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Color Computer Graphics as Applied to Introductory Calculus Instruction

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COLOR COMPUTER GRAPHICS AS APPLIED TO INTRODUCTORY CALCULUS INSTRUCTION

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ABSTRACT

The use of computer graphics to support and enhance the presentation of introductory calculus concepts is described.

Computer graphics provides more accurate graph sketching, consistent presentations and the ability to develop mathematical models incrementally. The addition of extensive use of color aids even more, adding contrast, color keying, dimensionality, and interest to an illustration.

Ten lessons have been designed, developed, and evaluated. They employ a set of subroutines which interface to the NEC APC microcomputer graphics software. These lessons as developed and evaluated may be used interactively in the classroom or by individuals, or noninteractively in the classroom by the use of photographic slides.

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I. INTRODUCTION

The need for pictures to aid understanding in mathematics has been apparent ever since man began applying functions and mathematical theorems to the world around him. To quote Descartes, "It is very helpful to represent these things in this fashion since nothing enters the mind more readily than geometric figures."(1) Figures are particularly helpful to students when each stage of a problem or solution can be presented separately. This allows the "mental situation of the problem solver" to evolve as the investigated geometric figure evolves before his eyes. "At each stage, the problem solver has a mental picture of the geometric figure he explores, but this picture changes in transition to the next stage; some details may recede into the background, other details come to our attention, new details are added."(2)

The teaching of a first class in calculus is a prime example of the need for illustrations. Much more than numbers and symbols are being manipulated; concepts with meaningful graphical representations are being presented.

Computer graphics has developed rapidly in the past few years, and is an increasingly available technology. A new generation of microcomputers has arrived; they combine capabilities for both graphics and text. The availability of these systems promises to fuel the demand for graphics capabilities, which in turn will fuel this availability.(3)

Thus the marriage of these two areas, computer graphics and a first course in calculus, seems only natural. On a good day, when the board has been recently cleaned and long pieces of chalk are still available, the average professor can create chalkboard drawings of the basic curves and/or concepts easily enough. But on any day, given sufficient power, computer-generated color pictures of these same concepts and more complex illustrations will always be accurate, quickly drawn, correctly labelled, attention-getters.

In addition, the use of color can add to the depth of comprehension by students. Color may be used to aid students in mentally sorting out the various pieces of a more complex illustration. When several labels and lines or curves appear together, the same key color may be used for all labels associated with the line or curve of the same color. "Color systems were state of the art a few years ago but are now numerous and affordable."(4)

However, very few projects have previously been undertaken to utilize the advantages of color computer graphics for the instruction of introductory calculus. A review of literature and communication with the chair of the SIGGRAPH Education Committee (5) found none that were readily available for interactive use and implemented on a portable higher resolution system. The graphical illustrations developed throughout this paper and discussed in detail in Chapter V provide this service.

If a large screen projector is available for the computer system used, pictures may be generated and developed interactively on the screen as the instructor gives an accompanying verbal explanation. Or, these graphs may be created beforehand with photographic slides made and presented to the class by the instructor.

The system can also be made available to students out of the classroom, so that they may review, at their own pace, the lessons presented in class. For this purpose, more text can be added to accompany the diagrams and guide the student through them.

II. REVIEW OF LITERATURE

A. COMPUTERS AND EDUCATION

Numerous applications of the computer and its abilities have been developed since its inception in the late 1930's. However, the early pioneers in this field were devoted almost exclusively to its use in research. Very few undergraduates were ever allowed to experience any form of contact with the new mechanism. Meanwhile, some groundwork was being made to broaden computer usage in education at several isolated institutions, notably Purdue, Carnegie-Mellon (then Carnegie Tech), and Stanford. "The National Science Foundation during this period also made significant contributions through its program of undergraduate instructional equipment."(6) And, in 1967, the Pierce Panel of the President's Science Advisory Committee published its Computers in Higher Education report which helped to serve "as a guide and focal point for persons working on the subject of computers in education." This report's recommendations concentrated mainly on the provision of computing service to the undergraduate and graduate community.

With the provision of these computing services on college and university campuses, old and new users alike began to discover more innovative applications. In 1971, at the Second Conference on Computers in the Undergraduate Curricula, Alfred Bork presented his six basic theses on computers in a learning environment, the second of which states:

> "We are only beginning the task of learning how to use computers in education.
>
> I worry greatly about teachers who feel that they _already_ know all the answers. We have a long way to go, and theoretical analysis will not tell us how to employ computers effectively. Hence we want to maintain flexibility, and we should be prepared for long years of trial and error while using computers in learning."(7)

Mirroring the dynamism of the high tech era, education has continued to change and to explore the possibilities of the computer. The three major uses of computers in the instructional process have continued to be: 1) as an aid in the delivery system for instruction, 2) programmed by the student himself for purposes of learning from the implementation of an algorithm, or 3) studied by students for the purpose of a job in a computer-related field.(8) It is the first of these three, as an aid in the delivery system, that has developed into what is commonly referred to as computer-aided instruction or CAI.

B. _CALCULUS AND THE COMPUTER_

1. _Without Graphics_. The computer, without the aid of graphics, has been used in several ways to teach calculus. However, the ideas behind these methods may be improved upon by the use of graphics as is done in later sections of this paper. The major uses of computers without graphics in teaching calculus have been either in a laboratory course as the main tool, or for interactive student lessons and/or learning games. The first of these primarily takes advantage of the computational abilities of the computer.

One such technique requires some student knowledge of coding and running programs, but "is an introduction neither to numerical analysis nor to computer science."(9) Only enough knowledge of programming techniques is needed in order for the students to code algorithms which are given them, and they do not have enough time available to become "proficient at constructing complicated algorithms, although they will be able to work through a few moderately difficult ones."

This approach was used with the CRICISAM experimental text at Northern Illinois University. One example of a lesson here is integration of a function on a given interval. The students evaluated a few integrals by explicitly calculating Riemann sums and taking limits. This was done using non-monotonic continuous functions with the fineness of the partitions serving as the stopping signal in a calculation. It is interesting to note that a geometric figure was used to explain this concept in the paper describing it, but the students only dealt with numbers and calculations, and still may not be able to relate to this concept geometrically.

At Gettysburg College, in Gettysburg, Pennsylvania, students worked with the epsilon-delta definition of a limit.(10) They programmed an algorithm to find all values of x "close" to some x_i value such that the function at x_i is within a small epsilon value of the function at x. This was done with fairly routine functions so that the student could "concentrate on the algorithm and not be distracted by an unnecessarily complicated function statement." Again, the student dealt only with numbers and calculations, not a pictorial approach.

This same epsilon-delta definition of a limit was taught in a different manner at Carleton College in Northfield, Minnesota. (11) Instead of the student doing actual programming, programs for interactive student use were developed. Two games, "Me Epsilon You Delta" and "You Epsilon Me Delta" were used in which the student plays against the computer to learn the definition of $\lim_{x \to a} f(x) = L$. These games pit student against computer in alternately choosing values of D and E for specified a, L, and function f(x) such that $|x-a| < D$ and $|f(x)-L| < E$. One could see that a similar approach using graphics could convey the same idea, but leave a much bigger impact. This game has some of the characteristics (interactivity, flexibility) of the more recent forms of CAI.

An example of modern CAI for calculus is the following. At the University of Puerto Rico, a complete course of CAI modules for first semester calculus was developed.(12) This was done with the basic philosophy and objective of overcoming "certain experimental and analytical deficiencies in the students' *preparation* (as well as to offer a challenge to the more highly motivated students)." The Math Department also desired "to overcome, to whatever extent possible in such a limited program, the tendency toward rote learning and memorization as the basis of learning." At first they had hoped to be able to utilize previous CAI programs by merely translating them into Spanish.

> "At the start of the project we wrote to numerous publishers and universities in order to obtain information on texts and already existing CAI for calculus courses. Although we then found some interesting supplementary texts, it soon became apparent that very little had been done for calculus like the interactive programs that we had envisioned. Algebra, statistics, and linear algebra seemed to be the areas that had attracted the most effort for a variety of reasons running from the size of the student populace to the appropriateness of the material for programming."

The result was their development of eight modules intended as strictly supplementary to, and not replacing, the textbook or the lecture. Each module presupposes that the student already has some familiarity with the topic presented. The general form of a module is an introduction followed by questions intended to determine the readiness of the student for this lesson; if not ready to benefit from this module, he may be asked to review before continuing. If ready, he then participates in working through a program-selected example. Following this, "the student is given the opportunity to make up and enter his own examples in the same format as in 3 [the program-selected example]. The computer provides the data and necessary prompting. The option to terminate is offered before each example."

These eight modules consist of the following: 1) Intro - Basic how-to's of system use. 2) Number - Needed for successful use and interpretation of remaining modules: covers round-off and reading of mathematical tables.

3) Limit - Uses a table with rational functions and values making their denominator values approach zero. 4) Secant - Illustrates the slope of secant line approaching slope of a corresponding tangent. 5) Graf - Calculates values of a function and its derivatives to aid in graph plotting. 6) Curve - Evaluates a function, finds intercepts, critical points, and points of inflection. 7) Newton - Finds roots of a given polynomial. 8) Area - Calculates the approximate integral by various methods.

While most of these could be enhanced by the use of graphics, this is particularly true of the fifth and sixth modules. This was also a comment by the developers:

> "It is our continuing disappointment that these modules contain very few graphs. Despite the fact that we had proposed to buy some graphics terminals and even a plotter, we were never able to do so. Several programs were developed to produce graphs with alphabetic characters but were never actually incorporated into the final modules. It is still not clear whether a graphing routine can make a straight line look like a straight line that will be very convincing for students."

The lack of special symbols, such as an integral sign and lower case alphanumerics, was also listed as a handicap. Nonetheless, it was felt that the project was reasonably successful. Though difficulties were encountered in collecting measurements for formal evaluation, it was found that all students who used five or more modules earned at least a grade of C in the course, while of these students, only one had obtained above 700 on the SAT math achievement.

Another attempt at CAI for calculus implemented on the IBM PC was evaluated by this author. Examination showed that this set of drill and practice lessons consisted primarily of skeleton problems which used randomly generated values as parameters. These values were then substituted into the problems at the time of program invocation. Except for the change of numerical values in the problems, this lesson was similar to questions and problems found at the end of the sections of an ordinary calculus textbook. No graphics and very little interactivity were used.

2. With Graphics. Computer graphics is the creation, storage, and manipulation of models of objects and their pictures via computer.(13) The principal leaders in this field have not historically focused much attention on the use of graphics for educational purposes. As recently as 1982, the ACM/SIGGRAPH Bibliography listed only 6 references for CAI, and this was under the main heading of "Miscellaneous." This seeming indifference was not due to the fact that graphics had no place in education; indeed, the opposite is true. Computer graphics, particularly interactive computer graphics, has the potential to be a fine illustrative and motivational tool. At the NICOGRAPH '82 conference held in Japan, this was one point made at the symposium on the future impact of computer graphics. Alfred Bork stated with reference to education, that "Perhaps it is a little unfortunate that we separate all computer graphics as if it is a separate problem. The problem is communication, and we need both words and pictures."(14) But unfortunately, in the past as well as today, the budgets

of most educational institutions are under strain enough without the additional high cost of graphics systems. Today though, we find that "personal computers have put interactive graphics within the reach of practically any interested educator."(15) As availability increases, so will the quality of software for graphics and for calculus.

At present however, while numerous lessons have been developed for CAI in other areas, there are relatively few to be found for introductory calculus. Of these, the use of graphics is limited.

One such set of calculus lessons available is contained as a part of the Plato system, developed at the University of Illinois, and supported by funds from the National Science Foundation. This system currently has available to students approximately 5 lessons in various stages of completion; they employ high resolution graphics on a monochrome system. Topics include slope of a tangent line to a curve, limits, Newton's method, and minimum and maximum theories.(16) These lessons are interactive and intended primarily for individual student use. They are available through the University of Illinois Plato network and require a large expensive computing system supporting CDC display terminals for use.

A set of four interactive lessons on limits was developed at the College of the Virgin Islands.(17) These are used on an Apple II Plus, a portable, yet low resolution system. They "attempt to treat the single topic of limits in some detail and from several perspectives." The four lessons include: 1) *Finding Limits* - Studies the behavior of the graph of a function near its limit by repeatedly magnifying its graph near the limit. 2) Epsilons & Deltas - Uses the graph of a function and a neighborhood of the limit for user to graphically determine what satisfies the definition of a limit. 3) *Limit Problems* - Student selects type and difficulty level for randomly generated user practice in limits. 4) Graphic Problems - Asks questions about one- and two-sided limits and continuity at points and on intervals for a given graph.

A similar system was presented at the 6th Annual National Educational Computing Conference in June of 1984. At the University of Southern California, calculus on the computer was offered for the first time in the fall of 1983. This course utilized a multi-user microcomputer system with some graphical capabilities. This is an interactive system designed for individual use.(18)

The three systems mentioned above are all geared to individual use; they are all also rather limited in the amount of material developed. This type of CAI, complete with text, and leading the student through an entire lesson which uses graphics, can be very effective, but also requires an enormous fee in man-hours to create. "It may take as much as 200 hours of an expert's time, with some additional programmer time, to program one hour of effective tested-out computer-aided instruction."(19) These systems also require the scheduling of individual students on available graphics terminals. Other possible uses should still be considered, such as the collective presentation of graphical calculus lessons.

> "The role of the classroom teacher frequently gets lost in the excitement over computers. But in addition to its great value as a learning tool in the hands of students, the computer also holds considerable promise as an aid for the teacher in illustrating lectures and especially in supporting interactive classroom discussions. Nowhere else is this better illustrated than in mathematics..."(15)

At the Conservatoire National des Arts et Metiers of Paris, France, drawings are produced to be used in such a manner. An Apple II microcomputer is used, and lessons are presented on ordinary television sets in black and white. This has been found to increase student interest, and help to "improve their understanding of the notion explained because concepts, which were very formal to them before, are, in a way, put into a concrete form..." The teacher also benefits not only from knowing that his students can understand more easily, but finds a source of pedagogical inspiration for new illustrations, exercises, and remarks.(20) A sample of their work consists of ten programs related to the conics (drawings of ellipses, eccentricity, plane sections of cones, etc.). One such program draws conics according to their eccentricity, which efficiently and pleasantly presents students with this definition.

Researchers at Pennsylvania State University use a library of classroom demonstration programs on an Apple II

Plus to sketch graphs and interactively illustrate concepts. This allows teachers to overcome the lack of time to carefully sketch graphs or perform "laborious calculations to illustrate such concepts as the delta-epsilon definition of limits."(21) Their only completed section relevant to a first course in calculus is a derivative demonstration. This illustrates the relationship between the graph of a function and its derivative by plotting each on separate halves of the screen; vertical lines are drawn to connect the two at critical points.

Similar work has been done at Duke University.(22) Classroom demonstrations on an Apple II with a 19" color monitor used the graphics capabilities to implement an "electronic blackboard." Using programs already available, prior to class, the instructor can create a graphic he would like to show his students, but cannot draw very well. This is then saved on a diskette, the computer wheeled on a cart to the classroom, and the picture recalled during class.

The preceding projects have made use of graphics to aid instruction of calculus by use of available low-resolution graphics systems. More impact and better "student eye training" can be made with even higher resolution and higher quality illustrations.

The Graphics Education Research Group of the Colorado Computing Center developed a series of high precision images to be used in an introductory calculus course.(23) These computer images were developed and displayed in a batch environment using a compositional modeling system. From these images, classroom visual aids in the form of large prints, slides, and transparencies were produced for commercial distribution. This group was formed "to investigate and develop economically viable computer visualization techniques for use in the educational process." They felt that a first course in calculus would be an ideal topic since textbook visuals are often vague or imprecise and few visual aids have been produced in this area. A real-time interactive graphics medium was ruled out due to precision requirements, the developmental time span, and the general imbalance of interactive graphics devices among educational institutions. Thus, since all such institutions have portable audio-visual equipment available, the group concentrated on non-interactive display techniques for high quality images.

This review of literature did not disclose any color graphics systems of higher resolution that were currently available for interactive use. Clearly, the combination of color graphics, better resolution, and interactivity as designed and developed in this project can be used to produce an effective didactic tool for a first course in calculus.

B. SCREEN DESIGN

Whether interactive or non-interactive, intended for classroom or individual use, graphical and textual output must be structured on the screen. The display must be well designed in order for the visual presentation to be effective. Following are guidelines for good ergonomics and good screen design.

1) Planning - Before anything else, the designer must realize what he desires to create, as well as what is within his power to create. What is the size of the screen? How many characters are available? Can the product be developed on a machine other than the one it is to be used on? If so, the designer may be able to take advantage of some special features, and should also be aware of the differences.(24)

Screens should be designed in a manner analogous to that used for textbook illustrations, but keeping in mind the special features of the new medium. If possible, it is even recommended that the original author of material should not be the screen designer. Rather, the author should lay out the raw ideas which are then converted by a competent graphic designer, the person who might do layout for a magazine ad.(24)

2) Consistency - A certain degree of consistency should be used within a single layout, and throughout a set of related illustrations. Too many changes of type style or line style can be distracting, and draw the viewer's attention away from the content of the presentation. To maintain this consistency, it is helpful to use a style sheet similar to that used by textbook producers for specifying typographic parameters.(25) This style sheet

consists of a matrix to be filled in by the designer. The rows indicate the parts of a unit to be treated specially, such as titles, labels, or bodies of text. The columns indicate the typographic parameters used; for example, the color, font and line styles, character size, and justification of text. Entries in this matrix are either checkmarks or numeric values.

3] Variety in textual styles - While some consistency must be maintained, a balanced variety of textual styles can spark new interest. Occasional use of right-justified text instead of the more common left-justified can create a fresh look without the confusion of a completely different text font or size.

4] Grammar - The rules of grammar must be followed for all text on the screen. Words should not be split in a sentence without hyphenating. Commas, apostrophes, and ends of sentences should all be appropriately placed.(26)

5] Phrases - Natural phrases should be kept together on a line, if possible. Try to keep a single thought together, rather than splitting it across several lines.

6] Short lines - Short lines avoid reading errors.

7] Text graphics - The graphic aspects of the text itself must not be forgotten. Particularly for mathematical illustrations, the use of special characters, subscripts, and superscripts is important. Common mathematical notation should be used rather than "cheap" substitutions. For example, though it is often easier to type a function involving division or fractions on a single line, such as 2/4 = 1/2, the vertical fraction method of writing is much cleaner and easier to read, as $\frac{2}{4} = \frac{1}{2}$. Special symbols should be taken advantage of, such as a smoothly curved integral sign appearing before f(x)dx instead of INT(f(x)dx).

8) Spacing - The amount of spacing on different computers varies. It may be necessary to double space to enhance readability.

9) Blank space - Computer "white" space is free; textbook "white" space is not. This is one advantage of using this mechanical marvel. Liberal use of blank with a small amount of essential information on each screen aids the mental digestive process.

10) Unnecessary information - Any information which is not relevant to the current idea should be removed. "The screen should not be a history of what has happened in the last five minutes."(27)

11) Variety in layout - Avoid monotony for the viewer. "Machines thrive on repetition. People thrive on variety."(28) If each text screen looks different, the eye can more quickly detect more information there. Borders, shadows, arrows, boxing, spacing may all create variety. Keep in mind however, the difference between a monotonous display and a consistent user interface. If the student must interact with the program, his interface should be consistent throughout.

12) Material on one screen- The graphics should be scaled to fit all on one screen. It should not "scroll" or move up the screen.

13) Subdivide complex illustrations - Too much information should not be included in one illustration. Complex concepts should be broken into smaller units and presented individually or one step at a time prior to presentation of the total picture.

14) Movement and animation - If animation is used, it should be related to the instructional message, and not merely for flashy display. But used correctly, animation can be a very effective technique.(26) For example, a moving object to illustrate velocity from point A to point B is more effective than simply illustrating point A and point B while noting that it took 3 seconds to cover the distance between the locations.

15) Perceptual guidelines - Words can help the viewer mentally organize a display. Use phrases such as " Note the point on the left of the screen..." or "This comes before..."

17) Emphasize key words and phrases - Key points can stand out from a body of text or illustration by use of highlights such as type style, blinking, color, or timing in display.

17) Timing - The element of timing can add to a presentation. For readability, a pause with a prompt can be

inserted between sentences or paragraphs of text. Graphic illustrations can also be added to in this manner.

18) Eye movement - Movement on the screen should follow the normal eye movement whenever possible. This is from left to right and top to bottom. Complicated movements should be avoided. For example, text should not be written in the top left corner, move to the lower part of the screen, and back to the top again. The student will probably miss the text on the lower part of the screen with this pattern.(26)

19) Know your audience - Present the material with a level of detail appropriate for the intended audience. For college-level material, more detail and smaller text sizes are acceptable if necessary; if the audience were to be elementary students, simpler illustrations and larger text would be recommended.

20) Review and evaluation - Once complete, don't be afraid to make revisions. Changes may be suggested after review directly at the screen by other designers, or if implementation reveals previously unnoticed flaws.

III. USE OF COLOR

A. COLOR AND EDUCATION

People react to color. Information content can almost always be conveyed by use of black and white, but "it is color that attracts, conveys meaning, elicits emotional response, and fosters the retention of information."(29) Current research on the use of color has found several situations in which color has been found to play a key role; studies of some other areas of color use have been nonconclusive. However, it is agreed that the emotional impact of color is greater than that of black and white. In fact, in a 1979 study by Johnson and Roberson, when subjects were informed that their particular study had shown no advantage of using color for instructional films, the subjects were upset because they liked the color presentations more.(30) This is one illustration of the idea that color can be used to invoke a positive response to material presented.

It has been shown that when color is selectively used to direct attention to specific material, increased comprehension of that material takes place. It has also been shown that the use of color can aid in focusing attention; however, the positive effects of color use possess a curvilinear relationship to the number of colors used. Research has demonstrated that there is an initial advantage in comprehension as the first few colors are added to an uncolored display; this however, is followed by a decrease as more colors are added.

A separate experiment revealed that the use of color versus black and white presentations of slides increased students' recall of peripheral visual material (that is, content which was irrelevant to the plot or theme), but did not significantly affect recall of the central material. It should be noted here that the slides consisted of multi-colored posters and comic book stories; color was not used on any particular segment to focus attention on that segment.

Thus, care should be taken to ensure that, where color is intended as more than an attention-getter, the eagerness to use this rainbow of color does not result in a fogging of the original material content. When a limited amount of color is used in a graphics display, the student can more ably "locate key points that can be used to partition the material to be searched."(31) But the advantages of this partitioning can be overcome if the number of color-coded partitions becomes too large.

Color coding for information location has also been found to help improve overall visual search performance. This keying of colors to information has proved most effective when many categories of information are to be presented, highly distinguishable colors and peripheral vision are used, and the quantity of objects in each category is kept reasonably small.(32)

In addition to effectiveness of selective color for immediate attention and understanding, the use of color also results in more accurate recall of material than the same material presented in black and white.

Research at San Francisco State College showed that when two-color slide images of red and green were used for the presentation of visual information, student test scores rose as much as two times as high as test scores following black and white slide presentations. The use of three-color slides, red and green on a blue background, increased the test scores by thirty percent.(33)

Similar studies at other locations have also shown that recall and comprehension can be increased by use of color. In general then, color schemes for understanding and remembering material should be developed as well as schemes to provide clarity in the presentation of an idea. "Color should be selected to supplement the idea, the organization, and the style - in that order."(34)

B. EFFECTIVE USE OF COLOR

In order to avoid a frivolous use of color, guidelines for its use should be followed. It should be considered as a tool with certain capabilities which are effective only when used properly. There are three considerations for forming such guidelines here: the implications of the hardware used to produce the colors, the physiological perception of the eye, and human psychology.

The first of these three is based on the relationship between the additive primaries of red, green, blue and their complements. These complements are cyan (turquoise), magenta, and yellow, respectively, and are also known as the subtractive primaries: they are the colors perceived by removal of their complement from white light.(13)(29) Figure 1 illustrates these relationships. Complements are shown exactly opposite each other. A color is composed by the colors at opposite ends of the arrows pointing into it; red, green, and blue light are used in this manner to create the eight available colors on the CRT screen of the NEC APC used for this project. The colors formed by a mixture of the additive primaries will appear brighter or of higher intensity because the mixed color has the intensity of two electron beams. Thus, yellow will appear brighter than red or green, cyan will appear brighter than blue or green, magenta brighter than red or blue, and white the brightest of all.

Without going into a detailed description of the physiology of the eye, the following generalizations may be made. Currently, color perception is based on the theory of three opponent receptions: blue/yellow, green/red, and white/black. These opponent-color combinations are considered to be too difficult to perceive when used directly together, and should be avoided. It has also been found that "warmer" colors, such as red, white, or yellow,

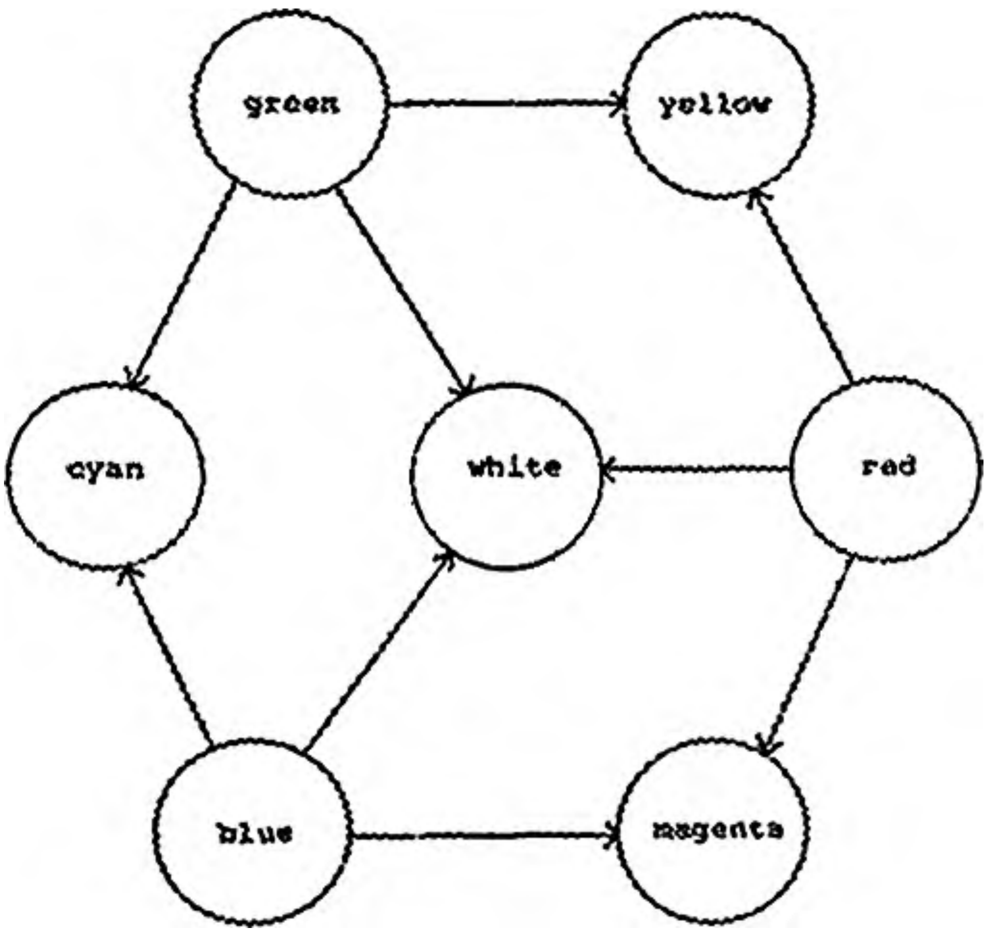


Figure 1. Additive Color Relationships

tend to attract attention, while blue has low visual activity and may go unnoticed.

Humans also react to color based on psychological factors. For example, red may be used to denote "stop" or "danger" as we are used to seeing such warning signs in red.

The following guidelines employ all three of the aforementioned considerations.

Use high color contrast for character/background pairs. Alphanumeric information should be displayed in the more noticeable colors of red, yellow, or white when used on a black background as for this project.

Blue should not be used for text or small, significant areas. It works best as a background color.

Color combinations should be compatible. It is recommended in literature to avoid the opponent-color teams of red/green or blue/yellow. However, some moderate use of these color combinations on this project were noted to be acceptable; in fact, one viewer even stated that he particularly liked these color combinations and they helped to attract his attention.

Since red and green are not easily seen at the peripheral of the eye's visual field and blue tends to fade into the background no matter where it is placed, segments to be perceived in this region should be done in white.

If color coding is used to enforce relationships pertinent to the content of material, keep the coding consistent throughout the lesson to reinforce the same relationships. As the number of colors increases, the size of the color-coded objects should also be increased to maintain comprehension.

Also, when color-coding, it should be remembered that about 6 to 10 percent of the male population is color-blind.(35) This normally affects the perception of red and green differences. In consideration of this, and for communication increase by use of redundancy, shape should be incorporated, whenever possible, into the same coding scheme.

From five to nine colors are about all that can be perceived effectively in one presentation.(29) Five is usually the "magic number" for short-term memory of a quantity of related objects. Thus, a display of only four colors should be near optimum to insure some memory space is conserved for other functions while colors are being "brain processed."

Beware of overpowering other colors with use of the brighter subtractive primary colors on the CRT. Though adjacent areas of color may in actuality be the same size, if one area is of higher intensity than another, it will appear to be of a slightly larger size.

Adjacency can also affect the perception of a color's true value. When complementary colors appear together, assimilation occurs, and they appear more alike if the items are small. The complementary color will take on the look of the stronger color surrounding it.

Warmer colors, such as red, will also appear to be closer and light in weight compared to darker colors, such as green, which appear more distant, smaller, and heavier.

Outlining colors in white or black accentuates contrast and tends to make an image appear busier. Used incorrectly, outlining may create confusion in an illustration rather than aid clarity. Depending on the width of the outline, it could also create other contrast effects on the object being outlined.

It was found on this project that the use of outlining was desired for such purposes as separating the function graphs from the color-filled area between the curve and an axis or another curve. In some situations, it was necessary to use a special plotting routine which drew a heavier line; otherwise, the curve appeared to fade into its adjacent color-filled area. For example, the use of a cyan border with its complement, red, used for area fill, caused the cyan to fade away to white. Figure 2 shows an example of this poor color combination on the left side; the combination on the right side does not have "fade-away" colors.

Strongly contrasting colors, such as a fully saturated red and fully saturated blue, should be used adjacently with caution when both are to cover much area. Though the contrast can be effective to convey differences, too much of these strong colors can cause eye strain. Again, this was perceived in several of the preliminary illustrations for

Figure 2. Use of Adjacent Complementary Colors

Figure 2. Use of Adjacent Complementary Colors

this project, resulting in necessary changes for some color schemes.

These basic guidelines were applied to all layouts used to create the color illustrations on this project. Some changes were implemented based on comments of reviewers and the tastes of the designer; these were required in order to produce the best possible end product.

IV. DESIGN CONCEPTS

A. GENERAL IMPLEMENTATION CONCEPTS

The 10 main segments developed are designed to be presented as supplementary visual aids in a first course in calculus classroom environment. In order to be used with an entire class, a large projection screen must be available; this should have a screen size of at least 4' x 6' to provide legibility at the back of large classrooms. The images, produced by the computer and projected onto the screen, can then become the primary lecture aid (replacing the traditional chalkboard or overhead projector). The instructor retains control over the display by his responsibility of loading the proper program and interactive use of an executing program.

Four of the ten main segments have corresponding programs which include a complete textual development analogous to a lecturer's description. These are designed for individual student use to complement the lecture. Students who desire a review of the classroom presentation may use these lessons for that purpose, though they should still be encouraged to seek personal help from an instructor if questions remain. These individual lessons can reduce the monotony of answering routine questions for an instructor, as well as giving the teacher valuable time to help students with more serious difficulties. In addition, students who may be apprehensive about asking for help from an instructor will usually not have the same reluctance toward using a terminal for assistance.

In the event that a large projection screen is not available for use with the computer system, photographic slides may be produced. The graphs and images drawn for interactive classroom use are photographed at pre-determined "pause" points in the programs. They may then be used in the classroom with a slide projector and traditional projection screen non-interactively. However, a sense of watching the illustrations evolve is still possible. They should be shown, in order, with each successive slide in a set illustrating a new stage in the development of a final picture. This can still be effective in much the same way as the interactive method of presentation.

B. DEVELOPMENT ENVIRONMENT

1. Supporting System. The illustrations and lessons prepared were developed on an NEC Advanced Personal Computer, a microcomputer system with two floppy disk drives. Graphics were implemented by use of the GRAFDRAW unit and a Font Compiler under the UCSD p-System. The GRAFDRAW unit is a collection of Pascal procedures which comprise a subset of the SIGGRAPH CORE standards. The display consists of 640 x 480 pixels, presently considered quite high for a microcomputer system. Eight colors are available, including black and white. The Font Compiler was used to redefine several rarely used keyboard characters to

special text characters, such as infinity, delta, integral sign, etc.

This particular system was purchased for the Computer Science Department of the University of Missouri-Rolla, and is presently located on the third floor of the Math-Computer Science Building. It was chosen for this project because of its color capabilities and higher resolution necessary to prepare effective illustrations.

The Pascal language is supported in this environment. This language worked well as it supports the concepts of structured programming; many subroutines were used to avoid redundant code. Most of these subroutines were kept in a library utility file supported by the p-System. This allowed access by all programs and quicker compilation.

2. Illustration Preparation. Each lesson topic is compiled as a separate program, employing the two graphic utility libraries. These programs include hardcoded stopping points or pauses, triggered by depressing the "return" key of the terminal console. A pause is included at a logical point in the development of each illustration.

In order to prepare the slides used for a non-interactive presentation, photographs are taken at each pause in the lesson using a 35 mm camera and tripod. The set of slides developed was photographed in a completely darkened (except for the terminal display) room at 1/4 second, aperture f8 using 200 ASA Ektachrome film.

C. PRODUCTION STEPS

The creation of these computer generated images and text may be viewed as a process of interrelated phases of development and analysis. The various stages of production are described in the following paragraphs.

To produce these lessons, the first step was a review of the topics presented in an introductory calculus course. The primary source for this was the text in use for several years at the University of Missouri-Rolla, Calculus with Analytic Geometry by Munem and Foulis. Chapters 0-7 cover the material commonly found in a university or college's first semester calculus course. This normally begins with a pre-calculus review of geometrical concepts, followed by the introduction of limits, and including the definitions and rules for elementary differentiation and integration with an introduction to their applications. The lessons to be illustrated were selected based on the criteria of potential gain from automation of illustration. These gains were considered in terms of accuracy, use of color, rate of presentation, repetitiveness of drawings, and use of interactivity.

After suitable topics were chosen, the concept and goal of each lesson was clarified. The functions which would most clearly illustrate each topic were sketched and preliminary drawings of the final forms were done on paper.

These drawings showed the order in which illustrations were to be generated, and notes were made on appropriate "pause" points.

A library of subroutines called TOOLS was developed, created, compiled, and added to as necessary. This library contains subroutines which could be commonly used by two or more of the programs. This includes procedures for such fundamentals as drawing the axes, plotting the graph of a function, or performing a text line feed for subscripting. It also includes subroutines used to create specialized illustrations. For example, algorithms are included to rotate a point about the x or y axis, to partition a given interval into n segments, or draw a line through two given points. A more complete description of this library is given in Section D of this chapter.

The programs also made use of the Grafdraw graphics utility library provided as a supplement to the NEC APC p-System implementation. These routines consisted of two distinct classes. The first type are graphic output primitives, functions which create graphic output or control the system's current screen position (in x-y coordinates); these result in display processing unit (DPU) instructions. The second class of routines is attribute settings, or functions which determine line style and color.[13]

Text was displayed on the screen by use of the Grafdraw "Text" command in order to maintain control over position, size, color, and style. The basic 256 character text font provided with the system was altered. This allowed special characters to be defined and easily used. The new font definition was then compiled, and characters for summation, delta, right-pointing arrow, integral sign, pi, and infinity were readily available for use. Subscripting and superscripting were accomplished by use of a "half" line feed subroutine included in the TOOLS library.

Programs were then coded, compiled, tested, and evaluated. Several students examined the programs and used the four individual ones. Their comments were recorded, resulting in several changes to the presentation style, as well as correction of a few typographical errors.

D. TOOLS SUBROUTINE LIBRARY

1. Miscellaneous Subroutines. The subroutines which were used to compose the TOOLS utility library are briefly described in the following paragraphs. Several of the more complicated concepts are explained in the following two sections.

Two subroutines help preserve the current state of the graphics core variables such as the current text color, text position, and fill color. "Save_Core" and "Restore_Core" were used primarily for the four individual student lessons when graphics and a body of text appear together on the screen.

Four subroutines were employed to ease the use of windowing. These are described in Section 2.

Control of prompts, pauses, and erasures was accomplished via "GoOn", "GoOn_Slides", "GoOn_and_Erase", and "EraseT". These used either current screen postion or accepted as input x and y values in screen coordinates to position a prompt message, and to erase selected regions of the screen.

"Ret" and "Ret2" controlled the line feed or carriage return of the graphical text fonts. They were used for subscripting and superscripting as well as standard line feeding in text bodies.

Control of the line style was performed with "Dot", "Dash", and "Solid". These subroutines consisted of hard-coded calls to GRAFDRAW attribute setting subroutines.

The mathematical functions were plotted on graphs in the window coordinate system. The axes for these graphs were first drawn with a call to "Axes", which accepted as input the variables used to determine the lengths of the axes in each of the four directions from the origin, as well as an input variable to determine the color of the axes. Tick marks could be requested to be drawn at integral locations on the axes, or they could be suppressed. The points for each function were then plotted by use of either "Plot" or "Heavy_Plot". The first of these takes as input the quantity of points to be plotted and the color in which the function is to be drawn. The latter makes use of the former to quickly plot the function at four additional positions varying by one pixel in each of the four directions from the origin; this results in a heavier line for greater contrast with surrounding regions.

"Y_For_X" and "X_For_Y" are used to determine the corresponding coordinate, given one of either x or y, for the current function.

The point-slope form of a line is used by "Line_Thru" to sketch a line through two points which are given it as inputs. The endpoints of this line are determined by input of an interval given in x coordinates.

"Line_Pt" draws a line whose path begins at a point given in screen coordinates as input, and ends at a point in window coordinates also given as input. This is used as a perceptual guideline to point to specific points on the graph, normally from a body of text.

Two subroutines are used to plot a line from one of the axes to a point on the graph, which is given as input. "Show_Horiz" draws such a line between a point on the graph of the current function and the point on the y axis of the same height. "Show_Vert" performs a similar function using the x axis.

Whenever a particular region between a curve and the x axis must be partitioned into n equal intervals, "Segments" is called. This uses a given interval on the x axis and a number, n, for the quantity of partitions as input. It also requires that a color be input for use as the partition line color, and a second color input if color fill of the segments is desired.

The five remaining subroutines developed for the TOOLS utility library are used for three-dimensional rotations and projections. "Init_for_Revolv" initializes the variables needed for a given angle increment. The four other routines are described in the paragraphs of Section 3 under this heading.

2. Windowing. The computer graphing system used defines all output in terms of an x-y coordinate system with origin at the lower left corner; it extends a distance of 640 pixels in the positive x direction and 480 pixels in the positive y direction. This project makes extensive use of graphs of mathematical functions. These graphs are plotted using the standard x-y Cartesian coordinate system. In order to completely and appropriately graph the desired functions, windowing must be implemented.

Windowing is the process of mapping the given "world" coordinates to screen coordinates.[13] "World" coordinates refers to the true numerical values in Cartesian coordinates of any point on the graph. Windowing allows the origin and scaling of this graph to differ from that of the screen coordinate system. In each of the programs, the variables for X_Origin, Y_Origin, and scale must be assigned values in screen coordinates before any part of the function's graph may be drawn.

Limits are set for the window coordinates at the time the axes are drawn; the length of the axis in each of the four directions from the origin is given. These lengths must be considered by the programmer whenever referring to a position on the graph.

All routines in TOOLS which refer to graphs use the world coordinates as input. Four of the TOOLS routines are adjustments to the four most basic DPU controlling routines of the GRAFDRAW unit. They are: Move_IGW_Abs, Move_IGW_Rel, Line_IGW_Abs, and Line_IGW_Rel. (IGW denotes In Graph Window) These allow control of the DPU as if it were operating in the world coordinate system. The two "Abs" procedures convert world coordinates to screen coordinates by the following equations:

X_Screen = Round (X_Origin + scale*X_World)

Y_Screen = Round (Y_Origin + scale*Y_World)

The two "Rel" procedures operate similarly, but without the addition of the origin values.

3. Three-Dimensional Rotations and Projections. For several of the lessons developed, such as volumes of solids of revolution, it was necessary to create three-dimensional projected illustrations. Functions were always specified in the two-dimensional coordinate system. In order to treat them in three dimensions, they were considered to originally lie in the z=0 plane.

To illustrate a solid formed by rotation of a curve about an axis, two forms of rotation were necessary. These are the rotation of an individual point on the curve and the rotation of a segment of the curve itself, each to be described in succeeding paragraphs. Both used the standard rotation equations for a right-handed coordinate system, with positive counterclockwise rotations.[13] Rotations were always performed in the window coordinate system.

Consider a point (x,y) to be rotated through an angle of θ. The transformation equations resulting in (x',y',z') would then be as follows:

Rotation about the x axis:

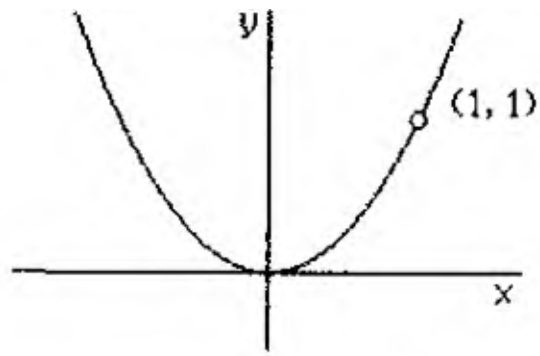
$$[\ x'\ y'\ z'\ 1\ ] = [\ x\ y\ z\ 1\ ] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation about the y axis:

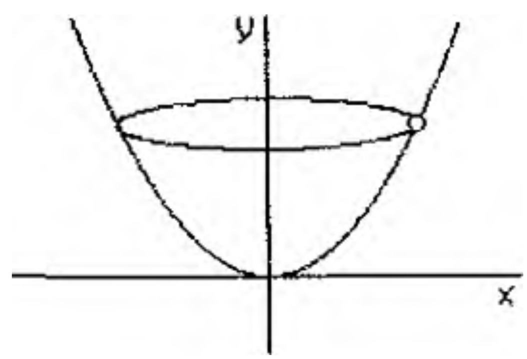
$$[\ x'\ y'\ z'\ 1\ ] = [\ x\ y\ z\ 1\ ] \begin{bmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The curve $y = x^2$, as shown in Figure 3.a, is displayed in Figure 3.b with the path traced by a complete revolution of the point at (1,1) about the y axis. Rotations of a particular point on a curve were required in order to show the flat surface(s) on the end(s) of an object, as well as for use in crosshatching.

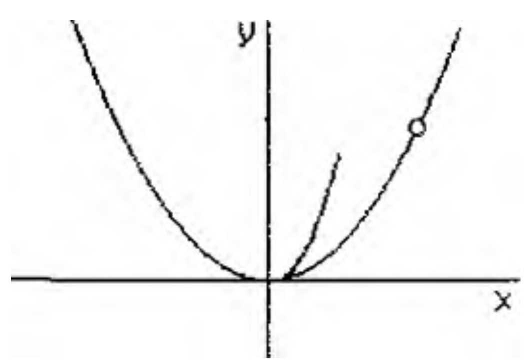
The subroutines Rot_X_Pt and Rot_Y_Pt included in TOOLS accomplished this form of rotation. Input parameters require the initial and final angle of rotation for the specified point: for a complete revolution, these are 0 and 360 degrees, respectively. For the developed programs, an increment of 1 degree was used to calculate successive values and fill arrays of the specified point rotated



a) Graph of the curve $y = x^2$ in the $z = 0$ plane



b) Rotation of the point (1,1) about the y axis



c) Rotation of the curve 60 degrees about the y axis on the interval from $x = 0$ to $x = 1$

Figure 3. Rotation of a Curve

through angles of the initial angle, initial angle + 1, ..., final angle - 1, final angle.

Figure 3.c shows the same curve on the interval [0,1] from Figure 3.a rotated 60 degrees about the y axis. Rotations of a section of a curve were used in order to sketch the mirror image of the curve on the opposite side of the axis of rotation. For Figure 3.c, this would have been a 180 degree rotation rather than a 60 degree one. Again, these curves may also be used for crosshatching.

The subroutines Rot_X_Curve and Rot_Y_Curve included in TOOLS provided this form of rotation. Input parameters require the specification of the interval endpoints and the angle through which the curve on this interval is to be rotated. Each previously calculated point on the curve and within the interval is then rotated the specified angle.

The four rotation routines perform projection also. This is done to provide a two-dimensional representation of a three-dimensional object.(36) The rotated three-dimensional world coordinates must be mapped into a projection plane world coordinate system. This new system provides the viewer with a projection plane on which the object can still be perceived as having depth.

Oblique projections were used for these three-dimensional drawings. This type of representation portrays three dimensions by drawing lines parallel to the third (z) axis at a consistent angle to the horizontal (x axis).(37) The x and y axes are perpendicular to each other. Thus the z coordinate, which is used after rotation of a point out of the z=0 plane, must be used to calculate the projected point onto the two-dimensional screen. So after the aforementioned rotations are performed, the new (x',y') point in projected window coordinates must be found. This is done by use of the following equations applied to the previously rotated (x,y,z) point.

Projection of points rotated about the x axis:

$$x' = x + z/2$$

$$y' = y + z/2$$

Projection of points rotated about the y axis:

$$x' = x + z/6$$

$$y' = y + z/6$$

The difference in the divisors of z is required in order for the perspective of the viewer to be adjusted for a more "normal" appearance.

The computer allows all these calculations to be performed relatively quickly. Points may then be plotted to accurately portray the solid objects to the viewer.

V. GRAPHICS PROGRAMS FOR INTRODUCTORY CALCULUS

A. ADVANTAGES OF THE SYSTEM

Whether used interactively or non-interactively, there are still many advantages to the use of this computerized "blackboard" system. Highlights of these are enumerated in the following paragraphs.

First of all, a computer-generated graph will be much more accurate than one drawn by hand on a chalkboard. Students can see the true representation of a function on a graph without the confusion of wondering such things as whether an inflection point occurs at x=1 or x=2. "This accuracy has also the merit of training the eye of the students."(19) Accuracy is further enhanced by the use of a higher resolution system such as the one used. Curves with very small or very large curvature may still be represented acceptably; this has been a problem for users of the Apple II or IBM PC, two common systems currently used. Curves on a lower resolution screen appear to be jagged or stairstep, and "the discretisation of the picture also produces the intermingling of close points"(19) for which the teacher must comment upon the reasons for this, in order to avoid wrong interpretations by the students.

Graphical representations of mathematical functions will also be created consistently every time, circles will not look like ellipses, hyperbolas will not be confused with parabolas, and division of an interval into n equal partitions will truly be n equal partitions.

Legibility is also increased when computer-generated text is used. Machine generated lettering and mathematical symbols will be consistently written every time. This text will be much more legible than the same information written on a blackboard or overhead projector.(38) Again, the higher resolution of the NEC APC allows an increased amount of detail to be more clearly shown than on the more common IBM PC or Apple II systems.

Hand in hand with accuracy and consistency is the ability to create difficult pictures that the instructor previously carried about in his head, but simply found too difficult or time-consuming to sketch in class. This includes such things as solids of revolution and successive graphs of area for Riemann sums, broken into more rectangles with each succeeding picture. The computer allows these to be created relatively easily. For the more difficult drawings, text book illustrations reflecting precise draftsmanship have usually been the primary source (23); however, the quantity is limited and the quality in some texts still may not be up to the standards of similar illustrations generated interactively by the computer.

The use of color is also a super plus for these illustrations. Though some textbooks do employ color, it is usually only one color with perhaps a lighter shade of the same color used for contrast. And, obviously, switching between different colors of chalk at a board can be confusing as well as time-consuming for an instructor. Color can increase and influence attention. "By carefully utilizing color to manipulate attention, the user can partition material at key points, organize it, and code it. Techniques that direct attention increase the likelihood that the attended to information will be processed."(39) Techniques of color utilization may include graphing several functions on the same set of axes in different colors, using a solid area of color "fill" to show a specific region, or highlighting specific segments. A unique feature of the system used is that it also allows text to be written in any of the eight available colors; this permits color-keying of the text and labels to corresponding graphic segments.

Along with the use of color, the computer offers several other features unavailable with conventional chalk-board methods. A variety of line widths or combinations of dots, dashes, and solids can be used to emphasize different sections of a diagram. This provides consistency for such concepts as hidden lines or connecting specific points to their coordinates on the axes. Different text styles may be defined and used, as well as changing the scaling of the height and width of existing or new text fonts. This focuses attention in a manner similar to the way different colors do; different text style or sizes may be keyed to different uses. It also relieves monotony for the viewer.

All of the above advantages are used together to create graphical representations which may be presented as a single complete illustration with all segments and labels in position, or as a series of progressive illustrations. Just as the solution to a problem is derived step by step, or a concept is presented by building one idea on top of others, the graphical representations can be shown in successive stages of development. The illustration can evolve on a screen and be built upon as it evolves in the viewer's mind. This is superior to textbook illustrations where the reader is normally presented with the final illustration to use as a reference even at the beginning of a presentation; this often leads to confusion on the part of the student, who cannot always sort out the relevant parts of a diagram.

A final advantage, but certainly not the least of these, is that the concentration of the instructor can be turned to the students and the explanation of the concepts being presented rather than focusing his attention on the creation of appropriate chalkboard images. Graphs and illustrations can be quickly and conveniently sketched with no more effort than it takes for the touch of a button. Computations are performed quickly and painlessly at successive stages of iterations or for different points along the graph of a function; the numerical results may be listed on the screen alongside the corresponding graphical represention. The instructor also gains from knowing that the comprehension and understanding of his students is improved.

B. LESSONS DEVELOPED

The ten programs which have been developed for interactive classroom presentations are briefly described here. A sample program use is given in Section C of this chapter. The four programs for individual use are discussed with their corresponding classroom version.

1) Inequalities - This graphically shows the solution set of a given inequality. The solution interval is found algebraically and highlighted on the illustration with a bold overlay of the existing function graph in a contrasting color. Integral values of x and the function at x are also calculated and displayed in a table, one set at a time as the corresponding points on the graph are illustrated. This program has a corresponding lesson for individual use.

2) Slope of a Tangent to a Curve - This lesson makes use of the definition of the derivative with limits to illustrate the derivation of the slope of a tangent line to a curve. The tangent to a given curve is drawn at a fixed point. A succession of neighboring points, gradually closer and closer to the fixed point, are shown. The secant lines between these neighboring points and the fixed point are drawn in color. The delta x distance is also pointed out as it approaches a distance of 0. The formulas for slope are shown at the side of the graphical diagram. This is also available for individual use.

3) Parabola - A menu at the beginning of this program allows use of two different versions. The first displays the same basic parabola $y = x^2$ with the coefficient of the x^2 term changed and the curve displayed in a different color for each different coefficient function. The corresponding focal points and directrices are drawn in the same color as the curve they define. The second version simply illustrates translation of the same parabola along the y axis.

4) Hyperbola - This lesson is similar to that for the parabola. There are two choices available in the menu. The first draws two hyperbolas in contrasting colors with different values in the denominator of the standard equations; this illustrates the effect of increasing or decreasing these values. The second version displays the asymptotes and dimensions of the rectangle used to sketch these asymptotes.

5) Linear Approximation - This lesson shows how a function may be approximately evaluated at a given point. If the given point does not work nicely in the function, a nearby point which does evaluate easily may be used for this approximation. A curve which represents a given function is shown, with just such a point for which it is difficult to evaluate the function. A neighboring point is shown, for which the function may easily be calculated. The tangent to the curve at this point is drawn, and the difference between the tangent's value at the given point's x value and the

curve's value at this point is shown to be quite small.

6) Area Between Two Curves - This lesson has two possible sets of curves to be drawn. In both, two curves are sketched and the distance between each and the x axis is illustrated by use of a color region fill. The area which is not bounded by the two curves can then be erased to the background color.

7) Derivatives and Graphing - A polynomial curve is shown with the curve of its derivative on the same graph in contrasting colors. This is done in two illustrations: one for the first derivative and one for the second derivative. Below the graphs, the function and its derivatives are shown and the characteristics of the function (increasing, decreasing, concave, convex) are shown with the corresponding characteristics of the derivative (positive, negative, zero).

8) Riemann Sums - This lesson shows how the area between a non-monotone continuous function and the x axis can be approximated by use of rectangles. Successive illustrations break the interval up into an increasing number of partitions, showing how the increase in quantity of partitions makes the area calculated more nearly approximate the total area desired. A corresponding individual lesson is also available for this program.

Figure 4. Solid of Revolution - Method of Cylindrical Shells

9) Volume of Solid of Revolution by Method of Disks - This program draws a curve, illustrates a given interval bounded by this curve and one of the axes, and then shows the solid generated by rotation of the curve about the axis. The solid may be displayed as cut into many disks, or a single sample disk may be selected in order to more clearly point out the dimensions used to calculate the volume of each disk. An individual lesson with text is also available for this program.

10) Volume of Solid of Revolution by Method of Cylindrical Shells - This is similar to the presentation used for the Method of Disks above. The curve and area to be revolved are shown, and the solid object is generated. The volume is then shown as being broken into many cylindrical shells as shown in Figure 4, or one cylindrical shell may be shown in order to more clearly illustrate the dimensions used to find the volume of each individual cylindrical shell.

C. SAMPLE LESSON USE

The following paragraphs describe a sample program to be used in a classroom session with a large screen projector available. Statements regarding the instructor's comments and actions roughly correspond to the text used in the individualized version of this lesson illustration set. This particular program illustrates the use of Riemann sums to find the value of a definite integral.

The first screen to appear upon program invocation is the title of the lesson to be presented, "Riemann Sums," in a large red text font. A short menu is presented which allows the user to request a general curve illustration of area between a curve and the x axis, or Riemann sums calculated for the specific curve, $y = 9 - x^2$, or the "quit" option. After selection by the user of the desired section, the program continues with the request. Upon completion of one of the two non-"quit" choices, the same menu reappears. Also used throughout the program, execution is "paused" at certain points where the instructor may wish to make comments; this pause will cause the continuous display of the current screen until depression of the return key causes the program to proceed to the next screen or add to the present screen.

The first choice in this sample program is intended primarily to be used as an introduction/review of the area between a curve and the x axis on a given interval. The first graph will then appear; this is a plot of a polynomial curve which will be used to illustrate the area between the curve and the x axis on an interval. The interval used is simply between two points which are chosen as "a" and "b" on the x axis. The curve is drawn in green, labelled merely as "y = f(x)." The points "a" and "b" are labelled on the x axis in cyan, with corresponding lines drawn from the points (a,0) and (b,0) up to the curve. The area is then split up into polygons, and the general notion of finding the area between a curve and the x axis is introduced and/or reviewed.

The second option creates a plot of the curve $y = 9 - x^2$ in the color red. The next "pause" is followed by the inscribing of 4 rectangles on the area between the curve and the x axis between x=1 and x=3. The rectangles are filled with color and the instructor points out the fact that the desired area is not completely covered by these four rectangles, but that this is a reasonably close approximation.

Next, the width of each rectangle is found. The difference in the two intervals' endpoints is illustrated by text to the side of the graph. The equation "3 - 1 = 2" is shown there, with dotted lines connecting the numerals 1 and 3 of this text to their corresponding points (1,0) and (3,0) on the x axis. The instructor then makes note of the total distance of 2 to be split among 4 equal partitions. The width of each rectangle, delta x, is illustrated on the screen.

The height of each rectangle is then determined. The point at which each rectangle contacts the curve is illustrated and labelled. The distances which determine each of these points' x and y coordinates are color-coded by use of yellow for the x values and green for the y, or f(x), values. After the next pause, a table illustrating these values is displayed, presented one line at a time by use of the pause control. The resulting screen is shown in Figure 5.

Figure 5. Inscribed Rectangles with Formula Table

Generalizations are then made about the formulas to be used when finding the area of the kth rectangle. The formulas for x_k and y_k are shown and color-coded to display the formula used to find the area of the kth rectangle. The summation formula for this rectangle is then shown, with the numeric value of this area.

A menu then prompts for input of a value, "n", to be used for the number of partitions in the next illustration; the choice to skip this option is also given. If a value is given, the same curve is sketched and the requested number of partitions is drawn; the numerical value of the represented area is written below the graph. This allows the instructor to play with the concept of different values of "n" before more formal presentation of this idea. This segment repeats with as many different inputs of "n" as desired until the segment's quit option is taken.

The next screen presents three graphs of the same curve with partitions of n=2, n=4, and n=8. The corresponding area summation formulas are shown below, with their respective values for the represented areas. After a pause, the general formula with the limit of n approaching infinity is given along with a graph illustrating the entire interval area shaded in. This screen is shown in Figure 6.

The initial menu then reappears and the session may be terminated, the introduction section reviewed, or this specific curve and its illustrations may be repeated.

Figure 6. Riemann Sums with Varying Values of "n"

D. <u>POSSIBLE EXTENSIONS AND IMPROVEMENTS</u>

Although a complete set of interactive graphics illustrations for each individual topic presented in an introductory calculus course is beyond the scope of this project, this is completely feasible. A few suggestions for future work are recommended.

The programs for the ten topics chosen could be adapted to cover different material which requires similar type illustrations. For example, the program for Riemann sums may be used as the basis for development of a lesson on the approximation of definite integrals by both the Simpson and Trapezoidal Rules. Volumes of solids of revolution illustrations could be modified to illustrate surface area of solids of revolution. Graph sketching, max and min points of a function could be expanded upon using the derivatives and graphing lesson.

Corresponding individual lessons may be developed from the classroom presentation programs currently without these. Since the concentration of this project was on the color graphics illustrations, following work could be done to develop more of the individualized lessons with an increased amount of interactivity. This would require the development of more conventional CAI programs, with student input analyzed by a series of branching instructions. For this type of presentation, many possible student answers must be considered and a detailed evaluation and test should be done.

For the classroom presentations, it is also recommended that with the purchase of a large projection screen, another purchase be made. A native code generator could be used to speed the creation of some of the more complicated illustrations for interactive presentations. In particular, the solids of revolution programs require many time-consuming calculations and thus are somewhat slow to draw on the screen. In order to revolve a point completely about an axis, approximately 10 seconds waiting time is required for calculations before the plotting of the revolved point appears on-screen. This slowness is due to the fact that code is generated as UCSD p-Code, an intermediate code used to make the p-System more flexible. However, p-Code is not the optimal code to be run on the machine. A native code generator could be installed in order to change this code to assembly language code instead. This could result in improvement in computationally related code by a factor varying from 20 to 70. For example, a simple manipulation such as incrementing a loop index by 1 could be done 50 times faster. This software is currently available for a moderate cost.

VI. CONCLUSIONS

It has been shown that a color computer-generated illustration system could be used more effectively than traditional chalkboard techniques to create graphs and text for a first course in calculus. The interactivity of such a system used with a large screen projector has the advantage of developing these visuals incrementally before the eyes of the students. In addition, with good screen design, the use of color can be quite effective. It can aid comprehension as a color code showing relationships, or increase the recall ability and retention of students. Color also can serve as a motivator, and increase interest in the subject matter being presented.

The project presented in this paper provides a system which may be used for this purpose. It combines the attractive features of interactivity and color on a higher resolution, relatively inexpensive microcomputer. A review of current literature and communication with various persons in the field of computer-aided instruction revealed no other projects which combined all of these same features.

The use of a large projection screen with this system is recommended for its classroom use. Interactivity and the dynamic development of quality color illustrations as students watch the increasing detail can be valuable for comprehension of the material. The corresponding individual-presentation lessons with additional textual information are also useful for the student's review of the classroom presentations.

This system, as developed and evaluated, was intended to display the capabilities and promote interest in the possibilites of this type of educational tool. The ten lessons developed for classroom use are available in the University of Missouri-Rolla Computer Science Department for future research or use, as are the four corresponding lessons to be used for individual lesson review. A more complete set of lessons could easily be created by use of the subroutine library developed for this project, and by modification of the existing lessons.

The project also was designed to serve as an example and trial system to promote effective use of color computer graphics materials for educational purposes. The guidelines presented and summarized here should be useful for anyone who must prepare graphics displays, especially in the field of education.

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