sigma \widehat{u}(t,\xi)\chi_3(|\xi|)\big\|_{L^2} \lesssim \|u_0\|_{\dot{H}^\sigma} + \|u_1\|_{L^2},$$

and

$$\big\|e^{\frac{1}{2}|\xi|^\sigma t}\widehat{u}_t(t,\xi)\chi_3(|\xi|)\big\|_{L^2} \lesssim \|u_0\|_{\dot{H}^\sigma} + \|u_1\|_{L^2}.$$

Summarizing, we have proved the statements in Theorem 8.1.2. □

8.1.3. The case $\delta \in \left(\frac{\sigma}{2}, \sigma\right)$

Theorem 8.1.3. *Let us consider the Cauchy problem (8.1) with $\delta \in (\frac{\sigma}{2}, \sigma)$. The data (u_0, u_1) are supposed to belong to the space $\dot{H}^\sigma \times L^2$. Then, there is a smoothing effect in the sense, that the solutions belong to the Gevrey-Sobolev space and the Gevrey space, respectively, as follows:*

$$u(t,\cdot) \in \Gamma^{\frac{1}{2(\sigma-\delta)},\sigma} \text{ and } |D|^\sigma u(t,\cdot),\ u_t(t,\cdot) \in \Gamma^{\frac{1}{2(\sigma-\delta)},0} \quad \text{for all } t > 0.$$

Proof. We will follow the proof of Theorem 8.1.1 to prove Theorem 8.1.3. For large frequencies, we get

$$\lambda_1 \sim -|\xi|^{2(\sigma-\delta)}, \quad \lambda_2 \sim -|\xi|^{2\delta} \quad \text{and} \quad \lambda_1 - \lambda_2 \sim |\xi|^{2\delta}.$$

By using the asymptotic behavior of the characteristic roots, we find the following estimates for large frequencies:

$$\begin{aligned} \big|\widehat{K_0}(t,\xi)\big| &\lesssim e^{-c|\xi|^{2(\sigma-\delta)}t}, & \big|\widehat{K_1}(t,\xi)\big| &\lesssim |\xi|^{-2\delta} e^{-c|\xi|^{2(\sigma-\delta)}t},\\ \big|\partial_t\widehat{K_0}(t,\xi)\big| &\lesssim |\xi|^{2(\sigma-\delta)} e^{-c|\xi|^{2(\sigma-\delta)}t}, & \big|\partial_t\widehat{K_1}(t,\xi)\big| &\lesssim e^{-c|\xi|^{2(\sigma-\delta)}t}, \end{aligned}$$

for some positive constant c. Hence, in the same way as in the proof of Theorem 8.1.1 we may prove the statements in Theorem 8.1.3. □

8.2. Propagation of singularities

In this section we would like to discuss another property of solutions to (8.1), the so-called propagation of singularities along characteristics. In order to describe this property more precisely, we are interested in considering the so-called micro-local descriptions whose main ideas come from the notion of wave front set. Denoting by $U_0(x_0)$ and $V_0(\xi_0)$ a neighborhood of a point $x_0 \in \mathbb{R}^n$ and a conical neighborhood of a point $\xi_0 \in \mathbb{R}^n$, respectively, we recall the following definitions of wave front set.

Definition 8.2.1. Let us consider a distribution $g \in \mathcal{D}'$. Then, a point $(x_0, \xi_0) \in \mathbb{R}^n \times (\mathbb{R}^n \backslash \{0\})$ does not belong to the wave front set WFg if there exist two functions χ and ψ having the following properties:

- $\chi \in \mathcal{C}_0^\infty\big(U_0(x_0)\big)$ and $\chi \equiv 1$ on a neighborhood $U_1(x_0)$ of x_0, where $\overline{U_1(x_0)} \subset U_0(x_0)$,
- $\psi \in \mathcal{C}^\infty$ such that $\psi \equiv 1$ on a conical neighborhood $V_1(\xi_0)$ of ξ_0 and $\psi \equiv 0$ outside a conical neighborhood $V_0(\xi_0)$, where $\overline{V_1(\xi_0)} \subset V_0(\xi_0)$,
- $\psi(D)(\chi g) \in \mathcal{C}^\infty$, where $\psi(D)$ is a pseudo-differential operator and $\psi(D)(\chi g)$ is defined in the form

$$\psi(D)(\chi g)(x) = \mathfrak{F}^{-1}\big(\psi(\xi)\mathfrak{F}(\chi g)(\xi)\big)(x).$$

Definition 8.2.2. Let us consider a distribution $g \in \mathcal{D}'$. Then, a point $(x_0, \xi_0) \in \mathbb{R}^n \times (\mathbb{R}^n \backslash \{0\})$ does not belong to the wave front set WFg if there exists a function χ having the following properties:

- $\chi \in \mathcal{C}_0^\infty\big(U_0(x_0)\big)$ and $\chi \equiv 1$ on a neighborhood $U_1(x_0)$ of x_0, where $\overline{U_1(x_0)} \subset U_0(x_0)$,
- to each γ there exists a constant C_γ such that $\big|\mathfrak{F}(\chi g)(\xi)\big| \leqslant C_\gamma \langle\xi\rangle^{-\gamma}$ in $V_0(\xi_0)$.

To apply the above definitions of wave front set for solutions to structurally damped σ-evolution equations, we introduce the following lemma (see, for instance, [26, 27]) which explains the structure of wave front set for Fourier integral operators.

Lemma 8.2.1. *Let $g \in \mathcal{E}'(\mathbb{R}^n)$ be a distribution with compact support in $\mathbb{R}^n$. Let A be the Fourier integral operator in the form*

$$Ag(x) := \int_{\mathbb{R}^n} e^{i\phi(x,\xi)}\, a(x,\xi)\, \mathfrak{F}(g)(\xi) d\xi,$$

where $\phi = \phi(x,\xi)$ is a given phase function and the amplitude $a = a(x,\xi)$ belongs to the Hörmander class $S^r(\mathbb{R}^n)$ with $r \in \mathbb{R}$, that is, the following inequalities are true:

$$\big|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\big| \leqslant C_{\alpha,\beta}\langle\xi\rangle^{r-|\beta|}$$

for any multi-index α, β, some constants $C_{\alpha,\beta}$ and for all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$. Let us assume that the set

$$\big\{(y,\xi) \in WFg:\ \text{there exists } x \in \mathbb{R}^n \text{ such that } y = \nabla_\xi \phi(x,\xi),\ \nabla_x \phi(x,\xi) = 0\big\}$$

is empty. Then, it holds

$$WF\,Ag \subset \big\{(x,\xi):\ \text{there exists } \eta \in \mathbb{R}^n \backslash \{0\} \text{ such that } \xi = \nabla_x \phi(x,\eta),\ \big(\nabla_\eta \phi(x,\eta),\eta\big) \in WFg\big\}.$$

Now let us consider the propagation of microlocal singularities for solutions to (8.1) with $\delta = 0$, that is, for solutions to the Cauchy problem

$$u_{tt} + (-\Delta)^\sigma u + \mu u_t = 0, \quad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x). \tag{8.3}$$

We are going to prove the following result.

Theorem 8.2.1. *The wave front set of solutions to (8.2) is described as follows:*

$$WFu(t,\cdot) \subset \Big\{\Big(x \pm t\xi \frac{2\sigma|\xi|^{2\sigma-2}}{\sqrt{4|\xi|^{2\sigma} - \mu^2}}, \xi\Big) : (x,\xi) \in WFu_0 \cup WFu_1\Big\} \qquad \text{for all } t \neq 0.$$

Proof. The study of propagation of microlocal singularities for solutions allows to restrict our considerations for large frequencies in the extended phase space. From Section 3.4 we get the following reprentation of solutions to (8.3) by Fourier multipliers for large frequencies:

$$\begin{aligned}
u_{\chi_3}(t,x) &= \mathfrak{F}^{-1}\big(\hat{u}(t,\xi)\chi_3(|\xi|)\big)\\
&= \mathfrak{F}^{-1}\Big(\frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2}\chi_3(|\xi|)\widehat{u_0}(\xi) + \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}\chi_3(|\xi|)\widehat{u_1}(\xi)\Big)\\
&= \mathfrak{F}^{-1}\Big(e^{-\frac{1}{2}i\sqrt{4|\xi|^{2\sigma}-\mu^2}t}e^{-\frac{t}{2}}\Big(\frac{1}{2} - \frac{1}{2i\sqrt{4|\xi|^{2\sigma} - \mu^2}}\Big)\chi_3(|\xi|)\widehat{u_0}(\xi)\Big)\\
&\quad - \mathfrak{F}^{-1}\Big(e^{-\frac{1}{2}i\sqrt{4|\xi|^{2\sigma}-\mu^2}t}\frac{e^{-\frac{t}{2}}}{i\sqrt{4|\xi|^{2\sigma} - \mu^2}}\chi_3(|\xi|)\widehat{u_1}(\xi)\Big)\\
&\quad + \mathfrak{F}^{-1}\Big(e^{\frac{1}{2}i\sqrt{4|\xi|^{2\sigma}-\mu^2}t}e^{-\frac{t}{2}}\Big(\frac{1}{2} + \frac{1}{2i\sqrt{4|\xi|^{2\sigma} - \mu^2}}\Big)\chi_3(|\xi|)\widehat{u_0}(\xi)\Big)\\
&\quad + \mathfrak{F}^{-1}\Big(e^{\frac{1}{2}i\sqrt{4|\xi|^{2\sigma}-\mu^2}t}\frac{e^{-\frac{t}{2}}}{i\sqrt{4|\xi|^{2\sigma} - \mu^2}}\chi_3(|\xi|)\widehat{u_1}(\xi)\Big).
\end{aligned}$$

Hence, it is reasonable to take account of the following phase functions:

$$\phi_\pm(t,x,\xi) = x\cdot\xi \pm \frac{1}{2}t\sqrt{4|\xi|^{2\sigma} - \mu^2}$$

These phase functions satisfy the assumption of Lemma 8.2.1 since $\nabla_x \phi_\pm(t,x,\eta) = \eta = 0$ is excluded in Definitions 8.2.1 and 8.2.2. For this reason, with $\xi = \eta$ we may conclude from $(x_0,\xi_0) \in \mathrm{WF}\,u_0 \cap \mathrm{WF}\,u_1$ as follows:

$$\nabla_\xi \phi_\pm(t,x_0,\xi_0) = x_0 \pm t\xi_0 \frac{2\sigma|\xi_0|^{2\sigma-2}}{\sqrt{4|\xi_0|^{2\sigma} - \mu^2}}.$$

Therefore, this completes our proof. □

Remark 8.2.1. In Theorem 8.2.1 we want to explain that microlocal singularities of solutions to (8.3) are contained on the lateral surface of the characteristic cone with apex in the singularities of the data. This means, that we have no small neighborhood in those points, where the data are $\mathcal{C}^\infty$. More precisely, if $(x_0,\xi_0) \in \mathrm{WF}\,u_0 \cup \mathrm{WF}\,u_1$, i.e. the estimate from Definition 8.2.2 is not true in the direction ξ_0, then the wave front set $\mathrm{WF}\,u(t,\cdot)$ is contained in the set (with respect to x) of points on the lateral surface of the characteristic cone with apex in x_0 in the direction

$$\vartheta_0 := \xi_0 \frac{2\sigma|\xi_0|^{2\sigma-2}}{\sqrt{4|\xi_0|^{2\sigma} - \mu^2}}.$$

Hence, the ξ_0 direction is a bad one with respect to ξ .

8.3. Loss of regularity

The main purpose of this section is to study the loss of regularity for solutions to the following visco-elastic damped σ-evolution models:

$$u_{tt} + (-\Delta)^\sigma u + (-\Delta)^\sigma u_t = 0, \quad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) \tag{8.4}$$

with $\sigma \geqslant 1$. The study of the regularity of solutions allows to restrict our considerations for large frequencies in the extended phase space. We are going to prove the following result.

Theorem 8.3.1. *Let a, ℓ_1, $\ell_2 \geqslant 0$. The Sobolev solutions to (8.4) satisfy the following estimates:*

$$\begin{aligned}
\big\||D|^a u_{\chi_3}(t,\cdot)\big\|_{L^2} &\lesssim e^{-ct}\big(\|u_0\|_{H^{[a-2\sigma]^+}} + (1+t)^{-\frac{\ell_1}{2\sigma}}\|u_0\|_{H^{a+\ell_1}}\big)\\
&\qquad + e^{-ct}\big((1+t)^{-\frac{\ell_2}{2\sigma}}\|u_1\|_{H^{[a-2\sigma+\ell_2]^+}} + \|u_1\|_{H^{[a-2\sigma]^+}}\big),\\
\big\||D|^a \partial_t u_{\chi_3}(t,\cdot)\big\|_{L^2} &\lesssim e^{-ct}\big((1+t)^{-\frac{\ell_1}{2\sigma}}\|u_0\|_{H^{a+\ell_1}} + \|u_0\|_{H^a}\big)\\
&\qquad + e^{-ct}\big((1+t)^{-\frac{\ell_2}{2\sigma}}\|u_1\|_{H^{[a-2\sigma+\ell_2]^+}} + \|u_1\|_{H^a}\big),
\end{aligned}$$

where c is a suitable positive constant.

Proof. Using partial Fourier transformation to (8.4) we obtain the Cauchy problem for $\hat{u}(t,\xi) := \mathfrak{F}(u(t,x))$, $\widehat{u_0}(\xi) := \mathfrak{F}(u_0(x))$ and $\widehat{u_1}(\xi) := \mathfrak{F}(u_1(x))$ as follows:

$$\hat{u}_{tt} + |\xi|^{2\sigma}\hat{u}_t + |\xi|^{2\sigma}\hat{u} = 0, \quad \hat{u}(0,\xi) = \widehat{u_0}(\xi), \quad \hat{u}_t(0,\xi) = \widehat{u_1}(\xi). \tag{8.5}$$

The characteristic roots are

$$\lambda_{1,2} = \lambda_{1,2}(\xi) = \frac{1}{2}|\xi|^{2\sigma}\Big(-1 \pm \sqrt{1 - \frac{4}{|\xi|^{2\sigma}}}\Big).$$

The solutions to (8.5) are presented by the following formula (here we assume $\lambda_1 \neq \lambda_2$):

$$\hat{u}(t,\xi) = \frac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2}\widehat{u_0}(\xi) + \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}\widehat{u_1}(\xi) =: \widehat{K_0}(t,\xi)\widehat{u_0}(\xi) + \widehat{K_1}(t,\xi)\widehat{u_1}(\xi).$$

We introduce the smooth cut-off functions $\chi_k = \chi_k(|\xi|)$ with $k = 1,2,3$ as in Section 4.4. Taking account of the cases of large frequencies separately we have

$$\lambda_1 \sim -1, \quad \lambda_2 \sim -|\xi|^{2\sigma}, \quad \text{and} \quad \lambda_1 - \lambda_2 \sim |\xi|^{2\sigma}.$$

More precisely, using Newton's binomial theorem we re-write

$$\sqrt{1 - \frac{4}{|\xi|^{2\sigma}}} = 1 - \frac{2}{|\xi|^{2\sigma}} - \frac{2}{|\xi|^{4\sigma}} - o\big(|\xi|^{-4\sigma}\big)$$

for large frequencies. Consequently, we derive the following relations for large frequencies:

$$\begin{aligned}
\lambda_1 &= -1 - |\xi|^{-2\sigma} - o\big(|\xi|^{-2\sigma}\big),\\
\lambda_2 &= |\xi|^{2\sigma} + 1 + |\xi|^{-2\sigma} + o\big(|\xi|^{-2\sigma}\big),\\
\lambda_1 - \lambda_2 &= |\xi|^{2\sigma} - 2 - 2|\xi|^{-2\sigma} + o\big(|\xi|^{-2\sigma}\big).
\end{aligned}$$

Using the above asymptotic behavior of the characteristic roots we find the estimates for large frequencies

$$\begin{aligned}
|\xi|^a\big|\widehat{K_0}(t,\xi)\big| &\lesssim |\xi|^{a-2\sigma}e^{-c|\xi|^{2\sigma}t} + e^{-ct}|\xi|^a e^{-c|\xi|^{-2\sigma}t},\\
|\xi|^a\big|\widehat{K_1}(t,\xi)\big| &\lesssim e^{-ct}|\xi|^{a-2\sigma}e^{-c|\xi|^{-2\sigma}t} + |\xi|^{a-2\sigma}e^{-c|\xi|^{2\sigma}t},\\
|\xi|^a\big|\partial_t\widehat{K_0}(t,\xi)\big| &\lesssim e^{-ct}|\xi|^a e^{-c|\xi|^{-2\sigma}t} + |\xi|^a e^{-c|\xi|^{2\sigma}t},\\
|\xi|^a\big|\partial_t\widehat{K_1}(t,\xi)\big| &\lesssim e^{-ct}|\xi|^{a-2\sigma}e^{-c|\xi|^{-2\sigma}t} + |\xi|^a e^{-c|\xi|^{2\sigma}t}
\end{aligned}$$

for any $a \geqslant 0$, where c is a suitable positive constant. From these estimates, we observe that the decay properties of regularity-loss type appear. Indeed, let us now decompose the solutions to (8.4) into three parts localized separately to low, middle and high frequencies, that is,

$$u(t,x) = u_{\chi_1}(t,x) + u_{\chi_2}(t,x) + u_{\chi_3}(t,x),$$

where

$$u_{\chi_k}(t,x) = \mathfrak{F}^{-1}\big(\chi_k(|\xi|)\widehat{u}(t,\xi)\big) \qquad \text{with } k = 1,2,3.$$

By using the following estimates:

$$\sup_{|\xi|\geqslant 1}\big(|\xi|^{-\beta}e^{-c|\xi|^{-\alpha}t}\big) \lesssim (1+t)^{-\frac{\beta}{\alpha}},$$

with $\beta \geqslant 0$ and $\alpha,\ c > 0$ we may arrive at

$$\begin{aligned}
\big\||D|^a u_{\chi_3}(t,\cdot)\big\|_{L^2} &\lesssim \big\|\chi_3(|\xi|)|\xi|^{a-2\sigma}e^{-c|\xi|^{2\sigma}t}\widehat{u}_0(\xi)\big\|_{L^2} + e^{-ct}\big\|\chi_3(|\xi|)|\xi|^{-\ell_1}e^{-c|\xi|^{-2\sigma}t}|\xi|^{a+\ell_1}\widehat{u}_0(\xi)\big\|_{L^2}\\
&\quad + e^{-ct}\big\|\chi_3(|\xi|)|\xi|^{-\ell_2}e^{-c|\xi|^{-2\sigma}t}|\xi|^{a-2\sigma+\ell_2}\widehat{u}_1(\xi)\big\|_{L^2} + \big\|\chi_3(|\xi|)|\xi|^{a-2\sigma}e^{-c|\xi|^{2\sigma}t}\widehat{u}_1(\xi)\big\|_{L^2}\\
&\lesssim e^{-ct}\|u_0\|_{H^{[a-2\sigma]^+}} + e^{-ct}(1+t)^{-\frac{\ell_1}{2\sigma}}\|u_0\|_{H^{a+\ell_1}}\\
&\quad + e^{-ct}(1+t)^{-\frac{\ell_2}{2\sigma}}\|u_1\|_{H^{[a-2\sigma+\ell_2]^+}} + e^{-ct}\|u_1\|_{H^{[a-2\sigma]^+}},
\end{aligned}$$

and

$$\begin{aligned}
\big\||D|^a\partial_t u_{\chi_3}(t,\cdot)\big\|_{L^2} &\lesssim e^{-ct}\big\|\chi_3(|\xi|)|\xi|^{-\ell_1}e^{-c|\xi|^{2\sigma}t}|\xi|^{a+\ell_1}\widehat{u}_0(\xi)\big\|_{L^2} + \big\|\chi_3(|\xi|)e^{-c|\xi|^{-2\sigma}t}|\xi|^{a}\widehat{u}_0(\xi)\big\|_{L^2}\\
&\quad + e^{-ct}\big\|\chi_3(|\xi|)|\xi|^{-\ell_2}e^{-c|\xi|^{-2\sigma}t}|\xi|^{a-2\sigma+\ell_2}\widehat{u}_1(\xi)\big\|_{L^2} + \big\|\chi_3(|\xi|)e^{-c|\xi|^{2\sigma}t}|\xi|^{a}\widehat{u}_1(\xi)\big\|_{L^2}\\
&\lesssim e^{-ct}(1+t)^{-\frac{\ell_1}{2\sigma}}\|u_0\|_{H^{a+\ell_1}} + e^{-ct}\|u_0\|_{H^{a}}\\
&\quad + e^{-ct}(1+t)^{-\frac{\ell_2}{2\sigma}}\|u_1\|_{H^{[a-2\sigma+\ell_2]^+}} + e^{-ct}\|u_1\|_{H^{a}}.
\end{aligned}$$

Therefore, this completes the proof of Theorem 8.3.1. $\square$

9. Blow-up result

The main goal of this chapter is to discuss the critical exponent for the following Cauchy problem for semi-linear structurally damped σ-evolution models:

$$\begin{cases} u_{tt} + (-\Delta)^\sigma u + (-\Delta)^\delta u_t = |u|^p, & x \in \mathbb{R}^n,\, t > 0, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{9.1}$$

with some $\sigma \geqslant 1$, $\delta \in [0,\sigma)$ and a given real number $p > 1$. Here, critical exponent $p_{crit} = p_{crit}(n)$ means that for some range of admissible $p > p_{crit}$ there exists a global (in time) Sobolev solution for small initial data from a suitable function space. Moreover, one can find suitable small data such that there exists no global (in time) Sobolev solution if $1 < p \leqslant p_{crit}$. In other words, we have, in general, only local (in time) Sobolev solutions under this assumption for the exponent p.

For the local existence of Sobolev solutions to (9.1), we address the interested readers to Proposition 9.1 in the paper [9]. The proof of blow-up results in the present paper is based on a contradiction argument by using the test function method. The test function method is not influenced by higher regularity of the data. For this reason, we restrict ourselves to the critical exponent for (9.1) in the case, where the data are supposed to belong to the energy space. To deal with the fractional Laplacian $(-\Delta)^\sigma$ and $(-\Delta)^\delta$ as well-known non-local operators, a modified test function method is applied to prove a blow-up result in the subcritical case and in the critical case as well.

Let us now introduce the following two parameters:

$$\mathtt{k}^- := \min\{\sigma; 2\delta\} \quad \text{and} \quad \mathtt{k}^+ := \max\{\sigma; 2\delta\}.$$

9.1. Main theorem

In order to state our blow-up result, we recall the global (in time) existence result of small data energy solutions to (9.1) in the following theorem.

Theorem 9.1.1 (Global existence). *Let $m \in [1,2)$ and $n > m_0\mathtt{k}^-$ with $\frac{1}{m_0} = \frac{1}{m} - \frac{1}{2}$. We assume the conditions*

$$\begin{aligned} &\frac{2}{m} \leqslant p < \infty && \text{if } n \leqslant 2\mathtt{k}^+, \\ &\frac{2}{m} \leqslant p \leqslant \frac{n}{n-2\mathtt{k}^+} && \text{if } n \in \Big(2\mathtt{k}^+, \frac{4\mathtt{k}^+}{2-m}\Big]. \end{aligned}$$

Moreover, we suppose the following condition:

$$p > 1 + \frac{m(\mathtt{k}^+ + \sigma)}{n - m\mathtt{k}^-}. \tag{9.2}$$

Then, there exists a constant $\varepsilon_0 > 0$ such that for any small data

$$(u_0, u_1) \in \big(L^m \cap H^{\mathtt{k}^+}\big) \times \big(L^m \cap L^2\big) \text{ satisfying the assumption } \|u_0\|_{L^m \cap H^{\mathtt{k}^+}} + \|u_1\|_{L^m \cap L^2} \leqslant \varepsilon_0,$$

we have a uniquely determined global (in time) small data energy solution

$$u \in C\big([0,\infty), H^{\mathtt{k}^+}\big) \cap C^1\big([0,\infty), L^2\big)$$

to (9.1). Moreover, the following estimates hold:

$$\begin{aligned} \|u(t,\cdot)\|_{L^2} &\lesssim (1+t)^{-\frac{n}{2(\mathtt{k}^+-\delta)}(\frac{1}{m}-\frac{1}{2})+\frac{\mathtt{k}^-}{2(\mathtt{k}^+-\delta)}}\big(\|u_0\|_{L^m \cap H^{\mathtt{k}^+}} + \|u_1\|_{L^m \cap L^2}\big), \\ \big\||D|^{\mathtt{k}^+} u(t,\cdot)\big\|_{L^2} &\lesssim (1+t)^{-\frac{n}{2(\mathtt{k}^+-\delta)}(\frac{1}{m}-\frac{1}{2})-\frac{\mathtt{k}^+-\mathtt{k}^-}{2(\mathtt{k}^+-\delta)}}\big(\|u_0\|_{L^m \cap H^{\mathtt{k}^+}} + \|u_1\|_{L^m \cap L^2}\big), \\ \|u_t(t,\cdot)\|_{L^2} &\lesssim (1+t)^{-\frac{n}{2(\mathtt{k}^+-\delta)}(\frac{1}{m}-\frac{1}{2})-\frac{\sigma-\mathtt{k}^-}{\mathtt{k}^+-\delta}}\big(\|u_0\|_{L^m \cap H^{\mathtt{k}^+}} + \|u_1\|_{L^m \cap L^2}\big). \end{aligned}$$

We are going to prove the following result.

Theorem 9.1.2 (Blow-up). *Let $\sigma \geqslant 1$, $\delta \in [0,\sigma)$ and $n > \mathtt{k}^-$. We assume that we choose the initial data $u_0 = 0$ and $u_1 \in L^1$ satisfying the following relation:*

$$\int_{\mathbb{R}^n} u_1(x)dx > \epsilon, \tag{9.3}$$

where ϵ is a suitable nonnegative constant. Moreover, we suppose the condition

$$p \in \Big(1, 1 + \frac{2\sigma}{n - \mathtt{k}^-}\Big]. \tag{9.4}$$

Then, there is no global (in time) Sobolev solution $u \in C\big([0,\infty), L^2\big)$ to (9.1).

Remark 9.1.1. We want to underline that the lifespan T_ε of Sobolev solutions to given data $(0, \varepsilon u_1)$ for any small positive constant ε in the subcritical case of Theorem 9.1.2 can be estimated as follows:

$$T_\varepsilon \leqslant C\varepsilon^{-\frac{(2\sigma-\mathtt{k}^-)(p-1)}{2\sigma-(n-\mathtt{k}^-)(p-1)}} \quad \text{with } C > 0. \tag{9.5}$$

Remark 9.1.2. If we choose $m = 1$ in Theorem 9.1.1, then from Theorem 9.1.2 it is clear that the critical exponent p_{crit} is given by

$$p_{crit}(n) = 1 + \frac{2\sigma}{n - 2\delta} \quad \text{if } \delta \in \Big[0, \frac{\sigma}{2}\Big] \text{ and } 4\delta < n \leqslant 4\sigma.$$

However, in the case $\delta \in (\frac{\sigma}{2}, \sigma)$ there appears a gap between the exponents given by $1 + \frac{2\delta+\sigma}{n-\sigma}$ from Theorem 9.1.1 and $1 + \frac{2\sigma}{n-\sigma}$ from Theorem 9.1.2 for $2\sigma < n \leqslant 8\delta$.

9.2. A modified test function

In this section, we collect some preliminary knowledge about a modified test function method needed in our proofs.

Definition 9.2.1 ([43, 70])**.** Let $s \in (0,1)$. Let X be a suitable set of functions defined on $\mathbb{R}^n$. Then, the fractional Laplacian $(-\Delta)^s$ in $\mathbb{R}^n$ is a non-local operator given by

$$(-\Delta)^s:\ v \in X \to (-\Delta)^s v(x) := C_{n,s}\ p.v.\int_{\mathbb{R}^n} \frac{v(x)-v(y)}{|x-y|^{n+2s}}dy$$

as long as the right-hand side exists, where $p.v.$ stands for Cauchy's principal value, $C_{n,s} := \frac{4^s\Gamma(\frac{n}{2}+s)}{\pi^{\frac{n}{2}}\Gamma(-s)}$ is a normalization constant and Γ denotes the Gamma function.

Lemma 9.2.1. *Let $\langle x\rangle := (1+|x|^2)^{\frac{1}{2}}$ for all $x \in \mathbb{R}^n$ and $q > 0$. Then, the following estimate holds for any multi-index α satisfying $|\alpha| \geqslant 1$:*

$$\big|\partial_x^\alpha \langle x\rangle^{-q}\big| \lesssim \langle x\rangle^{-q-|\alpha|} \quad \textit{for all } x \in \mathbb{R}^n.$$

Proof. First, we recall the following formula of derivatives of composed functions from Lemma B.6.2 for $|\alpha| \geqslant 1$:

$$\partial_x^\alpha h\big(f(x)\big) = \sum_{k=1}^{|\alpha|} h^{(k)}\big(f(x)\big)\left(\sum_{\substack{\gamma_1+\cdots+\gamma_k\leqslant\alpha\\ |\gamma_1|+\cdots+|\gamma_k|=|\alpha|,\ |\gamma_i|\geqslant 1}} \big(\partial_x^{\gamma_1} f(x)\big)\cdots\big(\partial_x^{\gamma_k} f(x)\big)\right),$$

where $h = h(z)$ and $h^{(k)}(z) = \frac{d^k h(z)}{dz^k}$. Applying this formula with $h(z) = z^{-\frac{q}{2}}$ and $f(x) = 1 + |x|^2$ we obtain

$$\begin{aligned}
\left|\partial_x^\alpha \langle x\rangle^{-q}\right| &\leqslant \sum_{k=1}^{|\alpha|}(1+|x|^2)^{-\frac{q}{2}-k}\left(\sum_{\substack{\gamma_1+\cdots+\gamma_k\leqslant\alpha\\ |\gamma_1|+\cdots+|\gamma_k|=|\alpha|,\,|\gamma_i|\geqslant 1}}\left|\partial_x^{\gamma_1}(1+|x|^2)\right|\cdots\left|\partial_x^{\gamma_k}(1+|x|^2)\right|\right)\\
&\leqslant C_1\sum_{k=1}^{|\alpha|}(1+|x|^2)^{-\frac{q}{2}-k}\begin{cases}1 & \text{if } 0\leqslant |x|\leqslant 1,\\ \left(\displaystyle\sum_{\substack{\gamma_1+\cdots+\gamma_k\leqslant\alpha\\ |\gamma_1|+\cdots+|\gamma_k|=|\alpha|,\,|\gamma_i|\geqslant 1}}|x|^{2-|\gamma_1|}\cdots|x|^{2-|\gamma_k|}\right) & \text{if } |x|\geqslant 1,\end{cases}\\
&\leqslant C_2\sum_{k=1}^{|\alpha|}(1+|x|^2)^{-\frac{q}{2}-k}\begin{cases}1 & \text{if}\quad 0\leqslant |x|\leqslant 1,\\ |x|^{2k-|\alpha|} & \text{if}\quad |x|\geqslant 1,\end{cases}\\
&\leqslant\begin{cases}C_2|\alpha|\langle x\rangle^{-q-2} & \text{if}\quad 0\leqslant |x|\leqslant 1,\\ C_2|\alpha|\langle x\rangle^{-q}|x|^{-|\alpha|} & \text{if}\quad |x|\geqslant 1,\end{cases}
\end{aligned}$$

where C_1 and C_2 are some suitable constants. This completes the proof. □

Lemma 9.2.2. *Let $m\in\mathbb{Z}$, $s\in(0,1)$ and $\gamma:=m+s$. If $v\in H^{2\gamma}(\mathbb{R}^n)$, then it holds*

$$(-\Delta)^\gamma v(x) = (-\Delta)^m\big((-\Delta)^s v(x)\big) = (-\Delta)^s\big((-\Delta)^m v(x)\big).$$

One can find the proof of Lemma 9.2.2 in Remark 3.2 in [1].

Lemma 9.2.3. *Let $\langle x\rangle := (1+|x|^2)^{\frac{1}{2}}$ for all $x\in\mathbb{R}^n$ and $q>0$. Let $m\in\mathbb{Z}$, $s\in(0,1)$ and $\gamma:=m+s$. Then, the following estimates hold for all $x\in\mathbb{R}^n$:*

$$\left|(-\Delta)^\gamma\langle x\rangle^{-q}\right| \lesssim \begin{cases}\langle x\rangle^{-q-2\gamma} & \textit{if}\quad 0<q+2m<n,\\ \langle x\rangle^{-n-2s}\log(e+|x|) & \textit{if}\quad q+2m=n,\\ \langle x\rangle^{-n-2s} & \textit{if}\quad q+2m>n.\end{cases} \tag{9.6}$$

Proof. We follow ideas from the proof of Lemma 1.5 in [29] devoting to the case $m=0$ and $s=\frac{1}{2}$, that is, the case $\gamma=\frac{1}{2}$ is generalized to any fractional number $\gamma>0$. To do this, for any $s\in(0,1)$ we shall divide the proof into two cases: $m=0$ and $m\geqslant 1$.

Let us consider the first case $m=0$. Denoting by $\psi=\psi(x):=\langle x\rangle^{-q}$ we write $(-\Delta)^s\langle x\rangle^{-q} = (-\Delta)^s(\psi)(x)$. According to Definition 9.2.1 of fractional Laplacian as a singular integral operator, we have

$$(-\Delta)^s(\psi)(x) := C_{n,\delta}\ p.v.\int_{\mathbb{R}^n}\frac{\psi(x)-\psi(y)}{|x-y|^{n+2s}}dy.$$

A standard change of variables leads to

$$\begin{aligned}
(-\Delta)^s(\psi)(x) &= -\frac{C_{n,s}}{2}\ p.v.\int_{\mathbb{R}^n}\frac{\psi(x+y)+\psi(x-y)-2\psi(x)}{|y|^{n+2s}}dy\\
&= -\frac{C_{n,s}}{2}\lim_{\varepsilon\to 0+}\int_{\varepsilon\leqslant|y|\leqslant 1}\frac{\psi(x+y)+\psi(x-y)-2\psi(x)}{|y|^{n+2s}}dy\\
&\quad -\frac{C_{n,s}}{2}\int_{|y|\geqslant 1}\frac{\psi(x+y)+\psi(x-y)-2\psi(x)}{|y|^{n+2s}}dy.
\end{aligned}$$

To deal with the first integral, after using a second order Taylor expansion for ψ, we arrive at

$$\frac{|\psi(x+y)+\psi(x-y)-2\psi(x)|}{|y|^{n+2s}} \lesssim \frac{\|\partial_x^2\psi\|_{L^\infty}}{|y|^{n+2s-2}}.$$

Thanks to the above estimate and $s\in(0,1)$, we may remove the principal value of the integral at the origin to conclude

$$(-\Delta)^s(\psi)(x) = -\frac{C_{n,s}}{2}\int_{\mathbb{R}^n}\frac{\psi(x+y)+\psi(x-y)-2\psi(x)}{|y|^{n+2s}}dy.$$

To prove the desired estimates, we shall divide our considerations into two subcases. In the first subcase $\{x : |x| \leqslant 1\}$, we can proceed as follows:

$$\begin{aligned}\big|(-\Delta)^s(\psi)(x)\big| &\lesssim \int_{|y|\leqslant 1} \frac{|\psi(x+y)+\psi(x-y)-2\psi(x)|}{|y|^{n+2s}}dy + \int_{|y|\geqslant 1} \frac{|\psi(x+y)+\psi(x-y)-2\psi(x)|}{|y|^{n+2s}}dy \\ &\lesssim \|\partial_x^2\psi\|_{L^\infty} \int_{|y|\leqslant 1} \frac{1}{|y|^{n+2s-2}}dy + \|\psi\|_{L^\infty} \int_{|y|\geqslant 1} \frac{1}{|y|^{n+2s}}dy.\end{aligned}$$

Due to the boundedness of the above two integrals, it follows immediately

$$\big|(-\Delta)^s(\psi)(x)\big| \lesssim 1 \quad \text{for all } |x| \leqslant 1. \tag{9.7}$$

In order to deal with the second subcase $\{x : |x| \geqslant 1\}$, we can re-write

$$\begin{aligned}(-\Delta)^s(\psi)(x) = &-\frac{C_{n,s}}{2} \int_{|y|\geqslant 2|x|} \frac{\psi(x+y)+\psi(x-y)-2\psi(x)}{|y|^{n+2s}}dy \\ &- \frac{C_{n,s}}{2} \int_{\frac{1}{2}|x|\leqslant|y|\leqslant 2|x|} \frac{\psi(x+y)+\psi(x-y)-2\psi(x)}{|y|^{n+2s}}dy \\ &- \frac{C_{n,s}}{2} \int_{|y|\leqslant\frac{1}{2}|x|} \frac{\psi(x+y)+\psi(x-y)-2\psi(x)}{|y|^{n+2s}}dy.\end{aligned} \tag{9.8}$$

For the first integral, we notice that the relations $|x+y| \geqslant |y|-|x| \geqslant |x|$ and $|x-y| \geqslant |y|-|x| \geqslant |x|$ hold for $|y| \geqslant 2|x|$. Since ψ is a decreasing function, we obtain the following estimate:

$$\begin{aligned}&\Big|\int_{|y|\geqslant 2|x|} \frac{\psi(x+y)+\psi(x-y)-2\psi(x)}{|y|^{n+2s}}dy\Big| \\ &\quad\leqslant 4|\psi(x)| \int_{|y|\geqslant 2|x|} \frac{1}{|y|^{n+2s}}dy \lesssim \langle x\rangle^{-q} \int_{|y|\geqslant 2|x|} \frac{1}{|y|^{1+2s}}d|y| \\ &\quad\lesssim \langle x\rangle^{-q}|x|^{-2s} \lesssim \langle x\rangle^{-q-2s} \qquad \big(\text{due to } |x| \approx \langle x\rangle \text{ for all } |x| \geqslant 1\big).\end{aligned} \tag{9.9}$$

It is clear that $|y| \approx |x|$ in the second integral domain. Moreover, it follows

$$\Big\{y : \frac{1}{2}|x| \leqslant |y| \leqslant 2|x|\Big\} \subset \big\{y : |x+y| \leqslant 3|x|\big\}, \tag{9.10}$$

$$\Big\{y : \frac{1}{2}|x| \leqslant |y| \leqslant 2|x|\Big\} \subset \big\{y : |x-y| \leqslant 3|x|\big\}. \tag{9.11}$$

For this reason, we arrive at

$$\begin{aligned}&\Big|\int_{\frac{1}{2}|x|\leqslant|y|\leqslant 2|x|} \frac{\psi(x+y)+\psi(x-y)-2\psi(x)}{|y|^{n+2s}}dy\Big| \\ &\quad\lesssim |x|^{-n-2s}\Big(\int_{|x+y|\leqslant 3|x|} \psi(x+y)dy + \int_{|x-y|\leqslant 3|x|} \psi(x-y)dy + \psi(x)\int_{\frac{1}{2}|x|\leqslant|y|\leqslant 2|x|} 1dy\Big) \\ &\quad\lesssim |x|^{-n-2s}\Big(\int_{|x+y|\leqslant 3|x|} \psi(x+y)dy + \langle x\rangle^{-q}|x|^n\Big),\end{aligned} \tag{9.12}$$

where we used the relation

$$\int_{|x+y|\leqslant 3|x|} \psi(x+y)dy = \int_{|x-y|\leqslant 3|x|} \psi(x-y)dy.$$

By the change of variables $r = |x+y|$, we apply the inequality $1+r^2 \geqslant \frac{(1+r)^2}{2}$ to get

$$\begin{aligned}\int_{|x+y|\leqslant 3|x|} \psi(x+y)dy &\lesssim \int_{r\leqslant 3|x|} (1+r^2)^{-\frac{q}{2}}\, r^{n-1}dr \lesssim \int_{r\leqslant 3|x|} (1+r)^{n-q-1}dr \\ &\lesssim \begin{cases} (1+3|x|)^{n-q} & \text{if} \quad 0<q<n, \\ \log(e+3|x|) & \text{if} \quad q=n, \\ 1 & \text{if} \quad q>n. \end{cases}\end{aligned} \tag{9.13}$$

By $|x| \approx \langle x \rangle$ for all $|x| \geqslant 1$, combining (9.12) and (9.13) leads to

$$\Big|\int_{\frac{1}{2}|x|\leqslant|y|\leqslant 2|x|} \frac{\psi(x+y)+\psi(x-y)-2\psi(x)}{|y|^{n+2s}}dy\Big| \lesssim \begin{cases} \langle x \rangle^{-q-2s} & \text{if} \quad 0<q<n, \\ \langle x \rangle^{-n-2s}\log(e+3|x|) & \text{if} \quad q=n, \\ \langle x \rangle^{-n-2s} & \text{if} \quad q>n. \end{cases} \tag{9.14}$$

For the third integral in (9.8), using again the second order Taylor expansion for ψ we obtain

$$\begin{aligned} &\Big|\int_{|y|\leqslant\frac{1}{2}|x|} \frac{\psi(x+y)+\psi(x-y)-2\psi(x)}{|y|^{n+2s}}dy\Big| \\ &\quad\leqslant \int_{|y|\leqslant\frac{1}{2}|x|} \frac{|\psi(x+y)+\psi(x-y)-2\psi(x)|}{|y|^{n+2s}}dy \lesssim \int_{|y|\leqslant\frac{1}{2}|x|} \max_{\theta\in[0,1]} \big|\partial_x^2\psi(x\pm\theta y)\big|\frac{1}{|y|^{n+2s-2}}dy \\ &\quad\lesssim \int_{|y|\leqslant\frac{1}{2}|x|} \max_{\theta\in[0,1]} \langle x\pm\theta y \rangle^{-q-2}\frac{1}{|y|^{n+2s-2}}dy \lesssim \langle x \rangle^{-q-2}\int_{|y|\leqslant\frac{1}{2}|x|} |y|^{1-2s}d|y| \lesssim \langle x \rangle^{-q-2s}. \end{aligned} \tag{9.15}$$

Here we used the relation $|x\pm\theta y| \geqslant |x|-\theta|y| \geqslant |x|-\frac{1}{2}|x| = \frac{1}{2}|x|$. From (9.8), (9.9), (9.14) and (9.15) we arrive at the following estimates for all $|x| \geqslant 1$:

$$\big|(-\Delta)^s(\psi)(x)\big| \lesssim \begin{cases} \langle x \rangle^{-q-2s} & \text{if} \quad 0<q<n, \\ \langle x \rangle^{-n-2s}\log(e+3|x|) & \text{if} \quad q=n, \\ \langle x \rangle^{-n-2s} & \text{if} \quad q>n. \end{cases} \tag{9.16}$$

Finally, combining (9.7) and (9.16) we may conclude all desired estimates for $m=0$.

Next let us turn to the second case $m \geqslant 1$. First, a straight-forward calculation gives the following relation:

$$-\Delta\langle x \rangle^{-r} = r\Big((n-r-2)\langle x \rangle^{-r-2}+(r+2)\langle x \rangle^{-r-4}\Big) \quad \text{for any } r>0. \tag{9.17}$$

By induction argument, carrying out m steps of (9.17) we obtain the following formula for any $m \geqslant 1$:

$$\begin{aligned} (-\Delta)^m\langle x \rangle^{-q} = (-1)^m \prod_{j=0}^{m-1}(q+2j)\Big(&\prod_{j=1}^{m}(-n+q+2j)\langle x \rangle^{-q-2m} \\ &- C_m^1\prod_{j=2}^{m}(-n+q+2j)(q+2m)\langle x \rangle^{-q-2m-2} \\ &+ C_m^2\prod_{j=3}^{m}(-n+q+2j)(q+2m)(q+2m+2)\langle x \rangle^{-q-2m-4} \\ &+\cdots+(-1)^m\prod_{j=0}^{m-1}(q+2m+2j)\langle x \rangle^{-q-4m}\Big). \end{aligned} \tag{9.18}$$

Then, thanks to Lemma 9.2.2, we derive

$$\begin{aligned} (-\Delta)^\gamma\langle x \rangle^{-q} &= (-\Delta)^s\big((-\Delta)^m\langle x \rangle^{-q}\big) \\ &= (-1)^m \prod_{j=0}^{m-1}(q+2j)\Big(\prod_{j=1}^{m}(-n+q+2j)\,(-\Delta)^s\langle x \rangle^{-q-2m} \\ &\qquad - C_m^1\prod_{j=2}^{m}(-n+q+2j)(q+2m)\,(-\Delta)^s\langle x \rangle^{-q-2m-2} \\ &\qquad + C_m^2\prod_{j=3}^{m}(-n+q+2j)(q+2m)(q+2m+2)\,(-\Delta)^s\langle x \rangle^{-q-2m-4} \\ &\qquad +\cdots+(-1)^m\prod_{j=0}^{m-1}(q+2m+2j)\,(-\Delta)^s\langle x \rangle^{-q-4m}\Big). \end{aligned} \tag{9.19}$$

For this reason, in order to conclude the desired estimates, we only indicate the following estimates for $k = 0, \cdots, m$:

$$\big|(-\Delta)^s \langle x\rangle^{-q-2(m+k)}\big| \lesssim \begin{cases} \langle x\rangle^{-q-2\gamma} & \text{if} \quad 0 < q+2m < n, \\ \langle x\rangle^{-n-2s}\log(e+|x|) & \text{if} \quad q+2m = n, \\ \langle x\rangle^{-n-2s} & \text{if} \quad q+2m > n. \end{cases} \tag{9.20}$$

Indeed, substituting q by $q + 2(m + k)$ with $k = 0, \cdots, m$ and $\gamma = s$ into (9.6) leads to

$$\big|(-\Delta)^s \langle x\rangle^{-q-2(m+k)}\big| \lesssim \begin{cases} \langle x\rangle^{-q-2\gamma} & \text{if} \quad 0 < q+2(m+k) < n, \\ \langle x\rangle^{-n-2s}\log(e+|x|) & \text{if} \quad q+2(m+k) = n, \\ \langle x\rangle^{-n-2s} & \text{if} \quad q+2(m+k) > n. \end{cases}$$

From these estimates, it follows immediately (9.20) to conclude (9.6) for any $m \geqslant 1$. Summarizing, the proof of Lemma 9.2.3 is completed. □

Lemma 9.2.4. *Let $s \in (0,1)$. Let ψ be a smooth function satisfying $\partial_x^2\psi \in L^\infty$. For any $R > 0$, let ψ_R be a function defined by*

$$\psi_R(x) := \psi\big(R^{-1}x\big)$$

for all $x \in \mathbb{R}^n$. Then, $(-\Delta)^s(\psi_R)$ satisfies the following scaling properties for all $x \in \mathbb{R}^n$:

$$(-\Delta)^s(\psi_R)(x) = R^{-2s}\big((-\Delta)^s\psi\big)\big(R^{-1}x\big).$$

Proof. Thanks to the assumption $\partial_x^2\psi \in L^\infty$, following the proof of Lemma 9.2.3 we may remove the principal value of the integral at the origin to conclude

$$\begin{aligned} (-\Delta)^s(\psi_R)(x) &= -\frac{C_{n,s}}{2}\int_{\mathbb{R}^n}\frac{\psi_R(x+y)+\psi_R(x-y)-2\psi_R(x)}{|y|^{n+2s}}dy \\ &= -\frac{C_{n,s}}{2}R^{-2s}\int_{\mathbb{R}^n}\frac{\psi\big(R^{-1}x+R^{-1}y\big)+\psi\big(R^{-1}x-R^{-1}y\big)-2\psi\big(R^{-1}x\big)}{|R^{-1}y|^{n+2s}}d(R^{-1}y) \\ &= R^{-2s}\big((-\Delta)^s\psi\big)\big(R^{-1}x\big). \end{aligned}$$

This completes the proof. □

Lemma 9.2.5 (One mapping property in the scale of fractional spaces $\{H^s\}_{s\in\mathbb{R}}$)**.** *Let γ, $s \in \mathbb{R}$. Then, the fractional Laplacian*

$$(-\Delta)^\gamma:\ f \to (-\Delta)^\gamma f = \big((-\Delta)^\gamma f\big)(x) := \mathfrak{F}^{-1}\big(|\xi|^{2\gamma}\widehat{f}(\xi)\big)(x)$$

maps isomorphically the space H^s onto $H^{s-2\gamma}$.

This result can be found in Section 2.3.8 in [72].

Lemma 9.2.6. *Let $f = f(x) \in H^s$ and $g = g(x) \in H^{-s}$ with $s \in \mathbb{R}$. Then, the following estimate holds:*

$$\Big|\int_{\mathbb{R}^n} f(x)\,g(x)dx\Big| \leqslant \|f\|_{H^s}\,\|g\|_{H^{-s}}.$$

The proof of Lemma 9.2.6 can be found in Theorem 16 in [27].

Lemma 9.2.7. *Let $s \in \mathbb{R}$. Let $v_1 = v_1(x) \in H^s$ and $v_2 = v_2(x) \in H^{-s}$. Then, the following relation holds:*

$$\int_{\mathbb{R}^n} v_1(x)\,v_2(x)dx = \int_{\mathbb{R}^n}\widehat{v}_1(\xi)\,\widehat{v}_2(\xi)d\xi.$$

Proof. We present the proof from Theorem 16 in [27] to make the paper self-contained. Since the space $\mathcal{S}$ is dense in H^s and H^{-s}, there exist sequences $\{v_{1,k}\}_k$ and $\{v_{2,k}\}_k$ with $v_{1,k} = v_{1,k}(x) \in \mathcal{S}$ and $v_{2,k} = v_{2,k}(x) \in \mathcal{S}$ such that

$$\|v_{1,k} - v_1\|_{H^s} \to 0 \quad \text{and} \quad \|v_{2,k} - v_2\|_{H^{-s}} \to 0 \quad \text{as } k \to \infty.$$

On the one hand, as $k \to \infty$ we have the relations

$$\widehat{V}_{1,k}(\xi) := (1+|\xi|^2)^{\frac{s}{2}}\,\widehat{v}_{1,k}(\xi) \to \widehat{V}_1(\xi) := (1+|\xi|^2)^{\frac{s}{2}}\,\widehat{v}_1(\xi) \quad \text{in } L^2,$$
$$\widehat{V}_{2,k}(\xi) := (1+|\xi|^2)^{-\frac{s}{2}}\,\widehat{v}_{2,k}(\xi) \to \widehat{V}_2(\xi) := (1+|\xi|^2)^{-\frac{s}{2}}\,\widehat{v}_2(\xi) \quad \text{in } L^2.$$

On the other hand, by Parseval-Plancherel formula we arrive at

$$\begin{aligned}\int_{\mathbb{R}^n} v_{1,k}(x)\,v_{2,k}(x)\,dx &= \big(v_{1,k},\, v_{2,k}\big)_{L^2} = \big(\widehat{v}_{1,k},\, \widehat{v}_{2,k}\big)_{L^2} = \int_{\mathbb{R}^n} \widehat{v}_{1,k}(\xi)\,\widehat{v}_{2,k}(\xi)\,d\xi \\ &= \int_{\mathbb{R}^n} (1+|\xi|^2)^{\frac{s}{2}}\,\widehat{v}_{1,k}(\xi)\,(1+|\xi|^2)^{-\frac{s}{2}}\,\widehat{v}_{2,k}(\xi)\,d\xi = \int_{\mathbb{R}^n} \widehat{V}_{1,k}(\xi)\,\widehat{V}_{2,k}(\xi)\,d\xi, \end{aligned} \tag{9.21}$$

where $(\cdot,\cdot)_{L^2}$ stands for the scalar product in L^2. Moreover, applying Lemma 9.2.6 we may estimate

$$\begin{aligned}&\Big|\int_{\mathbb{R}^n} \big(v_{1,k}(x)\,v_{2,k}(x) - v_1(x)\,v_2(x)\big)\,dx\Big| \\ &\quad\leqslant \Big|\int_{\mathbb{R}^n} \big(v_{1,k}(x) - v_1(x)\big)\,v_{2,k}(x)dx\Big| + \Big|\int_{\mathbb{R}^n} v_1(x)\,\big(v_{2,k}(x) - v_2(x)\big)\,dx\Big| \\ &\quad\leqslant \|v_{1,k} - v_1\|_{H^s}\,\|v_{2,k}\|_{H^{-s}} + \|v_1\|_{H^s}\,\|v_{2,k} - v_2\|_{H^{-s}} \to 0 \quad \text{as } k \to \infty.\end{aligned}$$

This is equivalent to

$$\int_{\mathbb{R}^n} v_{1,k}(x)\,v_{2,k}(x)\,dx \to \int_{\mathbb{R}^n} v_1(x)\,v_2(x)\,dx \quad \text{as } k \to \infty. \tag{9.22}$$

In the same way we also derive

$$\int_{\mathbb{R}^n} \widehat{V}_{1,k}(\xi)\,\widehat{V}_{2,k}(\xi)\,d\xi \to \int_{\mathbb{R}^n} \widehat{V}_1(\xi)\,\widehat{V}_2(\xi)\,d\xi \quad \text{as } k \to \infty. \tag{9.23}$$

Summarizing from (9.21) to (9.23) we may conclude

$$\int_{\mathbb{R}^n} v_1(x)\,v_2(x)\,dx = \int_{\mathbb{R}^n} \widehat{V}_1(\xi)\,\widehat{V}_2(\xi)\,d\xi = \int_{\mathbb{R}^n} \widehat{v}_1(\xi)\,\widehat{v}_2(\xi)\,d\xi.$$

Therefore, the proof of Lemma 9.2.7 is completed. □

9.3. Proof of the main theorem

We divide the proof of Theorem 9.1.2 into several cases.

9.3.1. The case that both parameters σ and δ are integers

Proof. The proof of this case can be found in the paper [9]. □

9.3.2. The case that the parameter σ is integer and the parameter δ is fractional from $(0,1)$

Proof. First, we introduce the function $\varphi = \varphi(|x|) := \langle x\rangle^{-n-2\delta}$ and the function $\eta = \eta(t)$ having the following properties:

$$1. \quad \eta \in \mathcal{C}_0^\infty([0,\infty)) \text{ and } \eta(t) = \begin{cases} 1 & \text{if} \quad 0 \leqslant t \leqslant \frac{1}{2}, \\ \text{decreasing} & \text{if} \quad \frac{1}{2} \leqslant t \leqslant 1, \\ 0 & \text{if} \quad t \geqslant 1, \end{cases}$$

$$2. \quad \eta^{-\frac{p'}{p}}(t)\big(|\eta'(t)|^{p'} + |\eta''(t)|^{p'}\big) \leqslant C \quad \text{for any } t \in \Big[\frac{1}{2}, 1\Big], \tag{9.24}$$

where p' is the conjugate of $p > 1$ and C is a suitable positive constant. Let R be a large parameter in $[0,\infty)$. We define the following test function:

$$\phi_R(t,x) := \eta_R(t)\varphi_R(x),$$

where $\eta_R(t) := \eta\big(R^{-\alpha}t\big)$ and $\varphi_R(x) := \varphi\big(R^{-1}x\big)$ with a fixed parameter $\alpha := 2\sigma - \mathtt{k}^-$. We define the functionals

$$I_R := \int_0^\infty \int_{\mathbb{R}^n} |u(t,x)|^p \phi_R(t,x)\,dxdt = \int_0^{R^\alpha} \int_{\mathbb{R}^n} |u(t,x)|^p \phi_R(t,x)\,dxdt$$

and

$$I_{R,t} := \int_{\frac{R^\alpha}{2}}^{R^\alpha} \int_{\mathbb{R}^n} |u(t,x)|^p \phi_R(t,x)\,dxdt.$$

Let us assume that $u = u(t,x)$ is a global (in time) Sobolev solution from $C\big([0,\infty), L^2\big)$ to (9.1). After multiplying the equation (9.1) by $\phi_R = \phi_R(t,x)$, we carry out partial integration to derive

$$\begin{aligned}
0 \leqslant I_R &= -\int_{\mathbb{R}^n} u_1(x)\varphi_R(x)\,dx + \int_{\frac{R^\alpha}{2}}^{R^\alpha} \int_{\mathbb{R}^n} u(t,x)\partial_t^2\eta_R(t)\varphi_R(x)\,dxdt \\
&\quad + \int_0^\infty \int_{\mathbb{R}^n} \eta_R(t)\varphi_R(x)\,(-\Delta)^\sigma u(t,x)\,dxdt - \int_{\frac{R^\alpha}{2}}^{R^\alpha} \int_{\mathbb{R}^n} \partial_t\eta_R(t)\varphi_R(x)\,(-\Delta)^\delta u(t,x)\,dxdt \\
&=: -\int_{\mathbb{R}^n} u_1(x)\varphi_R(x)\,dx + J_1 + J_2 - J_3. \qquad (9.25)
\end{aligned}$$

Applying Hölder's inequality with $\frac{1}{p} + \frac{1}{p'} = 1$ we may estimate as follows:

$$\begin{aligned}
|J_1| &\leqslant \int_{\frac{R^\alpha}{2}}^{R^\alpha} \int_{\mathbb{R}^n} |u(t,x)|\,\big|\partial_t^2\eta_R(t)\big|\varphi_R(x)\,dxdt \\
&\lesssim \Big(\int_{\frac{R^\alpha}{2}}^{R^\alpha} \int_{\mathbb{R}^n} \Big|u(t,x)\phi_R^{\frac{1}{p}}(t,x)\Big|^p\,dxdt\Big)^{\frac{1}{p}} \Big(\int_{\frac{R^\alpha}{2}}^{R^\alpha} \int_{\mathbb{R}^n} \Big|\phi_R^{-\frac{1}{p}}(t,x)\partial_t^2\eta_R(t)\varphi_R(x)\Big|^{p'}\,dxdt\Big)^{\frac{1}{p'}} \\
&\lesssim I_{R,t}^{\frac{1}{p}} \Big(\int_{\frac{R^\alpha}{2}}^{R^\alpha} \int_{\mathbb{R}^n} \eta_R^{-\frac{p'}{p}}(t)\big|\partial_t^2\eta_R(t)\big|^{p'}\varphi_R(x)\,dxdt\Big)^{\frac{1}{p'}}.
\end{aligned}$$

By the change of variables $\tilde{t} := R^{-\alpha}t$ and $\tilde{x} := R^{-1}x$, a straight-forward calculation gives

$$|J_1| \lesssim I_{R,t}^{\frac{1}{p}}\, R^{-2\alpha + \frac{n+\alpha}{p'}} \Big(\int_{\mathbb{R}^n} \langle\tilde{x}\rangle^{-n-2\delta}\,d\tilde{x}\Big)^{\frac{1}{p'}}. \qquad (9.26)$$

Here we used $\partial_t^2\eta_R(t) = R^{-2\alpha}\eta''(\tilde{t})$ and the assumption (9.24). Now let us turn to estimate J_2 and J_3. First, by using $\varphi_R \in H^{2\sigma}$ and $u \in C\big([0,\infty), L^2\big)$ we apply Lemma 9.2.7 to conclude the following relations:

$$\begin{aligned}
\int_{\mathbb{R}^n} \varphi_R(x)\,(-\Delta)^\sigma u(t,x)\,dx &= \int_{\mathbb{R}^n} |\xi|^{2\sigma}\widehat{\varphi}_R(\xi)\,\widehat{u}(t,\xi)\,d\xi = \int_{\mathbb{R}^n} u(t,x)\,(-\Delta)^\sigma\varphi_R(x)\,dx, \\
\int_{\mathbb{R}^n} \varphi_R(x)\,(-\Delta)^\delta u(t,x)\,dx &= \int_{\mathbb{R}^n} |\xi|^{2\delta}\widehat{\varphi}_R(\xi)\,\widehat{u}(t,\xi)\,d\xi = \int_{\mathbb{R}^n} u(t,x)\,(-\Delta)^\delta\varphi_R(x)\,dx.
\end{aligned}$$

Hence, we obtain

$$J_2 = \int_0^\infty \int_{\mathbb{R}^n} \eta_R(t)\varphi_R(x)\,(-\Delta)^\sigma u(t,x)\,dxdt = \int_0^\infty \int_{\mathbb{R}^n} \eta_R(t)u(t,x)\,(-\Delta)^\sigma\varphi_R(x)\,dxdt,$$

and

$$J_3 = \int_{\frac{R^\alpha}{2}}^{R^\alpha} \int_{\mathbb{R}^n} \partial_t\eta_R(t)\varphi_R(x)\,(-\Delta)^\delta u(t,x)\,dxdt = \int_{\frac{R^\alpha}{2}}^{R^\alpha} \int_{\mathbb{R}^n} \partial_t\eta_R(t)u(t,x)\,(-\Delta)^\delta\varphi_R(x)\,dxdt.$$

Applying Hölder's inequality again as we estimated J_1 leads to

$$|J_2| \leqslant I_R^{\frac{1}{p}} \Big(\int_0^{R^\alpha} \int_{\mathbb{R}^n} \eta_R(t)\varphi_R^{-\frac{p'}{p}}(x)\,\big|(-\Delta)^\sigma\varphi_R(x)\big|^{p'}\,dxdt\Big)^{\frac{1}{p'}},$$

and

$$|J_3| \leqslant I_{R,t}^{\frac{1}{p}} \Big(\int_{\frac{R^\alpha}{2}}^{R^\alpha} \int_{\mathbb{R}^n} \eta_R^{-\frac{p'}{p}}(t)\big|\partial_t\eta_R(t)\big|^{p'}\varphi_R^{\frac{-p'}{p}}(x)\,\big|(-\Delta)^\delta\varphi_R(x)\big|^{p'}\,dxdt\Big)^{\frac{1}{p'}}.$$

In order to control the above two integrals, the key tools rely on the results from Lemmas 9.2.1, 9.2.3 and 9.2.4. Namely, at first carrying out the change of variables $\tilde{t} := R^{-\alpha}t$ and $\tilde{x} := R^{-1}x$ we arrive at

$$\begin{aligned}|J_2| &\lesssim I_R^{\frac{1}{p}}\, R^{-2\sigma+\frac{n+\alpha}{p'}} \Big(\int_0^1 \int_{\mathbb{R}^n} \eta(\tilde{t})\varphi^{-\frac{p'}{p}}(\tilde{x})\,\big|(-\Delta)^\sigma(\varphi)(\tilde{x})\big|^{p'}\, d\tilde{x}d\tilde{t}\Big)^{\frac{1}{p'}}\\ &\lesssim I_R^{\frac{1}{p}}\, R^{-2\sigma+\frac{n+\alpha}{p'}} \Big(\int_{\mathbb{R}^n} \varphi^{-\frac{p'}{p}}(\tilde{x})\,\big|(-\Delta)^\sigma(\varphi)(\tilde{x})\big|^{p'}\, d\tilde{x}\Big)^{\frac{1}{p'}},\end{aligned}$$

where we note (σ is an integer) that $(-\Delta)^\sigma\varphi_R(x) = R^{-2\sigma}(-\Delta)^\sigma\varphi(\tilde{x})$. Using Lemma 9.2.1 implies the following estimate:

$$|J_2| \lesssim I_R^{\frac{1}{p}}\, R^{-2\sigma+\frac{n+\alpha}{p'}} \Big(\int_{\mathbb{R}^n} \langle\tilde{x}\rangle^{-n-2\delta-2\sigma p'}\, d\tilde{x}\Big)^{\frac{1}{p'}}. \tag{9.27}$$

Next carrying out again the change of variables $\tilde{t} := R^{-\alpha}t$ and $\tilde{x} := R^{-1}x$ and employing Lemma 9.2.4 we can proceed J_3 as follows:

$$\begin{aligned}|J_3| &\lesssim I_{R,t}^{\frac{1}{p}}\, R^{-2\delta-\alpha+\frac{n+\alpha}{p'}} \Big(\int_{\frac{1}{2}}^1 \int_{\mathbb{R}^n} \eta^{-\frac{p'}{p}}(\tilde{t})\big|\eta'(\tilde{t})\big|^{p'}\varphi^{-\frac{p'}{p}}(\tilde{x})\,\big|(-\Delta)^\delta(\varphi)(\tilde{x})\big|^{p'}\, d\tilde{x}d\tilde{t}\Big)^{\frac{1}{p'}}\\ &\lesssim I_{R,t}^{\frac{1}{p}}\, R^{-2\delta-\alpha+\frac{n+\alpha}{p'}} \Big(\int_{\mathbb{R}^n} \varphi^{-\frac{p'}{p}}(\tilde{x})\,\big|(-\Delta)^\delta(\varphi)(\tilde{x})\big|^{p'}\, d\tilde{x}\Big)^{\frac{1}{p'}}.\end{aligned}$$

Here we used $\partial_t\eta_R(t) = R^{-\alpha}\eta'(\tilde{t})$ and the assumption (9.24). To deal with the last integral, we apply Lemma 9.2.3 with $q = n + 2\delta$ and $\gamma = \delta$, that is, $m = 0$ and $s = \delta$ to get

$$|J_3| \lesssim I_{R,t}^{\frac{1}{p}} R^{-2\delta-\alpha+\frac{n+\alpha}{p'}} \Big(\int_{\mathbb{R}^n} \langle\tilde{x}\rangle^{-n-2\delta}\, d\tilde{x}\Big)^{\frac{1}{p'}}. \tag{9.28}$$

Because of the assumption (9.3), there exists a sufficiently large constant $R_0 > 0$ such that it holds

$$\int_{\mathbb{R}^n} u_1(x)\varphi_R(x)\, dx > 0 \tag{9.29}$$

for all $R > R_0$. Combining the estimates from (9.25) to (9.29) we may arrive at

$$\begin{aligned}0 < \int_{\mathbb{R}^n} u_1(x)\varphi_R(x)\, dx &\lesssim I_{R,t}^{\frac{1}{p}}\big(R^{-2\alpha+\frac{n+\alpha}{p'}} + R^{-\alpha-2\delta+\frac{n+\alpha}{p'}}\big) + I_R^{\frac{1}{p}}\, R^{-2\sigma+\frac{n+\alpha}{p'}} - I_R\\ &\lesssim I_R^{\frac{1}{p}}\, R^{-2\sigma+\frac{n+\alpha}{p'}} - I_R\end{aligned} \tag{9.30}$$

for all $R > R_0$. Moreover, applying the inequality

$$A\, y^\gamma - y \leqslant A^{\frac{1}{1-\gamma}} \quad \text{for any } A > 0,\ y \geqslant 0 \text{ and } 0 < \gamma < 1$$

leads to

$$0 < \int_{\mathbb{R}^n} u_1(x)\varphi_R(x)dx \lesssim R^{-2\sigma p'+n+\alpha} \tag{9.31}$$

for all $R > R_0$. It is clear that the assumption (9.4) is equivalent to $-2\sigma p' + n + \alpha \leqslant 0$. For this reason, in the subcritical case, that is, $-2\sigma p' + n + \alpha < 0$ letting $R \to \infty$ in (9.31) we obtain

$$\int_{\mathbb{R}^n} u_1(x)\, dx = 0.$$

This is a contradiction to the assumption (9.3).

Let us turn the critical case $p = 1 + \frac{2\sigma}{n-\mathrm{k}^-}$. It follows immediately $-2\sigma + \frac{n+\alpha}{p'} = 0$. Then, repeating some arguments as we did in the subcritical case we may conclude the following estimate:

$$0 < C_0 := \int_{\mathbb{R}^n} u_1(x)\varphi_R(x)\, dx \leqslant C_1 I_R^{\frac{1}{p}} - I_R, \quad \text{where } C_1 := \Big(\int_{\mathbb{R}^n} \langle\tilde{x}\rangle^{-n-2\delta}\, d\tilde{x}\Big)^{\frac{1}{p'}},$$

that is,

$$C_0 + I_R \leqslant C_1 I_R^{\frac{1}{p}}. \tag{9.32}$$

From (9.32) it is obvious that $I_R \leqslant C_1 I_R^{\frac{1}{p}}$ and $C_0 \leqslant C_1 I_R^{\frac{1}{p}}$. Hence, we obtain

$$I_R \leqslant C_1^{p'} \tag{9.33}$$

and

$$I_R \geqslant \frac{C_0^p}{C_1^p}, \tag{9.34}$$

respectively. By substituting (9.34) into the left-hand side of (9.32) and calculating straightforwardly, we get

$$I_R \geqslant \frac{C_0^{p^2}}{C_1^{p+p^2}}.$$

For any integer $j \geqslant 1$, an iteration argument leads to

$$I_R \geqslant \frac{C_0^{p^j}}{C_1^{p+p^2+\cdots+p^j}} = \frac{C_0^{p^j}}{C_1^{\frac{p^{j+1}-p}{p-1}}} = C_1^{\frac{p}{p-1}} \Big(\frac{C_0}{C_1^{\frac{p}{p-1}}} \Big)^{p^j}. \tag{9.35}$$

Now we choose the constant

$$\epsilon = \int_{\mathbb{R}^n} \langle \tilde{x} \rangle^{-n-2\delta} \, d\tilde{x}$$

in the assumption (9.3). Then, there exists a sufficiently large constant $R_1 > 0$ so that

$$\int_{\mathbb{R}^n} u_1(x) \varphi_R(x) \, dx > \epsilon$$

for all $R > R_1$. This is equivalent to

$$C_0 > C_1^{p'} = C_1^{\frac{p}{p-1}}, \quad \text{that is,} \quad \frac{C_0}{C_1^{\frac{p}{p-1}}} > 1.$$

Therefore, letting $j \to \infty$ in (9.35) we derive $I_R \to \infty$, which is a contradiction to (9.33). Summarizing, the proof is completed. □

Let us now consider the case of subcritical exponent to explain the estimate for lifespan T_ε of solutions in Remark 9.1.1. We assume that $u = u(t,x)$ is a local (in time) Sobolev solution to (9.1) in $[0,T) \times \mathbb{R}^n$. In order to prove the lifespan estimate, we replace the initial data $(0, u_1)$ by $(0, \varepsilon u_1)$ with a small constant $\varepsilon > 0$, where $u_1 \in L^1$ satisfies the assumption (9.3). Hence, there exists a sufficiently large constant $R_2 > 0$ so that we have

$$\int_{\mathbb{R}^n} u_1(x) \varphi_R(x) \, dx \geqslant c > 0$$

for any $R > R_2$. Repeating the steps in the above proofs we arrive at the following estimate:

$$\varepsilon \leqslant C \, R^{-2\sigma p' + n + \alpha} \leqslant C \, T^{-\frac{2\sigma p' - n - \alpha}{\alpha}}$$

with $R = T^{\frac{1}{\alpha}}$. Finally, letting $T \to T_\varepsilon^-$ we may conclude (9.5).

Remark 9.3.1. We want to underline that in the special case $\sigma = 1$ an $\delta = \frac{1}{2}$ the authors in [12] have investigated the critical exponent $p_{crit} = p_{crit}(n) = 1 + \frac{2}{n-1}$. If we plug $\sigma = 1$ and $\delta = \frac{1}{2}$ into the statements of Theorem 9.1.2, then the obtained results for the critical exponent p_{crit} coincide.

9.3.3. The case that the parameter σ is integer and the parameter δ is fractional from $(1, \sigma)$

Proof. We follow ideas from the proof of Section 9.3.2. At first, we denote $s_\delta := \delta - [\delta]$. Let us introduce test functions $\eta = \eta(t)$ as in Section 9.3.2 and $\varphi = \varphi(x) := \langle x \rangle^{-n-2s_\delta}$. We can repeat exactly the estimates for J_1 and J_2 as we did in the proof of Section 9.3.2 to conclude

$$|J_1| \lesssim I_{R,t}^{\frac{1}{p}} \, R^{-2\alpha + \frac{n+\alpha}{p'}}, \tag{9.36}$$

$$|J_2| \lesssim I_R^{\frac{1}{p}} \, R^{-2\sigma + \frac{n+\alpha}{p'}}. \tag{9.37}$$

Let us turn to estimate J_3, where δ is any fractional number in $(1,\sigma)$. In the first step, applying Lemma 9.2.7 and Hölder's inequality lead to

$$|J_3| \leqslant I_{R,t}^{\frac{1}{p}} \Big(\int_{\frac{R^\alpha}{2}}^{R^\alpha} \int_{\mathbb{R}^n} \eta_R^{-\frac{p'}{p}}(t) \big|\partial_t \eta_R(t)\big|^{p'} \varphi_R^{-\frac{p'}{p}}(x) \,\big|(-\Delta)^\delta \varphi_R(x)\big|^{p'} \,dxdt \Big)^{\frac{1}{p'}}.$$

Now we can re-write $\delta = m_\delta + s_\delta$, where $m_\delta := \lfloor \delta \rfloor \geqslant 1$ is integer and s_δ is a fractional number in $(0,1)$. Employing Lemma 9.2.2 we derive

$$(-\Delta)^\delta \varphi_R(x) = (-\Delta)^{s_\delta} \big((-\Delta)^{m_\delta} \varphi_R(x) \big).$$

By the change of variables $\tilde{x} := R^{-1}x$ we also notice that

$$(-\Delta)^{m_\delta} \varphi_R(x) = R^{-2m_\delta} (-\Delta)^{m_\delta} (\varphi)(\tilde{x})$$

since m_δ is an integer. Using the formula (9.18) we re-write

$$\begin{aligned}
(-\Delta)^{m_\delta} \varphi_R(x) = (-1)^{m_\delta} R^{-2m_\delta} \prod_{j=0}^{m_\delta - 1} (q+2j) \Big(& \prod_{j=1}^{m_\delta} (-n+q+2j) \langle \tilde{x} \rangle^{-q-2m_\delta} \\
& - C_{m_\delta}^1 \prod_{j=2}^{m_\delta} (-n+q+2j)(q+2m_\delta) \langle \tilde{x} \rangle^{-q-2m_\delta-2} \\
& + C_{m_\delta}^2 \prod_{j=3}^{m_\delta} (-n+q+2j)(q+2m_\delta)(q+2m_\delta+2) \langle \tilde{x} \rangle^{-q-2m_\delta-4} \\
& + \cdots + (-1)^{m_\delta} \prod_{j=0}^{m_\delta-1} (q+2m_\delta+2j) \langle \tilde{x} \rangle^{-q-4m_\delta} \Big),
\end{aligned}$$

where $q := n + 2s_\delta$. For simplicity, we introduce the following functions:

$$\varphi_k(x) := \langle x \rangle^{-q-2m_\delta-2k} \quad \text{and} \quad \varphi_{k,R}(x) := \varphi_k(R^{-1}x) = \langle \tilde{x} \rangle^{-q-2m_\delta-2k}$$

with $k = 0, \cdots, m_\delta$. As a result, by Lemma 9.2.4 we arrive at

$$\begin{aligned}
(-\Delta)^\delta \varphi_R(x) &= (-1)^{m_\delta} R^{-2m_\delta} \prod_{j=0}^{m_\delta-1} (q+2j) \Big(\prod_{j=1}^{m_\delta} (-n+q+2j)\, (-\Delta)^{s_\delta} (\varphi_{0,R})(x) \\
&\qquad - C_{m_\delta}^1 \prod_{j=2}^{m_\delta} (-n+q+2j)(q+2m_\delta)\, (-\Delta)^{s_\delta} (\varphi_{1,R})(x) \\
&\qquad + C_{m_\delta}^2 \prod_{j=3}^{m_\delta} (-n+q+2j)(q+2m_\delta)(q+2m_\delta+2)\, (-\Delta)^{s_\delta} (\varphi_{2,R})(x) \\
&\qquad + \cdots + (-1)^{m_\delta} \prod_{j=0}^{m_\delta-1} (q+2m_\delta+2j)\, (-\Delta)^{s_\delta} (\varphi_{m_\delta,R})(x) \Big) \\
&= (-1)^{m_\delta} R^{-2m_\delta-2s_\delta} \prod_{j=0}^{m_\delta-1} (q+2j) \Big(\prod_{j=1}^{m_\delta} (-n+q+2j)\, (-\Delta)^{s_\delta} (\varphi_0)(\tilde{x}) \\
&\qquad - C_{m_\delta}^1 \prod_{j=2}^{m_\delta} (-n+q+2j)(q+2m_\delta)\, (-\Delta)^{s_\delta} (\varphi_1)(\tilde{x}) \\
&\qquad + C_{m_\delta}^2 \prod_{j=3}^{m_\delta} (-n+q+2j)(q+2m_\delta)(q+2m_\delta+2)\, (-\Delta)^{s_\delta} (\varphi_2)(\tilde{x}) \\
&\qquad + \cdots + (-1)^{m_\delta} \prod_{j=0}^{m_\delta-1} (q+2m_\delta+2j)\, (-\Delta)^{s_\delta} (\varphi_{m_\delta})(\tilde{x}) \Big) \\
&= R^{-2\delta} (-\Delta)^\delta (\varphi)(\tilde{x}).
\end{aligned}$$

For this reason, performing the change of variables $\tilde{t} := R^{-\alpha}t$ we obtain

$$\begin{aligned} |J_3| &\lesssim I_{R,t}^{\frac{1}{p}} R^{-2\delta-\alpha+\frac{n+\alpha}{p'}} \Big(\int_{\frac{1}{2}}^{1} \int_{\mathbb{R}^n} \eta^{-\frac{p'}{p}}(\tilde{t}) \big|\eta'(\tilde{t})\big|^{p'} \varphi^{-\frac{p'}{p}}(\tilde{x}) \big|(-\Delta)^{\delta}(\varphi)(\tilde{x})\big|^{p'} d\tilde{x} d\tilde{t} \Big)^{\frac{1}{p'}} \\ &\lesssim I_{R,t}^{\frac{1}{p}} R^{-2\delta-\alpha+\frac{n+\alpha}{p'}} \Big(\int_{\mathbb{R}^n} \varphi^{-\frac{p'}{p}}(\tilde{x}) \big|(-\Delta)^{\delta}(\varphi)(\tilde{x})\big|^{p'} d\tilde{x} \Big)^{\frac{1}{p'}}. \end{aligned}$$

Here we used $\partial_t \eta_R(t) = R^{-\alpha}\eta'(\tilde{t})$ and the assumption (9.24). After applying Lemma 9.2.3 with $q = n + 2s_\delta$ and $\gamma = \delta$, i.e. $m = m_\delta$ and $s = s_\delta$, we may conclude

$$|J_3| \lesssim I_{R,t}^{\frac{1}{p}} R^{-2\delta-\alpha+\frac{n+\alpha}{p'}} \Big(\int_{\mathbb{R}^n} \langle \tilde{x} \rangle^{-n-2s_\delta} d\tilde{x} \Big)^{\frac{1}{p'}} \lesssim I_{R,t}^{\frac{1}{p}} R^{-2\delta-\alpha+\frac{n+\alpha}{p'}}. \tag{9.38}$$

Finally, combining (9.36) to (9.38) and repeating arguments as in Section 9.3.2 we may complete the proof of Theorem 9.1.2. □

9.3.4. The case that the parameter σ is fractional from $(1,\infty)$ and the parameter δ is integer

Proof. We follow ideas from the proofs of Sections 9.3.2 and 9.3.3. At first, we denote $s_\sigma := \sigma - \lfloor \sigma \rfloor$. Let us introduce test functions $\eta = \eta(t)$ as in Section 9.3.2 and $\varphi = \varphi(x) := \langle x \rangle^{-n-2s_\sigma}$. Then, repeating the proof of Sections 9.3.2 and 9.3.3 we may conclude what we wanted to prove. □

9.3.5. The case that the parameter σ is fractional from $(1,\infty)$ and the parameter δ is fractional from $(0,1)$

Proof. We follow ideas from the proofs of Sections 9.3.2 and 9.3.4. At first, we denote $s_\sigma := \sigma - \lfloor \sigma \rfloor$. Next, we put $s^* := \min\{s_\sigma, \delta\}$. It is obvious that s^* is fractional from $(0,1)$. Let us introduce test functions $\eta = \eta(t)$ as in Section 9.3.2 and $\varphi = \varphi(x) := \langle x \rangle^{-n-2s^*}$. Then, repeating the proof of Sections 9.3.2 and 9.3.4 we may conclude what we wanted to prove. □

9.3.6. The case that the parameter σ is fractional from $(1,\infty)$ and the parameter δ is fractional from $(1,\sigma)$

Proof. We follow ideas from the proofs of Sections 9.3.2 and 9.3.5. At first, we denote $s_\sigma := \sigma - \lfloor \sigma \rfloor$ and $s_\delta := \delta - \lfloor \delta \rfloor$. Next, we put $s^* := \min\{s_\sigma, s_\delta\}$. It is obvious that s^* is fractional from $(0,1)$. Let us introduce test functions $\eta = \eta(t)$ as in Section 9.3.2 and $\varphi = \varphi(x) := \langle x \rangle^{-n-2s^*}$. Then, repeating the proof of Sections 9.3.2 and 9.3.5 we may conclude what we wanted to prove. □

A. Notation- Guide to the reader

Symbols used throughout the thesis

$\lceil a \rceil = \text{ceiling}(a)$	smallest integer greater than or equal to $a \in \mathbb{R}$;
$\lfloor a \rfloor = \text{floor}(a)$	largest integer less than or equal to $a \in \mathbb{R}$;
$\{a\} = a - \lfloor a \rfloor$	fractional of $a \in \mathbb{R}$;
$[a]^+ = \max\{a, 0\}$	positive part of $a \in \mathbb{R}$;
$\operatorname{Re} z$	real part of $z \in \mathbb{C}$;
$\operatorname{Im} z$	imaginary part of $z \in \mathbb{C}$;
$\arg z$	argument of $z \in \mathbb{C}$;
$\|x\|$	Euclidean norm of $x \in \mathbb{R}^n$;
$\langle x \rangle = \sqrt{1 + \|x\|^2}$	Japanese bracket of $x \in \mathbb{R}^n$;
$\|\alpha\| = \alpha_1 + \cdots + \alpha_n$	length of the multi-index $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$;
$\\|u\\|_X$	the norm of a function $u \in X$;
$\mathfrak{F}_{x \to \xi}(u)$	Fourier transform of u;
$\mathfrak{F}^{-1}_{\xi \to x}(\hat{u})$	inverse Fourier transform of $\hat{u}$;
∇, ∇_x	spatial gradient;
Δ, Δ_x	Laplacian with respect to the spatial variables;
div	divergence with respect to the spatial variables;
$\|D\|^\sigma$, $\|D_x\|^\sigma$	pseudo-differential operator with symbol $\|\xi\|^\sigma$;
$\langle D \rangle^\sigma$, $\langle D_x \rangle^\sigma$	pseudo-differential operator with symbol $\langle \xi \rangle^\sigma$;
$f \lesssim g$	if there exists a positive constant C such that $f \leqslant Cg$;
$f \approx g$	if $f \lesssim g$ and $g \lesssim f$;
$f \simeq g$	if $f = Cg$ for some constant $C > 0$;
$f = o(g)$	if $\limsup_{x \to \infty} \frac{\|f(x)\|}{\|g(x)\|} = 0$;
$f = \mathcal{O}(g)$	if $\limsup_{x \to \infty} \frac{\|f(x)\|}{\|g(x)\|} < \infty$;
$\operatorname{supp} u$	support of the function u;
$f * g$	convolution between f and g;
$f *_{(x)} g$	convolution between f and g with respect to the spatial variables;
$\mathcal{J}_\mu(z)$	Bessel function of first kind of order μ;
$\tilde{\mathcal{J}}_\mu(z)$	modified Bessel function of first kind of order μ;
$p_{Fuj}(n)$	Fujita exponent;
$K_j(t, x)$	fundamental solutions to the σ-evolution equation with structural damping and visco-elastic damping;
$\widehat{K}_j(t, \xi)$	Fourier transform of fundamental solutions to the σ-evolution equation with structural damping and visco-elastic damping;
$\theta = \frac{\frac{1}{p_0} - \frac{1}{p} + \frac{s}{n}}{\frac{1}{p_0} - \frac{1}{p_1} + \frac{\sigma}{n}}$	exponent which appears in the fractional Gagliardo-Nirenberg inequality for $0 \leqslant s < \sigma$;
$B_1 \hookrightarrow B_2$	continuous embedding of B_1 in B_2.

Function spaces

We collect function spaces which are frequently used within this thesis.

$\mathcal{C}^k = \mathcal{C}^k(\mathbb{R}^n)$	spaces of k times continuously differentiable functions;
$\mathcal{C}_0^k = \mathcal{C}_0^k(\mathbb{R}^n)$	spaces of k times continuously differentiable functions with compact support;
$\mathcal{C}^\infty = \mathcal{C}^\infty(\mathbb{R}^n)$	spaces of infinitely continuously differentiable functions;
$\mathcal{C}_0^\infty = \mathcal{C}_0^\infty(\mathbb{R}^n)$	spaces of infinitely continuously differentiable functions with compact support;
$\mathcal{C}_b = \mathcal{C}_b(\mathbb{R}^n)$	spaces of bounded continuous functions;
$\mathcal{D}' = \mathcal{D}'(\mathbb{R}^n)$	spaces of distributions;
$\mathcal{E}' = \mathcal{E}'(\mathbb{R}^n)$	spaces of distributions with compact support;
$\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$	Schwartz spaces of rapidly decaying functions;
$\mathcal{S}' = \mathcal{S}'(\mathbb{R}^n)$	spaces of tempered distributions;
$\mathcal{Z} = \mathcal{Z}(\mathbb{R}^n)$	spaces of Schwartz functions with all moments vanishing;
$\mathcal{P}$	set of all polynomial functions in n variables;
$\mathcal{Z}' = \mathcal{Z}'(\mathbb{R}^n)$	topological dual of $\mathcal{Z}(\mathbb{R}^n)$ which can be canonically identified with the factor spaces $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}$;
$L^p = L^p(\mathbb{R}^n)$	Lebesgue spaces, $1 \leqslant p \leqslant \infty$;
$L^p_{\text{loc}} = L^p_{\text{loc}}(\mathbb{R}^n)$	spaces of locally p-summable functions, $1 \leqslant p < \infty$;
$L^{p,\infty} = L^{p,\infty}(\mathbb{R}^n)$	weak Lebesgue spaces, $0 < p \leqslant \infty$;
$W^{m,p} = W^{m,p}(\mathbb{R}^n)$	Sobolev spaces based on $L^p(\mathbb{R}^n)$, $1 \leqslant p \leqslant \infty$, $m \in \mathbb{N}$;
$H^s_p = H^s_p(\mathbb{R}^n) = \langle D \rangle^{-s} L^p(\mathbb{R}^n)$	Bessel potential spaces, $1 \leqslant p < \infty$, $s \in \mathbb{R}$;
$\dot{H}^s_p = \dot{H}^s_p(\mathbb{R}^n) = \lvert D \rvert^{-s} L^p(\mathbb{R}^n)$	homogeneous Bessel potential spaces, $1 \leqslant p < \infty$, $s \in \mathbb{R}$;
$H^s = H^s(\mathbb{R}^n) = H^s_2(\mathbb{R}^n)$	Sobolev spaces based on $L^2(\mathbb{R}^n)$, $s \in \mathbb{R}$;
$\dot{H}^s = \dot{H}^s(\mathbb{R}^n) = \dot{H}^s_2(\mathbb{R}^n)$	homogeneous Sobolev spaces based on $L^2(\mathbb{R}^n)$, $s \in \mathbb{R}$;
$F^s_{p,q} = F^s_{p,q}(\mathbb{R}^n)$	Triebel-Lizorkin spaces, $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leqslant \infty$;
$\dot{F}^s_{p,q} = \dot{F}^s_{p,q}(\mathbb{R}^n)$	homogeneous Triebel-Lizorkin spaces, $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leqslant \infty$;
$\dot{H}^{s,\alpha}_q = \dot{H}^{s,\alpha}_q(\mathbb{R}^n)$	weighted homogeneous Sobolev spaces of potential type, $s > 0$, $\alpha \in \mathbb{R}$, $1 < q < \infty$;
$L\big(L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)\big)$	spaces of linear continuous operators mapping $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$, $1 \leqslant p,\, q \leqslant \infty$;
$\Gamma^{a,s,\rho} = \Gamma^{a,s,\rho}(\mathbb{R}^n)$	Gevrey-Sobolev spaces, $0 < a,\, s < \infty$, $\rho \in \mathbb{R}$;
$\Gamma^{s,\rho} = \Gamma^{s,\rho}(\mathbb{R}^n)$	Gevrey-Sobolev spaces as the the inductive limit of all spaces $\Gamma^{a,s,\rho}$, i.e. $\Gamma^{s,\rho} := \bigcup_{a>0} \Gamma^{a,s,\rho}$, $0 < s < \infty$, $\rho \in \mathbb{R}$;
$\Gamma^{a,s} = \Gamma^{a,s}(\mathbb{R}^n)$	Gevrey spaces, $0 < a,\, s < \infty$;
$\Gamma^s = \Gamma^s(\mathbb{R}^n)$	Gevrey spaces as the the inductive limit of all spaces $\Gamma^{a,s}$, i.e. $\Gamma^s := \bigcup_{a>0} \Gamma^{a,s}$, $0 < s < \infty$.

B. Basic tools

B.1. Young's convolution inequality

Proposition B.1.1. *Let $f \in L^r$ and $g \in L^p$ be two given functions. Then, the following estimates hold for the convolution $u := f * g$:*

$$\|u\|_{L^q} \leqslant \|f\|_{L^r}\,\|g\|_{L^p} \quad \text{for all } 1 \leqslant p \leqslant q \leqslant \infty \text{ and } 1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{p}.$$

B.2. Riesz-Thorin interpolation theorem

Proposition B.2.1. *Let $1 \leqslant p_0, p_1, q_0, q_1 \leqslant \infty$. If T is a linear continuous operator from*

$$L\big(L^{p_0} \to L^{q_0}\big) \cap L\big(L^{p_1} \to L^{q_1}\big),$$

then T belongs to

$$L\big(L^{p_\theta} \to L^{q_\theta}\big) \quad \text{for each } \theta \in (0,1),$$

too, where

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Moreover, the following norm estimates are true:

$$\|T\|_{L(L^{p_\theta} \to L^{q_\theta})} \leqslant \|T\|^{1-\theta}_{L(L^{p_0} \to L^{q_0})}\,\|T\|^{\theta}_{L(L^{p_1} \to L^{q_1})}.$$

B.3. Modified Bessel functions

Let $\mathcal{J}_\mu = \mathcal{J}_\mu(s)$ be the Bessel function of order $\mu \in (-\infty, +\infty)$. Then, $\tilde{\mathcal{J}}_\mu(s) := \frac{\mathcal{J}_\mu(s)}{s^\mu}$ is called the modified Bessel function, where μ is a non-negative integer.

Proposition B.3.1. *Let $f \in L^p$, $p \in [1,2]$, be a radial function. Then, the Fourier transform $F(f)$ is also a radial function and it satisfies*

$$F_n(\xi) := \mathfrak{F}(f)(\xi) = c\int_0^\infty g(r) r^{n-1} \tilde{\mathcal{J}}_{\frac{n}{2}-1}(r|\xi|)dr, \text{ where } g(|x|) := f(x),$$

that is,

$$f(x) = \int_0^\infty F_n(r) r^{n-1} \tilde{\mathcal{J}}_{\frac{n}{2}-1}(r|x|)dr.$$

Proposition B.3.2. *Assume that μ is a non-negative integer. The following properties hold:*

1. $s\, d_s \tilde{\mathcal{J}}_\mu(s) = \tilde{\mathcal{J}}_{\mu-1}(s) - 2\mu \tilde{\mathcal{J}}_\mu(s)$,
2. $d_s \tilde{\mathcal{J}}_\mu(s) = -s \tilde{\mathcal{J}}_{\mu+1}(s)$,
3. $\tilde{\mathcal{J}}_{-\frac{1}{2}}(s) = \sqrt{\frac{2}{\pi}} \cos s$ *and* $\tilde{\mathcal{J}}_{\frac{1}{2}}(s) = \sqrt{\frac{2}{\pi}} \frac{\sin s}{s}$,
4. $|\tilde{\mathcal{J}}_\mu(s)| \leqslant Ce^{\pi|\mathrm{Im}\mu|}$ *if* $s \leqslant 1$, *and* $\tilde{\mathcal{J}}_\mu(s) = Cs^{-\frac{1}{2}} \cos\big(s - \frac{\mu}{2}\pi - \frac{\pi}{4}\big) + \mathcal{O}(|s|^{-\frac{3}{2}})$ *if* $|s| \geqslant 1$,
5. $\tilde{\mathcal{J}}_{\mu+1}(r|x|) = -\frac{1}{r|x|^2}\partial_r \tilde{\mathcal{J}}_\mu(r|x|)$, $r \neq 0$, $x \neq 0$.

B.4. Faà di Bruno's formula

Proposition B.4.1. *Let $h(g(x)) = (h \circ g)(x)$ with $x \in \mathbb{R}$. Then, we have*

$$\frac{d^n}{dx^n} h(g(x)) = \sum \frac{n!}{m_1!1!^{m_1} m_2!2!^{m_2} \cdots m_n!n!^{m_n}} h^{(m_1+m_2+\cdots+m_n)}(g(x)) \prod_{j=1}^{n} \left(g^{(j)}(x)\right)^{m_j},$$

where the sum is taken over all n- tuples of non-negative integers $(m_1, m_2, \cdots, m_n)$ satisfying the constraint of the following Diophantine equation:

$$1 \cdot m_1 + 2 \cdot m_2 + \cdots + n \cdot m_n = n.$$

B.5. A variant of Mikhlin-Hörmander multiplier theorem

Proposition B.5.1. *Let $q \in (1, \infty)$, $k = [\frac{n}{2}] + 1$ and $b \geqslant 0$. Suppose that $m \in C^k(\mathbb{R}^n \backslash \{0\})$ satisfying $m(\xi) = 0$ if $|\xi| \leqslant 1$ and*

$$\left|\partial_\xi^\alpha m(\xi)\right| \leqslant C|\xi|^{-nb\left|\frac{1}{q}-\frac{1}{2}\right|}\left(A|\xi|^{b-1}\right)^{|\alpha|},$$

for all $|\alpha| \leqslant k$, $|\xi| \geqslant 1$ and with some constant $A \geqslant 1$. Then, the operator $T_m = \mathfrak{F}^{-1}\big(m(t,\xi)\big) *_{(x)}$, defined by the action $T_m f(t,x) := \mathfrak{F}^{-1}_{\xi \to x}\Big(m(t,\xi)\mathfrak{F}_{y\to\xi}\big(f(y)\big)\Big)$, is continuously bounded from L^q into itself and satisfies the following estimate:

$$\|T_m f(t,\cdot)\|_{L^q} \leqslant C A^{n\left|\frac{1}{q}-\frac{1}{2}\right|}\|f\|_{L^q}.$$

The proof of this lemma can be found in [49] (Theorem 1) and [9] (Theorem 10).

B.6. Useful lemmas

Lemma B.6.1. *Let $\alpha, \beta \in \mathbb{R}$. Then, the following inequalities are satisfied:*

$$I(t) := \int_0^t (1+t-\tau)^{-\alpha}(1+\tau)^{-\beta} d\tau \lesssim \begin{cases} (1+t)^{-\min\{\alpha,\beta\}} & \text{if } \max\{\alpha,\beta\} > 1, \\ (1+t)^{-\min\{\alpha,\beta\}}\log(2+t) & \text{if } \max\{\alpha,\beta\} = 1, \\ (1+t)^{1-\alpha-\beta} & \text{if } \max\{\alpha,\beta\} < 1. \end{cases}$$

Proof. Let us divide the interval $[0,t]$ into $[0,t/2]$ and $[t/2,t]$. It holds

$$\frac{1}{2}(1+t) \leqslant 1+t-s \leqslant 1+t \quad \text{for any } s \in [0,t/2],$$
$$\frac{1}{2}(1+t) \leqslant 1+s \leqslant 1+t \quad \text{for any } s \in [t/2,t].$$

Hence, using the change of variables when needed we get

$$\begin{aligned} I(t) &\approx (1+t)^{-\alpha}\int_0^{t/2}(1+\tau)^{-\beta}d\tau + (1+t)^{-\beta}\int_{t/2}^{t}(1+t-\tau)^{-\alpha}d\tau \\ &= (1+t)^{-\alpha}\int_0^{t/2}(1+\tau)^{-\beta}d\tau + (1+t)^{-\beta}\int_0^{t/2}(1+\tau)^{-\alpha}d\tau \\ &\approx (1+t)^{-\min\{\alpha,\beta\}}\int_0^{t/2}(1+\tau)^{-\max\{\alpha,\beta\}}d\tau. \end{aligned}$$

Therefore, the proof of Lemma B.6.1 is completed.

Lemma B.6.2. *The following formula of derivative of composed function holds for any multi-index α:*

$$\partial_\xi^\alpha h\big(f(\xi)\big) = \sum_{k=1}^{|\alpha|} h^{(k)}\big(f(\xi)\big)\Big(\sum_{\substack{\gamma_1+\cdots+\gamma_k \leqslant \alpha \\ |\gamma_1|+\cdots+|\gamma_k|=|\alpha|,\, |\gamma_i|\geqslant 1}} \big(\partial_\xi^{\gamma_1} f(\xi)\big)\cdots\big(\partial_\xi^{\gamma_k} f(\xi)\big)\Big),$$

where $h = h(s)$ and $h^{(k)}(s) = \frac{d^k h(s)}{ds^k}$.

The result can be found in [63] at the page 202.

C. Some inequalities in fractional Sobolev spaces

In the Appendix we list some results of Harmonic Analysis which are important tools for proving results on the global (in time) existence of small data Sobolev solutions to semi-linear damped σ-evolution models with power non-linearities. In particular, these tools concern the fractional calculus which allows to estimate power non-linearities in Sobolev spaces of fractional order (see [61]).

First of all, we recall the Bessel and Riesz potential spaces. Let $s \in \mathbb{R}$ and $1 < p < \infty$. Then,

$$H^s_p(\mathbb{R}^n) = \big\{ u \in \mathcal{S}'(\mathbb{R}^n) : \|\langle D\rangle^s u\|_{L^p(\mathbb{R}^n)} = \|u\|_{H^s_p(\mathbb{R}^n)} < \infty \big\},$$
$$\dot{H}^s_p(\mathbb{R}^n) = \big\{ u \in \mathcal{Z}'(\mathbb{R}^n) : \||D|^s u\|_{L^p(\mathbb{R}^n)} = \|u\|_{\dot{H}^s_p(\mathbb{R}^n)} < \infty \big\}$$

are called Bessel and Riesz potential spaces, respectively.

C.1. Fractional Gagliardo-Nirenberg inequality

The first inequality that we present is a generalization of the classical Gagliardo-Nirenberg inequality to the case of Sobolev spaces of fractional order. Hence, we will refer to the following result as *the fractional Gagliardo-Nirenberg inequality.*

Proposition C.1.1. *Let $1 < p, p_0, p_1 < \infty$, $\sigma > 0$ and $s \in [0, \sigma)$. Then, it holds the following fractional Gagliardo-Nirenberg inequality for all $u \in L^{p_0} \cap \dot{H}^\sigma_{p_1}$:*

$$\|u\|_{\dot{H}^s_p} \lesssim \|u\|^{1-\theta}_{L^{p_0}} \|u\|^{\theta}_{\dot{H}^\sigma_{p_1}},$$

where $\theta = \theta_{s,\sigma}(p, p_0, p_1) = \frac{\frac{1}{p_0} - \frac{1}{p} + \frac{s}{n}}{\frac{1}{p_0} - \frac{1}{p_1} + \frac{\sigma}{n}}$ and $\frac{s}{\sigma} \leqslant \theta \leqslant 1$.

For the proof one can see [35].

Corollary C.1.1. *Let $1 < p, m < \infty$, $\sigma > 0$ and $s \in [0, \sigma)$. Then, we have the following inequality for all $u \in H^\sigma$:*

$$\big\||D|^s u\big\|_{L^p} \lesssim \|u\|^{1-\theta}_{L^m} \big\||D|^\sigma u\big\|^{\theta}_{L^m},$$

where $\theta = \theta_{s,\sigma}(p, m) = \frac{n}{\sigma}\big(\frac{1}{m} - \frac{1}{p} + \frac{s}{n}\big)$ and $\frac{s}{\sigma} \leqslant \theta_{s,\sigma}(p, m) \leqslant 1$.

Corollary C.1.2. *Let $q > 1$, $s_1, s_2 \geqslant 0$ and $\theta \in (0, 1)$. We assume $s \in [s_1, s_2]$ satisfying $s = (1-\theta)s_1 + \theta s_2$. Then, the following inequalities hold:*

$$\|u\|_{\dot{H}^s_q} \lesssim \|u\|^{1-\theta}_{\dot{H}^{s_1}_q} \|u\|^{\theta}_{\dot{H}^{s_2}_q}, \tag{C.1}$$

for any $u \in \dot{H}^{s_1}_q \cap \dot{H}^{s_2}_q$, and

$$\|u\|_{H^s_q} \lesssim \|u\|^{1-\theta}_{H^{s_1}_q} \|u\|^{\theta}_{H^{s_2}_q}, \tag{C.2}$$

for any $u \in H^{s_1}_q \cap H^{s_2}_q$.

Proof. From the statement of Proposition C.1.1, we obtain

$$\|v\|_{\dot{H}^a_q} \lesssim \|v\|^{1-\theta}_{L^q} \|v\|^{\theta}_{\dot{H}^\sigma_q},$$

where $\theta = \frac{a}{\sigma}$. Then, in order to prove (C.1), we will replace $a = s - s_1$, v by $|D|^{s_1} u$ and $\sigma = s_2 - s_1$. Consequently, from (C.1) we may conclude (C.2) by using the relation $\|u\|_{H^a_q} = \|u\|_{\dot{H}^a_q} + \|u\|_{L^q}$ with $a = s$, s_1 and s_2. $\square$

C.2. Fractional Leibniz rule

Proposition C.2.1. *Let us assume $s > 0$, $1 \leqslant r \leqslant \infty$ and $1 < p_1, p_2, q_1, q_2 < \infty$ satisfying the relation*

$$\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2}.$$

Then, the following fractional Leibniz rules hold:

$$\big\||D|^s(u\,v)\big\|_{L^r} \lesssim \big\||D|^s u\big\|_{L^{p_1}} \|v\|_{L^{p_2}} + \|u\|_{L^{q_1}} \big\||D|^s v\big\|_{L^{q_2}}$$

for any $u \in \dot{H}^s_{p_1} \cap L^{q_1}$ and $v \in \dot{H}^s_{q_2} \cap L^{p_2}$,

$$\big\|\langle D\rangle^s(u\,v)\big\|_{L^r} \lesssim \big\|\langle D\rangle^s u\big\|_{L^{p_1}} \|v\|_{L^{p_2}} + \|u\|_{L^{q_1}} \big\|\langle D\rangle^s v\big\|_{L^{q_2}}$$

for any $u \in H^s_{p_1} \cap L^{q_1}$ and $v \in H^s_{q_2} \cap L^{p_2}$.

These results can be found in [32].

C.3. Fractional chain rule

Proposition C.3.1. *Let us choose $s \in (0,1)$, $1 < r, r_1, r_2 < \infty$ and a C^1 function F satisfying for any $\tau \in [0,1]$ and $u, v \in \mathbb{R}$ the inequality*

$$|F'(\tau u + (1-\tau)v)| \leqslant \mu(\tau)\big(G(u) + G(v)\big),$$

for some continuous and non-negative function G and some non-negative function $\mu \in L^1([0,1])$. Under these assumptions, the following estimate is true:

$$\|F(u)\|_{\dot{H}^s_r} \lesssim \|G(u)\|_{L^{r_1}} \|u\|_{\dot{H}^s_{r_2}}$$

for any $u \in \dot{H}^s_{r_2}$ such that $G(u) \in L^{r_1}$, provided that

$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}.$$

For the proof of this result one can see [6] or the proof in a slightly modified version in [61].

In particular we may apply Proposition C.3.1 for $F(u) = |u|^p$ or $F(u) = \pm u|u|^{p-1}$. After choosing $G(u) = |F'(u)|$ and μ as a positive constant, the next result follows immediately.

Corollary C.3.1. *Let $F(u) = |u|^p$ or $F(u) = \pm u|u|^{p-1}$ for $p > 1$, $s \in (0,1)$ and $r, r_1, r_2 \in (1,\infty)$. Then, it holds*

$$\|F(u)\|_{\dot{H}^s_r} \lesssim \|u\|^{p-1}_{L^{r_1}} \|u\|_{\dot{H}^s_{r_2}}$$

for any $u \in L^{r_1} \cap H^s_{r_2}$, provided that

$$\frac{1}{r} = \frac{p-1}{r_1} + \frac{1}{r_2}.$$

The following result shows that there is no necessity to assume $s \in (0,1)$ in the last corollary.

Proposition C.3.2. *Let us choose $s > 0$, $p > \lceil s \rceil$ and $1 < r, r_1, r_2 < \infty$ satisfying*

$$\frac{1}{r} = \frac{p-1}{r_1} + \frac{1}{r_2}.$$

Let us denote by $F(u)$ one of the functions $|u|^p$, $\pm|u|^{p-1}u$. Then, it holds the following fractional chain rule:

$$\big\||D|^s F(u)\big\|_{L^r} \lesssim \|u\|^{p-1}_{L^{r_1}} \big\||D|^s u\big\|_{L^{r_2}}$$

for any $u \in L^{r_1} \cap \dot{H}^s_{r_2}$.

The proof can be found in [61].

C.4. Fractional powers

We apply a result from [62] for the fractional powers rule.

Proposition C.4.1. *Let $p > 1$, $1 < r < \infty$ and $u \in H^s_r$, where $s \in \big(\frac{n}{r}, p\big)$. Let us denote by $F(u)$ one of the functions $|u|^p$, $\pm|u|^{p-1}u$. Then, the following estimate holds:*

$$\|F(u)\|_{H^s_r} \leqslant C\|u\|_{H^s_r}\,\|u\|^{p-1}_{L^\infty}.$$

In particular, if $s \in \mathbb{N}$, one may weaken the condition on p to $p > s - \frac{1}{r}$.

We shall use the following corollary from Proposition C.4.1.

Corollary C.4.1. *Under the assumptions of Proposition C.4.1, it holds*

$$\|F(u)\|_{\dot{H}^s_r} \leqslant C\|u\|_{\dot{H}^s_r}\,\|u\|^{p-1}_{L^\infty}.$$

Proof. Let us prove it for $F(u) = |u|^p$. We write the estimate from Proposition C.4.1 in the form

$$\big\||u|^p\big\|_{\dot{H}^s_r} + \big\||u|^p\big\|_{L^r} \leqslant C\big(\|u\|_{\dot{H}^s_r} + \|u\|_{L^r}\big)\|u\|^{p-1}_{L^\infty}.$$

Using instead of u the dilation $u_\lambda(\cdot) := u(\lambda\cdot)$ in the last inequality we obtain the desired inequality after taking into consideration

$$\|u_\lambda\|_{\dot{H}^s_r} = \lambda^{s-\frac{n}{r}}\|u\|_{\dot{H}^s_r} \text{ and } \|u_\lambda\|_{L^r} = \lambda^{-\frac{n}{r}}\|u\|_{L^r}$$

and letting $\lambda \to \infty$. □

Proposition C.4.2. *Let $r \in (1, \infty)$ and $\sigma > 0$. Then, the following inequality holds:*

$$\|u\,v\|_{H^\sigma_r} \lesssim \|u\|_{H^\sigma_r}\|v\|_{L^\infty} + \|u\|_{L^\infty}\|v\|_{H^\sigma_r}$$

for any $u, v \in H^\sigma_r \cap L^\infty$.

Corollary C.4.2. *Let $r \in (1, \infty)$ and $\sigma > 0$. Then, the following inequality holds:*

$$\|u\,v\|_{\dot{H}^\sigma_r} \lesssim \|u\|_{\dot{H}^\sigma_r}\,\|v\|_{L^\infty} + \|u\|_{L^\infty}\,\|v\|_{\dot{H}^\sigma_r}$$

for any $u, v \in \dot{H}^\sigma_r \cap L^\infty$.

C.5. A fractional Sobolev embedding

Proposition C.5.1. *Let $n \geqslant 1$, $0 < s < n$, $1 < q \leqslant r < \infty$, $\alpha < \frac{n}{q'}$ where q' denotes conjugate number of q, and $\gamma > -\frac{n}{r}$, $\alpha \geqslant \gamma$ satisfying $\frac{1}{r} = \frac{1}{q} + \frac{\alpha-\gamma-s}{n}$. Then, it holds:*

$$\big\||x|^\gamma|D|^{-s}u\big\|_{L^r} \lesssim \big\||x|^\alpha u\big\|_{L^q},\ \textit{that is },\ \big\||x|^\gamma u\big\|_{L^r} \lesssim \big\||x|^\alpha\,|D|^s u\big\|_{L^q}$$

for any $u \in \dot{H}^{s,\alpha}_q$, where $\dot{H}^{s,\alpha}_q = \{u\,:\,|D|^s u \in L^q(\mathbb{R}^n,\,|x|^{\alpha q})\}$ is the weighted homogeneous Sobolev space of potential type with the norm $\|u\|_{\dot{H}^{s,\alpha}_q} = \big\||x|^\alpha\,|D|^s u\big\|_{L^q}$.

The proof can be found in [71].

Corollary C.5.1. *Let $1 < q < \infty$ and $0 < s_1 < \frac{n}{q} < s_2$. Then, for any function $u \in \dot{H}^{s_1}_q \cap \dot{H}^{s_2}_q$ we have*

$$\|u\|_{L^\infty} \lesssim \|u\|_{\dot{H}^{s_1}_q} + \|u\|_{\dot{H}^{s_2}_q}.$$

Proof. By choosing $\alpha = \gamma = 0$ and $s = s_1$ in Proposition C.5.1 we get

$$\|u\|_{L^r} \lesssim \big\||D|^{s_1}u\big\|_{L^q}, \text{ where } \frac{1}{r} = \frac{1}{q} - \frac{s_1}{n}.$$

Since $s_2 - s_1 > \frac{n}{r}$, we may conclude

$$\|u\|_{L^\infty} \lesssim \|u\|_{H^{s_2-s_1}_r} \lesssim \|u\|_{L^r} + \big\||D|^{s_2-s_1}u\big\|_{L^r} \lesssim \big\||D|^{s_1}u\big\|_{L^q} + \big\||D|^{s_2}u\big\|_{L^q}.$$

Hence, the proof of Corollary C.5.1 is completed. □

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