A New Wave Quantum Relativistic Equation from Quaternionic Representation of Maxwell-Dirac Isomorphism as an Alternative to Barut-Dirac Equation

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Abstract: It is known that Barut's equation could predict lepton and hadron mass with remarkable precision. Recently some authors have extended this equation, resulting in Barut-Dirac equation. In the present article we argue that it is possible to derive a new wave equation as alternative to Barut-Dirac's equation from the known exact correspondence (isomorphism) between Dirac equation and Maxwell electromagnetic equations via biquaternionic representation. Furthermore, in the present note we submit the viewpoint that it would be more conceivable if we interpret the *vierbein* of this equation in terms of superfluid velocity, which in turn brings us to the notion of topological electronic liquid. Some implications of this proposition include quantization of celestial systems. We also argue that it is possible to find some signatures of Bose-Einstein cosmology, which thus far is not explored sufficiently in the literature. Further experimental observation to verify or refute this proposition is recommended.

1. Introduction

It is known that Barut's equation could predict lepton and hadron mass with remarkable precision [1]. A plausible extension of Barut's equation is by using Barut-Dirac 's model via inclusion of electron self-field. Furthermore, a number of authors has extended

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this equation using non-linear field theory [2a][5][5a]. Barut's equation is as follows [5a]:

$$\left[i\gamma_\nu \partial_\nu - a\partial_\mu^2/m + \kappa \right] \Psi = 0 \tag{1}$$

where $\partial_\nu = \partial/\partial x_\nu$ and repeated indices imply a summation [5a]. The remaining parameters come from substitution of variables: $m = \kappa/\alpha_1$ and $a/m = -\alpha_2/\alpha_1$ [5a]. In the meantime Barut-Dirac-Vigier's equation could be written as:

$$\left[c\alpha.p - E + \beta(mc^2 + \in e^2/r) \right] \Psi = -[(\in \alpha\hbar e^2)/(4\pi mc^2 r^2)]i\beta\alpha\Psi \tag{2}$$

Despite this apparently remarkable result of Barut's equation, nonetheless there is question concerning the physical meaning of his equation, in particular from the viewpoint of non-linear field theory [2a]. This question seems very interesting, in particular considering the unsolved question concerning the physical meaning of *wavefunction* in Quantum Mechanics [4a]. It is known that some proponents of 'realism' interpretation of Quantum Mechanics predict that there should be a complete 'realism' description of physical model of electron, where non-local hidden variables could be included [4][1a]. We consider that this question remains open for discussion, in particular in the context of plausible analog between classical electrodynamics and non-local quantum interference effect, via Aharonov-Casher effect [8].

In the present article we argue that it is possible to derive a new wave quantum relativistic equation as an alternative to Barut-Dirac-Vigier's equation. Our description is based on the known exact correspondence (isomorphism) between Dirac equation and Maxwell electromagnetic equations via biquaternionic representation. In fact, we will discuss five approaches as alternative to Barut-Dirac equation. And we would argue that the question of which of these approaches is the most consistent with experimental data remains open. Our proposition of alternative to Barut(-Dirac) equation was based on characteristics of Barut equation:

- it is a second-order differential equation (1);
- it shall include the physical meaning of vierbein in quantum mechanical equation;
- it has neat linkage with other known equations in Quantum Mechanics including Dirac equation [5a], while its solution could be different from Dirac approach [11];
- our observation asserts that it shall also include a proper introduction of Lorentz force, and acceleration from relativistic fluid dynamics.

Furthermore, in the present note we submit the viewpoint that it would be more conceivable if we interpret the *vierbein* of this equation in terms of superfluid velocity [12][13], which in turn brings us to the notion of topological electronic liquid [27]. Its implications to quantization of celestial systems lead us to argue in favor of signatures of Bose-Einstein cosmology, which thus far has not been explored sufficiently yet in the literature [49a][49b].

What we would argue in the present note is that one could expect to extend further this quaternion representation into the form of unified wave equation, in particular using Ulrych's representation [7]. While such an attempt to interpret *vierbein* of Dirac equation has been made by de Broglie (in terms of 'Dirac fluid' [41]), it seems that an

exact representation in terms of superfluid velocity has never been made before. From this viewpoint one could argue that the superfluid vierbein interpretation will make the picture resembles superfluid bivacuum model of Kaivarainen [20][21]. Furthermore, this proposition seems to support previous hypothetical argument by Prof. J-P. Vigier on the further development of theoretical Quantum Mechanics [6]:

"..a revival, in modern *covariant form*, of the ether concept of the founding fathers of the theory of light (Maxwell, Lorentz, Einstein, *etc.*). This is a crucial question, and it now appears that the vacuum is a real physical medium, which presents surprising properties (superfluid, *i.e.* negligible resistance to inertial motions) . . . "

Provided this proposition of unified wave equation in terms of superfluid velocity *vierbein* corresponds to the observed facts, and then it could be used to predict some new observations, in particular in the context of condensed-matter analog of astrophysics [16][17][18]. Therefore in the last section we will extend this proposition to argue in favor of signatures of Bose-Einstein cosmology, including some recent relevant observation supporting this argument.

While quaternionic Quantum Mechanics has been studied before by Adler etc. [14c][28], and also biquaternionic Quantum Mechanics [2][3], it seems that interpreting the right-hand-side of the unified wave equation as superfluid 4-velocity has not been considered before, at least not yet in the context of cylindrical relativistic fluid of Carter and Sklarz-Horwitz.

In deriving these equations we will not rely on exactitude of the solutions, because as we shall see the known properties, like fine structure constant of hydrogen, can be derived from different approaches [11][15][19][22a]. Instead, we will use 'correspondence between physical theories' as a guiding principle, i.e. we argue that it is possible to derive some alternatives to Barut equation via generalization of various wave equations known in Quantum Mechanics. More linkage between these equations implies consistency.

Further experimental observation to verify or refute this proposition is recommended.

2. Biquaternion, Imaginary algebra, Unified relativistic Wave Equation

Before we discuss biquaternionic Maxwell equations from unified wave equation, first we should review Ulrych's method [7] by defining imaginary number representation as follows [7]:

$$x = x_0 + j.x_1, \quad j^2 = -1 \tag{3}$$

This leads to the multiplication and addition (or substraction) rules for any number, which is composed of real part and imaginary number:

$$(x \pm y) = (x_0 \pm y_0) + j.(x_1 \pm y_1), \tag{4}$$

$$(xy) = (x_0 y_0 + x_1 y_1) + j.(x_0 y_1 + x_1 y_0). \tag{5}$$

From these basic imaginary numbers, Ulrych [7] argues that it is possible to find a new *relativistic algebra*, which could be regarded as modified form of standard quaternion representation.

Once we define this imaginary number, it is possible to define further some relations as follows [14]. Given $w = x_0 + j.x_1$, then its D-conjugate of w could be written as:

$$\bar{w} = x_0 - j.x_1 \tag{6}$$

Also for any given two imaginary numbers $w_1, w_2 \in D$, we get the following relations [14]:

$$\overline{w_1 + w_2} = \bar{w}_1 + \bar{w}_2 \tag{7}$$

$$\overline{w_1 \bullet w_2} = \bar{w}_1 \bullet \bar{w}_2 \tag{8}$$

$$|w|^2 = \bar{w} \bullet w = x_0^2 - x_1^2 \tag{9}$$

$$|w_1 \bullet w_2|^2 = |w_1|^2 \bullet |w_2|^2 \tag{10}$$

All of these provide us nothing new. For extension of these imaginary numbers in Quantum Mechanics, see [33]. Now we will review a few elementary definitions of quaternions and biquaternions, which are proved to be useful.

It is known that biquaternions could describe Maxwell equations in its original form, and some of the use of biquaternions was discussed in [2][34].

Quaternion number, Q is defined by [33][60]:

$$Q = a + b.i + c.j + d.k \quad a, b, c, d \in R, \tag{11a}$$

where

$$i^2 = j^2 = k^2 = ijk = -1 \tag{11b}$$

Alternatively, one could extend this quaternion number to Clifford algebra [3a][3][6][25][41], because higher-dimensions Clifford algebra and analysis give the possibility to generalize the factorisations into higher spatial dimensions and even to space-time domains [70a]. In this regard quaternions H $\sim$ $C\ell_{0,2}$, while standard imaginary numbers C$\sim$ $C\ell_{0,1}$ [70a].

Biquaternion is an extension of this quaternion number, and it is described here using Hodge-bracket operator, in lieu of known Hodge operator ($** = -1$) [5a]:

$$\{Q\}* = (a + iA) + (b + iB).i + (c + iC).j + (d + iD).k, \tag{12a}$$

where the second part (A,B,C,D) is normally set to zero in standard quaternions [33].

For quaternion differential operator, we define quaternion Nabla operator:

$$\nabla^q \equiv c^{-1}.\partial/\partial t + (\partial/\partial x)i + (\partial/\partial y)j + (\partial/\partial z)k = c^{-1}.\partial/\partial t + \vec{i}.\vec{\nabla} \tag{12b}$$

And for biquaternion differential operator, we define a quaternion Nabla-Hodge-bracket operator:

$$\{\nabla^q\}* \equiv (c^{-1}.\partial/\partial t + c^{-1}.i\partial/\partial t) + \{\vec{\nabla}\}* \tag{12c}$$

where Nabla-Hodge-bracket operator is defined as:

$$\{\vec{\nabla}\}* \equiv (\partial/\partial x + i\partial/\partial X).i + (\partial/\partial y + i\partial/\partial Y).j + (\partial/\partial z + i\partial/\partial Z).k. \tag{13a}$$

It is worthnoting here that equations (4)-(10) are also applicable for biquaternion number. While equations (3)-(12a) are known in the existing literature [33][59], and sometimes called 'biparavector' (Baylis), we prefer to call it 'imaginary algebra' with emphasis on the use of Hodge-bracket operator. It is known that determinant and differentiation of quaternionic equations are different from standard differential equations [59], therefore solution for this problem has only been developed in recent years.

The Hodge-bracket operator proposed herein could become more useful if we introduce quaternion number (11a) in the paravector form [70]:

$$\vec{q} = \sum_{k=0}^{3} q_k.e_k \text{ when } \{q_k\} \subset C, \{e_k \,|\, k = 1, 2, 3\} \tag{13b}$$

and e_0 is the unit. Therefore, biquaternion number could be written in the same form [70]:

$$\{\vec{q}\}* = \vec{q} + i\vec{q} = \sum_{k=0}^{3} q_k.e_k + i\{\sum_{k=0}^{3} q_k.e_k\} \tag{13c}$$

Now we are ready to discuss Ulrych's method to describe unified wave equation [7], which argues that it is possible to define a unified wave equation in the form [7]:

$$D\phi(x) = m_\phi^2.\phi(x), \tag{14}$$

where unified (wave) differential operator D is defined as:

$$D = \left[(P - qA)_\mu \left(\bar{P} - qA \right)^\mu \right]. \tag{15}$$

To derive Maxwell equations from this unified wave equation, he uses free photon fields expression [7]:

$$DA(x) = 0, \tag{16}$$

where potential A(x) is given by:

$$A(x) = A^0(x) + jA^1(x), \tag{17}$$

and with electromagnetic fields:

$$E^i(x) = -\partial^0 A^i(x) - \partial^i A^0(x), \tag{18}$$

$$B^i(x) = \in^{ijk} \partial_j A_k(x). \tag{19}$$

Inserting these equations (17)-(19) into (16), one finds Maxwell electromagnetic equation [7]:

$$
\begin{aligned}
&-\nabla \bullet E(x) - \partial^0 C(x) \\
&+ij\nabla \bullet B(x) \\
&-j(\nabla x B(x) - \partial^0 E(x) - \nabla C(x)) \\
&-i(\nabla x E(x) + \partial^0 B(x)) = 0
\end{aligned}
\tag{20}
$$

The gauge transformation of the vector potential A(x) is given by [7]:

$$A'(x) = A(x) + \nabla\Lambda(x)/e, \tag{21}$$

where $\Lambda(x)$ is a scalar field. As equations (17)-(18) only use simple definitions of imaginary numbers (3)-(5), then an extension from (20) and (21) to biquaternionic form of Maxwell equations is possible [2][34].

In order to define biquaternionic representation of Maxwell equations, we could extend Ulrych's definition of *unified differential operator* [7] to its biquaternion counterpart, by using equation (12a), to become:

$$\{D\}* \equiv \left[(\{P\}*-q\{A\}*)_\mu \left(\{\bar{P}\}*-q\{A\}*\right)^\mu \right], \tag{22a}$$

or by definition $P = -i\hbar\nabla$ and (13a), equation (22a) could be written as:

$$\{D\}* \equiv \left[(-\hbar\{\nabla\}*-q\{A\}*)_\mu \left(-\hbar\{\nabla\}*-q\{A\}*\right)^\mu \right], \tag{22b}$$

where each component is now defined in its biquaternionic representation. Therefore the biquaternionic form of unified wave equation takes the form:

$$\{D\} * \phi(x) = m_\phi^2.\phi(x), \tag{23}$$

if we assume the wavefunction is not biquaternionic, and

$$\{D\} * \{\phi(x)\}* = m_\phi^2.\{\phi(x)\} *. \tag{24}$$

if we suppose that the wavefunction also takes the same biquaternionic form.

Now, biquaternionic representation of free photon fields could be written in the same way with (16), as follows:

$$\{D\} * A(x) = 0 \tag{25}$$

We will not explore here complete solution of this biquaternion equation, as it has been discussed in various literatures aforementioned above, including [2][33][34][59]. However, immediate implications of this biquaternion form of Ulrych's unified equation can be described as follows.

Ulrych's fermion wave equation in the presence of electromagnetic field reads [7]:

$$\left[(P - qA)_\mu \left(\bar{P} - qA\right)^\mu \psi \right] = -m^2.\psi, \tag{26}$$

which asserts c=1 (conventionally used to write wave equations). In accordance with Ulrych [7] this equation implies that the differential operator of the quantum wave equation (LHS) is composed of the momentum operator P multiplied by its dual operator, and taking into consideration electromagnetic field effect qA. And by using definition of momentum operator:

$$P = -i\hbar\nabla. \tag{27}$$

So we get three-dimensional relativistic wave equation [7]:

$$\left[\left(-i\hbar\nabla_\mu - qA_\mu\right)\left(-i\hbar\nabla^\mu - qA^\mu\right)\psi\right] = -m^2.c^2.\psi. \tag{28}$$

which is Klein-Gordon equation. Its 1-dimensional version has also been derived by Nottale [67, p,29]. A plausible extension of equation (28) using biquaternion differential operator defined above (22a) yields:

$$\left[\left(-\hbar\{\nabla_\mu\} * -q\{A_\mu\}*\right)\left(-\hbar\{\nabla^\mu\} * -q\{A^\mu\}*\right)\psi\right] = -m^2.c^2.\psi, \tag{29}$$

which could be called as 'biquaternionic' Klein-Gordon equation.

Therefore we conclude that there is neat correspondence between Ulrych's fermion wave equation and Klein Gordon equation, in particular via biquaternionic representation. It is also worthnoting that it could be shown that Schrodinger equation could be derived from Klein-Gordon equation [11], and Klein-Gordon equation also neatly corresponds to Duffin-Kemmer-Petiau equation. Furthermore it could be proved that modified (quaternion) Klein-Gordon equation could be related to Dirac equation [7]. All of these linkages seem to support argument by Gursey and Hestenes who find plenty of interesting features using quaternionic Dirac equation. In this regard, Meessen has proposed a method to describe elementary particle from Klein-Gordon equation [30].

By assigning imaginary numbers to each component [7, p.26], equation (26) could be rewritten as follows (by writing c=1):

$$\left[(P - qA)_\mu (P - qA)^\mu - eE^i ij\sigma_i - eB^i\sigma_i + m^2\right]\psi = 0, \tag{30}$$

where Pauli matrices σ_i are written explicitly. Now it is possible to rewrite equation (30) in complete tensor formalism [7], if Pauli matrices and electromagnetic fields are expressed with antisymmetric tensor, so we get:

$$\left[(P - qA)_\mu \left(\bar{P} - qA\right)^\mu - e\sigma_{\mu\nu}F^{\mu\nu} + m^2\right]\psi = 0, \tag{31}$$

where

$$F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu). \tag{32}$$

Note that equation (31) is formal identical to *quadratic form* of Dirac equation [7], which supports argument suggesting that modified (quaternion) Klein-Gordon equation could be related to Dirac equation. Interestingly, equation (31) is also known in the literature as Feynman-Gell-Mann's equation, and its implications will be discussed in subsequent section [5]. Interestingly, if we neglect contribution of the electromagnetic field (q and e) component, and using only 1-dimensional of the partial differentiation, one gets a wave equation from Feynman rules [56, p.6]:

$$\left(\partial_\mu\partial^\mu + m^2\right)\Psi = 0, \tag{33}$$

which has been used to describe quantum-electrodynamics without renormalization [56].

Further extension of equation (28) could be made by expressing it in terms of 4-velocity:

$$[(-i\hbar\nabla_\mu - qA_\mu)(-i\hbar\nabla^\mu - qA^\mu)\psi] = -p_\mu p^\mu.\psi. \tag{34}$$

In the context of relativistic fluid [10][11], one could argue that this 4-velocity corresponds to superfluid vierbein [13][16][17]. Therefore we could use Carter-Langlois' equation [12]:

$$\mu_\rho.\mu^\rho = -c^2.\mu^2, \tag{35}$$

by replacing m with the effective mass variable μ. This equation has the meaning of cylindrically symmetric superfluid with known metric [12]:

$$g_{\rho\sigma}.dx^\rho.dx^\sigma = -c^2.dt^2 + dz^2 + r^2.d\phi^2 + dr^2. \tag{36}$$

Further extension of equation (35) is possible, as discussed by Fischer [13], where the effective mass variable term also appears in the LHS of velocity equation, by defining momentum of the continuum as:

$$p_\alpha = \mu.u_\alpha. \tag{37}$$

Therefore equation (35) now becomes:

$$\mu^2.u_\alpha.u^\alpha = -c^2.\mu^2, \tag{38}$$

where the effective mass variable now acquires the meaning of chemical potential [13]:

$$\mu = \partial \in /\partial\rho, \tag{39}$$

and

$$\rho.p_\alpha/\mu = (K/\hbar^2) p_\alpha = j_\alpha, \tag{40}$$

$$K = \hbar^2 (\rho/\mu). \tag{41a}$$

The quantity K is defined as the stiffness coefficient against variations of the order parameter phase. Alternatively, from macroscopic dynamics of Bose-Einstein condensate containing vortex lattice, one could write the chemical potential in the form [57]:

$$\mu = \mu_0.\left[1 - (\Omega_0/\omega_\perp)^2\right]^{2/5} \tag{41b}$$

where the quantity Ω corresponds to the angular frequency of the sample and is assumed to be uniform, ω is the oscillator frequency, and chemical potential in the absence of rotation is given by [57]:

$$\mu_0 = (\hbar\omega_{ho}/2)(Na/0.0667a_{ho})^{2/5} \tag{41c}$$

and N represents the number of atoms and a is the corresponding oscillator length [57]:

$$a_{ho} = \sqrt{\hbar/M\omega_{ho}} \tag{41d}$$

Now the sound speed c_s could be related to the equations above, for a barotropic fluid [13], as:

$$c_s = d\left(\ln\mu\right)/d\left(\ln\rho\right) = \left(K/\hbar^2\right) d^2 \in /d\rho^2. \qquad (42)$$

Using this definition, then equation (42) could be rewritten as follows:

$$p_\alpha = \left(K^{-1}\hbar^2\right) j_\alpha = (j_\alpha/c_s).d^2 \in /d\rho^2, \qquad (43)$$

Introducing this result (43) into equation (34), we get:

$$[(-i\hbar\nabla_\mu - qA_\mu)(-i\hbar\nabla^\mu - qA^\mu)\psi] = -\left((j_\alpha/c_s).d^2 \in /d\rho^2\right)^2 .\psi \qquad (44)$$

which is an alternative expression of relativistic wavefunction in terms of superfluid sound speed, c_s. Note that this equation could appear only if we interpret 4-velocity in terms of *superfluid vierbein* [11][12]. Therefore this equation is Klein-Gordon equation, where vierbein is defined in terms of superfluid velocity. Alternatively, in condition without electromagnetic charge, then we can rewrite equation (44) in the known form of standard Klein-Gordon equation [36]:

$$[D_\mu D^\mu \psi] = -\left((j_\alpha/c_s).d^2 \in /d\rho^2\right)^2 .\psi. \qquad (45)$$

Therefore, this alternative representation of Klein-Gordon equation (45) has the physical meaning of relativistic wave equation for superfluid phonon [37][38].

A plausible extension of (44) is also possible using our definition of biquaternionic differential operator (22a):

$$\boxed{\{D\} * \psi = -\left((j_\alpha/c_s).d^2 \in /d\rho^2\right)^2 \psi \qquad (46)}$$

which is an alternative expression from Ulrych's [7] unified relativistic wave equation, where the vierbein is defined in terms of superfluid sound speed, c_s. This is the main result of this section. As alternative, equation (46) could be written in compact form:

$$[\{D\} * +\Gamma]\Psi = 0, \qquad (47)$$

where the operator Γ is defined according to the quadratic of equation (43):

$$\Gamma = \left((j_\alpha/c_s).d^2 \in /d\rho^2\right)^2 . \qquad (48a)$$

For the solution of equation (44)-(47), one could refer for instance to alternative description of quarks and leptons via SU(4) symmetry [28][58]. As we note above, equation (31) is also known in the literature as Feynman-Gell-Mann's equation, and it has been argued that it has neat linkage with Barut equation [5]. This assertion could made more conceivable by noting that equation (31) is quadratic form of Dirac equation. In this regard, recently Kruglov has considered a plausible generalization of Barut equation via third-order differential extension of Dirac equation [60]:

$$\left(\gamma_\mu\partial_\mu + m_1\right)\left(\gamma_\nu\partial_\nu + m_2\right)\left(\gamma_\alpha\partial_\alpha + m_3\right)\psi = 0. \qquad (48b)$$

It is also interesting to note that in his previous work, Kruglov [60a] has argued in favor of Dirac-Kahler equation:

$$(d - \delta + m)\,\psi = 0, \tag{48c}$$

where the operator $(d - \delta)$ is the analog of Dirac operator $\gamma_\mu \partial^\mu$. It seems plausible, therefore, in the context of Kruglov's recent attempt to generalize Barut equation [60] to argue that further generalization to biquaternionic form is possible by rewriting equation (47) in the third-order equation, by using our definition (12c):

$$[\{\vec{\nabla}^q_\mu\} * + p_\mu][\{\vec{\nabla}^q_\nu\} * + p_\nu][\{\vec{\nabla}^q_\alpha\} * + p_\alpha]\Psi = 0. \tag{48d}$$

Therefore, we could consider this equation as the first alternative to (generalized) Barut equation. Note that we use here equation (12c) instead of (22a), in accordance with Kruglov [60] definition:

$$\partial_\nu = \partial/\partial x_\nu = (\partial/\partial x_m, \partial/\partial(it)) \tag{48e}$$

In subsequent sections, we will consider a number of other plausible alternatives to Barut-Dirac's equation, in particular from the viewpoint of superfluid vierbein.

3. Alternative #2: Barut-Dirac-Feynman-Gell-Mann Equation

It is argued [5, p. 4] that Barut equation is the sum of Dirac equation and Feynman-Gell-Mann's equation (31). But from the aforementioned argument, it should be clear that the Feynman-Gell-Mann's equation is nothing more than Ulrych's fermion wave equation, which is indeed a quadratic of Dirac equation. Therefore, it seems that there should be other route to derive Barut-Dirac type equation. In this regard, we submit the viewpoint that the introduction of electron self-field would lead to an alternative of Barut equation.

First, let us rewrite equation (31) with assigning the real c in lieu of c=1:

$$\left[(P - qA)_\mu \left(\bar{P} - qA\right)^\mu - e\sigma_{\mu\nu}F^{\mu\nu} + m^2 c^2\right]\psi = 0, \tag{49}$$

By using equation (34), then Feynman-Gell-Mann's equation becomes:

$$\left[(-i\hbar\nabla_\mu - qA_\mu)\left(-i\hbar\nabla^\mu - qA^\mu\right) - e\sigma_{\mu\nu}F^{\mu\nu} + p_\mu p^\mu\right]\Psi = 0, \tag{50}$$

or

$$\left[(-i\hbar\nabla_\mu - qA_\mu)\left(-i\hbar\nabla^\mu - qA^\mu\right) + p_\mu p^\mu\right]\Psi = (e\sigma_{\mu\nu}F^{\mu\nu})\Psi, \tag{51}$$

which can be called Feynman-Gell-Mann's equation with superfluid vierbein interpretation, in particular if we then introduce equation (43) into the LHS.

In this regard, we can introduce Ibison's description of electron self-energy from ZPE [38]:

$$e\sigma_{\mu\nu}F^{\mu\nu} = m_0 a^\mu - m_0 \tau_0 \left[da^\lambda/d\tau + a^\lambda a_\lambda u^\mu/c^2\right] \tag{52}$$

where

$$\tau_0 = e^2/6\pi\varepsilon_0 m_0 c^3 \tag{53}$$

The first term in the right hand side of equation (52) could be written in the Lorentz form [42] [24a, p.12]:

$$m_0 a^\mu = m[dv/dt] = e[E + vxB] \tag{54}$$

where:

$$E = -\nabla\phi, \tag{55}$$

$$B = \nabla xA. \tag{56}$$

Therefore, by defining a new parameter [24a, p.12]:

$$\forall = e[E + vxB]^\mu - m_0(e^2/6\pi\varepsilon_0 m_0 c^3)\left[da^\lambda/d\tau + a^\lambda a_\lambda u^\mu/c^2\right], \tag{57}$$

one could rewrite equation (51) in term of equation (43):

$$\left[\left(-i\hbar\nabla_\mu - qA_\mu\right)\left(-i\hbar\nabla^\mu - qA^\mu\right) + \left((j_\alpha/c_s).d^2\in/d\rho^2\right)^2\right]\Psi = \forall\Psi, \tag{58}$$

which could be regarded as a second alternative expression of Barut equation. Therefore we propose to call it Barut-Dirac-Feynman-Gell-Mann equation. Implications of this equation should be verified via experiments, in particular with condensed-matter physics.

4. Alternative #3: Second Order Differential Form of Schrödinger-Type Equation

It is known that Barut equation is a typical second-order differential equation, which is therefore non-linear. Therefore a good alternative to Barut equation could be derived from similar approach with Schrödinger's original equation, but this time it should be differentiated twice.

In this regard, it seems worthnoting here that it is more proper to use Noether's expression of total energy in lieu of standard derivation of Schrödinger's equation ($E = \vec{p}^2/2m$). According to Noether's theorem [39], the total energy of the system corresponding to the time translation invariance is given by:

$$E = mc^2 + (cw/2).\int_0^\infty \left(\gamma^2.4\pi r^2.dr\right) = k\mu c^2 \tag{59}$$

where k is *dimensionless* function. It could be shown, that for low-energy state the total energy could be far less than $E = mc^2$. Interestingly Bakhoum [22] has also argued in favor of using $E = mv^2$ for expression of total energy, which expression could be traced back to Leibniz. Therefore it seems possible to argue that expression $E = mv^2$ is more generalized than the standard expression of special relativity, in particular because the total energy now depends on actual velocity [39].

From this new expression, it is plausible to rederive quantum relativistic wave equation in second-order differential expression, and it turns out the new equation should also include a Lorentz-force term in the same way of equation (57). This feature is seemingly interesting, because these equations are derived from different approach from (57).

We start with Bakhoum's assertion that it is more appropriate to use $E = mv^2$, instead of more convenient form $E = mc^2$. This assertion would imply [22]:

$$H^2 = p^2.c^2 - m_o^2.c^2.v^2. \tag{60}$$

Therefore, for phonon speed (c_s) in the limit $p \to 0$, we write [37]:

$$E(p) \equiv c_s. \, |p| \,. \tag{61}$$

A bit remark concerning Bakhoum's expression, it does not mean to imply or to interpret $E = mv^2$ as an assertion that it implies zero energy for a rest mass. Actually the problem comes from 'mixed' interpretation of what we mean with 'velocity'. In original Einstein's paper (1905) it is defined as 'kinetic velocity', which can be measured when standard 'steel rod' has velocity approximates the speed of light. But in quantum mechanics, we are accustomed to make use it deliberately to express 'photon speed'=c. According to Bakhoum, to get a consistent interpretation between special relativity and quantum mechanics, we should treat this definition of velocity according to its context, in particular to its linkage with electromagnetic field. Therefore, in special relativity 1905 paper, it should be better to interpret it as 'speed of free electron', which approximates c. For muon, Spohn [42] has obtained v=0.9997c which is very near to c, but not exactly =c. For hydrogen atom with 1 electron, the electron occupies the first excitation (quantum number n=1), which implies that their speed also approximate c, which then it is quite safe to assume $E \sim mc^2$. But for atoms with large amount of electrons occupying large quantum numbers, as Bakhoum showed that electron speed could be far less than c, therefore it will be more exact to use $E = mv^2$, where here v should be defined as 'average electron speed'. Furthermore, in the context of relativistic fluid, we could use $E_\alpha = \mu.u_\alpha u_\alpha$ from equation (37).

In the first approximation of relativistic wave equation, we could derive Klein-Gordon-type relativistic equation from equation (60), as follows. By introducing a new parameter:

$$\zeta = i(v/c), \tag{62}$$

then we can rewrite equation (60) in the known procedure of Klein-Gordon equation:

$$E^2 = p^2.c^2 + \zeta^2 m_o^2.c^4, \tag{63}$$

where $E = mv^2$. [22] By using known substitution:

$$E = i\hbar.\partial/\partial t, \quad p = \hbar\nabla/i, \tag{64}$$

and dividing by $(\hbar c)^2$, we get Klein-Gordon-type relativistic equation:

$$-c^{-2}\partial\Psi/\partial t + \nabla^2\Psi = k_o^{'\,2}\Psi, \tag{65}$$

where

$$k_o^{'} = \zeta m_o c/\hbar. \tag{66}$$

One could derive Dirac-type equation using similar method. But the use of new parameter (62) seems to be indirect, albeit it simplifies the solution because here we can use the same solution from Klein-Gordon equation [30].

Alternatively, one could derive a new quantum relativistic equation, by noting that expression of total energy $E = mv^2$ is already relativistic equation. We will derive here two approaches to get relativistic wave equation from this expression of total energy.

The first approach, is using Ulrych's [7] method as follows:

$$E = mv^2 = p.v \tag{67}$$

Taking square of this expression, we get:

$$E^2 = p^2.v^2 \tag{68}$$

or

$$p^2 = E^2/v^2 \tag{69}$$

Now we use Ulrych's substitution [7]:

$$\left[(P - qA)_\mu \left(\bar{P} - qA \right)^\mu \right] = p^2, \tag{70}$$

and introducing standard substitution in Quantum Mechanics (64), one gets:

$$\left[(P - qA)_\mu \left(\bar{P} - qA \right)^\mu \right] \Psi = v^{-2}.(i\hbar.\partial/\partial t)^2 \Psi, \tag{71}$$

or

$$\left[(-i\hbar\nabla_\mu - qA_\mu) \left(-i\hbar\nabla^\mu - qA^\mu \right) - (i\hbar/v.\partial/\partial t)^2 \right] \Psi = 0. \tag{72a}$$

which can be called as Noether-Ulrych-Feynman-Gell-Mann's (NUFG) equation. This is the third alternative to Barut-Dirac equation.

Alternatively, by using standard definition p=m.v, we can rewrite equation (71) in form of equation (43):

$$\left[(P - qA)_\mu \left(\bar{P} - qA \right)^\mu \right] \Psi = m^2 \left((j_\alpha/c_s).d^2 \in /d\rho^2 \right)^{-2} .(i\hbar.\partial/\partial t)^2 \Psi. \tag{72b}$$

In order to verify that we can use the same method with Schrödinger equation to derive nonlinear wave equation, let us consider Oleinik's nonlinear wave equation. It is argued that the proper equation of motion is not the Dirac or Schrödinger equation, but an equation with a new self-energy term [24]. This would mean that there is a pair wavefunction to include electron interaction with its surrounding medium. Therefore, the standard Schrödinger equation becomes nonlinear equations of motion [24]:

$$\left[i\partial/\partial t + \bar{\nabla}^2/2m - U(x) \right] \left(\frac{\Psi(x)}{\bar{\Psi}(x)} \right) = 0 \tag{73}$$

where we use $\hbar = 1$ for convenience.

From this equation, one can get the relativistic version corresponding to Dirac equation [24]. Interestingly, Froelich [66] has considered equation of motion for the few-body

systems associated with the hydrogen-antihydrogen pairs using radial Schrödinger-type equation. Therefore, it seems interesting to consider equation (73) also in the context of hydrogen-antihydrogen molecule.

And because equation (73) is derived from the standard definition of total energy $E = \vec{p}^2/2m$, then our method to use equation (60) seems to be a logical extension of Oleinik's method. To get nonlinear version similar to equation (73), then we could rewrite equation (72a) as:

$$\left[(-i\hbar\nabla_\mu - qA_\mu)(-i\hbar\nabla^\mu - qA^\mu) - (i\hbar/v.\partial/\partial t)^2 \right] \left(\frac{\Psi(x)}{\bar{\Psi}(x)} \right) = 0. \qquad (74)$$

What's more interesting here, is that Oleinik [24a, p.12] has shown that equation (73) could lead to an expression of Newtonian-Lorentz force similar to equation (54):

$$m_0 a^\mu = m[d^2r/dt^2] = e[E + v \times B] \qquad (75)$$

This verifies our aforementioned proposition that a good alternative to Barut's equation should include a Lorentz-force term in wave equation. In other words, from equation (73) we find neat linkage between Schrödinger equation, nonlinear wave, and Lorentz-force. We will use this linkage in the following section. It turns out that we can find a proper generalization of Barut's equation via introduction of Newtonian-acceleration from velocity of the relativistic fluid in similar form of Lorentz force.

5.   Alternative #4: Lorentz-force & Newtonian Acceleration Method

For the fourth method, we will introduce Leibniz rule [40] into equation (67) via differentiation with respect to time, which yields:

$$dE/dt = d[p.v]/dt = v.[dp/dt] + p.[dv/dt] \qquad (76)$$

The next step is taking derivation of the known substitution in QM:

$$dE/dt = i\hbar.\partial^2/\partial t^2, \qquad (77)$$

$$dp/dt = d(-i\hbar\nabla)/dt = -i\hbar\dot{\nabla}$$

Now, substituting back equation (77) and (64) into equation (76), we get:

$$(i\hbar.\partial^2/\partial t^2)\Psi = (v.[-i\hbar\dot{\nabla}] - [dv/dt].i\hbar\nabla)\Psi. \qquad (78)$$

At this point, we note that the second term in the right hand side of equation (78) could be written in the Lorentz force form [42], and following equation (54):

$$[dv/dt] = e/m.(E + vxB) \qquad (79)$$

where:

$$E = -\nabla\phi, \qquad (80)$$

$$B = \nabla x A. \tag{81}$$

Therefore, we can rewrite equation (78) in the form:

$$(i\hbar.\partial^2/\partial t^2)\Psi = (v.[-i\hbar\dot{\nabla}] - e/m.[E + vxB].i\hbar\nabla)\Psi, \tag{82}$$

which is a new wave relativistic quantum equation as alternative to Barut equation. To our present knowledge, this alternative wave equation (82) has never been derived elsewhere.

As an alternative to equation (79), we can rewrite Lorentz form in term of Newtonian acceleration. In this regard, it is worthnoting that the definition of acceleration of relativistic fluid is not widely accepted yet [10]. Therefore we will use here result from relativistic field equations from Poisson process [46], from which we get an expression of acceleration [46]:

$$dv/dt = \hbar/2m.(\partial^2 u/\partial x^2) - v.\partial u/\partial x + u.\partial v/\partial x - m^{-1}.\partial V/\partial x = \exists \tag{83}$$

Therefore, by substituting this equation into (78), we get:

$$(i\hbar.\partial^2/\partial t^2)\Psi = (v.[-i\hbar\dot{\nabla}] - \exists.i\hbar\nabla)\Psi, \tag{84}$$

which can be considered as a better alternative to equation (82).

6. Alternative #5: Schrödinger-Ginzburg-Landau Equation and Quantization of Celestial Systems

In the preceding section (#4), we have found the neat linkage between Schrödinger equation, nonlinear wave, and Lorentz-force, which indicates a possibility to be considered as alternative to Barut equation. Now, as the fifth alternative method, it will be shown that we can expect to generalize Schrödinger equation to describe quantization of celestial sytems. While this notion of macro-quantization is not widely accepted yet, as we will see the logarithmic nature of Schrödinger equation is sufficient to ensure its applicability to larger systems. As alternative, we will also discuss an outline for deriving Schrödinger equation from simplification of Ginzburg-Landau equation. It is known that Ginzburg-Landau equation exhibits fractal character.

First, let us rewrite Schrödinger equation (73) in its common form:

$$\left[i\partial/\partial t + \bar{\nabla}^2/2m - U(x)\right]\Psi = 0 \tag{85}$$

where we use $\hbar = 1$ for convenience, or

$$(i\partial/\partial t)\Psi = H.\Psi \tag{86}$$

Now, it is worthnoting here that Englman & Yahalom [4a] argue that this equation exhibits logarithmic character:

$$\ln\Psi(x,t) = \ln\left(|\Psi(x,t)|\right) + i.\arg(\Psi(x,t)) \tag{87}$$

Schrödinger already knew this expression in 1926, which then he used it to propose his equation called '*eigentliche Wellengleichung*' [4a]. Therefore equation (85) can be rewritten as follows:

$$2m(\partial \ln |\Psi| / \partial t) + 2\bar{\nabla} \ln |\Psi| . \bar{\nabla} \arg[\Psi] + \bar{\nabla}.\bar{\nabla} \arg[\Psi] = 0 \qquad (88)$$

Interestingly, Nottale's scale-relativistic method [43][44] was also based on generalization of Schrödinger equation to describe quantization of celestial systems. It is known that Nottale-Schumacher's method [45] could predict new exoplanets in good agreement with observed data. Nottale's scale-relativistic method is essentially based on the use of first-order scale-differentiation method defined as follows [43][44]:

$$\partial V/\partial(\ln \delta t) = \beta(V) = a + bV + ... \qquad (89)$$

Now it seems clear that the logarithmic derivation, which is essential in scale-relativity approach, also has been described properly in Schrödinger's original equation [4a]. In other word, its logarithmic form ensures applicability of Schrödinger equation to describe macroquantization of celestial systems.

To emphasize this assertion of the possibility to describe quantization of celestial systems, let us return for a while to the preceding section where we use Fischer' description [13] of relativistic momentum of 4-velocity (37)-(38). Interestingly Fischer [13] argues that the circulation leading to equation (37)-(38) is in the relativistic dense superfluid, defined as the integral of the momentum:

$$\gamma_s = \oint p_\mu dx^\mu = 2\pi.N_v\hbar, \qquad (90)$$

and is quantized into multiples of Planck's quantum of action. This equation is the co-variant Bohr-Sommerfeld quantization of γ_s. And then Fischer [13] concludes that the *Maxwell equations of ordinary electromagnetism can be cast into the form of conservation equations of relativistic perfect fluid hydrodynamics* [10], in good agreement with Vigier's guess as mentioned above. Furthermore, the topological character of equation (90) corresponds to the notion of topological electronic liquid, where compressible electronic liquid represents superfluidity [27].

It is worthnoting here, because here vortices are defined as elementary objects in the form of stable topological excitations [13], then equation (90) could be interpreted as signatures of Bohr-Sommerfeld quantization from topological quantized vortices. Fischer [13] also remarks that equation (90) is quite interesting for the study of superfluid rotation in the context of gravitation. Interestingly, application of Bohr-Sommerfeld quantization to celestial systems is known in literature [47][48], which here in the context of Fischer's arguments it seems plausible to suggest that quantization of celestial systems actually corresponds to superfluid-quantized vortices at large-scale [27]. In our opinion, this result supports known experiments suggesting neat correspondence between condensed matter physics and various cosmology phenomena [16]-[19].

To make the conclusion that quantization of celestial systems actually corresponds to superfluid-quantized vortices at large-scale a bit conceivable, let us consider an illustration of quantization of celestial orbit in solar system.

In order to obtain planetary orbit prediction from this hypothesis we could begin with the Bohr-Sommerfeld's conjecture of quantization of angular momentum. This conjecture may originate from the fact that according to BCS theory, superconductivity can exhibit macroquantum phenomena [16][65]. In principle, this hypothesis starts with observation that in quantum fluid systems like superfluidity, it is known that such vortexes are subject to quantization condition of integer multiples of 2π, or $\oint v_s.dl = 2\pi.n\hbar/m_4$. As we know, for the wavefunction to be well defined and unique, the momenta must satisfy Bohr-Sommerfeld's quantization condition:

$$\oint_{\Gamma} p.dx = 2\pi.n\hbar \tag{91}$$

for any closed classical orbit Γ. For the free particle of unit mass on the unit sphere the left-hand side is [49]:

$$\int_{0}^{T} v^2.d\tau = \omega^2.T = 2\pi.\omega \tag{92}$$

where T=$2\pi/\omega$ is the period of the orbit. Hence the quantization rule amounts to quantization of the rotation frequency (the angular momentum):$\omega = n\hbar$. Then we can write the force balance relation of Newton's equation of motion [49]:

$$GMm/r^2 = mv^2/r \tag{93}$$

Using Bohr-Sommerfeld's hypothesis of quantization of angular momentum, a new constant g was introduced:

$$mvr = ng/2\pi \tag{94}$$

Just like in the elementary Bohr theory (before Schrödinger), this pair of equations yields a known simple solution for the orbit radius for any quantum number of the form [49]:

$$r = n^2.g^2/(4\pi^2.GM.m^2) \tag{95}$$

which can be rewritten in the known form [43][44]:

$$r = n^2.GM/v_o^2 \tag{96}$$

where r, n, G, M, v_o represents orbit radii, quantum number (n=1,2,3,...), Newton gravitation constant, and mass of the nucleus of orbit, and specific velocity, respectively. In this equation (96), we denote:

$$v_o = (2\pi/g).GMm \tag{97}$$

The value of m is an adjustable parameter (similar to g). [43][44]

Using this equation (96), we could predict quantization of celestial orbits in the solar system, where for Jovian planets we use least-square method and define M in terms of reduced mass $\mu = (M_1.M_2)/(M_1 + M_2)$. From this viewpoint the result is shown in Table 1 below [49]:

Table 1. Comparison of prediction and observed orbit distance of
planets in Solar system (in 0.1 AU unit) [49]

Object	No.	Bode	Nottale	CSV	Observed	$\Delta(\%)$
	1		0.4	0.428		
	2		1.7	1.71		
Mercury	3	4	3.9	3.85	3.87	0.52
Venus	4	7	6.8	6.84	7.32	6.50
Earth	5	10	10.7	10.70	10.00	-6.95
Mars	6	16	15.4	15.4	15.24	-1.05
Hungarias	7		21.0	20.96	20.99	0.14
Asteroid	8		27.4	27.38	27.0	1.40
Camilla	9		34.7	34.6	31.5	-10.00
Object	No.	Bode	Nottale	CSV	Observed	$\Delta(\%)$
Jupiter	2	52		45.52	52.03	12.51
Saturn	3	100		102.4	95.39	-7.38
Uranus	4	196		182.1	191.9	5.11
Neptune	5			284.5	301	5.48
Pluto	6	388		409.7	395	-3.72
2003EL61	7			557.7	520	-7.24
Sedna	8	722		728.4	760	4.16
2003UB31	9			921.8	970	4.96
Unobserved	10			1138.1		
Unobserved	11			1377.1		

For comparison purpose, we also include some recent observation by M. Brown *et al.* from Caltech [50][51][52][53]. It is known that Brown *et al.* have reported not less than four new planetoids in the outer side of Pluto orbit, including 2003EL61 (at 52AU), 2005FY9 (at 52AU), 2003VB12 (at 76AU, dubbed as *Sedna.*) And recently Brown and his team reported a new planetoid finding, called 2003UB31 (97AU). This is not to include *Quaoar* (42AU), which has orbit distance more or less near Pluto (39.5AU), therefore this object is excluded from our discussion. It is interesting to remark here that all of those

new 'planetoids' are within 8% bound from our prediction of celestial quantization based on the above Bohr-Sommerfeld quantization hypothesis (Table 1). While this prediction is not so precise compared to the observed data, one could argue that the 8% bound limit also corresponds to the remaining planets, including inner planets. Therefore this 8% uncertainty could be attributed to macroquantum uncertainty and other local factors.

While our previous prediction only limits new planet finding until n=9 of Jovian planets (outer solar system), it seems that there are enough reasons to suppose that more planetoids are to be found in the near future. Therefore it is recommended to extend further the same quantization method to larger n values. For prediction purpose, we include in Table 1 new expected orbits based on the same quantization procedure we outlined before. For Jovian planets corresponding to quantum number n=10 and n=11, our method suggests that it is likely to find new orbits around 113.81 AU and 137.71 AU, respectively. It is recommended therefore, to find new planetoids around these predicted orbits.

As an interesting alternative method supporting this proposition of quantization from superfluid-quantized vortices (90), it is worthnoting here that Kiehn has argued in favor of re-interpreting the square of the wavefunction of Schrödinger equation as the vorticity distribution (including topological vorticity defects) in the fluid [61]. From this viewpoint, Kiehn suggests that there is *exact mapping* from Schrödinger equation to Navier-Stokes equation, using the notion of quantum vorticity [61]. Interestingly, de Andrade & Sivaram [62] also suggest that there exists formal analogy between Schrödinger equation and the Navier-Stokes viscous dissipation equation:

$$\partial V/\partial t = \nu.\nabla^2 V \qquad (98)$$

where ν is the kinematic viscosity. Their argument was based on propagation torsion model for quantized vortices [62]. While Kiehn's argument was intended for ordinary fluid, nonetheless the neat linkage between Navier-Stokes equation and superfluid turbulence is known in literature [63][64][21].

Therefore, it seems interesting to consider a plausible generalization of Schrödinger equation in particular in the context of viscous dissipation method. First, we could write Schrödinger equation for a charged particle interacting with an external electromagnetic field [61] in the form of equation (28) and (85):

$$[(-i\hbar\nabla_\mu - qA_\mu)(-i\hbar\nabla^\mu - qA^\mu)\Psi] = [-i2m.\partial/\partial t + 2mU(x)]\Psi. \qquad (99)$$

In the presence of electromagnetic potential [69], one could include another term into the LHS of equation (99):

$$[(-i\hbar\nabla_\mu - qA_\mu)(-i\hbar\nabla^\mu - qA^\mu) + eA_o]\Psi = 2m[-i\partial/\partial t + U(x)]\Psi. \qquad (100)$$

This equation has the physical meaning of Schrödinger equation for a charged particle interacting with an external electromagnetic field, which takes into consideration Aharonov effect [69]. Topological phase shift becomes its immediate implication, as already considered by Kiehn [61].

Therefore, in the context of quaternionic representation of Schrödinger equation [70], one could write equation (100) in terms of equation [22a]:

$$[\{D\} * + eA_o]\,\Psi = 2m\,[-i\partial/\partial t + U(x)]\,\Psi. \tag{101}$$

In the context of topological phase shift [69], it would be interesting therefore to find the scalar part of equation (101) in experiments [8].

As described above, one could also derive equation (96) from scale- relativistic Schrödinger equation [43][44]. It should be noted here, however, that Nottale's method [43][44] differs appreciably from the viscous dissipative Navier-Stokes approach of Kiehn, because Nottale only considers his equation in the Euler-Newton limit [67][68]. Nonetheless, as we shall see, it is possible to find a generalization of Schrödinger equation from Nottale's approach in similar form with equation (101). In order to do so, first we could rewrite Nottale's generalized Schrödinger equation via diffusion method [67][71]:

$$
\begin{aligned}
i2m\gamma\,&\big[-(i\gamma + a(t)/2)\,(\partial\psi/\partial x)^2\,\psi^{-2} + \partial\ln\psi/\partial t\big]\\
&+i\gamma a(t).\,(\partial^2\psi/\partial x^2)\,/\psi = \Phi + a(x)
\end{aligned}
\tag{102}
$$

where $\psi, a(x)$, Φ, γ each represents classical wave function, an arbitrary constant, scalar potential, and a constant, respectively. If the function f(t) is such that

$$a(t) = -i2\gamma, \quad \alpha(x) = 0, \tag{103}$$

$$\gamma = \hbar/2m \tag{104}$$

then one recovers the original Schrödinger equation (85).

Further generalization is possible if we rewrite equation (102) in quaternion form similar to equation (101):

$$
\begin{aligned}
i2m\gamma\,&\big[-(i\gamma + a(t)/2)\,(\{\nabla\}*)^2\psi^{-2} + \partial\ln\psi/\partial t\big]\\
&+i\gamma.a(t).\,(\{\nabla'\}*)\,/\psi = \Phi + a(x)
\end{aligned}
\tag{105}
$$

Alternatively, with respect to our superfluid dynamics interpretation [13], one could also get Schrödinger equation from simplification of Ginzburg-Landau equation. This method will be discussed subsequently. It is known that Ginzburg-Landau equation can be used to explain various aspects of superfluid dynamics [16][17][18].

According to Gross, Pitaevskii, Ginzburg, wavefunction of N bosons of a reduced mass m* can be described as [55]:

$$-(\hbar^2/2m*).\nabla^2\psi + \kappa\,|\psi|^2\,\psi = i\hbar.\partial\psi/\partial t \tag{106}$$

For some conditions (where the temperature dependence of the density of Cooper pairs, n_s, is just the square of order parameter. Or $|\psi|^2 \approx n_s = A(T_c - T)$), then it is possible

to replace the potential energy term in equation (106) with Hulthen potential. This substitution yields:

$$-(\hbar^2/2m*).\nabla^2\psi + V_{Hulthen}.\psi = i\hbar.\partial\psi/\partial t \tag{107}$$

where

$$V_{Hulthen}(r) = \kappa\,|\psi|^2 \approx -Ze^2.\delta.e^{-\delta r}/(1 - e^{-\delta r}) \tag{108}$$

This equation (108) has a pair of exact solutions. It could be shown that for small values of δ, the Hulthen potential (108) approximates the effective Coulomb potential, in particular for large radius [14b]:

$$V_{Coulomb}^{eff} = -e^2/r + \ell(\ell+1).\hbar^2/(2mr^2) \tag{109}$$

Therefore equation (109) could be rewritten as:

$$-\hbar^2\nabla^2\psi/2m* + \left[-e^2/r + \ell(\ell+1).\hbar^2/(2mr^2)\right].\psi = i\hbar.\partial\psi/\partial t \tag{110}$$

For large radii, second term in the square bracket of LHS of equation (110) reduces to zero [54],

$$\ell(\ell+1).\hbar^2/(2mr^2) \to 0 \tag{111}$$

so we can write equation (110) as:

$$(-\hbar^2\nabla^2\psi/2m* + U).\psi = i\hbar.\partial\psi/\partial t \tag{112}$$

where Coulomb potential can be written as:

$$U = -e^2/r \tag{113}$$

This equation (112) is nothing but Schrödinger equation (85). Therefore we have re-derived Schrödinger equation from simplification of Ginzburg-Landau equation, in the limit of small screening parameter. Calculation shows that introducing this Hulthen effect (108) into equation (107) will yield different result only at the order of 10^{-39} m compared to prediction using equation (110), which is of course negligible. Therefore, we conclude that for most celestial quantization problems the result of TDGL-Hulthen (110) is *essentially* the same with the result derived from equation (85). Now, to derive equation (96) from Schrödinger equation, the reader is advised to see Nottale's scale-relativistic method [43][44].

What we would emphasize here is that this derivation of Schrödinger equation from Ginzburg-Landau equation is in good agreement with our previous conjecture that equation (90) implies macroquantization corresponding to superfluid-quantized vortices. This conclusion is the main result of this section. It is also worthnoting here that there is recent attempt to introduce Ginzburg-Landau equation in the context of microtubule dynamics [72], which implies wide applicability of this equation.

In the following section, we would extend this argument by noting that macroquantization of celestial systems implies the *topological* character of superfluid-quantized vortices, and cosmic microwave background radiation is also an indication of such topological superfluid vortices.

7.   Further Note: Signatures of Bose-Einstein Cosmology

It is known that CMBR temperature (2.73K) is conventionally assumed to come from the hot early Universe, which then cools adiabatically to the present epoch. Nonetheless this description is not without problems, such as how to consider the small temperature fluctuations of CMBR as the seeds that give rise to large-scale structure such as galaxy formation [73]. Furthermore it is known that CMBR follows Planck radiation law with high precision, so one could argue whether it also indicates that large-scale structures obey quantum-mechanical principles. Therefore we will consider here some alternative hypothesis, which support the idea of low-energy quantum mechanics corresponding to superfluid vortices described in the preceding section.

In recent years, there are alternative arguments suggesting that the Universe indeed resembles the dynamics of N number of Planckian oscillators. Using similar assumption, for instance Antoniadis *et al.* [74] argue that CMBR temperature could be derived using *conformal invariance* symmetry, instead of using Harrison-Zel'dovich spectrum. Other has derived CMBR temperature from Weyl framework [74a]. Furthermore, if the CMBR temperature 2.73K could be interpreted as low-energy part of the Planck distribution law, then it seems to indicate that the Universe resembles Bose-Einstein condensate [75]. Pervushin *et al.* also argued that CMBR temperature could be derived from conformal cosmology with relative units [76]. These arguments seem to support Winterberg's hypothesis that superfluid phonon-roton aether could explain the origin of cosmic microwave background radiation [18][19].

Of course, it does not mean that CMBR data fits perfectly with Planck distribution law. It has been argued that CMBR data more corresponds to q-deformed Planck radiation distribution [77]. However, this argument requires further analysis. What interests us here is that there are reasons to believe that a quantum universe based on Planck scale is not merely a pure hypothetical notion, in particular if we consider known analogy between superfluidity and various cosmology phenomena [16][17].

Another argument comes from fractality argument. It has been discovered by Feynman that the typical quantum mechanical paths are non-differentiable and fractal [67]. In this regard, it has been argued that the Universe is embedded in Cantorian fractal spacetime having non-integer Hausdorff dimension [78], and from this viewpoint it could be inferred that the correlated fluctuations of the fractal spacetime is analogous to the Bose-Einstein condensate phenomenon. Interestingly, there is also hypothesis suggesting that Hausdorff dimension could be related to temperature of ideal Bose gas [79].

From these aforementioned arguments, it seems plausible to suppose that that CMBR temperature 2.73K could be interpreted as a signature of Bose-Einstein condensate cosmology. In particular, one could consider [22b] that "this relationship comes directly from Boltzmann's law N= B.k.T, where N is the background noise power; T is the background temperature in degrees Kelvin; and B is the bandwidth of the background radiation. It follows that the ratio (N/kB) for the cosmic background radiation is approximately equal to "e", because we usually convert the equation to decibels by taking natural logarithm.

The relationship is a solid one in fact." From this viewpoint, it seems quite conceivable to explain why CMBR temperature 2.73K is near enough to known number e= 2.71828..., which seems to suggest that the logarithmic form of Schrödinger equation ('eigentliche Wellengleichung') [4a] *may have* a deep linkage with this number e= 2.71828...

Nonetheless, we recognize that this proposition requires further analysis before we could regard it as conclusive. But we can describe here some arguments to support the new interpretation supporting this Bose-Einstein cosmology argument:

- From Fischer's argument [13] we know that Bohr-Sommerfeld quantization from superfluid vortice could exhibit at all scales, including celestial quantization. This proposition comes directly from his assertion of *the topological character* of superfluid vortices, because superfluid is topological electronic liquid [27].
- Extending further the aforementioned hypothesis of topological superfluid vortices, then it seems interesting to compare it with topological analysis of COBE-DMR data. G. Rocha *et al.* [80] argue using wavelet approach with Mexican Hat potential that it is possible to interpret the data as clue for a finite torus Universe, albeit not conclusive enough.
- Interestingly, this conjecture could be related to Bulgadaev's argument [81] suggesting that topological quantum number could be related to torus structure as stable *soliton* [81a]. In effect, this seems to imply that the basic structure of physical phenomena throughout all scales could take the form of topological torus.

In other words, the topological character of superfluid vortices implies that it is possible to generalize superfluid vortices to large scales. And the topological character of CMBR data seems to support our proposition that the universe indeed exhibits topological structures. It follows then that CMBR temperature is topological [80] in the sense that the superfluid nature of background temperature [18][19] could be explained from topological superfluid vortices.

Interestingly, similar argument has been pointed out by a number of authors by mentioning non-Gaussian part of CMBR spectrum. However, further discussion on this issue requires another note.

8.   Concluding Remarks

It is known that Barut equation could predict lepton mass (and also hadron mass) with remarkable precision. Therefore, in the present article, we attempt to find plausible linkage between Dirac-Maxwell's isomorphism and Barut-Dirac-Vigier equation. From this proposition we could find a unified wave equation in terms of superfluid velocity (vierbein), which then could be used as basis to derive some alternative descriptions of Barut equation. Further experiment is required to verify which equation is the most reliable.

In the present note we submit the viewpoint that it would be more conceivable if we interpret the *vierbein* of the unified wave equation in terms of superfluid velocity, which in turn brings us to the notion of topological electronic liquid. Nonetheless, the proposed

imaginary algebra discussed herein is only at its elementary form, and it requires further analysis in particular in the context of [5a][7][14][28]. It is likely that this subject will become the subject of subsequent paper.

Furthermore, the notion of topological electronic liquid could lead to topological superfluid vortices, which may explain the origin of macroquantization of celestial systems and perhaps also topological character of Cosmic Microwave Background Radiations. Nonetheless, such a proposition requires further analysis before it can be considered as conclusive.

Provided the aforementioned propositions of using superfluid velocity (*vierbein*) to describe unified wave equation correspond to the observed facts, and then in principle it seems to support arguments in favor of possibility to observe condensed-matter hadronic reaction.

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