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Computational Complexity of Geometric Symmetry Detection in Graphs

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Abstract

Constructing a visually informative drawing of an abstract graph is a problem of considerable practical importance, and has recently been the focus of much investigation. Displaying symmetry has emerged as one of the foremost criteria for achieving good drawings. Linear-time algorithms are already known for the detection and display of symmetry in trees, outerplanar graphs, and embedded planar graphs. The central results of this paper show that for general graphs, however, detecting the presence of even a single axial or rotational symmetry is NP-complete. A number of related results are also established, including the #P-completeness of counting the axial or rotational symmetries of a graph.

1. Introduction

Mathematically, an abstract graph simply consists of two sets, V and $E \subseteq V \times V$, and as such is completely specified by an enumeration of these sets or by adjacency lists or an adjacency matrix [1]. Such textual representations, however, convey very little structural information, and so graphs are instead often presented using drawings. Every given abstract graph has several different drawings, all equally "correct" but some certainly "better" than others in the sense that they clearly display important structural properties of the graph. For example, both drawings in Figure 1 represent the same graph, but properties such as planarity, biconnectivity, symmetry, diameter, and even bipartition are revealed by the second drawing but not by the first.

As discussed in [9], there are many different and sometimes mutually incompatible criteria for guiding the construction of good drawings of a graph. Amongst these, the best overall *general* criteria appear to be the display of axial symmetry and, to a somewhat lesser extent, the display of rotational symmetry. This stems primarily from the fact that by means of a symmetric drawing, an understanding of the entire graph may be built up from that of a smaller subgraph, replicated a number of times.

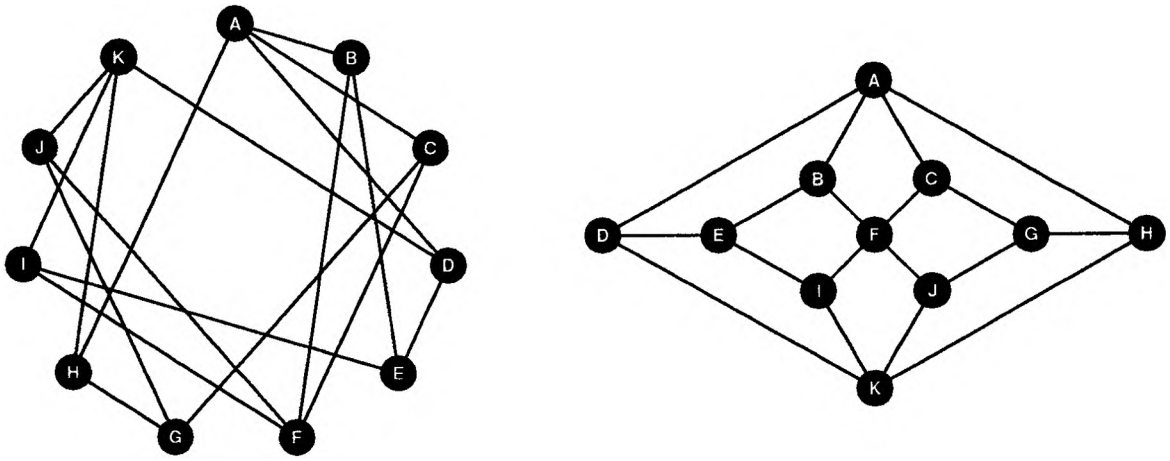

Figure 1: *Two drawings of the same graph*

Note that axial and rotational symmetry are considered inherent properties of the *abstract graph*, independent of any particular *drawing*. In fact these properties can be defined in a purely algebraic manner in terms of automorphisms of the graph [6]. However, it is more convenient here to work with the equivalent geometric concepts: an abstract graph is defined to have an axial (rotational) symmetry if there is *some* drawing of the graph having that axial (rotational) symmetry in the geometric sense. Here, a *drawing* of a graph consists of a set of distinct disks in the plane, one disk for each vertex, with continuous curves joining disks which represent adjacent vertices. For directed graphs, these curves carry arrowheads indicating edge direction. A curve and a disk must not intersect unless the corresponding edge and vertex are incident. For simplicity, all edge curves may be restricted to straight line segments; this ensures that every drawing of a graph is specified merely by the positions of its vertex disks, yet does not conflict with the display of either symmetry [6] or planarity [2].

In light of its relevance to the practical problem of graph drawing, it is important to develop computationally efficient algorithms for finding symmetry in abstract graphs. Linear-time algorithms (optimal) have been obtained for the detection and display of both axial and rotational symmetry in trees [7], outerplanar graphs [8], and embedded planar graphs [6]. Algorithms for the restricted class of *perfectly drawable* graphs are outlined in [4], although these appear to be inefficient as they involve the seemingly intractable problem of determining the automorphism group of the graph.

For general graphs, it was shown in [7] that symmetry detection is computationally at least as hard as graph isomorphism. This result is of fairly limited power, however, since the exact complexity status of graph isomorphism remains unknown.

The results below essentially resolve the complexity status of symmetry detection. The theorems which follow reveal that the basic problems of detecting if a general graph has even a single axial or rotational symmetry, as well as several refinements, are all NP-Complete. Under the widely-held assumption that $P \neq NP$, these results then preclude the existence of efficient general symmetry-detection algorithms.

2. NP-Completeness of Geometric Symmetry Detection

Let G be an arbitrary graph. For simplicity, G is assumed to be undirected; however, all the following results are readily extended to directed graphs by uniformly replacing each arc by a small undirected graph which encodes its orientation.

Throughout this section, the term *rotational symmetry* always refers to *non-trivial* rotational symmetry, thus excluding the identity rotation through $0°$. A vertex of G is said to be *fixed* under a symmetry if it is mapped onto itself by that symmetry. Clearly, G can have only zero or one vertices fixed under any rotational symmetry, whereas there are no such restrictions in the case of an axial symmetry. Recall that a *central symmetry* is a rotational symmetry through $180°$. An axial, rotational, or central symmetry of a graph is called a *geometric* symmetry, to help distinguish it from a general *symmetry,* which usually denotes any automorphism of the graph.

Several problems in the area of geometric symmetry detection are considered:

DAS : does G have any axial symmetry?
DAS-0 : does G have any axial symmetry with no fixed vertex?
DRS : does G have any rotational symmetry?
DRS-0 : does G have any rotational symmetry with no fixed vertex?
DRS-1 : does G have any rotational symmetry with one fixed vertex?
DCS : does G have any central symmetry?
DCS-0 : does G have any central symmetry with no fixed vertex?
DCS-1 : does G have any central symmetry with one fixed vertex?

The remainder of this section shows that each of these problems is NP-complete. Note, at the outset, that each of the problems lies in NP, since a polynomial-time algorithm can non-deterministically select a permutation of the vertices and then deterministically check that it yields a symmetry of the desired type.

Theorem 1 : *DAS is NP-Complete.*

Proof: A polynomial-time reduction from *3-SAT* [1] is presented: given a Boolean expression C in 3-CNF, a graph G is constructed in polynomial time such that:

$$C \text{ is SATISFIABLE} \iff G \text{ has an AXIAL SYMMETRY.}$$

Let $C = c_1 \wedge c_2 \wedge \ldots \wedge c_m$, where, without loss of generality, each c_j is a *distinct* disjunct of *exactly* three *distinct* literals; for $1 \leq j \leq m$, let $c_j = (x_{j,1} \vee x_{j,2} \vee x_{j,3})$. Let $a_1, a_2, \ldots, a_n$ be the variables appearing in C; thus each $x_{j,r}$ is some a_i or $\bar{a}_i$.

The corresponding graph G is constructed from the n subgraphs H_i $(1 \leq i \leq n)$ and the $8m$ vertices $z_{j,k}$ $(1 \leq j \leq m,\ 0 \leq k \leq 7)$. H_i is shown (twice) in Figure 2, where the broken line between v_i and w_i denotes a path of $i+1$ vertices, and the dotted vertical lines indicate axes of symmetry. Each vertex $z_{j,k}$ is joined to one vertex in each of three distinct H_i as follows: if $k = b_1b_2b_3$ in binary (so $k = 4b_1 + 2b_2 + b_3$) then $z_{j,k}$ is joined to $x_{j,1:b_1}$, $x_{j,2:b_2}$, and $x_{j,3:b_3}$. Thus, for example, if $c_4 = a_2 \vee \bar{a}_7 \vee a_9$ then $z_{4,6}$ is joined to $a_{2:1}$, $\bar{a}_{7:1}$, and $a_{9:0}$, since $6 = 110_2$.

Clearly, for each C the corresponding G may be constructed in polynomial time.

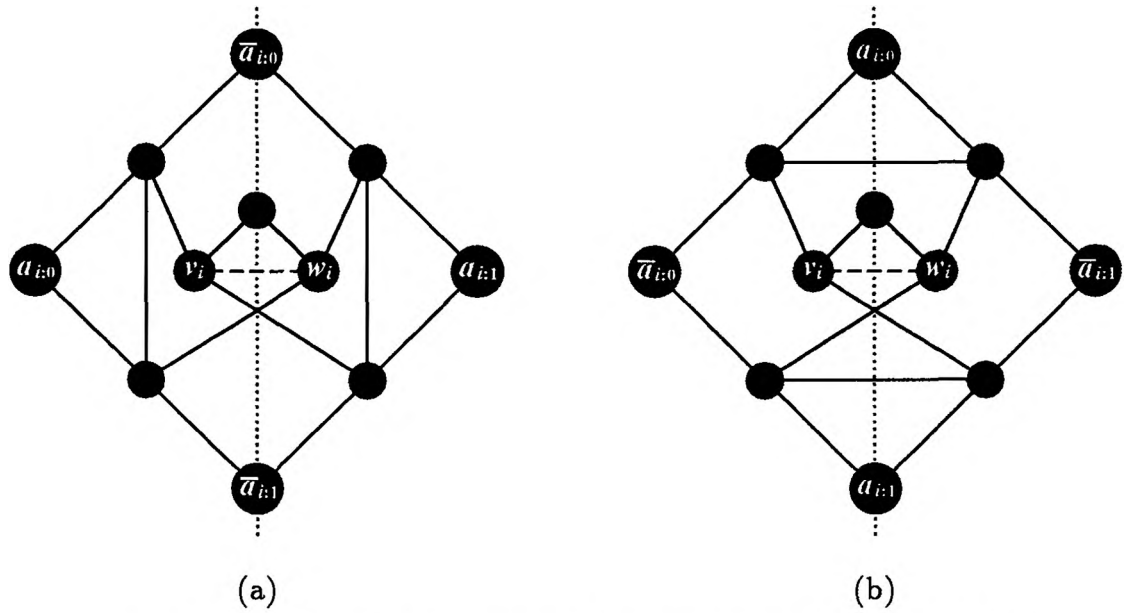

Figure 2: *The two axially-symmetric drawings of the subgraph H_i*

<u>C is SATISFIABLE $\Longrightarrow$ G has an AXIAL SYMMETRY:</u>

Let $\varphi : \{a_1, \ldots, a_n\} \longrightarrow \{T, F\}$ be a satisfying truth-assignment for C, and extend φ by setting $\varphi(\bar{a}_i) = \overline{\varphi(a_i)}$, for each i. Construct a drawing of G as follows:

(a) Pick an arbitrary straight line L in the plane.

(b) For each $1 \leq i \leq n$, draw H_i as in Figure 2(a) or Figure 2(b), according as $\varphi(a_i) = T$ or $\varphi(a_i) = F$, and place it symmetrically astride of L.

(c) For each $1 \leq j \leq m$, the clause $c_j = (x_{j,1} \vee x_{j,2} \vee x_{j,3})$ is satisfied; thus $\varphi(x_{j,r}) = T$ for some least $1 \leq r \leq 3$, so vertices $x_{j,r:0}$ and $x_{j,r:1}$ are placed *off* L in step (b). Vertices z_{j,k_1} and z_{j,k_2} are termed *partners* if the 3-bit binary representations of k_1 and k_2 differ only in the rth bit. For each $0 \leq k \leq 7$ ($k = b_1 b_2 b_3$ in binary) place vertex $z_{j,k}$ on the same side of L as vertex $x_{j,r:b_r}$, partners being placed symmetrically opposite one another, relative to L. Complete the drawing by now joining all vertices $z_{j,k}$ to their neighbors in G by means of straight line segments.

It is readily verified that L is an axis of symmetry in this drawing of G.

<u>G has an AXIAL SYMMETRY $\Longrightarrow$ C is SATISFIABLE:</u>

Let L be an axis of symmetry in a drawing of G. For each $1 \leq i \leq n$, the subgraph H_i must be mapped onto itself by the symmetry, since the path of $i+1$ degree-2 vertices between v_i and w_i is unique in G. A detailed but straightforward inspection reveals that the *only* axial symmetries of H_i are those displayed in Figure 2. Construct a truth-assignment $\varphi : \{a_1, \ldots, a_n\} \longrightarrow \{T, F\}$ by setting $\varphi(a_i) = T$ or $\varphi(a_i) = F$, according as L induces the symmetry of Figure 2(a) or Figure 2(b) in H_i, and extend φ to $\{\bar{a}_1, \ldots, \bar{a}_n\}$ as before. Note that $\varphi(a_i) = T$ iff vertices $a_{i:0}$ and $a_{i:1}$ lie off L, while $\varphi(\bar{a}_i) = T$ iff vertices $\bar{a}_{i:0}$ and $\bar{a}_{i:1}$ lie off L.

For each $1 \leq j \leq m$, consider now any vertex $z_{j,k}$ $(0 \leq k \leq 7)$. Since the entire drawing of G is symmetric about L and all clauses of C are distinct, it follows that $z_{j,k}$ must have at least one neighbor, say $x_{j,r:b}$, which lies off L. Then $\varphi(x_{j,r}) = T$, so φ satisfies clause c_j, and since j is arbitrary, φ thus satisfies C. □

Inspiration for the rather esoteric definition of the graph G above was drawn from a somewhat similar construction found in an elegant proof of Lubiw [5].

Throughout the remainder of the current section, the term *fixed-point-free* (FPF) refers to a symmetry which fixes no vertex of the graph in question. Furthermore, axial, rotational, and central symmetry are abbreviated AS, RS, and CS, respectively.

Theorem 2 : *DAS-0 is NP-Complete.*

Proof: In the proof of Theorem 1, replace each subgraph H_i by the corresponding $\widehat{H_i}$ shown (twice) in Figure 3, where the path between v_i and w_i now contains $2i$ vertices:

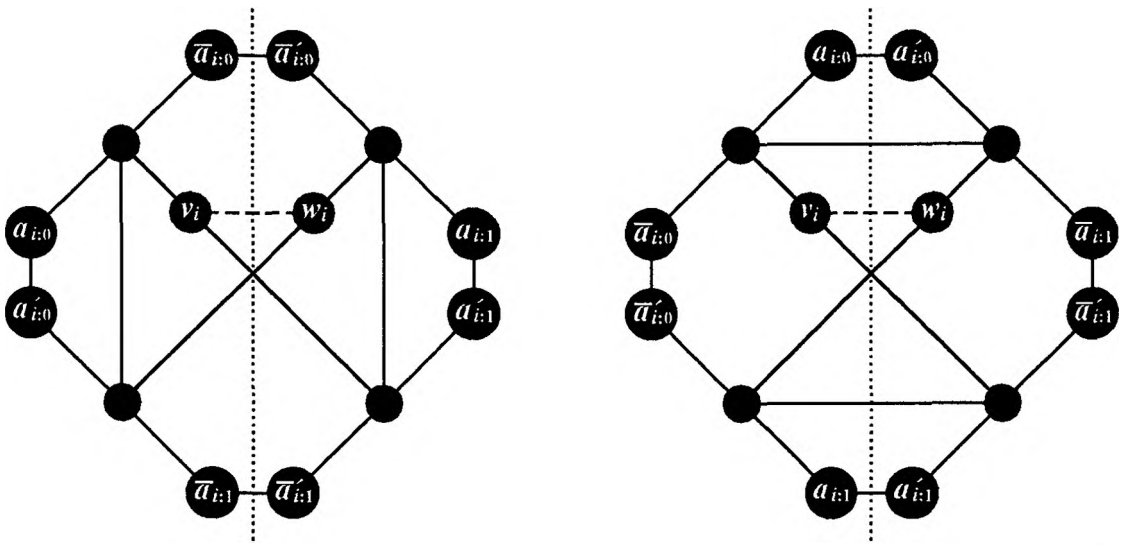

Figure 3: *The two fixed-point-free axially-symmetric drawings of the subgraph* $\widehat{H_i}$

In addition, replace each edge between a vertex $z_{j,k}$ and a vertex u_i in H_i by edges between $z_{j,k}$ and both u_i and u'_i in $\widehat{H_i}$. Any axial symmetry of the modified graph must clearly be FPF. The proof of Theorem 2 otherwise parallels that of Theorem 1. □

Theorem 2 also follows from the result of Lubiw [5] that the detection of order-2 FPF graph automorphisms is NP-Complete; using algebraic definitions of symmetry, it is shown in [6] that automorphisms of this type are precisely FPF axial symmetries. However, the present proof has the advantage of being easily adapted, as shown below, to produce corresponding results regarding rotational and central symmetries.

Recall that AS, RS, and CS are essentially graph automorphisms, and as such are inherent properties of *abstract* graphs. The following technical Lemma establishes a simple but useful equality by showing that the concepts of FPF AS and FPF CS, although distinct geometrically, are in fact identical from a graph-theoretic viewpoint.

Lemma 1 : *A graph automorphism is a FPF AS iff it is a FPF CS.*

Proof: Any drawing which displays a FPF AS can be transformed to display the same abstract symmetry as a FPF CS, simply by taking that portion of the graph lying on one side of the axis of symmetry and reflecting it in any line perpendicular to the axis (Figure 4 (a) → (b)). Conversely, any straight line passing through the center point of a FPF CS drawing, but avoiding all vertex disks, becomes the axis for a FPF AS upon a similar reflection (Figure 4 (b) → (c)). (A more rigorous proof of this Lemma, based on the formal algebraic definitions of symmetry, is presented in [6].) □

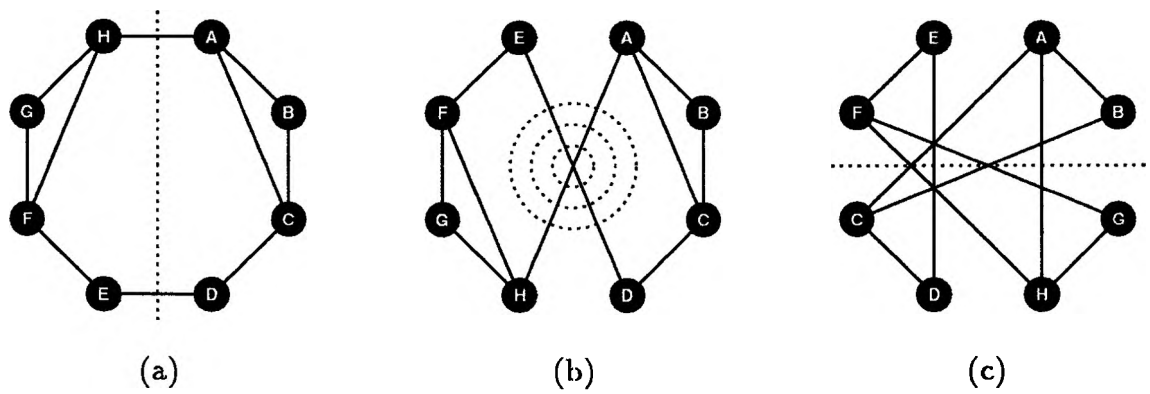

Figure 4: *Example of the equality of FPF AS and FPF CS*

Lemma 2 : *Let $\widehat{G}$ denote the modified graph constructed in the proof of Theorem 2. Then any (non-trivial) RS of $\widehat{G}$ is a FPF CS (and, by definition of CS, conversely).*

Proof: A non-trivial RS can fix at most one vertex. Now for each i, vertices v_i and w_i are distinguished in $\widehat{G}$, so such a RS must clearly interchange them. By the nature of rotational symmetry, it must then interchange *all* other vertices in pairs, except for possibly one fixed vertex, so the RS is a CS. But $\widehat{G}$ has an *even* number of vertices of any degree, so there can be no such fixed vertex; hence the CS is FPF. □

Lemmas 1 and 2 thus show that any FPF AS of $\widehat{G}$ is also a RS, a FPF RS, a CS, and a FPF CS, and conversely. Combining this with Theorem 2 yields the following:

Theorem 3 : *DRS is NP-Complete.* □

Theorem 4 : *DRS-0 is NP-Complete.* □

Theorem 5 : *DCS is NP-Complete.* □

Theorem 6 : *DCS-0 is NP-Complete.* □

Construct a graph $\widetilde{G}$ from $\widehat{G}$ by adding a new vertex y, and joining it to every vertex in $\widehat{G}$. Clearly, y is distinguished in $\widetilde{G}$ and so must be fixed under any symmetry. In particular, note that $\widetilde{G}$ has a RS or CS (fixing y) iff $\widehat{G}$ has a FPF RS or FPF CS, respectively. Combining this with Theorems 4 and 6 produces the further results:

Theorem 7 : *DRS-1 is NP-Complete.* □

Theorem 8 : *DCS-1 is NP-Complete.* □

3. #P-Completeness of Geometric Symmetry Counting

For a given problem Π, the associated *decision problem* asks whether there exists any solution for Π, while the *counting problem* asks how many such solutions exist. The class of *#P-Complete* problems can be characterized informally as consisting of all counting problems whose time complexities are polynomially equivalent to that of counting the number of satisfying truth-assignments for a general 3-CNF formula. For further details regarding #P-Completeness, consult Garey and Johnson [3].

Each of the counting problems corresponding to the decision problems of Section 2 is #P-Complete, since the reductions from 3-SAT are all *parsimonious* (see [3]):

Theorem 9 : *The problems of counting the number of axial symmetries, axial symmetries fixing no vertex, rotational symmetries, rotational symmetries fixing no vertex, rotational symmetries fixing one vertex, central symmetries, central symmetries fixing no vertex, and central symmetries fixing one vertex of a given graph, are all #P-Complete.* □

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