Maximal abelian sets of roots

R. Lawther

Author address:

Department of Pure Mathematics and Mathematical Statistics, Centre for Mathematical Sciences, Cambridge University, Wilberforce Road, Cambridge CB3 0WB

E-mail address: `ril10@cam.ac.uk`

Contents

Abstract

In this work we let Φ be an irreducible root system, with Coxeter group W. We consider subsets of Φ which are *abelian*, meaning that no two roots in the set have sum in $\Phi \cup \{0\}$. We classify all maximal abelian sets (i.e., abelian sets properly contained in no other) up to the action of W: for each W-orbit of maximal abelian sets we provide an explicit representative X, identify the (setwise) stabilizer W_X of X in W, and decompose X into W_X-orbits.

Abelian sets of roots are closely related to abelian unipotent subgroups of simple algebraic groups, and thus to abelian p-subgroups of finite groups of Lie type over fields of characteristic p. Parts of the work presented here have been used to confirm the p-rank of $E_8(p^n)$, and (somewhat unexpectedly) to obtain for the first time the 2-ranks of the Monster and Baby Monster sporadic groups, together with the double cover of the latter.

Root systems of classical type are dealt with quickly here; the vast majority of the present work concerns those of exceptional type. In these root systems we introduce the notion of a *radical* set; such a set corresponds to a subgroup of a simple algebraic group lying in the unipotent radical of a certain maximal parabolic subgroup. The classification of radical maximal abelian sets for the larger root systems of exceptional type presents an interesting challenge; it is accomplished by converting the problem to that of classifying certain graphs modulo a particular equivalence relation.

2010 *Mathematics Subject Classification.* Primary 17B22.

Key words and phrases. root system.

For RL, who wondered if it could be done,
from RL, who decided to try

CHAPTER 1

Introduction

Throughout this work, we let Φ be an irreducible root system, and W be the Coxeter group of Φ. A subset X of Φ is called *abelian* if no two of its elements have sum in $\Phi \cup \{0\}$. We shall classify all maximal abelian sets in Φ (i.e., abelian sets properly contained in no other) up to the action of W: for each W-orbit of maximal abelian sets we shall provide an explicit representative X, identify the (setwise) stabilizer W_X of X in W, and decompose X into W_X-orbits.

In this introductory chapter we begin with some background material and provide some motivation for studying these sets; we then establish notation and prove some preliminary results.

1.1. Background and motivation

The first work in this area was that of Mal'cev in [**10**]; he sought to determine the abelian subalgebras of maximal dimension in simple Lie algebras over $\mathbb{C}$ (and the corresponding subgroups of simple Lie groups, thereby generalizing a result of Schur). He began by quickly reducing to the case of abelian subalgebras consisting of nilpotent elements, where his approach was to translate the problem to that of finding abelian sets of roots of maximal size in irreducible root systems; because he could assume that the abelian subalgebra lay in the nilpotent radical of a given maximal soluble subalgebra, it sufficed to consider sets of positive roots, in which the possibility of summing to 0 did not arise. He found all such sets; however, his argument for the root system of type E_7 rested upon a combinatorial lemma the proof of which was omitted, while for the root system of type E_8 the result was simply stated without proof.

One important application of Mal'cev's results is to the p-ranks of finite groups of Lie type. In [**4**, 3.3], Gorenstein, Lyons and Solomon used Mal'cev's reduction technique to show that, in an untwisted finite group of Lie type in characteristic p, abelian p-subgroups correspond to abelian sets of roots, and consequently the p-rank is simply the value given in [**10**]. In the interest of completeness, they gave proofs for all cases except that of E_8 (although those for E_6 and E_7 were described as sketches, the latter of which employed the theory of modules with quadratic action to obtain initial information about the sets to be considered).

Later, in [**5, 6**] Guralnick and Malle provided a classification of 2F-modules for simple groups, which was needed for the completion of the classification of finite simple groups. Their work left undecided a small number of cases, which were subsequently treated in [**7**] and [**8**]. In [**7**] the smallest-dimensional non-trivial modules for the groups $F_4(q)$ (for q a power of a prime larger than 3) and $E_7(q)$ were shown not to be 2F-modules; the proof proceeded by locating both group and module inside a parabolic subgroup of a larger group, and using modifications of Mal'cev's technique to reduce to a problem about abelian sets of roots. Somewhat unexpectedly, a similar approach was successfully employed in [**8**] to handle the cases of the sporadic groups Co_1 and Co_2 acting on their smallest-dimensional non-trivial modules in characteristic 2. In these cases the role of the larger group containing both group and module was played by the Monster M and Baby Monster B respectively. Indeed, it proved possible to extend the arguments so as to obtain the 2-ranks of M, B and $2.B$, thereby completing the determination of p-ranks of sporadic groups as listed in [**4**, Table 5.6.1]; this extension required a portion of the results of the present work on maximal abelian sets in root systems of type E_6.

Recently, in [**11**] Pevtsova and Stark investigated the projective varieties of elementary subalgebras of maximal dimension in simple Lie algebras, under mild characteristic restrictions. They followed Mal'cev's method, quoting his results on abelian sets of roots of maximal size; for certain root systems, including those of type E_n, they used a computer program to verify that the lists of sets which he gave were correct.

In this work we consider not merely abelian sets of roots of maximal size, but more generally maximal abelian sets as defined above. If we take a simple Lie algebra $\mathfrak{G}$ with root system Φ, and let $\mathfrak{N}$ be the nilpotent subalgebra of $\mathfrak{G}$ spanned by the positive root vectors (for some choice of positive system in Φ), then each such maximal abelian set X consisting of positive roots naturally gives rise to an abelian subalgebra $\mathfrak{G}_X$ of $\mathfrak{N}$, by taking the span of the corresponding root vectors; provided the field characteristic does not divide any of the structure constants, such a subalgebra is easily seen to be maximal (in the sense of containment used here) among abelian subalgebras of $\mathfrak{N}$. It is however not clear that an arbitrary maximal abelian subalgebra of $\mathfrak{N}$ is necessarily the image of some $\mathfrak{G}_X$ under an automorphism of $\mathfrak{G}$. (Similar comments apply to maximal abelian unipotent subgroups of simple algebraic groups.)

The author would like to express his sincere gratitude to Bob Guralnick, without whose encouragement this work would not have been published.

1.2. Preliminary results

In the root system Φ let Π be a simple system and Φ^+ be the corresponding set of positive roots. For each root α write w_α for the corresponding reflection in W, so that for all $\beta \in \Phi$ we have $w_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha$, where $(\ ,\ )$ is a fixed inner product on $\mathbb{R}\Phi$; then $W = \langle w_\alpha : \alpha \in \Pi \rangle$.

If there are two root lengths, let Φ_l and Φ_s be the sets of long and short roots, and $\Pi_l = \Pi \cap \Phi_l$ and $\Pi_s = \Pi \cap \Phi_s$ be the sets of long and short simple roots; let $c \in \{2, 3\}$ be the ratio of squared root lengths. If instead there is a single root length, let $\Phi_l = \Phi$, $\Pi_l = \Pi$ and $\Phi_s = \Pi_s = \emptyset$, so that all roots are regarded as long, and let $c = 1$.

Lemma 1.1. *If $\gamma \in \Phi_l$ and $\alpha \in \Pi_s$, then $w_\alpha(\gamma) \in \{\gamma + c\alpha, \gamma, \gamma - c\alpha\}$.*

Proof. We have $w_\alpha(\gamma) = \gamma - \frac{2(\gamma,\alpha)}{(\alpha,\alpha)}\alpha$; since $\frac{2(\alpha,\gamma)}{(\gamma,\gamma)} \in \mathbb{Z}$ and $(\gamma, \gamma) = c(\alpha, \alpha)$, we have $\frac{2(\gamma,\alpha)}{(\alpha,\alpha)} = \frac{2c(\alpha,\gamma)}{(\gamma,\gamma)} \in c\mathbb{Z}$. Since the length of the α-string through γ is at most $c + 1$, the result follows. □

We may now give a simple characterization of long roots. Recall that any root may be written in the form $\sum_{\alpha\in\Pi} n_\alpha \alpha$ with each $n_\alpha \in \mathbb{Z}$; if the root is positive then each n_α is non-negative.

LEMMA 1.2. *Given $\gamma \in \Phi$, write $\gamma = \sum_{\alpha\in\Pi} n_\alpha \alpha$; then $\gamma \in \Phi_l$ if and only if c divides n_α for each $\alpha \in \Pi_s$.*

PROOF. First suppose $\gamma \in \Phi_l$; then γ lies in the W-orbit of some long simple root (for which the divisibility condition is immediate). Since each simple reflection only affects the coefficient of the simple root concerned, and Lemma 1.1 shows that if $\alpha \in \Pi_s$ then the coefficient of α can only be changed by a multiple of c, it follows that γ satisfies the divisibility condition as required.

Now suppose that γ satisfies the divisibility condition; let $\delta \in \Phi_l$, and consider the ratio $\frac{(\gamma,\gamma)}{(\delta,\delta)}$, with the numerator expanded out using the expression $\gamma = \sum_{\alpha\in\Pi} n_\alpha \alpha$. Each term involving a long simple root α is either $\frac{{n_\alpha}^2(\alpha,\alpha)}{(\delta,\delta)} = {n_\alpha}^2$ or a cross term $\frac{2n_\alpha n_\beta(\alpha,\beta)}{(\delta,\delta)} = n_\alpha n_\beta \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$; each in which both simple roots are short is either $\frac{{n_\alpha}^2(\alpha,\alpha)}{(\delta,\delta)} = \frac{{n_\alpha}^2}{c}$ or a cross term $\frac{2n_\alpha n_\beta(\alpha,\beta)}{(\delta,\delta)} = \frac{n_\alpha n_\beta}{c}\frac{2(\beta,\alpha)}{(\alpha,\alpha)}$. Thus each term is an integer; so $\frac{(\gamma,\gamma)}{(\delta,\delta)} \in \mathbb{Z}$, and we must have $\gamma \in \Phi_l$. $\square$

Given $\gamma, \delta \in \Phi$ we shall write $\gamma \preceq \delta$ as usual to mean that $\delta - \gamma = \sum_{\alpha\in\Pi} n_\alpha \alpha$ with each $n_\alpha \geq 0$.

LEMMA 1.3. *If $\gamma, \delta \in \Phi$ are roots of the same length with $\gamma \prec \delta$, then there is a simple root $\beta \in \Pi$ such that $\gamma \prec w_\beta(\gamma) \preceq \delta$.*

PROOF. Write $\delta - \gamma = \sum_{\alpha\in\Pi} n_\alpha \alpha$, and let $\Pi' = \{\alpha \in \Pi : n_\alpha > 0\}$. Since

$$\begin{aligned}(\gamma,\gamma) = (\delta,\delta) &= \left(\gamma + \sum_{\alpha\in\Pi'} n_\alpha\alpha, \gamma + \sum_{\alpha\in\Pi'} n_\alpha\alpha\right)\\ &= (\gamma,\gamma) + 2\left(\gamma, \sum_{\alpha\in\Pi'} n_\alpha\alpha\right) + \left(\sum_{\alpha\in\Pi'} n_\alpha\alpha, \sum_{\alpha\in\Pi'} n_\alpha\alpha\right)\\ &> (\gamma,\gamma) + 2\sum_{\alpha\in\Pi'} n_\alpha(\gamma,\alpha),\end{aligned}$$

for some $\beta \in \Pi'$ we must have $(\gamma, \beta) < 0$. Thus $w_\beta(\gamma) = \gamma + m\beta$ for some $m \in \mathbb{N}$, and so $\gamma \prec w_\beta(\gamma)$; moreover we also have $w_\beta(\gamma) \preceq \delta$ provided $m \leq n_\beta$. If $m = 1$ this is clear, so we may assume $m > 1$, which forces β to be short and γ (and hence δ) long, with $m = c$. By Lemma 1.2, n_β must be a multiple of m, and so $m \leq n_\beta$ as required. $\square$

COROLLARY 1.4. *If $\gamma, \delta \in \Phi^+$ are roots of the same length with $\gamma \prec \delta$, then there is a sequence $\alpha_1, \ldots, \alpha_r$ of simple roots such that $\gamma \prec w_{\alpha_1}(\gamma) \prec w_{\alpha_2}w_{\alpha_1}(\gamma) \cdots \prec w_{\alpha_r} \ldots w_{\alpha_2}w_{\alpha_1}(\gamma) = \delta$.*

PROOF. Use Lemma 1.3 and induction on $\operatorname{ht}\delta - \operatorname{ht}\gamma$. $\square$

We now show that in considering abelian sets we may restrict our attention to positive roots.

LEMMA 1.5. *If $X \subset \Phi$ is an abelian set, then some W-translate of X lies in Φ^+.*

PROOF. We use induction on the number n of negative roots in X; the statement is trivial if $n = 0$, so suppose $\delta \in \Phi^+$ with $-\delta \in X$. Let γ be a simple root of the same length as δ such that $\gamma \preceq \delta$. If $\delta \neq \gamma$, by Lemma 1.3 we may find $\beta \in \Pi$

with $-\delta \prec w_\beta(-\delta) \preceq -\gamma$; then $-\delta + \beta = -(\delta - \beta) \in \Phi$, so that $\beta \notin X$. Thus as the only positive root taken to a negative root by w_β is β itself, we see that replacing X by its image under w_β does not increase n. Replacing $-\delta$ by $w_\beta(-\delta)$, and repeating the process, we eventually obtain $\delta = \gamma$; this time apply w_γ (noting that $\gamma \notin X$) to reduce n by 1. By induction the result follows. □

Indeed we may refine this result as follows.

LEMMA 1.6. *If $X \subset \Phi$ is an abelian set, then some W-translate of X lies in Φ^+ and meets* Π.

PROOF. By Lemma 1.5 we may assume $X \subseteq \Phi^+$; use induction on the minimal height j of a root in X. If $j = 1$ then X itself meets Π, so assume $j > 1$; choose a root $\delta \in X$ of height j, and as in Corollary 1.4 find a simple root β with $w_\beta(\delta) \prec \delta$. Thus $w_\beta(X)$ lies in Φ^+ and contains a root of height less than j; the result follows. □

We may also show that a maximal abelian set must contain a basis of $\mathbb{R}\Phi$.

LEMMA 1.7. *If $X \subset \Phi$ is a maximal abelian set, then the roots in X span* $\mathbb{R}\Phi$.

PROOF. By Lemma 1.5 we may assume $X \subseteq \Phi^+$; take $\alpha \in \Phi^+ \setminus X$. Since by assumption $X \cup \{\alpha\}$ is not abelian there must be some $\beta_1 \in X$ with $\alpha + \beta_1 \in \Phi$; set $\alpha_1 = \alpha + \beta_1$. If $\alpha_1 \notin X$, there must similarly be some $\beta_2 \in X$ with $\alpha_1 + \beta_2 \in \Phi$; set $\alpha_2 = \alpha_1 + \beta_2$. Continuing in this fashion, for some j we must have $\alpha_j \in X$, because $\operatorname{ht}\alpha < \operatorname{ht}\alpha_1 < \cdots$; then $\alpha = \alpha_j - (\beta_1 + \cdots + \beta_j) \in \mathbb{R}X$. Thus each root in Φ is a linear combination of those in X as required. □

In [**9**] the notion of the *long height* of a long root was introduced: given $\gamma = \sum_{\alpha\in\Pi} n_\alpha \alpha \in \Phi_l$, we set $\operatorname{lht}\gamma = \sum_{\alpha\in\Pi_l} n_\alpha + \frac{1}{c}\sum_{\alpha\in\Pi_s} n_\alpha$ (note that by Lemma 1.2 $\operatorname{lht}\gamma \in \mathbb{Z}$). Thus in the case of a single root length, the long height of a root is simply its height. Write ρ for the highest root of Φ; recall that ρ is long, and that for all positive roots γ we have $\gamma \preceq \rho$.

LEMMA 1.8. *Given a long root $\gamma \in \Phi^+$, the number of positive roots β with $\gamma + \beta \in \Phi$ is* $\operatorname{lht}\rho - \operatorname{lht}\gamma$.

PROOF. We use induction on $\operatorname{lht}\rho - \operatorname{lht}\gamma$; the result is clearly true if $\operatorname{lht}\rho - \operatorname{lht}\gamma = 0$, so take $\gamma \neq \rho$. Applying Lemma 1.3 to the roots γ and ρ, we may then choose $\alpha \in \Pi$ with $\gamma \prec w_\alpha(\gamma)$; if $\alpha \in \Pi_l$ we must have $w_\alpha(\gamma) = \gamma + \alpha$, while by Lemma 1.1 if $\alpha \in \Pi_s$ then $w_\alpha(\gamma) = \gamma + c\alpha$. In either case $\operatorname{lht} w_\alpha(\gamma) = \operatorname{lht}\gamma + 1$; by inductive hypothesis the number of positive roots which may be added to $w_\alpha(\gamma)$ is $\operatorname{lht}\rho - \operatorname{lht} w_\alpha(\gamma) = \operatorname{lht}\rho - \operatorname{lht}\gamma - 1$. Now the positive roots which may be added to γ are the images under w_α of those which may be added to $w_\alpha(\gamma)$, together with α; thus they number $(\operatorname{lht}\rho - \operatorname{lht}\gamma - 1) + 1 = \operatorname{lht}\rho - \operatorname{lht}\gamma$ as required. □

Let Ψ be the subsystem consisting of roots orthogonal to ρ; we may now determine the size of Ψ.

LEMMA 1.9. $|\Psi| = |\Phi| - 4\operatorname{lht}\rho + 2$.

PROOF. Take $\alpha \in \Pi_l$; then given $\beta \in \Phi \setminus \{\pm\alpha\}$ there are three mutually exclusive possibilities: (i) $\alpha + \beta \in \Phi$; (ii) $\alpha - \beta \in \Phi$; (iii) $(\alpha, \beta) = 0$. The map $\beta \mapsto -\beta$ gives a bijection between the sets satisfying (i) and (ii), while $\beta \mapsto -(\alpha+\beta)$

gives a bijection between the sets of positive and negative roots satisfying (i). Since by Lemma 1.8 the number of positive roots satisfying (i) is $\operatorname{lht}\rho - 1$, we have

$$|\Phi| - 2 = 2(\operatorname{lht}\rho - 1) + 2(\operatorname{lht}\rho - 1) + |\Psi|,$$

and the result follows. □

We shall require the following piece of terminology: we say that one root *excludes* another if their sum is also a root. Thus if X is an abelian set in some root system, the assumption that a particular root β lies in X means that X can contain no root excluded by β.

In the following chapter we shall consider the root systems of classical type. We shall then turn to those of exceptional type, which will occupy the bulk of the present work.

CHAPTER 2

Root systems of classical type

We shall work through the different types of classical root system in turn. In contrast to the work to follow on exceptional root systems, where we shall express roots as linear combinations of simple roots, here we shall take the standard realization of the root system in terms of an orthonormal basis $\epsilon_1, \dots, \epsilon_d$ of Euclidean space $\mathbb{R}^d$ for an appropriate value of d.

2.1. Root systems of type A_n

Let $d = n+1$, and set $\Phi = \{\epsilon_i - \epsilon_j : i \neq j\}$. Suppose $X \subset \Phi$ is an abelian set; define $f : \{1, \dots, n+1\} \to \{\pm 1\}$ by $f(j) = -1$ if X contains a root $\epsilon_i - \epsilon_j$, and $f(j) = 1$ otherwise. Then if $f(j) = -1$ we cannot have a root $\epsilon_j - \epsilon_k$ in X. Using W we may assume that $\{j : f(j) = 1\} = \{1, \dots, m\}$ for some m; so X is contained in the maximal abelian set

$$X_m = \{\epsilon_i - \epsilon_j : i \leq m < j\},$$

of size $m(n+1-m)$. It is clear that no two of these sets lie in the same W-orbit (although the diagram automorphism θ interchanges X_m and X_{n+1-m}), and that the stabilizer W_{X_m} is the group $\langle w_\alpha : \alpha = \epsilon_i - \epsilon_{i+1},\ i \neq m\rangle \cong S_m \times S_{n+1-m}$, which is transitive on X_m.

2.2. Root systems of type C_n

Let $d = n$, and set $\Phi = \{\pm\epsilon_i \pm \epsilon_j : i \neq j\} \cup \{\pm 2\epsilon_i\}$. Suppose $X \subset \Phi$ is an abelian set; define $f : \{1, \dots, n\} \to \{\pm 1\}$ by $f(j) = -1$ if X contains a root $\pm\epsilon_i - \epsilon_j$ or $-2\epsilon_j$, and $f(j) = 1$ otherwise. Then if $f(j) = -1$ we cannot have a root $\pm\epsilon_i + \epsilon_j$ or $2\epsilon_j$ in X. Using the subgroup $\langle w_\alpha : \alpha = 2\epsilon_i\rangle$ of W we may assume that $f(j) = 1$ for all j; so X is contained in the maximal abelian set

$$X_0 = \{\epsilon_i + \epsilon_j : 1 \leq i < j \leq n\} \cup \{2\epsilon_i : 1 \leq i \leq n\},$$

of size $\frac{1}{2}n(n+1)$. The stabilizer W_{X_0} is the group $\langle w_\alpha : \alpha = \epsilon_i - \epsilon_{i+1}\rangle \cong S_n$, which is transitive on the sets of long and short roots of X_0.

2.3. Root systems of type D_n

Take $d = n$, and set $\Phi = \{\pm\epsilon_i \pm \epsilon_j : i \neq j\}$. Here it will be convenient to take the positive system $\Phi^+ = \{\epsilon_i \pm \epsilon_j : i < j\}$, and to restrict ourselves to maximal abelian sets lying in Φ^+. Let θ be the diagram automorphism which interchanges the simple roots $\epsilon_{n-1} - \epsilon_n$ and $\epsilon_{n-1} + \epsilon_n$ and fixes the others; if $n = 4$ let η be a diagram automorphism of order 3.

Take $1 \leq r \leq n$, and let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a partition of $n - r$ satisfying $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0$; for $i = 0, 1, \ldots, r$ set $\mu_i = r + \lambda_1 + \cdots + \lambda_i$. Set

$$X_{(r,\lambda)} = \{\epsilon_i + \epsilon_j : 1 \leq i < j \leq r\} \cup \{\epsilon_i \pm \epsilon_j : 1 \leq i \leq r,\ \mu_{i-1} < j \leq \mu_i\}.$$

Then $X_{(r,\lambda)}$ is an abelian set of size $\frac{1}{2}r(r-1) + 2(n-r)$. Let $\beta = \pm\epsilon_k \pm \epsilon_l$ be a root lying outside X, and assume $k < l$. If $l > r$ then β is excluded by some root $\epsilon_i \mp \epsilon_l$; so assume $l \leq r$, in which case at least one of the signs must be minus (and we must have $r \geq 2$). Provided $r > 2$ there exists $i \leq r$ with $i \neq k, l$, and then β is excluded by $\epsilon_i + \epsilon_k$ or $\epsilon_i + \epsilon_l$; if $r = 2$ then $-\epsilon_1 \pm \epsilon_2$ is excluded by $\epsilon_1 + \epsilon_3$, while $\epsilon_1 - \epsilon_2$ is excluded by $\epsilon_2 + \epsilon_n$ unless $\lambda = (n-2, 0)$. Thus $X_{(r,\lambda)}$ is maximal abelian provided $(r, \lambda) \neq (2, (n-2, 0))$ (note that $X_{(2,(n-2,0))} = \{\epsilon_1 + \epsilon_2, \epsilon_1 \pm \epsilon_j : j > 2\} \subset \{\epsilon_1 \pm \epsilon_j : j > 1\} = X_{(1,(n-1))}$). In similar fashion set

$$X_0 = \{\epsilon_i + \epsilon_j : 1 \leq i < j < n\} \cup \{\epsilon_i - \epsilon_n : 1 \leq i < n\};$$

then X_0 is maximal abelian of size $\frac{1}{2}n(n-1)$. Indeed X_0 is the image of $X_{(n,(0^n))}$ under θ, while for $(r, \lambda) \neq (n, (0^n))$ the set $X_{(r,\lambda)}$ is θ-stable. (If $n = 4$ the diagram automorphism η cycles $X_{(1,(3))}$, $X_{(4,(0^4))}$ and X_0, and fixes each of $X_{(2,(1^2))}$ and $X_{(3,(1,0^2))}$.)

Now let X be any maximal abelian set lying in Φ^+. Take $i > 1$, and set $N_i(X) = \{\beta \in X : (\beta, \epsilon_i) \neq 0\}$ and $n_i = |N_i(X)|$. We cannot have $n_i = 0$, since then the root $\epsilon_1 + \epsilon_i$ could not be excluded; we cannot have $n_i = 1$, since if $N_i(X) = \{\epsilon_i - \epsilon_j\}$ then $\epsilon_1 + \epsilon_i$ could again not be excluded, while if $N_i(X) = \{\epsilon_j + \delta\epsilon_i\}$ with $\delta \in \{\pm 1\}$ then $\epsilon_j - \delta\epsilon_i$ could not be excluded. Thus $n_i \geq 2$. If all roots in $N_i(X)$ have the same inner product with ϵ_i, let δ_i be the common value; if instead there are two roots in $N_i(X)$ which have opposite inner products with ϵ_i, in which case we must have $N_i(X) = \{\epsilon_j \pm \epsilon_i\}$ for some j, let $\delta_i = 0$.

In similar fashion set $\delta_1 = 1$; then using the subgroup $\langle w_\alpha : \alpha = \epsilon_i - \epsilon_{i+1} \rangle$ of W we may assume that δ_i is equal to 1 for $1 \leq i \leq s$, to -1 for $s + 1 \leq i \leq r$ and to 0 for $r + 1 \leq i \leq n$. If $r - s \geq 2$ we may apply $w_\alpha w_{\alpha'}$ with $\alpha, \alpha' = \epsilon_{s+1} \pm \epsilon_{s+2}$ to increase s by 2, so we may assume $s \in \{r-1, r\}$. Likewise if $s = r - 1$ and $r < n$ we may apply $w_\alpha w_{\alpha'}$ with $\alpha, \alpha' = \epsilon_{s+1} \pm \epsilon_n$ to give $s = r$; on the other hand if $s = r - 1$ and $r = n$ we may apply θ to give $s = r$. Now maximality of X ensures that it contains all roots $\epsilon_i + \epsilon_j$ for $1 \leq i < j \leq r$. For $1 \leq i \leq r$ let λ_i be the number of j such that $\epsilon_i - \epsilon_j \in X$; again using the subgroup $\langle w_\alpha : \alpha = \epsilon_i - \epsilon_{i+1} \rangle$ of W we may assume that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0$, and that $\epsilon_i - \epsilon_j \in X$ for $r + \lambda_1 + \cdots + \lambda_{i-1} < j \leq r + \lambda_1 + \cdots + \lambda_i$. Thus $X = X_{(r,\lambda)}$. As it is clear that no two of the sets $X_{(r,\lambda)}$ are W-translates of each other, it follows that the $X_{(r,\lambda)}$ for $(r, \lambda) \neq (2, (n-2, 0))$ together with X_0 form a set of representatives of the W-orbits of maximal abelian sets.

To consider stabilizers, it is convenient to work in $W.2 = \langle W, \theta \rangle \cong S_2 \wr S_n$, and to write elements of this group as signed permutations as in [**2**]. Given $X = X_{(r,\lambda)}$, write $\lambda = ((n-r)^{a_{n-r}}, (n-r-1)^{a_{n-r-1}}, \ldots, 1^{a_1}, 0^{a_0})$. We then have the following generators for $(W.2)_X$:

$\overline{k}$	for $k \in \{r+1, \ldots, n\}$,
$(k\ \ k+1)$	for $k \in \{r+1, \ldots, n\} \setminus \{\mu_1, \ldots, \mu_r\}$,
$(i\ \ i+1) \prod_{j=1}^{\lambda_i} (\mu_{i-1} + j\ \ \mu_i + j)$	for $i \in \{1, \ldots, r-1\}$ with $\lambda_i = \lambda_{i+1}$.

Consequently $(W.2)_X \cong \prod_{i=1}^{n-r} ((S_2 \wr S_i) \wr S_{a_i})$. The $(W.2)_X$-orbits in X are as follows: for each pair (k, l) with $0 \leq l < k \leq n-r$ we have the (possibly empty) $(W.2)_X$-orbit

$\{\epsilon_i+\epsilon_j : \lambda_i = k,\ \lambda_j = l\}$; for each k with $0 \le k \le n-r$ we have the (possibly empty) $(W.2)_X$-orbit $\{\epsilon_i \pm \epsilon_j : \lambda_i = k,\ \mu_{i-1} < j \le \mu_i\}$. The subgroup W_X is of index 2 in $(W.2)_X$ unless $(r, \lambda) = (n, (0^n))$, in which case we have $W_X = (W.2)_X$; the only difference between $(W.2)_X$-orbits and W_X-orbits occurs if $(r, \lambda) = (n-1, (1, 0^{n-2}))$, in which case the $(W.2)_X$-orbit $\{\epsilon_1 \pm \epsilon_n\}$ breaks into the two W_X-orbits $\{\epsilon_1 + \epsilon_n\}$ and $\{\epsilon_1 - \epsilon_n\}$. Finally, for $X = X_0$ the stabilizer W_X is the conjugate under θ of that of $X_{(n,(0^n))}$; thus it is generated by $(i\ \ i+1)$ for $1 \le i \le n-2$ together with $(n-1\ \ \overline{n})$, and is transitive on X.

2.4. Root systems of type B_n

Let $d = n$, and set $\Phi = \{\pm\epsilon_i \pm \epsilon_j : i \ne j\} \cup \{\pm\epsilon_i\}$. Much as before we take the positive system $\Phi^+ = \{\epsilon_i \pm \epsilon_j : i < j\} \cup \{\epsilon_i\}$.

Let $X \subset \Phi$ be a maximal abelian set. As a W-translate of X lies in Φ^+, where the only positive roots which can exclude ϵ_1 are short, X must contain some short root; since each short root excludes all of the others, X must contain exactly one short root. Let X^- be the set obtained by removing this short root from X; then X^- is an abelian set in the subsystem of long roots, which is the D_n system above. Thus, in the notation of section 2.3, replacing X by a W-translate if necessary we must have $X^- \subseteq X_{(r,\lambda)}$ for some (r, λ) (note that W contains the diagram automorphism θ of the D_n system, so we need not consider the possibility $X^- \subseteq X_0$). Now consider the short root in X. If it is $\pm\epsilon_j$ for some $j > r$, by applying w_{ϵ_j} if necessary (which leaves the containment $X^- \subseteq X_{(r,\lambda)}$ unaffected) we may assume it is ϵ_j; then X can contain no root $\epsilon_i - \epsilon_j$, so any roots $\epsilon_i + \epsilon_j$ for $1 \le i \le r$ cannot be excluded, whence by maximality we must have $(r, \lambda) = (1, (n-1))$. Applying $w_{\epsilon_2-\epsilon_j}$ if necessary we may further assume $j = 2$, in which case we have $X^- \subseteq X_{(2,(n-2,0))}$. Thus we may assume the short root is $\pm\epsilon_i$ for some $i \le r$. If it is $-\epsilon_i$ then X can contain no roots $\epsilon_i \pm \epsilon_j$, so applying w_{ϵ_i} will leave the containment $X^- \subseteq X_{(r,\lambda)}$ unaffected; so we may assume the short root is ϵ_i. Finally if $i > 1$ and $\lambda_{i-1} = \lambda_i$, we may apply the element of $W_{X_{(r,\lambda)}}$ of section 2.3 which interchanges $i-1$ and i. Thus if we set $I(\lambda) = \{1\} \cup \{2 \le i \le r : \lambda_{i-1} > \lambda_i\}$, then for some $i_0 \in I(\lambda)$ we have that X is the maximal abelian set

$$X_{(r,\lambda,i_0)} = X_{(r,\lambda)} \cup \{\epsilon_{i_0}\},$$

of size $\frac{1}{2}r(r-1) + 2(n-r) + 1$. Again it is clear that no two of the sets $X_{(r,\lambda,i_0)}$ are W-translates of each other, so that they form a set of representatives of the W-orbits of maximal abelian sets.

The stabilizer of the set $X_{(r,\lambda,i_0)}$ is clearly the intersection of those of $X_{(r,\lambda)}$ and ϵ_{i_0}; thus its generators are those listed in section 2.3 with the single exception of the element $(i_0\ \ i_0+1)\prod_{j=1}^{\lambda_{i_0}}(\mu_{i_0-1}+j\ \ \mu_{i_0}+j)$, and if we write $d = \lambda_{i_0}$ then its isomorphism type is obtained from that listed there by replacing the term $(S_2 \wr S_d) \wr S_{a_d}$ in the direct product by $(S_2 \wr S_d) \wr S_{a_d-1} \times (S_2 \wr S_d)$. Accordingly the W_X-orbits are the sets listed in section 2.3 with the following exceptions: for each $0 \le l \le n-r$ with $l \ne d$, the set $\{\epsilon_i + \epsilon_j : \lambda_i = d,\ \lambda_j = l\}$ splits into the two W_X-orbits $\{\epsilon_{i_0} + \epsilon_j : \lambda_j = l\}$ and $\{\epsilon_i + \epsilon_j : \lambda_i = d,\ i \ne i_0,\ \lambda_j = l\}$ (of which one or both may be empty); the set $\{\epsilon_i \pm \epsilon_j : \lambda_i = d,\ \mu_{i-1} < j \le \mu_i\}$ splits into the two W_X-orbits $\{\epsilon_{i_0} \pm \epsilon_j : \mu_{i_0-1} < j \le \mu_{i_0}\}$ and $\{\epsilon_i \pm \epsilon_j : \lambda_i = d,\ i \ne i_0,\ \mu_{i-1} < j \le \mu_i\}$ (of which one or both may be empty); finally there is the additional W_X-orbit $\{\epsilon_{i_0}\}$.

CHAPTER 3

The strategy for root systems of exceptional type

For the remainder of this work we suppose Φ is of exceptional type. We describe the strategy to be followed in the succeeding chapters in classifying the W-orbits of maximal abelian sets in Φ. We begin by defining certain types of abelian set. We then explain our method of proving the completeness of our classification in each case. Finally we consider the structure of each of the maximal abelian sets.

3.1. Radical and near-radical sets

Recall that ρ is the highest root of Φ, and Ψ is the subsystem consisting of roots orthogonal to ρ. By inspection we see that there exist simple roots δ and δ' such that δ is long, $\Psi \cap \Pi = \Pi \setminus \{\delta\}$, and every simple root in $\Pi \setminus \{\delta, \delta'\}$ is orthogonal to δ; moreover, if $\gamma = \sum_{\alpha \in \Pi} n_\alpha \alpha$ is a root, then $n_\delta \leq 2$ with equality if and only if $\gamma = \rho$, and $n_{\delta'} \leq 3$ with equality if and only if $\gamma \in \{\rho, \rho - \delta\}$. We set

$$\Xi = \left\{ \sum_{\alpha \in \Pi} n_\alpha \alpha \in \Phi : n_\delta = 1 \right\}, \qquad \Omega = \Xi \cup \{\rho\}.$$

Then $\Omega = \Phi^+ \setminus \Psi$; accordingly, in any simple algebraic group with Φ as root system, Ω is the set of roots of the unipotent radical of the maximal parabolic subgroup given by $\Pi \setminus \{\delta\}$. We shall call a subset of Ω a *radical* set.

By Lemma 1.9 we have $|\Xi| = \frac{1}{2}(|\Phi| - 2 - |\Psi|) = 2(\operatorname{lht} \rho - 1)$. For each $\beta \in \Xi$, we have $\rho - \beta = -w_\rho(\beta) \in \Xi$; thus Ξ consists of $\operatorname{lht} \rho - 1$ pairs $\{\beta, \rho - \beta\}$. Moreover, given $\beta \in \Xi$ the only root in Ω which may be added to β is $\rho - \beta$. Thus a set which is maximal among radical abelian sets consists of ρ together with one root from each pair $\{\beta, \rho - \beta\}$; such a set has size $\operatorname{lht} \rho$, and there are $2^{\operatorname{lht} \rho - 1}$ of these maximal radical abelian sets. Not all of them are necessarily maximal abelian, as there may be roots outside Ω which are not excluded; we shall need to decide which maximal radical abelian sets are in fact radical maximal abelian sets.

For the G_2 and F_4 root systems the classification of radical maximal abelian sets will be fairly simple. For the larger root systems it turns out to be useful to represent maximal radical abelian sets by certain graphs; the condition of maximality (among all abelian sets rather than just radical ones) and the action of $\operatorname{stab}_W(\rho)$ may be interpreted graphically, and we shall produce lists of equivalence classes of graphs representing the different W-orbits of radical maximal abelian sets.

For the two largest root systems, it will also turn out to be useful to consider sets which are 'almost radical': a subset of Φ^+ will be called a *near-radical* set if it contains exactly one root outside Ω for Φ of type E_7, or either one or two roots outside Ω for Φ of type E_8. The classification of near-radical maximal abelian sets will again involve graphs, but will be considerably more straightforward than that of the radical maximal abelian sets in these cases.

3.2. Proving completeness

Unless Φ is of type G_2 (in which case this turns out to be unnecessary), we shall begin by providing a short list of maximal abelian sets which are not radical (or near-radical, for Φ of type E_7 or E_8). We shall then write $\mathcal{S}(\Phi)$ for the collection of maximal abelian sets listed to this point, and describe a set as *known* if it is a W-translate of one of the sets in $\mathcal{S}(\Phi)$. Our goal will then be to show that any maximal abelian set is known; in other words, that the set $\mathcal{S}(\Phi)$ is complete.

Our basic approach will be as follows. Let X be a maximal abelian set. By Lemma 1.6 we may assume that X consists of positive roots and contains some simple root α; we shall work through the possibilities for α in turn. We shall build up the set X in stages by successively choosing roots to lie in it; each such choice will exclude certain other roots, and we shall call a positive root *available* at a given stage if it has not at that point been either chosen or excluded. Subsequent choices must then be made from among the available roots. At each stage we keep track of the roots excluded by the choices made, until we obtain an abelian set whose maximality can be checked; if it is maximal we shall then identify it as a W-translate of one of those in $\mathcal{S}(\Phi)$. We may then backtrack by returning to the last choice made and assuming the converse; ensuring in this way that all possibilities are considered will mean that all maximal abelian sets are obtained.

There are various ways in which this basic approach can be refined. Clearly any maximal abelian set of positive roots must contain ρ, so we shall assume without further comment that $\rho \in X$. As we work through the possibilities for α we shall be able to use some rather technical results (see Lemma 3.1 and Corollary 3.2 below) to assume that additional roots lie in X. At any given stage we may use the stabilizer in W of the roots chosen by that point to treat various available roots as equivalent, thereby reducing the number of possibilities to be considered. If at some point we find that an available root β cannot be added to any of the remaining available roots, for X to be maximal abelian it must contain β; in this case we shall say that $\beta \in X$ *by default*. On the other hand, for X to be maximal abelian it must exclude all negative roots as well as the positive roots outside X. Thus suppose at some point there is a negative root γ which has not been excluded. If γ can be added to just one available root β, then we must have $\beta \in X$ to exclude γ; on the other hand if γ cannot be added to any available root, we cannot obtain a maximal abelian set and so need not pursue the current line of investigation. Likewise if at some point the union of the sets of chosen and available roots is a W-translate of a subset of Ω, there is no need to pursue the line of investigation as the radical maximal abelian sets have already been treated (and for Φ of type E_7 or E_8 similar considerations apply to near-radical sets). Finally, whenever we reach a point where there are no further available roots, provided all negative roots have been excluded the set must be maximal abelian.

We end this section with the two rather technical results referred to above; these will be used repeatedly in the arguments showing the completeness of $\mathcal{S}(\Phi)$.

LEMMA 3.1. *Assume that Π_1 is a subset of Π and $\beta \in \Pi \setminus \Pi_1$ such that all roots in $\langle \Pi_1, \beta \rangle$ have the same length; write Z for the set of roots in $\langle \Pi_1, \beta \rangle$ of the form $\sum_{\alpha \in \Pi} n_\alpha \alpha$ with $n_\beta = 1$. Let Y be a set of positive roots preserved by $\langle w_\alpha : \alpha \in \Pi_1 \cup \{\beta\} \rangle$. If X is a set of positive roots with $Y \subset X$ such that X contains some but not all roots in Z, there is a positive W-translate of X which contains Y and meets $\langle \Pi_1 \rangle$.*

Proof. If X meets $\langle\Pi_1\rangle$ the result is clear, so assume $X \cap \langle\Pi_1\rangle = \emptyset$; choose $\gamma \in X \cap Z$. By Corollary 1.4, we may take simple roots $\alpha_1, \dots, \alpha_r$ such that $\beta \prec w_{\alpha_1}(\beta) \prec \dots \prec w_{\alpha_r} \dots w_{\alpha_1}(\beta) = \gamma$; so $\alpha_i \in \Pi_1$ for all i. Thus if we write $w = w_{\alpha_1} \dots w_{\alpha_r}$, then as X contains Y and consists of positive roots lying outside $\langle\Pi_1\rangle$ the same is true of $w(X)$, and $\beta = w(\gamma) \in w(X)$. Since w lies in $\langle w_\alpha : \alpha \in \Pi_1\rangle$ it stabilizes Z, so that $w(X)$ also contains some but not all of these roots. Thus by replacing X by $w(X)$ we may assume $\beta \in X$.

Now take $\gamma_1 \in Z \setminus X$, and as before take simple roots $\alpha_1, \dots, \alpha_r$ such that $\beta \prec w_{\alpha_1}(\beta) \prec \dots \prec w_{\alpha_r} \dots w_{\alpha_1}(\beta) = \gamma_1$; so again $\alpha_i \in \Pi_1$ for all i. Again, each $w_{\alpha_j} \dots w_{\alpha_r}(X)$ for $1 \le j \le r$ contains Y and consists of positive roots lying outside $\langle\Pi_1\rangle$, and as $\beta = w_{\alpha_1} \dots w_{\alpha_r}(\gamma_1)$ we have $\beta \notin w_{\alpha_1} \dots w_{\alpha_r}(X)$. Since $\beta \in X$ there exists a maximal j with $\beta \in w_{\alpha_{j+1}} \dots w_{\alpha_r}(X)$ but $\beta \notin w_{\alpha_j} \dots w_{\alpha_r}(X)$. Set $w' = w_{\alpha_{j+1}} \dots w_{\alpha_r}$ and $X' = w'(X)$; then $\beta \in X' \setminus w_{\alpha_j}(X')$. As β and α_j have the same length, we must therefore have $w_{\alpha_j}(\beta) = \beta + \alpha_j$; then $w_\beta w_{\alpha_j}(X')$ is a set of positive roots containing both Y and $w_\beta w_{\alpha_j}(\beta) = w_\beta(\beta + \alpha_j) = \alpha_j \in \langle\Pi_1\rangle$. □

The point of this result is the following. Suppose we are considering the possibilities for X containing the given set Y; we have already dealt with those which meet the set $\langle\Pi_1\rangle$, and we now wish to treat those which meet the set $\langle\Pi_1, \beta\rangle$. It is usually the case that Z is the set of positive roots in $\langle\Pi_1, \beta\rangle \setminus \langle\Pi_1\rangle$; by Lemma 3.1 we may then assume that X actually contains all the roots in Z, since otherwise by replacing X by a W-translate we revert to the situation already dealt with.

The second of these results will only be required in the treatment of the larger exceptional root systems.

Corollary 3.2. *Assume that all roots in Φ have the same length. Assume that Π_1 is a subset of Π and $\beta \in \Pi \setminus \Pi_1$; let Y be a set of positive roots preserved by $\langle w_\alpha : \alpha \in \Pi_1 \cup \{\beta\}\rangle$. Let $\beta' \in \Pi$ be orthogonal to all roots in $\Pi_1 \cup Y$ but not orthogonal to β, and set $\gamma = \beta + \beta' \in \Phi$. Let Π_2 be a subset of Π all of whose roots are orthogonal to $\Pi_1 \cup \{\beta\} \cup Y$, and set $\Pi_3 = \Pi_1 \cup \Pi_2$; for $i = 1, 2, 3$ set $W_i = \langle w_\alpha : \alpha \in \Pi_i\rangle$, so that $W_3 = W_1 \times W_2$. Suppose X is an abelian set of positive roots containing β and Y with $X \cap \langle\Pi_3\rangle = \emptyset$. If $X \cap W_3(\gamma)$ is not W_1-stable, there is a positive W-translate of X which contains Y and meets $\langle\Pi_1\rangle$.*

Proof. If $X \cap W_3(\gamma)$ is not W_1-stable, there exist $w \in W_3$ and $w' \in W_1$ with $w(\gamma) \in X$ but $w'w(\gamma) \notin X$. Since $W_3 = W_1 \times W_2$, we may write $w = w_{(1)}w_{(2)}$ with $w_{(i)} \in W_i$ for $i = 1, 2$; then as $w_{(1)}$ fixes β' and $w_{(2)}$ fixes β, we have

$$(\beta, w(\beta')) = (\beta, w_{(1)}w_{(2)}(\beta')) = ({w_{(2)}}^{-1}(\beta), w_{(1)}(\beta')) = (\beta, \beta') < 0.$$

Thus as X is abelian and contains β we must have $w(\beta') \notin X$. Set $X' = w^{-1}(X)$; then X' is an abelian set, consisting of positive roots (since $X \cap \langle\Pi_3\rangle = \emptyset$) and containing Y (since Y is preserved by W_1 and orthogonal to all roots in Π_2), and we have $\gamma \in X'$ but $\beta', w''(\gamma) \notin X'$ where $w'' = w^{-1}w'w \in W_1$. Set $X'' = w_{\beta'}(X')$; then X'' is similarly an abelian set consisting of positive roots, containing Y (since Y is preserved by $w_{\beta'}$), and we have $\beta = w_{\beta'}(\gamma) \in X''$ but $w''(\beta) = w''(w_{\beta'}(\gamma)) = w_{\beta'}(w''(\gamma)) \notin X''$ (since β' is orthogonal to the roots in Π_1). Since $w''(\beta)$ lies in the set Z of Lemma 3.1, that result may now be applied. □

In fact in many applications of these results the set Y will be empty, but the more general forms presented here will be needed on several occasions.

3.3. Identifying stabilizers and structure

Having determined the maximal abelian sets up to the action of W, we shall conclude by considering their structure: for each $X \in \mathcal{S}(\Phi)$, we shall identify its (setwise) stabilizer W_X in W and decompose X into W_X-orbits. Moreover, it is here that we shall prove conclusively that no two of the sets in $\mathcal{S}(\Phi)$ are W-translates of each other.

The key to all of this is the internal geometry of the set X; in particular we shall consider the relation of orthogonality on X. For $\beta \in X$ we define the *orthogonality count* $o(\beta)$ to be the number of roots in X which are orthogonal to β (for Φ of type G_2 or F_4 we refine this by counting long and short roots separately, so that $o(\beta)$ is an ordered pair). Clearly W_X must fix setwise the set of roots having any given orthogonality count. Indeed more generally if Z_1 and Z_2 are subsets of X which must be fixed setwise by W_X, then for any $n \leq |Z_2|$ the set of roots in Z_1 orthogonal to exactly n roots in Z_2 must also be fixed setwise by W_X.

Our method of determining W_X is then as follows. We shall produce certain elements of W which visibly stabilize X, and let G be the group they generate, so that $G \leq W_X$; we shall seek to show that in fact $W_X = G$. To do this we shall often find a subset X_1 of X which W_X must preserve and on which G acts transitively; we take $\beta_1 \in X_1$ and consider the stabilizers $(W_X)_{\beta_1}$ and G_{β_1}. Since

$$|W_X : (W_X)_{\beta_1}| = |X_1| = |G : G_{\beta_1}|,$$

it suffices to show that $(W_X)_{\beta_1} = G_{\beta_1}$. As $(W_X)_{\beta_1}$ must preserve the set of roots orthogonal to β_1, this frequently leads to a refinement of the partition of X. We may then seek a subset of X which $(W_X)_{\beta_1}$ must preserve and on which G_{β_1} acts transitively, and repeat the process. If we reach a point where we have stabilized enough roots to span $\mathbb{R}\Phi$ (note that X contains spanning sets by Lemma 1.7; in particular, we are in this situation if all roots of X are stabilized), then as W acts linearly the pointwise stabilizer in W_X must be trivial, and we may conclude that $W_X = G$ as desired.

This method may be modified in certain cases. For example, if there are sets which are visibly W_X-stable on which G acts as a direct product of symmetric groups, we may shorten the argument by taking the simultaneous stabilizer of all roots concerned.

In the final chapter of this work we shall provide tables detailing the results obtained for each of the exceptional root systems. In particular we shall give what we call the *signature* of each maximal abelian set X; this consists of the sequence of orthogonality counts of the roots in X, grouped into W_X-orbits. It will turn out that there are relatively few cases of two sets in the tables having the same signature; since the action of W clearly preserves signatures, in most cases this will provide an immediate proof that the set X is not in the same W-orbit as any other set in the table. For those instances of sets having the same signature, the orders of the stabilizers will sometimes suffice to show that the W-orbits are distinct; in other cases it will be necessary to consider the geometry more carefully (in addition, in E_6 there are three pairs of sets interchanged by the diagram automorphism, where separate arguments will be needed to obtain the desired conclusion).

Our final result in this section gives a particular feature possessed by maximal radical abelian sets in those exceptional root systems in which all roots have the same length. Recall that δ is the unique simple root not orthogonal to ρ, and δ' is the unique simple root having negative inner product with δ.

LEMMA 3.3. *Let Φ be of type E_6, E_7 or E_8, and X be a maximal radical abelian set; then $o(\beta)$ has the same parity for all $\beta \in X \setminus \{\rho\}$.*

PROOF. First consider the maximal radical abelian set $X = X^{(2)} \cup X^{(3)}$, where we write $X^{(j)} = \{\sum_{\alpha\in\Pi} m_\alpha \alpha \in \Phi : m_{\delta'} = j\}$ (so that $X^{(3)} = \{\rho, \rho - \delta\}$). The stabilizer of this set clearly contains $\langle w_\alpha : \alpha \in \Pi \setminus \{\delta'\}\rangle$, which acts transitively on $X^{(2)}$; thus all roots in $X^{(2)}$ have the same orthogonality count. Now $\sum_{\beta\in X} o(\beta)$ is clearly even (as orthogonality is a symmetric relation), and $o(\rho) = 0$; by inspection we see that $\operatorname{lht}\rho - 1 = |X^{(2)} \cup \{\rho - \delta\}|$ is also even. It follows that $o(\rho - \delta)$ must have the same parity as $o(\beta)$ for $\beta \in X^{(2)}$; thus the claim holds for this particular maximal radical abelian set.

Now consider the effect of taking a maximal radical abelian set X in which the equal parity condition holds and replacing one root β_0 by $\rho - \beta_0$. For all $\beta \in X\setminus\{\beta_0\}$ we have $(\beta, \rho) = (\beta, \beta_0) + (\beta, \rho - \beta_0)$; thus β is orthogonal to precisely one of β_0 and $\rho - \beta_0$. Replacing β_0 by $\rho - \beta_0$ will therefore change the orthogonality count of β by 1; so in the new set all roots apart from ρ and $\rho - \beta_0$ have the same parity of orthogonality count, and as above the final root $\rho - \beta_0$ must then also have the same parity for its orthogonality count. Since any maximal radical abelian set may be obtained from the set $X^{(2)} \cup X^{(3)}$ treated above by a succession of such replacements, the result holds. □

We conclude this section with some comments on notation. We write $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$, where ℓ is the rank of Φ and the simple roots are numbered as in [**1**]. For convenience, we shall often write w_i for the simple root reflection w_{α_i}; in addition, we shall write w_0 for the long word in the Weyl group, the unique element of W which maps each positive root to a negative root.

We shall write roots as linear combinations of simple roots, and represent them as ℓ-tuples of coefficients arranged as in a Dynkin diagram; thus for example according as Φ is of type G_2, F_4, E_6, E_7 or E_8 the high root ρ is denoted 32, 2342, $\begin{smallmatrix}1&2&3&2&1\\&&2&&\end{smallmatrix}$, $\begin{smallmatrix}2&3&4&3&2&1\\&&2&&&\end{smallmatrix}$ or $\begin{smallmatrix}2&4&6&5&4&3&2\\&&3&&&&\end{smallmatrix}$. Moreover, we shall often use dots to denote undetermined coefficients, our convention being that they may be replaced by any integers which yield a root; thus for example if Φ is of type F_4 the statement "$012\cdot \in X$" means that the set X contains all three roots 0120, 0121 and 0122, while "$12\cdot1 \notin X$" means that X contains neither 1221 nor 1231.

For the convenience of the reader, in an appendix at the end of this work we provide what we call the 'root tree' of each root system of exceptional type; this contains all the positive roots, appropriately arranged, and indicates the effect upon them of each of the elements w_i.

Finally, our notation for maximal abelian sets in root systems of exceptional type will be such that the size of a set is given by its subscript, while sets of the same size are distinguished by superscripts.

CHAPTER 4

The root system of type G_2

Let Φ be of type G_2; thus Φ has simple roots α_1, α_2 numbered as in [**1**], so that α_1 is short and α_2 is long.

4.1. Radical maximal abelian sets

We have $\rho = 32$, and the roots in Ξ are $\cdot 1$; there are 2 pairs of roots in Ξ summing to ρ, namely $\{01, 31\}$ and $\{11, 21\}$. Using $\mathrm{stab}_W(\rho) = \langle w_1 \rangle$ we may assume $31 \in X$, $01 \notin X$; thus up to the action of W there are 2 maximal radical abelian sets, each of which is easily seen to be maximal abelian:

$$X_3^1 = \{3\cdot, 21\},$$
$$X_3^2 = \{3\cdot, 11\}.$$

4.2. Determination of maximal abelian sets

We set

$$\mathcal{S}(G_2) = \{X_3^1, X_3^2\}.$$

As in section 3.2, we let X be any maximal abelian set consisting of positive roots and containing a simple root α; we seek to show that X is known, i.e., a W-translate of a set in $\mathcal{S}(G_2)$.

We work through the possibilities for the simple root α contained in X.

Lemma 4.1. *If $01 \in X$ then X is known.*

Proof. Take $01 \in X$; this excludes $10, 31$, so X is radical and hence known. □

Lemma 4.2. *If $10 \in X$ then X is known.*

Proof. Take $10 \in X$; this excludes $01, 11, 21$, giving $31 \in X$ by default; so

$$X = \{3\cdot, 10\} = w_2(X_3^2).$$

This proves the lemma. □

Combining Lemmas 4.1 and 4.2 we have proved the following.

Theorem 4.3. *If X is a maximal abelian set in a root system of type G_2, then a W-translate of X lies in $\mathcal{S}(G_2)$.*

4.3. Stabilizers and structure of maximal abelian sets

For each set $X \in \mathcal{S}(G_2)$ we shall determine its stabilizer W_X in W, and find the W_X-orbits in X. For $\beta \in X$ the orthogonality count $o(\beta)$ is the ordered pair (i, j), where i and j are the numbers of long and short roots in X which are orthogonal to β, respectively; for convenience we shall write the pair (i, j) simply as ij. We shall follow the basic approach explained in section 3.3. We shall take each possibility for X in turn.

If $X = X_3^1$ we must fix $\{3\cdot\}$ (being the set of roots β with β long and $o(\beta) = 00$) and 21 (β short, $o(\beta) = 00$). We set

$$G = \langle w_2 \rangle;$$

then G acts as S_2 on $\{3\cdot\}$, so we may fix both of these roots, and thus $W_X = G$.

If $X = X_3^2$ we must fix 32, 31 (β long, $o(\beta) = 00, 01$ respectively) and 11 (β short, $o(\beta) = 10$). We set

$$G = 1;$$

since all roots are fixed, we have $W_X = G$.

The results found here are presented in tabular form in the final chapter of this work. Since the sets in $\mathcal{S}(G_2)$ have distinct signatures, we have thus shown the following.

THEOREM 4.4. *The* 2 *sets in* $\mathcal{S}(G_2)$ *represent different* W*-orbits.*

CHAPTER 5

The root system of type F_4

Let Φ be of type F_4; thus Φ has simple roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ numbered as in [**1**], so that α_1, α_2 are long and α_3, α_4 are short, and α_2 is not orthogonal to α_3.

5.1. Radical maximal abelian sets

We have $\rho = 2342$, and the roots in Ξ are $1\cdots$; there are 3 pairs of short roots and 4 pairs of long roots in Ξ summing to ρ. The pairs of short roots in Ξ are

$$\{1110, 1232\}, \qquad \{1111, 1231\}, \qquad \{1121, 1221\}.$$

Using $\mathrm{stab}_W(\rho) = \langle w_2, w_3, w_4 \rangle$ we may assume $1232 \in X$, $1110 \notin X$; then using $\langle w_2, w_3 \rangle$ we may assume $1231 \in X$, $1111 \notin X$; finally using $\langle w_2 \rangle$ we may assume $1221 \in X$, $1121 \notin X$. Having thus determined the short roots in X, and noting that $\langle w_3, w_4 \rangle$ is the setwise stabilizer in $\mathrm{stab}_W(\rho)$ of the roots chosen so far, we turn to the long roots. The pairs of long roots in Ξ are

$$\{1000, 1342\}, \qquad \{1100, 1242\}, \qquad \{1120, 1222\}, \qquad \{1122, 1220\}.$$

Using $\langle w_3, w_4 \rangle$, we see that the set X is determined by (a) whether it contains 1342 or 1000, and (b) the number of long roots $12\cdot\cdot$ it contains; so up to the action of W there are $2.4 = 8$ maximal radical abelian sets.

We now consider which of these sets are maximal abelian. As -2342 and the roots $-1\cdots$ are excluded by 2342, it suffices to consider roots β of the form $0\cdots$. Each short root $0\cdots$ is excluded by four of the six short roots in Ξ, so certainly it is excluded by at least one short root in X. However, each long root $0\cdots$ is excluded by four long roots and one short root in Ξ, with the long roots concerned lying one in each pair (for example the roots in Ξ which exclude 0100 are $1000, 112\cdot, 1242$). Thus up to the action of W there is a single maximal radical abelian set which is not maximal abelian, namely that containing 1342 and exactly two long roots $12\cdot\cdot$ (for example if $X = \{\cdot\cdot 42, 123\cdot, 1221, 1\cdot 22\}$ then X is not maximal abelian as it lies in $X \cup \{0122\}$). We therefore have the radical maximal abelian sets

$$\begin{aligned}
X_8^1 &= \{\cdot 342, 12\cdot\cdot\},\\
X_8^2 &= \{\cdot\cdot 42, 123\cdot, 1221, 1122, 1120\},\\
X_8^3 &= \{\cdot 342, 123\cdot, 1221, 1122, 1120, 1100\},\\
X_8^4 &= \{2342, 12\cdot\cdot, 1000\},\\
X_8^5 &= \{2342, 12\cdot 2, 12\cdot 1, 1122, 1000\},\\
X_8^6 &= \{2342, 1242, 123\cdot, 1221, 1122, 1120, 1000\},\\
X_8^7 &= \{2342, 123\cdot, 1221, 1122, 1120, 1\cdot 00\}.
\end{aligned}$$

5.2. Determination of maximal abelian sets

We begin by giving one further maximal abelian set which is not radical (indeed, it is the set found above showing that one of the maximal radical abelian sets was not maximal abelian): we set

$$X_9 = \{\cdot\cdot\cdot 2, 12{\cdot}1\}.$$

We then set

$$\mathcal{S}(F_4) = \{X_8^1, \ldots, X_8^7, X_9\}.$$

As in section 3.2, we let X be any maximal abelian set consisting of positive roots and containing a simple root α; we seek to show that X is known, i.e., a W-translate of a set in $\mathcal{S}(F_4)$.

We work through the possibilities for the simple root α contained in X.

Lemma 5.1. *If* $1000 \in X$ *then* X *is known.*

Proof. Take $1000 \in X$; this excludes $01{\cdot\cdot}, 1342$. Unless X contains some root $00{\cdot\cdot}$ the set will be radical. Using Lemma 3.1 (with $Y = \{1000\}$) we may assume that one of the following holds: (i) $0010 \in X$; (ii) $0010 \notin X$, $00{\cdot}1 \in X$.

First assume (i) holds; this excludes the roots $0001, 1100, 111{\cdot}, 1221, 1222, 1232$, giving $1120, 1121, 1242 \in X$ by default. To exclude -0100 we must have $1220 \in X$, which then excludes $0011, 1122$, giving $1231 \in X$ by default; so

$$X = \{2342, 1242, 1231, 1{\cdot}20, 1121, 1000, 0010\} = w_2w_4w_3w_2w_1w_2(X_8^2).$$

Now assume (ii) holds; this excludes $0010, 11{\cdot}0, 1220, 1{\cdot\cdot}1$, giving $1122, 12{\cdot}2 \in X$ by default, so

$$X = \{2342, 12{\cdot}2, 1122, 00{\cdot}1, 1000\} = w_2w_3w_2w_1w_2w_3w_2(X_8^5).$$

This proves the lemma. □

Lemma 5.2. *If* $0100 \in X$ *then* X *is known.*

Proof. By Lemma 5.1 we may assume $1000 \notin X$. Take $0100 \in X$; then by Lemma 3.1 we may assume ${\cdot}100 \in X$ (otherwise we revert to the case covered in Lemma 5.1), which excludes $001{\cdot}, {\cdot}12{\cdot}, 1242$, giving $1222, 1342 \in X$ by default. First assume $0001 \in X$; this excludes ${\cdot}110, 1220, 12{\cdot}1$, giving $1232 \in X$ by default, and up to the action of $\langle w_1 \rangle$ we have

$$X = \{{\cdot}342, 1232, 1222, 1111, {\cdot}100, 0001\} = w_3w_2w_1w_3w_2(X_8^6).$$

Thus assume $0001 \notin X$, which gives $1220, 1221 \in X$ by default. If X contains no root ${\cdot}11{\cdot}$ then by default we have $123{\cdot} \in X$, so

$$X = \{{\cdot}342, 123{\cdot}, 122{\cdot}, {\cdot}100\} = w_3w_4(X_9).$$

So assume X contains some root ${\cdot}11{\cdot}$; using $\langle w_1, w_4 \rangle$ we may assume $1111 \in X$, which excludes $011{\cdot}, 1231$. If $1110 \in X$ this excludes 1232, so

$$X = \{{\cdot}342, 122{\cdot}, 111{\cdot}, {\cdot}100\} = w_3w_4w_2w_3w_2(X_9).$$

If on the other hand $1110 \notin X$ this gives $1232 \in X$ by default, so

$$X = \{{\cdot}342, 1232, 122{\cdot}, 1111, {\cdot}100\} = w_3w_4w_2(X_9).$$

This proves the lemma. □

Lemma 5.3. *If* $0001 \in X$ *then* X *is known.*

PROOF. By Lemmas 5.1, 5.2 we may assume $\cdot\cdot 00 \notin X$. Take $0001 \in X$; this excludes $\cdot\cdot 10$, $\cdot\cdot 20$, $\cdot\cdot 21$, 1231, giving $0122, 1\cdot\cdot 2 \in X$ by default. Using $\langle w_1, w_2 \rangle$ we may assume $0011 \in X$; this excludes $\cdot 111$, so

$$X = \{\cdot\cdot\cdot 2, 00\cdot 1\} = w_2w_3w_1w_2w_3w_1w_2(X_9).$$

This proves the lemma. □

LEMMA 5.4. *If* $0010 \in X$ *then* X *is known.*

PROOF. By Lemmas 5.1, 5.2, 5.3 we may assume $\cdot\cdot 00, 0001 \notin X$. Take $0010 \in X$; by Lemma 3.1 we may in fact assume $001\cdot \in X$. This excludes $\cdot 11\cdot$, $122\cdot$, $123\cdot$, giving $1\cdot 42 \in X$ by default, so

$$X \subset \{\cdot\cdot 42, \cdot 12\cdot, 001\cdot\} \subset w_2w_1(\Omega).$$

This proves the lemma. □

Combining Lemmas 5.1, 5.2, 5.3 and 5.4 we have proved the following.

THEOREM 5.5. *If* X *is a maximal abelian set in a root system of type* F_4*, then a* W*-translate of* X *lies in* $\mathcal{S}(F_4)$.

5.3. Stabilizers and structure of maximal abelian sets

For each set $X \in \mathcal{S}(F_4)$ we shall determine its stabilizer W_X in W, and find the W_X-orbits in X. For $\beta \in X$ the orthogonality count $o(\beta)$ is the ordered pair (i, j), where i and j are the numbers of long and short roots in X which are orthogonal to β, respectively; for convenience we shall write the pair (i, j) simply as ij. We shall follow the basic approach explained in section 3.3; in fact in each case we shall end up stabilizing the roots $2342, 1232, 1231, 1221$, which span $\mathbb{R}\Phi$. We shall work through the possibilities for X in turn.

If $X = X_8^1$ we must fix $\{\cdot 342\}, \{1242, 1222, 1220\}$ (being the sets of roots β with β long and $o(\beta) = 00, 21$ respectively) and $\{123\cdot, 1221\}$ (β short, $o(\beta) = 10$). We set

$$G = \langle w_1, w_3, w_4 \rangle;$$

then G acts as S_3 on $\{123\cdot, 1221\}$ and independently as S_2 on $\{\cdot 342\}$, so we may fix all of these roots, and thus $W_X = G$.

If $X = X_8^2$ we must fix $2342, 1342, 1242, \{1122, 1120\}$ (β long, $o(\beta) = 00, 20, 01, 22$ respectively) and $\{123\cdot\}, 1221$ (β short, $o(\beta) = 10, 30$ respectively). We set

$$G = \langle w_4 \rangle;$$

then G acts as S_2 on $\{123\cdot\}$, so we may fix both of these roots, and thus $W_X = G$.

If $X = X_8^3$ we must fix $2342, 1342, \{1122, 1120, 1100\}$ (β long, $o(\beta) = 00, 30, 32$ respectively) and $\{123\cdot, 1221\}$ (β short, $o(\beta) = 20$). We set

$$G = \langle w_3, w_4 \rangle;$$

then G acts as S_3 on $\{123\cdot, 1221\}$, so we may fix all of these roots, and thus $W_X = G$.

If $X = X_8^4$ we must fix $2342, \{1242, 1222, 1220\}, 1000$ (β long, $o(\beta) = 00, 31, 33$ respectively) and $\{123\cdot, 1221\}$ (β short, $o(\beta) = 20$). We set

$$G = \langle w_3, w_4 \rangle;$$

then G acts as S_3 on $\{123\cdot, 1221\}$, so we may fix all of these roots, and thus $W_X = G$.

If $X = X_8^5$ we must fix $\scriptstyle 2342, \{1242, 1222\}, 1122, 1000$ (β long, $o(\beta) = 00, 21, 02, 23$ respectively) and $\scriptstyle 1232, \{12\cdot1\}$ (β short, $o(\beta) = 10, 30$ respectively). We set

$$G = \langle w_3 \rangle;$$

then G acts as S_2 on $\{\scriptstyle 12\cdot1\}$, so we may fix both of these roots, and thus $W_X = G$.

If $X = X_8^6$ we must fix $\scriptstyle 2342, 1242, \{1122, 1120\}, 1000$ (β long, $o(\beta) = 00, 11, 12, 13$ respectively) and $\{\scriptstyle 123\cdot\}, 1221$ (β short, $o(\beta) = 20, 40$ respectively). We set

$$G = \langle w_4 \rangle;$$

then G acts as S_2 on $\{\scriptstyle 123\cdot\}$, so we may fix both of these roots, and thus $W_X = G$.

If $X = X_8^7$ we must fix $\scriptstyle 2342, \{1122, 1120, 1100\}, 1000$ (β long, $o(\beta) = 00, 22, 03$ respectively) and $\{\scriptstyle 123\cdot, 1221\}$ (β short, $o(\beta) = 30$). We set

$$G = \langle w_3, w_4 \rangle;$$

then G acts as S_3 on $\{\scriptstyle 123\cdot, 1221\}$, so we may fix all of these roots, and thus $W_X = G$.

If $X = X_9$ we must fix $\{\scriptstyle \cdot342\}, \{1242, 1222\}, \{\cdot122\}$ (β long, $o(\beta) = 10, 11, 12$ respectively) and $\scriptstyle 1232, \{12\cdot1\}$ (β short, $o(\beta) = 00, 30$ respectively). We set

$$G = \langle w_1, w_3 \rangle;$$

then G acts as S_2 on each of $\{\scriptstyle \cdot342\}$ and $\{\scriptstyle 12\cdot1\}$ independently, so we may fix all of these roots, and thus $W_X = G$.

The results found here are presented in tabular form in the final chapter of this work. In each case we may observe that each set $\{\beta \in X : o(\beta) = ij\}$ is in fact a single W_X-orbit. Since all the sets in $\mathcal{S}(F_4)$ have distinct signatures, we have thus shown the following.

THEOREM 5.6. *The* 8 *sets in* $\mathcal{S}(F_4)$ *represent different* W*-orbits.*

CHAPTER 6

The root system of type E_6

Let Φ be of type E_6; thus Φ has simple roots $\alpha_1, \ldots, \alpha_6$ numbered as in [**1**]. We write θ for the diagram automorphism which fixes α_2 and α_4, and interchanges α_1 and α_3 with α_6 and α_5 respectively.

6.1. Radical maximal abelian sets

We have $\rho = \begin{smallmatrix}12321\\2\end{smallmatrix}$, and the roots in Ξ are $\begin{smallmatrix}\cdot\cdot\cdot\cdot\cdot\\1\end{smallmatrix}$; there are 10 pairs of roots in Ξ summing to ρ, namely $\{\begin{smallmatrix}00000\\1\end{smallmatrix}, \begin{smallmatrix}12321\\1\end{smallmatrix}\}$ and 9 of the form $\{\begin{smallmatrix}\cdot\cdot1\cdot\cdot\\1\end{smallmatrix}, \begin{smallmatrix}\cdot\cdot2\cdot\cdot\\1\end{smallmatrix}\}$. We may arrange these as follows.

$$\{\begin{smallmatrix}11111\\1\end{smallmatrix}, \begin{smallmatrix}01210\\1\end{smallmatrix}\}$$

$$\{\begin{smallmatrix}11110\\1\end{smallmatrix}, \begin{smallmatrix}01211\\1\end{smallmatrix}\} \qquad \{\begin{smallmatrix}01111\\1\end{smallmatrix}, \begin{smallmatrix}11210\\1\end{smallmatrix}\}$$

$$\{\begin{smallmatrix}11100\\1\end{smallmatrix}, \begin{smallmatrix}01221\\1\end{smallmatrix}\} \qquad \{\begin{smallmatrix}01110\\1\end{smallmatrix}, \begin{smallmatrix}11211\\1\end{smallmatrix}\} \qquad \{\begin{smallmatrix}00111\\1\end{smallmatrix}, \begin{smallmatrix}12210\\1\end{smallmatrix}\}$$

$$\{\begin{smallmatrix}01100\\1\end{smallmatrix}, \begin{smallmatrix}11221\\1\end{smallmatrix}\} \qquad \{\begin{smallmatrix}00110\\1\end{smallmatrix}, \begin{smallmatrix}12211\\1\end{smallmatrix}\}$$

$$\{\begin{smallmatrix}00100\\1\end{smallmatrix}, \begin{smallmatrix}12221\\1\end{smallmatrix}\}$$

$$\{\begin{smallmatrix}00000\\1\end{smallmatrix}, \begin{smallmatrix}12321\\1\end{smallmatrix}\}$$

We shall regard the pair at the bottom of this array as isolated. We shall call a root in Ξ *odd* or *even* according to the parity of its α_4-coefficient; thus each pair consists of an odd root and an even root. We may then specify a maximal radical abelian set X by simply giving the parity of the root selected in each of the 10 pairs; for convenience we shall represent the set X graphically.

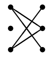
To begin with, we note that $w_0 w_\rho$ preserves ρ and maps each root β of Ξ to $-\theta w_\rho(\beta) = \rho - \theta(\beta)$; in particular it interchanges the two roots in the isolated pair. It therefore suffices to consider maximal radical abelian sets containing $\begin{smallmatrix}12321\\1\end{smallmatrix}$. We may then identify the 9 non-isolated pairs with unordered pairs $\{i, j\}$ with $i \in \{1, 2, 3\}$ and $j \in \{4, 5, 6\}$; we shall write the unordered pair $\{i, j\}$ simply as ij, and we give the correspondence by the following array.

$$\begin{array}{ccccc} & & 34 & & \\ & 24 & & 35 & \\ 14 & & 25 & & 36 \\ & 15 & & 26 & \\ & & 16 & & \end{array}$$

We may then represent a maximal radical abelian set X containing $\begin{smallmatrix}12321\\1\end{smallmatrix}$ by a bipartite graph Γ_X with vertex set $\{1, 2, 3\} \cup \{4, 5, 6\}$, where the choice in X of the odd or even root in the pair identified with ij is denoted by the presence or absence in Γ_X of the edge ij. (Alternatively, we may regard this as the choice of a black edge or a white edge.) We shall arrange the vertices as follows:

$$\begin{matrix}3 & 4 & \\ 2 & 5 & \\ 1 & 6 & \cdot\end{matrix}$$

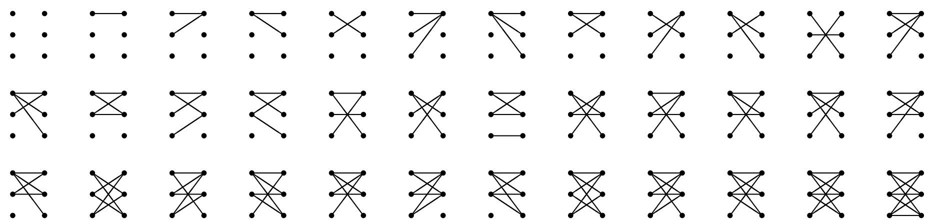
For example, if $X = \{\substack{12\cdot\cdot1\\ \cdot}, \substack{11211\\ 1}, \substack{0121\cdot\\ 1}, \substack{0\cdot111\\ 1}, \substack{\cdot1100\\ 1}\}$ then Γ_X is

 .

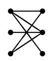
We now consider the action of $\mathrm{stab}_W(\rho) = \langle w_1, w_3, w_4, w_5, w_6\rangle$ on the roots in Ξ. Within this group the (pointwise) stabilizer W' of the isolated pair is $\langle w_1, w_3, w_5, w_6\rangle$, which does not affect α_4-coefficients and therefore permutes edges without changing colours; in fact the notation chosen for the pairs means that the generating elements of $W' \cong S_3 \times S_3$ act as permutations of the vertices as follows:

$$w_1 = (4\ 5),\ w_3 = (5\ 6),\ w_5 = (1\ 2),\ w_6 = (2\ 3).$$

Thus two maximal radical abelian sets containing $\substack{12321\\ 1}$ lie in the same W'-orbit if and only if their graphs are isomorphic (where we require an isomorphism to preserve rather than interchange the vertex sets $\{1, 2, 3\}$ and $\{4, 5, 6\}$). The bipartite graphs up to isomorphism are as follows.

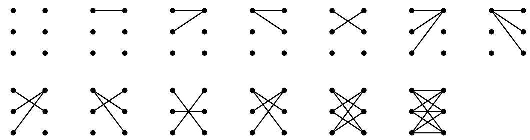


The remaining generator of $\mathrm{stab}_W(\rho)$ is w_4. In determining its effect, we shall assume that in the graph Γ_X being considered the edge 16 is absent (or white), so that the corresponding set X contains $\substack{12221\\ 1}$ and hence $w_4(X)$ still contains $\substack{12321\\ 1}$. (Since W' acts transitively on edges, this assumption excludes only the last of the 36 graphs above.) If we set $\{i, k\} = \{2, 3\}$ and $\{j, \ell\} = \{4, 5\}$, each edge ij in Γ_X gives rise to the edge $k\ell$ in $\Gamma_{w_4(X)}$ of the opposite colour (alternatively, the presence or absence of ij gives rise to the absence or presence of $k\ell$ respectively). Thus for example with the set X as above, in the graph Γ_X the edge 35 is present (or black), and so in $\Gamma_{w_4(X)}$ the edge 24 is absent (or white); on the other hand in Γ_X the edge 34 is absent (or white), and so in $\Gamma_{w_4(X)}$ the edge 25 is present (or black). Treating all four edges between $\{2, 3\}$ and $\{4, 5\}$ thus, we see that the graph $\Gamma_{w_4(X)}$ is

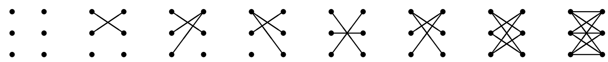
 ,

corresponding to $w_4(X) = \{\substack{12\cdot\cdot1\\ \cdot}, \substack{01211\\ 1}, \substack{\cdot\cdot111\\ 1}, \substack{01110\\ 1}, \substack{\cdot1100\\ 1}\}$.

Thus if one of the graphs above contains a white edge such that among the remaining four vertices there are at least three black edges, we may use the group $\langle w_1, w_3, w_5, w_6\rangle$ to move the white edge to 16 and then apply w_4 to reduce the number of black edges; thus the set is a W-translate of one appearing earlier on the list, and so need be considered no further. This disposes of the eighth and twelfth in the top row, all but the fifth, sixth and seventh in the middle row and all but the second and twelfth in the bottom row. Moreover, the fifth and sixth in the middle row are in fact interchanged by w_4; also the seventh in the middle row may be transformed to the twelfth in the bottom row by applying first w_4 and then the element $w_0 w_\rho$ mentioned above. We thus have the following 13 graphs representing maximal radical abelian sets up to the action of W.



We now need to decide which of these sets are maximal abelian. The roots $-\substack{12321\\2}, -\substack{\cdot\cdot\cdot\cdot\cdot\\1}$ are excluded by $\substack{12321\\2}$, while $-\substack{\cdot\cdot 1\cdot\cdot\\0}$ are excluded by $\substack{12321\\1}$. To exclude $\substack{10000\\0}$ we must have some root $\substack{01\cdot\cdot\cdot\\1}$ in X; using $\langle w_1, w_3\rangle$ we see that to exclude the roots $\substack{\cdot\cdot 000\\0}$ there cannot be two vertices in $\{4,5,6\}$ such that all edges from one are black and all from the other are white, which disposes of the sixth set. Likewise to exclude the roots $\substack{000\cdot\cdot\\0}$ there cannot be two such vertices in $\{1,2,3\}$, which disposes of the seventh set. Finally, to exclude $\substack{00100\\0}$ we must have $\substack{00000\\1}$, $\substack{12221\\1}$ or some root $\substack{\cdot 111\cdot\\1}$ in X; using $\langle w_1, w_3, w_5, w_6\rangle$ we see that to exclude the roots $\substack{\cdot\cdot 1\cdot\cdot\\0}$ the graph must not contain a black edge such that the four non-adjacent edges are all white, which disposes of the second, third and fourth sets. We are thus left with the following 8 graphs representing radical maximal abelian sets.



We therefore set

$$
\begin{aligned}
X_{11}^1 &= \{\substack{12321\\\cdot}, \substack{\cdot\cdot 2\cdot\cdot\\1}\},\\
X_{11}^2 &= \{\substack{12\cdot 21\\\cdot}, \substack{1221\cdot\\1}, \substack{\cdot 1221\\1}, \substack{11211\\1}, \substack{01210\\1}, \substack{11110\\1}, \substack{01111\\1}\},\\
X_{11}^3 &= \{\substack{12\cdot\cdot\cdot\\\cdot}, \substack{112\cdot 1\\1}, \substack{01210\\1}, \substack{111\cdot 0\\1}, \substack{01111\\1}\},\\
X_{11}^4 &= \{\substack{\cdot\cdot\cdot 21\\\cdot}, \substack{1\cdot 211\\1}, \substack{01210\\1}, \substack{0\cdot 111\\1}, \substack{11110\\1}\},\\
X_{11}^5 &= \{\substack{12\cdot 21\\\cdot}, \substack{12211\\1}, \substack{11221\\1}, \substack{11210\\1}, \substack{01211\\1}, \substack{01\cdot 10\\1}, \substack{11100\\1}, \substack{00111\\1}\},\\
X_{11}^6 &= \{\substack{12321\\\cdot}, \substack{1\cdot 2\cdot 1\\1}, \substack{01210\\1}, \substack{111\cdot 0\\1}, \substack{0\cdot 111\\1}\},\\
X_{11}^7 &= \{\substack{12\cdot 21\\\cdot}, \substack{11211\\1}, \substack{01210\\1}, \substack{111\cdot 0\\1}, \substack{0\cdot 111\\1}, \substack{01100\\1}, \substack{00110\\1}\},\\
X_{11}^8 &= \{\substack{12321\\\cdot}, \substack{\cdot\cdot 1\cdot\cdot\\1}\}.
\end{aligned}
$$

6.2. Determination of maximal abelian sets

We begin by giving some maximal abelian sets which are not radical; we set

$$
\begin{aligned}
X_{12} &= \{\substack{1\cdot\cdot\cdot 1\\\cdot}, \substack{1\cdot 210\\1}, \substack{012\cdot 1\\1}\},\\
X_{13}^1 &= \{\substack{12\cdot\cdot\cdot\\\cdot}, \substack{\cdot 1\cdot\cdot 1\\\cdot}\},\\
X_{13}^2 &= \{\substack{\cdot\cdot\cdot 21\\\cdot}, \substack{1\cdot\cdot 1\cdot\\\cdot}\},\\
X_{16}^1 &= \{\substack{1\cdot\cdot\cdot\cdot\\\cdot}\},\\
X_{16}^2 &= \{\substack{\cdot\cdot\cdot\cdot 1\\\cdot}\},
\end{aligned}
$$

so that X_{13}^2 and X_{16}^2 are the images under θ of X_{13}^1 and X_{16}^1 respectively. We set

$$\mathcal{S}(E_6) = \{X_{11}^1, \ldots, X_{11}^8, X_{12}, X_{13}^1, X_{13}^2, X_{16}^1, X_{16}^2\}.$$

As in section 3.2, we let X be any maximal abelian set consisting of positive roots and containing a simple root α; we seek to show that X is known, i.e., a W-translate of a set in $\mathcal{S}(E_6)$. Here we shall be able to make use of the automorphism θ to shorten the argument in places (noting that the set $\mathcal{S}(E_6)$ is stable under θ).

We work through the possibilities for the simple root α contained in X.

Lemma 6.1. *If* $\substack{00000\\1} \in X$ *then* X *is known.*

Proof. Assume $\substack{00000\\1} \in X$; this excludes $\substack{\cdot\cdot 1\cdot\cdot\\0}$, $\substack{12321\\1}$. Unless X contains some root $\substack{\cdot\cdot 000\\0}$ or $\substack{000\cdot\cdot\\0}$ the set will be radical; using θ and Lemma 3.1 (with $Y = \{\substack{00000\\1}\}$) we may assume that one of the following holds: (i) $\substack{01000\\0} \in X$; (ii) $\substack{01000\\0}, \substack{00010\\0} \notin X$, $\substack{1\cdot 000\\0} \in X$.

First assume (i) holds; this excludes $\substack{10000\\0}$, $\substack{001\cdot\cdot\\1}$, $\substack{112\cdot\cdot\\1}$, giving $\substack{01111\\1}, \substack{12221\\1} \in X$ by default. Using Lemma 3.1 again (with $Y = \{\substack{00000\\1}, \substack{01000\\0}\}$) we have three possibilities to consider: (a) $\substack{00010\\0} \in X$; (b) $\substack{00010\\0} \notin X$, $\substack{000\cdot 1\\0} \in X$; (c) $\substack{000\cdot\cdot\\0} \notin X$. If (a) holds, this excludes $\substack{00001\\0}$, $\substack{\cdot 1100\\1}$, $\substack{01211\\1}$, $\substack{12211\\1}$, giving $\substack{\cdot 1110\\1} \in X$ by default. To exclude $-\substack{00100\\0}$ we must have $\substack{01210\\1} \in X$, which excludes $\substack{11000\\0}$, $\substack{00011\\0}$, $\substack{11111\\1}$, giving $\substack{12210\\1}, \substack{01221\\1} \in X$ by default; so

$$\begin{aligned} X &= \{\substack{12321\\2}, \substack{12221\\1}, \substack{12210\\1}, \substack{01221\\1}, \substack{01\cdot 10\\1}, \substack{01111\\1}, \substack{11110\\1}, \substack{00000\\1}, \substack{01000\\0}, \substack{00010\\0}\} \\ &= w_4 w_1 w_6 w_3 w_5 w_4 w_2 (X_{11}^2). \end{aligned}$$

If instead (b) holds, this excludes $\substack{\cdot 11\cdot 0\\1}$, $\substack{01210\\1}$, $\substack{12210\\1}$, giving $\substack{11111\\1}, \substack{12211\\1} \in X$ by default. To exclude $-\substack{00100\\0}$ we must have $\substack{01211\\1} \in X$, which excludes $\substack{11000\\0}$, giving $\substack{01221\\1} \in X$ by default; so

$$X = \{\substack{12321\\2}, \substack{122\cdot 1\\1}, \substack{012\cdot 1\\1}, \substack{\cdot 1111\\1}, \substack{000\cdot 1\\0}, \substack{00000\\1}, \substack{01000\\0}\} = w_1 w_4 w_3 w_5 w_4 w_2 (X_{11}^4).$$

Finally if (c) holds, then $\substack{011\cdot 0\\1}, \substack{1221\cdot\\1} \in X$ by default. To exclude $-\substack{00100\\0}$ we must have some root $\substack{0121\cdot\\1}$ present; using $\langle w_6 \rangle$ we may assume $\substack{01210\\1} \in X$, which excludes $\substack{11000\\0}$, $\substack{11111\\1}$. To exclude $-\substack{00111\\0}$ we must have some root $\substack{012\cdot 1\\1}$ present; using $\langle w_5 \rangle$ we may assume $\substack{01211\\1} \in X$, which excludes $\substack{11110\\1}$. To exclude $-\substack{10000\\0}$ we must have $\substack{11100\\1} \in X$, which excludes $\substack{01221\\1}$; so

$$X = \{\substack{12321\\2}, \substack{122\cdot\cdot\\1}, \substack{01\cdot 1\cdot\\1}, \substack{\cdot 1100\\1}, \substack{00000\\1}, \substack{01000\\0}\} = w_4 w_5 w_4 w_1 w_6 w_3 w_5 w_4 (X_{12}).$$

Thus we now assume instead that (ii) holds; this excludes $\substack{0\cdot 1\cdot\cdot\\1}$, $\substack{012\cdot\cdot\\1}$, giving $\substack{11111\\1}, \substack{1\cdot 2\cdot 1\\1} \in X$ by default. Using Lemma 3.1 again (with $Y = \{\substack{00000\\1}, \substack{1\cdot 000\\0}\}$) we have two possibilities to consider: (a) $\substack{000\cdot 1\\0} \in X$; (b) $\substack{000\cdot 1\\0} \notin X$. If (a) holds, this excludes $\substack{1\cdot\cdot\cdot 0\\1}$, so

$$X = \{\substack{12321\\2}, \substack{1\cdot 2\cdot 1\\1}, \substack{11111\\1}, \substack{1\cdot 000\\0}, \substack{000\cdot 1\\0}, \substack{00000\\1}\} = w_4 w_3 w_5 w_4 w_2 (X_{11}^6).$$

If instead (b) holds, then $\substack{1\cdot\cdot\cdot 0\\1} \in X$ by default, so

$$X = \{\substack{12321\\2}, \substack{1\cdot 2\cdot 1\\1}, \substack{11111\\1}, \substack{1\cdot\cdot\cdot 0\\1}, \substack{1\cdot 000\\0}, \substack{00000\\1}\} = w_4 w_3 w_5 w_6 w_4 w_5 (X_{13}^2).$$

This proves the lemma. □

Lemma 6.2. *If* $\substack{00100\\0} \in X$ *then* X *is known.*

PROOF. As before we assume $\substack{00000\\1} \notin X$, $\substack{00100\\\cdot} \in X$; this excludes $\substack{\cdot 1000\\0}$, $\substack{0001\cdot\\0}$, $\substack{\cdot 111\cdot\\\cdot}$, $\substack{12221\\1}$, giving $\substack{11211\\1}$, $\substack{12321\\1} \in X$ by default. Using θ we may divide into three possibilities: (i) $\substack{10000\\0}$, $\substack{00001\\0} \in X$; (ii) $\substack{10000\\0} \in X$, $\substack{00001\\0} \notin X$; (iii) $\substack{10000\\0}$, $\substack{00001\\0} \notin X$.

First assume (i) holds; this excludes the roots $\substack{01100\\\cdot}$, $\substack{00110\\\cdot}$, $\substack{012\cdot\cdot\\1}$, $\substack{1\cdot 210\\1}$, giving $\substack{12211\\1}$, $\substack{11221\\1} \in X$ by default, so

$$X \subset \{\substack{12321\\\cdot}, \substack{1\cdot 211\\1}, \substack{11221\\1}, \substack{11100\\\cdot}, \substack{00111\\\cdot}, \substack{00100\\\cdot}, \substack{10000\\0}, \substack{00001\\0}\} \subset w_3w_5w_4w_2(\Omega).$$

Now assume (ii) holds; this excludes $\substack{01100\\\cdot}$, $\substack{012\cdot\cdot\\1}$, giving $\substack{11210\\1}$, $\substack{11221\\1} \in X$ by default. To exclude $-\substack{01000\\0}$ we must have some root $\substack{1221\cdot\\1}$ present; using $\langle w_6 \rangle$ we may assume $\substack{12210\\1} \in X$, which excludes $\substack{00111\\\cdot}$. If $\substack{12211\\1} \notin X$ we have

$$X \subset \{\substack{12321\\\cdot}, \substack{112\cdot\cdot\\1}, \substack{12210\\1}, \substack{11100\\\cdot}, \substack{001\cdot 0\\\cdot}, \substack{10000\\0}\} \subset w_6w_3w_5w_4w_2(\Omega);$$

so we may assume $\substack{12211\\1} \in X$, which excludes $\substack{00110\\\cdot}$, giving $\substack{11100\\\cdot} \in X$ by default, so

$$X = \{\substack{12321\\\cdot}, \substack{112\cdot\cdot\\1}, \substack{1221\cdot\\1}, \substack{11100\\\cdot}, \substack{00100\\\cdot}, \substack{10000\\0}\} = w_3w_5w_6w_4w_5w_2w_4(X_{12}).$$

Finally assume (iii) holds; then $\substack{0121\cdot\\1}$, $\substack{11210\\1} \in X$ by default. If $\substack{\cdot 1100\\\cdot}$, $\substack{0011\cdot\\\cdot} \notin X$, then $\substack{1221\cdot\\1}$, $\substack{\cdot 1221\\1} \in X$ by default, so

$$X = \{\substack{12321\\\cdot}, \substack{\cdot 1221\\1}, \substack{\cdot\cdot 21\cdot\\1}, \substack{00100\\\cdot}\} = w_3w_5w_1w_6(X_{12}).$$

So we may assume X contains some root $\substack{\cdot 1100\\\cdot}$ or $\substack{0011\cdot\\\cdot}$; using θ and $\langle w_1 \rangle$ we may assume it contains some root $\substack{01100\\\cdot}$. However, unless we actually have $\substack{01100\\\cdot} \in X$ we could apply w_3 to produce a positive set meeting $\{\substack{00\cdot 00\\\cdot}\}$ in a proper non-empty subset of $\{\substack{00100\\\cdot}\}$, and then by Lemmas 3.1 and 6.1 X would be known. Thus we may assume $\substack{01100\\\cdot} \in X$, which excludes $\substack{0011\cdot\\\cdot}$, $\substack{11221\\1}$, giving $\substack{1221\cdot\\1} \in X$ by default; to exclude $-\substack{00010\\0}$ we must have $\substack{01221\\1} \in X$, which excludes $\substack{11100\\\cdot}$, so

$$X = \{\substack{12321\\\cdot}, \substack{01221\\1}, \substack{\cdot\cdot 21\cdot\\1}, \substack{0\cdot 100\\\cdot}\} = w_1w_3w_5w_6(X^1_{13}).$$

This proves the lemma. □

LEMMA 6.3. *If* $\substack{01000\\0} \in X$ *or* $\substack{00010\\0} \in X$ *then* X *is known.*

PROOF. Using θ, as before we assume $\substack{00\cdot 00\\\cdot} \notin X$, $\substack{01\cdot 00\\\cdot} \in X$; this excludes $\substack{10000\\0}$, $\substack{00\cdot 1\cdot\\\cdot}$, $\substack{1111\cdot\\\cdot}$, $\substack{112\cdot\cdot\\1}$, giving $\substack{12\cdot\cdot 1\\1} \in X$ by default. Unless we have $\substack{01110\\\cdot} \in X$ we could apply w_5 to produce a positive set meeting $\{\substack{0\cdot\cdot 00\\\cdot}\}$ in a proper non-empty subset of $\{\substack{01\cdot 00\\\cdot}\}$, and then by Lemmas 3.1, 6.1 and 6.2 X would be known. Thus we may assume $\substack{01110\\\cdot} \in X$, which excludes $\substack{00001\\0}$, $\substack{11100\\\cdot}$, giving $\substack{01111\\\cdot}$, $\substack{12210\\1} \in X$ by default. To exclude $-\substack{10000\\0}$ we must have $\substack{11000\\0} \in X$, which excludes $\substack{012\cdot\cdot\\1}$; so

$$X = \{\substack{12\cdot\cdot\cdot\\\cdot}, \substack{011\cdot\cdot\\\cdot}, \substack{\cdot 1000\\0}\} = w_4w_2w_5w_6w_4w_5w_1(X^2_{13}).$$

This proves the lemma. □

LEMMA 6.4. *If* $\substack{10000\\0} \in X$ *or* $\substack{00001\\0} \in X$ *then* X *is known.*

PROOF. Using θ, as before we assume $\substack{0\cdot\cdot\cdot 0\\\cdot} \notin X$, $\substack{1\cdot\cdot\cdot 0\\\cdot} \in X$; this excludes $\substack{0\cdot\cdot\cdot 1\\\cdot}$, giving $\substack{11111\\0}$, $\substack{1\cdot\cdot\cdot 1\\1} \in X$ by default, so

$$X = \{\substack{1\cdot\cdot\cdot\cdot\\\cdot}\} = X^1_{16}.$$

This proves the lemma. □

Combining Lemmas 6.1, 6.2, 6.3 and 6.4 we have proved the following.

THEOREM 6.5. *If X is a maximal abelian set in a root system of type E_6, then a W-translate of X lies in $\mathcal{S}(E_6)$.*

6.3. Stabilizers and structure of maximal abelian sets

For each set $X \in \mathcal{S}(E_6)$ we shall determine its stabilizer W_X in W, and find the W_X-orbits on X. Recall that for $\beta \in X$ the orthogonality count $o(\beta)$ is simply the number of roots in X which are orthogonal to β. If in fact X is radical, we may read off the orthogonality counts from the graph Γ_X. Here two roots represented by (black or white) edges in Γ_X are orthogonal if and only if the edges either meet at a vertex and are of different colours, or do not meet and are of the same colour. Suppose Γ_X has e (black) edges. We then have $o(\genfrac{}{}{0pt}{}{12321}{2}) = 0$ and $o(\genfrac{}{}{0pt}{}{12321}{1}) = e$; for any $\beta \in X$ represented by a (black) edge in Γ_X which meets t others, we have $o(\beta) = 1 + (4 - t) + (e - t - 1) = 4 + e - 2t$; for any $\beta \in X$ represented by an absent (white) edge in Γ_X which meets t (black) edges, we have $o(\beta) = (4 - (e - t)) + t = 4 - e + 2t$. (We observe that, as given by Lemma 3.3, $o(\beta)$ therefore has the same parity for all $\beta \in X \setminus \{\rho\}$.)

We shall again follow the basic approach explained in section 3.3; however, there are modifications which may apply in certain cases. It is sometimes helpful to make use of θ, by considering the stabilizer of X in $W.2 = \langle W, \theta \rangle$ and then taking the intersection with W. Also, we note that in many cases we must fix the root $\genfrac{}{}{0pt}{}{12321}{2}$, so we need only work within its stabilizer, which is $\langle w_1, w_3, w_4, w_5, w_6 \rangle$. Finally, in any given situation there may be a convenient argument of another kind. We shall work through the possibilities for X in turn; note that for the pairs of sets interchanged by θ we need only consider one set, as the stabilizers will also be interchanged by θ.

If $X = X_{11}^1$ we must fix $\{\genfrac{}{}{0pt}{}{12321}{\cdot}\}, \{\genfrac{}{}{0pt}{}{\cdot\cdot2\cdot\cdot}{1}\}$ (being the roots β with $o(\beta) = 0, 4$ respectively). We set

$$G = \langle w_1, w_2, w_3, w_5, w_6 \rangle;$$

then G is transitive on $\{\genfrac{}{}{0pt}{}{12321}{\cdot}\}$, so we may fix both roots. Since the stabilizer of $\genfrac{}{}{0pt}{}{12321}{1}$ in $\mathrm{stab}_W(\genfrac{}{}{0pt}{}{12321}{2})$ is $\langle w_1, w_3, w_5, w_6 \rangle$, we have $W_X = G$.

If $X = X_{11}^2$ we must fix $\genfrac{}{}{0pt}{}{12321}{2}, \{\genfrac{}{}{0pt}{}{12\cdot21}{1}\}, \{\genfrac{}{}{0pt}{}{1221\cdot}{1}, \genfrac{}{}{0pt}{}{\cdot1221}{1}\}, \{\genfrac{}{}{0pt}{}{11211}{1}, \genfrac{}{}{0pt}{}{01210}{1}, \genfrac{}{}{0pt}{}{11110}{1}, \genfrac{}{}{0pt}{}{01111}{1}\}$ ($o(\beta) = 0, 2, 4, 6$ respectively). We set

$$G = \langle w_1 w_4, w_4 w_6 \rangle,$$

and $G.2 = \langle G, \theta \rangle$; then $G.2$ is transitive on $\{\genfrac{}{}{0pt}{}{11211}{1}, \genfrac{}{}{0pt}{}{01210}{1}, \genfrac{}{}{0pt}{}{11110}{1}, \genfrac{}{}{0pt}{}{01111}{1}\}$, so we may fix $\beta_1 = \genfrac{}{}{0pt}{}{11211}{1}$. We must then fix $\genfrac{}{}{0pt}{}{12221}{1}, \{\genfrac{}{}{0pt}{}{12210}{1}, \genfrac{}{}{0pt}{}{01221}{1}\}$ (by orthogonality to β_1) and hence $\genfrac{}{}{0pt}{}{12321}{1}, \{\genfrac{}{}{0pt}{}{12211}{1}, \genfrac{}{}{0pt}{}{11221}{1}\}$; we must then fix $\{\genfrac{}{}{0pt}{}{11110}{1}, \genfrac{}{}{0pt}{}{01111}{1}\}$ (by orthogonality to $\genfrac{}{}{0pt}{}{12321}{1}$) and hence $\genfrac{}{}{0pt}{}{01210}{1}$. Inside $\mathrm{stab}_{G.2}(\beta_1)$ we have $\langle \theta \rangle$ giving transitivity on $\{\genfrac{}{}{0pt}{}{12211}{1}, \genfrac{}{}{0pt}{}{11221}{1}\}$, so we may fix $\beta_2 = \genfrac{}{}{0pt}{}{12211}{1}$. We must then fix $\genfrac{}{}{0pt}{}{01221}{1}, \genfrac{}{}{0pt}{}{11110}{1}$ (by orthogonality to β_2), by which point all roots are fixed. Thus the stabilizer in $W.2$ is $G.2 = \langle G, \theta \rangle$; as G is θ-stable we have $W_X = G$. (In this case we note also that the set of roots with $o(\beta) = 4$ is in fact a union of the two W_X-orbits $\{\genfrac{}{}{0pt}{}{1221\cdot}{1}\}$ and $\{\genfrac{}{}{0pt}{}{\cdot1221}{1}\}$.)

If $X = X_{11}^3$ we must fix $\substack{12321\\2}$, $\{\substack{12\cdot\cdot\cdot\\1}\}$, $\{\substack{112\cdot1\\1}, \substack{111\cdot0\\1}\}$, $\{\substack{01210\\1}, \substack{01111\\1}\}$ ($o(\beta) = 0, 3, 5, 7$ respectively). We set

$$G = \langle w_5, w_4w_6 \rangle;$$

then G is transitive on $\{\substack{12\cdot\cdot\cdot\\1}\}$, so we may fix $\beta_1 = \substack{12321\\1}$. We must then fix $\{\substack{111\cdot0\\1}\}$, $\substack{01111\\1}$ (by orthogonality to β_1) and hence $\{\substack{112\cdot1\\1}\}$, $\substack{01210\\1}$; we must then fix $\substack{12210\\1}$ (by orthogonality to $\substack{01111\\1}$) and hence $\{\substack{122\cdot1\\1}\}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_5 \rangle$ giving transitivity on $\{\substack{122\cdot1\\1}\}$, so we may fix $\beta_2 = \substack{12221\\1}$. We must then fix $\substack{11211\\1}$, $\substack{11100\\1}$ (by orthogonality to β_2), by which point all roots are fixed; so $W_X = G$.

If $X = X_{11}^5$ we must fix $\substack{12321\\2}$, $\substack{12321\\1}$, $\{\substack{122\cdot1\\1}, \substack{11221\\1}, \substack{11210\\1}, \substack{0121\cdot\\1}\}$, $\{\substack{01110\\1}, \substack{11100\\1}, \substack{00111\\1}\}$ ($o(\beta) = 0, 3, 5, 7$ respectively). We set

$$G = \langle w_1w_5, w_3w_6 \rangle,$$

and $G.2 = \langle G, \theta \rangle$; then $G.2$ is transitive on $\{\substack{122\cdot1\\1}, \substack{11221\\1}, \substack{11210\\1}, \substack{0121\cdot\\1}\}$, so we may fix $\beta_1 = \substack{12221\\1}$. We must then fix $\{\substack{11210\\1}, \substack{0121\cdot\\1}\}$, $\{\substack{11100\\1}, \substack{00111\\1}\}$ (by orthogonality to β_1) and hence $\{\substack{12211\\1}, \substack{11221\\1}\}$, $\substack{01110\\1}$; we must then fix $\{\substack{11210\\1}, \substack{01211\\1}\}$ (by orthogonality to $\substack{01110\\1}$) and hence $\substack{01210\\1}$. Inside $\mathrm{stab}_{G.2}(\beta_1)$ we have $\langle \theta \rangle$ giving transitivity on $\{\substack{12211\\1}, \substack{11221\\1}\}$, so we may fix $\beta_2 = \substack{12211\\1}$. We must then fix $\substack{11210\\1}$, $\substack{00111\\1}$ (by orthogonality to β_2), by which point all roots are fixed. Thus the stabilizer in $W.2$ is $G.2 = \langle G, \theta \rangle$; as G is θ-stable we have $W_X = G$.

If $X = X_{11}^6$ we must fix $\substack{12321\\2}$, $\{\substack{111\cdot0\\1}, \substack{0\cdot111\\1}\}$, $\substack{01210\\1}$ ($o(\beta) = 0, 6, 8$ respectively), $\substack{12321\\1}$ ($o(\beta) = 4$, orthogonal to all of $\{\substack{111\cdot0\\1}, \substack{0\cdot111\\1}\}$), $\{\substack{1\cdot2\cdot1\\1}\}$ ($o(\beta) = 4$, orthogonal to two of $\{\substack{111\cdot0\\1}, \substack{0\cdot111\\1}\}$). We set

$$G = \langle w_3, w_5 \rangle,$$

and $G.2 = \langle G, \theta \rangle$; then $G.2$ is transitive on $\{\substack{1\cdot2\cdot1\\1}\}$, so we may fix $\beta_1 = \substack{12221\\1}$. We must then fix $\substack{11211\\1}$, $\{\substack{11100\\1}, \substack{00111\\1}\}$ (by orthogonality to β_1) and hence $\{\substack{12211\\1}, \substack{11221\\1}\}$, $\{\substack{11110\\1}, \substack{01111\\1}\}$. Inside $\mathrm{stab}_{G.2}(\beta_1)$ we have $\langle \theta \rangle$ giving transitivity on $\{\substack{12211\\1}, \substack{11221\\1}\}$, so we may fix $\beta_2 = \substack{12211\\1}$. We must then fix $\substack{11110\\1}$, $\substack{00111\\1}$ (by orthogonality to β_2), by which point all roots are fixed. Thus the stabilizer in $W.2$ is $G.2 = \langle G, \theta \rangle$; as G is θ-stable we have $W_X = G$. (In this case we note also that the set of roots with $o(\beta) = 6$ is in fact a union of the two W_X-orbits $\{\substack{111\cdot0\\1}\}$ and $\{\substack{0\cdot111\\1}\}$.)

If $X = X_{11}^7$ we must fix $\substack{12321\\2}$ and $\{\substack{12\cdot21\\1}, \substack{11211\\1}, \substack{01210\\1}, \substack{111\cdot0\\1}, \substack{0\cdot111\\1}, \substack{01100\\1}, \substack{00110\\1}\}$ ($o(\beta) = 0, 6$ respectively). We set

$$G = \langle w_1w_4, w_3w_5, w_4w_6 \rangle,$$

and $G.2 = \langle G, \theta \rangle$; then $G.2$ is transitive on $\{\substack{12\cdot21\\1}, \substack{11211\\1}, \substack{01210\\1}, \substack{111\cdot0\\1}, \substack{0\cdot111\\1}, \substack{01100\\1}, \substack{00110\\1}\}$, so we may fix $\beta_1 = \substack{12321\\1}$. We must then fix $\{\substack{111\cdot0\\1}, \substack{0\cdot111\\1}, \substack{01100\\1}, \substack{00110\\1}\}$ (by orthogonality to β_1) and hence $\{\substack{12221\\1}, \substack{11211\\1}, \substack{01210\\1}\}$. Inside $\mathrm{stab}_{G.2}(\beta_1)$ we have $\langle w_1w_6, w_3w_5, \theta \rangle$ giving transitivity on $\{\substack{111\cdot0\\1}, \substack{0\cdot111\\1}, \substack{01100\\1}, \substack{00110\\1}\}$, so we may fix $\beta_2 = \substack{11110\\1}$. We must then fix $\{\substack{11211\\1}, \substack{01210\\1}\}$, $\{\substack{0\cdot111\\1}, \substack{01100\\1}\}$ (by orthogonality to β_2) and hence $\substack{12221\\1}$, $\{\substack{11100\\1}, \substack{00110\\1}\}$; we must then fix $\{\substack{00111\\1}, \substack{01100\\1}\}$ (by orthogonality to $\substack{12221\\1}$) and hence $\substack{01111\\1}$. Inside $\mathrm{stab}_{G.2}(\beta_1, \beta_2)$ we have $\langle w_1w_6\theta \rangle$ giving transitivity on $\{\substack{11100\\1}, \substack{00110\\1}\}$, so we may fix $\beta_3 = \substack{11100\\1}$. We must then fix $\substack{01210\\1}$, $\substack{00111\\1}$ (by orthogonality to β_3), by which point all roots are fixed. Thus the stabilizer in $W.2$ is $G.2 = \langle G, \theta \rangle$; as G is θ-stable we have $W_X = G$.

If $X = X_{11}^8$ we must fix $\begin{smallmatrix}12321\\2\end{smallmatrix}$, $\{\begin{smallmatrix}\cdot\cdot1\cdot\cdot\\1\end{smallmatrix}\}$, $\begin{smallmatrix}12321\\1\end{smallmatrix}$ ($o(\beta) = 0, 5, 9$ respectively). We set

$$G = \langle w_1, w_3, w_5, w_6\rangle;$$

since the stabilizer of $\begin{smallmatrix}12321\\1\end{smallmatrix}$ in $\mathrm{stab}_W(\begin{smallmatrix}12321\\2\end{smallmatrix})$ is $\langle w_1, w_3, w_5, w_6\rangle$, we have $W_X = G$.

If $X = X_{12}$ we must fix $\{\begin{smallmatrix}12321\\\cdot\end{smallmatrix}\}$, $\{\begin{smallmatrix}1\cdot2\cdot1\\1\end{smallmatrix}\}$, $\{\begin{smallmatrix}11111\\\cdot\end{smallmatrix}\}$, $\{\begin{smallmatrix}1\cdot210\\1\end{smallmatrix}, \begin{smallmatrix}012\cdot1\\1\end{smallmatrix}\}$ ($o(\beta) = 1, 3, 5, 6$ respectively). We set

$$G = \langle w_2, w_3, w_5\rangle,$$

and $G.2 = \langle G, \theta\rangle$; then $G.2$ is transitive on $\{\begin{smallmatrix}1\cdot2\cdot1\\1\end{smallmatrix}\}$, so we may fix $\beta_1 = \begin{smallmatrix}12221\\1\end{smallmatrix}$. We must then fix $\begin{smallmatrix}11211\\1\end{smallmatrix}$, $\{\begin{smallmatrix}11210\\1\end{smallmatrix}, \begin{smallmatrix}01211\\1\end{smallmatrix}\}$ (by orthogonality to β_1) and hence $\{\begin{smallmatrix}12211\\1\end{smallmatrix}, \begin{smallmatrix}11221\\1\end{smallmatrix}\}$, $\{\begin{smallmatrix}12210\\1\end{smallmatrix}, \begin{smallmatrix}01221\\1\end{smallmatrix}\}$. Inside $\mathrm{stab}_{G.2}(\beta_1)$ we have $\langle w_2, \theta\rangle$ giving transitivity on $\{\begin{smallmatrix}12321\\\cdot\end{smallmatrix}\}$, so we may fix $\beta_2 = \begin{smallmatrix}12321\\2\end{smallmatrix}$. We must then fix $\begin{smallmatrix}11111\\0\end{smallmatrix}$ (by orthogonality to β_2) and hence $\begin{smallmatrix}11111\\1\end{smallmatrix}$. Inside $\mathrm{stab}_{G.2}(\beta_1, \beta_2)$ we have $\langle\theta\rangle$ giving transitivity on $\{\begin{smallmatrix}12211\\1\end{smallmatrix}, \begin{smallmatrix}11221\\1\end{smallmatrix}\}$, so we may fix $\beta_3 = \begin{smallmatrix}12211\\1\end{smallmatrix}$. We must then fix $\begin{smallmatrix}11210\\1\end{smallmatrix}$, $\begin{smallmatrix}01221\\1\end{smallmatrix}$ (by orthogonality to β_3), by which point all roots are fixed. Thus the stabilizer in $W.2$ is $G.2 = \langle G, \theta\rangle$; as G is θ-stable we have $W_X = G$. (In this case we note also that the set of roots with $o(\beta) = 6$ is in fact a union of the two W_X-orbits $\{\begin{smallmatrix}1\cdot210\\1\end{smallmatrix}\}$ and $\{\begin{smallmatrix}012\cdot1\\1\end{smallmatrix}\}$.)

If $X = X_{13}^1$ we must fix $\{\begin{smallmatrix}12\cdot\cdot1\\\cdot\end{smallmatrix}\}$, $\{\begin{smallmatrix}\cdot1\cdot\cdot1\\\cdot\end{smallmatrix}\}$, $\begin{smallmatrix}12210\\1\end{smallmatrix}$ ($o(\beta) = 2, 5, 8$ respectively). We set

$$G = \langle w_1, w_2, w_4, w_5\rangle;$$

then G is transitive on $\{\begin{smallmatrix}\cdot1\cdot\cdot1\\\cdot\end{smallmatrix}\}$, so we may fix $\beta_1 = \begin{smallmatrix}11221\\1\end{smallmatrix}$. We must then fix $\begin{smallmatrix}12211\\1\end{smallmatrix}$, $\{\begin{smallmatrix}01\cdot11\\\cdot\end{smallmatrix}\}$ (by orthogonality to β_1) and hence $\{\begin{smallmatrix}12\cdot21\\\cdot\end{smallmatrix}\}$, $\{\begin{smallmatrix}01221\\1\end{smallmatrix}, \begin{smallmatrix}11\cdot11\\\cdot\end{smallmatrix}\}$; we must then fix $\begin{smallmatrix}01221\\1\end{smallmatrix}$ (by orthogonality to $\begin{smallmatrix}12211\\1\end{smallmatrix}$) and hence $\{\begin{smallmatrix}11\cdot11\\\cdot\end{smallmatrix}\}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_2, w_4\rangle$ acting as S_3 on $\{\begin{smallmatrix}12\cdot21\\\cdot\end{smallmatrix}\}$, so we may fix all three roots; we then have fixed all of the roots $\begin{smallmatrix}12\cdot\cdot\cdot\\\cdot\end{smallmatrix}$, $\begin{smallmatrix}11221\\1\end{smallmatrix}$, which span $\mathbb{R}\Phi$, so $W_X = G$.

If $X = X_{16}^1$ all roots β have $o(\beta) = 5$. We set

$$G = \langle w_2, w_3, w_4, w_5, w_6\rangle;$$

since any element of W outside G has a reduced expression ending with w_1, it makes the root $\begin{smallmatrix}10000\\0\end{smallmatrix}$ negative and so cannot stabilize X, so $W_X = G$.

The results found here are presented in tabular form in the final chapter of this work. In most instances we may immediately see that the set is not a W-translate of any of the others, since the signature uniquely identifies it; the only cases requiring further consideration are the three pairs X_{11}^3 and X_{11}^4, X_{13}^1 and X_{13}^2, and X_{16}^1 and X_{16}^2, where the two sets of each pair are interchanged by θ. We shall show that in each case X and $\theta(X)$ lie in different W-orbits.

If $X = X_{16}^1$ the result is clear from the identification of W_X. Thus let $X = X_{11}^3$ or X_{13}^1, and suppose we had $w \in W$ with $w(X) = \theta(X)$. The element w would then have to take $\{\begin{smallmatrix}12\cdot\cdot\cdot\\\cdot\end{smallmatrix}\}$ to $\{\begin{smallmatrix}\cdot\cdot\cdot21\\\cdot\end{smallmatrix}\}$; by adjusting w by an element of W_X we could ensure that it fixed firstly $\begin{smallmatrix}12321\\2\end{smallmatrix}$ (which is automatic anyway in the case of X_{11}^3), then $\begin{smallmatrix}12321\\1\end{smallmatrix}$, and then $\begin{smallmatrix}12221\\1\end{smallmatrix}$ (note that in the case of X_{11}^3 it would have to fix $\begin{smallmatrix}01210\\1\end{smallmatrix}$ by non-orthogonality to $\begin{smallmatrix}12321\\1\end{smallmatrix}$, and thus take $\{\begin{smallmatrix}122\cdot1\\1\end{smallmatrix}\}$ to $\{\begin{smallmatrix}1\cdot221\\1\end{smallmatrix}\}$ by orthogonality to $\begin{smallmatrix}01210\\1\end{smallmatrix}$). Thus w would have to lie in the pointwise stabilizer in W of the three roots $\begin{smallmatrix}12\cdot21\\\cdot\end{smallmatrix}$, which is $\langle w_1, w_6\rangle$; but no element in this group sends X to $\theta(X)$.

We have thus shown the following.

THEOREM 6.6. *The* 13 *sets in* $\mathcal{S}(E_6)$ *represent different* W*-orbits.*

CHAPTER 7

The root system of type E_7

Let Φ be of type E_7; thus Φ has simple roots $\alpha_1, \dots, \alpha_7$ numbered as in [**1**].

7.1. Radical maximal abelian sets

We have $\rho = \genfrac{}{}{0pt}{}{234321}{2}$, and the roots in Ξ are $\genfrac{}{}{0pt}{}{1\cdot\cdot\cdot\cdot\cdot}{\cdot}$; there are 16 pairs of roots in Ξ summing to ρ, namely $\{\genfrac{}{}{0pt}{}{100000}{0}, \genfrac{}{}{0pt}{}{134321}{2}\}$ and 15 of the form $\{\genfrac{}{}{0pt}{}{11\cdot\cdot\cdot\cdot}{\cdot}, \genfrac{}{}{0pt}{}{12\cdot\cdot\cdot\cdot}{\cdot}\}$. We may arrange these as follows.

$$
\begin{array}{lllll}
\{\genfrac{}{}{0pt}{}{112221}{1},\genfrac{}{}{0pt}{}{122100}{1}\} & & & & \\
 & \{\genfrac{}{}{0pt}{}{112211}{1},\genfrac{}{}{0pt}{}{122110}{1}\} & & & \\
\{\genfrac{}{}{0pt}{}{112210}{1},\genfrac{}{}{0pt}{}{122111}{1}\} & & \{\genfrac{}{}{0pt}{}{112111}{1},\genfrac{}{}{0pt}{}{122210}{1}\} & & \\
 & \{\genfrac{}{}{0pt}{}{112110}{1},\genfrac{}{}{0pt}{}{122211}{1}\} & & \{\genfrac{}{}{0pt}{}{111111}{1},\genfrac{}{}{0pt}{}{123210}{1}\} & \\
\{\genfrac{}{}{0pt}{}{112100}{1},\genfrac{}{}{0pt}{}{122221}{1}\} & & \{\genfrac{}{}{0pt}{}{111110}{1},\genfrac{}{}{0pt}{}{123211}{1}\} & & \{\genfrac{}{}{0pt}{}{111111}{0},\genfrac{}{}{0pt}{}{123210}{2}\} \\
 & \{\genfrac{}{}{0pt}{}{111100}{1},\genfrac{}{}{0pt}{}{123221}{1}\} & & \{\genfrac{}{}{0pt}{}{111110}{0},\genfrac{}{}{0pt}{}{123211}{2}\} & \\
\{\genfrac{}{}{0pt}{}{111000}{1},\genfrac{}{}{0pt}{}{123321}{1}\} & & \{\genfrac{}{}{0pt}{}{111100}{0},\genfrac{}{}{0pt}{}{123221}{2}\} & & \\
 & \{\genfrac{}{}{0pt}{}{111000}{0},\genfrac{}{}{0pt}{}{123321}{2}\} & & & \\
\{\genfrac{}{}{0pt}{}{110000}{0},\genfrac{}{}{0pt}{}{124321}{2}\} & & & & \\
 & & & & \\
\{\genfrac{}{}{0pt}{}{100000}{0},\genfrac{}{}{0pt}{}{134321}{2}\} & & & &
\end{array}
$$

As with the case of E_6, we shall regard the pair at the bottom of this array as isolated (at least for now; the full picture will not in fact emerge until we consider E_8). We shall again call a root in Ξ *odd* or *even*, this time according to the parity of its α_3-coefficient, so that each pair consists of an odd root and an even root; as before, a maximal radical abelian set may then be specified by simply giving the parity of the root selected in each pair, and we shall represent these sets graphically.

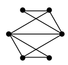
To begin with, we note that $w_0 w_\rho$ preserves ρ and maps each root β of Ξ to $-w_\rho(\beta) = \rho - \beta$, and thus interchanges the two roots in each pair; it therefore suffices to consider maximal radical abelian sets containing $\genfrac{}{}{0pt}{}{134321}{2}$. We may identify the 15 non-isolated pairs with unordered pairs from the set $\{1, 2, 3, 4, 5, 6\}$; we shall write the unordered pair $\{i, j\}$ simply as ij, and we give the correspondence by the following array.

$$\begin{array}{ccccc}
12 & & & & \\
 & 13 & & & \\
23 & & 14 & & \\
 & 24 & & 15 & \\
34 & & 25 & & 16 \\
 & 35 & & 26 & \\
45 & & 36 & & \\
 & 46 & & & \\
56 & & & &
\end{array}$$

We may then represent a maximal radical abelian set X containing $\begin{smallmatrix}134321\\2\end{smallmatrix}$ by a graph Γ_X with vertex set $\{1,2,3,4,5,6\}$, where the choice in X of the odd or even root in the pair identified with ij is denoted by the presence or absence in Γ_X of the edge ij. (Alternatively, we may regard this as the choice of a black edge or a white edge.) We shall arrange the vertices in a regular hexagon as follows:

$$\begin{array}{cccc}
 & 3 & 4 & \\
2 & & & 5 \\
 & 1 & 6 &
\end{array}.$$

For example, if $X = \{ \begin{smallmatrix}\cdot\cdot\cdot\cdot21\\2\end{smallmatrix}, \begin{smallmatrix}112221\\1\end{smallmatrix}, \begin{smallmatrix}1121\cdot0\\1\end{smallmatrix}, \begin{smallmatrix}122111\\1\end{smallmatrix}, \begin{smallmatrix}122\cdot10\\1\end{smallmatrix}, \begin{smallmatrix}11111\cdot\\\cdot\end{smallmatrix}, \begin{smallmatrix}111\cdot00\\1\end{smallmatrix} \}$ then Γ_X is

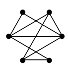

.

We now consider the action of $\mathrm{stab}_W(\rho) = \langle w_2, w_3, w_4, w_5, w_6, w_7\rangle$ on the roots in Ξ. Within this group the (pointwise) stabilizer W' of the isolated pair is $\langle w_2, w_4, w_5, w_6, w_7\rangle$, which does not affect α_3-coefficients and therefore permutes edges without changing colours; in fact the notation chosen for the pairs means that the generating elements of $W' \cong S_6$ act as permutations of the vertices as follows:

$$w_2 = (5\ 6),\ w_4 = (4\ 5),\ w_5 = (3\ 4),\ w_6 = (2\ 3),\ w_7 = (1\ 2).$$

Thus two maximal radical abelian sets containing $\begin{smallmatrix}134321\\2\end{smallmatrix}$ lie in the same W'-orbit if and only if their graphs are isomorphic. By [**12**], up to isomorphism there are 156 graphs on 6 vertices, which we list in Figure 7.1; since there are $2^{16} = 65536$ maximal radical abelian sets, this already gives a significant reduction.

The remaining generator of $\mathrm{stab}_W(\rho)$ is w_3. In determining its effect, we shall assume that in the graph Γ_X being considered the edge 56 is absent (or white), so that the corresponding set X contains $\begin{smallmatrix}124321\\2\end{smallmatrix}$ and hence $w_3(X)$ still contains $\begin{smallmatrix}134321\\2\end{smallmatrix}$. (Since W' acts transitively on edges, this assumption excludes only the last of the 156 graphs in Figure 7.1.) If we set $\{i,j,k,\ell\} = \{1,2,3,4\}$, each edge ij in Γ_X gives rise to the edge $k\ell$ in $\Gamma_{w_3(X)}$ of the opposite colour (alternatively, the presence or absence of ij gives rise to the absence or presence of $k\ell$ respectively). Thus for example with the set X as above, in the graph Γ_X the edge 12 is present (or black), and so in $\Gamma_{w_3(X)}$ the edge 34 is absent (or white); on the other hand in Γ_X the edge 13 is absent (or white), and so in $\Gamma_{w_3(X)}$ the edge 24 is present (or black). Treating all six edges among the vertices $1,2,3,4$ thus, we see that the graph $\Gamma_{w_3(X)}$ is


,

corresponding to $w_3(X) = \{ \begin{smallmatrix}\cdot\cdot\cdot\cdot21\\2\end{smallmatrix}, \begin{smallmatrix}122221\\1\end{smallmatrix}, \begin{smallmatrix}1221\cdot0\\1\end{smallmatrix}, \begin{smallmatrix}11\cdot111\\\cdot\end{smallmatrix}, \begin{smallmatrix}11\cdot\cdot10\\\cdot\end{smallmatrix}, \begin{smallmatrix}111\cdot00\\1\end{smallmatrix} \}$.

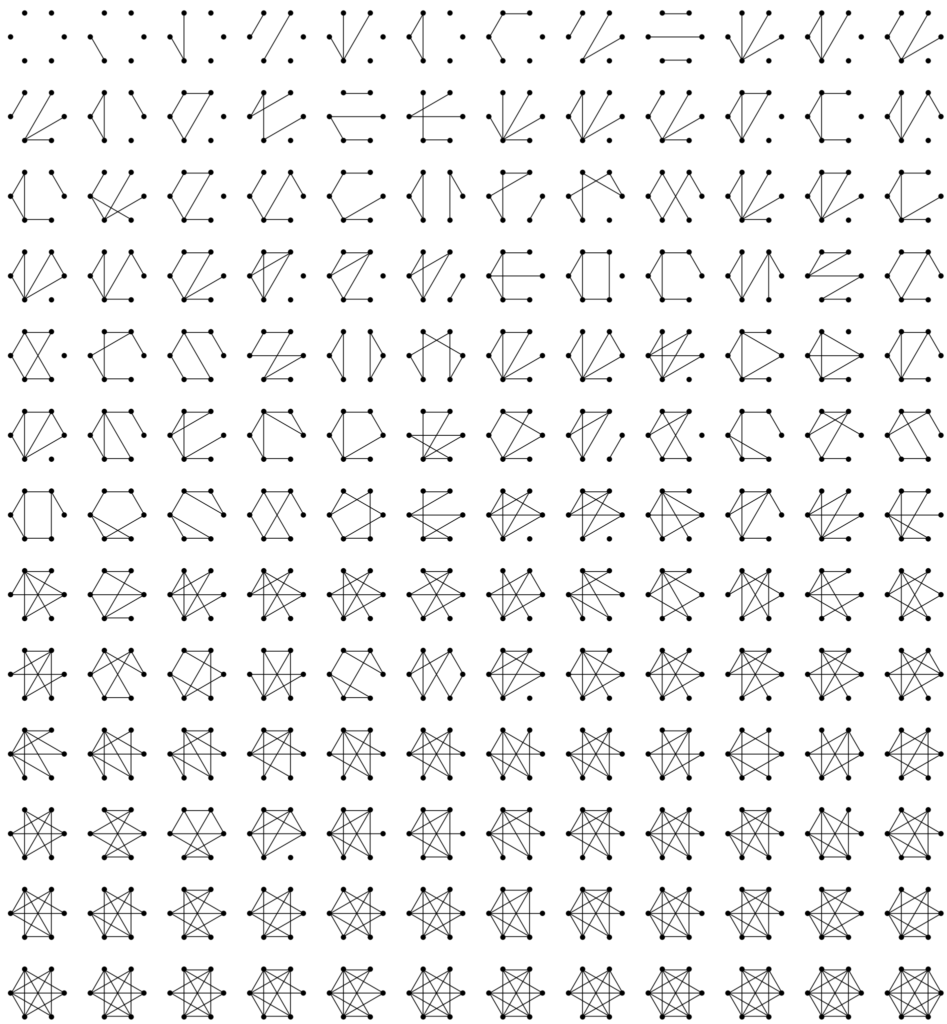


FIGURE 7.1. Graphs on 6 vertices up to isomorphism

Now if in the graph Γ_X at least four of the six edges among the vertices $1, 2, 3, 4$ are present (and 56 is absent), then the graph $\Gamma_{w_3(X)}$ will have fewer edges than Γ_X, which therefore need be considered no further. (Likewise, if Γ_X is the complete graph then applying $w_3 w_4 w_5 w_2 w_4 w_3$ to X produces a set still containing $\substack{134321\\2}$ whose graph has just seven edges; so the complete graph requires no further consideration.) This immediately disposes of 115 of the 156 graphs in Figure 7.1; moreover if we temporarily indicate the graph in row x and column y of Figure 7.1 by (x, y), then applying w_3 to $(3, 4)$, $(5, 3)$, $(7, 2)$ and $(7, 3)$ produces graphs isomorphic to $(3, 5)$, $(4, 11)$, $(7, 6)$ and $(7, 5)$ respectively, so that we are left with just 37 graphs to consider. For convenience we list these graphs separately in Figure 7.2, where we call two graphs *equivalent* if they correspond to sets which are W-translates of each other.

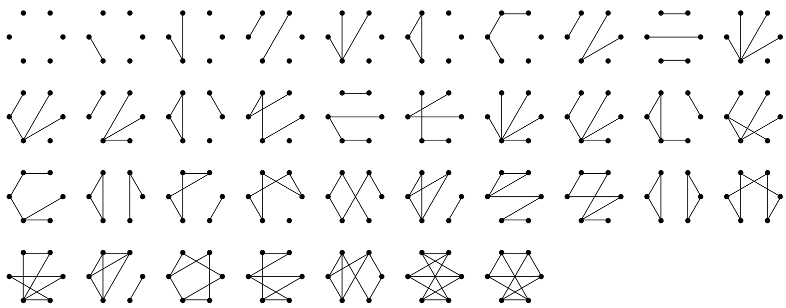
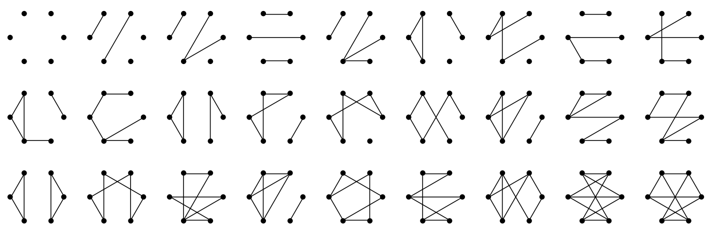


FIGURE 7.2. Graphs on 6 vertices up to equivalence

(Note that at this stage we have not yet shown that the graphs in Figure 7.2 are all inequivalent; for those which correspond to maximal abelian sets, the fact that the sets concerned all lie in different W-orbits will be shown in section 7.4.)

We must now consider which of the graphs in Figure 7.2 represent maximal abelian sets. The roots $-\genfrac{}{}{0pt}{}{234321}{2}$, $-\genfrac{}{}{0pt}{}{1\cdot\cdot\cdot\cdot\cdot}{\cdot}$ are excluded by $\genfrac{}{}{0pt}{}{234321}{2}$, while $-\genfrac{}{}{0pt}{}{01\cdot\cdot\cdot\cdot}{\cdot}$ are excluded by $\genfrac{}{}{0pt}{}{134321}{2}$. To exclude $\genfrac{}{}{0pt}{}{000000}{1}$ we must have some root $\genfrac{}{}{0pt}{}{111\cdot\cdot\cdot}{0}$ or $\genfrac{}{}{0pt}{}{123\cdot\cdot\cdot}{1}$ in X; using W' we see that to exclude the roots $\genfrac{}{}{0pt}{}{00\cdot\cdot\cdot\cdot}{\cdot}$ the graph must not have two vertices of which one is joined to all of the other four and the other is joined to none of the other four, which disposes of the seventh graph in the second row. Finally, to exclude $\genfrac{}{}{0pt}{}{010000}{0}$ we must have $\genfrac{}{}{0pt}{}{124321}{2}$ or some root $\genfrac{}{}{0pt}{}{112\cdot\cdot\cdot}{1}$; using W' we see that to exclude the roots $\genfrac{}{}{0pt}{}{01\cdot\cdot\cdot\cdot}{\cdot}$ the graph must not have an edge present such that all edges which are not incident with it are absent, which disposes of the second, third, fifth, sixth, seventh and tenth graphs in the first row and the first, eighth and tenth in the second row. We are therefore left with the following 27 graphs representing radical maximal abelian sets.



In fact in three of the 27 cases we shall choose to take not the set given by the graph above, but instead its image under w_3; the graphs concerned are the seventh in the second row and the fourth and seventh in the third row. Note that in these graphs the edges present comprise 56, at least five of the six edges among the vertices $1, 2, 3, 4$, and in the third of these cases two other edges. Thus in each case the original set X contains the root $\genfrac{}{}{0pt}{}{110000}{0}$, so its image $w_3(X)$ contains $\genfrac{}{}{0pt}{}{100000}{0}$ rather than $\genfrac{}{}{0pt}{}{134321}{2}$ from the isolated pair; in addition, in the first two of these three cases $w_3(X)$ contains at most one root $\genfrac{}{}{0pt}{}{11\cdot\cdot\cdot\cdot}{\cdot}$, while in the third case $w_3(X)$ contains three such roots. In each case we make this choice because it will in due course lead to a more convenient form for the stabilizer. We therefore set

$$X_{17}^{1} = \{ \substack{\cdot 34321\\2}, \substack{12\cdot\cdot\cdot\cdot\\\cdot} \},$$
$$X_{17}^{2} = \{ \substack{\cdot\cdot 4321\\2}, \substack{123\cdot\cdot\cdot\\\cdot}, \substack{1222\cdot 1\\1}, \substack{1221\cdot 0\\1}, \substack{112210\\1}, \substack{112111\\1} \},$$
$$X_{17}^{3} = \{ \substack{\cdot\cdot\cdot\cdot\cdot\cdot\\2}, \substack{123321\\1}, \substack{12\cdot 2\cdot 1\\1}, \substack{1221\cdot 0\\1}, \substack{112210\\1}, \substack{11\cdot 111\\1} \},$$
$$X_{17}^{4} = \{ \substack{\cdot\cdot\cdot\cdot\cdot 1\\2}, \substack{123\cdot 21\\1}, \substack{122\cdot 11\\1}, \substack{12\cdot\cdot\cdot 0\\1}, \substack{112100\\1}, \substack{111110\\1}, \substack{111111\\0} \},$$
$$X_{17}^{5} = \{ \substack{\cdot\cdot\cdot 321\\\cdot}, \substack{12\cdot 2\cdot 1\\\cdot}, \substack{1221\cdot 0\\1}, \substack{112210\\1}, \substack{11\cdot 111\\\cdot} \},$$
$$X_{17}^{6} = \{ \substack{\cdot\cdot\cdot\cdot\cdot\cdot\\2}, \substack{1\cdot\cdot 2\cdot\cdot\\1}, \substack{111000\\1} \},$$
$$X_{17}^{7} = \{ \substack{\cdot\cdot\cdot\cdot\cdot\cdot\\2}, \substack{123\cdot\cdot 1\\1}, \substack{122221\\1}, \substack{122210\\1}, \substack{11221\cdot\\1}, \substack{122100\\1}, \substack{112110\\1}, \substack{111111\\1} \},$$
$$X_{17}^{8} = \{ \substack{\cdot\cdot\cdot\cdot\cdot 1\\2}, \substack{123\cdot 21\\1}, \substack{123210\\1}, \substack{122\cdot 1\cdot\\1}, \substack{112221\\1}, \substack{112100\\1}, \substack{111110\\1}, \substack{111111\\0} \},$$
$$X_{17}^{9} = \{ \substack{\cdot\cdot\cdot\cdot\cdot 1\\2}, \substack{12\cdot\cdot 21\\1}, \substack{12\cdot 210\\1}, \substack{122111\\1}, \substack{112211\\1}, \substack{122100\\1}, \substack{11\cdot 110\\1}, \substack{111111\\0} \},$$
$$X_{17}^{10} = \{ \substack{\cdot\cdot\cdot\cdot\cdot 1\\2}, \substack{1\cdot\cdot 2\cdot\cdot\\1}, \substack{111111\\0}, \substack{111000\\1} \},$$
$$X_{17}^{11} = \{ \substack{\cdot\cdot\cdot 321\\\cdot}, \substack{1232\cdot 1\\\cdot}, \substack{122211\\1}, \substack{112221\\1}, \substack{1\cdot 2210\\1}, \substack{122110\\1}, \substack{112100\\1}, \substack{111111\\\cdot} \},$$
$$X_{17}^{12} = \{ \substack{\cdot\cdot 4321\\2}, \substack{1\cdot\cdot 2\cdot\cdot\\\cdot}, \substack{111000\\\cdot} \},$$
$$X_{17}^{13} = \{ \substack{\cdot 34321\\2}, \substack{123\cdot\cdot\cdot\\\cdot}, \substack{122210\\1}, \substack{122111\\1}, \substack{1122\cdot 1\\1}, \substack{1121\cdot 0\\1}, \substack{110000\\0} \},$$
$$X_{17}^{14} = \{ \substack{\cdot\cdot\cdot\cdot\cdot\cdot\\2}, \substack{12321\cdot\\1}, \substack{122221\\1}, \substack{122111\\1}, \substack{122210\\1}, \substack{1122\cdot 1\\1}, \substack{112110\\1}, \substack{111\cdot 00\\1} \},$$
$$X_{17}^{15} = \{ \substack{\cdot\cdot\cdot 321\\2}, \substack{1\cdot\cdot 221\\1}, \substack{12321\cdot\\\cdot}, \substack{122211\\1}, \substack{122110\\1}, \substack{112210\\1}, \substack{112111\\1}, \substack{111000\\1}, \substack{111100\\0} \},$$
$$X_{17}^{16} = \{ \substack{234321\\2}, \substack{12\cdot\cdot 21\\\cdot}, \substack{12\cdot\cdot 1\cdot\\\cdot}, \substack{112221\\1}, \substack{100000\\0} \},$$
$$X_{17}^{17} = \{ \substack{\cdot\cdot\cdot\cdot\cdot 1\\2}, \substack{123\cdot 21\\1}, \substack{1\cdot 2\cdot\cdot 0\\1}, \substack{11111\cdot\\1}, \substack{111111\\0} \},$$
$$X_{17}^{18} = \{ \substack{\cdot\cdot\cdot\cdot\cdot 1\\2}, \substack{123\cdot 21\\1}, \substack{122211\\1}, \substack{11\cdot 111\\\cdot}, \substack{112210\\1}, \substack{122110\\1}, \substack{1\cdot 2100\\1}, \substack{111110\\1} \},$$
$$X_{17}^{19} = \{ \substack{\cdot 34321\\2}, \substack{1\cdot\cdot 2\cdot\cdot\\\cdot}, \substack{11\cdot 000\\\cdot} \},$$
$$X_{17}^{20} = \{ \substack{\cdot 34321\\2}, \substack{123321\\1}, \substack{1232\cdot\cdot\\2}, \substack{122221\\1}, \substack{123211\\1}, \substack{12\cdot 210\\1}, \substack{122111\\1}, \substack{1122\cdot 1\\1}, \substack{112110\\1}, \substack{111100\\1}, \substack{11\cdot 000\\0} \},$$
$$X_{17}^{21} = \{ \substack{\cdot\cdot 4321\\2}, \substack{123\cdot 21\\\cdot}, \substack{1\cdot 2\cdot 11\\1}, \substack{1\cdot 2100\\1}, \substack{11111\cdot\\\cdot} \},$$
$$X_{17}^{22} = \{ \substack{234321\\2}, \substack{12\cdot\cdot\cdot\cdot\\\cdot}, \substack{100000\\0} \},$$
$$X_{17}^{23} = \{ \substack{\cdot\cdot 4321\\2}, \substack{123321\\1}, \substack{123221\\2}, \substack{123211\\\cdot}, \substack{1\cdot 2221\\1}, \substack{1\cdot 2\cdot 10\\1}, \substack{111111\\\cdot}, \substack{111100\\1}, \substack{111000\\0} \},$$
$$X_{17}^{24} = \{ \substack{\cdot\cdot 4321\\2}, \substack{123\cdot 21\\\cdot}, \substack{122210\\1}, \substack{122111\\1}, \substack{112211\\1}, \substack{112110\\1}, \substack{11111\cdot\\\cdot}, \substack{1\cdot 2100\\1} \},$$
$$X_{17}^{25} = \{ \substack{234321\\2}, \substack{12\cdot 321\\2}, \substack{12\cdot\cdot 1\cdot\\\cdot}, \substack{1\cdot\cdot 221\\1}, \substack{111000\\1}, \substack{111100\\0}, \substack{100000\\0} \},$$
$$X_{17}^{26} = \{ \substack{\cdot\cdot\cdot 321\\\cdot}, \substack{1\cdot\cdot 1\cdot\cdot\\\cdot} \},$$
$$X_{17}^{27} = \{ \substack{\cdot\cdot\cdot 321\\2}, \substack{123221\\1}, \substack{122211\\1}, \substack{1221\cdot 0\\1}, \substack{11\cdot 111\\\cdot}, \substack{112210\\1}, \substack{112100\\1}, \substack{111110\\\cdot}, \substack{111100\\0}, \substack{111000\\1} \}.$$

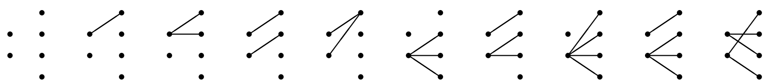
7.2. Near-radical maximal abelian sets

As explained in section 3.1, here it will also be useful to consider subsets X of Φ^+ satisfying $|X \setminus \Omega| = 1$; we call these near-radical sets. Suppose that the maximal abelian set X consists of $\substack{234321\\2}$, one root of the form $\substack{0\cdot\cdot\cdot\cdot\cdot\\\cdot}$ and various roots of the form $\substack{1\cdot\cdot\cdot\cdot\cdot\\\cdot}$. Using the group $\langle w_2, w_3, w_4, w_5, w_6, w_7 \rangle$ we may assume the second root is $\substack{012221\\1}$; this excludes $\substack{1\cdot\cdot\cdot 00\\\cdot}$, giving $\substack{1\cdot\cdot\cdot 21\\\cdot} \in X$ by default. We therefore have $\substack{\cdot\cdot\cdot\cdot21\\ \cdot} \in X$, and the remaining roots of X are to be chosen one from each of 8 pairs summing to ρ, of the form $\{\substack{11\cdot\cdot1\cdot\\ \cdot}, \substack{12\cdot\cdot1\cdot\\ \cdot}\}$.

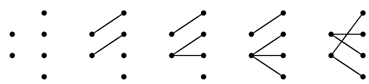
In terms of the identification with unordered pairs in section 7.1, the pairs in which a choice must be made are ij where $i \in \{1,2\}$ and $j \in \{3,4,5,6\}$. We may thus represent the set X by a bipartite graph Γ'_X with edges between $\{1,2\}$ and $\{3,4,5,6\}$, where as before the presence or absence of an edge corresponds to the choice of an odd or even root respectively. We shall arrange the vertices as follows:

$$\begin{array}{cc} & 3\\ 2 & 4\\ 1 & 5\\ & 6 \end{array}.$$

The subgroup of W stabilizing both $\substack{234321\\2}$ and $\substack{012221\\1}$ is $\langle w_2, w_3, w_4, w_5, w_7\rangle$, whose generators act as before: $\langle w_2, w_4, w_5, w_7\rangle$ acts as $S_2 \times S_4$ on the vertices of Γ'_X, while if we set $\{i,k\} = \{1,2\}$ and $\{j,\ell\} = \{3,4\}$, then the presence or absence of the edge ij in Γ'_X gives rise to the absence or presence of the edge $k\ell$ in $\Gamma'_{w_3(X)}$. Thus if a graph has more than 2 edges between $\{1,2\}$ and some subset $\{a,b\}$ of $\{3,4,5,6\}$, we may use the group $\langle w_2, w_4, w_5\rangle$ to move $\{a,b\}$ to $\{3,4\}$ and then apply w_3 to produce a graph with fewer edges than the original, which therefore need be considered no further. As a result we may obtain the following list of 10 graphs which do require consideration.



We must now consider which of these graphs correspond to maximal abelian sets. The roots $\substack{\cdot\cdot\cdot\cdot21\\ \cdot}$ between them exclude all negative roots apart from $-\substack{000001\\0}$, and all positive roots $\substack{0\cdot\cdot\cdot00\\ \cdot}$. To exclude $\substack{000001\\0}$ we must have some root $\substack{11\cdot\cdot10\\ \cdot}$ or $\substack{12\cdot\cdot10\\ \cdot}$ in X, while to exclude $-\substack{000001\\0}$ we must have some root $\substack{11\cdot\cdot11\\ \cdot}$ or $\substack{12\cdot\cdot11\\ \cdot}$; thus the graph must not have one of the vertices $\{1,2\}$ joined to all of the other four and the other joined to none of them, which disposes of the eighth graph. To exclude $\substack{000010\\0}$ we must have $\substack{112211\\1}$ or some root $\substack{12\cdot211\\ \cdot}$ in X; using the group $\langle w_2, w_3, w_4, w_5, w_7\rangle$ we see that to exclude the roots $\substack{0\cdot\cdot\cdot1\cdot\\ \cdot}$ the graph must not have an edge of one colour such that the 3 non-adjacent edges are all of the other colour, which disposes of the second, third, fifth and sixth graphs. We are thus left with the following 5 graphs representing near-radical maximal abelian sets.



We therefore set

$$\begin{aligned}
X^1_{18} &= \{\substack{\cdot\cdot\cdot\cdot21\\ \cdot}, \substack{12\cdot\cdot1\cdot\\ \cdot}\},\\
X^2_{18} &= \{\substack{\cdot\cdot\cdot\cdot21\\ \cdot}, \substack{12321\cdot\\ \cdot}, \substack{122211\\1}, \substack{112111\\1}, \substack{112210\\1}, \substack{122110\\1}\},\\
X^3_{18} &= \{\substack{\cdot\cdot\cdot\cdot21\\ \cdot}, \substack{12321\cdot\\2}, \substack{12\cdot211\\1}, \substack{11\cdot111\\1}, \substack{112210\\1}, \substack{122110\\1}\},\\
X^4_{18} &= \{\substack{\cdot\cdot\cdot\cdot21\\ \cdot}, \substack{12\cdot211\\ \cdot}, \substack{11\cdot111\\ \cdot}, \substack{112210\\1}, \substack{122110\\1}\},\\
X^5_{18} &= \{\substack{\cdot\cdot\cdot\cdot21\\ \cdot}, \substack{123211\\2}, \substack{12\cdot210\\1}, \substack{112211\\1}, \substack{122111\\1}, \substack{11\cdot110\\1}, \substack{111111\\0}\}.
\end{aligned}$$

7.3. Determination of maximal abelian sets

We begin by giving some maximal abelian sets which are neither radical nor near-radical; we set

$$\begin{aligned}
X_{14} &= \{ \substack{\cdot\cdot\cdot\cdot\cdot\cdot\\2}, \substack{123321\\1}, \substack{122111\\1}, \substack{112110\\1}, \substack{012100\\1}, \substack{111100\\1}, \substack{011110\\1}, \substack{001111\\1} \},\\
X_{17}^{28} &= \{ \substack{\cdot\cdot4321\\2}, \substack{123\cdot\cdot\cdot\\\cdot}, \substack{122221\\1}, \substack{112211\\1}, \substack{122100\\1}, \substack{112110\\1}, \substack{012210\\1}, \substack{012111\\1} \},\\
X_{19} &= \{ \substack{\cdot\cdot\cdot321\\\cdot}, \substack{12\cdot2\cdot\cdot\\\cdot}, \substack{\cdot122\cdot1\\1}, \substack{122111\\1} \},\\
X_{20}^{1} &= \{ \substack{\cdot\cdot4321\\2}, \substack{123\cdot\cdot\cdot\\\cdot}, \substack{\cdot\cdot2\cdot\cdot1\\1} \},\\
X_{20}^{2} &= \{ \substack{\cdot\cdot\cdot321\\\cdot}, \substack{\cdot\cdot\cdot2\cdot\cdot\\\cdot} \},\\
X_{22} &= \{ \substack{\cdot\cdot\cdot\cdot\cdot\cdot\\2}, \substack{\cdot\cdot\cdot\cdot\cdot1\\1} \},\\
X_{27} &= \{ \substack{\cdot\cdot\cdot\cdot\cdot1\\\cdot} \}.
\end{aligned}$$

We then set

$$\mathcal{S}(E_7) = \{X_{14}, X_{17}^1, \ldots, X_{17}^{27}, X_{17}^{28}, X_{18}^1, \ldots, X_{18}^5, X_{19}, X_{20}^1, X_{20}^2, X_{22}, X_{27}\}.$$

As in section 3.2, we let X be any maximal abelian set consisting of positive roots and containing a simple root α; we seek to show that X is known, i.e., a W-translate of a set in $\mathcal{S}(E_7)$. Here we note that if at some point the union of the sets of chosen and available roots is a W-translate of a set with at most one root outside Ω, there will be no need to continue the line of investigation since we have determined the radical and near-radical maximal abelian sets.

We work through the possibilities for the simple root α contained in X. In the first of these we take $\alpha = \substack{100000\\0}$. Since the radical sets have been treated we may also assume that X contains some root $\substack{0\cdot\cdot\cdot\cdot\cdot\\\cdot}$; as α excludes the roots $\substack{01\cdot\cdot\cdot\cdot\\\cdot}$, we may assume that X contains some root of the form $\substack{00\cdot\cdot\cdot\cdot\\\cdot}$, and hence some simple root α' of this form. For convenience we shall subdivide this first step of the analysis according to the possibilities for α'.

Lemma 7.1. *If* $\substack{100000\\0}, \substack{000000\\1} \in X$ *then* X *is known.*

Proof. We assume $\substack{100000\\0}, \substack{000000\\1} \in X$, which excludes $\substack{01\cdot\cdot\cdot\cdot\\\cdot}, \substack{001\cdot\cdot\cdot\\0}, \substack{111\cdot\cdot\cdot\\0}, \substack{123\cdot\cdot\cdot\\1}, \substack{134321\\2}$, giving $\substack{111111\\1}, \substack{123321\\2} \in X$ by default. Using Lemma 3.1 (with $Y = \{\substack{100000\\0}, \substack{000000\\1}\}$) we may assume that one of the following holds: (i) $\substack{000100\\0} \in X$; (ii) $\substack{000100\\0} \notin X$, $\substack{000\cdot10\\0} \in X$; (iii) $\substack{000\cdot\cdot0\\0} \notin X$, $\substack{000\cdot\cdot1\\0} \in X$; (iv) $\substack{000\cdot\cdot\cdot\\0} \notin X$.

First assume (i) holds; this excludes $\substack{00001\cdot\\0}, \substack{001000\\1}, \substack{111000\\1}, \substack{1\cdot211\cdot\\1}, \substack{123221\\2}$, giving $\substack{111100\\1}, \substack{112211\\1} \in X$ by default. To exclude $-\substack{011000\\0}$ we must have $\substack{122100\\1} \in X$, which excludes $\substack{00011\cdot\\0}, \substack{00111\cdot\\1}, \substack{112221\\1}$, giving $\substack{122211\\1}, \substack{123211\\2} \in X$ by default. If $\substack{110000\\0} \in X$, this excludes $\substack{001100\\1}, \substack{124321\\2}$; to exclude $-\substack{001000\\0}$ we must have $\substack{112100\\1} \in X$, which excludes $\substack{122221\\1}$; to exclude $-\substack{000010\\0}$ we must have $\substack{111110\\1} \in X$, which excludes $\substack{000001\\0}$, giving $\substack{1\cdot2210\\1}, \substack{123210\\2} \in X$ by default; so

$$\begin{aligned}
X &= \{ \substack{234321\\2}, \substack{123321\\2}, \substack{12321\cdot\\2}, \substack{1\cdot221\cdot\\1}, \substack{11111\cdot\\1}, \substack{1\cdot\cdot100\\1}, \substack{1\cdot0000\\0}, \substack{000000\\1}, \substack{000100\\0} \}\\
&= w_4 w_3 w_6 w_5 w_4 w_2 w_7 w_6 w_5 w_4 w_3 w_1 (X_{17}^2).
\end{aligned}$$

We may therefore assume $\substack{110000\\0} \notin X$, giving $\substack{124321\\2} \in X$ by default. If $\substack{000001\\0} \in X$ this excludes $\substack{111110\\1}, \substack{1\cdot2210\\1}, \substack{123210\\2}$; to exclude $-\substack{000010\\0}$ we must have $\substack{122221\\1} \in X$, which excludes $\substack{001100\\1}$, $\substack{112100\\1}$; so

$$X = \{\substack{234321\\2}, \substack{12\cdot321\\2}, \substack{123211\\2}, \substack{122221\\1}, \substack{1\cdot2211\\1}, \substack{122100\\1}, \substack{111111\\1}, \substack{111100\\1}, \substack{100000\\0}, \substack{000000\\1}, \substack{000100\\0}, \substack{000001\\0}\}$$
$$= w_4w_3w_6w_5w_4w_2w_5(X_{14}).$$

We may therefore assume $\substack{000001\\0} \notin X$, giving $\substack{111110\\1}$, $\substack{1\cdot2210\\1}$, $\substack{123210\\2} \in X$ by default. If $\substack{122221\\1} \in X$ this excludes $\substack{001100\\1}$, $\substack{112100\\1}$; so

$$X = \{\substack{234321\\2}, \substack{12\cdot321\\2}, \substack{12321\cdot\\2}, \substack{1222\cdot\cdot\\1}, \substack{11221\cdot\\1}, \substack{1111\cdot\cdot\\1}, \substack{122100\\1}, \substack{100000\\0}, \substack{000000\\1}, \substack{000100\\0}\}$$
$$= w_4w_3w_6w_5w_4w_2w_7w_6w_5w_4w_3(X_{17}^{28}).$$

Finally we may therefore assume $\substack{122221\\1} \notin X$, giving $\substack{001100\\1}$, $\substack{112100\\1} \in X$ by default; so

$$X = \{\substack{234321\\2}, \substack{12\cdot321\\2}, \substack{12321\cdot\\2}, \substack{1\cdot221\cdot\\1}, \substack{1111\cdot\cdot\\1}, \substack{1\cdot2100\\1}, \substack{001100\\1}, \substack{100000\\0}, \substack{000000\\1}, \substack{000100\\0}\}$$
$$= w_4w_3w_6w_5w_4w_2w_5w_7w_6w_5w_4w_3w_1(X_{18}^{2}).$$

Next assume (ii) holds; this excludes $\substack{000001\\0}$, $\substack{001\cdot00\\1}$, $\substack{1\cdot\cdot\cdot00\\1}$, $\substack{1\cdot2\cdot11\\1}$, $\substack{123211\\2}$, giving $\substack{111110\\1}$, $\substack{1\cdot2221\\1}$, $\substack{123221\\2} \in X$ by default. To exclude $-\substack{011000\\0}$ we must have $\substack{122110\\1} \in X$, which excludes $\substack{000111\\0}$, $\substack{001111\\1}$, giving $\substack{112110\\1} \in X$ by default; to exclude $-\substack{011100\\0}$ we must have $\substack{122210\\1} \in X$, which excludes $\substack{000011\\0}$, giving $\substack{112210\\1}$, $\substack{123210\\2} \in X$ by default; so

$$X \subset \{\substack{234321\\2}, \substack{12\cdot\cdot21\\2}, \substack{123210\\2}, \substack{1\cdot2221\\1}, \substack{1\cdot2\cdot10\\1}, \substack{11111\cdot\\1}, \substack{001110\\1}, \substack{1\cdot0000\\0}, \substack{000000\\1}, \substack{000\cdot10\\0}\}$$
$$\subset w_4w_3w_5w_4w_2w_7w_6w_5w_4w_3w_1(\Omega \cup \{\substack{012211\\1}\}).$$

Next assume (iii) holds; this excludes $\substack{001\cdot\cdot0\\1}$, $\substack{111\cdot\cdot0\\1}$, $\substack{1\cdot2\cdot\cdot0\\1}$, $\substack{123210\\2}$, giving $\substack{1\cdot2\cdot\cdot1\\1}$, $\substack{1232\cdot1\\2} \in X$ by default; so

$$X \subset \{\substack{234321\\2}, \substack{12\cdot\cdot\cdot1\\2}, \substack{1\cdot2\cdot\cdot1\\1}, \substack{111111\\1}, \substack{001111\\1}, \substack{1\cdot0000\\0}, \substack{000000\\1}, \substack{000\cdot\cdot1\\0}\}$$
$$\subset w_4w_3w_5w_4w_2w_6w_5w_4w_3w_1(\Omega \cup \{\substack{012211\\1}\}).$$

Finally assume (iv) holds; this gives $\substack{111\cdot\cdot0\\1}$, $\substack{1232\cdot\cdot\\2} \in X$ by default. If X is not to be a subset of $\Omega\cup\{\substack{000000\\1}\}$ it must contain some root $\substack{001\cdot\cdot\cdot\\1}$; using $\langle w_5, w_6, w_7\rangle$ we may assume $\substack{001000\\1} \in X$, which excludes $\substack{110000\\0}$, $\substack{1222\cdot\cdot\\1}$, giving $\substack{1121\cdot\cdot\\1}$, $\substack{124321\\2} \in X$ by default. To exclude $-\substack{011000\\0}$ we must have some root $\substack{1221\cdot\cdot\\1}$ present; using $\langle w_6, w_7\rangle$ we may assume $\substack{122100\\1} \in X$, which excludes $\substack{00111\cdot\\1}$, $\substack{112221\\1}$. Now to exclude the two roots $-\substack{01111\cdot\\0}$ we must have $\substack{12211\cdot\\1} \in X$, which excludes $\substack{001100\\1}$, $\substack{11221\cdot\\1}$; so

$$X = \{\substack{234321\\2}, \substack{12\cdot\cdot\cdot\cdot\\2}, \substack{1\cdot\cdot1\cdot\cdot\\1}, \substack{111000\\1}, \substack{100000\\0}, \substack{00\cdot000\\1}\}$$
$$= w_5w_4w_3w_6w_5w_4w_2w_7w_6w_5w_4w_3w_6w_7(X_{19}).$$

This proves the lemma. □

LEMMA 7.2. *If* $\substack{100000\\0}$, $\substack{001000\\0} \in X$ *then* X *is known.*

PROOF. As before we assume $\substack{000000\\1} \notin X$, $\substack{100000\\0}$, $\substack{001000\\\cdot} \in X$, which excludes $\substack{01\cdot\cdot\cdot\cdot\\\cdot}$, $\substack{0001\cdot\cdot\\0}$, $\substack{110000\\0}$, $\substack{1111\cdot\cdot\\\cdot}$, $\substack{1222\cdot\cdot\\1}$, $\substack{123321\\\cdot}$, $\substack{134321\\2}$, giving $\substack{112111\\1}$, $\substack{124321\\2} \in X$ by default. Using Lemma 3.1 (with $Y = \{\substack{100000\\0}, \substack{001000\\\cdot}\}$) we may assume that one of the following holds: (i) $\substack{000010\\0} \in X$; (ii) $\substack{000010\\0} \notin X$, $\substack{0000\cdot1\\0} \in X$; (iii) $\substack{0000\cdot\cdot\\0} \notin X$.

First assume (i) holds; this excludes $\begin{smallmatrix}000001\\0\end{smallmatrix}$, $\begin{smallmatrix}001100\\\cdot\end{smallmatrix}$, $\begin{smallmatrix}1\cdot2100\\1\end{smallmatrix}$, $\begin{smallmatrix}112211\\1\end{smallmatrix}$, $\begin{smallmatrix}123211\\\cdot\end{smallmatrix}$, giving $\begin{smallmatrix}112110\\1\end{smallmatrix}$, $\begin{smallmatrix}112221\\1\end{smallmatrix}$, $\begin{smallmatrix}123221\\\cdot\end{smallmatrix} \in X$ by default. To exclude $-\begin{smallmatrix}000100\\0\end{smallmatrix}$ we must have $\begin{smallmatrix}112210\\1\end{smallmatrix} \in X$, which excludes $\begin{smallmatrix}000011\\0\end{smallmatrix}$, $\begin{smallmatrix}122111\\1\end{smallmatrix}$; to exclude $-\begin{smallmatrix}010000\\0\end{smallmatrix}$ we must have $\begin{smallmatrix}122110\\1\end{smallmatrix} \in X$, which excludes $\begin{smallmatrix}001111\\\cdot\end{smallmatrix}$, giving $\begin{smallmatrix}123210\\\cdot\end{smallmatrix} \in X$ by default; so

$$\begin{aligned} X &\subset \{\begin{smallmatrix}234321\\2\end{smallmatrix}, \begin{smallmatrix}124321\\2\end{smallmatrix}, \begin{smallmatrix}123221\\\cdot\end{smallmatrix}, \begin{smallmatrix}123210\\\cdot\end{smallmatrix}, \begin{smallmatrix}112221\\1\end{smallmatrix}, \begin{smallmatrix}122110\\1\end{smallmatrix}, \begin{smallmatrix}112\cdot10\\1\end{smallmatrix}, \begin{smallmatrix}112111\\1\end{smallmatrix}, \begin{smallmatrix}111000\\\cdot\end{smallmatrix}, \begin{smallmatrix}001110\\\cdot\end{smallmatrix}, \\ &\qquad \begin{smallmatrix}001000\\\cdot\end{smallmatrix}, \begin{smallmatrix}100000\\0\end{smallmatrix}, \begin{smallmatrix}000010\\0\end{smallmatrix}\} \\ &\subset w_3w_5w_4w_2w_7w_6w_5w_4w_3w_1(\Omega). \end{aligned}$$

Next assume (ii) holds; this excludes $\begin{smallmatrix}0011\cdot0\\\cdot\end{smallmatrix}$, $\begin{smallmatrix}1\cdot21\cdot0\\1\end{smallmatrix}$, $\begin{smallmatrix}112210\\1\end{smallmatrix}$, $\begin{smallmatrix}123210\\\cdot\end{smallmatrix}$, giving $\begin{smallmatrix}122111\\1\end{smallmatrix}$, $\begin{smallmatrix}1122\cdot1\\1\end{smallmatrix}$, $\begin{smallmatrix}1232\cdot1\\\cdot\end{smallmatrix} \in X$ by default; so

$$\begin{aligned} X &\subset \{\begin{smallmatrix}234321\\2\end{smallmatrix}, \begin{smallmatrix}124321\\2\end{smallmatrix}, \begin{smallmatrix}1232\cdot1\\\cdot\end{smallmatrix}, \begin{smallmatrix}112\cdot\cdot1\\1\end{smallmatrix}, \begin{smallmatrix}122111\\1\end{smallmatrix}, \begin{smallmatrix}001111\\\cdot\end{smallmatrix}, \begin{smallmatrix}111000\\\cdot\end{smallmatrix}, \begin{smallmatrix}001000\\\cdot\end{smallmatrix}, \begin{smallmatrix}0000\cdot1\\0\end{smallmatrix}, \begin{smallmatrix}100000\\0\end{smallmatrix}\} \\ &\subset w_3w_5w_4w_2w_6w_5w_4w_3w_1(\Omega). \end{aligned}$$

Finally assume (iii) holds; this gives $\begin{smallmatrix}1121\cdot0\\1\end{smallmatrix} \in X$ by default. To exclude $-\begin{smallmatrix}010000\\0\end{smallmatrix}$ we must have some root $\begin{smallmatrix}1221\cdot\cdot\\1\end{smallmatrix}$ present; using $\langle w_6, w_7\rangle$ we may assume $\begin{smallmatrix}122100\\1\end{smallmatrix} \in X$, which excludes $\begin{smallmatrix}00111\cdot\\\cdot\end{smallmatrix}$, $\begin{smallmatrix}112221\\1\end{smallmatrix}$, giving $\begin{smallmatrix}12321\cdot\\\cdot\end{smallmatrix} \in X$ by default. To exclude the two roots $-\begin{smallmatrix}00011\cdot\\0\end{smallmatrix}$ we must have $\begin{smallmatrix}11221\cdot\\1\end{smallmatrix} \in X$, which excludes $\begin{smallmatrix}12211\cdot\\1\end{smallmatrix}$. If neither root $\begin{smallmatrix}001100\\\cdot\end{smallmatrix}$ is present then $\begin{smallmatrix}111000\\\cdot\end{smallmatrix}$, $\begin{smallmatrix}123221\\\cdot\end{smallmatrix} \in X$ by default, so

$$\begin{aligned} X &= \{\begin{smallmatrix}234321\\2\end{smallmatrix}, \begin{smallmatrix}124321\\2\end{smallmatrix}, \begin{smallmatrix}1232\cdot\cdot\\\cdot\end{smallmatrix}, \begin{smallmatrix}112\cdot1\cdot\\1\end{smallmatrix}, \begin{smallmatrix}1\cdot2100\\1\end{smallmatrix}, \begin{smallmatrix}111000\\\cdot\end{smallmatrix}, \begin{smallmatrix}001000\\\cdot\end{smallmatrix}, \begin{smallmatrix}100000\\0\end{smallmatrix}\} \\ &= w_3w_5w_6w_7w_4w_2w_5w_6w_4w_2w_5w_4w_3(X_{19}). \end{aligned}$$

So we may assume X contains some root $\begin{smallmatrix}001100\\\cdot\end{smallmatrix}$; however, unless we actually have $\begin{smallmatrix}001100\\\cdot\end{smallmatrix} \in X$ we could apply w_5 to produce a positive set meeting $\{\begin{smallmatrix}00\cdot000\\\cdot\end{smallmatrix}\}$ in a proper non-empty subset of $\{\begin{smallmatrix}001000\\\cdot\end{smallmatrix}\}$, and then by Lemmas 3.1 and 7.1 X would be known. Thus we may assume $\begin{smallmatrix}001100\\\cdot\end{smallmatrix} \in X$, which excludes $\begin{smallmatrix}111000\\\cdot\end{smallmatrix}$, $\begin{smallmatrix}123221\\\cdot\end{smallmatrix}$; so

$$\begin{aligned} X &= \{\begin{smallmatrix}234321\\2\end{smallmatrix}, \begin{smallmatrix}124321\\2\end{smallmatrix}, \begin{smallmatrix}12321\cdot\\\cdot\end{smallmatrix}, \begin{smallmatrix}112\cdot1\cdot\\1\end{smallmatrix}, \begin{smallmatrix}1\cdot2100\\1\end{smallmatrix}, \begin{smallmatrix}001\cdot00\\\cdot\end{smallmatrix}, \begin{smallmatrix}100000\\0\end{smallmatrix}\} \\ &= w_3w_6w_5w_4w_2w_7w_6w_5w_4w_3w_1w_3w_6w_5(X_{17}^2). \end{aligned}$$

This proves the lemma. □

LEMMA 7.3. *If* $\begin{smallmatrix}100000\\0\end{smallmatrix}$, $\begin{smallmatrix}000100\\0\end{smallmatrix} \in X$ *then* X *is known.*

PROOF. As before we assume $\begin{smallmatrix}00\cdot000\\\cdot\end{smallmatrix} \notin X$, $\begin{smallmatrix}100000\\0\end{smallmatrix}$, $\begin{smallmatrix}00\cdot100\\\cdot\end{smallmatrix} \in X$, which excludes $\begin{smallmatrix}01\cdot\cdot\cdot\cdot\\\cdot\end{smallmatrix}$, $\begin{smallmatrix}00001\cdot\\0\end{smallmatrix}$, $\begin{smallmatrix}11\cdot000\\\cdot\end{smallmatrix}$, $\begin{smallmatrix}1\cdot\cdot11\cdot\\\cdot\end{smallmatrix}$, $\begin{smallmatrix}12\cdot221\\\cdot\end{smallmatrix}$, $\begin{smallmatrix}134321\\2\end{smallmatrix}$, giving $\begin{smallmatrix}112211\\1\end{smallmatrix}$, $\begin{smallmatrix}12\cdot321\\\cdot\end{smallmatrix} \in X$ by default. We begin by supposing $\begin{smallmatrix}000001\\0\end{smallmatrix} \in X$, which excludes $\begin{smallmatrix}00\cdot110\\\cdot\end{smallmatrix}$, $\begin{smallmatrix}1\cdot\cdot210\\\cdot\end{smallmatrix}$, giving $\begin{smallmatrix}12\cdot211\\\cdot\end{smallmatrix} \in X$ by default. To exclude $-\begin{smallmatrix}000010\\0\end{smallmatrix}$ we must have $\begin{smallmatrix}112221\\1\end{smallmatrix} \in X$, which excludes $\begin{smallmatrix}122100\\1\end{smallmatrix}$; so

$$\begin{aligned} X &\subset \{\begin{smallmatrix}234321\\2\end{smallmatrix}, \begin{smallmatrix}12\cdot321\\\cdot\end{smallmatrix}, \begin{smallmatrix}12\cdot211\\\cdot\end{smallmatrix}, \begin{smallmatrix}1122\cdot1\\1\end{smallmatrix}, \begin{smallmatrix}11\cdot100\\\cdot\end{smallmatrix}, \begin{smallmatrix}00\cdot111\\\cdot\end{smallmatrix}, \begin{smallmatrix}00\cdot100\\\cdot\end{smallmatrix}, \begin{smallmatrix}100000\\0\end{smallmatrix}, \begin{smallmatrix}000001\\0\end{smallmatrix}\} \\ &\subset w_3w_4w_2w_6w_5w_4w_3w_1(\Omega). \end{aligned}$$

We may therefore assume $\begin{smallmatrix}000001\\0\end{smallmatrix} \notin X$, giving $\begin{smallmatrix}112210\\1\end{smallmatrix} \in X$ by default. To exclude $-\begin{smallmatrix}000010\\0\end{smallmatrix}$ we must have $\begin{smallmatrix}112221\\1\end{smallmatrix}$ or some root $\begin{smallmatrix}00\cdot110\\\cdot\end{smallmatrix}$ present; thus X cannot contain $\begin{smallmatrix}122100\\1\end{smallmatrix}$, so that $\begin{smallmatrix}112221\\1\end{smallmatrix} \in X$ by default. To exclude $-\begin{smallmatrix}011000\\1\end{smallmatrix}$ we must have some root $\begin{smallmatrix}12321\cdot\\2\end{smallmatrix}$ present; using $\langle w_7\rangle$ we may assume $\begin{smallmatrix}123211\\2\end{smallmatrix} \in X$, which excludes $\begin{smallmatrix}00\cdot110\\0\end{smallmatrix}$. If we had $\begin{smallmatrix}001110\\1\end{smallmatrix} \in X$ we could apply w_6 to produce a positive set meeting $\{\begin{smallmatrix}00\cdot\cdot00\\\cdot\end{smallmatrix}\}$ in a proper non-empty subset of $\{\begin{smallmatrix}00\cdot100\\\cdot\end{smallmatrix}\}$, and then by Lemmas 3.1, 7.1 and 7.2 X

would be known; so we may assume $\genfrac{}{}{0pt}{}{001110}{1} \notin X$, giving $\genfrac{}{}{0pt}{}{12\cdot211}{1} \in X$ by default. If no root $\genfrac{}{}{0pt}{}{00\cdot111}{\cdot}$ is present then $\genfrac{}{}{0pt}{}{11\cdot100}{\cdot}, \genfrac{}{}{0pt}{}{12\cdot210}{\cdot} \in X$ by default, so

$$\begin{aligned} X &= \{\genfrac{}{}{0pt}{}{234321}{2}, \genfrac{}{}{0pt}{}{12\cdot321}{\cdot}, \genfrac{}{}{0pt}{}{12\cdot21\cdot}{\cdot}, \genfrac{}{}{0pt}{}{1122\cdot\cdot}{1}, \genfrac{}{}{0pt}{}{11\cdot100}{\cdot}, \genfrac{}{}{0pt}{}{00\cdot100}{\cdot}, \genfrac{}{}{0pt}{}{100000}{0}\} \\ &= w_3w_4w_2w_6w_5w_4w_3w_7w_6w_5w_4(X^1_{20}). \end{aligned}$$

So we may assume X contains some root $\genfrac{}{}{0pt}{}{00\cdot111}{\cdot}$; however, unless we actually have $\genfrac{}{}{0pt}{}{00\cdot111}{\cdot} \in X$ we could apply w_6w_7 to produce a positive set meeting $\{\genfrac{}{}{0pt}{}{00\cdot\cdot00}{\cdot}\}$ in a proper non-empty subset of $\{\genfrac{}{}{0pt}{}{00\cdot100}{\cdot}\}$, and then by Lemmas 3.1, 7.1 and 7.2 X would be known. Thus we may assume $\genfrac{}{}{0pt}{}{00\cdot111}{\cdot} \in X$, which excludes $\genfrac{}{}{0pt}{}{11\cdot100}{\cdot}, \genfrac{}{}{0pt}{}{12\cdot210}{\cdot}$; so

$$\begin{aligned} X &= \{\genfrac{}{}{0pt}{}{234321}{2}, \genfrac{}{}{0pt}{}{12\cdot321}{\cdot}, \genfrac{}{}{0pt}{}{12\cdot211}{\cdot}, \genfrac{}{}{0pt}{}{1122\cdot\cdot}{1}, \genfrac{}{}{0pt}{}{00\cdot111}{\cdot}, \genfrac{}{}{0pt}{}{00\cdot100}{\cdot}, \genfrac{}{}{0pt}{}{100000}{0}\} \\ &= w_3w_4w_2w_6w_5w_4w_3w_1w_3w_4w_5(X^6_{17}). \end{aligned}$$

This proves the lemma. □

Lemma 7.4. *If* $\genfrac{}{}{0pt}{}{100000}{0}, \genfrac{}{}{0pt}{}{000010}{0} \in X$ *then* X *is known.*

Proof. As before we assume $\genfrac{}{}{0pt}{}{00\cdot\cdot00}{\cdot} \notin X$, $\genfrac{}{}{0pt}{}{100000}{0}, \genfrac{}{}{0pt}{}{00\cdot\cdot10}{\cdot} \in X$, which excludes $\genfrac{}{}{0pt}{}{01\cdot\cdot\cdot\cdot}{\cdot}, \genfrac{}{}{0pt}{}{000001}{0}, \genfrac{}{}{0pt}{}{11\cdot\cdot00}{\cdot}, \genfrac{}{}{0pt}{}{122100}{1}, \genfrac{}{}{0pt}{}{1\cdot\cdot\cdot11}{\cdot}, \genfrac{}{}{0pt}{}{134321}{2}$, giving $\genfrac{}{}{0pt}{}{112221}{1}, \genfrac{}{}{0pt}{}{12\cdot\cdot21}{\cdot} \in X$ by default. To exclude $-\genfrac{}{}{0pt}{}{000001}{0}$ we must have some root $\genfrac{}{}{0pt}{}{00\cdot\cdot11}{\cdot}$ present; however, unless we actually have $\genfrac{}{}{0pt}{}{00\cdot\cdot11}{\cdot} \in X$ we could apply w_7 to produce a positive set meeting $\{\genfrac{}{}{0pt}{}{00\cdot\cdot\cdot0}{\cdot}\}$ in a proper non-empty subset of $\{\genfrac{}{}{0pt}{}{00\cdot\cdot10}{\cdot}\}$, and then by Lemmas 3.1, 7.1, 7.2 and 7.3 X would be known. Thus we may assume $\genfrac{}{}{0pt}{}{00\cdot\cdot11}{\cdot} \in X$, which excludes $\genfrac{}{}{0pt}{}{1\cdot\cdot\cdot10}{\cdot}$; so

$$\begin{aligned} X &= \{\genfrac{}{}{0pt}{}{234321}{2}, \genfrac{}{}{0pt}{}{12\cdot\cdot21}{\cdot}, \genfrac{}{}{0pt}{}{112221}{1}, \genfrac{}{}{0pt}{}{00\cdot\cdot1\cdot}{\cdot}, \genfrac{}{}{0pt}{}{100000}{0}\} \\ &= w_3w_4w_5w_2w_4w_3w_1w_3w_4w_2w_5w_4w_3(X^{16}_{17}). \end{aligned}$$

This proves the lemma. □

Lemma 7.5. *If* $\genfrac{}{}{0pt}{}{100000}{0}, \genfrac{}{}{0pt}{}{000001}{0} \in X$ *then* X *is known.*

Proof. As before we assume $\genfrac{}{}{0pt}{}{00\cdot\cdot\cdot0}{\cdot} \notin X$, $\genfrac{}{}{0pt}{}{100000}{0}, \genfrac{}{}{0pt}{}{00\cdot\cdot\cdot1}{\cdot} \in X$, which excludes $\genfrac{}{}{0pt}{}{01\cdot\cdot\cdot\cdot}{\cdot}, \genfrac{}{}{0pt}{}{11\cdot\cdot\cdot0}{\cdot}, \genfrac{}{}{0pt}{}{12\cdot\cdot\cdot0}{\cdot}, \genfrac{}{}{0pt}{}{134321}{2}$, giving $\genfrac{}{}{0pt}{}{11\cdot\cdot\cdot1}{\cdot}, \genfrac{}{}{0pt}{}{12\cdot\cdot\cdot1}{\cdot} \in X$ by default; so

$$X = \{\genfrac{}{}{0pt}{}{234321}{2}, \genfrac{}{}{0pt}{}{12\cdot\cdot\cdot1}{\cdot}, \genfrac{}{}{0pt}{}{11\cdot\cdot\cdot1}{\cdot}, \genfrac{}{}{0pt}{}{00\cdot\cdot\cdot1}{\cdot}, \genfrac{}{}{0pt}{}{100000}{0}\} = w_3w_4w_5w_6w_2w_4w_3w_5w_4w_2(X_{22}).$$

This proves the lemma. □

This completes the treatment of the sets containing $\genfrac{}{}{0pt}{}{100000}{0}$. We therefore move on to consider the other possibilities in turn for the simple root α lying in X. In the analysis to follow, at some points we shall write $X = X_c \cup X_a$, where X_c is the set of roots which have been chosen by then (including those known to be in X by default), and X_a is a subset (to be determined) of the available roots; if we can find $w \in W$ which sends one element of X_c to $\genfrac{}{}{0pt}{}{100000}{0}$ while preserving the positivity of both X_c and the set of all available roots, there will be no need to pursue the line of reasoning further.

Lemma 7.6. *If* $\genfrac{}{}{0pt}{}{010000}{0} \in X$ *then* X *is known.*

Proof. As before we assume $\begin{smallmatrix}100000\\0\end{smallmatrix} \notin X$, $\begin{smallmatrix}\cdot 10000\\0\end{smallmatrix} \in X$; this excludes the roots $\begin{smallmatrix}001\cdots\\\cdot\end{smallmatrix}$, $\begin{smallmatrix}\cdot 12\cdots\\1\end{smallmatrix}$, $\begin{smallmatrix}124321\\2\end{smallmatrix}$, giving $\begin{smallmatrix}122221\\1\end{smallmatrix}$, $\begin{smallmatrix}134321\\2\end{smallmatrix} \in X$ by default. Using Lemma 3.1 we may assume that one of the following holds: (i) $\begin{smallmatrix}000100\\0\end{smallmatrix} \in X$; (ii) $\begin{smallmatrix}000100\\0\end{smallmatrix} \notin X$, $\begin{smallmatrix}000\cdot 10\\0\end{smallmatrix} \in X$; (iii) $\begin{smallmatrix}000\cdot\cdot 0\\0\end{smallmatrix} \notin X$, $\begin{smallmatrix}000\cdot\cdot 1\\0\end{smallmatrix} \in X$; (iv) $\begin{smallmatrix}000\cdots\\0\end{smallmatrix} \notin X$.

First assume (i) holds; this excludes $\begin{smallmatrix}00001\cdot\\0\end{smallmatrix}$, $\begin{smallmatrix}\cdot 11000\\\cdot\end{smallmatrix}$, $\begin{smallmatrix}12211\cdot\\1\end{smallmatrix}$, $\begin{smallmatrix}123221\\\cdot\end{smallmatrix}$, giving $\begin{smallmatrix}122211\\1\end{smallmatrix}$, $\begin{smallmatrix}123321\\2\end{smallmatrix} \in X$ by default. Suppose $\begin{smallmatrix}000001\\0\end{smallmatrix} \in X$; this excludes $\begin{smallmatrix}000110\\0\end{smallmatrix}$, $\begin{smallmatrix}\cdot 11110\\\cdot\end{smallmatrix}$, $\begin{smallmatrix}12\cdot 210\\\cdot\end{smallmatrix}$, giving $\begin{smallmatrix}123211\\2\end{smallmatrix} \in X$ by default. To exclude $-\begin{smallmatrix}001000\\0\end{smallmatrix}$ we must have $\begin{smallmatrix}123211\\1\end{smallmatrix} \in X$, which excludes $\begin{smallmatrix}000000\\1\end{smallmatrix}$, giving $\begin{smallmatrix}123321\\1\end{smallmatrix} \in X$ by default. Thus $X = X_c \cup X_a$ where

$$X_c = \{\begin{smallmatrix}\cdot 34321\\2\end{smallmatrix}, \begin{smallmatrix}123321\\\cdot\end{smallmatrix}, \begin{smallmatrix}123211\\\cdot\end{smallmatrix}, \begin{smallmatrix}1222\cdot 1\\1\end{smallmatrix}, \begin{smallmatrix}\cdot 10000\\0\end{smallmatrix}, \begin{smallmatrix}000100\\0\end{smallmatrix}, \begin{smallmatrix}000001\\0\end{smallmatrix}\},$$
$$X_a \subset \{\begin{smallmatrix}122100\\1\end{smallmatrix}, \begin{smallmatrix}\cdot 11111\\\cdot\end{smallmatrix}, \begin{smallmatrix}\cdot 11100\\\cdot\end{smallmatrix}, \begin{smallmatrix}000111\\0\end{smallmatrix}\};$$

set

$$w = w_3w_4w_5w_6w_7w_1w_3w_4w_5w_6w_2w_4,$$

then we have $w(X) \subset \Phi^+$, and $\begin{smallmatrix}100000\\0\end{smallmatrix} = w(\begin{smallmatrix}000001\\0\end{smallmatrix}) \in w(X_c)$. So suppose instead $\begin{smallmatrix}000001\\0\end{smallmatrix} \notin X$, giving $\begin{smallmatrix}122210\\1\end{smallmatrix} \in X$ by default. To exclude $-\begin{smallmatrix}001000\\0\end{smallmatrix}$ we must have some root $\begin{smallmatrix}12321\cdot\\1\end{smallmatrix}$ present; using $\langle w_7\rangle$ we may assume $\begin{smallmatrix}123211\\1\end{smallmatrix} \in X$, which excludes $\begin{smallmatrix}000000\\1\end{smallmatrix}$, $\begin{smallmatrix}000110\\0\end{smallmatrix}$, $\begin{smallmatrix}\cdot 11110\\1\end{smallmatrix}$, giving $\begin{smallmatrix}123321\\1\end{smallmatrix} \in X$ by default. Thus $X = X_c \cup X_a$ where

$$X_c = \{\begin{smallmatrix}\cdot 34321\\2\end{smallmatrix}, \begin{smallmatrix}123321\\\cdot\end{smallmatrix}, \begin{smallmatrix}123211\\1\end{smallmatrix}, \begin{smallmatrix}1222\cdot\cdot\\1\end{smallmatrix}, \begin{smallmatrix}\cdot 10000\\0\end{smallmatrix}, \begin{smallmatrix}000100\\0\end{smallmatrix}\},$$
$$X_a \subset \{\begin{smallmatrix}12321\cdot\\2\end{smallmatrix}, \begin{smallmatrix}123210\\1\end{smallmatrix}, \begin{smallmatrix}122100\\1\end{smallmatrix}, \begin{smallmatrix}\cdot 11111\\\cdot\end{smallmatrix}, \begin{smallmatrix}\cdot 11110\\0\end{smallmatrix}, \begin{smallmatrix}\cdot 11100\\\cdot\end{smallmatrix}, \begin{smallmatrix}000111\\0\end{smallmatrix}\};$$

set

$$w = w_3w_4w_5w_6w_2w_1w_3w_4w_5w_6w_2w_4,$$

then we have $w(X) \subset \Phi^+$, and $\begin{smallmatrix}100000\\0\end{smallmatrix} = w(\begin{smallmatrix}000100\\0\end{smallmatrix}) \in w(X_c)$.

Next assume (ii) holds; this excludes the roots $\begin{smallmatrix}000001\\0\end{smallmatrix}$, $\begin{smallmatrix}\cdot 11\cdot 00\\\cdot\end{smallmatrix}$, $\begin{smallmatrix}122100\\1\end{smallmatrix}$, $\begin{smallmatrix}12\cdot\cdot 11\\\cdot\end{smallmatrix}$, giving $\begin{smallmatrix}123\cdot 21\\2\end{smallmatrix} \in X$ by default. Thus $X = X_c \cup X_a$ where

$$X_c = \{\begin{smallmatrix}\cdot 34321\\2\end{smallmatrix}, \begin{smallmatrix}123\cdot 21\\2\end{smallmatrix}, \begin{smallmatrix}122221\\1\end{smallmatrix}, \begin{smallmatrix}\cdot 10000\\0\end{smallmatrix}, \begin{smallmatrix}000\cdot 10\\0\end{smallmatrix}\},$$
$$X_a \subset \{\begin{smallmatrix}123\cdot 21\\1\end{smallmatrix}, \begin{smallmatrix}12\cdot\cdot 10\\\cdot\end{smallmatrix}, \begin{smallmatrix}\cdot 1111\cdot\\\cdot\end{smallmatrix}, \begin{smallmatrix}000\cdot 11\\0\end{smallmatrix}, \begin{smallmatrix}000000\\1\end{smallmatrix}\};$$

set

$$w = w_3w_4w_5w_6w_1w_3w_4w_5w_2w_4,$$

then we have $w(X) \subset \Phi^+$, and $\begin{smallmatrix}100000\\0\end{smallmatrix} = w(\begin{smallmatrix}000010\\0\end{smallmatrix}) \in w(X_c)$.

Next assume (iii) holds; this excludes the roots $\begin{smallmatrix}\cdot 11\cdot\cdot 0\\\cdot\end{smallmatrix}$, $\begin{smallmatrix}12\cdot\cdot\cdot 0\\\cdot\end{smallmatrix}$, giving $\begin{smallmatrix}\cdot 11111\\1\end{smallmatrix}$, $\begin{smallmatrix}122\cdot 11\\1\end{smallmatrix}$, $\begin{smallmatrix}123\cdot\cdot 1\\2\end{smallmatrix} \in X$ by default. To exclude $-\begin{smallmatrix}001000\\0\end{smallmatrix}$ we must have some root $\begin{smallmatrix}1232\cdot 1\\1\end{smallmatrix}$ present, so $\begin{smallmatrix}000000\\1\end{smallmatrix} \notin X$, giving $\begin{smallmatrix}\cdot 11111\\0\end{smallmatrix}$, $\begin{smallmatrix}123\cdot\cdot 1\\1\end{smallmatrix} \in X$ by default; so

$$\begin{aligned} X &= \{\begin{smallmatrix}\cdot 34321\\2\end{smallmatrix}, \begin{smallmatrix}123\cdot\cdot 1\\\cdot\end{smallmatrix}, \begin{smallmatrix}122\cdot\cdot 1\\1\end{smallmatrix}, \begin{smallmatrix}\cdot 11111\\\cdot\end{smallmatrix}, \begin{smallmatrix}000\cdot\cdot 1\\0\end{smallmatrix}, \begin{smallmatrix}\cdot 10000\\0\end{smallmatrix}\} \\ &= w_4w_2w_5w_4w_3w_1w_6w_5w_4w_3w_2w_4(X^1_{20}). \end{aligned}$$

Finally assume (iv) holds; this gives $\begin{smallmatrix}122\cdot 11\\1\end{smallmatrix}$, $\begin{smallmatrix}122\cdot\cdot 0\\1\end{smallmatrix} \in X$ by default. To exclude $-\begin{smallmatrix}001000\\0\end{smallmatrix}$ we must have some root $\begin{smallmatrix}1232\cdot\cdot\\1\end{smallmatrix}$ or some root $\begin{smallmatrix}\cdot 11000\\0\end{smallmatrix}$ present, so $\begin{smallmatrix}000000\\1\end{smallmatrix} \notin X$. By Corollary 3.2 and the previous lemmas in this section, we may assume $X \cap \{\begin{smallmatrix}\cdot 11\cdots\\\cdot\end{smallmatrix}\}$ is stable under $\langle w_1\rangle$. If no root $\begin{smallmatrix}\cdot 11\cdots\\\cdot\end{smallmatrix}$ is present then $\begin{smallmatrix}123\cdots\\\cdot\end{smallmatrix} \in X$ by default; so

$$X = \{\begin{smallmatrix}\cdot 34321\\2\end{smallmatrix}, \begin{smallmatrix}123\cdots\\\cdot\end{smallmatrix}, \begin{smallmatrix}122\cdots\\1\end{smallmatrix}, \begin{smallmatrix}\cdot 10000\\0\end{smallmatrix}\} = w_4w_5w_6w_7w_2w_4w_5w_6(X^1_{18}).$$

So we may assume X contains some root $\substack{\cdot 11\cdot\cdot\cdot\\ \cdot}$; using $\langle w_2, w_5, w_6, w_7\rangle$ we may assume $\substack{\cdot 11000\\ 0} \in X$, which excludes $\substack{\cdot 111\cdot\cdot\\ 1}$, $\substack{123321\\ 2}$, giving $\substack{1232\cdot\cdot\\ 1} \in X$ by default. If $\substack{\cdot 11000\\ 1} \in X$ this excludes $\substack{\cdot 111\cdot\cdot\\ 0}$, $\substack{123321\\ 1}$, giving $\substack{1232\cdot\cdot\\ 2} \in X$ by default; so

$$X = \{\substack{\cdot 34321\\ 2}, \substack{1232\cdot\cdot\\ \cdot}, \substack{122\cdot\cdot\cdot\\ 1}, \substack{\cdot 1\cdot 000\\ \cdot}\} = w_5w_6w_7w_4w_5w_6w_2w_4w_5(X^2_{20}).$$

So we may assume $\substack{\cdot 11000\\ 1} \notin X$, giving $\substack{123321\\ 1} \in X$ by default. To exclude $-\substack{000000\\ 1}$ we must have some root $\substack{1232\cdot\cdot\\ 2}$ present; using $\langle w_6, w_7\rangle$ we may assume $\substack{123210\\ 2} \in X$, which excludes $\substack{\cdot 11111\\ 0}$. To exclude $-\substack{001111\\ 1}$ we must have some root $\substack{1232\cdot 1\\ 2}$ present; using $\langle w_6\rangle$ we may assume $\substack{123211\\ 2} \in X$, which excludes $\substack{\cdot 11110\\ 0}$. If $\substack{123221\\ 2} \in X$ this excludes $\substack{\cdot 11100\\ 0}$; so

$$X = \{\substack{\cdot 34321\\ 2}, \substack{1232\cdot\cdot\\ 2}, \substack{123\cdot\cdot\cdot\\ 1}, \substack{122\cdot\cdot\cdot\\ 1}, \substack{\cdot 1\cdot 000\\ 0}\} = w_5w_6w_7w_4w_5w_6w_2w_4w_5w_6w_7(X_{19}).$$

So we may assume $\substack{123221\\ 2} \notin X$, giving $\substack{\cdot 11100\\ 0} \in X$ by default; so

$$X = \{\substack{\cdot 34321\\ 2}, \substack{12321\cdot\\ 2}, \substack{123\cdot\cdot\cdot\\ 1}, \substack{122\cdot\cdot\cdot\\ 1}, \substack{\cdot 1\cdot\cdot 00\\ 0}\} = w_6w_7w_5w_6w_4w_5w_2w_4w_5w_6w_7(X^1_{20}).$$

This proves the lemma. □

As was the case with $\substack{100000\\ 0}$, to show that a set is known it will now suffice to show that it is a W-translate of a subset of Φ^+ containing $\substack{010000\\ 0}$.

LEMMA 7.7. *If* $\substack{001000\\ 0} \in X$ *then* X *is known.*

PROOF. As before we assume $\substack{\cdot\cdot 0000\\ 0} \notin X$, $\substack{\cdot\cdot 1000\\ 0} \in X$; this excludes the roots $\substack{000000\\ 1}$, $\substack{0001\cdot\cdot\\ 0}$, $\substack{\cdot\cdot 11\cdot\cdot\\ 1}$, $\substack{\cdot\cdot 22\cdot\cdot\\ 1}$, $\substack{123321\\ 2}$, giving $\substack{123221\\ 1}$, $\substack{1\cdot 4321\\ 2} \in X$ by default. To exclude $-\substack{000100\\ 0}$ we must have $\substack{123321\\ 1}$ or some root $\substack{\cdot\cdot 1100\\ 0}$ present, so some root $\substack{\cdot\cdot 1000\\ 1}$ is absent; if we also had some root $\substack{\cdot\cdot 1000\\ 1}$ present, we could apply w_2 to produce a positive set meeting $\{\substack{\cdot\cdot\cdot 000\\ 0}\}$ in a proper non-empty subset of $\{\substack{\cdot\cdot 1000\\ 0}\}$, whence by Lemma 3.1 and the previous lemmas in this section X would be known. Thus we must have $\substack{\cdot\cdot 1000\\ 1} \notin X$, giving $\substack{123321\\ 1} \in X$ by default. Using Lemma 3.1 we may assume that one of the following holds: (i) $\substack{000010\\ 0} \in X$; (ii) $\substack{000010\\ 0} \notin X$, $\substack{0000\cdot 1\\ 0} \in X$; (iii) $\substack{0000\cdot\cdot\\ 0} \notin X$.

First assume (i) holds; this excludes the roots $\substack{000001\\ 0}$, $\substack{\cdot\cdot 1100\\ 0}$, $\substack{\cdot\cdot 2100\\ 1}$, $\substack{123211\\ \cdot}$, giving $\substack{123221\\ 2} \in X$ by default. Thus $X = X_c \cup X_a$ where

$$X_c = \{\substack{\cdot\cdot 4321\\ 2}, \substack{123\cdot 21\\ 1}, \substack{123221\\ 2}, \substack{\cdot\cdot 1000\\ 0}, \substack{000010\\ 0}\},$$
$$X_a \subset \{\substack{123210\\ \cdot}, \substack{\cdot\cdot 211\cdot\\ 1}, \substack{\cdot\cdot 111\cdot\\ 0}, \substack{000011\\ 0}\};$$

set

$$w = w_4w_5w_6w_3w_4w_2w_5,$$

then we have $w(X) \subset \Phi^+$, and $\substack{010000\\ 0} = w(\substack{000010\\ 0}) \in w(X_c)$.

Next assume (ii) holds; this excludes $\substack{\cdot\cdot 11\cdot 0\\ 0}$, $\substack{\cdot\cdot 21\cdot 0\\ 1}$, $\substack{123210\\ \cdot}$, giving $\substack{\cdot\cdot 1111\\ 0}$, $\substack{\cdot\cdot 2111\\ 1}$, $\substack{123211\\ \cdot}$, $\substack{123221\\ 2} \in X$ by default; so

$$X = \{\substack{\cdot\cdot 4321\\ 2}, \substack{123321\\ 1}, \substack{1232\cdot 1\\ \cdot}, \substack{\cdot\cdot 2111\\ 1}, \substack{\cdot\cdot 1111\\ 0}, \substack{\cdot\cdot 1000\\ 0}, \substack{0000\cdot 1\\ 0}\}$$
$$= w_5w_6w_2w_4w_5w_3w_4w_2w_1w_3w_4w_5(X_{19}).$$

Finally assume (iii) holds; this gives $\substack{12321\cdot\\ 1} \in X$ by default. By Corollary 3.2 and the previous lemmas in this section, we may assume $X \cap \{\substack{\cdot\cdot 11\cdot\cdot\\ 0}\}$ is stable under $\langle w_1, w_3\rangle$. If no root $\genfrac{}{}{0pt}{}{\cdot\cdot11\cdot\cdot}{0}$ is present then $\genfrac{}{}{0pt}{}{\cdot\cdot21\cdot\cdot}{1}, \genfrac{}{}{0pt}{}{1232\cdot\cdot}{2} \in X$ by default; so

$$X = \{ \genfrac{}{}{0pt}{}{\cdot\cdot4321}{2}, \genfrac{}{}{0pt}{}{1232\cdot\cdot}{2}, \genfrac{}{}{0pt}{}{123\cdot\cdot\cdot}{1}, \genfrac{}{}{0pt}{}{\cdot\cdot21\cdot\cdot}{1}, \genfrac{}{}{0pt}{}{\cdot\cdot1000}{0} \} = w_2w_5w_6w_7(X_{22}).$$

So using $\langle w_6, w_7\rangle$ we may assume $\genfrac{}{}{0pt}{}{\cdot\cdot1100}{0} \in X$, which excludes $\genfrac{}{}{0pt}{}{\cdot\cdot211\cdot}{1}, \genfrac{}{}{0pt}{}{123221}{2}$. If no root $\genfrac{}{}{0pt}{}{\cdot\cdot111\cdot}{0}$ is present then $\genfrac{}{}{0pt}{}{\cdot\cdot2100}{1}, \genfrac{}{}{0pt}{}{12321\cdot}{2} \in X$ by default; so

$$X = \{ \genfrac{}{}{0pt}{}{\cdot\cdot4321}{2}, \genfrac{}{}{0pt}{}{123\cdot\cdot\cdot}{1}, \genfrac{}{}{0pt}{}{12321\cdot}{2}, \genfrac{}{}{0pt}{}{\cdot\cdot2100}{1}, \genfrac{}{}{0pt}{}{\cdot\cdot1\cdot00}{0} \} = w_6w_5w_7w_6w_2w_4w_3w_5w_4w_1w_3(X_{18}^1).$$

So using $\langle w_7\rangle$ we may assume $\genfrac{}{}{0pt}{}{\cdot\cdot1110}{0} \in X$, which excludes $\genfrac{}{}{0pt}{}{\cdot\cdot2100}{1}, \genfrac{}{}{0pt}{}{123211}{2}$. To exclude $-\genfrac{}{}{0pt}{}{000000}{1}$ we must have $\genfrac{}{}{0pt}{}{123210}{2} \in X$, which excludes $\genfrac{}{}{0pt}{}{\cdot\cdot1111}{0}$; so

$$\begin{aligned} X &= \{ \genfrac{}{}{0pt}{}{\cdot\cdot4321}{2}, \genfrac{}{}{0pt}{}{123\cdot\cdot\cdot}{1}, \genfrac{}{}{0pt}{}{123210}{2}, \genfrac{}{}{0pt}{}{\cdot\cdot1\cdot\cdot0}{0} \} \\ &= w_2w_7w_6w_5w_4w_3w_1w_5w_4w_3w_6w_5w_4w_7w_6w_5(X_{17}^6). \end{aligned}$$

This proves the lemma. □

Lemma 7.8. *If $\genfrac{}{}{0pt}{}{000000}{1} \in X$ then X is known.*

Proof. As before we assume $\genfrac{}{}{0pt}{}{\cdot\cdot\cdot000}{0} \notin X$, $\genfrac{}{}{0pt}{}{\cdot\cdot\cdot000}{1} \in X$; this excludes $\genfrac{}{}{0pt}{}{0001\cdot\cdot}{0}$, $\genfrac{}{}{0pt}{}{\cdot\cdot11\cdot\cdot}{0}, \genfrac{}{}{0pt}{}{\cdot\cdot22\cdot\cdot}{1}, \genfrac{}{}{0pt}{}{123\cdot\cdot\cdot}{1}$, giving $\genfrac{}{}{0pt}{}{1\cdot\cdot\cdot21}{2} \in X$ by default. If some root $\genfrac{}{}{0pt}{}{\cdot\cdot1100}{1}$ were absent, we could apply w_5 to produce a positive set meeting $\{ \genfrac{}{}{0pt}{}{\cdot\cdot\cdot000}{\cdot} \}$ in a proper non-empty subset of $\{ \genfrac{}{}{0pt}{}{\cdot\cdot\cdot000}{1} \}$, whence as before X would be known; so we must have $\genfrac{}{}{0pt}{}{\cdot\cdot1100}{1} \in X$, which excludes $\genfrac{}{}{0pt}{}{00001\cdot}{0}, \genfrac{}{}{0pt}{}{\cdot\cdot211\cdot}{1}$, giving $\genfrac{}{}{0pt}{}{123211}{2} \in X$ by default. Similarly if some root $\genfrac{}{}{0pt}{}{\cdot12100}{1}$ or $\genfrac{}{}{0pt}{}{122100}{1}$ were absent, we could apply w_5w_4 or $w_5w_4w_3$ and argue in the same fashion to deduce that X would be known; so we must have $\genfrac{}{}{0pt}{}{\cdot\cdot2100}{1} \in X$, which excludes $\genfrac{}{}{0pt}{}{\cdot\cdot111\cdot}{1}$; so

$$X \subset \{ \genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot\cdot\cdot}{2}, \genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot00}{1}, \genfrac{}{}{0pt}{}{000001}{0} \} \subset w_6w_5w_4w_3w_1(\Omega).$$

This proves the lemma. □

Lemma 7.9. *If $\genfrac{}{}{0pt}{}{000100}{0} \in X$ then X is known.*

Proof. As before we assume $\genfrac{}{}{0pt}{}{\cdot\cdot\cdot000}{\cdot} \notin X$, $\genfrac{}{}{0pt}{}{\cdot\cdot\cdot100}{\cdot} \in X$; this excludes the roots $\genfrac{}{}{0pt}{}{00001\cdot}{0}, \genfrac{}{}{0pt}{}{\cdot\cdot\cdot11\cdot}{\cdot}, \genfrac{}{}{0pt}{}{\cdot\cdot\cdot221}{\cdot}$, giving $\genfrac{}{}{0pt}{}{\cdot\cdot\cdot211}{\cdot}, \genfrac{}{}{0pt}{}{1\cdot\cdot321}{\cdot} \in X$ by default. However, now $-\genfrac{}{}{0pt}{}{000010}{0}$ cannot be excluded; so no sets require consideration, and the lemma is proved. □

Lemma 7.10. *If $\genfrac{}{}{0pt}{}{000010}{0} \in X$ then X is known.*

Proof. As before we assume $\genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot00}{\cdot} \notin X$, $\genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot10}{\cdot} \in X$; this excludes the roots $\genfrac{}{}{0pt}{}{000001}{0}, \genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot11}{\cdot}$, giving $\genfrac{}{}{0pt}{}{012221}{1}, \genfrac{}{}{0pt}{}{1\cdot\cdot\cdot21}{\cdot} \in X$ by default. However, now $-\genfrac{}{}{0pt}{}{000001}{0}$ cannot be excluded; so no sets require consideration, and the lemma is proved. □

Lemma 7.11. *If $\genfrac{}{}{0pt}{}{000001}{0} \in X$ then X is known.*

Proof. As before we assume $\genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot\cdot0}{\cdot} \notin X$, $\genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot\cdot1}{\cdot} \in X$; so

$$X = \{ \genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot\cdot1}{\cdot} \} = X_{27}.$$

This proves the lemma. □

Combining the various lemmas in this section we have proved the following.

Theorem 7.12. *If X is a maximal abelian set in a root system of type E_7, then a W-translate of X lies in $\mathcal{S}(E_7)$.*

7.4. Stabilizers and structure of maximal abelian sets

For each set $X \in \mathcal{S}(E_7)$ we shall determine its stabilizer W_X in W, and find the W_X-orbits on X. Recall that for $\beta \in X$ the orthogonality count $o(\beta)$ is simply the number of roots in X which are orthogonal to β. If in fact X is radical, we may read off the orthogonality counts from the graph Γ_X. Here two roots represented by (black or white) edges in Γ_X are orthogonal if and only if the edges either meet at a vertex and are of different colours, or do not meet and are of the same colour. Suppose Γ_X has e (black) edges. We then have $o(\substack{234321\\2}) = 0$ and $o(\substack{134321\\2}) = e$; for any $\beta \in X$ represented by a (black) edge in Γ_X which meets t others, we have $o(\beta) = 1 + (8 - t) + (e - t - 1) = 8 + e - 2t$; for any $\beta \in X$ represented by an absent (white) edge in Γ_X which meets t (black) edges, we have $o(\beta) = (6 - (e - t)) + t = 6 - e + 2t$. (Again we observe that, as given by Lemma 3.3, $o(\beta)$ therefore has the same parity for all $\beta \in X \setminus \{\rho\}$.)

Here we have also the near-radical sets to consider. For such a set X represented by the bipartite graph Γ'_X with e (black) edges, we have $o(\substack{234321\\2}) = 1$, $o(\substack{134321\\2}) = 1 + e$, $o(\substack{112221\\1}) = 9 - e$ and $o(\substack{012221\\1}) = 9$. This is because $\substack{234321\\2}$ is orthogonal to $\substack{012221\\1}$, which is also orthogonal to the eight roots represented by the (black or white) edges of Γ'_X; and $\substack{134321\\2}$ is orthogonal to $\substack{112221\\1}$ and the roots represented by the (black) edges of Γ'_X, while $\substack{112221\\1}$ is also orthogonal to those represented by the absent (white) edges. For the remaining roots the orthogonality count is most simply obtained by adding the (black) edge 12 to Γ'_X to represent the presence of $\substack{112221\\1}$ and treating the resulting graph as above, except that for roots β represented by (black or white) edges ij with $i \in \{1, 2\}$ and $j \in \{3, 4, 5, 6\}$ the value of $o(\beta)$ must be increased by one because of the presence of $\substack{012221\\1}$.

We use the same method as employed in the E_6 analysis; there will however be many more cases in which a set $\{\beta \in X : o(\beta) = i\}$ breaks into a union of W_X-orbits. In most of the sets we shall see that W_X must fix $\substack{234321\\2}$, and thus must lie in $\langle w_2, w_3, w_4, w_5, w_6, w_7 \rangle$. Once more we shall work through the possibilities for X in turn.

If $X = X_{14}$ we must fix $\{\substack{\cdot\cdot\cdot\cdot\cdot\cdot\\2}\}, \{\substack{123321\\1}, \substack{122111\\1}, \substack{112110\\1}, \substack{012100\\1}, \substack{111100\\1}, \substack{011110\\1}, \substack{001111\\1}\}$ ($o(\beta) = 3, 9$ respectively). Here we first note that each of the roots in the first set is orthogonal to three in the second, in such a way that we may identify the two sets with the points and lines of the Fano plane; consequently W_X is isomorphic to a subgroup of $L_3(2)$. We set

$$G = \langle w_1 w_4, w_3 w_7, w_4 {w_6}^{w_5} \rangle;$$

then G is 2-transitive on $\{\substack{\cdot\cdot\cdot\cdot\cdot\cdot\\2}\}$ and the stabilizer of both $\substack{234321\\2}$ and $\substack{134321\\2}$ contains $\langle w_4 w_6, w_4 {w_6}^{w_5} \rangle$ (note that $w_4 w_6 = [w_1 w_4, w_4 {w_6}^{w_5}] \in G$), so that $|G| \geq 7.6.4 = 168 = |L_3(2)|$. Thus $W_X = G$.

If $X = X^1_{17}$ we must fix $\{\substack{\cdot 34321\\2}\}, \{\substack{12\cdot\cdot\cdot\cdot\\\cdot}\}$ ($o(\beta) = 0, 6$ respectively). We set

$$G = \langle w_1, w_2, w_4, w_5, w_6, w_7 \rangle;$$

then G is transitive on $\{\substack{\cdot 34321\\2}\}$, so we may fix $\beta_1 = \substack{234321\\2}$ and thus $\beta_2 = \substack{134321\\2}$. In $\mathrm{stab}_W(\beta_1) = \langle w_2, w_3, w_4, w_5, w_6, w_7 \rangle$ the stabilizer of β_2 is $\langle w_2, w_4, w_5, w_6, w_7 \rangle$ (since this subgroup has index 32, which equals the number of roots $\substack{1\cdot\cdot\cdot\cdot\cdot\\\cdot}$ to which β_2 may be taken under $\mathrm{stab}_W(\beta_1)$); so $W_X = G$.

If $X = X_{17}^2$ we must fix ${}^{234321}_{2}$, ${}^{134321}_{2}$, ${}^{124321}_{2}$, $\{{}^{123\cdots}_{\cdot}\}$, $\{{}^{1222\cdot1}_{1}, {}^{1221\cdot0}_{1}\}$, $\{{}^{112210}_{1}, {}^{112111}_{1}\}$ ($o(\beta) = 0, 2, 4, 6, 8, 10$ respectively). We set

$$G = \langle w_2, w_6, w_5w_7\rangle;$$

then G is transitive on $\{{}^{123\cdots}_{\cdot}\}$, so we may fix $\beta_1 = {}^{123321}_{2}$. We must then fix $\{{}^{1232\cdot\cdot}_{1}\}, \{{}^{1221\cdot0}_{1}\}$, ${}^{112111}_{1}$ (by orthogonality to β_1) and hence $\{{}^{123321}_{1}, {}^{1232\cdot\cdot}_{2}\}$, $\{{}^{1222\cdot1}_{1}\}$, ${}^{112210}_{1}$; we must then fix ${}^{123210}_{1}$ and $\{{}^{123321}_{1}, {}^{123210}_{2}\}$ (by orthogonality to ${}^{112111}_{1}$) and hence $\{{}^{1232\cdot1}_{1}\}, \{{}^{1232\cdot1}_{2}\}$; we must then fix ${}^{123321}_{1}$ (by orthogonality to both of $\{{}^{1221\cdot0}_{1}\}$) and hence ${}^{123210}_{2}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_6\rangle$ giving transitivity on $\{{}^{1221\cdot0}_{1}\}$, so we may fix $\beta_2 = {}^{122110}_{1}$. We must then fix ${}^{123211}_{2}$, ${}^{123211}_{1}$, ${}^{122211}_{1}$ (by orthogonality to β_2), by which point all roots are fixed; so $W_X = G$.

If $X = X_{17}^3$ we must fix ${}^{234321}_{2}$, ${}^{134321}_{2}$, ${}^{112210}_{1}$ ($o(\beta) = 0, 3, 11$ respectively), $\{{}^{1232\cdot1}_{2}\}, \{{}^{12\cdot2\cdot1}_{1}\}$ ($o(\beta) = 5, 7$ respectively, orthogonal to ${}^{112210}_{1}$), $\{{}^{12\cdot321}_{2}\}$ ($o(\beta) = 5$, not orthogonal to ${}^{112210}_{1}$), $\{{}^{11\cdot111}_{1}\}$ ($o(\beta) = 9$, orthogonal to ${}^{134321}_{2}$), $\{{}^{1221\cdot0}_{1}\}$ ($o(\beta) = 9$, not orthogonal to ${}^{134321}_{2}$), ${}^{123210}_{2}$ ($o(\beta) = 7$, orthogonal to all of $\{{}^{12\cdot2\cdot1}_{1}\}$), ${}^{123321}_{1}$ ($o(\beta) = 7$, orthogonal to none of $\{{}^{12\cdot2\cdot1}_{1}\}$). We set

$$G = \langle w_4, w_6\rangle;$$

then G acts as S_2 on each of $\{{}^{12\cdot321}_{2}\}$ and $\{{}^{1232\cdot1}_{2}\}$ independently, so we may fix all of these roots. We then have fixed all of the roots ${}^{\cdots\cdots}_{2}$, which span $\mathbb{R}\Phi$; so $W_X = G$.

If $X = X_{17}^4$ we must fix ${}^{234321}_{2}$, ${}^{134321}_{2}$, $\{{}^{12\cdot\cdot\cdot1}_{2}, {}^{123\cdot21}_{1}, {}^{122\cdot11}_{1}, {}^{12\cdot\cdot\cdot0}_{1}\}$, $\{{}^{112100}_{1}, {}^{111110}_{1}, {}^{111111}_{0}\}$ ($o(\beta) = 0, 3, 7, 11$ respectively). We set

$$G = \langle w_5, w_4w_6, w_2w_7\rangle;$$

then G is transitive on $\{{}^{12\cdot\cdot\cdot1}_{2}, {}^{123\cdot21}_{1}, {}^{122\cdot11}_{1}, {}^{12\cdot\cdot\cdot0}_{1}\}$, so we may fix $\beta_1 = {}^{124321}_{2}$. We must then fix $\{{}^{122\cdot1\cdot}_{1}, {}^{122100}_{1}\}, \{{}^{111110}_{1}, {}^{111111}_{0}\}$ (by orthogonality to β_1) and hence $\{{}^{123\cdot21}_{\cdot}, {}^{123211}_{2}, {}^{123210}_{1}\}$, ${}^{112100}_{1}$; we must then fix $\{{}^{123\cdot21}_{\cdot}\}, \{{}^{122\cdot1\cdot}_{1}\}$ (by orthogonality to ${}^{112100}_{1}$) and hence $\{{}^{123211}_{2}, {}^{123210}_{1}\}$, ${}^{122100}_{1}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_5, w_2w_7\rangle$ giving transitivity on $\{{}^{123\cdot21}_{\cdot}\}$, so we may fix $\beta_2 = {}^{123321}_{2}$. We must then fix ${}^{123221}_{1}$, ${}^{123210}_{1}$, $\{{}^{12211\cdot}_{1}\}$, ${}^{111111}_{0}$ (by orthogonality to β_2) and hence $\{{}^{123321}_{1}, {}^{123221}_{2}\}$, ${}^{123211}_{2}$, $\{{}^{12221\cdot}_{1}\}$, ${}^{111110}_{1}$; we must then fix ${}^{123221}_{2}$, ${}^{122210}_{1}$, ${}^{122110}_{1}$ (by orthogonality to ${}^{111111}_{0}$), by which point all roots are fixed; so $W_X = G$.

If $X = X_{17}^5$ we must fix ${}^{234321}_{2}$, ${}^{134321}_{2}$, $\{{}^{11\cdot111}_{\cdot}\}, \{{}^{1221\cdot0}_{1}\}$, ${}^{112210}_{1}$ ($o(\beta) = 0, 4, 8, 10, 12$ respectively), $\{{}^{12\cdot2\cdot1}_{\cdot}\}$ ($o(\beta) = 6$, orthogonal to ${}^{112210}_{1}$), $\{{}^{12\cdot321}_{\cdot}\}$ ($o(\beta) = 6$, not orthogonal to ${}^{112210}_{1}$). We set

$$G = \langle w_2, w_4, w_6\rangle;$$

then G acts as S_3 on $\{{}^{12\cdot321}_{\cdot}\}$ and independently as S_2 on $\{{}^{1221\cdot0}_{1}\}$, so we may fix all of these roots. We then have fixed all of the roots ${}^{\cdots321}_{\cdot}$, ${}^{1221\cdot0}_{1}$, which span $\mathbb{R}\Phi$; so $W_X = G$.

If $X = X_{17}^6$ we must fix ${}^{234321}_{2}$, $\{{}^{1\cdot\cdot321}_{2}\}, \{{}^{1232\cdot\cdot}_{2}\}, \{{}^{1\cdot\cdot2\cdot\cdot}_{1}\}$, ${}^{111000}_{1}$ ($o(\beta) = 0, 4, 6, 8, 12$ respectively). We set

$$G = \langle w_3, w_4, w_6, w_7\rangle;$$

then G acts as S_3 on each of $\{\genfrac{}{}{0pt}{}{1\cdot\cdot321}{2}\}$ and $\{\genfrac{}{}{0pt}{}{1232\cdot\cdot}{2}\}$ independently, so we may fix all of these roots. We then have fixed all of the roots $\genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot\cdot\cdot}{2}$, which span $\mathbb{R}\Phi$; so $W_X = G$.

If $X = X_{17}^7$ we must fix $\genfrac{}{}{0pt}{}{234321}{2}, \{\genfrac{}{}{0pt}{}{1\cdot\cdot321}{2}\}, \{\genfrac{}{}{0pt}{}{1232\cdot1}{1}, \genfrac{}{}{0pt}{}{122221}{1}, \genfrac{}{}{0pt}{}{122210}{1}, \genfrac{}{}{0pt}{}{11221\cdot}{1}\}, \{\genfrac{}{}{0pt}{}{122100}{1}, \genfrac{}{}{0pt}{}{112110}{1}, \genfrac{}{}{0pt}{}{111111}{1}\}$ ($o(\beta) = 0, 4, 8, 10$ respectively), $\genfrac{}{}{0pt}{}{123321}{1}$ ($o(\beta) = 6$, orthogonal to all of $\{\genfrac{}{}{0pt}{}{122100}{1}, \genfrac{}{}{0pt}{}{112110}{1}, \genfrac{}{}{0pt}{}{111111}{1}\}$), $\{\genfrac{}{}{0pt}{}{1232\cdot\cdot}{2}\}$ ($o(\beta) = 6$, orthogonal to one of $\{\genfrac{}{}{0pt}{}{122100}{1}, \genfrac{}{}{0pt}{}{112110}{1}, \genfrac{}{}{0pt}{}{111111}{1}\}$). We set

$$G = \langle w_3 w_6, w_4 w_7 \rangle;$$

then G is transitive on $\{\genfrac{}{}{0pt}{}{1232\cdot1}{1}, \genfrac{}{}{0pt}{}{122221}{1}, \genfrac{}{}{0pt}{}{122210}{1}, \genfrac{}{}{0pt}{}{11221\cdot}{1}\}$, so we may fix $\beta_1 = \genfrac{}{}{0pt}{}{123221}{1}$. We must then fix $\genfrac{}{}{0pt}{}{123321}{2}, \{\genfrac{}{}{0pt}{}{12321\cdot}{2}\}, \{\genfrac{}{}{0pt}{}{122210}{1}, \genfrac{}{}{0pt}{}{11221\cdot}{1}\}, \{\genfrac{}{}{0pt}{}{122100}{1}, \genfrac{}{}{0pt}{}{111111}{1}\}$ (by orthogonality to β_1) and hence $\{\genfrac{}{}{0pt}{}{1\cdot4321}{2}\}, \genfrac{}{}{0pt}{}{123221}{2}, \{\genfrac{}{}{0pt}{}{123211}{1}, \genfrac{}{}{0pt}{}{122221}{1}\}, \genfrac{}{}{0pt}{}{112110}{1}$; we must then fix $\genfrac{}{}{0pt}{}{123211}{1}, \genfrac{}{}{0pt}{}{122100}{1}$ (by orthogonality to $\genfrac{}{}{0pt}{}{123321}{2}$) and hence $\genfrac{}{}{0pt}{}{122221}{1}, \genfrac{}{}{0pt}{}{111111}{1}$; we must then fix $\genfrac{}{}{0pt}{}{134321}{2}, \genfrac{}{}{0pt}{}{123211}{2}, \{\genfrac{}{}{0pt}{}{122210}{1}, \genfrac{}{}{0pt}{}{112211}{1}\}$ (by orthogonality to $\genfrac{}{}{0pt}{}{112110}{1}$) and hence $\genfrac{}{}{0pt}{}{124321}{2}, \genfrac{}{}{0pt}{}{123210}{2}, \genfrac{}{}{0pt}{}{112210}{1}$; we must then fix $\genfrac{}{}{0pt}{}{112211}{1}$ (by orthogonality to $\genfrac{}{}{0pt}{}{134321}{2}$), by which point all roots are fixed; so $W_X = G$.

If $X = X_{17}^8$ we must fix $\genfrac{}{}{0pt}{}{234321}{2}, \genfrac{}{}{0pt}{}{134321}{2}, \{\genfrac{}{}{0pt}{}{111110}{1}, \genfrac{}{}{0pt}{}{111111}{0}\}, \genfrac{}{}{0pt}{}{112100}{1}$ ($o(\beta) = 0, 4, 10, 12$ respectively), $\genfrac{}{}{0pt}{}{124321}{2}, \{\genfrac{}{}{0pt}{}{123211}{2}, \genfrac{}{}{0pt}{}{123210}{1}\}$ ($o(\beta) = 6, 8$ respectively, orthogonal to both of $\{\genfrac{}{}{0pt}{}{111110}{1}, \genfrac{}{}{0pt}{}{111111}{0}\}$), $\{\genfrac{}{}{0pt}{}{123\cdot21}{\cdot}\}, \{\genfrac{}{}{0pt}{}{122\cdot1\cdot}{1}\}$ ($o(\beta) = 6, 8$ respectively, orthogonal to one of $\{\genfrac{}{}{0pt}{}{111110}{1}, \genfrac{}{}{0pt}{}{111111}{0}\}$), $\genfrac{}{}{0pt}{}{112221}{1}$ ($o(\beta) = 8$, orthogonal to neither of $\{\genfrac{}{}{0pt}{}{111110}{1}, \genfrac{}{}{0pt}{}{111111}{0}\}$). We set

$$G = \langle w_5, w_2 w_7 \rangle;$$

then G is transitive on $\{\genfrac{}{}{0pt}{}{122\cdot1\cdot}{1}\}$, so we may fix $\beta_1 = \genfrac{}{}{0pt}{}{122211}{1}$. We must then fix $\{\genfrac{}{}{0pt}{}{123221}{\cdot}\}, \genfrac{}{}{0pt}{}{123210}{1}, \genfrac{}{}{0pt}{}{122110}{1}, \genfrac{}{}{0pt}{}{111110}{1}$ (by orthogonality to β_1) and hence $\{\genfrac{}{}{0pt}{}{123321}{\cdot}\}, \genfrac{}{}{0pt}{}{123211}{2}, \{\genfrac{}{}{0pt}{}{122210}{1}, \genfrac{}{}{0pt}{}{122111}{1}\}, \genfrac{}{}{0pt}{}{111111}{0}$; we must then fix $\genfrac{}{}{0pt}{}{123321}{1}, \genfrac{}{}{0pt}{}{123221}{1}, \genfrac{}{}{0pt}{}{122111}{1}$ (by orthogonality to $\genfrac{}{}{0pt}{}{111110}{1}$), by which point all roots are fixed; so $W_X = G$.

If $X = X_{17}^9$ we must fix $\genfrac{}{}{0pt}{}{234321}{2}, \genfrac{}{}{0pt}{}{134321}{2}, \{\genfrac{}{}{0pt}{}{123211}{2}, \genfrac{}{}{0pt}{}{12\cdot210}{1}, \genfrac{}{}{0pt}{}{122111}{1}\}$ ($o(\beta) = 0, 4, 8$ respectively), $\{\genfrac{}{}{0pt}{}{112211}{1}, \genfrac{}{}{0pt}{}{11\cdot110}{1}, \genfrac{}{}{0pt}{}{111111}{0}\}$ ($o(\beta) = 10$, orthogonal to three of $\{\genfrac{}{}{0pt}{}{123211}{2}, \genfrac{}{}{0pt}{}{12\cdot210}{1}, \genfrac{}{}{0pt}{}{122111}{1}\}$), $\genfrac{}{}{0pt}{}{122100}{1}$ ($o(\beta) = 10$, orthogonal to none of $\{\genfrac{}{}{0pt}{}{123211}{2}, \genfrac{}{}{0pt}{}{12\cdot210}{1}, \genfrac{}{}{0pt}{}{122111}{1}\}$), $\{\genfrac{}{}{0pt}{}{12\cdot321}{2}, \genfrac{}{}{0pt}{}{12\cdot221}{1}\}$ ($o(\beta) = 6$, orthogonal to two roots in $\{\genfrac{}{}{0pt}{}{112211}{1}, \genfrac{}{}{0pt}{}{11\cdot110}{1}, \genfrac{}{}{0pt}{}{111111}{0}\}$ which are orthogonal to each other), $\{\genfrac{}{}{0pt}{}{123321}{1}, \genfrac{}{}{0pt}{}{123221}{2}\}$ ($o(\beta) = 6$, orthogonal to two roots in $\{\genfrac{}{}{0pt}{}{112211}{1}, \genfrac{}{}{0pt}{}{11\cdot110}{1}, \genfrac{}{}{0pt}{}{111111}{0}\}$ which are not orthogonal to each other). We set

$$G = \langle w_4, w_2 w_5 w_7 \rangle;$$

then G is transitive on $\{\genfrac{}{}{0pt}{}{12\cdot321}{2}, \genfrac{}{}{0pt}{}{12\cdot221}{1}\}$, so we may fix $\beta_1 = \genfrac{}{}{0pt}{}{124321}{2}$. We must then fix $\genfrac{}{}{0pt}{}{122221}{1}, \{\genfrac{}{}{0pt}{}{122210}{1}, \genfrac{}{}{0pt}{}{122111}{1}\}, \{\genfrac{}{}{0pt}{}{111110}{1}, \genfrac{}{}{0pt}{}{111111}{0}\}$ (by orthogonality to β_1) and hence $\{\genfrac{}{}{0pt}{}{123321}{2}, \genfrac{}{}{0pt}{}{123221}{1}\}, \{\genfrac{}{}{0pt}{}{123211}{2}, \genfrac{}{}{0pt}{}{123210}{1}\}$ and $\{\genfrac{}{}{0pt}{}{112211}{1}, \genfrac{}{}{0pt}{}{112110}{1}\}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_2 w_5 w_7 \rangle$ giving transitivity on $\{\genfrac{}{}{0pt}{}{123321}{1}, \genfrac{}{}{0pt}{}{123221}{2}\}$, so we may fix $\beta_2 = \genfrac{}{}{0pt}{}{123321}{1}$. We must then fix $\genfrac{}{}{0pt}{}{123211}{2}, \genfrac{}{}{0pt}{}{122111}{1}, \genfrac{}{}{0pt}{}{112110}{1}, \genfrac{}{}{0pt}{}{111110}{1}$ (by orthogonality to β_2) and hence $\genfrac{}{}{0pt}{}{123210}{1}, \genfrac{}{}{0pt}{}{122210}{1}, \genfrac{}{}{0pt}{}{112211}{1}, \genfrac{}{}{0pt}{}{111111}{0}$; we must then fix $\genfrac{}{}{0pt}{}{123321}{2}$ (by orthogonality to $\genfrac{}{}{0pt}{}{111111}{0}$), by which point all roots are fixed; so $W_X = G$.

If $X = X_{17}^{10}$ we must fix $\genfrac{}{}{0pt}{}{234321}{2}, \{\genfrac{}{}{0pt}{}{1\cdot\cdot321}{2}\}, \genfrac{}{}{0pt}{}{111000}{1}$ ($o(\beta) = 0, 5, 13$ respectively), $\{\genfrac{}{}{0pt}{}{1\cdot\cdot2\cdot1}{1}\}, \{\genfrac{}{}{0pt}{}{1\cdot\cdot210}{1}\}$ ($o(\beta) = 7, 9$ respectively, orthogonal to one of $\{\genfrac{}{}{0pt}{}{1\cdot\cdot321}{2}\}$), $\{\genfrac{}{}{0pt}{}{1232\cdot1}{2}\}$ ($o(\beta) = 7$, orthogonal to none of $\{\genfrac{}{}{0pt}{}{1\cdot\cdot321}{2}\}$), $\genfrac{}{}{0pt}{}{111111}{0}$ ($o(\beta) = 9$, orthogonal to all of $\{\genfrac{}{}{0pt}{}{1\cdot\cdot321}{2}\}$). We set

$$G = \langle w_3, w_4, w_6 \rangle;$$

then G acts as S_3 on $\{{}^{1\cdot\cdot321}_{2}\}$ and independently as S_2 on $\{{}^{1232\cdot1}_{2}\}$, so we may fix all of these roots. We then have fixed all of the roots ${}^{\cdot\cdot\cdot\cdot\cdot1}_{2}$, ${}^{111000}_{1}$, which span $\mathbb{R}\Phi$; so $W_X = G$.

If $X = X_{17}^{11}$ we must fix ${}^{234321}_{2}$, $\{{}^{122110}_{1}, {}^{112100}_{1}\}$ ($o(\beta) = 0, 11$ respectively), $\{{}^{123321}_{\cdot}\}, \{{}^{122211}_{1}, {}^{112221}_{1}\}, \{{}^{111111}_{\cdot}\}$ ($o(\beta) = 5, 7, 9$ respectively, orthogonal to both of $\{{}^{122110}_{1}, {}^{112100}_{1}\}$), $\{{}^{1\cdot4321}_{2}\}, \{{}^{1232\cdot1}_{\cdot}\}, \{{}^{1\cdot2210}_{1}\}$ ($o(\beta) = 5, 7, 9$ respectively, orthogonal to one of $\{{}^{122110}_{1}, {}^{112100}_{1}\}$). We set

$$G = \langle w_2, w_3 w_6 \rangle;$$

then G is transitive on $\{{}^{1232\cdot1}_{\cdot}\}$, so we may fix $\beta_1 = {}^{123221}_{2}$. We must then fix ${}^{123321}_{1}$, ${}^{123211}_{1}$, ${}^{122211}_{1}$, ${}^{111111}_{0}$, ${}^{112100}_{1}$ (by orthogonality to β_1) and hence ${}^{123321}_{2}$, $\{{}^{123221}_{1}, {}^{123211}_{2}\}$, ${}^{112221}_{1}$, ${}^{111111}_{1}$, ${}^{122110}_{1}$; we must then fix ${}^{134321}_{2}$, ${}^{123211}_{2}$, ${}^{122210}_{1}$ (by orthogonality to ${}^{112221}_{1}$), by which point all roots are fixed; so $W_X = G$.

If $X = X_{17}^{12}$ we must fix ${}^{234321}_{2}$, $\{{}^{1\cdot4321}_{2}\}, \{{}^{1232\cdot\cdot}_{\cdot}\}, \{{}^{1\cdot22\cdot\cdot}_{1}\}, \{{}^{111000}_{\cdot}\}$ ($o(\beta) = 0, 5, 7, 9, 11$ respectively). We set

$$G = \langle w_2, w_3, w_6, w_7 \rangle;$$

then G is transitive on $\{{}^{1232\cdot\cdot}_{\cdot}\}$, so we may fix $\beta_1 = {}^{123221}_{2}$. We must then fix $\{{}^{12321\cdot}_{1}\}, \{{}^{1\cdot221\cdot}_{1}\}$, ${}^{111000}_{0}$ (by orthogonality to β_1) and hence $\{{}^{12321\cdot}_{2}, {}^{123221}_{1}\}$, $\{{}^{1\cdot2221}_{1}\}$, ${}^{111000}_{1}$; we must then fix $\{{}^{12321\cdot}_{2}\}$ (by orthogonality to ${}^{111000}_{0}$) and hence ${}^{123221}_{1}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_3, w_7 \rangle$, which acts as S_2 on each of $\{{}^{1\cdot4321}_{2}\}$ and $\{{}^{12321\cdot}_{2}\}$ independently, so we may fix all of these roots. We then have fixed all of the roots ${}^{\cdot\cdot4321}_{2}$, ${}^{1232\cdot\cdot}_{2}$, ${}^{111000}_{0}$, which span $\mathbb{R}\Phi$; so $W_X = G$.

If $X = X_{17}^{13}$ we must fix ${}^{234321}_{2}$, ${}^{134321}_{2}$, $\{{}^{123\cdot\cdot\cdot}_{\cdot}\}$, ${}^{110000}_{0}$ ($o(\beta) = 0, 5, 7, 13$ respectively), $\{{}^{1122\cdot1}_{1}, {}^{1121\cdot0}_{1}\}$ ($o(\beta) = 9$, orthogonal to ${}^{110000}_{0}$), $\{{}^{122210}_{1}, {}^{122111}_{1}\}$ ($o(\beta) = 9$, not orthogonal to ${}^{110000}_{0}$). We set

$$G = \langle w_2, w_6, w_5 w_7 \rangle;$$

then G is transitive on $\{{}^{123\cdot\cdot\cdot}_{\cdot}\}$, so we may fix $\beta_1 = {}^{123321}_{2}$. We must then fix $\{{}^{1232\cdot\cdot}_{1}\}$, ${}^{122111}_{1}$, $\{{}^{1121\cdot0}_{1}\}$ (by orthogonality to β_1) and hence $\{{}^{123321}_{1}, {}^{1232\cdot\cdot}_{2}\}$, ${}^{122210}_{1}$, $\{{}^{1122\cdot1}_{1}\}$; we must then fix $\{{}^{123321}_{1}, {}^{123210}_{2}\}$, ${}^{123210}_{1}$ (by orthogonality to ${}^{122111}_{1}$) and hence $\{{}^{1232\cdot1}_{2}\}, \{{}^{1232\cdot1}_{1}\}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_6 \rangle$ giving transitivity on $\{{}^{1232\cdot1}_{2}\}$, so we may fix $\beta_2 = {}^{123221}_{2}$. We must then fix ${}^{123321}_{1}$, ${}^{123211}_{1}$, ${}^{112211}_{1}$, ${}^{112100}_{1}$ (by orthogonality to β_2), by which point all roots are fixed; so $W_X = G$.

If $X = X_{17}^{14}$ we must fix ${}^{234321}_{2}$, $\{{}^{1\cdot\cdot\cdot\cdot\cdot}_{2}\}, \{{}^{12321\cdot}_{1}, {}^{122221}_{1}, {}^{122111}_{1}, {}^{122210}_{1}, {}^{1122\cdot1}_{1}, {}^{112110}_{1}, {}^{111\cdot00}_{1}\}$ ($o(\beta) = 0, 5, 9$ respectively). We set

$$G = \langle w_3 w_5, w_4 w_6, w_5 w_7 \rangle;$$

then G is transitive on $\{{}^{12321\cdot}_{1}, {}^{122221}_{1}, {}^{122111}_{1}, {}^{122210}_{1}, {}^{1122\cdot1}_{1}, {}^{112110}_{1}, {}^{111\cdot00}_{1}\}$, so we may fix $\beta_1 = {}^{111000}_{1}$. We must then fix $\{{}^{1\cdot\cdot321}_{2}\}, \{{}^{12321\cdot}_{1}, {}^{122221}_{1}, {}^{122210}_{1}, {}^{1122\cdot1}_{1}\}$ (by orthogonality to β_1) and hence $\{{}^{1232\cdot\cdot}_{2}\}, \{{}^{122111}_{1}, {}^{112110}_{1}, {}^{111100}_{1}\}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_3 w_7, w_4 w_6 \rangle$ giving transitivity on $\{{}^{12321\cdot}_{1}, {}^{122221}_{1}, {}^{122210}_{1}, {}^{1122\cdot1}_{1}\}$, so we may fix $\beta_2 = {}^{123210}_{1}$. We must then fix ${}^{123321}_{2}$, $\{{}^{1232\cdot1}_{2}\}, \{{}^{122221}_{1}, {}^{1122\cdot1}_{1}\}, \{{}^{122111}_{1}, {}^{111100}_{1}\}$ (by orthogonality to β_2) and hence $\{{}^{1\cdot4321}_{2}\}$, ${}^{123210}_{2}$, $\{{}^{123211}_{1}, {}^{122210}_{1}\}$, ${}^{112110}_{1}$; we must then fix ${}^{134321}_{2}$, ${}^{123211}_{2}$ (by orthogonality to ${}^{112110}_{1}$) and hence ${}^{124321}_{2}$, ${}^{123221}_{2}$. We then have fixed all of the roots ${}^{\cdot\cdot\cdot\cdot\cdot\cdot}_{2}$, which span $\mathbb{R}\Phi$; so $W_X = G$.

If $X = X_{17}^{15}$ we must fix $\substack{234321\\2}$, $\{\substack{1\cdot4321\\2}\}$, $\{\substack{111000\\1}, \substack{111100\\0}\}$ ($o(\beta) = 0, 5, 11$ respectively), $\{\substack{123321\\2}, \substack{123221\\1}\}$, $\{\substack{1\cdot2221\\1}\}$ ($o(\beta) = 7, 9$ respectively, orthogonal to both of $\{\substack{111000\\1}, \substack{111100\\0}\}$), $\{\substack{12321\cdot\\\cdot}\}$, $\{\substack{122211\\1}, \substack{122110\\1}, \substack{112210\\1}, \substack{112111\\1}\}$ ($o(\beta) = 7, 9$ respectively, orthogonal to one of $\{\substack{111000\\1}, \substack{111100\\0}\}$). We set

$$G = \langle w_3w_7, w_2w_5w_7\rangle;$$

then G is transitive on $\{\substack{12321\cdot\\\cdot}\}$, so we may fix $\beta_1 = \substack{123211\\2}$. We must then fix $\substack{123221\\1}$, $\substack{123210\\1}$, $\{\substack{122110\\1}, \substack{112210\\1}\}$, $\substack{111100\\0}$ (by orthogonality to β_1) and hence $\substack{123321\\2}$, $\{\substack{123210\\2}, \substack{123211\\1}\}$, $\{\substack{122211\\1}, \substack{112111\\1}\}$, $\substack{111000\\1}$; we must then fix $\substack{123210\\2}$, $\substack{112111\\1}$, $\substack{122110\\1}$ (by orthogonality to $\substack{111100\\0}$) and hence $\substack{123211\\1}$, $\substack{122211\\1}$, $\substack{112210\\1}$; we must then fix $\substack{124321\\2}$, $\substack{112221\\1}$ (by orthogonality to $\substack{122211\\1}$), by which point all roots are fixed; so $W_X = G$.

If $X = X_{17}^{16}$ we must fix $\substack{234321\\2}$, $\{\substack{12\cdot\cdot21\\\cdot}\}$, $\substack{100000\\0}$ ($o(\beta) = 0, 6, 14$ respectively), $\{\substack{12\cdot\cdot1\cdot\\\cdot}\}$ ($o(\beta) = 8$, orthogonal to $\substack{100000\\0}$), $\substack{112221\\1}$ ($o(\beta) = 8$, not orthogonal to $\substack{100000\\0}$). We set

$$G = \langle w_2, w_4, w_5, w_7\rangle;$$

then G is transitive on $\{\substack{12\cdot\cdot1\cdot\\\cdot}\}$, so we may fix $\beta_1 = \substack{123211\\2}$. We must then fix $\{\substack{12\cdot\cdot21\\1}\}$, $\{\substack{12\cdot\cdot10\\1}\}$ (by orthogonality to β_1) and hence $\{\substack{12\cdot\cdot21\\2}\}$, $\{\substack{123210\\2}, \substack{12\cdot\cdot11\\1}\}$; we must then fix $\substack{123210\\2}$ (by orthogonality to all of $\{\substack{12\cdot\cdot21\\1}\}$) and hence $\{\substack{12\cdot\cdot11\\1}\}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_4, w_5\rangle$, which acts as S_3 on $\{\substack{12\cdot\cdot21\\2}\}$, so we may fix all of these roots. We then have fixed all of the roots $\substack{234321\\2}$, $\substack{12\cdot\cdot\cdot\cdot\\2}$, $\substack{100000\\0}$, which span $\mathbb{R}\Phi$; so $W_X = G$.

If $X = X_{17}^{17}$ we must fix $\substack{234321\\2}$, $\substack{111111\\0}$ ($o(\beta) = 0, 12$ respectively), $\{\substack{1\cdot2100\\1}\}$ ($o(\beta) = 10$, orthogonal to $\substack{111111\\0}$), $\{\substack{123\cdot21\\1}\}$, $\substack{111111\\1}$ ($o(\beta) = 8, 10$ respectively, not orthogonal to $\substack{111111\\0}$), $\{\substack{1\cdot4321\\2}\}$ ($o(\beta) = 6$, orthogonal to $\substack{111111\\1}$), $\{\substack{123\cdot21\\2}\}$ ($o(\beta) = 6$, not orthogonal to $\substack{111111\\1}$), $\substack{123211\\2}$ ($o(\beta) = 8$, orthogonal to neither of $\{\substack{1\cdot2100\\1}\}$), $\substack{111110\\1}$ ($o(\beta) = 8$, orthogonal to both of $\{\substack{1\cdot4321\\2}\}$), $\{\substack{1\cdot2\cdot10\\1}\}$ ($o(\beta) = 8$, orthogonal to one of $\{\substack{123\cdot21\\1}\}$). We set

$$G = \langle w_3, w_5\rangle;$$

then G acts as S_2 on each of $\{\substack{1\cdot4321\\2}\}$ and $\{\substack{123\cdot21\\2}\}$ independently, so we may fix all of these roots. We then have fixed all of the roots $\substack{\cdot\cdot\cdot\cdot\cdot1\\2}$, $\substack{111111\\0}$, which span $\mathbb{R}\Phi$; so $W_X = G$.

If $X = X_{17}^{18}$ we must fix $\substack{234321\\2}$, $\{\substack{1\cdot\cdot\cdot\cdot1\\2}\}$, $\{\substack{123\cdot21\\1}, \substack{122211\\1}, \substack{11\cdot111\\1}\}$ ($o(\beta) = 0, 6, 8$ respectively), $\{\substack{112210\\1}, \substack{122110\\1}, \substack{1\cdot2100\\1}, \substack{111110\\1}\}$ ($o(\beta) = 10$, orthogonal to three of $\{\substack{1\cdot\cdot\cdot\cdot1\\2}\}$), $\substack{111111\\0}$ ($o(\beta) = 10$, orthogonal to all of $\{\substack{1\cdot\cdot\cdot\cdot1\\2}\}$). We set

$$G = \langle w_3w_5, w_4w_6\rangle;$$

then G is transitive on $\{\substack{123\cdot21\\1}, \substack{122211\\1}, \substack{11\cdot111\\1}\}$, so we may fix $\beta_1 = \substack{111111\\1}$. We must then fix $\{\substack{1\cdot4321\\2}\}$, $\{\substack{123\cdot21\\1}\}$, $\{\substack{112210\\1}, \substack{122110\\1}, \substack{1\cdot2100\\1}\}$ (by orthogonality to β_1) and hence $\{\substack{123\cdot\cdot1\\2}\}$, $\{\substack{122211\\1}, \substack{112111\\1}\}$, $\substack{111110\\1}$; we must then fix $\substack{123211\\2}$, $\{\substack{1\cdot2100\\1}\}$ (by orthogonality to $\substack{111110\\1}$) and hence $\{\substack{123\cdot21\\2}\}$, $\{\substack{112210\\1}, \substack{122110\\1}\}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_3w_5\rangle$ giving transitivity on $\{\substack{112210\\1}, \substack{122110\\1}\}$, so we may fix $\beta_2 = \substack{112210\\1}$. We must then fix $\substack{134321\\2}$, $\substack{123221\\2}$, $\substack{123221\\1}$, $\substack{122100\\1}$ (by orthogonality to β_2) and hence $\substack{124321\\2}$, $\substack{123321\\2}$, $\substack{123321\\1}$, $\substack{112100\\1}$; we must then fix $\substack{112111\\1}$ (by orthogonality to $\substack{134321\\2}$), by which point all roots are fixed; so $W_X = G$.

If $X = X_{17}^{19}$ we must fix $\genfrac{}{}{0pt}{}{234321}{2}, \genfrac{}{}{0pt}{}{134321}{2}, \{\genfrac{}{}{0pt}{}{12\cdot2\cdot\cdot}{\cdot}\}, \{\genfrac{}{}{0pt}{}{1122\cdot\cdot}{1}, \genfrac{}{}{0pt}{}{11\cdot000}{\cdot}\}$ ($o(\beta) = 0, 6, 8, 10$ respectively). We set

$$G = \langle w_2, w_4, w_2w_5{w_7}^{w_4w_6w_5} \rangle;$$

then G is transitive on $\{\genfrac{}{}{0pt}{}{12\cdot2\cdot\cdot}{\cdot}\}$, so we may fix $\beta_1 = \genfrac{}{}{0pt}{}{123221}{2}$. We must then fix $\{\genfrac{}{}{0pt}{}{12\cdot21\cdot}{1}\}, \{\genfrac{}{}{0pt}{}{11221\cdot}{1}, \genfrac{}{}{0pt}{}{11\cdot000}{0}\}$ (by orthogonality to β_1) and hence $\{\genfrac{}{}{0pt}{}{12321\cdot}{2}, \genfrac{}{}{0pt}{}{12\cdot221}{1}\}$, $\{\genfrac{}{}{0pt}{}{112221}{1}, \genfrac{}{}{0pt}{}{111000}{1}\}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_4, w_2w_5{w_7}^{w_4w_6w_5} \rangle$ giving transitivity on $\{\genfrac{}{}{0pt}{}{12321\cdot}{2}, \genfrac{}{}{0pt}{}{12\cdot221}{1}\}$, so we may fix $\beta_2 = \genfrac{}{}{0pt}{}{123211}{2}$. We must then fix $\{\genfrac{}{}{0pt}{}{12\cdot221}{1}\}, \{\genfrac{}{}{0pt}{}{12\cdot210}{1}\}$, $\genfrac{}{}{0pt}{}{112221}{1}, \{\genfrac{}{}{0pt}{}{112210}{1}, \genfrac{}{}{0pt}{}{11\cdot000}{0}\}$ (by orthogonality to β_2) and hence $\genfrac{}{}{0pt}{}{123210}{2}, \{\genfrac{}{}{0pt}{}{12\cdot211}{1}\}, \genfrac{}{}{0pt}{}{111000}{1}$, $\genfrac{}{}{0pt}{}{112211}{1}$; we must then fix $\{\genfrac{}{}{0pt}{}{11\cdot000}{0}\}$ (by orthogonality to $\genfrac{}{}{0pt}{}{112211}{1}$) and hence $\genfrac{}{}{0pt}{}{112210}{1}$. Inside $\mathrm{stab}_G(\beta_1, \beta_2)$ we then have $\langle w_4 \rangle$ giving transitivity on $\{\genfrac{}{}{0pt}{}{11\cdot000}{0}\}$, so we may fix $\beta_3 = \genfrac{}{}{0pt}{}{110000}{0}$. We must then fix $\genfrac{}{}{0pt}{}{123221}{1}, \genfrac{}{}{0pt}{}{123211}{1}, \genfrac{}{}{0pt}{}{123210}{1}$ (by orthogonality to β_3), by which point all roots are fixed; so $W_X = G$.

If $X = X_{17}^{20}$ we must fix $\genfrac{}{}{0pt}{}{234321}{2}, \genfrac{}{}{0pt}{}{134321}{2}, \{\genfrac{}{}{0pt}{}{123321}{1}, \genfrac{}{}{0pt}{}{1232\cdot\cdot}{2}, \genfrac{}{}{0pt}{}{122221}{1}, \genfrac{}{}{0pt}{}{123211}{1}, \genfrac{}{}{0pt}{}{12\cdot210}{1}, \genfrac{}{}{0pt}{}{122111}{1}\}, \{\genfrac{}{}{0pt}{}{1122\cdot1}{1}, \genfrac{}{}{0pt}{}{112110}{1}, \genfrac{}{}{0pt}{}{111100}{1}, \genfrac{}{}{0pt}{}{11\cdot000}{0}\}$ ($o(\beta) = 0, 6, 8, 10$ respectively). We set

$$G = \langle w_4w_6, w_2w_5w_7 \rangle;$$

then G is transitive on $\{\genfrac{}{}{0pt}{}{1122\cdot1}{1}, \genfrac{}{}{0pt}{}{112110}{1}, \genfrac{}{}{0pt}{}{111100}{1}, \genfrac{}{}{0pt}{}{11\cdot000}{0}\}$, so we may fix $\beta_1 = \genfrac{}{}{0pt}{}{110000}{0}$. We must then fix $\{\genfrac{}{}{0pt}{}{123321}{1}, \genfrac{}{}{0pt}{}{1232\cdot\cdot}{2}, \genfrac{}{}{0pt}{}{12321\cdot}{1}\}, \{\genfrac{}{}{0pt}{}{1122\cdot1}{1}, \genfrac{}{}{0pt}{}{112110}{1}\}$ (by orthogonality to β_1) and hence $\{\genfrac{}{}{0pt}{}{122221}{1}, \genfrac{}{}{0pt}{}{122210}{1}, \genfrac{}{}{0pt}{}{122111}{1}\}$ and $\{\genfrac{}{}{0pt}{}{111100}{1}, \genfrac{}{}{0pt}{}{111000}{0}\}$; we must then fix $\{\genfrac{}{}{0pt}{}{123321}{1}, \genfrac{}{}{0pt}{}{123221}{2}\}$, $\genfrac{}{}{0pt}{}{122221}{1}$ and $\genfrac{}{}{0pt}{}{112221}{1}$ (by orthogonality to both of $\{\genfrac{}{}{0pt}{}{111100}{1}, \genfrac{}{}{0pt}{}{111000}{0}\}$) and hence $\{\genfrac{}{}{0pt}{}{12321\cdot}{\cdot}\}, \{\genfrac{}{}{0pt}{}{122210}{1}, \genfrac{}{}{0pt}{}{122111}{1}\}, \{\genfrac{}{}{0pt}{}{112211}{1}, \genfrac{}{}{0pt}{}{112110}{1}\}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_2w_5w_7 \rangle$ giving transitivity on $\{\genfrac{}{}{0pt}{}{111100}{1}, \genfrac{}{}{0pt}{}{111000}{0}\}$, so we may fix $\beta_2 = \genfrac{}{}{0pt}{}{111000}{0}$. We must then fix $\{\genfrac{}{}{0pt}{}{12321\cdot}{2}\}, \genfrac{}{}{0pt}{}{122210}{1}, \genfrac{}{}{0pt}{}{112211}{1}$ (by orthogonality to β_2) and hence $\{\genfrac{}{}{0pt}{}{12321\cdot}{1}\}, \genfrac{}{}{0pt}{}{122111}{1}, \genfrac{}{}{0pt}{}{112110}{1}$; we must then fix $\genfrac{}{}{0pt}{}{123321}{1}, \genfrac{}{}{0pt}{}{123211}{2}, \genfrac{}{}{0pt}{}{123211}{1}$ (by orthogonality to $\genfrac{}{}{0pt}{}{112110}{1}$), by which point all roots are fixed; so $W_X = G$. (In this case we note also that the set of roots with $o(\beta) = 8$ is in fact a union of the two W_X-orbits $\{\genfrac{}{}{0pt}{}{123321}{1}, \genfrac{}{}{0pt}{}{1232\cdot1}{2}, \genfrac{}{}{0pt}{}{12\cdot210}{1}, \genfrac{}{}{0pt}{}{122111}{1}\}$ and $\{\genfrac{}{}{0pt}{}{122221}{1}, \genfrac{}{}{0pt}{}{123211}{1}, \genfrac{}{}{0pt}{}{123210}{2}\}$.)

If $X = X_{17}^{21}$ we must fix $\genfrac{}{}{0pt}{}{234321}{2}, \{\genfrac{}{}{0pt}{}{1\cdot4321}{2}, \genfrac{}{}{0pt}{}{123\cdot21}{\cdot}, \genfrac{}{}{0pt}{}{1\cdot2\cdot11}{1}, \genfrac{}{}{0pt}{}{111111}{\cdot}\}, \{\genfrac{}{}{0pt}{}{1\cdot2100}{1}, \genfrac{}{}{0pt}{}{111110}{\cdot}\}$ ($o(\beta) = 0, 7, 11$ respectively). We set

$$G = \langle w_2, w_5, w_2w_3{w_6}^{w_4} \rangle;$$

then G is transitive on $\{\genfrac{}{}{0pt}{}{1\cdot4321}{2}, \genfrac{}{}{0pt}{}{123\cdot21}{\cdot}, \genfrac{}{}{0pt}{}{1\cdot2\cdot11}{1}, \genfrac{}{}{0pt}{}{111111}{\cdot}\}$, so we may fix $\beta_1 = \genfrac{}{}{0pt}{}{134321}{2}$. We must then fix $\{\genfrac{}{}{0pt}{}{11\cdot\cdot11}{\cdot}\}, \{\genfrac{}{}{0pt}{}{112100}{1}, \genfrac{}{}{0pt}{}{111110}{\cdot}\}$ (by orthogonality to β_1) and hence $\{\genfrac{}{}{0pt}{}{124321}{2}, \genfrac{}{}{0pt}{}{123\cdot21}{\cdot}, \genfrac{}{}{0pt}{}{122\cdot11}{1}\}, \genfrac{}{}{0pt}{}{122100}{1}$; we must then fix $\{\genfrac{}{}{0pt}{}{124321}{2}, \genfrac{}{}{0pt}{}{123\cdot21}{\cdot}\}, \{\genfrac{}{}{0pt}{}{111110}{\cdot}\}$ (by orthogonality to $\genfrac{}{}{0pt}{}{122100}{1}$) and hence $\{\genfrac{}{}{0pt}{}{122\cdot11}{1}\}, \genfrac{}{}{0pt}{}{112100}{1}$; we must then fix $\{\genfrac{}{}{0pt}{}{123\cdot21}{\cdot}\}$ and $\{\genfrac{}{}{0pt}{}{111111}{\cdot}\}$ (by orthogonality to $\genfrac{}{}{0pt}{}{112100}{1}$) and hence $\genfrac{}{}{0pt}{}{124321}{2}, \{\genfrac{}{}{0pt}{}{112\cdot11}{1}\}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_2, w_5 \rangle$, which acts as S_2 on each of $\{\genfrac{}{}{0pt}{}{112\cdot11}{1}\}$ and $\{\genfrac{}{}{0pt}{}{111111}{\cdot}\}$ independently, so we may fix all of these roots. We then have fixed all of the roots $\genfrac{}{}{0pt}{}{\cdot\cdot4321}{2}, \genfrac{}{}{0pt}{}{11\cdot\cdot11}{\cdot}$, which span $\mathbb{R}\Phi$; so $W_X = G$.

If $X = X_{17}^{22}$ we must fix $\genfrac{}{}{0pt}{}{234321}{2}, \{\genfrac{}{}{0pt}{}{12\cdot\cdot\cdot\cdot}{\cdot}\}, \genfrac{}{}{0pt}{}{100000}{0}$ ($o(\beta) = 0, 7, 15$ respectively). We set

$$G = \langle w_2, w_4, w_5, w_6, w_7 \rangle;$$

since G is the stabilizer in $\mathrm{stab}_W(\genfrac{}{}{0pt}{}{234321}{2})$ of $\genfrac{}{}{0pt}{}{100000}{0}$ we must have $W_X = G$.

If $X = X_{17}^{23}$ we must fix $\substack{234321\\2}$, $\{\substack{1\cdot4321\\2}, \substack{123321\\1}, \substack{123221\\2}, \substack{1\cdot2221\\1}\}$, $\{\substack{123211\\\cdot}, \substack{1\cdot2\cdot10\\1}, \substack{111111\\\cdot}\}$, $\{\substack{111100\\1}, \substack{111000\\0}\}$ ($o(\beta) = 0, 7, 9, 11$ respectively). We set

$$G = \langle w_3, w_2w_5, w_2w_5{w_7}^{w_4}\rangle;$$

then G is transitive on $\{\substack{123211\\\cdot}, \substack{1\cdot2\cdot10\\1}, \substack{111111\\\cdot}\}$, so we may fix $\beta_1 = \substack{123211\\2}$. We must then fix $\{\substack{123321\\1}, \substack{1\cdot2221\\1}\}$, $\{\substack{1\cdot2\cdot10\\1}, \substack{111111\\0}\}$, $\substack{111000\\0}$ (by orthogonality to β_1) and hence $\{\substack{1\cdot4321\\2}, \substack{123221\\2}\}$, $\{\substack{123211\\1}, \substack{111111\\1}\}$, $\substack{111100\\1}$; we must then fix $\{\substack{1\cdot2210\\1}\}$, $\substack{111111\\1}$ (by orthogonality to $\substack{111000\\0}$) and hence $\{\substack{1\cdot2110\\1}, \substack{111111\\0}\}$, $\substack{123211\\1}$; we must then fix $\{\substack{1\cdot2221\\1}\}$, $\substack{123221\\2}$, $\{\substack{1\cdot2110\\1}\}$ (by orthogonality to $\substack{123211\\1}$) and hence $\substack{123321\\1}$, $\{\substack{1\cdot4321\\2}\}$, $\substack{111111\\0}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_3\rangle$ giving transitivity on $\{\substack{1\cdot2210\\1}\}$, so we may fix $\beta_2 = \substack{122210\\1}$. We must then fix $\substack{124321\\2}$, $\substack{112221\\1}$, $\substack{112110\\1}$ (by orthogonality to β_2), by which point all roots are fixed; so $W_X = G$. (In this case we note also that the set of roots with $o(\beta) = 7$ is in fact a union of the two W_X-orbits $\{\substack{1\cdot4321\\2}, \substack{1\cdot2221\\1}\}$ and $\{\substack{123321\\1}, \substack{123221\\2}\}$.)

If $X = X_{17}^{24}$ we must fix $\substack{234321\\2}$, $\{\substack{1\cdot2100\\1}\}$ ($o(\beta) = 0, 11$ respectively), $\{\substack{123\cdot21\\\cdot}\}$, $\{\substack{11111\cdot\\\cdot}\}$ ($o(\beta) = 7, 9$ respectively, orthogonal to both of $\{\substack{1\cdot2100\\1}\}$), $\{\substack{1\cdot4321\\2}\}$, $\{\substack{122210\\1}, \substack{122111\\1}, \substack{112211\\1}, \substack{112110\\1}\}$ ($o(\beta) = 7, 9$ respectively, orthogonal to one of $\{\substack{1\cdot2100\\1}\}$). We set

$$G = \langle w_2, w_3w_7, w_5w_7\rangle;$$

then G is transitive on $\{\substack{122210\\1}, \substack{122111\\1}, \substack{112211\\1}, \substack{112110\\1}\}$, so we may fix $\beta_1 = \substack{122210\\1}$. We must then fix $\substack{124321\\2}$, $\{\substack{123221\\\cdot}\}$, $\{\substack{111111\\\cdot}\}$, $\substack{112100\\1}$ (by orthogonality to β_1) and hence $\substack{134321\\2}$, $\{\substack{123321\\\cdot}\}$, $\{\substack{111110\\\cdot}\}$, $\substack{122100\\1}$; we must then fix $\substack{122111\\1}$ (by orthogonality to $\substack{112100\\1}$) and hence $\{\substack{112211\\1}, \substack{112110\\1}\}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_2\rangle$ giving transitivity on $\{\substack{111111\\\cdot}\}$, so we may fix $\beta_2 = \substack{111111\\1}$. We must then fix $\substack{123321\\1}$, $\substack{123221\\1}$, $\substack{112110\\1}$, $\substack{111110\\0}$ (by orthogonality to β_2), by which point all roots are fixed; so $W_X = G$.

If $X = X_{17}^{25}$ we must fix $\substack{234321\\2}$, $\{\substack{12\cdot321\\2}, \substack{12\cdot221\\1}, \substack{12\cdot\cdot1\cdot\\\cdot}\}$, $\{\substack{112221\\1}, \substack{111000\\1}, \substack{111100\\0}\}$, $\substack{100000\\0}$ ($o(\beta) = 0, 8, 10, 12$ respectively). We set

$$G = \langle w_7, w_2w_5, w_4{w_7}^{w_6w_5w_6}\rangle;$$

then G is transitive on $\{\substack{12\cdot321\\2}, \substack{12\cdot221\\1}, \substack{12\cdot\cdot1\cdot\\\cdot}\}$, so we may fix $\beta_1 = \substack{124321\\2}$. We must then fix $\{\substack{122221\\1}, \substack{122\cdot1\cdot\\1}\}$, $\{\substack{111000\\1}, \substack{111100\\0}\}$ (by orthogonality to β_1) and hence $\{\substack{123321\\2}, \substack{123221\\1}, \substack{12321\cdot\\\cdot}\}$, $\substack{112221\\1}$; we must then fix $\{\substack{12321\cdot\\\cdot}\}$, $\{\substack{122\cdot1\cdot\\1}\}$ (by orthogonality to $\substack{112221\\1}$) and hence $\{\substack{123321\\2}, \substack{123221\\1}\}$ and $\substack{122221\\1}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_7, w_2w_5\rangle$ giving transitivity on $\{\substack{12321\cdot\\\cdot}\}$, so we may fix $\beta_2 = \substack{123211\\2}$. We must then fix $\substack{123221\\1}$, $\substack{123210\\1}$, $\{\substack{122\cdot10\\1}\}$, $\substack{111100\\0}$ (by orthogonality to β_2) and hence $\substack{123321\\2}$, $\{\substack{123210\\2}, \substack{123211\\1}\}$, $\{\substack{122\cdot11\\1}\}$, $\substack{111000\\1}$; we must then fix $\substack{123210\\2}$, $\substack{122111\\1}$, $\substack{122110\\1}$ (by orthogonality to $\substack{111100\\0}$), by which point all roots are fixed; so $W_X = G$.

If $X = X_{17}^{26}$ we must fix $\substack{234321\\2}$, $\{\substack{1\cdot\cdot321\\\cdot}, \substack{1\cdot\cdot1\cdot\cdot\\\cdot}\}$ ($o(\beta) = 0, 9$ respectively). We set

$$G = \langle w_2, w_3, w_4, w_2w_5{w_7}^{w_4w_6w_5}\rangle;$$

then G is transitive on $\{\substack{1\cdot\cdot321\\\cdot}, \substack{1\cdot\cdot1\cdot\cdot\\\cdot}\}$, so we may fix $\beta_1 = \substack{134321\\2}$. We must then fix $\{\substack{11\cdot1\cdot\cdot\\\cdot}\}$ (by orthogonality to β_1) and hence $\{\substack{12\cdot321\\\cdot}, \substack{1221\cdot\cdot\\1}\}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_2, w_4, w_2w_5{w_7}^{w_6w_4w_5}\rangle$ giving transitivity on $\{\substack{11\cdot1\cdot\cdot\\\cdot}\}$, so we may fix $\beta_2 = \substack{111100\\0}$. We must then fix $\{\substack{12\cdot321\\2}, \substack{12211\cdot\\1}\}$, $\{\substack{11\cdot11\cdot\\1}\}$ (by orthogonality to β_2) and hence $\{\substack{123321\\1}, \substack{122100\\1}\}$ and $\{\substack{11\cdot100\\1}, \substack{11111\cdot\\0}\}$. Inside $\mathrm{stab}_G(\beta_1, \beta_2)$ we have

$\langle w_4, {w_2w_5w_7}^{w_6w_4w_5}\rangle$ giving transitivity on $\{\substack{11\cdot100\\1}, \substack{11111\cdot\\0}\}$, so we may fix $\beta_3 = \substack{111110\\0}$. We must then fix $\{\substack{12\cdot321\\2}, \substack{122111\\1}\}, \substack{122100\\1}, \{\substack{11\cdot111\\1}\}, \{\substack{11\cdot100\\1}\}$ (by orthogonality to β_3) and hence $\substack{122110\\1}, \substack{123321\\1}, \{\substack{11\cdot110\\1}\}, \substack{111111\\0}$; we must then fix $\{\substack{12\cdot321\\2}\}$ (by orthogonality to $\substack{111111\\0}$) and hence $\substack{122111\\1}$. Inside $\mathrm{stab}_G(\beta_1, \beta_2, \beta_3)$ we have $\langle w_4\rangle$ giving transitivity on $\{\substack{12\cdot321\\2}\}$, so we may fix $\beta_4 = \substack{124321\\2}$. We must then fix $\substack{111111\\1}, \substack{111110\\1}, \substack{111100\\1}$ (by orthogonality to β_4), by which point all roots are fixed; so $W_X = G$.

If $X = X^{27}_{17}$ we must fix $\substack{234321\\2}, \{\substack{1\cdot\cdot321\\2}, \substack{123221\\1}, \substack{122211\\1}, \substack{1221\cdot0\\1}, \substack{11\cdot111\\\cdot}, \substack{112210\\1}, \substack{112100\\1}, \substack{111110\\\cdot}, \substack{111100\\0}, \substack{111000\\1}\}$ ($o(\beta) = 0, 9$ respectively). We set

$$G = \langle w_3w_7, w_4w_6, w_2w_5w_7\rangle;$$

then G is transitive on $X \setminus \{\substack{234321\\2}\}$, so we may fix $\beta_1 = \substack{134321\\2}$. We must then fix $\{\substack{11\cdot111\\\cdot}, \substack{112210\\1}, \substack{112100\\1}, \substack{111110\\\cdot}, \substack{111100\\0}, \substack{111000\\1}\}$ (by orthogonality to β_1) and hence $\{\substack{12\cdot321\\2}, \substack{123221\\1}, \substack{122211\\1}, \substack{1221\cdot0\\1}\}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_4w_6, w_2w_5w_7\rangle$ giving transitivity on $\{\substack{12\cdot321\\2}, \substack{123221\\1}, \substack{122211\\1}, \substack{1221\cdot0\\1}\}$, so we may fix $\beta_2 = \substack{124321\\2}$. We must then fix $\{\substack{122211\\1}, \substack{1221\cdot0\\1}\}, \{\substack{11111\cdot\\\cdot}, \substack{111100\\0}, \substack{111000\\1}\}$ (by orthogonality to β_2) and hence $\{\substack{123321\\2}, \substack{123221\\1}\}, \{\substack{112111\\1}, \substack{112210\\1}, \substack{112100\\1}\}$; we must then fix $\substack{122100\\1}, \substack{112100\\1}, \{\substack{111100\\0}, \substack{111000\\1}\}$ (by orthogonality to both of $\{\substack{123321\\2}, \substack{123221\\1}\}$) and hence $\{\substack{122211\\1}, \substack{122110\\1}\}, \{\substack{112111\\1}, \substack{112210\\1}\}, \{\substack{11111\cdot\\\cdot}\}$. Inside $\mathrm{stab}_G(\beta_1, \beta_2)$ we have $\langle w_2w_5w_7\rangle$ giving transitivity on $\{\substack{123321\\2}, \substack{123221\\1}\}$, so we may fix $\beta_3 = \substack{123321\\2}$. We must then fix $\substack{122110\\1}, \substack{112111\\1}, \{\substack{11111\cdot\\0}\}$ (by orthogonality to β_3) and hence $\substack{122211\\1}, \substack{112210\\1}, \{\substack{11111\cdot\\1}\}$; we must then fix $\substack{111110\\1}, \substack{111110\\0}, \substack{111100\\0}$ (by orthogonality to $\substack{112111\\1}$), by which point all roots are fixed; so $W_X = G$.

If $X = X^{28}_{17}$ we must fix $\{\substack{\cdot\cdot4321\\2}\}, \{\substack{123\cdot\cdot\cdot\\\cdot}\}, \{\substack{122221\\1}, \substack{112211\\1}, \substack{122100\\1}, \substack{112110\\1}, \substack{012210\\1}, \substack{012111\\1}\}$ ($o(\beta) = 2, 6, 10$ respectively). We set

$$G = \langle w_2, w_1w_5, w_3w_6, w_5w_7\rangle;$$

then G is transitive on $\{\substack{122221\\1}, \substack{112211\\1}, \substack{122100\\1}, \substack{112110\\1}, \substack{012210\\1}, \substack{012111\\1}\}$, so we may fix $\beta_1 = \substack{122221\\1}$. We must then fix $\substack{124321\\2}, \{\substack{12321\cdot\\\cdot}\}$ (by orthogonality to β_1) and hence $\{\substack{\cdot34321\\2}\}, \{\substack{123\cdot21\\\cdot}\}$; we must then fix $\substack{122100\\1}$ (by orthogonality to $\substack{124321\\2}$) and hence $\{\substack{112211\\1}, \substack{112110\\1}, \substack{012210\\1}, \substack{012111\\1}\}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_2, w_1w_5, w_5w_7\rangle$ giving transitivity on $\{\substack{112211\\1}, \substack{112110\\1}, \substack{012210\\1}, \substack{012111\\1}\}$, so we may fix $\beta_2 = \substack{112211\\1}$. We must then fix $\substack{134321\\2}, \{\substack{123221\\\cdot}\}, \{\substack{123210\\\cdot}\}$ (by orthogonality to β_2) and hence $\substack{234321\\2}, \{\substack{123321\\\cdot}\}, \{\substack{123211\\\cdot}\}$; we must then fix $\substack{112110\\1}$ (by orthogonality to $\substack{134321\\2}$) and hence $\{\substack{012210\\1}, \substack{012111\\1}\}$; we must then fix $\substack{012111\\1}$ (by orthogonality to both of $\{\substack{123321\\\cdot}\}$) and hence $\substack{012210\\1}$. Inside $\mathrm{stab}_G(\beta_1, \beta_2)$ we have $\langle w_2\rangle$ giving transitivity on $\{\substack{123321\\\cdot}\}$, so we may fix $\beta_3 = \substack{123321\\2}$. We must then fix $\substack{123221\\1}, \substack{123211\\1}, \substack{123210\\1}$ (by orthogonality to β_3), by which point all roots are fixed; so $W_X = G$.

If $X = X^1_{18}$ we must fix $\{\substack{\cdot34321\\2}\}, \{\substack{12\cdot\cdot21\\\cdot}\}, \{\substack{12\cdot\cdot1\cdot\\\cdot}\}, \{\substack{\cdot12221\\1}\}$ ($o(\beta) = 1, 5, 8, 9$ respectively). We set

$$G = \langle w_1, w_2, w_4, w_5, w_7\rangle;$$

then G is transitive on $\{\substack{\cdot34321\\2}\}$, so we may fix $\beta_1 = \substack{234321\\2}$. We must then fix $\substack{012221\\1}$ (by orthogonality to β_1) and hence $\substack{112221\\1}$. Since the stabilizer in W of $\substack{234321\\2}, \substack{134321\\2}, \substack{012221\\1}$ is $\langle w_2, w_4, w_5, w_7\rangle$ we must have $W_X = G$.

If $X = X^2_{18}$ we must fix ${\substack{234321\\2}}, \{{\substack{1\cdot4321\\2}}\}, \{{\substack{123\cdot21\\\cdot}}\}, \{{\substack{1\cdot2221\\1}}\}, \{{\substack{12321\cdot\\\cdot}}\}, {\substack{012221\\1}}$, $\{{\substack{122211\\1}}, {\substack{112111\\1}}, {\substack{112210\\1}}, {\substack{122110\\1}}\}$ ($o(\beta) = 1, 3, 5, 7, 8, 9, 10$ respectively). We set

$$G = \langle w_2, w_3w_7, w_5w_7\rangle;$$

then G is transitive on $\{{\substack{123\cdot21\\\cdot}}\}$, so we may fix $\beta_1 = {\substack{123321\\2}}$. We must then fix ${\substack{123221\\1}}, \{{\substack{12321\cdot\\1}}\}, \{{\substack{112111\\1}}, {\substack{122110\\1}}\}$ (by orthogonality to β_1) and hence $\{{\substack{123321\\1}}, {\substack{123221\\2}}\}$, $\{{\substack{12321\cdot\\2}}\}, \{{\substack{122211\\1}}, {\substack{112210\\1}}\}$; we must then fix ${\substack{123321\\1}}$ (by orthogonality to both of $\{{\substack{12321\cdot\\2}}\}$) and hence ${\substack{123221\\2}}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_3w_7\rangle$ giving transitivity on $\{{\substack{1\cdot4321\\2}}\}$, so we may fix $\beta_2 = {\substack{134321\\2}}$. We must then fix ${\substack{112221\\1}}, {\substack{112210\\1}}, {\substack{112111\\1}}$ (by orthogonality to β_2) and hence ${\substack{122221\\1}}, {\substack{122211\\1}}, {\substack{122110\\1}}$; we must then fix ${\substack{123210\\2}}, {\substack{123210\\1}}$ (by orthogonality to ${\substack{112111\\1}}$), by which point all roots are fixed; so $W_X = G$.

If $X = X^3_{18}$ we must fix ${\substack{234321\\2}}, \{{\substack{1\cdot\cdot\cdot21\\2}}\}, \{{\substack{1\cdot\cdot\cdot21\\1}}\}, {\substack{123211\\2}}$ and $\{{\substack{112210\\1}}, {\substack{122110\\1}}\}$ ($o(\beta) = 1, 4, 6, 7, 11$ respectively), ${\substack{123210\\2}}$ ($o(\beta) = 9$, orthogonal to all of $\{{\substack{1\cdot\cdot\cdot21\\1}}\}$), $\{{\substack{12\cdot211\\1}}, {\substack{11\cdot111\\1}}\}$ ($o(\beta) = 9$, orthogonal to two of $\{{\substack{1\cdot\cdot\cdot21\\1}}\}$), ${\substack{012221\\1}}$ ($o(\beta) = 9$, orthogonal to none of $\{{\substack{1\cdot\cdot\cdot21\\1}}\}$). We set

$$G = \langle w_4, w_3w_5\rangle;$$

then G is transitive on $\{{\substack{1\cdot\cdot\cdot21\\1}}\}$, so we may fix $\beta_1 = {\substack{123321\\1}}$. We must then fix ${\substack{123221\\2}}, \{{\substack{11\cdot111\\1}}\}, {\substack{122110\\1}}$ (by orthogonality to β_1) and hence $\{{\substack{1\cdot\cdot321\\2}}\}, \{{\substack{12\cdot211\\1}}\}, {\substack{112210\\1}}$; we must then fix $\{{\substack{12\cdot321\\2}}\}, {\substack{112221\\1}}$ (by orthogonality to ${\substack{122110\\1}}$) and hence ${\substack{134321\\2}}$, $\{{\substack{12\cdot221\\1}}\}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_4\rangle$ giving transitivity on $\{{\substack{12\cdot321\\2}}\}$, so we may fix $\beta_2 = {\substack{124321\\2}}$. We must then fix ${\substack{122221\\1}}, {\substack{122211\\1}}, {\substack{111111\\1}}$ (by orthogonality to β_2), by which point all roots are fixed; so $W_X = G$.

If $X = X^4_{18}$ we must fix ${\substack{234321\\2}}, \{{\substack{1\cdot\cdot\cdot21\\\cdot}}\}, \{{\substack{12\cdot211\\\cdot}}, {\substack{11\cdot111\\\cdot}}\}, {\substack{012221\\1}}, \{{\substack{112210\\1}}, {\substack{122110\\1}}\}$ ($o(\beta) = 1, 5, 8, 9, 12$ respectively). We set

$$G = \langle w_2, w_4, w_3w_5\rangle;$$

then G is transitive on $\{{\substack{1\cdot\cdot\cdot21\\\cdot}}\}$, so we may fix $\beta_1 = {\substack{134321\\2}}$. We must then fix ${\substack{112221\\1}}, \{{\substack{11\cdot111\\\cdot}}\}, {\substack{112210\\1}}$ (by orthogonality to β_1) and hence $\{{\substack{12\cdot\cdot21\\\cdot}}\}, \{{\substack{12\cdot211\\\cdot}}\}, {\substack{122110\\1}}$; we must then fix $\{{\substack{12\cdot321\\\cdot}}\}$ (by orthogonality to ${\substack{122110\\1}}$) and hence $\{{\substack{12\cdot221\\\cdot}}\}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_2, w_4\rangle$, which acts as S_3 on $\{{\substack{12\cdot321\\\cdot}}\}$, so we may fix all of these roots. We then have fixed all of the roots ${\substack{\cdot\cdot\cdot321\\\cdot}}, {\substack{112210\\1}}, {\substack{122110\\1}}$, which span $\mathbb{R}\Phi$; so $W_X = G$.

If $X = X^5_{18}$ we must fix ${\substack{234321\\2}}, \{{\substack{1\cdot\cdot\cdot21\\\cdot}}\}, {\substack{012221\\1}}, \{{\substack{123211\\2}}, {\substack{12\cdot210\\1}}, {\substack{112211\\1}}, {\substack{122111\\1}}$, ${\substack{11\cdot110\\1}}, {\substack{111111\\0}}\}$ ($o(\beta) = 1, 5, 9, 10$ respectively). We set

$$G = \langle w_4, w_3w_5, w_2w_5w_7\rangle;$$

then G is transitive on $\{{\substack{1\cdot\cdot\cdot21\\\cdot}}\}$, so we may fix $\beta_1 = {\substack{134321\\2}}$. We must then fix ${\substack{112221\\1}}, \{{\substack{112211\\1}}, {\substack{11\cdot110\\1}}, {\substack{111111\\0}}\}$ (by orthogonality to β_1) and hence $\{{\substack{12\cdot\cdot21\\\cdot}}\}$ and $\{{\substack{123211\\2}}, {\substack{12\cdot210\\1}}, {\substack{122111\\1}}\}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_4, w_2w_5w_7\rangle$ giving transitivity on $\{{\substack{112211\\1}}, {\substack{11\cdot110\\1}}, {\substack{111111\\0}}\}$, so we may fix $\beta_2 = {\substack{111111\\0}}$. We must then fix $\{{\substack{12\cdot\cdot21\\2}}\}$, $\{{\substack{123211\\2}}, {\substack{12\cdot210\\1}}\}, \{{\substack{11\cdot110\\1}}\}$ (by orthogonality to β_2) and hence $\{{\substack{12\cdot\cdot21\\1}}\}, {\substack{122111\\1}}, {\substack{112211\\1}}$; we must then fix ${\substack{123221\\2}}, \{{\substack{12\cdot221\\1}}\}, \{{\substack{12\cdot210\\1}}\}$ (by orthogonality to ${\substack{112211\\1}}$) and hence $\{{\substack{12\cdot321\\2}}\}, {\substack{123321\\1}}, {\substack{123211\\2}}$. Inside $\mathrm{stab}_G(\beta_1, \beta_2)$ we have $\langle w_4\rangle$ giving transitivity on $\{{\substack{11\cdot110\\1}}\}$, so we may fix $\beta_3 = {\substack{111110\\1}}$. We must then fix ${\substack{124321\\2}}, {\substack{123221\\1}}, {\substack{123210\\1}}$ (by orthogonality to β_3), by which point all roots are fixed; so $W_X = G$.

If $X = X_{19}$ we must fix $\{\genfrac{}{}{0pt}{}{\cdot 34321}{2}\}, \{\genfrac{}{}{0pt}{}{12\cdot 321}{\cdot}\}, \{\genfrac{}{}{0pt}{}{12\cdot 2\cdot 1}{\cdot}\}, \{\genfrac{}{}{0pt}{}{\cdot 122\cdot 1}{1}\}$ ($o(\beta) = 2, 4, 7, 9$ respectively), $\genfrac{}{}{0pt}{}{122111}{1}$ ($o(\beta) = 10$, orthogonal to all of $\{\genfrac{}{}{0pt}{}{12\cdot 321}{\cdot}\}$), $\{\genfrac{}{}{0pt}{}{12\cdot 210}{\cdot}\}$ ($o(\beta) = 10$, orthogonal to one of $\{\genfrac{}{}{0pt}{}{12\cdot 321}{\cdot}\}$). We set

$$G = \langle w_1, w_2, w_4, w_6\rangle;$$

then G is transitive on $\{\genfrac{}{}{0pt}{}{12\cdot 2\cdot 1}{\cdot}\}$, so we may fix $\beta_1 = \genfrac{}{}{0pt}{}{123221}{2}$. We must then fix $\genfrac{}{}{0pt}{}{123321}{1}, \{\genfrac{}{}{0pt}{}{12\cdot 211}{1}\}, \{\genfrac{}{}{0pt}{}{12\cdot 210}{1}\}, \{\genfrac{}{}{0pt}{}{\cdot 12211}{1}\}$ (by orthogonality to β_1) and hence $\{\genfrac{}{}{0pt}{}{12\cdot 321}{2}\}$, $\{\genfrac{}{}{0pt}{}{12\cdot 221}{1}, \genfrac{}{}{0pt}{}{123211}{2}\}$, $\genfrac{}{}{0pt}{}{123210}{2}$ and $\{\genfrac{}{}{0pt}{}{\cdot 12221}{1}\}$; we must then fix $\genfrac{}{}{0pt}{}{123211}{2}$ (by orthogonality to $\genfrac{}{}{0pt}{}{123321}{1}$) and hence $\{\genfrac{}{}{0pt}{}{12\cdot 221}{1}\}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_1, w_4\rangle$, which acts as S_2 on each of $\{\genfrac{}{}{0pt}{}{\cdot 34321}{2}\}$ and $\{\genfrac{}{}{0pt}{}{12\cdot 321}{2}\}$ independently, so we may fix all of these roots. We then have fixed all of the roots $\genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot\cdot\cdot}{2}$, which span $\mathbb{R}\Phi$; so $W_X = G$.

If $X = X_{20}^1$ we must fix $\{\genfrac{}{}{0pt}{}{\cdot\cdot 4321}{2}\}, \{\genfrac{}{}{0pt}{}{123\cdot\cdot 1}{\cdot}\}, \{\genfrac{}{}{0pt}{}{\cdot\cdot 2\cdot\cdot 1}{1}\}, \{\genfrac{}{}{0pt}{}{123210}{\cdot}\}$ ($o(\beta) = 3, 6, 9, 12$ respectively). We set

$$G = \langle w_1, w_2, w_3, w_5, w_6\rangle;$$

then G is transitive on $\{\genfrac{}{}{0pt}{}{\cdot\cdot 2\cdot\cdot 1}{1}\}$, so we may fix $\beta_1 = \genfrac{}{}{0pt}{}{122221}{1}$. We must then fix $\genfrac{}{}{0pt}{}{124321}{2}, \{\genfrac{}{}{0pt}{}{123211}{\cdot}\}, \{\genfrac{}{}{0pt}{}{\cdot 12\cdot 11}{1}\}$ (by orthogonality to β_1) and hence $\{\genfrac{}{}{0pt}{}{\cdot 34321}{2}\}, \{\genfrac{}{}{0pt}{}{123\cdot 21}{\cdot}\}$, $\{\genfrac{}{}{0pt}{}{122\cdot 11}{1}, \genfrac{}{}{0pt}{}{\cdot 12221}{1}\}$; we must then fix $\{\genfrac{}{}{0pt}{}{122\cdot 11}{1}\}$ (by orthogonality to $\genfrac{}{}{0pt}{}{124321}{2}$) and hence $\{\genfrac{}{}{0pt}{}{\cdot 12221}{1}\}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_1, w_2, w_5\rangle$, which acts as S_2 on each of $\{\genfrac{}{}{0pt}{}{\cdot 34321}{2}\}$, $\{\genfrac{}{}{0pt}{}{123210}{\cdot}\}$ and $\{\genfrac{}{}{0pt}{}{122\cdot 11}{1}\}$ independently, so we may fix all of these roots. We then have fixed all of the roots $\genfrac{}{}{0pt}{}{\cdot 34321}{2}, \genfrac{}{}{0pt}{}{122\cdot\cdot 1}{1}, \genfrac{}{}{0pt}{}{123210}{\cdot}$, which span $\mathbb{R}\Phi$; so $W_X = G$.

If $X = X_{20}^2$ we must fix $\{\genfrac{}{}{0pt}{}{\cdot\cdot\cdot 321}{\cdot}\}, \{\genfrac{}{}{0pt}{}{\cdot\cdot\cdot 2\cdot\cdot}{\cdot}\}$ ($o(\beta) = 3, 9$ respectively). We set

$$G = \langle w_1, w_2, w_3, w_4, w_6, w_7\rangle;$$

then G is transitive on $\{\genfrac{}{}{0pt}{}{\cdot\cdot\cdot 2\cdot\cdot}{\cdot}\}$, so we may fix $\beta_1 = \genfrac{}{}{0pt}{}{123221}{2}$. We must then fix $\genfrac{}{}{0pt}{}{123321}{1}, \{\genfrac{}{}{0pt}{}{\cdot\cdot\cdot 21\cdot}{1}\}$ (by orthogonality to β_1) and hence $\{\genfrac{}{}{0pt}{}{\cdot\cdot\cdot 321}{2}\}, \{\genfrac{}{}{0pt}{}{12321\cdot}{2}, \genfrac{}{}{0pt}{}{\cdot\cdot\cdot 221}{1}\}$; we must then fix $\{\genfrac{}{}{0pt}{}{12321\cdot}{2}\}$ (by orthogonality to $\genfrac{}{}{0pt}{}{123321}{1}$) and hence $\{\genfrac{}{}{0pt}{}{\cdot\cdot\cdot 221}{1}\}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_1, w_3, w_4, w_7\rangle$, which acts as S_4 on $\{\genfrac{}{}{0pt}{}{\cdot\cdot\cdot 321}{2}\}$ and independently as S_2 on $\{\genfrac{}{}{0pt}{}{12321\cdot}{2}\}$, so we may fix all of these roots. We then have fixed all of the roots $\genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot\cdot\cdot}{2}$, which span $\mathbb{R}\Phi$; so $W_X = G$.

If $X = X_{22}$ we must fix $\{\genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot\cdot 1}{2}\}, \{\genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot\cdot 1}{1}\}, \genfrac{}{}{0pt}{}{123210}{2}$ ($o(\beta) = 5, 9, 15$ respectively). We set

$$G = \langle w_1, w_3, w_4, w_5, w_6\rangle;$$

then G acts as S_6 on $\{\genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot\cdot 1}{2}\}$, so we may fix all of these roots. We then have fixed all of the roots $\genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot\cdot\cdot}{2}$, which span $\mathbb{R}\Phi$; so $W_X = G$.

If $X = X_{27}$ all roots β have $o(\beta) = 10$. We set

$$G = \langle w_1, w_2, w_3, w_4, w_5, w_6\rangle;$$

since any element of W outside G has a reduced expression ending with w_7, it makes the root $\genfrac{}{}{0pt}{}{000001}{0}$ negative and so cannot stabilize X, so $W_X = G$.

The results found here are presented in tabular form in the final chapter of this work. As with the maximal abelian sets in the E_6 root system, in most instances we may immediately see that the set is not a W-translate of any of the others, since the signature uniquely identifies it; the only cases for which this is not true are X_{17}^{26} and X_{17}^{27}, whose stabilizers have orders 1152 and 192 respectively. We have thus shown the following.

THEOREM 7.13. *The 39 sets in $\mathcal{S}(E_7)$ represent different W-orbits.*

CHAPTER 8

The root system of type E_8

Let Φ be of type E_8; thus Φ has simple roots $\alpha_1, \dots, \alpha_8$ numbered as in [**1**].

8.1. Radical maximal abelian sets

We have $\rho = {}^{2465432}_{\;\;3}$, and the roots in Ξ are ${}^{\cdot\cdot\cdot\cdot\cdot\cdot 1}_{\;\;\cdot}$; there are 28 pairs of roots in Ξ summing to ρ, namely $\{{}^{0000001}_{\;\;0}, {}^{2465431}_{\;\;3}\}$ and 27 of the form $\{{}^{\cdot\cdot\cdot\cdot\cdot 11}_{\;\;\cdot}, {}^{\cdot\cdot\cdot\cdot\cdot 21}_{\;\;\cdot}\}$. We may arrange these as follows.

$$\{{}^{1233321}_{\;\;1}, {}^{1232111}_{\;\;2}\}$$

$$\{{}^{1233221}_{\;\;1}, {}^{1232211}_{\;\;2}\}$$

$$\{{}^{1233211}_{\;\;1}, {}^{1232221}_{\;\;2}\} \; \{{}^{1232221}_{\;\;1}, {}^{1233211}_{\;\;2}\}$$

$$\{{}^{1232211}_{\;\;1}, {}^{1233221}_{\;\;2}\} \; \{{}^{1222221}_{\;\;1}, {}^{1243211}_{\;\;2}\}$$

$$\{{}^{1232111}_{\;\;1}, {}^{1233321}_{\;\;2}\} \; \{{}^{1222211}_{\;\;1}, {}^{1243221}_{\;\;2}\} \; \{{}^{1122221}_{\;\;1}, {}^{1343211}_{\;\;2}\}$$

$$\{{}^{1222111}_{\;\;1}, {}^{1243321}_{\;\;2}\} \; \{{}^{1122211}_{\;\;1}, {}^{1343221}_{\;\;2}\} \; \{{}^{0122221}_{\;\;1}, {}^{2343211}_{\;\;2}\}$$

$$\{{}^{1221111}_{\;\;1}, {}^{1244321}_{\;\;2}\} \; \{{}^{1122111}_{\;\;1}, {}^{1343321}_{\;\;2}\} \; \{{}^{0122211}_{\;\;1}, {}^{2343221}_{\;\;2}\} \; \{{}^{2465431}_{\;\;3}, {}^{0000001}_{\;\;0}\}$$

$$\{{}^{1121111}_{\;\;1}, {}^{1344321}_{\;\;2}\} \; \{{}^{0122111}_{\;\;1}, {}^{2343321}_{\;\;2}\} \; \{{}^{2465421}_{\;\;3}, {}^{0000011}_{\;\;0}\}$$

$$\{{}^{1111111}_{\;\;1}, {}^{1354321}_{\;\;2}\} \; \{{}^{0121111}_{\;\;1}, {}^{2344321}_{\;\;2}\} \; \{{}^{2465321}_{\;\;3}, {}^{0000111}_{\;\;0}\}$$

$$\{{}^{0111111}_{\;\;1}, {}^{2354321}_{\;\;2}\} \; \{{}^{2464321}_{\;\;3}, {}^{0001111}_{\;\;0}\}$$

$$\{{}^{0011111}_{\;\;1}, {}^{2454321}_{\;\;2}\} \; \{{}^{2454321}_{\;\;3}, {}^{0011111}_{\;\;0}\}$$

$$\{{}^{2354321}_{\;\;3}, {}^{0111111}_{\;\;0}\}$$

$$\{{}^{1354321}_{\;\;3}, {}^{1111111}_{\;\;0}\}$$

Once more, we shall call a root in Ξ *odd* or *even*, this time according to the parity of its α_2-coefficient, so that each pair consists of an odd root and an even root; again a maximal radical abelian set may be specified by simply giving the parity of the root selected in each pair.

We may identify the 28 pairs with unordered pairs from the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ (where again we write the unordered pair $\{i, j\}$ simply as ij); we give the correspondence by the following array.

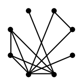
12						
	13					
23		14				
	24		15			
34		25		16		
	35		26		17	
45		36		27		18
	46		37		28	
56		47		38		
	57		48			
67		58				
	68					
78						

In similar fashion to before, we may then represent a maximal radical abelian set X by a graph Γ_X with vertex set $\{1, 2, 3, 4, 5, 6, 7, 8\}$, where the choice in X of the odd or even root in the pair identified with ij is denoted by the presence or absence in Γ_X of the edge ij (again, alternatively we may regard this as the choice of a black edge or a white edge). We shall arrange the vertices in a regular octagon as follows:

	4	5	
3			6
2			7
	1	8	

.

For example, if $X = \{ \genfrac{}{}{0pt}{}{246\cdot\cdot\cdot\cdot}{3}, \genfrac{}{}{0pt}{}{2\cdot54321}{2}, \genfrac{}{}{0pt}{}{\cdot\cdot4\cdot\cdot21}{2}, \genfrac{}{}{0pt}{}{1233\cdot\cdot1}{\cdot}, \genfrac{}{}{0pt}{}{\cdot\cdot22221}{1}, \genfrac{}{}{0pt}{}{1111111}{1}, \genfrac{}{}{0pt}{}{\cdot\cdot11111}{0} \}$ then Γ_X is



.

We now consider the action of $\mathrm{stab}_W(\rho) = \langle w_1, w_2, w_3, w_4, w_5, w_6, w_7 \rangle$ on the roots in Ξ. This group is $W(E_7)$; we shall write σ for the high root $\genfrac{}{}{0pt}{}{234321 0}{2}$ of the E_7 subsystem, and set $W' = \langle w_\sigma, w_1, w_3, w_4, w_5, w_6, w_7 \rangle$. Each generator of W' preserves the parity of α_2-coefficients, and therefore permutes edges without changing colours; in fact the notation chosen for the pairs means that the generating elements of $W' \cong S_8$ act as permutations of the vertices as follows:

$$w_\sigma = (7\ 8),\ w_1 = (6\ 7),\ w_3 = (5\ 6),\ w_4 = (4\ 5),\ w_5 = (3\ 4),\ w_6 = (2\ 3),\ w_7 = (1\ 2).$$

Thus two maximal radical abelian sets lie in the same W'-orbit if and only if their graphs are isomorphic. By [**12**], up to isomorphism there are 12346 graphs on 8 vertices; since there are $2^{28} = 268435456$ maximal radical abelian sets, this already gives a significant reduction. Nevertheless it is somewhat impractical to list all graphs as we did in the E_7 case.

The remaining generator of $\mathrm{stab}_W(\rho)$ is w_2. If we set $\{i, j, k, \ell\} = \{1, 2, 3, 4\}$ or $\{5, 6, 7, 8\}$, each edge ij in Γ_X gives rise to the edge $k\ell$ in $\Gamma_{w_2(X)}$ of the opposite colour (alternatively, the presence or absence of ij gives rise to the absence or presence of $k\ell$ respectively). Thus for example with the set X as above, in the graph Γ_X the edge 12 is present (or black), so in $\Gamma_{w_2(X)}$ the edge 34 is absent (or white); on the other hand in Γ_X the edge 14 is absent (or white), so in $\Gamma_{w_2(X)}$ the edge 23 is present (or black). Treating all twelve edges among the vertices $1, 2, 3, 4$ and among the vertices $5, 6, 7, 8$ thus, we see that the graph $\Gamma_{w_2(X)}$ is

,

corresponding to $w_2(X) = \{ \substack{2\cdot\cdot\cdot\cdot\cdot\cdot \\ 3}, \substack{\cdot\cdot4\cdot\cdot21 \\ 2}, \substack{1233\cdot\cdot1 \\ \cdot}, \substack{\cdot\cdot22221 \\ 1}, \substack{\cdot\cdot11111 \\ 1}, \substack{1111111 \\ 0} \}$.

Note that if in the graph Γ_X at least seven of the twelve edges among the vertices $1, 2, 3, 4$ and among the vertices $5, 6, 7, 8$ are present, then $\Gamma_{w_2(X)}$ will have fewer edges than Γ_X, which therefore need be considered no further.

Following [**3**, p.46], we shall call w_2 and its conjugates under W' *bifid maps*, and we see that the 72 cosets of W' in $W(E_7)$ are represented by 1, the 35 bifid maps, and the products of these with $w_0 w_\rho$, which is the longest element of $W(E_7)$. Since $w_0 w_\rho$ maps each root β of Ξ to $-w_\rho(\beta) = \rho - \beta$, it interchanges the two roots in each pair, and thus the graph $\Gamma_{w_0 w_\rho(X)}$ is the complement of Γ_X.

We introduce some further terminology and notation. As in the E_7 case, we call two graphs *equivalent* if they correspond to sets which are W-translates of each other. Let $e = e(\Gamma)$ be the number of edges of the graph Γ. A *partition* of a graph Γ is a pair of complementary subsets of four vertices each, and will be written e.g. $1234|5678$; the partitions thus naturally correspond to the bifid maps, which as in [**3**, p.46] will be written e.g. $(1234|5678)$ (and a bar over such a symbol will denote the product of the bifid map with complementation). A partition *contains* an edge of Γ if the two endpoints of the edge lie in the same subset; it is *heavy*, *full* or *light* if the number of edges it contains is more than 6, exactly 6 or less than $e - 8$ respectively. A graph is *reducible* if it is equivalent to one with a smaller value of e, and *irreducible* otherwise. The statement above about coset representatives shows that if two graphs are equivalent then one can be obtained from the other by applying at most one bifid map, possibly followed by complementation.

LEMMA 8.1. *A graph Γ is irreducible if and only if $e(\Gamma) \leq 14$ and Γ contains no partition which is either heavy or light.*

PROOF. Complementation gives a graph with $28 - e$ edges; if a partition contains n edges, the corresponding bifid map gives a graph with $12 + e - 2n$ edges, and then complementation produces one with $16 - e + 2n$ edges. The conditions for these to be less than e are $e > 14$, $n > 6$ and $n < e - 8$ respectively. ☐

We also need to consider when a graph represents a maximal abelian set; we shall call such a graph *relevant*. The roots $-\substack{2465432 \\ 3}, -\substack{\cdot\cdot\cdot\cdot\cdot\cdot1 \\ \cdot}$ are excluded by $\substack{2465432 \\ 3}$, so the only roots which need to be excluded by the other roots of X are those of the form $\substack{\cdot\cdot\cdot\cdot\cdot\cdot0 \\ \cdot}$. To exclude $\substack{2343210 \\ 2}$ we must have some root $\substack{0\cdot\cdot\cdot\cdot\cdot1 \\ \cdot}$ in X; using W' we see that to exclude the roots $\pm\substack{\cdot\cdot\cdot\cdot\cdot\cdot0 \\ 2}$ and $\substack{\cdot\cdot\cdot\cdot\cdot\cdot0 \\ 0}$ the graph must not have two vertices of which one is joined to all the other six and the other is joined to none of the other six. Similarly, to exclude $\substack{0000000 \\ 1}$ we must have some root $\substack{\cdot\cdot11111 \\ 0}$, some root $\substack{123\cdot\cdot\cdot1 \\ 1}$ or some root $\substack{\cdot\cdot54321 \\ 2}$ in X; using W' we see that to exclude the roots $\pm\substack{\cdot\cdot\cdot\cdot\cdot\cdot0 \\ 1}$ the graph must not have a partition in which all vertices in one set and none in the other are joined. Thus a graph is relevant if and only if no two of its vertices are joined to all and none of the other vertices respectively, and it has no partition with K_4 in one half and the empty graph in the other.

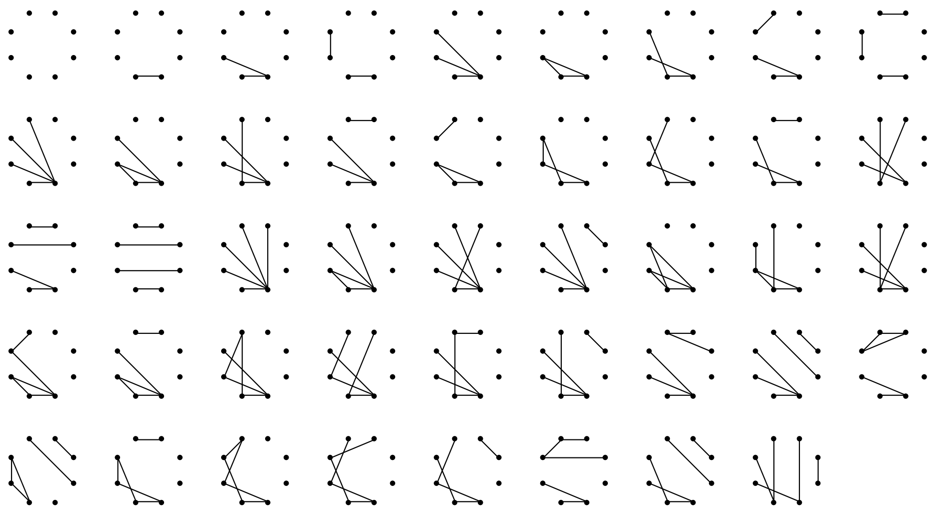


FIGURE 8.1. Graphs on 8 vertices with $e \leq 5$

Observe that no graph with a K_4 subgraph need be considered: if there are no edges among the other four vertices the graph is irrelevant, while otherwise the corresponding partition is heavy and the graph is reducible.

Our task is then to determine up to equivalence all relevant irreducible graphs (*rigs* for short). As seen above, it suffices to consider graphs with $e \leq 14$. From [**12**] we obtain the numbers of graphs for each value of e; we list these values below, together with the numbers of rigs up to equivalence (denoted $\sim$) which we claim.

e	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
# graphs	1	1	2	5	11	24	56	115	221	402	663	980	1312	1557	1646
# rigs/$\sim$	1	1	2	5	11	24	53	96	115	84	30	5	2	1	0

For values of e up to 9 we shall simply produce the list of all such graphs, eliminate any which are irrelevant or reducible, and determine any equivalences among the remainder. For larger values of e we shall instead produce a list of relevant irreducible graphs and argue that any such is equivalent to one of those listed. It may or may not be clear that there are no further equivalences among the final list of rigs; as before, the determination of stabilizers and orbit structure will settle this eventually. For each value of e, we shall give the numbering used for the rigs found.

We list in Figure 8.1 the graphs with $e \leq 5$; the above shows that all of these are relevant and irreducible, and clearly there can be no equivalences between them. We number them in accordance with the following scheme.

1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18
19	20	21	22	23	24	25	26	27
28	29	30	31	32	33	34	35	36
37	38	39	40	41	42	43	44	

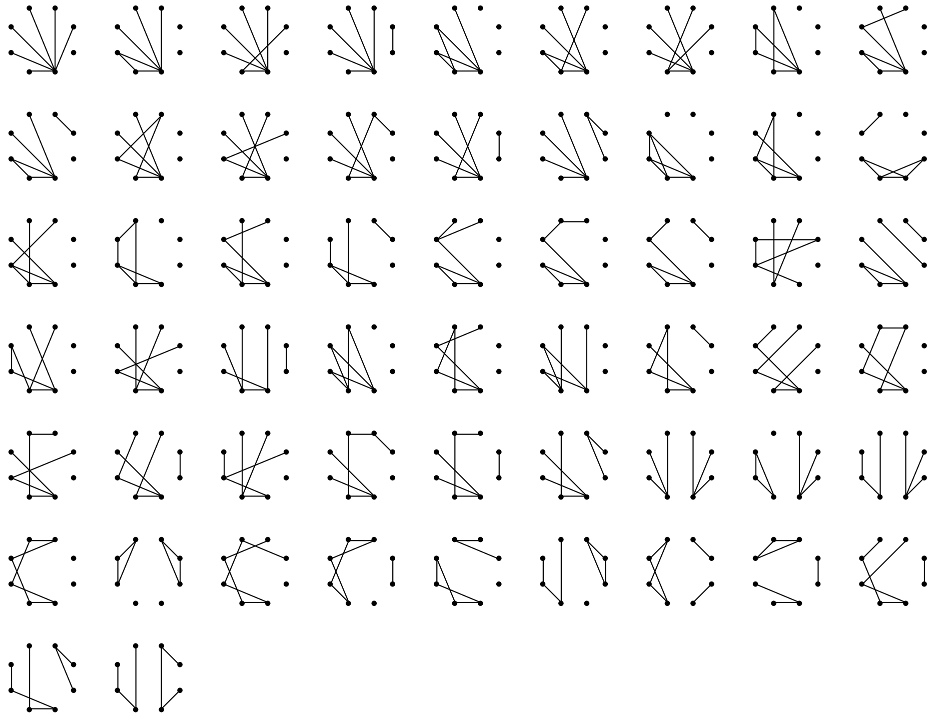


FIGURE 8.2. Graphs on 8 vertices with $e = 6$

In Figure 8.2 we list the graphs with $e = 6$. Here we see that the first in the first row is not relevant because of the vertices 8 and 7, which are joined to all and none of the other vertices respectively; neither is the seventh in the second row, because the partition $1238|4567$ has K_4 in the first half and the empty graph in the second; and the bifid map $(1238|4567)$ sends the fifth in the sixth row to the last in the third row. (In this case it is straightforward to see that there are no other equivalences in the list, since the only way to form an equivalent graph without increasing the number of edges is to apply a bifid map corresponding to a full partition, which must therefore contain all 6 edges of the graph.) Again the following scheme gives the numbering we use for these graphs, as well as recording the points above which enable us to disregard three of them.

irrel. 87	45	46	47	48	49	50	51	52
53	54	55	56	57	58	irrel. $1238 \mid 4567$	59	60
61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78
79	80	81	82	83	84	85	86	87
88	89	90	91	$(1238 \mid 4567) \mapsto 69$	92	93	94	95
96	97							

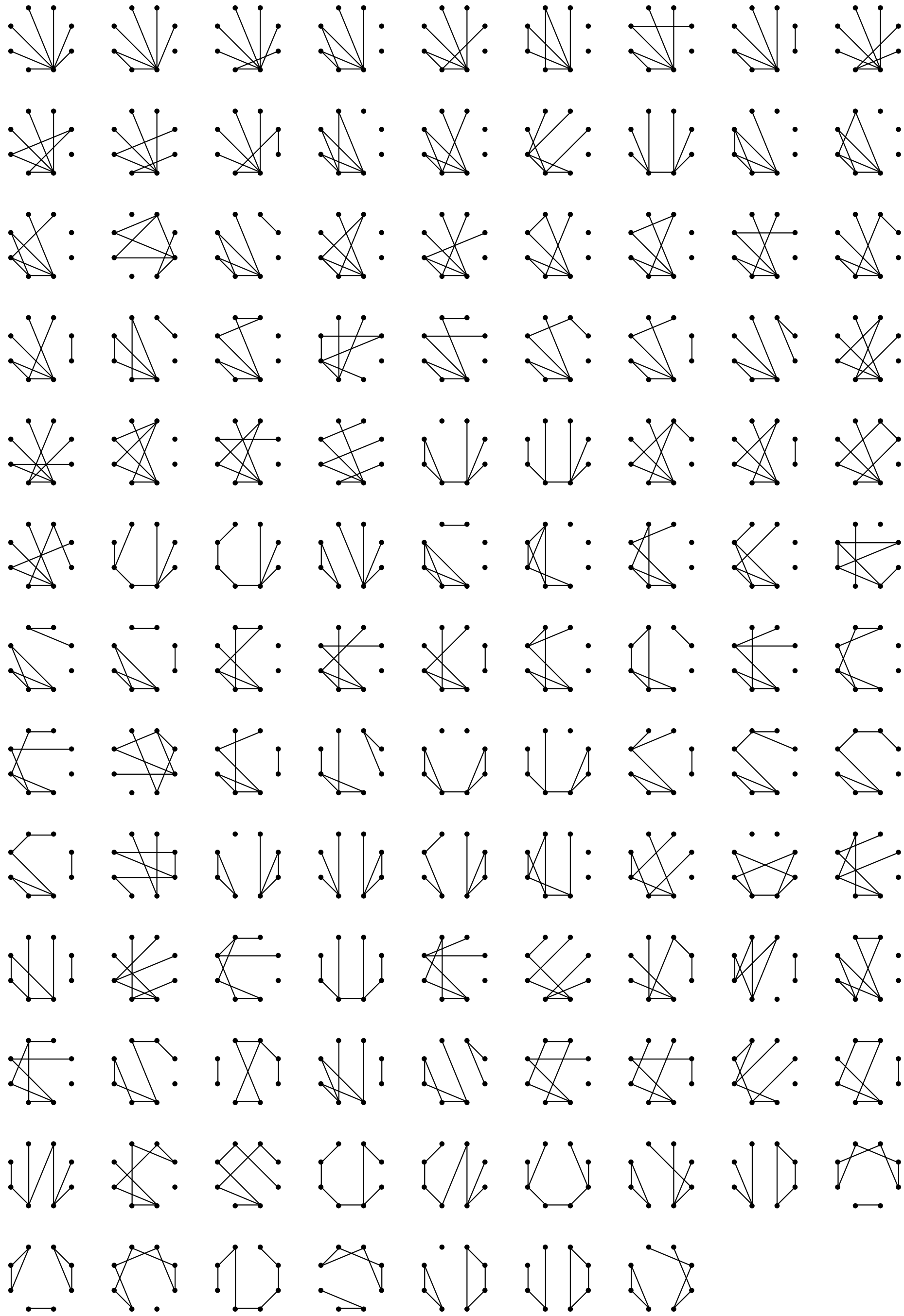


FIGURE 8.3. Graphs on 8 vertices with $e = 7$

In Figure 8.3 we list the graphs with $e = 7$. Here 3 graphs are not relevant; this time 9 are reducible, which we indicate in the following scheme by giving the appropriate bifid map with an arrow pointing downwards; and there are 7 equivalences among the remainder.

<table>
<tr><td>irrel.
87</td><td>irrel.
87</td><td>98</td><td>99</td><td>100</td><td>101</td><td>102</td><td>103</td><td>104</td></tr>
<tr><td>105</td><td>106</td><td>107</td><td>108</td><td>109</td><td>110</td><td>111</td><td>irrel.
1238|4567</td><td>112</td></tr>
<tr><td>113</td><td>114</td><td>115</td><td>116</td><td>117</td><td>118</td><td>119</td><td>120</td><td>121</td></tr>
<tr><td>122</td><td>123</td><td>124</td><td>125</td><td>126</td><td>127</td><td>128</td><td>129</td><td>130</td></tr>
<tr><td>131</td><td>132</td><td>133</td><td>134</td><td>135</td><td>136</td><td>137</td><td>138</td><td>139</td></tr>
<tr><td>140</td><td>141</td><td>142</td><td>143</td><td>(1238|4567)
↓</td><td>144</td><td>(1248|3567)
↦115</td><td>145</td><td>146</td></tr>
<tr><td>(1238|4567)
↓</td><td>(1238|4567)
↓</td><td>147</td><td>148</td><td>149</td><td>150</td><td>151</td><td>(1248|3567)
↦127</td><td>152</td></tr>
<tr><td>153</td><td>154</td><td>155</td><td>156</td><td>157</td><td>158</td><td>159</td><td>(1238|4567)
↦129</td><td>(1238|4567)
↦156</td></tr>
<tr><td>160</td><td>161</td><td>(1234|5678)
↓</td><td>(1234|5678)
↓</td><td>(1234|5678)
↓</td><td>162</td><td>163</td><td>164</td><td>165</td></tr>
<tr><td>166</td><td>167</td><td>168</td><td>169</td><td>(1248|3567)
↦128</td><td>170</td><td>171</td><td>172</td><td>173</td></tr>
<tr><td>174</td><td>(1238|4567)
↦155</td><td>175</td><td>176</td><td>(1238|4567)
↦160</td><td>177</td><td>178</td><td>179</td><td>180</td></tr>
<tr><td>181</td><td>182</td><td>183</td><td>184</td><td>185</td><td>186</td><td>187</td><td>(1234|5678)
↓</td><td>188</td></tr>
<tr><td>189</td><td>190</td><td>191</td><td>192</td><td>(1234|5678)
↓</td><td>(1234|5678)
↓</td><td>193</td><td></td><td></td></tr>
</table>

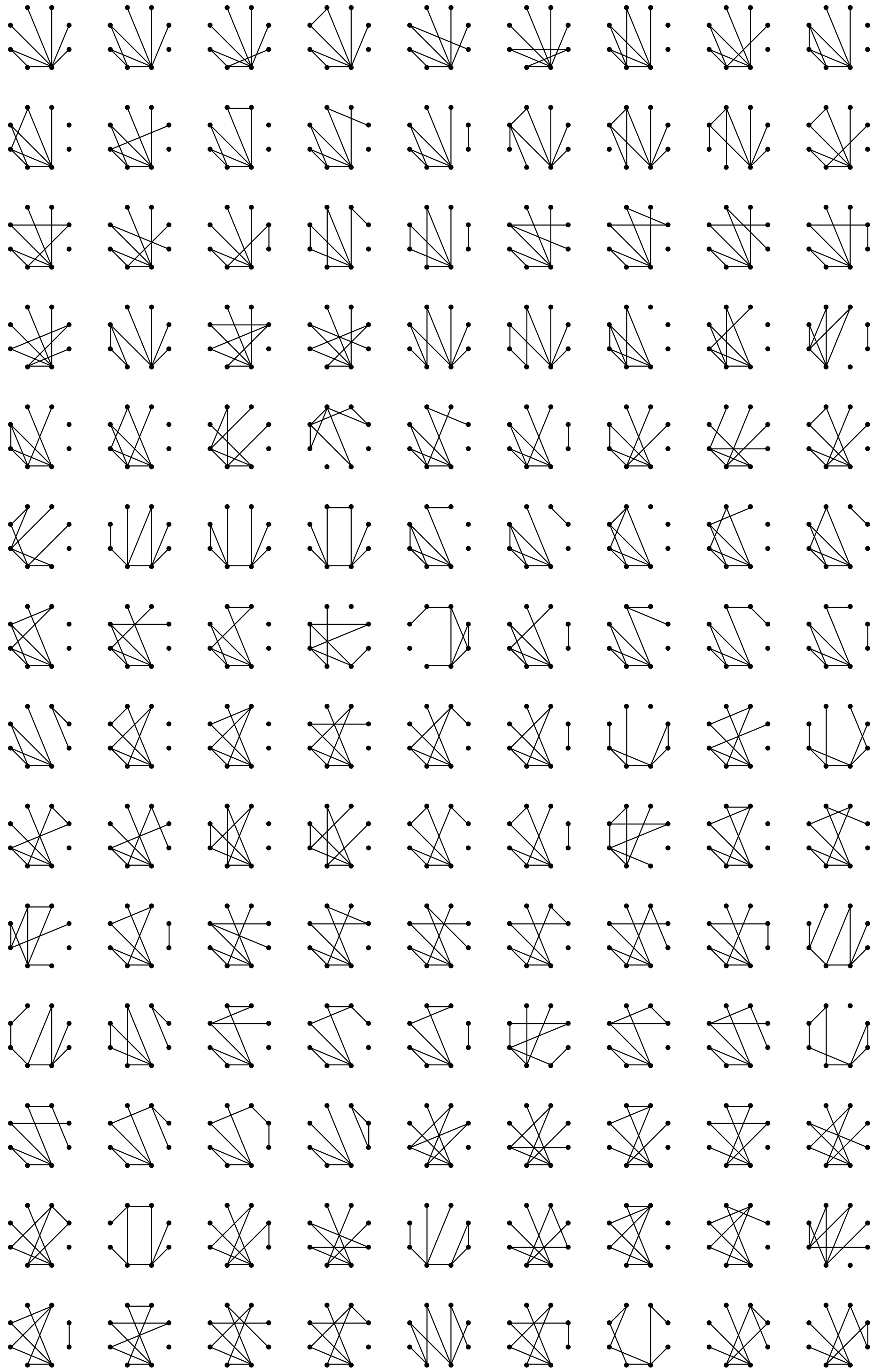


FIGURE 8.4. Graphs on 8 vertices with $e = 8$

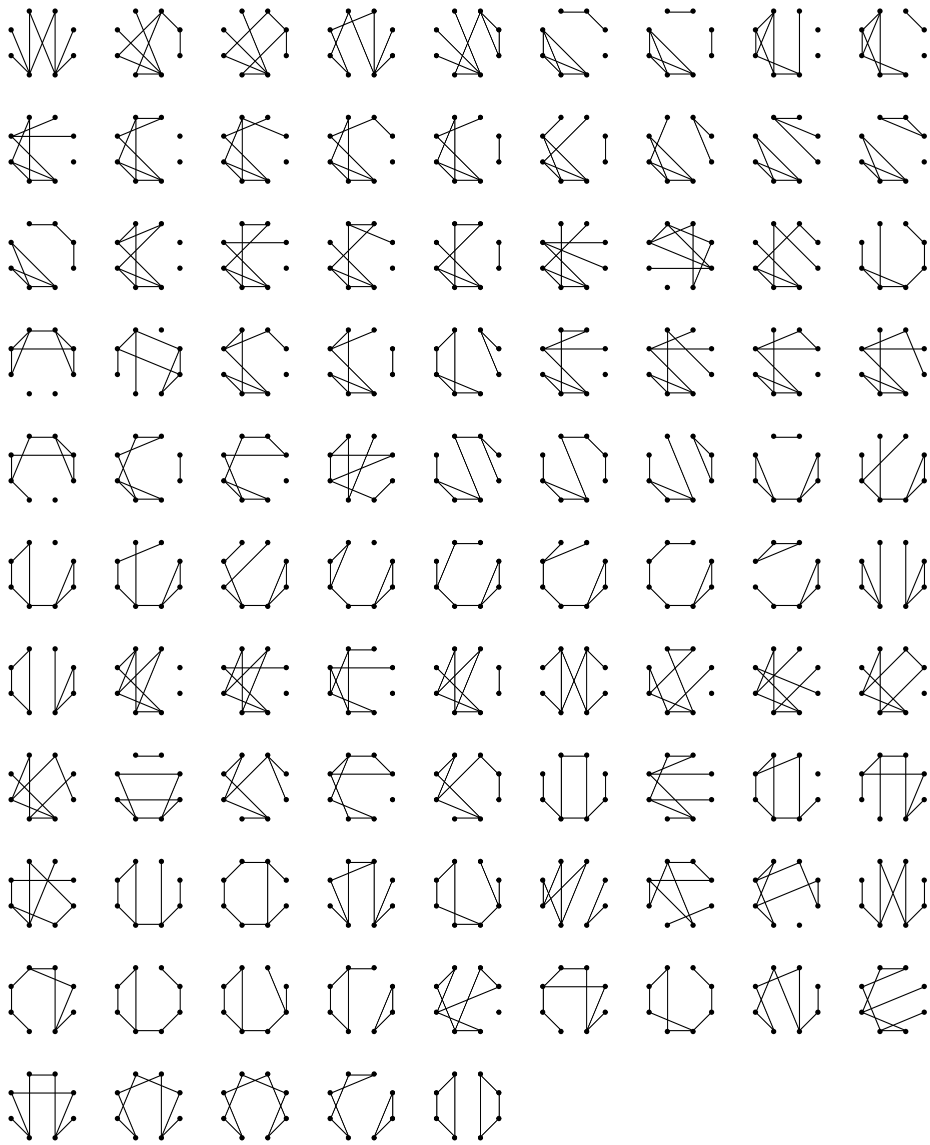


FIGURE 8.4. Graphs on 8 vertices with $e = 8$ (continued)

In Figure 8.4 we list the graphs with $e = 8$. Here 6 graphs are not relevant, 42 are reducible, and equivalences allow us to ignore a further 58 among the remainder. Again we indicate this information in the following scheme; the horizontal line denotes the break between pages.

irrel. 87	irrel. 87	194	irrel. 87	195	196	197	198	irrel. $1238\mid4567$
199	200	201	202	203	204	205	206	207
208	209	210	211	212	213	214	215	216
$(1267\mid3458)\mapsto 210$	217	218	219	220	221	irrel. $1238\mid4567$	$(1348\mid2567)\mapsto 201$	222
irrel. $1238\mid4567$	223	224	225	226	227	$(1567\mid2348)\mapsto 207$	228	$(1567\mid2348)\mapsto 213$
229	230	$(1234\mid5678)\downarrow$	231	$(1238\mid4567)\downarrow$	$(1238\mid4567)\downarrow$	232	$(1248\mid3567)\mapsto 202$	233
234	235	$(1238\mid4567)\mapsto 226$	236	$(1678\mid2345)\mapsto 245$	237	$(1238\mid4567)\downarrow$	$(1238\mid4567)\downarrow$	$(1238\mid4567)\downarrow$
$(1238\mid4567)\downarrow$	$(1258\mid3467)\mapsto 222$	238	$(1258\mid3467)\mapsto 233$	$(1256\mid3478)\mapsto 214$	239	240	241	242
$(1256\mid3478)\mapsto 215$	243	244	245	246	247	248	249	250
251	252	$(1258\mid3467)\mapsto 246$	$(1257\mid3468)\mapsto 237$	253	254	255	256	257
258	259	$(1278\mid3456)\mapsto 246$	$(1238\mid4567)\mapsto 257$	260	261	$(1248\mid3567)\downarrow$	$(1248\mid3567)\downarrow$	262
263	$(1248\mid3567)\downarrow$	$(1248\mid3567)\downarrow$	$(1238\mid4567)\downarrow$	$(1256\mid3478)\mapsto 212$	264	265	266	$(1256\mid3478)\mapsto 255$
$(1256\mid3478)\mapsto 216$	267	268	$(1456\mid2378)\mapsto 247$	269	$(1567\mid2348)\mapsto 268$	270	271	272
273	274	275	276	277	278	279	$(1567\mid2348)\downarrow$	$(1567\mid2348)\downarrow$

280	281	$(1348\mid2567)\mapsto 281$	282	$(1567\mid2348)\downarrow$	$(1238\mid4567)\downarrow$	$(1238\mid4567)\downarrow$	$(1234\mid5678)\mapsto 203$	$(1234\mid5678)\mapsto 239$
$(1248\mid3567)\downarrow$	$(1248\mid3567)\mapsto 227$	$(1248\mid3567)\mapsto 237$	$(1248\mid3567)\downarrow$	$(1248\mid3567)\downarrow$	$(1258\mid3467)\mapsto 274$	$(1238\mid4567)\downarrow$	$(1238\mid4567)\downarrow$	$(1238\mid4567)\downarrow$
$(1238\mid4567)\downarrow$	283	$(1245\mid3678)\mapsto 253$	$(1238\mid4567)\mapsto 254$	$(1245\mid3678)\mapsto 275$	$(1248\mid3567)\mapsto 262$	284	285	286
287	288	$(1248\mid3567)\mapsto 256$	$(1248\mid3567)\mapsto 278$	289	$(1248\mid3567)\mapsto 256$	$(1248\mid3567)\mapsto 263$	$(1248\mid3567)\downarrow$	$(1248\mid3567)\downarrow$
$(1238\mid4567)\mapsto 258$	290	$(1238\mid4567)\mapsto 286$	291	$(1238\mid4567)\downarrow$	$(1238\mid4567)\downarrow$	$(1238\mid4567)\downarrow$	292	$(1678\mid2345)\mapsto 259$
$(1234\mid5678)\downarrow$	$(1678\mid2345)\mapsto 289$	$(1678\mid2345)\mapsto 289$	$(1234\mid5678)\downarrow$	$(1678\mid2345)\downarrow$	$(1678\mid2345)\downarrow$	$(1678\mid2345)\downarrow$	$(1678\mid2345)\downarrow$	$(1234\mid5678)\downarrow$
$(1234\mid5678)\downarrow$	$\overline{(1236\mid4578)}\mapsto 231$	$(1245\mid3678)\mapsto 247$	$\overline{(1256\mid3478)}\mapsto 264$	$(1245\mid3678)\mapsto 260$	293	$(1367\mid2458)\mapsto 255$	294	295
$(1368\mid2457)\mapsto 297$	296	$(1567\mid2348)\mapsto 263$	$(1278\mid3456)\mapsto 285$	$(1567\mid2348)\mapsto 291$	$(1234\mid5678)\mapsto 300$	297	$(1234\mid5678)\mapsto 252$	298
299	$(1234\mid5678)\downarrow$	300	$(1345\mid2678)\mapsto 260$	301	$(1234\mid5678)\mapsto 292$	302	$\overline{(1456\mid2378)}\mapsto 275$	$(1458\mid2367)\mapsto 296$
$(1237\mid4568)\mapsto 290$	$(1234\mid5678)\downarrow$	$(1234\mid5678)\downarrow$	$(1234\mid5678)\downarrow$	303	$(1234\mid5678)\mapsto 305$	304	$(1234\mid5678)\mapsto 306$	305
$\overline{(1567\mid2348)}\mapsto 296$	306	307	308	$(1234\mid5678)\downarrow$				

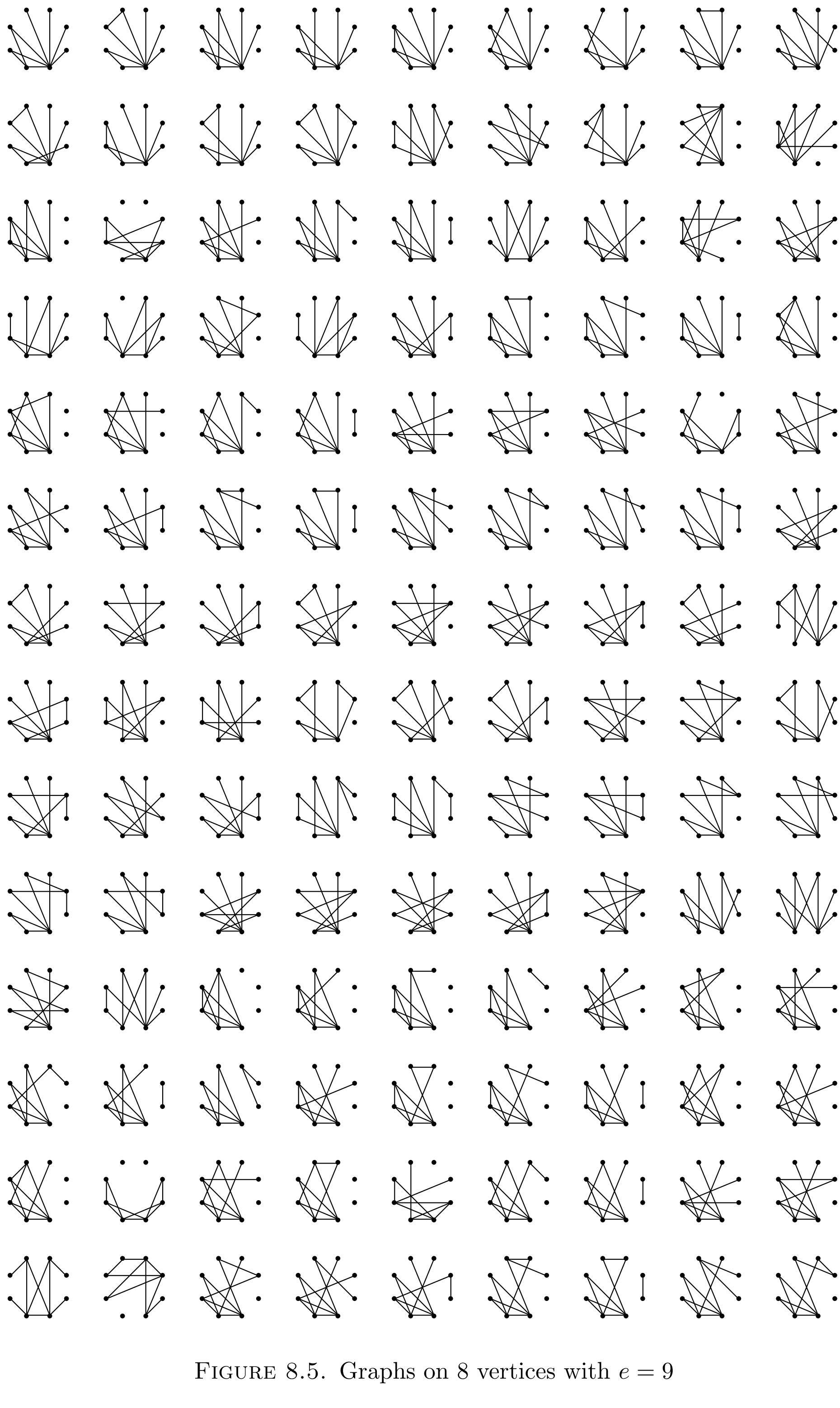


FIGURE 8.5. Graphs on 8 vertices with $e = 9$

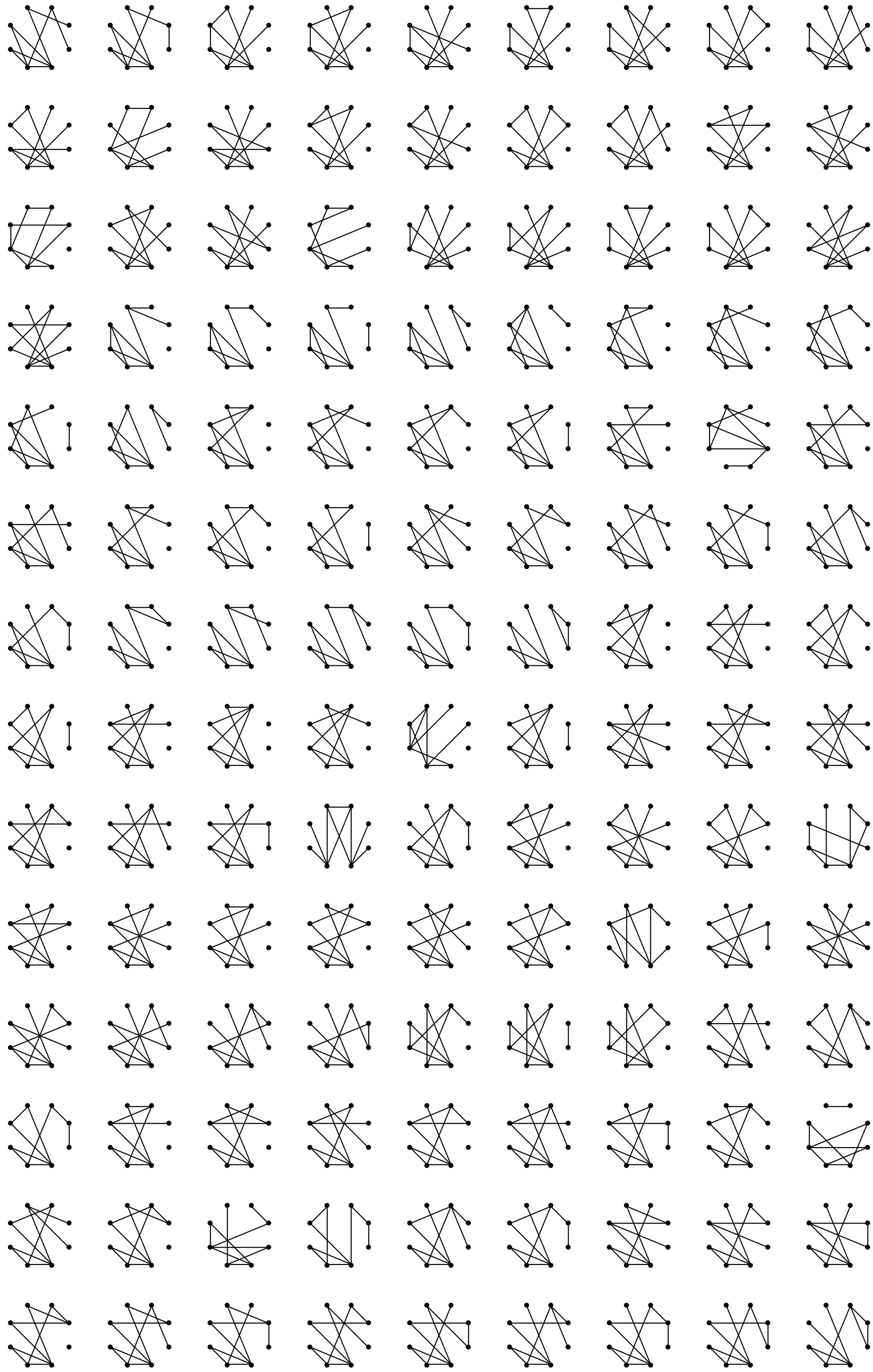


FIGURE 8.5. Graphs on 8 vertices with $e = 9$ (continued)

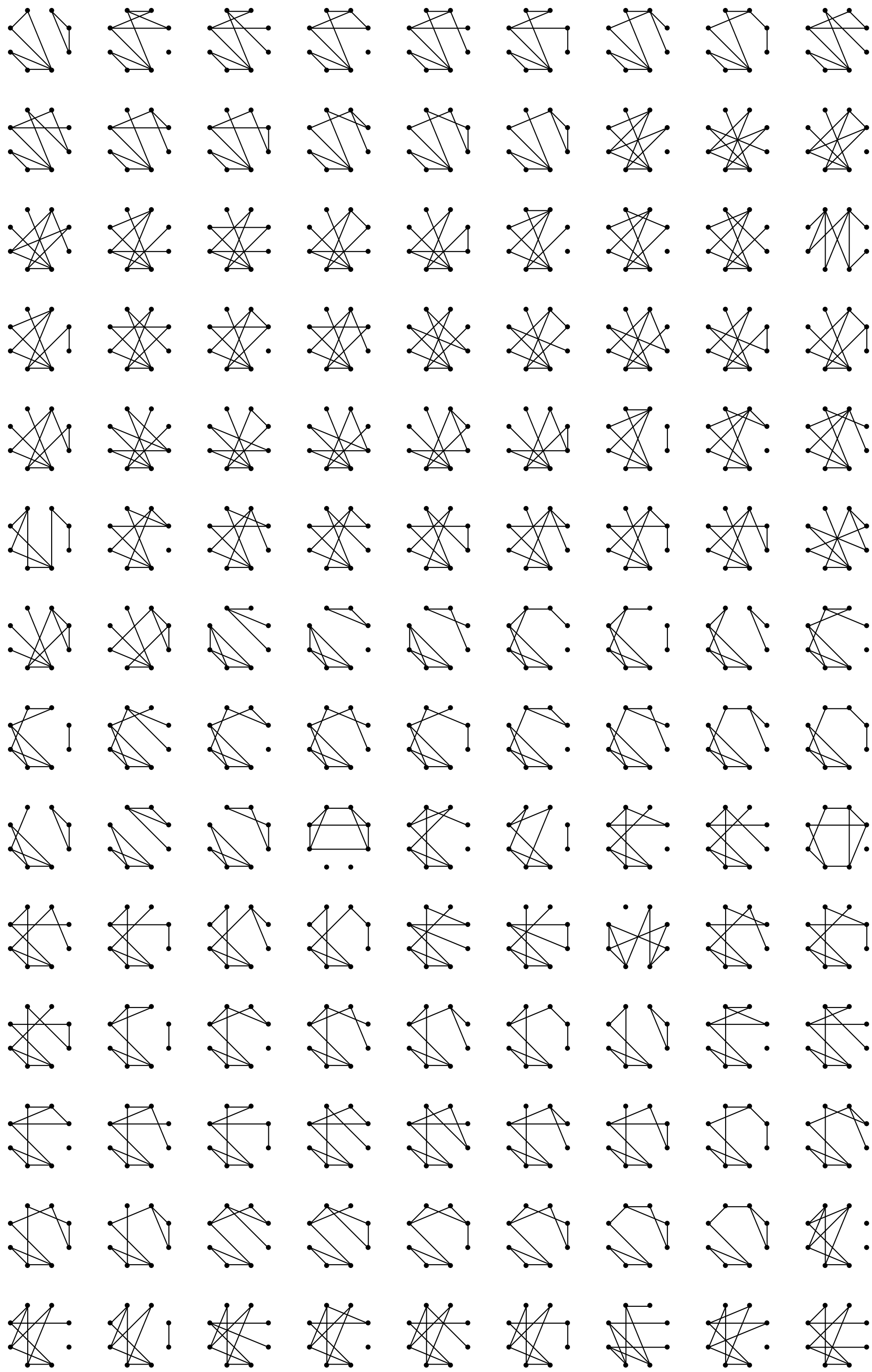


FIGURE 8.5. Graphs on 8 vertices with $e = 9$ (continued)

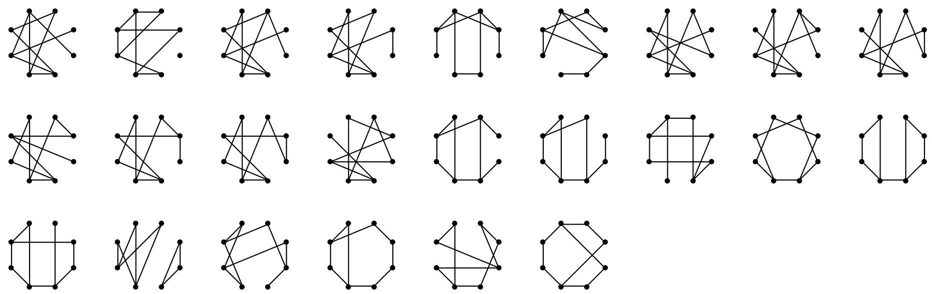


FIGURE 8.5. Graphs on 8 vertices with $e = 9$ (continued)

In Figure 8.5 we list the graphs with $e = 9$. This time 12 graphs are not relevant, 209 are reducible, and equivalences allow us to ignore a further 97 among the remainder. Again we indicate this information in the following scheme, where the horizontal lines denote the breaks between pages.

<table>
<tr><td>irrel.
87</td><td>irrel.
87</td><td>irrel.
87</td><td>309</td><td>irrel.
87</td><td>irrel.
87</td><td>310</td><td>irrel.
87</td><td>311</td></tr>
<tr><td>312</td><td>313</td><td>314</td><td>irrel.
87</td><td>315</td><td>316</td><td>317</td><td>318</td><td>319</td></tr>
<tr><td>irrel.
1238|4567</td><td>320</td><td>(1348|2567)
↦330</td><td>(1238|4567)
↦355</td><td>321</td><td>322</td><td>irrel.
1238|4567</td><td>323</td><td>(1236|4578)
↦312</td></tr>
<tr><td>324</td><td>325</td><td>326</td><td>327</td><td>328</td><td>(1238|4567)
↓</td><td>(1238|4567)
↓</td><td>(1238|4567)
↓</td><td>329</td></tr>
<tr><td>330</td><td>(1367|2458)
↦354</td><td>331</td><td>332</td><td>(1348|2567)
↦325</td><td>(1236|4578)
↦315</td><td>333</td><td>334</td><td>(1238|4567)
↦356</td></tr>
<tr><td>(1238|4567)
↦363</td><td>335</td><td>(1238|4567)
↓</td><td>(1238|4567)
↓</td><td>(1238|4567)
↓</td><td>(1238|4567)
↓</td><td>(1238|4567)
↓</td><td>(1238|4567)
↓</td><td>(1267|3458)
↓</td></tr>
<tr><td>(1267|3458)
↓</td><td>(1367|2458)
↦328</td><td>(1267|3458)
↓</td><td>(1267|3458)
↓</td><td>(1236|4578)
↦316</td><td>(1268|3457)
↦362</td><td>(1267|3458)
↓</td><td>(1267|3458)
↓</td><td>336</td></tr>
<tr><td>(1267|3458)
↓</td><td>337</td><td>338</td><td>339</td><td>340</td><td>(1267|3458)
↓</td><td>(1367|2458)
↦335</td><td>(1258|3467)
↦367</td><td>341</td></tr>
<tr><td>(1267|3458)
↦346</td><td>342</td><td>(1267|3458)
↦347</td><td>343</td><td>344</td><td>(1258|3467)
↓</td><td>(1258|3467)
↓</td><td>(1238|4567)
↦368</td><td>345</td></tr>
<tr><td>(1258|3467)
↓</td><td>(1258|3467)
↓</td><td>(1267|3458)
↓</td><td>346</td><td>347</td><td>(1267|3458)
↓</td><td>348</td><td>349</td><td>350</td></tr>
<tr><td>351</td><td>352</td><td>irrel.
1238|4567</td><td>irrel.
1238|4567</td><td>(1238|4567)
↓</td><td>(1238|4567)
↓</td><td>(1348|2567)
↓</td><td>(1248|3567)
↦325</td><td>(1248|3567)
↦334</td></tr>
<tr><td>(1348|2567)
↓</td><td>(1348|2567)
↓</td><td>(1238|4567)
↓</td><td>irrel.
1238|4567</td><td>(1238|4567)
↓</td><td>(1238|4567)
↓</td><td>(1238|4567)
↓</td><td>353</td><td>354</td></tr>
<tr><td>(1567|2348)
↦311</td><td>355</td><td>(1248|3567)
↦331</td><td>(1238|4567)
↦326</td><td>356</td><td>(1567|2348)
↦338</td><td>(1567|2348)
↦342</td><td>(1358|2467)
↦334</td><td>$\overline{(1678|2345)}$
↦333</td></tr>
<tr><td>357</td><td>358</td><td>(1358|2467)
↦363</td><td>(1256|3478)
↦359</td><td>(1238|4567)
↦365</td><td>(1238|4567)
↓</td><td>(1238|4567)
↓</td><td>(1238|4567)
↓</td><td>(1238|4567)
↓</td></tr>
</table>

<table>
<tr><td>(1238|4567)
↓</td><td>(1238|4567)
↓</td><td>(1567|2348)
↓</td><td>(1567|2348)
↦337</td><td>(1567|2348)
↦339</td><td>(1567|2348)
↦338</td><td>(1567|2348)
↦340</td><td>(1567|2348)
↓</td><td>(1567|2348)
↓</td></tr>
<tr><td>(1567|2348)
↦343</td><td>359</td><td>(1456|2378)
↦340</td><td>(1267|3458)
↦335</td><td>(1256|3478)
↓</td><td>(1256|3478)
↓</td><td>(1567|2348)
↓</td><td>(1356|2478)
↦332</td><td>(1268|3457)
↦371</td></tr>
<tr><td>360</td><td>(1256|3478)
↦361</td><td>(1256|3478)
↓</td><td>361</td><td>(1567|2348)
↓</td><td>(1567|2348)
↓</td><td>(1567|2348)
↓</td><td>(1567|2348)
↓</td><td>$\overline{(1234|5678)}$
↓</td></tr>
<tr><td>$\overline{(1234|5678)}$
↓</td><td>(1238|4567)
↓</td><td>(1238|4567)
↓</td><td>(1238|4567)
↓</td><td>(1238|4567)
↓</td><td>362</td><td>(1238|4567)
↦327</td><td>363</td><td>(1248|3567)
↓</td></tr>
<tr><td>(1248|3567)
↓</td><td>(1238|4567)
↓</td><td>(1238|4567)
↦328</td><td>(1235|4678)
↦340</td><td>(1238|4567)
↦335</td><td>(1235|4678)
↦345</td><td>$\overline{(1568|2347)}$
↦359</td><td>364</td><td>365</td></tr>
</table>

$(1348 \mid 2567) \mapsto 373$	$(1238 \mid 4567) \downarrow$	$(1238 \mid 4567) \downarrow$	$(1238 \mid 4567) \downarrow$	$(1238 \mid 4567) \downarrow$	$(1238 \mid 4567) \downarrow$	$(1238 \mid 4567) \downarrow$	$(1238 \mid 4567) \downarrow$	$(1238 \mid 4567) \downarrow$
$(1238 \mid 4567) \downarrow$	$(1238 \mid 4567) \downarrow$	$(1238 \mid 4567) \downarrow$	$(1238 \mid 4567) \downarrow$	$(1238 \mid 4567) \downarrow$	$(1238 \mid 4567) \downarrow$	$(1258 \mid 3467) \mapsto 321$	$(1258 \mid 3467) \downarrow$	$(1256 \mid 3478) \downarrow$
$(1258 \mid 3467) \downarrow$	$(1258 \mid 3467) \mapsto 332$	366	$(1235 \mid 4678) \mapsto 342$	367	$(1235 \mid 4678) \mapsto 351$	$(1258 \mid 3467) \downarrow$	$(1257 \mid 3468) \downarrow$	$(1258 \mid 3467) \downarrow$
$(1256 \mid 3478) \mapsto 342$	$(1257 \mid 3468) \downarrow$	$(1258 \mid 3467) \downarrow$	368	$(1256 \mid 3478) \mapsto 345$	$(1268 \mid 3457) \mapsto 365$	$(1256 \mid 3478) \downarrow$	$(1256 \mid 3478) \downarrow$	369
$\overline{(1234 \mid 5678)} \mapsto 360$	$(1268 \mid 3457) \mapsto 373$	$\overline{(1346 \mid 2578)} \mapsto 342$	$(1358 \mid 2467) \mapsto 364$	$(1268 \mid 3457) \mapsto 383$	$(1256 \mid 3478) \mapsto 341$	370	$(1248 \mid 3567) \mapsto 387$	$(1256 \mid 3478) \downarrow$
$(1256 \mid 3478) \downarrow$	$(1248 \mid 3567) \mapsto 383$	$(1256 \mid 3478) \mapsto 345$	$(1238 \mid 4567) \mapsto 384$	371	372	373	$(1257 \mid 3468) \downarrow$	$(1567 \mid 2348) \downarrow$
$(1567 \mid 2348) \downarrow$	$(1278 \mid 3456) \mapsto 371$	$(1278 \mid 3456) \mapsto 373$	$\overline{(1234 \mid 5678)} \mapsto 387$	$(1248 \mid 3567) \downarrow$	$(1248 \mid 3567) \downarrow$	$(1248 \mid 3567) \downarrow$	$\overline{(1346 \mid 2578)} \mapsto 346$	374
$(1258 \mid 3467) \mapsto 369$	$(1248 \mid 3567) \mapsto 370$	375	376	$(1248 \mid 3567) \downarrow$	$(1248 \mid 3567) \downarrow$	$(1258 \mid 3467) \downarrow$	$(1248 \mid 3567) \downarrow$	$(1248 \mid 3567) \downarrow$
$\overline{(1678 \mid 2345)} \mapsto 345$	$(1257 \mid 3468) \downarrow$	$(1258 \mid 3467) \downarrow$	$(1238 \mid 4567) \mapsto 386$	$(1258 \mid 3467) \downarrow$	$(1248 \mid 3567) \downarrow$	$(1248 \mid 3567) \downarrow$	$(1248 \mid 3567) \downarrow$	$(1248 \mid 3567) \downarrow$

$(1238 \mid 4567) \downarrow$	$(1278 \mid 3456) \downarrow$	$(1278 \mid 3456) \mapsto 369$	$(1278 \mid 3456) \downarrow$	$(1248 \mid 3567) \downarrow$	$(1248 \mid 3567) \downarrow$	$(1248 \mid 3567) \downarrow$	$(1238 \mid 4567) \downarrow$	$(1248 \mid 3567) \downarrow$
$(1248 \mid 3567) \downarrow$	$(1248 \mid 3567) \downarrow$	$(1248 \mid 3567) \downarrow$	$(1238 \mid 4567) \downarrow$	$(1238 \mid 4567) \downarrow$	$(1248 \mid 3567) \downarrow$	$\overline{(1234 \mid 5678)} \downarrow$	$(1256 \mid 3478) \downarrow$	$(1256 \mid 3478) \downarrow$
$(1256 \mid 3478) \mapsto 344$	$\overline{(1234 \mid 5678)} \downarrow$	$\overline{(1234 \mid 5678)} \downarrow$	$(1256 \mid 3478) \mapsto 344$	$\overline{(1234 \mid 5678)} \mapsto 385$	$\overline{(1234 \mid 5678)} \downarrow$	$\overline{(1234 \mid 5678)} \downarrow$	$\overline{(1234 \mid 5678)} \downarrow$	377
$(1467 \mid 2358) \mapsto 372$	$\overline{(1234 \mid 5678)} \downarrow$	$\overline{(1234 \mid 5678)} \mapsto 347$	$(1368 \mid 2457) \mapsto 375$	$(1256 \mid 3478) \downarrow$	$(1256 \mid 3478) \downarrow$	$(1256 \mid 3478) \mapsto 375$	$\overline{(1234 \mid 5678)} \mapsto 386$	$(1567 \mid 2348) \downarrow$
$(1567 \mid 2348) \downarrow$	$\overline{(1234 \mid 5678)} \downarrow$	$(1456 \mid 2378) \downarrow$	$(1456 \mid 2378) \mapsto 372$	$(1567 \mid 2348) \downarrow$	$(1567 \mid 2348) \downarrow$	378	$\overline{(1234 \mid 5678)} \mapsto 349$	379
380	$\overline{(1234 \mid 5678)} \mapsto 351$	$(1257 \mid 3468) \downarrow$	$(1258 \mid 3467) \mapsto 382$	$(1258 \mid 3467) \downarrow$	$(1248 \mid 3567) \mapsto 379$	$(1258 \mid 3467) \mapsto 374$	$(1258 \mid 3467) \mapsto 382$	$\overline{(1678 \mid 2345)} \mapsto 382$
$(1567 \mid 2348) \downarrow$	$(1567 \mid 2348) \downarrow$	$(1238 \mid 4567) \downarrow$	$(1238 \mid 4567) \downarrow$	$(1238 \mid 4567) \downarrow$	$(1238 \mid 4567) \downarrow$	$(1238 \mid 4567) \downarrow$	$(1238 \mid 4567) \downarrow$	$(1238 \mid 4567) \downarrow$
$(1238 \mid 4567) \downarrow$	$(1238 \mid 4567) \downarrow$	$(1238 \mid 4567) \downarrow$	$(1238 \mid 4567) \downarrow$	$(1238 \mid 4567) \downarrow$	$(1238 \mid 4567) \downarrow$	$(1238 \mid 4567) \downarrow$	$(1238 \mid 4567) \downarrow$	$(1238 \mid 4567) \downarrow$
$(1238 \mid 4567) \downarrow$	$(1238 \mid 4567) \downarrow$	$(1238 \mid 4567) \downarrow$	381	$(1258 \mid 3467) \mapsto 361$	382	$(1258 \mid 3467) \downarrow$	$(1258 \mid 3467) \downarrow$	383
$(1257 \mid 3468) \mapsto 386$	$(1258 \mid 3467) \downarrow$	$(1348 \mid 2567) \downarrow$	$(1348 \mid 2567) \downarrow$	$(1258 \mid 3467) \downarrow$	$(1258 \mid 3467) \downarrow$	384	$\overline{(1568 \mid 2347)} \mapsto 390$	$(1258 \mid 3467) \downarrow$
$(1258 \mid 3467) \downarrow$	$(1238 \mid 4567) \mapsto 380$	$(1278 \mid 3456) \downarrow$	$(1248 \mid 3567) \mapsto 376$	$(1248 \mid 3567) \downarrow$	$(1248 \mid 3567) \downarrow$	$(1238 \mid 4567) \downarrow$	$(1278 \mid 3456) \downarrow$	$(1238 \mid 4567) \mapsto 376$
$(1278 \mid 3456) \downarrow$	$(1248 \mid 3567) \downarrow$	$(1248 \mid 3567) \downarrow$	$(1248 \mid 3567) \downarrow$	$(1248 \mid 3567) \downarrow$	$(1248 \mid 3567) \downarrow$	$(1248 \mid 3567) \downarrow$	$(1248 \mid 3567) \downarrow$	$(1248 \mid 3567) \downarrow$
$(1248 \mid 3567) \downarrow$	$(1248 \mid 3567) \downarrow$	$(1278 \mid 3456) \downarrow$	$(1238 \mid 4567) \downarrow$	$(1278 \mid 3456) \downarrow$	$(1238 \mid 4567) \downarrow$	$(1238 \mid 4567) \downarrow$	$(1238 \mid 4567) \downarrow$	$\overline{(1237 \mid 4568)} \downarrow$
$\overline{(1237 \mid 4568)} \downarrow$	$\overline{(1237 \mid 4568)} \downarrow$	$(1245 \mid 3678) \downarrow$	$(1245 \mid 3678) \mapsto 369$	$\overline{(1237 \mid 4568)} \downarrow$	$(1245 \mid 3678) \downarrow$	385	$\overline{(1237 \mid 4568)} \downarrow$	386

$(1358 \mid 2467) \downarrow$	387	$\overline{(1237 \mid 4568)} \downarrow$	$(1358 \mid 2467) \downarrow$	388	389	$(1457 \mid 2368) \mapsto 390$	$(1248 \mid 3567) \mapsto 376$	$\overline{(1237 \mid 4568)} \downarrow$
$(1248 \mid 3567) \downarrow$	$(1248 \mid 3567) \downarrow$	$(1248 \mid 3567) \downarrow$	390	$(1234 \mid 5678) \downarrow$	$(1234 \mid 5678) \downarrow$	$(1458 \mid 2367) \mapsto 389$	391	$(1234 \mid 5678) \downarrow$
$(1234 \mid 5678) \downarrow$	$(1234 \mid 5678) \downarrow$	$(1234 \mid 5678) \downarrow$	$(1234 \mid 5678) \downarrow$	392	$\overline{(1357 \mid 2468)} \downarrow$			

This completes the lists of all graphs for given values of e. In Figure 8.6 we list the remaining graphs with which we shall be concerned; those with $e = 10$ occur in the first four rows, while those with $e \geq 11$ occupy the last. We number these

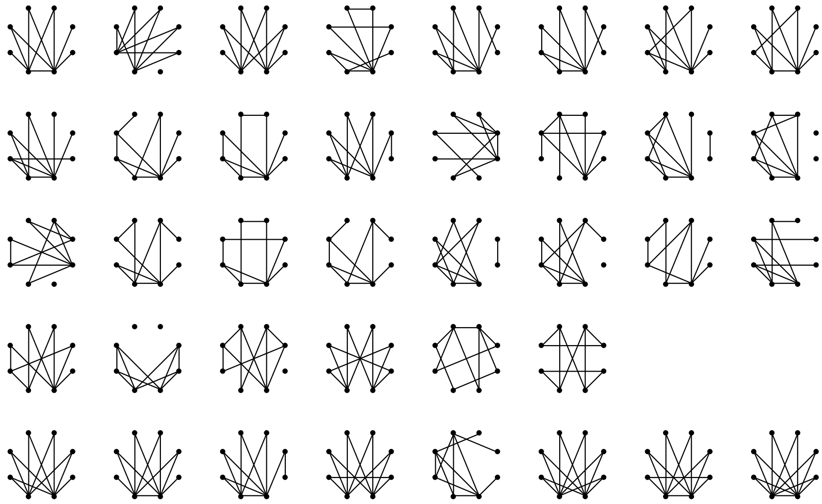


FIGURE 8.6. Rigs with $e \geq 10$ up to equivalence

graphs according to the following scheme.

393	394	395	396	397	398	399	400
401	402	403	404	405	406	407	408
409	410	411	412	413	414	415	416
417	418	419	420	421	422		
423	424	425	426	427	428	429	430

Our proof that the graphs in Figure 8.6 complete the list of equivalence classes of relevant graphs will take some time. In what follows the graph numbered i in the scheme above will be denoted Γ_i; we describe graphs equivalent to those in Figure 8.6 as *known*.

Thus let Γ be a rig with at least 10 edges. Recall that Γ can contain no heavy or light partition, and can have no K_4 subgraph. We shall call a set of edges *parallel* if no two of them have an endpoint in common; note that Γ must contain a pair of parallel edges, because otherwise any two edges would meet, and so if $12, 13$ were edges then any other edge would have to be either 23 or $1i$ for some $i \geq 4$, contrary to $e \geq 10$.

LEMMA 8.2. *If $e \geq 10$ and Γ does not have three parallel edges, then Γ is known.*

PROOF. Suppose $e \geq 10$ and Γ does not have three parallel edges; assume Γ has edges $12, 34$, so that it can have no edge among $\{5, 6, 7, 8\}$. Of the $e - 2$ other edges there can be at most three among $\{1, 2, 3, 4\}$ (to avoid a K_4), so there must be at least five from $\{1, 2, 3, 4\}$ to $\{5, 6, 7, 8\}$.

Suppose if possible both 1 and 2 were joined to $\{5, 6, 7, 8\}$; since we cannot have edges $1i, 2j$ with $i, j \geq 5$ and $i \neq j$ (else they and 34 would be parallel), we would have say $15, 25$ and no edge $1i$ or $2i$ for $i \geq 6$. Thus there must be at least three edges from $\{3, 4\}$ to $\{5, 6, 7, 8\}$, so by the same argument we cannot have both 3 and 4 joined to $\{5,6,7,8\}$; say there is no edge $4i$ with $i \geq 5$ and we have at least three edges from 3 to $\{5,6,7,8\}$. We cannot have the edge 14 as it would be parallel to both 25 and some edge $3j$ with $j \neq 5$; similarly 24 cannot be an edge. Thus we must have $e = 10$ and the remaining edges are $13, 23, 35, 36, 37, 38$; but then 3 and 4 are joined to all and none respectively of the other six vertices.

So we cannot have both 1 and 2 joined to $\{5,6,7,8\}$, and likewise we cannot have both 3 and 4 joined to $\{5,6,7,8\}$; say 1 and 4 are not joined to $\{5,6,7,8\}$. Thus we must have at least five edges from $\{2,3\}$ to $\{5,6,7,8\}$, of which two will be parallel, so we cannot have 14. Consequently all edges are either 23 or from $\{2,3\}$ to $\{1,4,5,6,7,8\}$; let z be the number of the latter which are missing. First assume 23 is an edge; then $z \leq 3$. If $z = 0$ we have Γ_{430}, so assume $2i$ is missing. Since then i is joined to no vertex with the possible exception of 3, for Γ to be relevant some edge $3j$ with $j \neq i$ must be missing; thus if $e = 11$ we have Γ_{424}, while if $e = 10$ the other missing edge cannot be $2j$ or $3i$ so we may assume it is $2k$ for some $k \notin \{i,j\}$, giving Γ_{393}. Thus we may assume 23 is not an edge, so that $z \leq 2$. If $z = 0$ we have Γ_{428}, so assume $2i$ is missing; if $z = 1$ we have Γ_{423}, so assume $z = 2$; if the other missing edge is $3i$ we have Γ_{394}, if it is $3j$ for $j \neq i$ we have Γ_{395}, while if it is $2j$ for $j \neq i$ then applying $\overline{(1klm|23ij)}$ (where $\{i,j,k,l,m\} = \{4,5,6,7,8\}$) gives Γ_{394}. □

We may therefore assume for the rest of the analysis that Γ has three parallel edges, which we may take to be $12, 34, 56$. As we proceed we shall frequently find that the presence of one particular edge or set of edges means that we may assume that another is absent, for example because it would make a particular partition heavy, or because if it were present we could reduce to a case already considered; we shall say that the first *excludes* the second. We shall use the symbol $\square|2$ for a full partition $abcd|efgh$ which contains edges ab, ac, bd, cd and two among $\{e,f,g,h\}$; note that applying the corresponding bifid map $(abcd|efgh)$ produces a graph containing four edges among $\{e,f,g,h\}$, of which two must be parallel, together with ad and bc, and so any graph with a $\square|2$ partition is equivalent to one with four parallel edges (and the same value of e).

We divide into two cases: $e = 10$ and $e \geq 11$. We begin with the case $e \geq 11$. Recall that we assume $12, 34, 56$ are edges. Consider the three partitions $1234|5678$, $1256|3478$ and $1278|3456$, which between them contain each of the other edges exactly once, with the possible exception of 78. If 78 were an edge we would have $e - 4 \geq 7$ other edges, so that one of the partitions would have to contain at least three of the other edges and thus be heavy. Consequently

$$\Gamma \text{ cannot have four parallel edges;}$$

therefore it can also contain no $\square|2$ partition. There are thus $e - 3$ other edges; as each of the partitions can contain at most three more edges, we have $e - 3 \leq 9$ and $e = 11$ or 12. We may therefore assume that $1256|3478$ and $1278|3456$ contain three more edges each, and $1234|5678$ contains the remaining $e - 9$. Note that there cannot be a pair of parallel edges among $\{1,2,7,8\}$, $\{3,4,7,8\}$ or $\{5,6,7,8\}$; in particular, each can contain at most two of the other edges. Let n be the number of edges from $\{5,6\}$ to $\{7,8\}$, so that $0 \leq n \leq 2$.

For $5 \leq i \leq 8$ let x_i and y_i be the numbers of edges from i to $\{1,2\}$ and $\{3,4\}$ respectively; thus $\sum_{i=5}^{8}(x_i + y_i) = 6$. Let $\{a,b,c,d\} = \{5,6,7,8\}$. If ab is an edge then each of $12ab|34cd$ and $12cd|34ab$ must contain three of the six edges between

$\{1,2,3,4\}$ and $\{5,6,7,8\}$, so that $x_a + x_b + y_c + y_d = 3 = y_a + y_b + x_c + x_d$. If ac is also an edge, we also have $x_a + x_c + y_b + y_d = 3 = y_a + y_c + x_b + x_d$, so that $x_b - y_b = x_c - y_c$ and $x_a - y_a = x_d - y_d$. We shall call $x_i - y_i$ the *bias* of the vertex i; thus if ab and ac are edges then a and d have the same bias, as do b and c.

We now work through the possible values of n.

LEMMA 8.3. *With the notation above, if $n = 2$ then Γ is known.*

PROOF. The two edges from $\{5,6\}$ to $\{7,8\}$, together with 56, ensure that all four of these vertices have the same bias, which cannot then be ± 2; interchanging $\{1,2\}$ and $\{3,4\}$ if necessary we may assume the common bias is 0 or 1. Moreover, as there are three edges among $\{5,6,7,8\}$, no two of which are parallel, there must be one from each parallel pair.

First assume the bias is 1. This requires a total of five edges from $\{5,6,7,8\}$ to $\{1,2\}$ and just one to $\{3,4\}$; we may assume the latter is $3a$, and then as a has bias 1 we must have both edges $1a$ and $2a$. Deleting the vertices 3 and 4 removes the edges 34 and $3a$, together with $e - 11$ edges from $\{1,2\}$ to $\{3,4\}$; this gives a graph Γ' on six vertices with nine edges, no three of which are parallel. In Γ' write $d'(z)$ for the degree of a vertex z, and let x be a vertex of minimal degree; then we cannot have $d'(x) = 0$, as nine edges on five vertices would force a K_4. If y is a vertex adjacent to x, then among the remaining four vertices there can be at most one edge present from each parallel pair; thus $d'(x) + d'(y) \geq 7$, and as $d'(y) \leq 5$ this forces $d'(x) \geq 2$. If we had $d'(x) \geq 3$ the degree sum in Γ' would be at least $7 + 4.3 = 19$, contrary to assumption; so we must have $d'(x) = 2$, and $d'(y_1) = d'(y_2) = 5$ where y_1 and y_2 are the two vertices adjacent to x. Thus if $x_1 = x, x_2, x_3, x_4$ are the four vertices other than y_1 and y_2, the edges are y_1y_2 and x_iy_j for all $i \leq 4$ and $j \leq 2$. Since in this graph no triangle is disjoint from an edge, after interchanging 5 and 6 if necessary the three edges among $\{5,6,7,8\}$ (which are disjoint from 12) must be $56, 57, 58$; so after interchanging 1 and 2 if necessary we have $\{x_1, x_2, x_3, x_4\} = \{2,6,7,8\}$, $\{y_1, y_2\} = \{1,5\}$. Now replace the deleted vertices and edges to recover Γ; since a was joined to both 1 and 2 as well as 3, we must have $a = 5$. Thus if $e = 11$ we have Γ_{425}, while if $e = 12$ there is also an edge from $\{1,2\}$ to $\{3,4\}$, which cannot be 13 or 24 (else $1458|2367$ would be light) or 23 (else $1245|3678$ would be light), so must be 14, giving Γ_{429}.

Now assume the bias is 0. First suppose some vertex, say a, is adjacent to all of $\{1,2,3,4\}$; then some other vertex, say b, must be adjacent to one each of $\{1,2\}$ and $\{3,4\}$, say to 2 and 3. Since $12, 3b, 4a$ are parallel they exclude cd, so we must have ab. However, the pair of remaining edges among $\{5,6,7,8\}$ cannot then be bc, bd (as then $124a|3bcd$ would be heavy), so we may assume we have ac, and the remaining edge is ad or bc; but then a and d are joined to all and none respectively of the other six vertices. Thus instead there must be three vertices, say a, b, c, each joined to one of $\{1,2\}$ and one of $\{3,4\}$. Next suppose some vertex of $\{1,2,3,4\}$, say 1, is joined to all three of a, b, c; interchanging 3 and 4 if necessary we must have edges $3a, 3b$ and either $3c$ or $4c$. However, if $4c$ were an edge then $12, 3b, 4c$ would be parallel, excluding ad and forcing bc to be an edge; but then $34bc|12ad$ would be $\square|2$. Thus we must have $3c$. Now the three edges among $\{5,6,7,8\}$ cannot be ab, ac, bc (else we would have a K_4 among $\{1, a, b, c\}$), ad, bd, cd (else $123d|4abc$ would be light), or bc, bd, cd (else $124a|3bcd$ would be heavy), so they must be ab, ac, ad. Since we cannot add the edge 13 (else we would have a K_4 among $\{1,3,a,b\}$), $23, 24$ (else $1acd|234b$ would be heavy) or 14 (else $124b|3acd$ would be heavy), we must have

$e = 11$, which gives Γ_{427}. Thus finally we may assume (after interchanging 1 and 2, and 3 and 4, if necessary), that $1, 2, 3, 4$ are joined to two, one, two and one respectively of $\{5, 6, 7, 8\}$; suppose the edges are $1a, 1b, 2c, 3a$ and either $3b, 4c$ or $3c, 4b$. We cannot have $3b, 4c$ (else $13ab|24cd$ would be heavy, since ab or cd must be present), so we must have $3c, 4b$; but then $1b, 2c, 34$ are parallel and so exclude ad, so we must have bc, and now $12bc|34ad$ is $\square|2$, a contradiction. □

Lemma 8.4. *With the notation above, if $n = 1$ then Γ is known.*

Proof. We may assume that the edges in $1234|5678$ are $12, 23, 34, 56, 67$, together with an extra edge if $e = 12$, which cannot be 14 (else $1234|5678$ would be $\square|2$) and so must be 13 or 24. Thus 5 and 7 have the same bias, as do 6 and 8. We know that among each of $\{1, 2, 7, 8\}$ and $\{3, 4, 7, 8\}$ we cannot have a pair of parallel edges, so that an edge $i8$ excludes $j7$, where $\{i, j\} = \{1, 2\}$ or $\{3, 4\}$; interchanging 5 and 7 we see that $i8$ also excludes $j5$. The partitions $1357|2468$ and $1368|2457$ each contain $e - 11$ of the edges in $1234|5678$, while the other six edges each lie in precisely one of them; thus for neither to be light, each must contain three of the edges between $\{1, 2, 3, 4\}$ and $\{5, 6, 7, 8\}$. Thus of the six edges ij with $i < 5 \leq j$, three have $i+j$ even and three have $i+j$ odd; we shall call this the *parity condition*.

First suppose 8 has positive degree; interchanging $1, 2$ and $4, 3$ if necessary, we may assume one of the following holds: (i) $28, 38$ are edges; (ii) 28 is not an edge but 38 is; (iii) $28, 38$ are not edges but $18, 48$ are; (iv) $18, 28, 38$ are not edges but 48 is.

Suppose (i) holds; then this excludes $15, 17, 45, 47$ (by parallels), $16, 48$ (else $1567|2348$ would be heavy), and $46, 18$ (else $1238|4567$ would be heavy), so that 8 has bias 0 and hence $x_6 = y_6 \leq 1$. We must have 26 or 36 (else $1578|2346$ would be light), so both 26 and 36 must be edges and we have two of $25, 27, 35, 37$, so we may assume 25 is an edge; but this excludes 27 (else $1348|2567$ would be heavy), 35 (else we would have a K_4 among $\{2, 3, 5, 6\}$) and 37 (else $1256|3478$ would be heavy), a contradiction.

Suppose (ii) holds; this excludes $45, 47$ (by parallels) and 18 (else $1238|4567$ would be $\square|2$). If 48 were also an edge this would exclude $35, 37$ (by parallels) and $15, 16, 17$ (else $1567|2348$ would be heavy); so by parity we must have $26, 46$, and hence not 25 (else $1256|3478$ would be heavy) but $27, 36$. However, this would mean that 5 and 7 do not have the same bias. Thus 48 cannot be an edge, and so $y_6 - x_6 = y_8 - x_8 = 1$; so we must have either $x_6 = 1$, $y_6 = 2$ or $x_6 = 0$, $y_6 = 1$.

First suppose $x_6 = 1$, $y_6 = 2$, so that we have $36, 46$ and either 16 or 26. If we have 26 this excludes 15 (else $1256|3478$ would be $\square|2$); by parity we must have either 25 or 27, so we may assume 25 is an edge, but this excludes 17 (by parallels), 35 (else we would have a K_4 among $\{2, 3, 5, 6\}$) and 37 (else $1256|3478$ would be heavy). So we must have 16 instead, which excludes $15, 17$ (else $1567|2348$ would be heavy) and $25, 27$ (by parity), and thus we must have $35, 37$. Here we must have $e = 11$, because adding 24 would make $1567|2348$ heavy and adding 13 would give 3 and 8 joined to all and none respectively of the other six vertices; applying $(1238|4567)$ gives Γ_{425}.

Now suppose $x_6 = 0$, $y_6 = 1$, so that $16, 26$ are not edges and we have one of $36, 46$. We thus have four of $15, 17, 25, 27, 35, 37$; since we cannot have both of $15, 17$ (else $1567|2348$ would be heavy) or both of $25, 27$ (else $2567|1348$ would be $\square|2$), we must have $35, 37$, together with one of $15, 17$ and one of $25, 27$. This excludes 46 by parity, so we must have 36. We may assume 15 is an edge; this excludes 25 (else $1245|3678$ would be heavy), so we must have 27. Again we must have $e = 11$, because adding 24 would make $1567|2348$ heavy and adding 13 would give 3 and 8 joined to all and none respectively of the other six vertices; applying $(1248|3567)$ gives Γ_{427}.

Suppose (iii) holds; this excludes $15, 17, 25, 27, 35, 37, 45, 47$ (by parallels), so we must have $16, 26, 36, 46$. Again we must have $e = 11$, because adding 13 or 24 would give a K_4 among $\{1, 2, 3, 6\}$ or $\{2, 3, 4, 6\}$; applying $(1236|4578)$ gives Γ_{427}.

Suppose (iv) holds; this excludes $15, 17, 35, 37$ (by parallels), so by parity we must have $26, 46$. Since $y_6 - x_6 = y_8 - x_8 = 1$ we must have 36 and not 16; thus we must have two of $25, 27, 45, 47$. We cannot have both of $25, 27$ (else $1348|2567$ would be heavy) or both of $45, 47$ (else $1238|4567$ would be heavy), so we must have one of $25, 27$ and one of $45, 47$. We may assume 25 is an edge; this excludes 47 (else $1256|3478$ would be heavy), so we must have 45. Once more we must have $e = 11$, because adding 13 would make $1236|4578$ heavy and adding 24 would give a K_4 among $\{2, 3, 4, 6\}$; so we have Γ_{427}.

Thus we may assume 8 has degree 0; so $x_6 - y_6 = x_8 - y_8 = 0$. Also 6 cannot now be joined to all vertices except 8, so as $56, 67$ are edges we must have $x_6 = y_6 \leq 1$. Again interchanging $1, 2$ and $4, 3$ if necessary, we may assume one of the following holds: (a) $26, 36$ are edges and $16, 46$ are not; (b) $16, 36$ are edges and $26, 46$ are not; (c) $16, 46$ are edges and $26, 36$ are not; (d) $16, 26, 36, 46$ are not edges.

Suppose (a) holds. We cannot have all of $15, 17, 45, 47$ (else $1457|2368$ would be heavy), so there is some edge from $\{2, 3\}$ to $\{5, 7\}$, which we may assume to be 25; this excludes 35 (else we would have a K_4 among $\{2, 3, 5, 6\}$). By parity we must have two of $15, 17, 37$, so as we cannot have both of $15, 37$ (else $1256|3478$ would be heavy) we must have 17, which excludes 15 (else $1567|2348$ would be $\Box|2$), 45 (else $1267|3458$ would be $\Box|2$) and 47 (else $1478|2356$ would be heavy), so we must have $27, 37$; but now we have a K_4 among $\{2, 3, 6, 7\}$.

Suppose (b) holds. By parity we need three of $15, 17, 35, 37$, and since we cannot have both of $15, 17$ (else $1567|2348$ would be heavy) we must have $35, 37$ and one of $15, 17$, which we may assume to be 15; this excludes 27 (else $1267|3458$ would be $\Box|2$), $25, 47$ (else $1256|3478$ would be heavy), so we must have 45. Once more we must have $e = 11$, because adding 24 would make $1678|2345$ heavy and adding 13 would give a K_4 among $\{1, 3, 5, 6\}$; applying $(1258|3467)$ gives Γ_{427}.

Suppose (c) holds. We cannot have both of $15, 17$ (else $1567|2348$ would be heavy) or both of $45, 47$ (else $1238|4567$ would be heavy), so we must have at least two of $25, 27, 35, 37$. We may assume 25 is an edge; this excludes $37, 47$ (else $1256|3478$ would be $\Box|2$). By parity we need two of $15, 17, 35$, so we must have 35, which excludes $17, 27$ (else $3456|1278$ would be $\Box|2$), so we must have $15, 45$; but now $1235|4678$ is heavy.

Finally suppose (d) holds; so we must have six of $15, 17, 25, 27, 35, 37, 45, 47$. If each of 5 and 7 is joined to three of $\{1, 2, 3, 4\}$, we may assume they each have bias 1, so we have edges $15, 17, 25, 27$ and one each of $35, 45$ and $37, 47$. By parity we must have 35 or 37, and 45 or 47, so we may assume the other two edges are $35, 47$; but now $1235|4678$ is heavy. Thus one of 5 and 7 is joined to all of $\{1, 2, 3, 4\}$; we may assume $15, 25, 35, 45$ are edges, which excludes 17 (else $1678|2345$ would be heavy) and 47 (else $1235|4678$ would be heavy), so we must have $27, 37$. Yet again we must have $e = 11$, because adding 13 or 24 would give a K_4 among $\{1, 2, 3, 5\}$ or $\{2, 3, 4, 5\}$; applying $(1345|2678)$ gives Γ_{427}. $\Box$

LEMMA 8.5. *With the notation above, if* $n = 0$ *then* Γ *is known.*

PROOF. Observe that if $e = 12$ and there is some edge from $\{7,8\}$ to $\{1,2,3,4\}$, we may reduce to an earlier case by interchanging $\{5,6\}$ and either $\{1,2\}$ or $\{3,4\}$. Thus we need only treat the possibility that $e = 12$ if both 7 and 8 have degree 0.

First consider the case where the edges in $1234|5678$ are $12, 14, 23, 34, 56$, together with an extra edge 13 or 24 if $e = 12$. The partitions $1357|2468$ and $1368|2457$ again each contain $e - 11$ of the edges in $1234|5678$, while the other six edges each lie in precisely one of them; thus for neither to be light, each must contain three of the edges between $\{1,2,3,4\}$ and $\{5,6,7,8\}$. For $5 \leq i \leq 8$ let o_i and e_i be the numbers of edges from i to $\{1,3\}$ and $\{2,4\}$ respectively; then we must have $o_5 + e_6 + o_7 + e_8 = 3 = e_5 + o_6 + e_7 + o_8$. Since we may interchange 7 and 8 here we also have $o_5 + e_6 + e_7 + o_8 = 3 = e_5 + o_6 + o_7 + e_8$; so $o_5 - e_5 = o_6 - e_6$ and $o_7 - e_7 = o_8 - e_8$. Similarly for $1 \leq i \leq 4$ let x_i and y_i be the numbers of edges from i to $\{5,6\}$ and $\{7,8\}$ respectively; then as $1256|3478$ and $1278|3456$ each contain three of the edges between $\{1,2,3,4\}$ and $\{5,6,7,8\}$ we must have $x_1 + x_2 + y_3 + y_4 = 3 = y_1 + y_2 + x_3 + x_4$. Since we may interchange 2 and 4 here we also have $x_1 + y_2 + y_3 + x_4 = 3 = y_1 + x_2 + x_3 + y_4$; so $x_1 - y_1 = x_3 - y_3$ and $x_2 - y_2 = x_4 - y_4$.

First suppose both 7 and 8 have degree 0. We then need six edges from $\{5,6\}$ to $\{1,2,3,4\}$, so at least two vertices in $\{1,2,3,4\}$, say i and j, are joined to both 5 and 6; then i and j cannot be adjacent (else we would have a K_4 among $\{i,j,5,6\}$), so we may assume $15, 16, 35, 36$ are edges. We may also assume one of the remaining edges is 25, which excludes 26 (else we would have a K_4 among $\{1,2,5,6\}$) and 45 (else $o_5 - e_5 \neq o_6 - e_6$), so we must have 46. Adding 13 would give a K_4 among $\{1,2,3,5\}$, so if $e = 12$ the extra edge must be 24; applying $(1235|4678)$ gives Γ_{425} if $e = 11$ and Γ_{429} if $e = 12$. Thus we may assume there is an edge between $\{7,8\}$ and $\{1,2,3,4\}$, which we may take to be 48, which excludes $17, 37$ (by parallels), giving $o_7 = 0$; by the above we need only consider $e = 11$ from now on.

Next suppose $o_8 > 0$; we may assume we have the edge 38, which excludes $27, 47$ (by parallels), so $e_7 = 0$ and thus $o_8 = e_8$. If $o_8 = e_8 = 2$ we would have $18, 28$ and two edges from $\{5,6\}$ to $\{1,2,3,4\}$; we could assume 15 was an edge, but then $1567|2348$ would be heavy. Thus we must have $o_8 = e_8 = 1$, so $18, 28$ are not edges and we have four edges from $\{5,6\}$ to $\{1,2,3,4\}$; thus $y_1 = y_2 = 0$, $y_3 = y_4 = 1$, so $x_3 = x_1 + 1$, $x_4 = x_2 + 1$. By interchanging $1,4$ and $2,3$ if necessary we may assume $x_3 = 2$, $x_1 = x_4 = 1$, $x_2 = 0$; so we have $35, 36$, one of $15, 16$ and one of $45, 46$. We may assume 15 is an edge, which gives $o_5 = 2$, $o_6 = 1$, so $e_5 = e_6 + 1$ and we must have 45; applying $(1567|2348)$ gives Γ_{427}. Thus we may assume $o_8 = 0$, so we do not have 18 or 38; thus $e_7 = e_8$, and $y_1 = y_3 = 0$, so that $x_1 = x_3$.

Suppose 28 is an edge, so that $e_8 = 2$; thus $e_7 = 2$ and we must have $27, 47$ together with two edges from $\{5,6\}$ to $\{1,2,3,4\}$. This excludes $15, 16$ (else $2348|1567$ would be $\square|2$) and $35, 36$ (else $1248|3567$ would be $\square|2$); we may assume we have the edge 25, which excludes 26 (else $x_2 - y_2 \neq x_4 - y_4$) and 45 (else $o_5 - e_5 \neq o_6 - e_6$), so we must have 46, giving Γ_{426}. Thus we may assume 28 is not an edge; so $e_7 = 1$, and we must have either 27 or 47 together with four edges from $\{5,6\}$ to $\{1,2,3,4\}$.

We cannot have $x_1 = x_3 = 2$, since this would require the edges $15, 16, 35, 36$ and then $1356|2478$ would be heavy. Suppose $x_1 = x_3 = 0$, so that we have none of $15, 16, 35, 36$; then we must have $25, 26, 45, 46$, and so $x_2 = x_4 = 2$, whence

$y_2 = y_4 = 1$ and we must have 27, making 1348|2567 heavy. Thus we must have $x_1 = x_3 = 1$; we may assume 15 is an edge, together with one of 35, 36. This excludes 26 (else 1256|3478 would be $\square|2$).

Suppose 47 is an edge, so that 27 is not; then $y_2 = 0$, $y_4 = 2$, so we must have $x_2 = 0$, $x_4 = 2$, giving the edges 45, 46. As $e_5 = e_6 = 1$ we must have 36 to ensure $o_5 = o_6$; applying (1456|2378) gives Γ_{427}. Thus we may assume instead 27 is an edge, which excludes 46 (else 1456|2378 would be $\square|2$), so we must have 25, 45; this excludes 36 (else 2356|1478 would be $\square|2$), so we must have 35, giving Γ_{427}.

This completes the case where the edges in 1234|5678 are 12, 14, 23, 34, 56, together with an extra edge if $e = 12$. From now on we may therefore assume instead that they are 12, 13, 23, 34, 56 (and note that we may also assume that $e = 11$, since an extra edge would give the configuration just handled).

First suppose there are two parallel edges from $\{1, 2\}$ to $\{5, 6\}$, say 15, 26; then the remaining edge in 1256|3478 cannot go between $\{3, 4\}$ and $\{7, 8\}$ (else 1256|3478 would be $\square|2$), so we may assume it is 25 (and 16 is not an edge). This excludes 35 (else we would have a K_4 among $\{1, 2, 3, 5\}$), and then we must have 46 (else 1246|3578 would be light), which excludes 17, 18 (else 2346|1578 would be $\square|2$). If there is no edge from 2 to $\{7, 8\}$ we must have 36, 45; applying (1235|4678) gives Γ_{425}. So we may assume we have 28, which excludes 36 (else 1258|3467 would be heavy) and 45 (else 1238|4567 would be heavy), so we must have 27; applying (1256|3478) gives Γ_{427}. Thus we may assume there are not two parallel edges from $\{1, 2\}$ to $\{5, 6\}$; in particular, there must be some edge from $\{3, 4\}$ to $\{7, 8\}$.

Next suppose there are two parallel edges from $\{3, 4\}$ to $\{5, 6\}$, say 35, 46; then the remaining edge in 1278|3456 cannot go between $\{1, 2\}$ and $\{7, 8\}$ (else 3456|1278 would be $\square|2$), so we must have one of 36, 45. However, if 45 is an edge this excludes 38, 47 (else 1238|4567 would be heavy) and 37, 48 (else 1237|4568 would be heavy), and so there is no edge from $\{3, 4\}$ to $\{7, 8\}$, contrary to the above; so we must instead have 36. If there is an edge from 3 to $\{7, 8\}$, say 37, this excludes 48 (by parallels); but then 3 and 8 are joined to all and none respectively of the other six vertices. So 37, 38 are not edges and we must have an edge from 4 to $\{7, 8\}$, say 48; this excludes 15, 25 (else 1235|4678 would be heavy), so we need two of 16, 26, 47; but now 1236|4578 is heavy. Thus we may assume there are not two parallel edges from $\{3, 4\}$ to $\{5, 6\}$; in particular, there must be some edge from $\{1, 2\}$ to $\{7, 8\}$, say 28, which excludes 17, 47 (by parallels).

Suppose 38 is an edge; this excludes 18, 45, 46 (else 1238|4567 would be heavy), so for 1278|3456 to contain six edges we must have two of 27, 35, 36, and thus we may assume 35 is an edge. If we had 36 and not 27, then 3 and 7 would be joined to all and none respectively of the other six vertices; so instead we must have 27 and not 36. If 37 is an edge this excludes 48 (by parallels), 15, 16 (else 1456|2378 would be heavy) and 25 (else 1578|2346 would be light), so we must have 26; applying (1278|3456) gives Γ_{425}. So we may assume 37 is not an edge. We cannot have 48 as this would exclude 15, 16 (else 1567|2348 would be heavy) and 25, 26 (else 1348|2567 would be heavy); likewise we cannot have 15 as this would exclude 26 (by parallels from $\{1, 2\}$ to $\{5, 6\}$), 16 (else 1567|2348 would be heavy) and 25 (else we would have a K_4 among $\{1, 2, 3, 5\}$); since we cannot have both 16, 25 by parallels from $\{1, 2\}$ to $\{5, 6\}$ we must have 26, but this excludes 16 (else 1245|3678 would be light) and 25 (else 1348|2567 would be heavy). Thus we may assume 38 is not an edge.

Suppose 48 is an edge; this excludes 37 (by parallels) and $15, 16$ (else $2348|1567$ would be $\square|2$), so for $1256|3478$ to contain six edges we must have $25, 26$, which excludes $18, 27$ (else $1348|2567$ would be heavy). So we must have two of $35, 36, 45, 46$. We cannot have both of $35, 36$ (else we would have a K_4 among $\{2, 3, 5, 6\}$), so we may assume 45 is an edge, which excludes 36 (by parallels from $\{3, 4\}$ to $\{5, 6\}$) and 46 (else $1238|4567$ would be heavy), so we must have 35; applying $(1238|4567)$ gives Γ_{427}. Thus we may assume 48 is not an edge; so the edge from $\{3, 4\}$ to $\{7, 8\}$ must be 37. Note that if 27 were an edge we could interchange 7 and 8 to obtain a graph with edges $28, 38$ as considered in the previous paragraph; so we may assume 27 is not an edge.

Suppose 18 is an edge; this excludes $45, 46$ (else $1238|4567$ would be heavy), so for $1278|3456$ to contain six edges we must have exactly one of $35, 36$, say 35. We must then have two edges from $\{1, 2\}$ to $\{5, 6\}$. We cannot have 15 as this would exclude 26 (by parallels from $\{1, 2\}$ to $\{5, 6\}$), 16 (else $1568|2347$ would be heavy) and 25 (else we would have a K_4 among $\{1, 2, 3, 5\}$); since we cannot have both of $16, 25$ (by parallels from $\{1, 2\}$ to $\{5, 6\}$) we must have 26, but this excludes 16 (else $1245|3678$ would be light) and 25 (else $1347|2568$ would be heavy). Thus we may assume 18 is not an edge.

Therefore we must have one of $15, 26$, one of $16, 25$, one of $35, 46$ and one of $36, 45$. We cannot have both of $45, 46$ (else $1238|4567$ would be heavy), so we may assume 35 is an edge. However, we cannot have both of $15, 25$ (else we would have a K_4 among $\{1, 2, 3, 5\}$) or both of $16, 26$ (else $1268|3457$ would be heavy); if we have both of $15, 16$ this excludes 36 (else we would have a K_4 among $\{1, 3, 5, 6\}$) and 45 (else $1456|2378$ would be heavy); likewise if we have both of $25, 26$ this excludes 36 (else we would have a K_4 among $\{2, 3, 5, 6\}$) and 45 (else $1268|3457$ would be heavy). This proves the lemma. $\square$

This completes the treatment of the case $e \geq 11$; we therefore assume from now on that $e = 10$. Here it is possible for Γ to have four parallel edges; we begin with this situation. Thus suppose $12, 34, 56, 78$ are edges; as before the three partitions $1234|5678$, $1256|3478$ and $1278|3456$ contain each of the other six edges exactly once, so for none to be heavy they must each contain two further edges. We shall call one of these three partitions *balanced* if the two further edges are in different halves of the partition, and *unbalanced* otherwise. Let u be the number of unbalanced partitions among the three, so that $u \in \{0, 1, 2, 3\}$. Moreover, let p be the number of these unbalanced partitions in which the two further edges are parallel, so that $p \leq u$.

Lemma 8.6. *With the notation above, if $u = 0$ then Γ is known.*

Proof. First suppose some vertex is incident with three of the further edges; we may assume we have $13, 15, 17$, so that the remaining edges go one each from $\{3, 4\}$ to $\{5, 6\}$, from $\{3, 4\}$ to $\{7, 8\}$ and from $\{5, 6\}$ to $\{7, 8\}$. For $1468|2357$ not to be light it must contain at least two of the remaining edges, so we may assume we have 37 or 48, and 57 or 68 (and if 7 is incident with just one of these two edges it is 37); for $1467|2358$ not to be light it must then contain at least one more edge, so the final edge must be 35 or 46. We cannot then have $48, 68$ as $1235|4678$ would be heavy; if we have $37, 68$ this excludes 46 (else $1237|4568$ would be heavy), so we must have 35, and applying $(1235|4678)$ gives Γ_{419}; if we have $37, 57$ this excludes

35 (else we would have a K_4 among $\{1, 3, 5, 7\}$), so we must have 46, and applying (1578|2346) again gives Γ_{419}.

Thus we may assume in each pair $\{1, 2\}$, $\{3, 4\}$, $\{5, 6\}$ and $\{7, 8\}$ one vertex is incident with two of the further edges while the other is incident with one; we may assume we have $13, 15, 27$. This time for 1467|2358 not to be light it must contain at least two of the remaining edges, so interchanging $3, 4$ and $5, 6$ if necessary we may assume we have 58 or 67, and at least one of $38, 47, 35, 46$. Thus if we had neither 35 nor 46 then 1468|2357 would be light; so we must have 35 or 46 together with an edge from $\{3, 4\}$ to $\{7, 8\}$. Suppose we have 58; this excludes $37, 48$ (else 1457|2368 would be light). If we also have 35 this excludes 47 (else 1356|2478 would be heavy), so we must have 38, and applying (1247|2358) gives Γ_{419}; if instead we also have 46 then the presence of 38 gives Γ_{422}, while that of 47 means that applying (1234|5678) gives Γ_{422}. So we may assume we have 67, which excludes 35 (else 1345|2678 would be heavy), so we must have 46, which excludes $37, 48$ (else 1457|2368 would be light) and 47 (else 1235|4678 would be heavy), so we must have 38; applying (1256|3478) gives Γ_{422}. □

Lemma 8.7. *With the notation above, if $u = 1$ then Γ is known.*

Proof. We may assume the unbalanced partition is 1278|3456. First suppose $p = 1$, so that the two further edges it contains are parallel; we may assume they are $17, 28$. Each of the remaining four edges is from $\{1, 2, 7, 8\}$ to $\{3, 4, 5, 6\}$. If one of $\{1, 2, 7, 8\}$ is incident with two of these edges, we may assume we have $13, 15$; for 1468|2357 not to be light it must then contain the other two edges, so we must have 37 or 48, and 57 or 68. If we have $37, 57$ then applying $\overline{(1358|2467)}$ gives Γ_{419}; we cannot have $37, 68$ as then 1347|2568 would be heavy, or $57, 48$ as then 1567|2348 would be heavy; and if we have $48, 68$ this gives Γ_{420}. Thus we may assume each of $\{1, 2, 7, 8\}$ is incident with one edge to $\{3, 4, 5, 6\}$; so we may assume we have $15, 23$, which excludes $38, 57$ (else 1567|2348 would be heavy). For 1368|2457 not to be light it must contain the other two edges, which must then be $47, 68$, giving Γ_{422}.

So now suppose $p = 0$, so that the two further edges in 1278|3456 are not parallel; we may assume they are $18, 28$. If one of $\{1, 2\}$ is incident with both the edges to $\{3, 4, 5, 6\}$ we may assume we have $13, 15$. For 1467|2358 not to be light it must contain at least one of the other edges; after interchanging $3, 4$ and $5, 6$ if necessary we may assume we have 38 or 47, which excludes 57 (else 1238|4567 would be heavy). For 1468|2357 not to be light it must contain the final edge, which must then be 68; choosing 38 gives Γ_{410} while 47 gives Γ_{421}. Thus we may assume each of $\{1, 2\}$ is incident with one edge to $\{3, 4, 5, 6\}$; so we may assume we have $15, 23$. For each of 1367|2458 and 1368|2457 not to be light we must have one of $37, 48, 58, 67$ and one of $38, 47, 57, 68$. If we have $37, 57$ then applying $\overline{(1368|2457)}$ gives Γ_{421}; if we had $37, 68$ then 1568|2347 would be heavy; if we had $48, 57$ then 1567|2348 would be heavy; if we have $48, 68$ this gives Γ_{411}; if we have $38, 58$ this gives Γ_{412}; if we had $38, 67$ then 1567|2348 would be heavy; if we had $47, 58$ then 1568|2347 would be heavy; and if we had $47, 67$ then 1238|4567 would be heavy. □

Lemma 8.8. *With the notation above, if $u = 2$ then Γ is known.*

Proof. We may assume the balanced partition is 1278|3456, and the four further edges in 1234|5678 and 1256|3478 are from $\{1, 2\}$ to $\{3, 4, 5, 6\}$. First suppose

$p \geq 1$; we may assume we have $13, 24$. If in fact $p = 2$, we may assume we also have $15, 26$, and may take one of the remaining edges to be 17; but now $1468|2357$ will be light. Thus we must have $p = 1$, so that the two edges from $\{1, 2\}$ to $\{5, 6\}$ must share a vertex v. We may assume the edges are $15, 16$ if $v \in \{1, 2\}$, or $15, 25$ if $v \in \{5, 6\}$; but in either case, for each of $1457|2368$, $1458|2367$, $1467|2358$ and $1468|2357$ not to be light, among the other two edges we must have one of $17, 28, 36, 45$, one of $18, 27, 36, 45$, one of $17, 28, 35, 46$ and one of $18, 27, 35, 46$, which is impossible with $1278|3456$ balanced.

Thus we may assume $p = 0$, so that in each of $1234|5678$ and $1256|3478$ the two further edges are not parallel. Let v_1 and v_2 be the vertices common to the two further edges in $1234|5678$ and $1256|3478$ respectively. If $v_1, v_2 \in \{3, 4, 5, 6\}$, we may assume we have $13, 23, 15, 25$; this excludes $35, 46$ (else $1235|4678$ would be heavy), so we may assume the other two edges are $36, 17$, and then applying $\overline{(1358|2467)}$ gives Γ_{411}. If instead $|\{v_1, v_2\} \cap \{3, 4, 5, 6\}| = 1$, we may assume we have $13, 14, 15, 25$; by interchanging 7 and 8, and 3 and 4, we may assume the other two edges are 17 or 27, and 35 or 36. We cannot have 27 as this excludes both 35 (else $1345|2678$ would be heavy) and 36 (else $1346|2578$ would be heavy), so we must have 17; according as we have 35 or 36 applying $(1278|3456)$ gives Γ_{410} or Γ_{411}. Thus we may assume $v_1, v_2 \in \{1, 2\}$. If $v_1 \neq v_2$ we may assume we have $13, 14, 25, 26$, together with $17, 35$; but now $1478|2356$ is heavy. Thus we may assume we have $13, 14, 15, 16$, together with 35, which excludes $27, 28$ (else $1345|2678$ would be heavy), so we may take the final edge to be 17, giving Γ_{398}. □

Lemma 8.9. *With the notation above, if $u = 3$ then Γ is known.*

Proof. First suppose $p \geq 2$; we may assume we have $13, 24, 15, 26$, with the other two edges in $1278|3456$. For each of $1467|2358$ and $1468|2357$ not to be light we must have two of $17, 28, 35, 46$ and two of $18, 27, 35, 46$, so the other two edges must be $35, 46$, and then applying $\overline{(1467|2358)}$ gives Γ_{420}.

Next suppose $p = 1$; we may assume we have $13, 24$. Note that $13, 24, 56, 78$ are also four parallel edges, so that each of $1356|2478$ and $1378|2456$ must contain two further edges; by the cases already considered we may assume each of these partitions is unbalanced with its two further edges not parallel. We may assume one of the other four edges is 25; thus the other edge in $1256|3478$ must be 15 or 26, while that in $1378|2456$ must be 45 or 26. If we have 15 we must therefore also have 45; now the remaining edge in $1278|3456$ must be 35 or 46 while that in $1356|2478$ must be 16 or 35, so we must have 35, giving Γ_{407}. If instead we have 26, the two remaining edges must lie in $1356|2478$; thus if they go from $\{1, 2\}$ to $\{7, 8\}$ they must be $27, 28$, and then applying $(1234|5678)$ gives Γ_{407}, while if they go from $\{3, 4\}$ to $\{5, 6\}$ they must be $35, 36$, and applying $(1278|3456)$ gives Γ_{417}.

Finally assume $p = 0$. Thus in each of our three partitions the two further edges meet at a vertex; let these vertices (which need not be distinct) be v_1, v_2, v_3. We may assume we have pairs of edges from $\{1, 2\}$ to $\{3, 4\}$ and from $\{1, 2\}$ to $\{5, 6\}$. First suppose the third pair is from $\{3, 4\}$ to $\{5, 6\}$. If say $\{v_1, v_2, v_3\} \cap \{1, 2\} = \emptyset$, we may assume we have $13, 23, 15, 25$, which excludes $35, 46$ (else $1235|4678$ would be heavy); but now the edges of the third pair must be parallel. So $\{v_1, v_2, v_3\}$ must meet each of $\{1, 2\}$, $\{3, 4\}$ and $\{5, 6\}$, and we may assume we have $13, 23, 35, 45, 15, 16$, giving Γ_{413}. Now suppose the third pair is from $\{1, 2\}$ to $\{7, 8\}$. If $v_i \in \{1, 2\}$ for at most one i, we may again assume we have $13, 23, 15, 25$. By interchanging 1 and 2, and 7 and 8, we may assume we have 17; if the other edge is 27 this gives Γ_{416}, while if it is 18 then applying (1235|4678) gives Γ_{416}. Thus we may assume $v_i \in \{1, 2\}$ for at least two values of i, and so may assume we have 13, 14; if we then had 25, 26 the third pair of edges would make 1347|2568 heavy, so we may assume we have 15, 16; if we had 27, 28 then 1345|2678 would be heavy, while if we had 17, 18 then 1 and 2 would be joined to all and none respectively of the other six vertices, so we may assume we have 17, 27, giving Γ_{396}. □

Thus from now on we assume Γ does not have four parallel edges; so 78 is not an edge, and as in the case $e \geq 11$ there can be no □|2 partition. Of the seven edges other than 12, 34, 56 we may assume 1234|5678 contains three, and thus is full. We observe that if a partition $abcd|efgh$ is full, has three parallel edges but not four, and is not □|2, then there are four possible configurations for the edges it contains: (I) ab, bc, cd, ef, eg, fg; (II) ab, bc, cd, ef, eg, eh; (III) ab, ac, bc, cd, ef, fg; (IV) ab, ac, bc, bd, cd, ef. We shall work through these four possibilities for 1234|5678 in turn; note that, as we select edges in tackling the later ones, if we ever see a partition of an earlier type we need proceed no further with the line of investigation, since no further edges may be added within the partition (else it would become heavy), and therefore we would be in a case already considered.

Lemma 8.10. *With the notation above, if* 1234|5678 *is of type (I) then* Γ *is known.*

Proof. We may assume the three further edges in 1234|5678 are 23, 57, 67. First suppose 48 is present; this excludes 15, 16, 17, 35, 36, 37 (by parallels) and 28, 38 (else 1567|2348 would be heavy). We cannot also have 18 as this would exclude the remaining possible edges (by parallels), so we must have three edges from $\{2, 4\}$ to $\{5, 6, 7\}$. By interchanging 1, 2 and 8, 4, we may assume we have at least two edges from 2 to $\{5, 6, 7\}$; but now 1348|2567 will be heavy. Thus 48 must be absent; similarly so must 18.

Suppose 38 is present; this excludes 15, 16, 17, 28 (else 1567|2348 would be heavy) and 45, 46, 47 (else 1238|4567 would be heavy), so we must have three edges from $\{2, 3\}$ to $\{5, 6, 7\}$. We cannot have at least two from 2 (else 1348|2567 would be heavy) or all three from 3 (else we would have a K_4 among $\{3, 5, 6, 7\}$), so we may assume we have 35, 36 and an edge from 2; if it is 25 or 26 then applying (1348|2567) gives Γ_{396}, while if it is 27 then applying (1248|3567) gives Γ_{407}. Thus we may assume 38 is absent; similarly we may assume 28 is absent.

Thus the remaining four edges are from $\{1, 2, 3, 4\}$ to $\{5, 6, 7\}$. We can have at most one from each of 1 and 4 (else 1567|2348 or 1238|4567 would be heavy), and at most two from each of 2 and 3 (else we would have a K_4 among $\{2, 5, 6, 7\}$ or $\{3, 5, 6, 7\}$). Up to interchanging 1, 2 and 4, 3 there are four possibilities: (i) two each from 2 and 3; (ii) two from 2, one from 3, one from $\{1, 4\}$; (iii) two from 2, one each from 1 and 4; (iv) one each from 1, 2, 3 and 4.

If (i) holds we may assume we have 25, 26, 35; this excludes 36 (else we would have a K_4 among $\{2, 3, 5, 6\}$) so we must have 37, and then applying (1248|3567) gives Γ_{398}.

If (ii) holds we may assume we have 25, 26. If we have 37 this excludes 15, 16, 47 (else 1256|3478 would be heavy); but now 1356|2478 will be light. Thus we may assume we have 35, which excludes 17, 47 (else 1267|3458 or 3457|1268 would be □|2). If we have 15 then applying (1567|2348) gives Γ_{406}; if we have 16 this gives

Γ_{414}; if we have 45 then applying (1268|3457) gives Γ_{406}; if we have 46 then applying (1234|5678) gives Γ_{414}.

If (iii) holds we may assume we have $25, 26$. If we have 17 this excludes $45, 46$ (else 2345|1678 or 2346|1578 would be $\square|2$), so we must have 47, and then applying (1348|2567) gives Γ_{413}. Thus we may assume we have 15, which excludes 46 (else 2346|1578 would be $\square|2$) and 47 (else 1256|3478 would be heavy), so we must have 45, and then applying (1348|2567) gives Γ_{414}.

Finally if (iv) holds we may assume we have 25. If we also have 35 we cannot have both 15 and 45 (else 5 and 8 would be joined to all and none respectively of the other six vertices), so we may assume we have 46; this excludes 15 (else 1235|4678 would be heavy) and 17 (else 3456|1278 would be $\square|2$), so we must have 16, and then applying (1678|2345) gives Γ_{415}. Thus we may assume we have 36. If we have 15 this excludes $46, 47$ (else 1258|3467 would be heavy), so we must have 45, giving Γ_{409}; if we have 16 this excludes $45, 47$ (else 1236|4578 would be $\square|2$), so we must have 46, giving Γ_{409}; if we have 17 this excludes $45, 47$ (else 2345|1678 or 2356|1478 would be $\square|2$) and 46 (else 1257|3468 would be heavy). $\square$

Lemma 8.11. *With the notation above, if* 1234|5678 *is of type (II) then* Γ *is known.*

Proof. We may assume the three further edges in 1234|5678 are $23, 57, 58$. First suppose there is some edge from $\{1, 4\}$ to $\{6, 7, 8\}$; we may assume we have 16, which excludes $27, 28, 47, 48$ (by parallels) and $17, 18, 36$ (else 1567|2348, 1568|2347 or 1236|4578 would be $\square|2$).

If we also have 46, this excludes $37, 38$ (by parallels) and 26 (else 2346|1578 would be $\square|2$), so we must have two edges from 5 to $\{1, 2, 3, 4\}$; note that we may interchange $1, 2$ and $4, 3$. If we have $15, 25$ or $15, 45$ this gives Γ_{415}, while if we have $15, 35$ or $25, 35$ then applying (1234|5678) or (1246|3578) gives Γ_{415}. So we may assume 46 is absent.

If we also have 45, this excludes 26 (else 1236|4578 would be heavy) and $37, 38$ (else 3457|1268 or 3458|1267 would be $\square|2$), so we must have two edges from 5 to $\{1, 2, 3\}$. Since we may interchange $1, 6$ and $3, 4$, we may assume we have 15; if we also have 25 this gives Γ_{403}, while if instead we have 35 then applying (1234|5678) gives Γ_{403}. So we may assume 45 is absent.

If we also have 25, this excludes $37, 38$ (else 1256|3478 would be $\square|2$), so we must have two of $15, 26, 35$. We cannot have both of $15, 26$ (else we would have a K_4 among $\{1, 2, 5, 6\}$), so we must have 35; this excludes 26 (else 3457|1268 would be of type (I)), so we must have 15, giving Γ_{402}. So we may assume 25 is absent. If we had $37, 38$ then 3578|1246 would be $\square|2$, so we may assume 38 is absent; thus we must have three of $15, 26, 35, 37$. If we had $15, 37$ then 1568|2347 would be heavy, so we must have $26, 35$; but then 3457|1268 is of type (I).

Thus we may assume there is no edge from $\{1, 4\}$ to $\{6, 7, 8\}$. If we had $15, 25, 35, 45$ then 5 and 6 would be joined to all and none respectively of the other six vertices; so we must have some edge from $\{2, 3\}$ to $\{6, 7, 8\}$, which we may take to be 26.

Suppose we have 15; this excludes $37, 38$ (else 1256|3478 would be $\square|2$) and 36 (else 1578|2346 would be heavy), so we must have two of $25, 27, 28, 35, 45$. If we have $27, 28$ this gives Γ_{404}, so we may assume 28 is absent. If we have 25 then according as the other edge is 27, 35 or 45 we have Γ_{405}, Γ_{401} or Γ_{400}, so we may assume 25 is absent; we cannot have $27, 35$ (else 1367|2458 would be light), so we must have 45; if the other edge is 35 this gives Γ_{399}, while if it is 27 then applying (1267|3458) gives Γ_{404}. Thus we may assume 15 is absent.

Suppose we have 45; this excludes 36 (else 1236|4578 would be heavy) and 37, 38 (else 3457|1268 or 3458|1267 would be $\square|2$), so we must have two of 25, 27, 28, 35. If we have 27, 28 then applying (1234|5678) gives Γ_{404}, so we may assume 28 is absent. We cannot have 27, 35 (else 1267|3458 would be heavy), so we must have 25; according as the other edge is 27 or 35 applying (1234|5678) gives Γ_{405} or Γ_{401}. Thus we may assume 45 is absent.

If we have 25, 35 this excludes 36 (else we would have a K_4 among $\{2,3,5,6\}$), so we may assume the other edge is from 7 to $\{2,3\}$; if it is 37 this gives Γ_{406}, while if it is 27 then applying (1348|2567) gives Γ_{408}. Thus we may assume at least one of 25, 35 is absent; so we must have at least three edges in total from $\{2,3\}$ to $\{6,7,8\}$, and thus may assume we have 27 in addition to 26. This excludes 38 (else 2567|1348 would be $\square|2$); note that we may now interchange 6 and 7.

If we have 25 we cannot have 35 (as above); if the other edge is 28 this gives Γ_{397}, while if it is 36 then applying (1346|2578) gives Γ_{414}. So we may assume 25 is absent. If we have 35 this excludes 28 (else 1245|3678 would be light), so we may assume the other edge is 36, and then applying (1236|4578) gives Γ_{409}. So we may assume 35 is absent. Thus we must have two of 28, 36, 37, so we may assume we have 36; if the other edge is 28 then applying (1346|2578) gives Γ_{413}, while if it is 37 then applying (1458|2367) gives Γ_{415}. □

LEMMA 8.12. *With the notation above, if* 1234|5678 *is of type (III) then* Γ *is known.*

PROOF. We may assume the three further edges in 1234|5678 are 13, 23, 67; this excludes 45, 47 (else 1238|4567 would be of type (I)). Note that we may interchange 5 and 7.

First suppose we have 38; this excludes 18, 28, 46 (else 1238|4567 would be heavy) and 15, 17, 25, 27 (else 1567|2348 or 2567|1348 would be of type (II)). We cannot then have 48, since this excludes 35, 37 (by parallels) and 16, 26 (else 1256|3478 would be of type (I)), leaving 36 as the only available edge; so we must have three of 16, 26, 35, 36, 37. If we had 16, 26, 36 then we would have a K_4 among $\{1,2,3,6\}$; thus we may assume we have 37, which excludes 16, 26 (else 1256|3478 would be of type (II)), so we must have 35, 36; but now 3 and 4 are joined to all and none respectively of the other six vertices. Thus we may assume 38 is absent.

Next suppose we have 48; this excludes 15, 17, 25, 27 (by parallels), 16, 26 (else 2348|1567 or 1348|2567 would be of type (II)), 18, 28 (else 1348|2567 or 2348|1567 would be $\square|2$) and 46 (else 4568|1237 would be of type (I)). Thus we must have 35, 36, 37, but now 1248|3567 is heavy. So we may assume 48 is absent.

Next suppose we have 46; this excludes 18, 28 (else 1238|4567 would be heavy). If we also have at least one of 35, 37 we may assume 37 is present, which excludes 15, 25 (else 3467|1258 would be $\square|2$) and 17, 27 (else 1237|4568 would be heavy), so we must have two of 16, 26, 35, 36. If we had 35, 36 then 3 and 8 would be joined to all and none respectively of the other six vertices, so we may assume we have 16, which excludes 35 (else 1457|2368 would be light); if the other edge is 26 then applying (1237|4568) gives Γ_{399}, while if it is 36 then applying $\overline{(1457|2368)}$ gives Γ_{401}. So we may assume 35, 37 are absent. If we had 16, 26, 36 then 6 and 8 would be joined to all and none respectively of the other six vertices, so we must have some edge from $\{1,2\}$ to $\{5,7\}$, which we may take to be 15; this excludes 17 (else 1567|2348 would be $\square|2$) and 25 (else 1235|4678 would be heavy). We cannot have 27 as this would exclude $16, 26, 36$ (else 2347|1568, 1345|2678 or 1257|3468 would be of type (I)), so we must have two of $16, 26, 36$; after interchanging $1, 5$ and $3, 4$ if necessary we may assume we have 16, and if the other edge is 26 then applying (1368|2457) gives Γ_{403}, while if it is 36 then applying (1234|5678) gives Γ_{402}. Thus we may assume 46 is absent.

Next suppose we have at least one of $18, 28$, which we may take to be 18; this excludes $25, 27$ (by parallels), 26 (else 1348|2567 would be of type (II)) and 28 (else 1238|4567 would be heavy), so we must have three edges from $\{1,3\}$ to $\{5,6,7\}$, and thus at least one from $\{1,3\}$ to $\{5,7\}$. Since we may interchange $1, 8$ and $3, 4$, and 5 and 7, we may assume we have 15; this excludes 17 (else 1567|2348 would be $\square|2$) and $36, 37$ (else 3467|1258 would be of type (II)), so we must have $16, 35$, but now 1678|2345 is of type (II). Thus we may assume $18, 28$ are absent.

Next suppose there is no edge from $\{1,2\}$ to $\{5,7\}$; then we must have four of $16, 26, 35, 36, 37$, and as we may interchange 1 and 2, and 5 and 7, we may assume we have $16, 37$. We cannot then have 36 as it would exclude 26 (else we would have a K_4 among $\{1,2,3,6\}$) and 35 (else 3 and 8 would be joined to all and none respectively of the other six vertices), so we must have $26, 35$, and then applying (1256|3478) gives Γ_{399}. Thus we may assume there is some edge from $\{1,2\}$ to $\{5,7\}$, which we may take to be 15; this excludes 17 (else 1567|2348 would be $\square|2$).

Suppose we have 16; this excludes 27 (else 2347|1568 would be of type (I)). Also, we may assume 37 is absent, since otherwise interchanging $2, 3, 4$ and $5, 6, 7$ would give a graph containing the six edges assumed at the start of this proof together with 46, which we have already handled. Thus we must have two edges from $\{2,3\}$ to $\{5,6\}$; we cannot have a pair of the form ik, jk (else we would have a K_4 among $\{1,i,j,k\}$), so they must be parallel; if we had $25, 36$ then 3467|1258 would be of type (I), so we must have $26, 35$, and then applying (1256|3478) gives Γ_{403}. Thus we may assume 16 is absent.

Suppose we have 25; this excludes 27 (else 2567|1348 would be $\square|2$), 35 (else we would have a K_4 among $\{1,2,3,5\}$) and 36 (else 3467|1258 would be of type (I)), so we must have $26, 37$, but now 1256|3478 is heavy. Thus we may assume 25 is absent.

Suppose we have 26; this excludes 27 (else 1345|2678 would be of type (I)) and 37 (else 1256|3478 would be $\square|2$), so we must have $35, 36$, and then applying (1235|4678) gives Γ_{402}. Thus we may assume 26 is absent.

Finally then we must have three of $27, 35, 36, 37$; if we had $35, 36, 37$ then 3 and 8 would be joined to all and none respectively of the other six vertices, so we must have 27. According as the missing edge from 3 to $\{5,6,7\}$ is 35, 36 or 37, applying (1568|2347), (1234|5678) or (1345|2678) gives Γ_{414}. $\square$

LEMMA 8.13. *With the notation above, if* 1234|5678 *is of type (IV) then* Γ *is known.*

PROOF. We may assume the three further edges in 1234|5678 are $13, 23, 24$. First suppose some edge from $\{1,4\}$ to $\{5,6\}$ is present, which we may take to be 15; this excludes $17, 18$ (else 1567|2348 or 1568|2347 would be of type (I)) and $27, 28, 37, 38, 47, 48$ (else 2347|1568 or 2348|1567 would be of type (III)). If 16 is also present, we must have two from $\{2,3,4\}$ to $\{5,6\}$. If we have $45, 46$ this gives Γ_{418}, so we may assume we have some edge from $\{2,3\}$ to $\{5,6\}$, which we may take to be 25; this excludes $26, 35$ (else we have a K_4 among $\{1,2,5,6\}$ or $\{1,2,3,5\}$); if we have 36 then applying (1256|3478) gives Γ_{397}, while if instead we have 45 or 46 then applying (1234|5678) gives Γ_{397} or Γ_{405}. So we may assume 16 is absent. If 46 is present, we may assume 45 is not, since otherwise interchanging $1, 5$ and $4, 6$ would give a graph containing the six edges assumed at the start of this proof, together with 15 and 16, which we have just handled. Thus we must have two edges from $\{2,3\}$ to $\{5,6\}$, so we may assume we have 25, which excludes 35 (else we would have a K_4 among $\{1,2,3,5\}$); if the other edge is 26 this gives Γ_{408}, while if it is 36 then applying (1235|4678) gives Γ_{405}. So we may assume 46 is absent. If we had $25, 35$ then we would have a K_4 among $\{1,2,3,5\}$; so we may assume 35 is missing. Thus we must have three of $25, 26, 36, 45$; we must have 25 (else 1467|2358 would be light). If the other two are $26, 36$ then applying (1256|3478) gives Γ_{401}; if they are $26, 45$ then applying (1234|5678) gives Γ_{401}; if they are $36, 45$ then applying (1234|5678) gives Γ_{400}. Thus we may assume $15, 16, 45, 46$ are absent.

Next suppose some edge from $\{1,4\}$ to $\{7,8\}$ is present, which we may take to be 17; this excludes $28, 38, 48$ (by parallels). If we also have 18 this excludes $27, 37, 47$ (by parallels), so we must have two from $\{2,3\}$ to $\{5,6\}$, and thus may assume we have 25; but now 2456|1378 is of type (II). Thus we may assume 18 is absent. If we have at least one of $27, 37$ we may assume we have 27, which excludes $35, 36$ (else 3456|1278 would be of type (I)) and 37 (else we would have a K_4 among $\{1,2,3,7\}$), so we must have two of $25, 26, 47$; we may assume we have 25, but this excludes $26, 47$ (else 1347|2568 would be of type (I) or $\square|2$). Thus we may assume $27, 37$ are absent. We cannot have 47 as this would exclude $25, 26, 35, 36$ (else 1347|2568 or 1247|3568 would be $\square|2$); so we must have three from $\{2,3\}$ to $\{5,6\}$, which we may take to be $25, 26, 35$, but now 1347|2568 is of type (I). Thus we may assume $17, 18, 47, 48$ are absent.

Thus we must have four edges from $\{2,3\}$ to $\{5,6,7,8\}$; we may assume we have at least two from 2. If we had $25, 26, 27, 28$ then 2 and 7 would be joined to all and none respectively of the other six vertices; so we must have at least one edge from 3. If there are three edges from 2, we may suppose the missing edge from 2 is 26 or 28; but if we have $25, 27, 28$, this excludes $35, 36, 37, 38$ (else 3456|1278, 2568|1347 or 2567|1348 would be of type (II)), while if we have $25, 26, 27$, this excludes 38 (else 1348|2567 would be heavy), and then 2 and 8 are joined to all and none respectively of the other six vertices. Thus we must have two edges from each of 2 and 3; we may assume 2 has at least as many edges to $\{5,6\}$ as 3 does. If we have $27, 28$ we must then have $37, 38$, and then applying (1234|5678) gives Γ_{418}; so we may assume we have 25. If we also have 26, this excludes $37, 38$ (else 1256|3478 would be of type (III)), so we must have $35, 36$; but now we have a K_4 among $\{2,3,5,6\}$. Thus we may assume we also have 27, which excludes 38 (else 2567|1348 would be of type (II)), so we must have 37; for 1457|2368 not to be light we must have 36 rather than 35, and then applying (1467|2358) gives Γ_{397}. □

This concludes the argument showing that the graphs in Figure 8.6 complete the list of relevant irreducible graphs. As with the E_7 root system earlier, in a few cases we take not the set corresponding to the graph given above but a W-translate thereof, as this leads to a more convenient form for the stabilizer. In each case this gives a reducible graph equivalent to the original; we call it a *replacement*. In

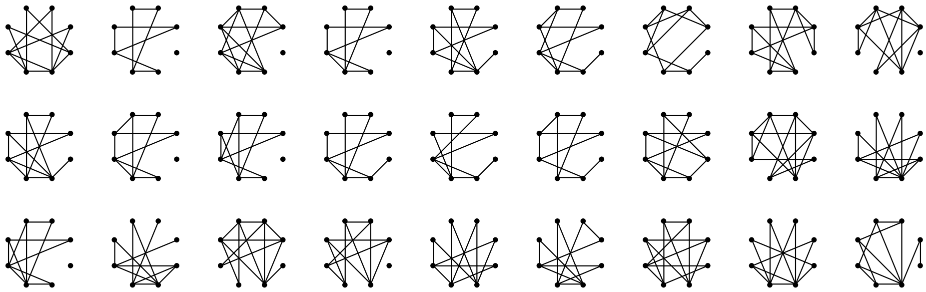


FIGURE 8.7. Replacement graphs

Figure 8.7 we list these replacements, and in the following scheme indicate in each case the original and an element of W sending it to its replacement.

60	68	114	125	146	154	161	175	225
(1347|2568)	(1458|2367)	$\overline{(1467\|2358)}$	(1458|2367)	(1458|2367)	$\overline{(1467\|2358)}$	$\overline{(1457\|2368)}$	(1457|2368)	(1568|2347)
236	248	251	261	284	291	299	302	320
(1458|2367)	(1458|2367)	(1457|2368)	(1458|2367)	$\overline{(1567\|2348)}$	(1458|2367)	(1458|2367)	(1568|2347)	(1458|2367)
323	356	358	364	374	375	389	405	409
(1458|2367)	(1458|2367)	(1348|2567)	$\overline{(1567\|2348)}$	(1458|2367)	(1458|2367)	$\overline{(1567\|2348)}$	$\overline{(1678\|2345)}$	$\overline{(1346\|2578)}$

We write X^i_{29} for the maximal abelian set corresponding to the graph Γ_i or its replacement, and therefore set

$X^1_{29} = \{\substack{2465432\\3}, \substack{\cdot\cdot\cdot\cdot\cdot\cdot1\\2}, \substack{\cdot\cdot\cdot\cdot\cdot\cdot1\\0}\}$,

$X^2_{29} = \{\substack{246543\cdot\\3}, \substack{\cdot\cdot\cdot\cdot\cdot\cdot1\\2}, \substack{\cdot\cdot\cdot\cdot\cdot11\\0}\}$,

$X^3_{29} = \{\substack{24654\cdot\cdot\\3}, \substack{\cdot\cdot\cdot\cdot\cdot\cdot1\\2}, \substack{\cdot\cdot\cdot\cdot111\\0}\}$,

$X^4_{29} = \{\substack{246543\cdot\\3}, \substack{\cdot\cdot\cdot4321\\2}, \substack{\cdot\cdot\cdot3\cdot\cdot1\\2}, \substack{1232\cdot11\\2}, \substack{1233211\\1}, \substack{\cdot\cdot\cdot\cdot\cdot11\\0}\}$,

$X^5_{29} = \{\substack{2465\cdot\cdot\cdot\\3}, \substack{\cdot\cdot\cdot\cdot\cdot\cdot1\\2}, \substack{\cdot\cdot\cdot1111\\0}\}$,

$X^6_{29} = \{\substack{24654\cdot\cdot\\3}, \substack{\cdot\cdot\cdot\cdot321\\2}, \substack{\cdot\cdot\cdot\cdot2\cdot1\\2}, \substack{1233321\\1}, \substack{\cdot\cdot\cdot\cdot111\\0}\}$,

$X^7_{29} = \{\substack{24654\cdot\cdot\\3}, \substack{\cdot\cdot\cdot4321\\2}, \substack{\cdot\cdot\cdot3\cdot\cdot1\\2}, \substack{1232221\\2}, \substack{1233221\\1}, \substack{1232111\\2}, \substack{\cdot\cdot\cdot\cdot111\\0}\}$,

$X^8_{29} = \{\substack{24654\cdot\cdot\\3}, \substack{\cdot\cdot\cdot4321\\2}, \substack{\cdot\cdot43\cdot\cdot1\\2}, \substack{123\cdot2\cdot1\\2}, \substack{1232111\\\cdot}, \substack{\cdot\cdot\cdot\cdot111\\0}\}$,

$X^9_{29} = \{\substack{246543\cdot\\3}, \substack{2454321\\2}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\2}, \substack{12\cdot3\cdot21\\2}, \substack{12\cdot\cdot\cdot11\\2}, \substack{1233211\\1}, \substack{1221111\\1}, \substack{\cdot\cdot\cdot\cdot\cdot11\\0}\}$,

$X^{10}_{29} = \{\substack{246\cdot\cdot\cdot\cdot\\3}, \substack{\cdot\cdot\cdot\cdot\cdot\cdot1\\2}, \substack{\cdot\cdot11111\\0}\}$,

$X^{11}_{29} = \{\substack{2465\cdot\cdot\cdot\\3}, \substack{\cdot\cdot\cdot\cdot321\\2}, \substack{\cdot\cdot\cdot\cdot2\cdot1\\2}, \substack{1233321\\1}, \substack{\cdot\cdot\cdot1111\\0}\}$,

$X^{12}_{29} = \{\substack{2465\cdot\cdot\cdot\\3}, \substack{\cdot\cdot\cdot\cdot\cdot21\\2}, \substack{\cdot\cdot43211\\2}, \substack{1232\cdot11\\2}, \substack{1232221\\1}, \substack{\cdot\cdot\cdot1111\\0}\}$,

$X^{13}_{29} = \{\substack{2465\cdot\cdot\cdot\\3}, \substack{2454321\\2}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\2}, \substack{12\cdot3\cdot\cdot1\\2}, \substack{1232\cdot\cdot1\\2}, \substack{1221111\\1}, \substack{\cdot\cdot\cdot1111\\0}\}$,

$X^{14}_{29} = \{\substack{24654\cdot\cdot\\3}, \substack{\cdot\cdot54321\\2}, \substack{\cdot\cdot4\cdot\cdot\cdot1\\2}, \substack{123\cdot2\cdot1\\2}, \substack{1233321\\1}, \substack{1232111\\1}, \substack{\cdot\cdot\cdot\cdot111\\0}\}$,

$X^{15}_{29} = \{\substack{24654\cdot\cdot\\3}, \substack{\cdot\cdot\cdot4321\\2}, \substack{\cdot\cdot\cdot3\cdot\cdot1\\2}, \substack{12332\cdot1\\1}, \substack{1232111\\2}, \substack{\cdot\cdot\cdot\cdot111\\0}\}$,

$X^{16}_{29} = \{\substack{24654\cdot\cdot\\3}, \substack{\cdot\cdot\cdot\cdot321\\2}, \substack{\cdot\cdot432\cdot1\\2}, \substack{1233211\\2}, \substack{1232221\\2}, \substack{1233221\\1}, \substack{1232111\\2}, \substack{1232211\\1}, \substack{\cdot\cdot\cdot\cdot111\\0}\}$,

$X^{17}_{29} = \{\substack{24654\cdot\cdot\\3}, \substack{2454321\\2}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\2}, \substack{12\cdot3\cdot\cdot1\\2}, \substack{1232221\\2}, \substack{1233221\\1}, \substack{1232111\\2}, \substack{1221111\\1}, \substack{\cdot\cdot\cdot\cdot111\\0}\}$,

$X^{18}_{29} = \{\substack{2465432\\3}, \substack{2465\cdot21\\3}, \substack{2454321\\2}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\2}, \substack{12\cdot\cdot\cdot21\\2}, \substack{1232\cdot11\\2}, \substack{12\cdot2221\\1}, \substack{\cdot\cdot\cdot1111\\0}, \substack{0000001\\0}\}$,

$$X_{29}^{19} = \{\substack{24654\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\2}, \substack{13\cdot4321\\2}, \substack{1\cdot\cdot\cdot2\cdot1\\2}, \substack{12\cdot3321\\2}, \substack{1232111\\2}, \substack{1221111\\1}, \substack{1122111\\1}, \substack{\cdot\cdot\cdot\cdot111\\0}\},$$

$$X_{29}^{20} = \{\substack{246543\cdot\\3}, \substack{2\cdot\cdot\cdot321\\2}, \substack{13\cdot4321\\2}, \substack{\cdot\cdot\cdot\cdot\cdot11\\2}, \substack{1\cdot\cdot\cdot221\\2}, \substack{12\cdot3321\\2}, \substack{1221111\\1}, \substack{1122111\\1}, \substack{0122211\\1}, \substack{\cdot\cdot\cdot\cdot\cdot11\\0}\},$$

$$X_{29}^{21} = \{\substack{24\cdot\cdot\cdot\cdot\cdot\\3}, \substack{\cdot\cdot\cdot\cdot\cdot\cdot1\\2}, \substack{\cdot111111\\0}\},$$

$$X_{29}^{22} = \{\substack{246\cdot\cdot\cdot\cdot\\3}, \substack{\cdot\cdot\cdot\cdot\cdot21\\2}, \substack{\cdot\cdot\cdot\cdot211\\2}, \substack{1233321\\1}, \substack{\cdot\cdot11111\\0}\},$$

$$X_{29}^{23} = \{\substack{246\cdot\cdot\cdot\cdot\\3}, \substack{2454321\\2}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\2}, \substack{12\cdot\cdot\cdot21\\2}, \substack{123\cdot\cdot11\\2}, \substack{1222221\\1}, \substack{\cdot\cdot11111\\0}\},$$

$$X_{29}^{24} = \{\substack{246\cdot\cdot\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\2}, \substack{1\cdot4\cdot\cdot\cdot1\\2}, \substack{123\cdot\cdot\cdot1\\2}, \substack{1111111\\1}, \substack{\cdot\cdot11111\\0}\},$$

$$X_{29}^{25} = \{\substack{2465\cdot\cdot\cdot\\3}, \substack{\cdot\cdot\cdot4321\\2}, \substack{\cdot\cdot\cdot3\cdot\cdot1\\2}, \substack{1233\cdot21\\1}, \substack{1232221\\2}, \substack{\cdot\cdot\cdot1111\\0}\},$$

$$X_{29}^{26} = \{\substack{24654\cdot\cdot\\3}, \substack{\cdot\cdot54321\\2}, \substack{\cdot\cdot4\cdot\cdot\cdot1\\2}, \substack{1233321\\\cdot}, \substack{1233221\\2}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{1232221\\1}, \substack{\cdot\cdot\cdot\cdot111\\0}\},$$

$$X_{29}^{27} = \{\substack{2465\cdot\cdot\cdot\\3}, \substack{2454321\\2}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\2}, \substack{12\cdot\cdot\cdot21\\2}, \substack{1232\cdot11\\2}, \substack{12\cdot2221\\1}, \substack{\cdot\cdot\cdot1111\\0}\},$$

$$X_{29}^{28} = \{\substack{2465\cdot\cdot\cdot\\3}, \substack{\cdot\cdot54321\\2}, \substack{\cdot\cdot4\cdot\cdot\cdot1\\2}, \substack{123\cdot2\cdot1\\2}, \substack{1233321\\1}, \substack{1232111\\1}, \substack{\cdot\cdot\cdot1111\\0}\},$$

$$X_{29}^{29} = \{\substack{2465\cdot\cdot\cdot\\3}, \substack{2454321\\2}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\2}, \substack{12\cdot3\cdot\cdot1\\2}, \substack{1233321\\1}, \substack{12322\cdot1\\2}, \substack{1221111\\1}, \substack{\cdot\cdot\cdot1111\\0}\},$$

$$X_{29}^{30} = \{\substack{2465\cdot\cdot\cdot\\3}, \substack{\cdot\cdot54321\\2}, \substack{\cdot\cdot4\cdot\cdot\cdot1\\2}, \substack{1233321\\2}, \substack{12322\cdot1\\\cdot}, \substack{1232111\\2}, \substack{\cdot\cdot\cdot1111\\0}\},$$

$$X_{29}^{31} = \{\substack{2465\cdot\cdot\cdot\\3}, \substack{2454321\\2}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\2}, \substack{12\cdot\cdot321\\2}, \substack{1243221\\2}, \substack{1233211\\2}, \substack{1232\cdot\cdot1\\2}, \substack{1232211\\1}, \substack{1222221\\1}, \substack{\cdot\cdot\cdot1111\\0}\},$$

$$X_{29}^{32} = \{\substack{2465\cdot\cdot\cdot\\3}, \substack{2454321\\2}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\2}, \substack{12\cdot3\cdot21\\2}, \substack{1243211\\2}, \substack{1232\cdot\cdot1\\2}, \substack{1232221\\1}, \substack{1221111\\1}, \substack{\cdot\cdot\cdot1111\\0}\},$$

$$X_{29}^{33} = \{\substack{2465\cdot\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\2}, \substack{1\cdot4\cdot\cdot\cdot1\\2}, \substack{123\cdot\cdot21\\2}, \substack{1232\cdot11\\2}, \substack{1232221\\1}, \substack{1111111\\1}, \substack{\cdot\cdot\cdot1111\\0}\},$$

$$X_{29}^{34} = \{\substack{2465\cdot\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\2}, \substack{1354321\\2}, \substack{1\cdot\cdot3\cdot\cdot1\\2}, \substack{1232\cdot\cdot1\\2}, \substack{1\cdot21111\\1}, \substack{\cdot\cdot\cdot1111\\0}\},$$

$$X_{29}^{35} = \{\substack{2465\cdot\cdot\cdot\\3}, \substack{2\cdot54321\\2}, \substack{1\cdot44321\\2}, \substack{\cdot\cdot\cdot3\cdot\cdot1\\2}, \substack{1232\cdot\cdot1\\2}, \substack{1111111\\1}, \substack{0121111\\1}, \substack{\cdot\cdot\cdot1111\\0}\},$$

$$X_{29}^{36} = \{\substack{24654\cdot\cdot\\3}, \substack{2454321\\2}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\2}, \substack{12\cdot\cdot2\cdot1\\2}, \substack{12\cdot\cdot111\\\cdot}, \substack{\cdot\cdot\cdot\cdot111\\0}\},$$

$$X_{29}^{37} = \{\substack{2465432\\3}, \substack{2\cdot54321\\2}, \substack{1\cdot44321\\2}, \substack{\cdot\cdot\cdot3\cdot\cdot1\\\cdot}, \substack{1111111\\1}, \substack{0121111\\1}, \substack{\cdot\cdot\cdot\cdot\cdot\cdot1\\0}\},$$

$$X_{29}^{38} = \{\substack{24654\cdot\cdot\\3}, \substack{2454321\\2}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\2}, \substack{12\cdot3\cdot\cdot1\\2}, \substack{12332\cdot1\\1}, \substack{1232111\\2}, \substack{1221111\\1}, \substack{\cdot\cdot\cdot\cdot111\\0}\},$$

$$X_{29}^{39} = \{\substack{24654\cdot\cdot\\3}, \substack{\cdot\cdot54321\\2}, \substack{\cdot\cdot4\cdot\cdot\cdot1\\2}, \substack{1233211\\2}, \substack{1232221\\2}, \substack{1233221\\1}, \substack{1232111\\2}, \substack{1232\cdot11\\1}, \substack{\cdot\cdot\cdot\cdot111\\0}\},$$

$$X_{29}^{40} = \{\substack{24654\cdot\cdot\\3}, \substack{2454321\\2}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\2}, \substack{1244321\\2}, \substack{12432\cdot1\\2}, \substack{1233321\\2}, \substack{1233211\\2}, \substack{1232221\\2}, \substack{1233221\\1}, \substack{1232111\\2}, \substack{1232211\\1}, \substack{1222111\\1}, \substack{\cdot\cdot\cdot\cdot111\\0}\},$$

$$X_{29}^{41} = \{\substack{24654\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\2}, \substack{1\cdot4\cdot\cdot\cdot1\\2}, \substack{1233321\\2}, \substack{1233211\\2}, \substack{1232221\\2}, \substack{1233221\\1}, \substack{1232111\\2}, \substack{1232211\\1}, \substack{1111111\\1}, \substack{\cdot\cdot\cdot\cdot111\\0}\},$$

$$X_{29}^{42} = \{\substack{24654\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\2}, \substack{13\cdot4321\\2}, \substack{1243321\\2}, \substack{1\cdot\cdot\cdot2\cdot1\\2}, \substack{1232111\\\cdot}, \substack{1221111\\1}, \substack{1122111\\1}, \substack{\cdot\cdot\cdot\cdot111\\0}\},$$

$$X_{29}^{43} = \{\substack{24654\cdot\cdot\\3}, \substack{2\cdot54321\\2}, \substack{1\cdot44321\\2}, \substack{\cdot\cdot\cdot3\cdot\cdot1\\2}, \substack{1232221\\2}, \substack{1233221\\1}, \substack{1232111\\2}, \substack{1111111\\1}, \substack{0121111\\1}, \substack{\cdot\cdot\cdot\cdot111\\0}\},$$

$$X_{29}^{44} = \{\substack{2465432\\3}, \substack{2465421\\3}, \substack{2454321\\3}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\2}, \substack{12\cdot\cdot\cdot21\\2}, \substack{1243211\\2}, \substack{123\cdot221\\1}, \substack{1232111\\2}, \substack{\cdot111111\\0}, \substack{0011111\\1}, \substack{000\cdot111\\0}, \substack{0000001\\0}\},$$

$$X_{29}^{45} = \{\substack{24\cdot\cdot\cdot\cdot\cdot\\\cdot}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\2}, \substack{12\cdot\cdot321\\\cdot}, \substack{12\cdot\cdot2\cdot1\\2}, \substack{\cdot111111\\0}\},$$

$$X_{29}^{46} = \{\substack{24\cdot\cdot\cdot\cdot\cdot\\\cdot}, \substack{23\cdot\cdot\cdot\cdot1\\2}, \substack{1\cdot\cdot\cdot\cdot21\\2}, \substack{12\cdot\cdot\cdot11\\2}, \substack{1122221\\1}, \substack{\cdot111111\\0}\},$$

$$X_{29}^{47} = \{\substack{24\cdot\cdot\cdot\cdot\cdot\\3}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\2}, \substack{12\cdot\cdot\cdot\cdot1\\2}, \substack{\cdot111111\\0}, \substack{0011111\\1}\},$$

$$X_{29}^{48} = \{ \substack{246\cdot\cdot\cdot\cdot\\ 3}, \substack{\cdot\cdot\cdot4321\\ 2}, \substack{\cdot\cdot\cdot3\cdot\cdot1\\ 2}, \substack{1233\cdot21\\ 1}, \substack{1232221\\ 2}, \substack{\cdot\cdot11111\\ 0} \},$$

$$X_{29}^{49} = \{ \substack{246\cdot\cdot\cdot\cdot\\ 3}, \substack{2454321\\ 2}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\ 2}, \substack{12\cdot\cdot\cdot21\\ 2}, \substack{1233321\\ 1}, \substack{123\cdot211\\ 2}, \substack{1222221\\ 1}, \substack{\cdot\cdot11111\\ 0} \},$$

$$X_{29}^{50} = \{ \substack{246\cdot\cdot\cdot\cdot\\ 3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\ 2}, \substack{1354321\\ 2}, \substack{1\cdot4\cdot\cdot21\\ 2}, \substack{123\cdot\cdot\cdot1\\ 2}, \substack{1\cdot22221\\ 1}, \substack{\cdot\cdot11111\\ 0} \},$$

$$X_{29}^{51} = \{ \substack{246\cdot\cdot\cdot\cdot\\ 3}, \substack{\cdot\cdot54321\\ 2}, \substack{\cdot\cdot4\cdot\cdot\cdot1\\ 2}, \substack{1233\cdot21\\ 2}, \substack{1232\cdot11\\ 2}, \substack{1233211\\ 1}, \substack{1232221\\ 1}, \substack{\cdot\cdot11111\\ 0} \},$$

$$X_{29}^{52} = \{ \substack{246\cdot\cdot\cdot\cdot\\ 3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\ 2}, \substack{13\cdot\cdot\cdot\cdot1\\ 2}, \substack{1244321\\ 2}, \substack{12\cdot\cdot2\cdot1\\ 2}, \substack{1233321\\ \cdot}, \substack{1222111\\ 1}, \substack{\cdot\cdot11111\\ 0} \},$$

$$X_{29}^{53} = \{ \substack{246\cdot\cdot\cdot\cdot\\ 3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\ 2}, \substack{1\cdot4\cdot\cdot\cdot1\\ 2}, \substack{1233321\\ \cdot}, \substack{123\cdot2\cdot1\\ 2}, \substack{1111111\\ 1}, \substack{\cdot\cdot11111\\ 0} \},$$

$$X_{29}^{54} = \{ \substack{246\cdot\cdot\cdot\cdot\\ 3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\ 2}, \substack{13\cdot\cdot\cdot\cdot1\\ 2}, \substack{12\cdot\cdot321\\ 2}, \substack{123\cdot2\cdot1\\ 2}, \substack{1232111\\ 2}, \substack{12222\cdot1\\ 1}, \substack{\cdot\cdot11111\\ 0} \},$$

$$X_{29}^{55} = \{ \substack{246\cdot\cdot\cdot\cdot\\ 3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\ 2}, \substack{1\cdot\cdot\cdot321\\ 2}, \substack{1343211\\ 2}, \substack{1243221\\ 2}, \substack{123\cdot2\cdot1\\ 2}, \substack{1232111\\ 2}, \substack{1222221\\ 1}, \substack{1122211\\ 1}, \substack{\cdot\cdot11111\\ 0} \},$$

$$X_{29}^{56} = \{ \substack{246\cdot\cdot\cdot\cdot\\ 3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\ 2}, \substack{134\cdot\cdot\cdot1\\ 2}, \substack{12\cdot\cdot\cdot21\\ 2}, \substack{123\cdot\cdot11\\ 2}, \substack{1222221\\ 1}, \substack{1111111\\ 1}, \substack{\cdot\cdot11111\\ 0} \},$$

$$X_{29}^{57} = \{ \substack{246\cdot\cdot\cdot\cdot\\ 3}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\ 2}, \substack{12\cdot\cdot\cdot21\\ 2}, \substack{123\cdot\cdot11\\ 2}, \substack{1222221\\ 1}, \substack{\cdot\cdot11111\\ 0}, \substack{0011111\\ 1} \},$$

$$X_{29}^{58} = \{ \substack{246\cdot\cdot\cdot\cdot\\ 3}, \substack{2454321\\ 2}, \substack{\cdot\cdot4\cdot\cdot\cdot1\\ 2}, \substack{123\cdot\cdot\cdot1\\ 2}, \substack{\cdot111111\\ 1}, \substack{\cdot\cdot11111\\ 0} \},$$

$$X_{29}^{59} = \{ \substack{2465\cdot\cdot\cdot\\ 3}, \substack{\cdot\cdot54321\\ 2}, \substack{\cdot\cdot4\cdot\cdot\cdot1\\ 2}, \substack{1233321\\ \cdot}, \substack{12322\cdot1\\ \cdot}, \substack{\cdot\cdot\cdot1111\\ 0} \},$$

$$X_{29}^{60} = \{ \substack{24654\cdot\cdot\\ 3}, \substack{2\cdot54321\\ \cdot}, \substack{1\cdot\cdot\cdot321\\ \cdot}, \substack{23432\cdot1\\ 2}, \substack{1\cdot43211\\ 2}, \substack{123\cdot221\\ \cdot}, \substack{1\cdot22211\\ 1}, \substack{012\cdot111\\ 1}, \substack{000\cdot111\\ 0} \},$$

$$X_{29}^{61} = \{ \substack{2465\cdot\cdot\cdot\\ 3}, \substack{2454321\\ 2}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\ 2}, \substack{12\cdot\cdot321\\ \cdot}, \substack{1243211\\ 2}, \substack{1233221\\ 2}, \substack{12322\cdot1\\ 2}, \substack{1232221\\ 1}, \substack{1222211\\ 1}, \substack{\cdot\cdot\cdot1111\\ 0} \},$$

$$X_{29}^{62} = \{ \substack{24654\cdot\cdot\\ 3}, \substack{\cdot\cdot54321\\ 2}, \substack{\cdot\cdot4\cdot\cdot\cdot1\\ 2}, \substack{1233221\\ 2}, \substack{1233321\\ 1}, \substack{1232211\\ 2}, \substack{1233211\\ 1}, \substack{1232221\\ 1}, \substack{1232111\\ 1}, \substack{\cdot\cdot\cdot\cdot111\\ 0} \},$$

$$X_{29}^{63} = \{ \substack{2465\cdot\cdot\cdot\\ 3}, \substack{2454321\\ 2}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\ 2}, \substack{1244321\\ 2}, \substack{12432\cdot1\\ 2}, \substack{1233321\\ \cdot}, \substack{123\cdot221\\ 2}, \substack{1232211\\ 2}, \substack{1232221\\ 1}, \substack{1222111\\ 1}, \substack{\cdot\cdot\cdot1111\\ 0} \},$$

$$X_{29}^{64} = \{ \substack{24654\cdot\cdot\\ 3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\ 2}, \substack{1\cdot4\cdot\cdot\cdot1\\ 2}, \substack{1233\cdot21\\ 2}, \substack{1233321\\ 1}, \substack{1232211\\ 2}, \substack{1233211\\ 1}, \substack{1232221\\ 1}, \substack{1111111\\ 1}, \substack{\cdot\cdot\cdot\cdot111\\ 0} \},$$

$$X_{29}^{65} = \{ \substack{2465\cdot\cdot\cdot\\ 3}, \substack{2454321\\ 2}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\ 2}, \substack{1244321\\ 2}, \substack{12\cdot\cdot2\cdot1\\ 2}, \substack{1233321\\ 1}, \substack{12\cdot2111\\ 1}, \substack{\cdot\cdot\cdot1111\\ 0} \},$$

$$X_{29}^{66} = \{ \substack{2465\cdot\cdot\cdot\\ 3}, \substack{2454321\\ 2}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\ 2}, \substack{1243321\\ 2}, \substack{12\cdot\cdot2\cdot1\\ 2}, \substack{1233321\\ 1}, \substack{1232111\\ 1}, \substack{1221111\\ 1}, \substack{\cdot\cdot\cdot1111\\ 0} \},$$

$$X_{29}^{67} = \{ \substack{2465\cdot\cdot\cdot\\ 3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\ 2}, \substack{1\cdot4\cdot\cdot\cdot1\\ 2}, \substack{123\cdot2\cdot1\\ 2}, \substack{1233321\\ 1}, \substack{1232111\\ 1}, \substack{1111111\\ 1}, \substack{\cdot\cdot\cdot1111\\ 0} \},$$

$$X_{29}^{68} = \{ \substack{24654\cdot\cdot\\ 3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\ 2}, \substack{13\cdot4321\\ 2}, \substack{12\cdot3\cdot21\\ 2}, \substack{1343211\\ 2}, \substack{1232\cdot11\\ 2}, \substack{1233211\\ 1}, \substack{12\cdot2221\\ 1}, \substack{1122\cdot11\\ 1}, \substack{1221111\\ 1}, \substack{\cdot\cdot\cdot\cdot111\\ 0} \},$$

$$X_{29}^{69} = \{ \substack{2465\cdot\cdot\cdot\\ 3}, \substack{2\cdot54321\\ 2}, \substack{1\cdot44321\\ 2}, \substack{\cdot\cdot\cdot3\cdot\cdot1\\ 2}, \substack{1233321\\ 1}, \substack{12322\cdot1\\ 2}, \substack{1111111\\ 1}, \substack{0121111\\ 1}, \substack{\cdot\cdot\cdot1111\\ 0} \},$$

$$X_{29}^{70} = \{ \substack{24654\cdot\cdot\\ 3}, \substack{2464321\\ 3}, \substack{2454321\\ 2}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\ 2}, \substack{1244321\\ 2}, \substack{12\cdot3\cdot21\\ 2}, \substack{1233211\\ 2}, \substack{12332\cdot1\\ 1}, \substack{1232111\\ 2}, \substack{1222221\\ 1}, \substack{\cdot\cdot11111\\ 0}, \substack{0000111\\ 0} \},$$

$$X_{29}^{71} = \{ \substack{2465\cdot\cdot\cdot\\ 3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\ 2}, \substack{1\cdot\cdot\cdot321\\ 2}, \substack{1343211\\ 2}, \substack{12\cdot3221\\ 2}, \substack{1232\cdot\cdot1\\ 2}, \substack{12\cdot2221\\ 1}, \substack{1122211\\ 1}, \substack{\cdot\cdot\cdot1111\\ 0} \},$$

$$X_{29}^{72} = \{ \substack{24654\cdot\cdot\\ 3}, \substack{2454321\\ 3}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\ 2}, \substack{12\cdot\cdot\cdot21\\ 2}, \substack{1243211\\ 2}, \substack{123\cdot221\\ 1}, \substack{1232111\\ 2}, \substack{\cdot111111\\ 0}, \substack{0011111\\ 1}, \substack{000\cdot111\\ 0} \},$$

$$X_{29}^{73} = \{ \substack{2465432\\ 3}, \substack{246\cdot\cdot21\\ 3}, \substack{\cdot\cdot54321\\ 2}, \substack{\cdot\cdot4\cdot\cdot\cdot1\\ 2}, \substack{123\cdot\cdot21\\ \cdot}, \substack{\cdot\cdot11111\\ 0}, \substack{0000001\\ 0} \},$$

$$X_{29}^{74}=\{\substack{2465\cdot\cdot\cdot\\3},\substack{2454321\\2},\substack{\cdot3\cdot\cdot\cdot\cdot1\\2},\substack{1244321\\2},\substack{12432\cdot1\\2},\substack{1233321\\2},\substack{12322\cdot1\\\cdot},\substack{1232111\\2},\substack{1222111\\1},\substack{\cdot\cdot\cdot1111\\0}\},$$

$$X_{29}^{75}=\{\substack{2465432\\3},\substack{2465\cdot21\\3},\substack{2454321\\\cdot},\substack{\cdot3\cdot\cdot\cdot\cdot1\\2},\substack{12\cdot\cdot\cdot21\\2},\substack{1243211\\2},\substack{123\cdot\cdot21\\1},\substack{\cdot111111\\0},\substack{0001111\\0},\substack{0000001\\0}\},$$

$$X_{29}^{76}=\{\substack{2465\cdot\cdot\cdot\\3},\substack{2\cdot\cdot\cdot\cdot\cdot1\\2},\substack{1\cdot4\cdot\cdot\cdot1\\2},\substack{1233321\\2},\substack{12322\cdot1\\\cdot},\substack{1232111\\2},\substack{1111111\\1},\substack{\cdot\cdot\cdot1111\\0}\},$$

$$X_{29}^{77}=\{\substack{2465\cdot\cdot\cdot\\3},\substack{2\cdot\cdot\cdot\cdot\cdot1\\2},\substack{13\cdot\cdot\cdot21\\2},\substack{124\cdot321\\2},\substack{12\cdot\cdot211\\2},\substack{123\cdot221\\2},\substack{1232111\\\cdot},\substack{1222211\\1},\substack{1122221\\1},\substack{\cdot\cdot\cdot1111\\0}\},$$

$$X_{29}^{78}=\{\substack{2465\cdot\cdot\cdot\\3},\substack{2454321\\2},\substack{\cdot3\cdot\cdot\cdot\cdot1\\2},\substack{12\cdot3321\\2},\substack{1243221\\2},\substack{1233211\\2},\substack{1232\cdot\cdot1\\2},\substack{1232211\\1},\substack{1222221\\1},\substack{1221111\\1},\substack{\cdot\cdot\cdot1111\\0}\},$$

$$X_{29}^{79}=\{\substack{2465\cdot\cdot\cdot\\3},\substack{2\cdot\cdot\cdot\cdot\cdot1\\2},\substack{13\cdot\cdot321\\2},\substack{12\cdot3\cdot21\\2},\substack{1\cdot43211\\2},\substack{1232\cdot\cdot1\\2},\substack{1232221\\1},\substack{1122211\\1},\substack{1221111\\1},\substack{\cdot\cdot\cdot1111\\0}\},$$

$$X_{29}^{80}=\{\substack{2465\cdot\cdot\cdot\\3},\substack{\cdot3\cdot\cdot\cdot\cdot1\\2},\substack{12\cdot\cdot321\\2},\substack{1243221\\2},\substack{1233211\\2},\substack{1232\cdot\cdot1\\2},\substack{1232211\\1},\substack{1222221\\1},\substack{\cdot\cdot\cdot1111\\0},\substack{0011111\\1}\},$$

$$X_{29}^{81}=\{\substack{24654\cdot\cdot\\3},\substack{2\cdot\cdot\cdot\cdot\cdot1\\2},\substack{1\cdot\cdot\cdot321\\2},\substack{1343211\\2},\substack{12\cdot3221\\2},\substack{1232\cdot11\\2},\substack{1233211\\1},\substack{12\cdot2221\\1},\substack{1122211\\1},\substack{\cdot\cdot\cdot\cdot111\\0}\},$$

$$X_{29}^{82}=\{\substack{2465\cdot\cdot\cdot\\3},\substack{2\cdot\cdot\cdot\cdot\cdot1\\2},\substack{134\cdot\cdot\cdot1\\2},\substack{1243\cdot\cdot1\\2},\substack{123\cdot\cdot21\\2},\substack{1232\cdot11\\2},\substack{1232221\\1},\substack{1221111\\1},\substack{1111111\\1},\substack{\cdot\cdot\cdot1111\\0}\},$$

$$X_{29}^{83}=\{\substack{2465\cdot\cdot\cdot\\3},\substack{\cdot3\cdot\cdot\cdot\cdot1\\2},\substack{12\cdot3\cdot21\\2},\substack{1243211\\2},\substack{1232\cdot\cdot1\\2},\substack{1232221\\1},\substack{1221111\\1},\substack{\cdot\cdot\cdot1111\\0},\substack{0011111\\1}\},$$

$$X_{29}^{84}=\{\substack{2465\cdot\cdot\cdot\\3},\substack{2454321\\2},\substack{\cdot\cdot4\cdot\cdot\cdot1\\2},\substack{123\cdot\cdot21\\2},\substack{1232\cdot11\\2},\substack{1232221\\1},\substack{\cdot111111\\1},\substack{\cdot\cdot\cdot1111\\0}\},$$

$$X_{29}^{85}=\{\substack{2465432\\3},\substack{\cdot\cdot54321\\\cdot},\substack{\cdot\cdot4\cdot\cdot\cdot1\\2},\substack{123\cdot\cdot21\\\cdot},\substack{000\cdot\cdot\cdot1\\0}\},$$

$$X_{29}^{86}=\{\substack{2465432\\3},\substack{\cdot\cdot54321\\\cdot},\substack{\cdot\cdot4\cdot\cdot\cdot1\\2},\substack{1233\cdot\cdot1\\\cdot},\substack{000\cdot\cdot\cdot1\\0}\},$$

$$X_{29}^{87}=\{\substack{2465432\\3},\substack{\cdot\cdot54321\\\cdot},\substack{\cdot\cdot4\cdot\cdot\cdot1\\2},\substack{1233321\\\cdot},\substack{1233221\\2},\substack{1232211\\2},\substack{1233211\\1},\substack{1232221\\1},\substack{000\cdot\cdot\cdot1\\0}\},$$

$$X_{29}^{88}=\{\substack{24654\cdot\cdot\\3},\substack{2\cdot\cdot\cdot\cdot\cdot1\\2},\substack{13\cdot\cdot\cdot\cdot1\\2},\substack{12432\cdot1\\2},\substack{1233321\\2},\substack{1233211\\2},\substack{1232221\\2},\substack{1233221\\1},\substack{1232111\\2},\substack{1232211\\1},\substack{122\cdot111\\1},\substack{\cdot\cdot\cdot\cdot111\\0}\},$$

$$X_{29}^{89}=\{\substack{2465432\\3},\substack{\cdot\cdot4\cdot\cdot\cdot1\\2},\substack{123\cdot\cdot11\\\cdot},\substack{\cdot\cdot11111\\1},\substack{\cdot\cdot\cdot\cdot\cdot\cdot1\\0}\},$$

$$X_{29}^{90}=\{\substack{24654\cdot\cdot\\3},\substack{2\cdot\cdot\cdot\cdot\cdot1\\2},\substack{1354321\\2},\substack{1343\cdot\cdot1\\2},\substack{1244321\\2},\substack{12432\cdot1\\2},\substack{1233321\\2},\substack{1233211\\2},\substack{1232221\\2},\substack{1233221\\1},\substack{1232111\\2},\substack{1232211\\1},\substack{1222111\\1},\substack{1121111\\1},\substack{\cdot\cdot\cdot\cdot111\\0}\},$$

$$X_{29}^{91}=\{\substack{2465432\\3},\substack{\cdot3\cdot\cdot\cdot\cdot1\\2},\substack{12432\cdot1\\2},\substack{1233321\\2},\substack{1233\cdot21\\1},\substack{1233211\\2},\substack{1232221\\2},\substack{1232211\\1},\substack{122\cdot111\\1},\substack{\cdot\cdot\cdot\cdot\cdot\cdot1\\0},\substack{0011111\\1}\},$$

$$X_{29}^{92}=\{\substack{2465432\\3},\substack{\cdot\cdot4\cdot\cdot\cdot1\\2},\substack{1233321\\\cdot},\substack{1233221\\2},\substack{1232211\\2},\substack{1233211\\1},\substack{1232221\\1},\substack{\cdot\cdot11111\\\cdot},\substack{000\cdot\cdot\cdot1\\0}\},$$

$$X_{29}^{93}=\{\substack{2465432\\3},\substack{2\cdot54321\\2},\substack{1354321\\3},\substack{\cdot\cdot4\cdot\cdot\cdot1\\2},\substack{1233\cdot21\\1},\substack{1233211\\2},\substack{1232221\\2},\substack{1232\cdot11\\1},\substack{1111111\\1},\substack{0\cdot\cdot\cdot\cdot\cdot1\\0}\},$$

$$X_{29}^{94}=\{\substack{24654\cdot\cdot\\3},\substack{\cdot3\cdot\cdot\cdot\cdot1\\2},\substack{12\cdot\cdot2\cdot1\\2},\substack{12\cdot\cdot111\\\cdot},\substack{\cdot\cdot\cdot\cdot111\\0},\substack{0011111\\1}\},$$

$$X_{29}^{95}=\{\substack{24654\cdot\cdot\\3},\substack{\cdot3\cdot\cdot\cdot\cdot1\\2},\substack{124\cdot321\\2},\substack{12\cdot3211\\2},\substack{123\cdot221\\2},\substack{1233221\\1},\substack{1232111\\\cdot},\substack{1222211\\1},\substack{\cdot\cdot\cdot\cdot111\\0},\substack{0011111\\1}\},$$

$$X_{29}^{96} = \{ \substack{24654\cdot\cdot\\3}, \substack{2454321\\2}, \substack{\cdot\cdot4\cdot\cdot\cdot1\\2}, \substack{1233\cdot21\\2}, \substack{1232\cdot11\\2}, \substack{1233211\\1}, \substack{1232221\\1}, \substack{\cdot111111\\1}, \substack{\cdot\cdot\cdot\cdot111\\0} \},$$

$$X_{29}^{97} = \{ \substack{2465432\\3}, \substack{2454321\\\cdot}, \substack{2354321\\2}, \substack{1354321\\3}, \substack{\cdot\cdot4\cdot\cdot\cdot1\\2}, \substack{1233321\\\cdot}, \substack{1233221\\2}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{1232221\\1}, \substack{1111111\\1}, \substack{0111111\\0}, \substack{000\cdot\cdot\cdot1\\0} \},$$

$$X_{29}^{98} = \{ \substack{246543\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot21\\\cdot}, \substack{1\cdot\cdot\cdot\cdot\cdot1\\2}, \substack{0122221\\1}, \substack{1111111\\0} \},$$

$$X_{29}^{99} = \{ \substack{24\cdot\cdot\cdot\cdot\cdot\\\cdot}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\2}, \substack{1244321\\2}, \substack{12\cdot3\cdot\cdot1\\2}, \substack{1233\cdot21\\1}, \substack{1232221\\2}, \substack{\cdot111111\\0} \},$$

$$X_{29}^{100} = \{ \substack{24\cdot\cdot\cdot\cdot\cdot\\\cdot}, \substack{23\cdot\cdot\cdot\cdot1\\2}, \substack{13\cdot\cdot\cdot21\\2}, \substack{12\cdot\cdot\cdot21\\2}, \substack{12\cdot\cdot211\\2}, \substack{1233321\\1}, \substack{1122221\\1}, \substack{\cdot111111\\0} \},$$

$$X_{29}^{101} = \{ \substack{24\cdot\cdot\cdot\cdot\cdot\\\cdot}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\2}, \substack{124\cdot\cdot\cdot1\\2}, \substack{1233\cdot21\\2}, \substack{1232\cdot11\\2}, \substack{1233211\\1}, \substack{1232221\\1}, \substack{\cdot111111\\0} \},$$

$$X_{29}^{102} = \{ \substack{24\cdot\cdot\cdot\cdot\cdot\\\cdot}, \substack{23\cdot\cdot\cdot\cdot1\\2}, \substack{1\cdot\cdot4321\\2}, \substack{1\cdot\cdot\cdot2\cdot1\\2}, \substack{12\cdot3321\\\cdot}, \substack{1122111\\1}, \substack{\cdot111111\\0} \},$$

$$X_{29}^{103} = \{ \substack{24\cdot\cdot\cdot\cdot\cdot\\3}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\2}, \substack{12\cdot\cdot321\\\cdot}, \substack{12\cdot\cdot2\cdot1\\2}, \substack{\cdot111111\\0}, \substack{0011111\\1} \},$$

$$X_{29}^{104} = \{ \substack{24\cdot\cdot\cdot\cdot\cdot\\\cdot}, \substack{\cdot3\cdot\cdot\cdot21\\2}, \substack{12\cdot\cdot\cdot\cdot1\\2}, \substack{\cdot122221\\1}, \substack{\cdot111111\\0} \},$$

$$X_{29}^{105} = \{ \substack{24\cdot\cdot\cdot\cdot\cdot\\\cdot}, \substack{23\cdot\cdot\cdot\cdot1\\2}, \substack{1\cdot\cdot\cdot321\\2}, \substack{12\cdot\cdot2\cdot1\\2}, \substack{1232111\\2}, \substack{11222\cdot1\\1}, \substack{\cdot111111\\0} \},$$

$$X_{29}^{106} = \{ \substack{24\cdot\cdot\cdot\cdot\cdot\\\cdot}, \substack{\cdot3\cdot\cdot321\\2}, \substack{2343221\\2}, \substack{12\cdot\cdot\cdot\cdot1\\2}, \substack{1343211\\2}, \substack{1122211\\1}, \substack{0122221\\1}, \substack{\cdot111111\\0} \},$$

$$X_{29}^{107} = \{ \substack{24\cdot\cdot\cdot\cdot\cdot\\3}, \substack{23\cdot\cdot\cdot\cdot1\\2}, \substack{13\cdot\cdot\cdot21\\2}, \substack{12\cdot\cdot\cdot\cdot1\\2}, \substack{1122221\\1}, \substack{\cdot111111\\0}, \substack{0011111\\1} \},$$

$$X_{29}^{108} = \{ \substack{246\cdot\cdot\cdot\cdot\\3}, \substack{\cdot\cdot54321\\2}, \substack{\cdot\cdot4\cdot\cdot\cdot1\\2}, \substack{123\cdot\cdot21\\\cdot}, \substack{\cdot\cdot11111\\0} \},$$

$$X_{29}^{109} = \{ \substack{246\cdot\cdot\cdot\cdot\\3}, \substack{2454321\\2}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\2}, \substack{12\cdot\cdot\cdot21\\2}, \substack{1233\cdot21\\1}, \substack{1233211\\2}, \substack{1222221\\1}, \substack{\cdot\cdot11111\\0} \},$$

$$X_{29}^{110} = \{ \substack{24654\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\2}, \substack{1\cdot\cdot\cdot321\\2}, \substack{1343221\\2}, \substack{12\cdot3211\\2}, \substack{1233\cdot21\\1}, \substack{1232221\\2}, \substack{12\cdot2211\\1}, \substack{1122221\\1}, \substack{\cdot\cdot\cdot\cdot111\\0} \},$$

$$X_{29}^{111} = \{ \substack{246543\cdot\\3}, \substack{\cdot\cdot54321\\\cdot}, \substack{\cdot\cdot4\cdot\cdot\cdot1\\2}, \substack{123\cdot\cdot21\\\cdot}, \substack{000\cdot\cdot11\\0} \},$$

$$X_{29}^{112} = \{ \substack{246\cdot\cdot\cdot\cdot\\3}, \substack{\cdot\cdot54321\\2}, \substack{\cdot\cdot4\cdot\cdot\cdot1\\2}, \substack{1233321\\\cdot}, \substack{1233211\\2}, \substack{1232221\\2}, \substack{1233221\\1}, \substack{1232211\\1}, \substack{\cdot\cdot11111\\0} \},$$

$$X_{29}^{113} = \{ \substack{246\cdot\cdot\cdot\cdot\\3}, \substack{2454321\\2}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\2}, \substack{12\cdot\cdot321\\\cdot}, \substack{12\cdot3211\\2}, \substack{123\cdot221\\2}, \substack{1233221\\1}, \substack{1222211\\1}, \substack{\cdot\cdot11111\\0} \},$$

$$X_{29}^{114} = \{ \substack{246\cdot\cdot\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\2}, \substack{1344321\\2}, \substack{1243\cdot21\\2}, \substack{1343211\\2}, \substack{1233\cdot21\\1}, \substack{1233211\\2}, \substack{1232221\\2}, \substack{1232\cdot11\\1}, \substack{1222221\\1}, \substack{1122\cdot11\\1}, \substack{1221111\\1}, \substack{1111111\\1}, \substack{\cdot\cdot11111\\0} \},$$

$$X_{29}^{115} = \{ \substack{246\cdot\cdot\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\2}, \substack{1\cdot4\cdot\cdot\cdot1\\2}, \substack{1233\cdot\cdot1\\2}, \substack{1233\cdot21\\1}, \substack{1232221\\2}, \substack{1111111\\1}, \substack{\cdot\cdot11111\\0} \},$$

$$X_{29}^{116} = \{ \substack{246\cdot\cdot\cdot\cdot\\3}, \substack{2454321\\2}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\2}, \substack{12\cdot\cdot321\\\cdot}, \substack{123\cdot2\cdot1\\2}, \substack{12222\cdot1\\1}, \substack{\cdot\cdot11111\\0} \},$$

$$X_{29}^{117} = \{ \substack{246\cdot\cdot\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\2}, \substack{1\cdot\cdot\cdot321\\2}, \substack{1343211\\2}, \substack{1243221\\2}, \substack{123\cdot2\cdot1\\2}, \substack{1233321\\1}, \substack{1222221\\1}, \substack{1122211\\1}, \substack{\cdot\cdot11111\\0} \},$$

$$X_{29}^{118} = \{ \substack{246\cdot\cdot\cdot\cdot\\3}, \substack{2454321\\2}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\2}, \substack{124\cdot\cdot21\\2}, \substack{123\cdot2\cdot1\\2}, \substack{1233321\\1}, \substack{1222221\\1}, \substack{1232111\\1}, \substack{\cdot\cdot11111\\0} \},$$

$$X_{29}^{119} = \{ \substack{246\cdot\cdot\cdot\cdot\\3}, \substack{2454321\\2}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\2}, \substack{1244321\\2}, \substack{1243221\\2}, \substack{1233321\\\cdot}, \substack{123\cdot2\cdot1\\2}, \substack{1222221\\1}, \substack{1222111\\1}, \substack{\cdot\cdot11111\\0} \},$$

$$X_{29}^{120} = \{ \substack{246\cdot\cdot\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\2}, \substack{1\cdot\cdot4321\\2}, \substack{13432\cdot1\\2}, \substack{12\cdot3321\\\cdot}, \substack{1243221\\2}, \substack{123\cdot2\cdot1\\2}, \substack{1222221\\1}, \substack{1122111\\1}, \substack{\cdot\cdot11111\\0} \},$$

$$X_{29}^{121} = \{ \substack{246\cdot\cdot\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\2}, \substack{134\cdot\cdot\cdot1\\2}, \substack{124\cdot\cdot21\\2}, \substack{1233321\\\cdot}, \substack{123\cdot2\cdot1\\2}, \substack{1222221\\1}, \substack{1111111\\1}, \substack{\cdot\cdot11111\\0} \},$$

$$X_{29}^{122} = \{ \substack{246\cdot\cdot\cdot\cdot\\3}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\2}, \substack{124\cdot\cdot21\\2}, \substack{1233321\\\cdot}, \substack{123\cdot2\cdot1\\2}, \substack{1222221\\1}, \substack{\cdot\cdot11111\\0}, \substack{0011111\\1} \},$$

$$X_{29}^{123} = \{ \substack{246\cdot\cdot\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\2}, \substack{1\cdot4\cdot\cdot\cdot1\\2}, \substack{1233\cdot21\\2}, \substack{1232\cdot11\\2}, \substack{1233211\\1}, \substack{1232221\\1}, \substack{1111111\\1}, \substack{\cdot\cdot11111\\0} \},$$

$$X_{29}^{124} = \{ \substack{246\cdot\cdot\cdot\cdot\\3}, \substack{2454321\\2}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\2}, \substack{12\cdot\cdot2\cdot1\\2}, \substack{1233321\\\cdot}, \substack{122\cdot111\\1}, \substack{\cdot\cdot11111\\0} \},$$

$$X_{29}^{125} = \{\substack{24654\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\2}, \substack{13\cdot4321\\2}, \substack{12\cdot3\cdot21\\2}, \substack{1343211\\2}, \substack{1233321\\1}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{12\cdot2221\\1}, \substack{1122\cdot11\\1}, \substack{1221111\\1}, \substack{\cdot\cdot\cdot\cdot111\\0}\},$$

$$X_{29}^{126} = \{\substack{246\cdot\cdot\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\2}, \substack{13\cdot4321\\2}, \substack{1\cdot\cdot\cdot2\cdot1\\2}, \substack{12\cdot3321\\\cdot}, \substack{1221111\\1}, \substack{1122111\\1}, \substack{\cdot\cdot11111\\0}\},$$

$$X_{29}^{127} = \{\substack{246\cdot\cdot\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\2}, \substack{134\cdot\cdot\cdot1\\2}, \substack{1244321\\2}, \substack{12\cdot\cdot2\cdot1\\2}, \substack{1233321\\\cdot}, \substack{1222111\\1}, \substack{1111111\\1}, \substack{\cdot\cdot11111\\0}\},$$

$$X_{29}^{128} = \{\substack{246\cdot\cdot\cdot\cdot\\3}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\2}, \substack{1244321\\2}, \substack{12\cdot\cdot2\cdot1\\2}, \substack{1233321\\\cdot}, \substack{1222111\\1}, \substack{\cdot\cdot11111\\0}, \substack{0011111\\1}\},$$

$$X_{29}^{129} = \{\substack{246\cdot\cdot\cdot\cdot\\3}, \substack{2454321\\2}, \substack{\cdot\cdot4\cdot\cdot\cdot1\\2}, \substack{1233321\\\cdot}, \substack{123\cdot2\cdot1\\2}, \substack{\cdot111111\\1}, \substack{\cdot\cdot11111\\0}\},$$

$$X_{29}^{130} = \{\substack{246\cdot\cdot\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\2}, \substack{13\cdot\cdot\cdot21\\2}, \substack{12\cdot\cdot321\\2}, \substack{123\cdot2\cdot1\\2}, \substack{1232111\\2}, \substack{1\cdot22221\\1}, \substack{1222211\\1}, \substack{\cdot\cdot11111\\0}\},$$

$$X_{29}^{131} = \{\substack{246\cdot\cdot\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot321\\2}, \substack{1\cdot\cdot\cdot\cdot21\\2}, \substack{2343211\\2}, \substack{123\cdot\cdot11\\2}, \substack{1\cdot22221\\1}, \substack{0122211\\1}, \substack{\cdot\cdot11111\\0}\},$$

$$X_{29}^{132} = \{\substack{246\cdot\cdot\cdot\cdot\\3}, \substack{\cdot\cdot\cdot4321\\2}, \substack{\cdot343\cdot\cdot1\\2}, \substack{123\cdot\cdot\cdot1\\2}, \substack{1222\cdot\cdot1\\1}, \substack{\cdot\cdot11111\\0}\},$$

$$X_{29}^{133} = \{\substack{246\cdot\cdot\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\2}, \substack{13\cdot4321\\2}, \substack{12\cdot\cdot321\\2}, \substack{13432\cdot1\\2}, \substack{123\cdot2\cdot1\\2}, \substack{1232111\\2}, \substack{12222\cdot1\\1}, \substack{1122111\\1}, \substack{\cdot\cdot11111\\0}\},$$

$$X_{29}^{134} = \{\substack{246\cdot\cdot\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot21\\2}, \substack{13\cdot\cdot321\\2}, \substack{1244321\\2}, \substack{1343211\\2}, \substack{12432\cdot1\\2}, \substack{123\cdot\cdot\cdot1\\2}, \substack{1222111\\1}, \substack{1122211\\1}, \substack{0122221\\1}, \substack{\cdot\cdot11111\\0}\},$$

$$X_{29}^{135} = \{\substack{246543\cdot\\3}, \substack{\cdot\cdot54321\\\cdot}, \substack{\cdot\cdot4\cdot\cdot\cdot1\\2}, \substack{1233\cdot\cdot1\\\cdot}, \substack{000\cdot\cdot11\\0}\},$$

$$X_{29}^{136} = \{\substack{246543\cdot\\3}, \substack{\cdot\cdot54321\\\cdot}, \substack{\cdot\cdot4\cdot\cdot\cdot1\\2}, \substack{1233321\\\cdot}, \substack{1233221\\2}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{1232221\\1}, \substack{000\cdot\cdot11\\0}\},$$

$$X_{29}^{137} = \{\substack{246\cdot\cdot\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\2}, \substack{134\cdot\cdot\cdot1\\2}, \substack{124\cdot321\\2}, \substack{123\cdot\cdot\cdot1\\2}, \substack{12222\cdot1\\1}, \substack{1111111\\1}, \substack{\cdot\cdot11111\\0}\},$$

$$X_{29}^{138} = \{\substack{246\cdot\cdot\cdot\cdot\\3}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\2}, \substack{124\cdot321\\2}, \substack{123\cdot\cdot\cdot1\\2}, \substack{12222\cdot1\\1}, \substack{\cdot\cdot11111\\0}, \substack{0011111\\1}\},$$

$$X_{29}^{139} = \{\substack{246\cdot\cdot\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\2}, \substack{1\cdot4\cdot321\\2}, \substack{1343221\\2}, \substack{123\cdot\cdot\cdot1\\2}, \substack{1243211\\2}, \substack{1222211\\1}, \substack{1122221\\1}, \substack{1111111\\1}, \substack{\cdot\cdot11111\\0}\},$$

$$X_{29}^{140} = \{\substack{246\cdot\cdot\cdot\cdot\\3}, \substack{2454321\\2}, \substack{234\cdot\cdot\cdot1\\2}, \substack{1\cdot\cdot\cdot321\\2}, \substack{1343211\\2}, \substack{12\cdot\cdot221\\2}, \substack{123\cdot\cdot11\\2}, \substack{1222221\\1}, \substack{1122211\\1}, \substack{\cdot\cdot11111\\0}, \substack{0111111\\1}\},$$

$$X_{29}^{141} = \{\substack{246543\cdot\\3}, \substack{\cdot\cdot54321\\\cdot}, \substack{\cdot\cdot4\cdot\cdot\cdot1\\2}, \substack{1233321\\\cdot}, \substack{123\cdot211\\\cdot}, \substack{000\cdot\cdot11\\0}\},$$

$$X_{29}^{142} = \{\substack{246543\cdot\\3}, \substack{\cdot\cdot54321\\\cdot}, \substack{\cdot\cdot4\cdot\cdot\cdot1\\2}, \substack{1233221\\2}, \substack{1233321\\1}, \substack{1233211\\\cdot}, \substack{1232211\\2}, \substack{1232111\\1}, \substack{000\cdot\cdot11\\0}\},$$

$$X_{29}^{143} = \{\substack{2465432\\3}, \substack{\cdot\cdot\cdot4321\\\cdot}, \substack{\cdot\cdot\cdot3\cdot\cdot1\\\cdot}, \substack{0000\cdot\cdot1\\0}\},$$

$$X_{29}^{144} = \{\substack{24654\cdot\cdot\\3}, \substack{\cdot\cdot54321\\2}, \substack{\cdot\cdot4\cdot\cdot\cdot1\\2}, \substack{123\cdot2\cdot1\\1}, \substack{1232111\\\cdot}, \substack{\cdot\cdot\cdot\cdot111\\0}\},$$

$$X_{29}^{145} = \{\substack{2465\cdot\cdot\cdot\\3}, \substack{2454321\\2}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\2}, \substack{124\cdot321\\2}, \substack{12\cdot3211\\2}, \substack{123\cdot221\\2}, \substack{1233\cdot21\\1}, \substack{1232111\\1}, \substack{1222211\\1}, \substack{\cdot\cdot\cdot1111\\0}\},$$

$$X_{29}^{146} = \{\substack{246\cdot\cdot\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\2}, \substack{13\cdot4321\\\cdot}, \substack{12\cdot3\cdot21\\2}, \substack{1343211\\2}, \substack{1232\cdot11\\2}, \substack{1233211\\1}, \substack{12\cdot2221\\1}, \substack{1122\cdot11\\1}, \substack{1221111\\1}, \substack{0\cdot11111\\0}\},$$

$$X_{29}^{147} = \{\substack{2465\cdot\cdot\cdot\\3}, \substack{2454321\\2}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\2}, \substack{12\cdot3321\\\cdot}, \substack{1243211\\2}, \substack{123\cdot221\\2}, \substack{1232211\\2}, \substack{1232221\\1}, \substack{1222211\\1}, \substack{1221111\\1}, \substack{\cdot\cdot\cdot1111\\0}\},$$

$$X_{29}^{148} = \{\substack{2465\cdot\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\2}, \substack{1\cdot\cdot4321\\2}, \substack{13432\cdot1\\2}, \substack{12\cdot3321\\\cdot}, \substack{1243211\\2}, \substack{1233221\\2}, \substack{12322\cdot1\\2}, \substack{1232221\\1}, \substack{1222211\\1}, \substack{1122111\\1}, \substack{\cdot\cdot\cdot1111\\0}\},$$

$$X_{29}^{149} = \{\substack{2465\cdot\cdot\cdot\\3}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\2}, \substack{12\cdot\cdot321\\\cdot}, \substack{1243211\\2}, \substack{123\cdot221\\2}, \substack{1232211\\2}, \substack{1232221\\1}, \substack{1222211\\1}, \substack{\cdot\cdot\cdot1111\\0}, \substack{0011111\\1}\},$$

$$X_{29}^{150} = \{ \substack{2465\cdot\cdot\cdot\\3}, \substack{2454321\\2}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\2}, \substack{1244321\\2}, \substack{12432\cdot1\\2}, \substack{123\cdot221\\2}, \substack{1233321\\1}, \substack{1232211\\2}, \substack{1232221\\1}, \substack{12\cdot2111\\1}, \substack{\cdot\cdot\cdot1111\\0} \},$$

$$X_{29}^{151} = \{ \substack{24654\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\2}, \substack{1\cdot4\cdot\cdot\cdot1\\2}, \substack{1233221\\2}, \substack{1233321\\1}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{1232221\\1}, \substack{1232111\\1}, \substack{1111111\\1}, \substack{\cdot\cdot\cdot\cdot111\\0} \},$$

$$X_{29}^{152} = \{ \substack{24654\cdot\cdot\\3}, \substack{2454321\\2}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\2}, \substack{12432\cdot1\\2}, \substack{1233321\\\cdot}, \substack{1233211\\2}, \substack{1232221\\2}, \substack{1233221\\1}, \substack{1232211\\1}, \substack{122\cdot111\\1}, \substack{\cdot\cdot\cdot\cdot111\\0} \},$$

$$X_{29}^{153} = \{ \substack{24654\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\2}, \substack{13\cdot4321\\2}, \substack{1\cdot432\cdot1\\2}, \substack{12\cdot3321\\2}, \substack{1233\cdot21\\1}, \substack{1233211\\2}, \substack{1232221\\2}, \substack{1232211\\1}, \substack{1221111\\1}, \substack{1122111\\1}, \substack{\cdot\cdot\cdot\cdot111\\0} \},$$

$$X_{29}^{154} = \{ \substack{24654\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\2}, \substack{13\cdot4321\\\cdot}, \substack{1243\cdot21\\2}, \substack{1343211\\2}, \substack{1233\cdot21\\1}, \substack{1233211\\2}, \substack{1232221\\2}, \substack{1232\cdot11\\1}, \substack{1222221\\1}, \substack{1122\cdot11\\1}, \substack{1221111\\1}, \substack{0\cdot\cdot\cdot111\\0} \},$$

$$X_{29}^{155} = \{ \substack{2465\cdot\cdot\cdot\\3}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\2}, \substack{1244321\\2}, \substack{12432\cdot1\\2}, \substack{1233321\\\cdot}, \substack{123\cdot221\\2}, \substack{1232211\\2}, \substack{1232221\\1}, \substack{1222111\\1}, \substack{\cdot\cdot\cdot1111\\0}, \substack{0011111\\1} \},$$

$$X_{29}^{156} = \{ \substack{24654\cdot\cdot\\3}, \substack{2454321\\2}, \substack{\cdot\cdot4\cdot\cdot\cdot1\\2}, \substack{1233321\\\cdot}, \substack{1233221\\2}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{1232221\\1}, \substack{\cdot111111\\1}, \substack{\cdot\cdot\cdot\cdot111\\0} \},$$

$$X_{29}^{157} = \{ \substack{246543\cdot\\3}, \substack{\cdot3\cdot4321\\\cdot}, \substack{\cdot\cdot\cdot3\cdot\cdot1\\\cdot}, \substack{1244321\\2}, \substack{00\cdot\cdot\cdot11\\\cdot} \},$$

$$X_{29}^{158} = \{ \substack{246543\cdot\\3}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\\cdot}, \substack{124\cdot\cdot\cdot1\\2}, \substack{1233321\\\cdot}, \substack{1233221\\2}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{1232221\\1}, \substack{00\cdot\cdot\cdot11\\\cdot} \},$$

$$X_{29}^{159} = \{ \substack{2465\cdot\cdot\cdot\\3}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\2}, \substack{1244321\\2}, \substack{12\cdot\cdot2\cdot1\\2}, \substack{1233321\\1}, \substack{12\cdot2111\\1}, \substack{\cdot\cdot\cdot1111\\0}, \substack{0011111\\1} \},$$

$$X_{29}^{160} = \{ \substack{2465\cdot\cdot\cdot\\3}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\2}, \substack{1243\cdot\cdot1\\2}, \substack{123\cdot2\cdot1\\2}, \substack{1233321\\1}, \substack{1232111\\1}, \substack{1221111\\1}, \substack{\cdot\cdot\cdot1111\\0}, \substack{0011111\\1} \},$$

$$X_{29}^{161} = \{ \substack{24654\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\2}, \substack{1354321\\3}, \substack{1343\cdot21\\2}, \substack{1244321\\2}, \substack{12\cdot3211\\2}, \substack{1232221\\2}, \substack{1233221\\1}, \substack{1232111\\2}, \substack{12\cdot2\cdot11\\1}, \substack{1122221\\1}, \substack{11\cdot1111\\1}, \substack{0\cdot\cdot\cdot111\\0} \},$$

$$X_{29}^{162} = \{ \substack{24654\cdot\cdot\\3}, \substack{2454321\\\cdot}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\2}, \substack{124\cdot\cdot\cdot1\\2}, \substack{1233321\\2}, \substack{123\cdot2\cdot1\\1}, \substack{1232111\\2}, \substack{\cdot111111\\0}, \substack{000\cdot111\\0} \},$$

$$X_{29}^{163} = \{ \substack{24654\cdot\cdot\\3}, \substack{2464321\\3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\2}, \substack{13\cdot\cdot\cdot21\\2}, \substack{12\cdot\cdot321\\2}, \substack{1243211\\2}, \substack{12332\cdot1\\\cdot}, \substack{1232111\\2}, \substack{1222211\\1}, \substack{1122221\\1}, \substack{\cdot\cdot11111\\0}, \substack{0000111\\0} \},$$

$$X_{29}^{164} = \{ \substack{246543\cdot\\3}, \substack{2454321\\2}, \substack{\cdot3\cdot4321\\\cdot}, \substack{2343221\\2}, \substack{1343321\\2}, \substack{12\cdot\cdot\cdot21\\2}, \substack{\cdot\cdot\cdot3211\\2}, \substack{1233\cdot21\\1}, \substack{1122211\\1}, \substack{0122111\\1}, \substack{00\cdot\cdot\cdot11\\0} \},$$

$$X_{29}^{165} = \{ \substack{2465\cdot\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\2}, \substack{13\cdot\cdot321\\2}, \substack{1244321\\2}, \substack{1\cdot43211\\2}, \substack{1243221\\2}, \substack{1233321\\2}, \substack{12322\cdot1\\\cdot}, \substack{1232111\\2}, \substack{1222111\\1}, \substack{1122211\\1}, \substack{\cdot\cdot\cdot1111\\0} \},$$

$$X_{29}^{166} = \{ \substack{246543\cdot\\3}, \substack{2465321\\3}, \substack{2454321\\3}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\2}, \substack{124\cdot\cdot\cdot1\\2}, \substack{1233\cdot21\\2}, \substack{1233321\\1}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{1232221\\1}, \substack{\cdot111111\\0}, \substack{0011111\\1}, \substack{0001111\\0}, \substack{0000011\\0} \},$$

$$X_{29}^{167} = \{ \substack{2465\cdot\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot21\\2}, \substack{1\cdot\cdot\cdot321\\2}, \substack{1\cdot43211\\2}, \substack{123\cdot221\\2}, \substack{1232\cdot11\\2}, \substack{1232221\\1}, \substack{1\cdot22211\\1}, \substack{0122221\\1}, \substack{\cdot\cdot\cdot1111\\0} \},$$

$$X_{29}^{168} = \{ \substack{24654\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\2}, \substack{13\cdot4321\\2}, \substack{1\cdot432\cdot1\\2}, \substack{1243321\\2}, \substack{1233211\\2}, \substack{1232221\\2}, \substack{1233221\\1}, \substack{1232111\\2}, \substack{1232\cdot11\\1}, \substack{1221111\\1}, \substack{1122111\\1}, \substack{\cdot\cdot\cdot\cdot111\\0} \},$$

$$X_{29}^{169} = \{ \substack{246543\cdot\\3}, \substack{2454321\\3}, \substack{23\cdot\cdot\cdot\cdot1\\2}, \substack{13\cdot\cdot\cdot\cdot1\\\cdot}, \substack{124\cdot\cdot\cdot1\\2}, \substack{1233321\\\cdot}, \substack{1233221\\2}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{1232221\\1}, \substack{0111111\\0}, \substack{0011111\\1}, \substack{000\cdot\cdot11\\0} \},$$

$$X_{29}^{170} = \{ \substack{2465\cdot\cdot\cdot\\3}, \substack{2454321\\2}, \substack{\cdot 3\cdot\cdot\cdot 21\\2}, \substack{124\cdot 321\\2}, \substack{1243211\\2}, \substack{123\cdot 2\cdot 1\\2}, \substack{1232111\\\cdot}, \substack{1222211\\1}, \substack{\cdot 122221\\1}, \substack{\cdot\cdot\cdot 1111\\0} \},$$

$$X_{29}^{171} = \{ \substack{2465\cdot\cdot\cdot\\3}, \substack{2354321\\2}, \substack{\cdot 34\cdot\cdot\cdot 1\\2}, \substack{12\cdot\cdot\cdot 21\\2}, \substack{1232\cdot 11\\2}, \substack{12\cdot 2221\\1}, \substack{1111111\\1}, \substack{\cdot\cdot\cdot 1111\\0}, \substack{0011111\\1} \},$$

$$X_{29}^{172} = \{ \substack{2465432\\3}, \substack{\cdot 3\cdot\cdot\cdot\cdot 1\\2}, \substack{12\cdot\cdot 321\\2}, \substack{12\cdot\cdot 2\cdot 1\\1}, \substack{1232111\\2}, \substack{\cdot\cdot\cdot\cdot\cdot\cdot 1\\0}, \substack{0011111\\1} \},$$

$$X_{29}^{173} = \{ \substack{2465432\\3}, \substack{246\cdot\cdot 21\\3}, \substack{2454321\\2}, \substack{\cdot 3\cdot\cdot\cdot\cdot 1\\2}, \substack{12\cdot 3\cdot 21\\\cdot}, \substack{1233211\\2}, \substack{1232221\\2}, \substack{1222221\\1}, \substack{1221111\\1}, \substack{\cdot\cdot 11111\\0}, \substack{0000001\\0} \},$$

$$X_{29}^{174} = \{ \substack{2465\cdot\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot 321\\2}, \substack{13\cdot 4321\\2}, \substack{\cdot\cdot 432\cdot 1\\2}, \substack{12\cdot 3321\\2}, \substack{12322\cdot 1\\\cdot}, \substack{1232111\\2}, \substack{1221111\\1}, \substack{1122111\\1}, \substack{\cdot\cdot\cdot 1111\\0} \},$$

$$X_{29}^{175} = \{ \substack{246\cdot\cdot\cdot\cdot\\3}, \substack{234\cdot\cdot\cdot 1\\2}, \substack{1344321\\2}, \substack{12\cdot 3\cdot 21\\2}, \substack{1343211\\2}, \substack{1232\cdot 11\\2}, \substack{1233211\\1}, \substack{12\cdot 2221\\1}, \substack{1122\cdot 11\\1}, \substack{1221111\\1}, \substack{\cdot\cdot 11111\\\cdot} \},$$

$$X_{29}^{176} = \{ \substack{2465432\\3}, \substack{2465\cdot 21\\3}, \substack{2454321\\3}, \substack{\cdot 3\cdot\cdot\cdot\cdot 1\\2}, \substack{124\cdot\cdot\cdot 1\\2}, \substack{123\cdot\cdot 21\\\cdot}, \substack{\cdot 111111\\0}, \substack{0011111\\1}, \substack{0001111\\0}, \substack{0000001\\0} \},$$

$$X_{29}^{177} = \{ \substack{2465\cdot\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot\cdot 1\\2}, \substack{13\cdot 4321\\2}, \substack{13432\cdot 1\\2}, \substack{1243\cdot 21\\2}, \substack{1233321\\2}, \substack{1233211\\2}, \substack{1232\cdot\cdot 1\\2}, \substack{1232211\\1}, \substack{1222221\\1}, \substack{1221111\\1}, \substack{1122111\\1}, \substack{\cdot\cdot\cdot 1111\\0} \},$$

$$X_{29}^{178} = \{ \substack{2465\cdot\cdot\cdot\\3}, \substack{23\cdot\cdot\cdot\cdot 1\\2}, \substack{1\cdot\cdot 4321\\2}, \substack{13432\cdot 1\\2}, \substack{1243\cdot 21\\2}, \substack{1233321\\2}, \substack{1233211\\2}, \substack{1232\cdot\cdot 1\\2}, \substack{1232211\\1}, \substack{1222221\\1}, \substack{1122111\\1}, \substack{\cdot\cdot\cdot 1111\\0}, \substack{0011111\\1} \},$$

$$X_{29}^{179} = \{ \substack{24654\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot\cdot 1\\2}, \substack{13\cdot\cdot\cdot 21\\2}, \substack{124\cdot 321\\2}, \substack{12\cdot 3211\\2}, \substack{1232221\\2}, \substack{1233221\\1}, \substack{1232111\\\cdot}, \substack{12\cdot 2211\\1}, \substack{1122221\\1}, \substack{\cdot\cdot\cdot\cdot 111\\0} \},$$

$$X_{29}^{180} = \{ \substack{2465\cdot\cdot\cdot\\3}, \substack{\cdot 3\cdot\cdot\cdot\cdot 1\\2}, \substack{12\cdot 3321\\2}, \substack{1243221\\2}, \substack{1233211\\2}, \substack{1232\cdot\cdot 1\\2}, \substack{1232211\\1}, \substack{1222221\\1}, \substack{1221111\\1}, \substack{\cdot\cdot\cdot 1111\\0}, \substack{0011111\\1} \},$$

$$X_{29}^{181} = \{ \substack{2465432\\3}, \substack{2454321\\\cdot}, \substack{\cdot 3\cdot\cdot\cdot\cdot 1\\\cdot}, \substack{12\cdot\cdot 321\\\cdot}, \substack{12\cdot 3221\\2}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{12\cdot 2221\\1}, \substack{000\cdot\cdot\cdot 1\\0} \},$$

$$X_{29}^{182} = \{ \substack{2465\cdot\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot\cdot 1\\2}, \substack{1343\cdot\cdot 1\\2}, \substack{12\cdot\cdot 321\\2}, \substack{1243211\\2}, \substack{1233221\\2}, \substack{1232\cdot\cdot 1\\2}, \substack{1232221\\1}, \substack{1222211\\1}, \substack{11\cdot 1111\\1}, \substack{\cdot\cdot\cdot 1111\\0} \},$$

$$X_{29}^{183} = \{ \substack{2465\cdot\cdot\cdot\\3}, \substack{2\cdot 54321\\2}, \substack{2343\cdot\cdot 1\\2}, \substack{134\cdot\cdot\cdot 1\\2}, \substack{124\cdot 321\\2}, \substack{12\cdot\cdot 211\\2}, \substack{123\cdot 221\\2}, \substack{1232111\\\cdot}, \substack{1222211\\1}, \substack{1111111\\1}, \substack{0121111\\1}, \substack{\cdot\cdot\cdot 1111\\0} \},$$

$$X_{29}^{184} = \{ \substack{246543\cdot\\3}, \substack{2454321\\\cdot}, \substack{2354321\\2}, \substack{1354321\\3}, \substack{\cdot\cdot 4\cdot\cdot\cdot 1\\2}, \substack{1233221\\2}, \substack{1233321\\1}, \substack{1233211\\\cdot}, \substack{1232211\\2}, \substack{1232111\\1}, \substack{1111111\\1}, \substack{0111111\\0}, \substack{000\cdot\cdot 11\\0} \},$$

$$X_{29}^{185} = \{ \substack{2465432\\3}, \substack{2454321\\\cdot}, \substack{\cdot 3\cdot\cdot\cdot\cdot 1\\\cdot}, \substack{124\cdot\cdot 21\\2}, \substack{1233221\\2}, \substack{1233321\\1}, \substack{1233211\\\cdot}, \substack{1232211\\2}, \substack{1222221\\1}, \substack{1232111\\1}, \substack{000\cdot\cdot\cdot 1\\0} \},$$

$$X_{29}^{186} = \{ \substack{246543\cdot\\3}, \substack{1354321\\\cdot}, \substack{\cdot\cdot 4\cdot\cdot\cdot 1\\2}, \substack{1233321\\\cdot}, \substack{123\cdot 211\\\cdot}, \substack{0\cdot 11111\\\cdot}, \substack{000\cdot\cdot 11\\0} \},$$

$$X_{29}^{187} = \{ \substack{2465432\\3}, \substack{2\cdot 54321\\\cdot}, \substack{1\cdot\cdot 4321\\\cdot}, \substack{\cdot\cdot\cdot 3\cdot\cdot 1\\\cdot}, \substack{0121111\\1}, \substack{000\cdot\cdot\cdot 1\\0} \},$$

$$X_{29}^{188} = \{ \substack{246543\cdot\\3}, \substack{2344321\\2}, \substack{1354321\\2}, \substack{\cdot 343\cdot 21\\2}, \substack{1244321\\2}, \substack{1243221\\2}, \substack{1233321\\2}, \substack{\cdot\cdot\cdot\cdot\cdot 11\\2}, \substack{123\cdot 211\\1}, \substack{1222111\\1}, \substack{1121111\\1}, \substack{\cdot\cdot\cdot\cdot\cdot 11\\0}, \substack{0\cdot 11111\\1} \},$$

$$X_{29}^{189} = \{ \substack{246543\cdot\\3}, \substack{\cdot\cdot 4\cdot\cdot\cdot 1\\2}, \substack{123\cdot\cdot 11\\\cdot}, \substack{\cdot\cdot 11111\\1}, \substack{\cdot\cdot\cdot\cdot\cdot 11\\0} \},$$

$$X_{29}^{190} = \{ \substack{2465432\\3}, \substack{234\cdot\cdot\cdot1\\2}, \substack{1354321\\2}, \substack{1343\cdot\cdot1\\2}, \substack{1244321\\2}, \substack{1243221\\2}, \substack{1233321\\2}, \substack{12\cdot3211\\2}, \substack{1233\cdot21\\1}, \substack{1232221\\2}, \substack{1232211\\1}, \substack{1222111\\1}, \substack{1121111\\1}, \substack{\cdot\cdot\cdot\cdot\cdot\cdot1\\0}, \substack{0\cdot11111\\1} \},$$

$$X_{29}^{191} = \{ \substack{246543\cdot\\3}, \substack{2354321\\2}, \substack{1354321\\3}, \substack{\cdot\cdot4\cdot\cdot\cdot1\\2}, \substack{1233221\\2}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{1232221\\1}, \substack{1232111\\\cdot}, \substack{1111111\\1}, \substack{0111111\\0}, \substack{00\cdot\cdot\cdot11\\\cdot} \},$$

$$X_{29}^{192} = \{ \substack{24654\cdot\cdot\\3}, \substack{2344321\\2}, \substack{1354321\\2}, \substack{\cdot343321\\2}, \substack{1244321\\2}, \substack{\cdot\cdot\cdot\cdot2\cdot1\\2}, \substack{12\cdot2111\\\cdot}, \substack{1121111\\1}, \substack{\cdot\cdot\cdot\cdot111\\0}, \substack{0\cdot11111\\1} \},$$

$$X_{29}^{193} = \{ \substack{2465432\\3}, \substack{2454321\\2}, \substack{\cdot354321\\3}, \substack{2344321\\2}, \substack{1354321\\2}, \substack{\cdot\cdot\cdot3\cdot\cdot1\\\cdot}, \substack{1244321\\2}, \substack{1121111\\1}, \substack{0111111\\1}, \substack{00\cdot\cdot\cdot\cdot1\\0} \},$$

$$X_{29}^{194} = \{ \substack{2\cdot\cdot\cdot\cdot\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot21\\2}, \substack{1\cdot\cdot\cdot321\\2}, \substack{1\cdot\cdot\cdot2\cdot1\\2}, \substack{1233321\\1}, \substack{0122221\\1}, \substack{1111111\\0} \},$$

$$X_{29}^{195} = \{ \substack{2\cdot\cdot\cdot\cdot\cdot\cdot\\3}, \substack{\cdot\cdot\cdot4321\\2}, \substack{\cdot\cdot\cdot\cdot2\cdot1\\2}, \substack{1\cdot\cdot3321\\\cdot}, \substack{0122111\\1}, \substack{1111111\\0} \},$$

$$X_{29}^{196} = \{ \substack{2\cdot\cdot\cdot\cdot\cdot\cdot\\3}, \substack{\cdot\cdot\cdot\cdot321\\2}, \substack{1\cdot\cdot\cdot2\cdot1\\2}, \substack{1232111\\2}, \substack{01222\cdot1\\1}, \substack{1111111\\0} \},$$

$$X_{29}^{197} = \{ \substack{24\cdot\cdot\cdot\cdot\cdot\\3}, \substack{\cdot\cdot54321\\2}, \substack{\cdot\cdot4\cdot\cdot\cdot1\\2}, \substack{123\cdot\cdot21\\\cdot}, \substack{\cdot111111\\0} \},$$

$$X_{29}^{198} = \{ \substack{24\cdot\cdot\cdot\cdot\cdot\\3}, \substack{\cdot\cdot\cdot\cdot\cdot21\\2}, \substack{2343211\\2}, \substack{12\cdot3211\\2}, \substack{1233\cdot21\\1}, \substack{1122221\\1}, \substack{\cdot111111\\0} \},$$

$$X_{29}^{199} = \{ \substack{24\cdot\cdot\cdot\cdot\cdot\\3}, \substack{\cdot\cdot54321\\2}, \substack{\cdot\cdot4\cdot\cdot\cdot1\\2}, \substack{1233321\\\cdot}, \substack{1233211\\2}, \substack{1232221\\2}, \substack{1233221\\1}, \substack{1232211\\1}, \substack{\cdot111111\\0} \},$$

$$X_{29}^{200} = \{ \substack{24\cdot\cdot\cdot\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\2}, \substack{1\cdot\cdot\cdot321\\2}, \substack{1343211\\2}, \substack{12\cdot32\cdot1\\2}, \substack{1233\cdot21\\1}, \substack{1232221\\2}, \substack{1122211\\1}, \substack{\cdot111111\\0} \},$$

$$X_{29}^{201} = \{ \substack{24\cdot\cdot\cdot\cdot\cdot\\\cdot}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\2}, \substack{12\cdot3\cdot\cdot1\\2}, \substack{1233\cdot21\\1}, \substack{1232221\\2}, \substack{1221111\\1}, \substack{\cdot111111\\0} \},$$

$$X_{29}^{202} = \{ \substack{24\cdot\cdot\cdot\cdot\cdot\\\cdot}, \substack{\cdot354321\\2}, \substack{2344321\\2}, \substack{\cdot\cdot\cdot3\cdot21\\\cdot}, \substack{1244321\\2}, \substack{\cdot\cdot\cdot3211\\2}, \substack{1232221\\2}, \substack{1121111\\1}, \substack{\cdot111111\\0} \},$$

$$X_{29}^{203} = \{ \substack{24\cdot\cdot\cdot\cdot\cdot\\3}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\2}, \substack{1244321\\2}, \substack{12\cdot3\cdot\cdot1\\2}, \substack{1233\cdot21\\1}, \substack{1232221\\2}, \substack{\cdot111111\\0}, \substack{0011111\\1} \},$$

$$X_{29}^{204} = \{ \substack{2465432\\3}, \substack{246\cdot321\\3}, \substack{\cdot\cdot54321\\\cdot}, \substack{\cdot\cdot4\cdot\cdot\cdot1\\2}, \substack{12332\cdot1\\\cdot}, \substack{1232111\\\cdot}, \substack{00000\cdot1\\0} \},$$

$$X_{29}^{205} = \{ \substack{2465432\\3}, \substack{246\cdot321\\3}, \substack{\cdot\cdot54321\\\cdot}, \substack{\cdot\cdot4\cdot\cdot\cdot1\\2}, \substack{123\cdot221\\\cdot}, \substack{1232111\\\cdot}, \substack{00000\cdot1\\0} \},$$

$$X_{29}^{206} = \{ \substack{2465432\\3}, \substack{246\cdot321\\3}, \substack{\cdot\cdot54321\\\cdot}, \substack{\cdot\cdot4\cdot\cdot\cdot1\\2}, \substack{1233221\\2}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{1232221\\1}, \substack{1232111\\\cdot}, \substack{00000\cdot1\\0} \},$$

$$X_{29}^{207} = \{ \substack{24\cdot\cdot\cdot\cdot\cdot\\\cdot}, \substack{\cdot3\cdot\cdot\cdot21\\2}, \substack{124\cdot321\\2}, \substack{2343211\\2}, \substack{12\cdot\cdot2\cdot1\\2}, \substack{1233321\\1}, \substack{1232111\\1}, \substack{1122221\\1}, \substack{\cdot111111\\0} \},$$

$$X_{29}^{208} = \{ \substack{24\cdot\cdot\cdot\cdot\cdot\\\cdot}, \substack{23\cdot\cdot\cdot\cdot1\\2}, \substack{1\cdot\cdot4321\\2}, \substack{1343221\\2}, \substack{12\cdot3321\\\cdot}, \substack{12\cdot\cdot2\cdot1\\2}, \substack{1122221\\1}, \substack{1122111\\1}, \substack{\cdot111111\\0} \},$$

$$X_{29}^{209} = \{ \substack{24\cdot\cdot\cdot\cdot\cdot\\3}, \substack{\cdot\cdot\cdot4321\\2}, \substack{23432\cdot1\\2}, \substack{1343\cdot21\\2}, \substack{12\cdot3321\\\cdot}, \substack{12\cdot\cdot2\cdot1\\2}, \substack{1122221\\1}, \substack{0122111\\1}, \substack{\cdot111111\\0} \},$$

$$X_{29}^{210} = \{ \substack{24\cdot\cdot\cdot\cdot\cdot\\3}, \substack{\cdot3\cdot\cdot\cdot21\\2}, \substack{12\cdot\cdot321\\\cdot}, \substack{2343211\\2}, \substack{12\cdot\cdot2\cdot1\\2}, \substack{1122221\\1}, \substack{\cdot111111\\0}, \substack{0011111\\1} \},$$

$$X_{29}^{211} = \{ \substack{24\cdot\cdot\cdot\cdot\cdot\\3}, \substack{2\cdot54321\\2}, \substack{\cdot\cdot4\cdot\cdot\cdot1\\2}, \substack{1233\cdot21\\2}, \substack{1232\cdot11\\2}, \substack{1233211\\1}, \substack{1232221\\1}, \substack{1111111\\1}, \substack{\cdot111111\\0} \},$$

$$X_{29}^{212} = \{ \substack{24\cdot\cdot\cdot\cdot\cdot\\3}, \substack{\cdot354321\\2}, \substack{\cdot\cdot4\cdot\cdot\cdot1\\2}, \substack{1233\cdot21\\2}, \substack{1232\cdot11\\2}, \substack{1233211\\1}, \substack{1232221\\1}, \substack{\cdot111111\\0}, \substack{0011111\\1} \},$$

$$X_{29}^{213} = \{ \substack{24\cdot\cdot\cdot\cdot\cdot\\3}, \substack{\cdot\cdot\cdot4321\\2}, \substack{\cdot\cdot\cdot\cdot2\cdot1\\2}, \substack{12\cdot3321\\\cdot}, \substack{\cdot122111\\1}, \substack{\cdot111111\\0} \},$$

$$X_{29}^{214} = \{ \substack{24\cdot\cdot\cdot\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot321\\2}, \substack{1354321\\2}, \substack{\cdot\cdot\cdot\cdot2\cdot1\\2}, \substack{12\cdot\cdot321\\\cdot}, \substack{112\cdot111\\1}, \substack{\cdot111111\\0} \},$$

$$X_{29}^{215} = \{ \substack{24\cdot\cdot\cdot\cdot\cdot\\\cdot}, \substack{\cdot354321\\2}, \substack{2343321\\2}, \substack{1344321\\2}, \substack{\cdot\cdot\cdot\cdot2\cdot1\\2}, \substack{12\cdot\cdot321\\\cdot}, \substack{1122111\\1}, \substack{0121111\\1}, \substack{\cdot111111\\0} \},$$

$$X_{29}^{216} = \{ \substack{24\cdot\cdot\cdot\cdot\cdot\\3}, \substack{\cdot3\cdot4321\\2}, \substack{2343321\\2}, \substack{\cdot\cdot\cdot\cdot2\cdot1\\2}, \substack{12\cdot\cdot321\\\cdot}, \substack{1122111\\1}, \substack{\cdot111111\\0}, \substack{0011111\\1} \},$$

$$X_{29}^{217} = \{ \substack{2465432\\3}, \substack{\cdot\cdot\cdot\cdot321\\\cdot}, \substack{\cdot\cdot\cdot32\cdot1\\\cdot}, \substack{00000\cdot1\\0} \},$$

$$X_{29}^{218} = \{ \substack{24\cdot\cdot\cdot\cdot\cdot\\3}, \substack{\cdot\cdot\cdot4321\\2}, \substack{2343\cdot\cdot1\\2}, \substack{12\cdot3\cdot\cdot1\\2}, \substack{1232\cdot\cdot1\\2}, \substack{1122\cdot\cdot1\\1}, \substack{\cdot111111\\0} \},$$

$$X_{29}^{219} = \{ \substack{24\cdot\cdot\cdot\cdot\cdot\\\cdot}, \substack{\cdot3\cdot4321\\2}, \substack{23432\cdot1\\2}, \substack{1343321\\2}, \substack{12\cdot\cdot\cdot\cdot1\\2}, \substack{11222\cdot1\\1}, \substack{0122111\\1}, \substack{\cdot111111\\0} \},$$

$$X_{29}^{220} = \{ {2465432 \atop 3}, {\cdot\cdot\cdot\cdot 321 \atop \cdot}, {\cdot\cdot 432\cdot 1 \atop 2}, {123\cdot 221 \atop \cdot}, {00000\cdot 1 \atop 0} \},$$

$$X_{29}^{221} = \{ {2465432 \atop 3}, {\cdot\cdot\cdot\cdot 321 \atop \cdot}, {\cdot\cdot 432\cdot 1 \atop 2}, {1233221 \atop 2}, {1232211 \atop 2}, {1233211 \atop 1}, {1232221 \atop 1}, {00000\cdot 1 \atop 0} \},$$

$$X_{29}^{222} = \{ {2465432 \atop 3}, {\cdot 3\cdot\cdot\cdot\cdot 1 \atop 2}, {12\cdot\cdot 321 \atop \cdot}, {12\cdot\cdot 2\cdot 1 \atop 1}, {\cdot\cdot\cdot\cdot\cdot\cdot 1 \atop 0}, {0011111 \atop 1} \},$$

$$X_{29}^{223} = \{ {246\cdot\cdot\cdot\cdot \atop 3}, {\cdot\cdot\cdot\cdot 321 \atop 2}, {\cdot 3432\cdot 1 \atop 2}, {1243221 \atop 2}, {1233\cdot 21 \atop 1}, {1233211 \atop 2}, {1232221 \atop 2}, {1232211 \atop 1}, {1222221 \atop 1}, {\cdot\cdot 11111 \atop 0} \},$$

$$X_{29}^{224} = \{ {2465\cdot\cdot\cdot \atop 3}, {2\cdot\cdot\cdot\cdot\cdot 1 \atop 2}, {13\cdot\cdot\cdot 21 \atop 2}, {12\cdot\cdot 321 \atop \cdot}, {1243211 \atop 2}, {12322\cdot 1 \atop \cdot}, {1222211 \atop 1}, {1122221 \atop 1}, {\cdot\cdot\cdot 1111 \atop 0} \},$$

$$X_{29}^{225} = \{ {2465432 \atop 3}, {2\cdot\cdot\cdot 321 \atop \cdot}, {23432\cdot 1 \atop 2}, {1343321 \atop 2}, {1244321 \atop 2}, {1\cdot 43221 \atop 2}, {123\cdot\cdot 11 \atop \cdot}, {1\cdot 22221 \atop 1}, {1222111 \atop 1}, {11\cdot 1111 \atop \cdot}, {00000\cdot 1 \atop 0} \},$$

$$X_{29}^{226} = \{ {246\cdot\cdot\cdot\cdot \atop 3}, {\cdot\cdot 54321 \atop 2}, {2344321 \atop 2}, {\cdot\cdot\cdot 3\cdot 21 \atop \cdot}, {1244321 \atop 2}, {\cdot 343211 \atop 2}, {1233211 \atop 2}, {1232221 \atop 2}, {1222221 \atop 1}, {1121111 \atop 1}, {\cdot\cdot 11111 \atop 0} \},$$

$$X_{29}^{227} = \{ {246\cdot\cdot\cdot\cdot \atop 3}, {\cdot 3\cdot\cdot\cdot\cdot 1 \atop 2}, {12\cdot\cdot\cdot 21 \atop 2}, {1233\cdot 21 \atop 1}, {1233211 \atop 2}, {1222221 \atop 1}, {\cdot 111111 \atop 0}, {0011111 \atop \cdot} \},$$

$$X_{29}^{228} = \{ {2465\cdot\cdot\cdot \atop 3}, {\cdot\cdot\cdot\cdot 321 \atop 2}, {2343211 \atop 2}, {1\cdot 43221 \atop 2}, {1233321 \atop 1}, {1233211 \atop 2}, {12322\cdot 1 \atop 2}, {1232211 \atop 1}, {1\cdot 22221 \atop 1}, {0122211 \atop 1}, {\cdot\cdot\cdot 1111 \atop 0} \},$$

$$X_{29}^{229} = \{ {24654\cdot\cdot \atop 3}, {2\cdot\cdot\cdot\cdot\cdot 1 \atop 2}, {13\cdot\cdot\cdot 21 \atop 2}, {124\cdot 321 \atop 2}, {12\cdot 3211 \atop 2}, {1233\cdot 21 \atop 1}, {1232221 \atop 2}, {1232\cdot 11 \atop 1}, {1222211 \atop 1}, {1122221 \atop 1}, {\cdot\cdot\cdot\cdot 111 \atop 0} \},$$

$$X_{29}^{230} = \{ {246543\cdot \atop 3}, {\cdot\cdot 54321 \atop \cdot}, {\cdot 34\cdot\cdot\cdot 1 \atop 2}, {124\cdot\cdot 21 \atop 2}, {1233321 \atop \cdot}, {1233221 \atop 2}, {1232211 \atop 2}, {1233211 \atop 1}, {12\cdot 2221 \atop 1}, {000\cdot\cdot 11 \atop 0} \},$$

$$X_{29}^{231} = \{ {246543\cdot \atop 3}, {\cdot\cdot 54321 \atop \cdot}, {\cdot 34\cdot\cdot\cdot 1 \atop 2}, {1243\cdot\cdot 1 \atop 2}, {123\cdot\cdot 21 \atop \cdot}, {1221111 \atop 1}, {000\cdot\cdot 11 \atop 0} \},$$

$$X_{29}^{232} = \{ {246\cdot\cdot\cdot\cdot \atop 3}, {\cdot\cdot 54321 \atop 2}, {\cdot\cdot 4\cdot\cdot\cdot 1 \atop 2}, {1233\cdot 21 \atop 1}, {1233211 \atop 2}, {1232221 \atop 2}, {1232\cdot 11 \atop 1}, {\cdot\cdot 11111 \atop 0} \},$$

$$X_{29}^{233} = \{ {246\cdot\cdot\cdot\cdot \atop 3}, {2\cdot\cdot\cdot\cdot\cdot 1 \atop 2}, {1\cdot 4\cdot\cdot\cdot 1 \atop 2}, {1233321 \atop \cdot}, {1233211 \atop 2}, {1232221 \atop 2}, {1233221 \atop 1}, {1232211 \atop 1}, {1111111 \atop 1}, {\cdot\cdot 11111 \atop 0} \},$$

$$X_{29}^{234} = \{ {246\cdot\cdot\cdot\cdot \atop 3}, {2454321 \atop 2}, {\cdot 3\cdot\cdot\cdot\cdot 1 \atop 2}, {1244321 \atop 2}, {1233\cdot 21 \atop \cdot}, {12\cdot 3211 \atop 2}, {1232221 \atop 2}, {1222\cdot 11 \atop 1}, {\cdot\cdot 11111 \atop 0} \},$$

$$X_{29}^{235} = \{ {246\cdot\cdot\cdot\cdot \atop 3}, {2\cdot\cdot\cdot\cdot\cdot 1 \atop 2}, {1\cdot\cdot 4321 \atop 2}, {1343221 \atop 2}, {1243321 \atop 2}, {1\cdot\cdot 3211 \atop 2}, {1233\cdot 21 \atop \cdot}, {1232221 \atop 2}, {1222211 \atop 1}, {1122111 \atop 1}, {\cdot\cdot 11111 \atop 0} \},$$

$$X_{29}^{236} = \{ {246\cdot\cdot\cdot\cdot \atop 3}, {2\cdot\cdot\cdot\cdot\cdot 1 \atop 2}, {13\cdot 4321 \atop \cdot}, {12\cdot 3\cdot 21 \atop 2}, {1343211 \atop 2}, {1233321 \atop 1}, {1232211 \atop 2}, {1233211 \atop 1}, {12\cdot 2221 \atop 1}, {1122\cdot 11 \atop 1}, {1221111 \atop 1}, {0\cdot 11111 \atop 0} \},$$

$$X_{29}^{237} = \{ {246\cdot\cdot\cdot\cdot \atop 3}, {\cdot 3\cdot\cdot\cdot\cdot 1 \atop 2}, {12\cdot\cdot 321 \atop \cdot}, {12\cdot 3211 \atop 2}, {123\cdot 221 \atop 2}, {1233221 \atop 1}, {1222211 \atop 1}, {\cdot 111111 \atop 0}, {0011111 \atop \cdot} \},$$

$$X_{29}^{238} = \{ {246\cdot\cdot\cdot\cdot \atop 3}, {2454321 \atop 2}, {\cdot 3\cdot\cdot\cdot\cdot 1 \atop 2}, {1244321 \atop 2}, {1233321 \atop \cdot}, {123\cdot 2\cdot 1 \atop 2}, {1222\cdot\cdot 1 \atop 1}, {\cdot\cdot 11111 \atop 0} \},$$

$$X_{29}^{239} = \{ {246\cdot\cdot\cdot\cdot \atop 3}, {\cdot 3\cdot\cdot\cdot\cdot 1 \atop 2}, {12\cdot\cdot 321 \atop \cdot}, {123\cdot 2\cdot 1 \atop 2}, {12222\cdot 1 \atop 1}, {\cdot 111111 \atop 0}, {0011111 \atop \cdot} \},$$

$$X_{29}^{240} = \{ {24654\cdot\cdot \atop 3}, {\cdot 3\cdot\cdot\cdot\cdot 1 \atop \cdot}, {124\cdot\cdot\cdot 1 \atop 2}, {1233321 \atop \cdot}, {1233221 \atop 2}, {1232211 \atop 2}, {1233211 \atop 1}, {1232221 \atop 1}, {00\cdot\cdot 111 \atop \cdot} \},$$

$$X_{29}^{241} = \{ {246\cdot\cdot\cdot\cdot \atop 3}, {2\cdot\cdot\cdot\cdot\cdot 1 \atop 2}, {13\cdot\cdot 321 \atop 2}, {1244321 \atop 2}, {1343211 \atop 2}, {1243221 \atop 2}, {1233321 \atop \cdot}, {123\cdot 2\cdot 1 \atop 2}, {1222221 \atop 1}, {1222111 \atop 1}, {1122211 \atop 1}, {\cdot\cdot 11111 \atop 0} \},$$

$$X_{29}^{242} = \{ {24654\cdot\cdot \atop 3}, {2454321 \atop 2}, {2354321 \atop 3}, {1354321 \atop \cdot}, {\cdot\cdot 4\cdot\cdot\cdot 1 \atop 2}, {1233321 \atop \cdot}, {1233221 \atop 2}, {1232211 \atop 2}, {1233211 \atop 1}, {1232221 \atop 1}, {0111111 \atop 1}, {00\cdot\cdot 111 \atop 0} \},$$

$$X_{29}^{243} = \{ \substack{246\cdot\cdot\cdot\cdot \\ 3}, \substack{2454321 \\ 2}, \substack{234\cdot\cdot\cdot1 \\ 2}, \substack{1\cdot\cdot\cdot321 \\ 2}, \substack{1343211 \\ 2}, \substack{1243221 \\ 2}, \substack{123\cdot2\cdot1 \\ 2}, \substack{1233321 \\ 1}, \substack{1222221 \\ 1}, \substack{1122211 \\ 1}, \substack{\cdot\cdot11111 \\ 0}, \substack{0111111 \\ 1} \},$$

$$X_{29}^{244} = \{ \substack{246\cdot\cdot\cdot\cdot \\ 3}, \substack{\cdot\cdot\cdot\cdot321 \\ 2}, \substack{\cdot3432\cdot1 \\ 2}, \substack{1233221 \\ 2}, \substack{1232\cdot11 \\ 2}, \substack{1233211 \\ 1}, \substack{1232221 \\ 1}, \substack{12222\cdot1 \\ 1}, \substack{\cdot\cdot11111 \\ 0} \},$$

$$X_{29}^{245} = \{ \substack{246\cdot\cdot\cdot\cdot \\ 3}, \substack{\cdot\cdot\cdot\cdot321 \\ 2}, \substack{23432\cdot1 \\ 2}, \substack{1343221 \\ 2}, \substack{1243211 \\ 2}, \substack{1233221 \\ 2}, \substack{1232\cdot11 \\ 2}, \substack{1233211 \\ 1}, \substack{1232221 \\ 1}, \substack{1222211 \\ 1}, \substack{1122221 \\ 1}, \substack{\cdot\cdot11111 \\ 0} \},$$

$$X_{29}^{246} = \{ \substack{246\cdot\cdot\cdot\cdot \\ 3}, \substack{2\cdot\cdot\cdot\cdot\cdot1 \\ 2}, \substack{134\cdot\cdot\cdot1 \\ 2}, \substack{124\cdot\cdot21 \\ 2}, \substack{123\cdot2\cdot1 \\ 2}, \substack{1233321 \\ 1}, \substack{1222221 \\ 1}, \substack{1232111 \\ 1}, \substack{1111111 \\ 1}, \substack{\cdot\cdot11111 \\ 0} \},$$

$$X_{29}^{247} = \{ \substack{246\cdot\cdot\cdot\cdot \\ 3}, \substack{\cdot3\cdot\cdot\cdot\cdot1 \\ 2}, \substack{124\cdot\cdot21 \\ 2}, \substack{123\cdot2\cdot1 \\ 2}, \substack{1233321 \\ 1}, \substack{1222221 \\ 1}, \substack{1232111 \\ 1}, \substack{\cdot111111 \\ 0}, \substack{0011111 \\ \cdot} \},$$

$$X_{29}^{248} = \{ \substack{24654\cdot\cdot \\ 3}, \substack{2\cdot\cdot\cdot\cdot\cdot1 \\ 2}, \substack{13\cdot4321 \\ 2}, \substack{1243\cdot21 \\ 2}, \substack{1343211 \\ 2}, \substack{1233221 \\ 2}, \substack{1233321 \\ 1}, \substack{1232211 \\ 2}, \substack{1233211 \\ 1}, \substack{12\cdot2221 \\ 1}, \substack{1232111 \\ 1}, \substack{1122\cdot11 \\ 1}, \substack{1221111 \\ 1}, \substack{\cdot\cdot\cdot\cdot111 \\ 0} \},$$

$$X_{29}^{249} = \{ \substack{246\cdot\cdot\cdot\cdot \\ 3}, \substack{2454321 \\ 2}, \substack{\cdot3\cdot\cdot\cdot\cdot1 \\ 2}, \substack{1243221 \\ 2}, \substack{1233321 \\ \cdot}, \substack{123\cdot2\cdot1 \\ 2}, \substack{1222221 \\ 1}, \substack{122\cdot111 \\ 1}, \substack{\cdot\cdot11111 \\ 0} \},$$

$$X_{29}^{250} = \{ \substack{246\cdot\cdot\cdot\cdot \\ 3}, \substack{2\cdot\cdot\cdot\cdot\cdot1 \\ 2}, \substack{1354321 \\ 2}, \substack{1343\cdot\cdot1 \\ 2}, \substack{1244321 \\ 2}, \substack{1243221 \\ 2}, \substack{1233321 \\ \cdot}, \substack{123\cdot2\cdot1 \\ 2}, \substack{1222221 \\ 1}, \substack{1222111 \\ 1}, \substack{1121111 \\ 1}, \substack{\cdot\cdot11111 \\ 0} \},$$

$$X_{29}^{251} = \{ \substack{24654\cdot\cdot \\ 3}, \substack{2\cdot\cdot\cdot\cdot\cdot1 \\ 2}, \substack{13\cdot4321 \\ 2}, \substack{12\cdot3321 \\ 2}, \substack{1343211 \\ 2}, \substack{1243221 \\ 2}, \substack{123\cdot2\cdot1 \\ 1}, \substack{1232111 \\ 2}, \substack{1222221 \\ 1}, \substack{1122\cdot11 \\ 1}, \substack{1221111 \\ 1}, \substack{\cdot\cdot\cdot\cdot111 \\ 0} \},$$

$$X_{29}^{252} = \{ \substack{246\cdot\cdot\cdot\cdot \\ 3}, \substack{\cdot3\cdot\cdot\cdot\cdot1 \\ 2}, \substack{1244321 \\ 2}, \substack{1243221 \\ 2}, \substack{1233321 \\ \cdot}, \substack{123\cdot2\cdot1 \\ 2}, \substack{1222221 \\ 1}, \substack{1222111 \\ 1}, \substack{\cdot111111 \\ 0}, \substack{0011111 \\ \cdot} \},$$

$$X_{29}^{253} = \{ \substack{246\cdot\cdot\cdot\cdot \\ 3}, \substack{\cdot\cdot54321 \\ 2}, \substack{2343321 \\ 2}, \substack{1344321 \\ 2}, \substack{\cdot3432\cdot1 \\ 2}, \substack{124\cdot\cdot21 \\ 2}, \substack{1233321 \\ \cdot}, \substack{123\cdot2\cdot1 \\ 2}, \substack{1222221 \\ 1}, \substack{1122111 \\ 1}, \substack{0121111 \\ 1}, \substack{\cdot\cdot11111 \\ 0} \},$$

$$X_{29}^{254} = \{ \substack{246\cdot\cdot\cdot\cdot \\ 3}, \substack{2\cdot\cdot\cdot\cdot\cdot1 \\ 2}, \substack{1344321 \\ 2}, \substack{124\cdot\cdot21 \\ 2}, \substack{13432\cdot1 \\ 2}, \substack{1233321 \\ \cdot}, \substack{123\cdot2\cdot1 \\ 2}, \substack{1222221 \\ 1}, \substack{1122111 \\ 1}, \substack{1111111 \\ 1}, \substack{\cdot\cdot11111 \\ 0} \},$$

$$X_{29}^{255} = \{ \substack{246\cdot\cdot\cdot\cdot \\ 3}, \substack{2454321 \\ 2}, \substack{234\cdot\cdot\cdot1 \\ 2}, \substack{1\cdot\cdot4321 \\ 2}, \substack{13432\cdot1 \\ 2}, \substack{1243\cdot21 \\ 2}, \substack{1233321 \\ \cdot}, \substack{123\cdot2\cdot1 \\ 2}, \substack{1222221 \\ 1}, \substack{1122111 \\ 1}, \substack{\cdot\cdot11111 \\ 0}, \substack{0111111 \\ 1} \},$$

$$X_{29}^{256} = \{ \substack{246\cdot\cdot\cdot\cdot \\ 3}, \substack{23\cdot\cdot\cdot\cdot1 \\ 2}, \substack{1\cdot\cdot4321 \\ 2}, \substack{13432\cdot1 \\ 2}, \substack{1243\cdot21 \\ 2}, \substack{1233321 \\ \cdot}, \substack{123\cdot2\cdot1 \\ 2}, \substack{1222221 \\ 1}, \substack{1122111 \\ 1}, \substack{\cdot\cdot11111 \\ 0}, \substack{0011111 \\ 1} \},$$

$$X_{29}^{257} = \{ \substack{246543\cdot \\ 3}, \substack{2454321 \\ \cdot}, \substack{\cdot3\cdot\cdot\cdot\cdot1 \\ \cdot}, \substack{124\cdot\cdot21 \\ 2}, \substack{1233321 \\ \cdot}, \substack{123\cdot211 \\ \cdot}, \substack{1222221 \\ 1}, \substack{000\cdot\cdot11 \\ 0} \},$$

$$X_{29}^{258} = \{ \substack{246543\cdot \\ 3}, \substack{2454321 \\ \cdot}, \substack{\cdot3\cdot\cdot\cdot\cdot1 \\ \cdot}, \substack{124\cdot\cdot21 \\ 2}, \substack{1233221 \\ 2}, \substack{1233321 \\ 1}, \substack{1233211 \\ \cdot}, \substack{1232211 \\ 2}, \substack{1222221 \\ 1}, \substack{1232111 \\ 1}, \substack{000\cdot\cdot11 \\ 0} \},$$

$$X_{29}^{259} = \{ \substack{246\cdot\cdot\cdot\cdot \\ 3}, \substack{2454321 \\ 2}, \substack{\cdot\cdot4\cdot\cdot\cdot1 \\ 2}, \substack{1233\cdot21 \\ 2}, \substack{1232\cdot11 \\ 2}, \substack{1233211 \\ 1}, \substack{1232221 \\ 1}, \substack{\cdot111111 \\ 1}, \substack{\cdot\cdot11111 \\ 0} \},$$

$$X_{29}^{260} = \{ \substack{246\cdot\cdot\cdot\cdot \\ 3}, \substack{\cdot3\cdot\cdot\cdot\cdot1 \\ 2}, \substack{12\cdot\cdot2\cdot1 \\ 2}, \substack{1233321 \\ \cdot}, \substack{122\cdot111 \\ 1}, \substack{\cdot\cdot11111 \\ 0}, \substack{0011111 \\ 1} \},$$

$$X_{29}^{261} = \{ \substack{24654\cdot\cdot \\ 3}, \substack{2\cdot\cdot\cdot\cdot\cdot1 \\ 2}, \substack{13\cdot4321 \\ \cdot}, \substack{12\cdot3\cdot21 \\ 2}, \substack{1343211 \\ 2}, \substack{1233321 \\ 1}, \substack{1232211 \\ 2}, \substack{1233211 \\ 1}, \substack{12\cdot2221 \\ 1}, \substack{1122\cdot11 \\ 1}, \substack{1221111 \\ 1}, \substack{0\cdot\cdot\cdot111 \\ 0} \},$$

$$X_{29}^{262} = \{ \substack{24654\cdot\cdot \\ 3}, \substack{\cdot3\cdot\cdot\cdot\cdot1 \\ \cdot}, \substack{124\cdot\cdot\cdot1 \\ 2}, \substack{1233221 \\ 2}, \substack{1232211 \\ 2}, \substack{1233211 \\ 1}, \substack{1232221 \\ 1}, \substack{1232111 \\ \cdot}, \substack{00\cdot\cdot111 \\ \cdot} \},$$

$$X_{29}^{263} = \{ \substack{246\cdot\cdot\cdot\cdot \\ 3}, \substack{2454321 \\ 2}, \substack{234\cdot321 \\ 2}, \substack{13\cdot4321 \\ 2}, \substack{\cdot\cdot\cdot\cdot2\cdot1 \\ 2}, \substack{12\cdot3321 \\ \cdot}, \substack{1221111 \\ 1}, \substack{1122111 \\ 1}, \substack{\cdot\cdot11111 \\ 0}, \substack{0111111 \\ 1} \},$$

$$X_{29}^{264} = \{\substack{246\cdot\cdot\cdot\cdot\\3}, \substack{\cdot\cdot\cdot\cdot321\\2}, \substack{2343211\\2}, \substack{1343221\\2}, \substack{123\cdot2\cdot1\\2}, \substack{1232111\\2}, \substack{12222\cdot1\\1}, \substack{1122221\\1}, \substack{0122211\\1}, \substack{\cdot\cdot11111\\0}\},$$

$$X_{29}^{265} = \{\substack{246\cdot\cdot\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\2}, \substack{13\cdot\cdot\cdot21\\2}, \substack{1243221\\2}, \substack{123\cdot\cdot\cdot1\\2}, \substack{1\cdot22221\\1}, \substack{122\cdot111\\1}, \substack{\cdot\cdot11111\\0}\},$$

$$X_{29}^{266} = \{\substack{246\cdot\cdot\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\2}, \substack{13\cdot4321\\2}, \substack{1343221\\2}, \substack{1243\cdot21\\2}, \substack{123\cdot\cdot\cdot1\\2}, \substack{1\cdot22221\\1}, \substack{1221111\\1}, \substack{1122111\\1}, \substack{\cdot\cdot11111\\0}\},$$

$$X_{29}^{267} = \{\substack{246543\cdot\\3}, \substack{2454321\\\cdot}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\\cdot}, \substack{1243\cdot\cdot1\\2}, \substack{1233221\\2}, \substack{1233321\\1}, \substack{1232221\\\cdot}, \substack{1232211\\2}, \substack{1232111\\1}, \substack{1221111\\1}, \substack{000\cdot\cdot11\\0}\},$$

$$X_{29}^{268} = \{\substack{246\cdot\cdot\cdot\cdot\\3}, \substack{23\cdot\cdot\cdot\cdot1\\2}, \substack{13\cdot\cdot\cdot21\\2}, \substack{124\cdot321\\2}, \substack{123\cdot\cdot\cdot1\\2}, \substack{1\cdot22221\\1}, \substack{1222211\\1}, \substack{\cdot\cdot11111\\0}, \substack{0011111\\1}\},$$

$$X_{29}^{269} = \{\substack{246543\cdot\\3}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\\cdot}, \substack{12\cdot\cdot321\\\cdot}, \substack{12\cdot3221\\2}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{12\cdot2221\\1}, \substack{00\cdot\cdot\cdot11\\\cdot}\},$$

$$X_{29}^{270} = \{\substack{246\cdot\cdot\cdot\cdot\\3}, \substack{2454321\\2}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\2}, \substack{123\cdot\cdot\cdot1\\2}, \substack{122\cdot\cdot\cdot1\\1}, \substack{\cdot\cdot11111\\0}\},$$

$$X_{29}^{271} = \{\substack{246\cdot\cdot\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\2}, \substack{1354321\\2}, \substack{1343\cdot\cdot1\\2}, \substack{1244321\\2}, \substack{123\cdot\cdot\cdot1\\2}, \substack{1222\cdot\cdot1\\1}, \substack{1121111\\1}, \substack{\cdot\cdot11111\\0}\},$$

$$X_{29}^{272} = \{\substack{2465432\\3}, \substack{\cdot\cdot\cdot\cdot321\\2}, \substack{2343211\\2}, \substack{1343221\\2}, \substack{12\cdot\cdot2\cdot1\\1}, \substack{1232111\\2}, \substack{1122221\\1}, \substack{0122211\\1}, \substack{\cdot\cdot\cdot\cdot\cdot\cdot1\\0}\},$$

$$X_{29}^{273} = \{\substack{246\cdot\cdot\cdot\cdot\\3}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\2}, \substack{1244321\\2}, \substack{123\cdot\cdot\cdot1\\2}, \substack{1222\cdot\cdot1\\1}, \substack{\cdot111111\\0}, \substack{0011111\\\cdot}\},$$

$$X_{29}^{274} = \{\substack{246\cdot\cdot\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\2}, \substack{13\cdot4321\\2}, \substack{1243\cdot21\\2}, \substack{1343211\\2}, \substack{123\cdot\cdot\cdot1\\2}, \substack{1222221\\1}, \substack{1122\cdot11\\1}, \substack{1221111\\1}, \substack{\cdot\cdot11111\\0}\},$$

$$X_{29}^{275} = \{\substack{246\cdot\cdot\cdot\cdot\\3}, \substack{\cdot\cdot54321\\2}, \substack{2343321\\2}, \substack{1344321\\2}, \substack{\cdot3432\cdot1\\2}, \substack{124\cdot321\\2}, \substack{123\cdot\cdot\cdot1\\2}, \substack{12222\cdot1\\1}, \substack{1122111\\1}, \substack{0121111\\1}, \substack{\cdot\cdot11111\\0}\},$$

$$X_{29}^{276} = \{\substack{246\cdot\cdot\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\2}, \substack{1\cdot44321\\2}, \substack{13432\cdot1\\2}, \substack{1243321\\2}, \substack{123\cdot\cdot\cdot1\\2}, \substack{12222\cdot1\\1}, \substack{1122111\\1}, \substack{1111111\\1}, \substack{\cdot\cdot11111\\0}\},$$

$$X_{29}^{277} = \{\substack{2465432\\3}, \substack{24\cdot\cdot321\\\cdot}, \substack{2354321\\3}, \substack{1354321\\2}, \substack{\cdot\cdot4\cdot\cdot\cdot1\\2}, \substack{123\cdot\cdot21\\\cdot}, \substack{1111111\\0}, \substack{0111111\\1}, \substack{00000\cdot1\\0}\},$$

$$X_{29}^{278} = \{\substack{246\cdot\cdot\cdot\cdot\\3}, \substack{23\cdot\cdot\cdot\cdot1\\2}, \substack{1\cdot\cdot4321\\2}, \substack{13432\cdot1\\2}, \substack{1243321\\2}, \substack{123\cdot\cdot\cdot1\\2}, \substack{12222\cdot1\\1}, \substack{1122111\\1}, \substack{\cdot\cdot11111\\0}, \substack{0011111\\1}\},$$

$$X_{29}^{279} = \{\substack{24654\cdot\cdot\\3}, \substack{2454321\\\cdot}, \substack{2354321\\2}, \substack{1354321\\3}, \substack{\cdot\cdot4\cdot\cdot\cdot1\\2}, \substack{1233211\\2}, \substack{1232221\\2}, \substack{1233221\\1}, \substack{1232111\\\cdot}, \substack{1232211\\1}, \substack{1111111\\1}, \substack{0111111\\0}, \substack{000\cdot111\\0}\},$$

$$X_{29}^{280} = \{\substack{2465432\\3}, \substack{24\cdot4321\\\cdot}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\\cdot}, \substack{12\cdot\cdot\cdot21\\\cdot}, \substack{0000\cdot\cdot1\\0}\},$$

$$X_{29}^{281} = \{\substack{246\cdot\cdot\cdot\cdot\\3}, \substack{2354321\\2}, \substack{\cdot34\cdot\cdot\cdot1\\2}, \substack{124\cdot321\\2}, \substack{123\cdot\cdot\cdot1\\2}, \substack{12222\cdot1\\1}, \substack{1111111\\1}, \substack{\cdot\cdot11111\\0}, \substack{0011111\\1}\},$$

$$X_{29}^{282} = \{\substack{2465432\\3}, \substack{\cdot\cdot\cdot4321\\\cdot}, \substack{\cdot343\cdot\cdot1\\2}, \substack{12432\cdot1\\2}, \substack{1233321\\\cdot}, \substack{1233211\\2}, \substack{1232221\\2}, \substack{1233221\\1}, \substack{1232211\\1}, \substack{1222111\\1}, \substack{0000\cdot\cdot1\\0}\},$$

$$X_{29}^{283} = \{\substack{2465\cdot\cdot\cdot\\3}, \substack{\cdot\cdot\cdot4321\\2}, \substack{\cdot343\cdot\cdot1\\2}, \substack{1243211\\2}, \substack{1233221\\2}, \substack{1233321\\1}, \substack{12322\cdot1\\2}, \substack{1232221\\1}, \substack{12\cdot2111\\1}, \substack{1222211\\1}, \substack{\cdot\cdot\cdot1111\\0}\},$$

$$X_{29}^{284} = \{\substack{24654\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\2}, \substack{13\cdot4321\\\cdot}, \substack{1243321\\2}, \substack{1\cdot43211\\2}, \substack{123\cdot\cdot21\\\cdot}, \substack{1\cdot22211\\1}, \substack{1221111\\1}, \substack{1122111\\1}, \substack{0\cdot\cdot\cdot111\\0}\},$$

$$X_{29}^{285} = \{\substack{2465\cdot\cdot\cdot\\3}, \substack{2\cdot54321\\2}, \substack{\cdot343\cdot\cdot1\\2}, \substack{1\cdot44321\\2}, \substack{12\cdot3321\\\cdot}, \substack{1243211\\2}, \substack{1233221\\2}, \substack{12322\cdot1\\2}, \substack{1232221\\1}, \substack{1222211\\1}, \substack{1111111\\1}, \substack{0121111\\1}, \substack{\cdot\cdot\cdot1111\\0}\},$$

$$X_{29}^{286} = \{ \substack{24654\cdot\cdot\\3}, \substack{2354321\\2}, \substack{1354321\\3}, \substack{\cdot\cdot4\cdot\cdot\cdot1\\2}, \substack{1233321\\\cdot}, \substack{1233221\\2}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{1232221\\1}, \substack{1111111\\1}, \substack{0111111\\0}, \substack{00\cdot\cdot111\\\cdot} \},$$

$$X_{29}^{287} = \{ \substack{2465432\\3}, \substack{234\cdot\cdot\cdot1\\2}, \substack{1344321\\2}, \substack{1\cdot432\cdot1\\2}, \substack{1243321\\2}, \substack{123\cdot\cdot11\\\cdot}, \substack{1221111\\1}, \substack{1122111\\1}, \substack{\cdot\cdot11111\\\cdot}, \substack{000\cdot\cdot\cdot1\\0} \},$$

$$X_{29}^{288} = \{ \substack{2465432\\3}, \substack{\cdot354321\\\cdot}, \substack{2344321\\2}, \substack{\cdot3432\cdot1\\2}, \substack{1343321\\2}, \substack{124\cdot\cdot\cdot1\\2}, \substack{1233221\\2}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{1232221\\1}, \substack{1232111\\\cdot}, \substack{1121111\\1}, \substack{0122111\\1}, \substack{00\cdot\cdot\cdot\cdot1\\\cdot} \},$$

$$X_{29}^{289} = \{ \substack{24654\cdot\cdot\\3}, \substack{2454321\\2}, \substack{\cdot\cdot4\cdot\cdot\cdot1\\2}, \substack{1233221\\2}, \substack{1233321\\1}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{1232221\\1}, \substack{1232111\\1}, \substack{\cdot111111\\1}, \substack{\cdot\cdot\cdot\cdot111\\0} \},$$

$$X_{29}^{290} = \{ \substack{24654\cdot\cdot\\3}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\2}, \substack{12432\cdot1\\2}, \substack{1233321\\\cdot}, \substack{1233211\\2}, \substack{1232221\\2}, \substack{1233221\\1}, \substack{1232211\\1}, \substack{122\cdot111\\1}, \substack{\cdot\cdot\cdot\cdot111\\0}, \substack{0011111\\1} \},$$

$$X_{29}^{291} = \{ \substack{24654\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\2}, \substack{13\cdot4321\\\cdot}, \substack{1243321\\2}, \substack{1343211\\2}, \substack{12\cdot3221\\2}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{12\cdot2221\\1}, \substack{1232111\\\cdot}, \substack{1122\cdot11\\1}, \substack{1221111\\1}, \substack{0\cdot\cdot\cdot111\\0} \},$$

$$X_{29}^{292} = \{ \substack{246543\cdot\\3}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\\cdot}, \substack{12\cdot3\cdot\cdot1\\\cdot}, \substack{1221111\\1}, \substack{00\cdot\cdot\cdot11\\\cdot} \},$$

$$X_{29}^{293} = \{ \substack{2465432\\3}, \substack{24\cdot4321\\\cdot}, \substack{2354321\\2}, \substack{1354321\\3}, \substack{\cdot34\cdot\cdot\cdot1\\2}, \substack{124\cdot\cdot21\\2}, \substack{1233221\\2}, \substack{1233321\\1}, \substack{12\cdot2221\\\cdot}, \substack{1232211\\2}, \substack{1232111\\1}, \substack{1111111\\1}, \substack{0111111\\0}, \substack{0000\cdot\cdot1\\0} \},$$

$$X_{29}^{294} = \{ \substack{2465\cdot\cdot\cdot\\3}, \substack{2\cdot\cdot4321\\2}, \substack{1\cdot\cdot\cdot321\\2}, \substack{23432\cdot1\\2}, \substack{1343221\\2}, \substack{1243211\\2}, \substack{12322\cdot1\\\cdot}, \substack{1232111\\2}, \substack{1222211\\1}, \substack{1122221\\1}, \substack{0122111\\1}, \substack{\cdot\cdot\cdot1111\\0} \},$$

$$X_{29}^{295} = \{ \substack{2465\cdot\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\2}, \substack{134\cdot\cdot21\\2}, \substack{12\cdot\cdot321\\2}, \substack{1243211\\2}, \substack{12322\cdot1\\\cdot}, \substack{1232111\\2}, \substack{1222211\\1}, \substack{1122221\\1}, \substack{1111111\\1}, \substack{\cdot\cdot\cdot1111\\0} \},$$

$$X_{29}^{296} = \{ \substack{246543\cdot\\3}, \substack{2454321\\2}, \substack{\cdot3\cdot4321\\\cdot}, \substack{2343321\\2}, \substack{\cdot\cdot\cdot3211\\2}, \substack{1343221\\2}, \substack{12\cdot3\cdot21\\\cdot}, \substack{1232221\\2}, \substack{1221111\\1}, \substack{1122111\\1}, \substack{0122211\\1}, \substack{00\cdot\cdot\cdot11\\0} \},$$

$$X_{29}^{297} = \{ \substack{2465\cdot\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot321\\2}, \substack{13\cdot4321\\2}, \substack{\cdot\cdot\cdot\cdot\cdot11\\2}, \substack{1\cdot43221\\2}, \substack{1233321\\2}, \substack{1232221\\2}, \substack{1232211\\1}, \substack{122\cdot111\\1}, \substack{1122111\\1}, \substack{0122211\\1}, \substack{\cdot\cdot\cdot1111\\0} \},$$

$$X_{29}^{298} = \{ \substack{2465432\\3}, \substack{\cdot\cdot54321\\\cdot}, \substack{234\cdot\cdot\cdot1\\2}, \substack{1344321\\2}, \substack{1\cdot432\cdot1\\2}, \substack{1243321\\2}, \substack{1233221\\2}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{1232221\\1}, \substack{1232111\\\cdot}, \substack{1221111\\1}, \substack{1122111\\1}, \substack{000\cdot\cdot\cdot1\\0} \},$$

$$X_{29}^{299} = \{ \substack{24654\cdot\cdot\\3}, \substack{2\cdot54321\\2}, \substack{13\cdot4321\\\cdot}, \substack{23432\cdot1\\2}, \substack{12\cdot3\cdot21\\2}, \substack{1343211\\2}, \substack{1233321\\1}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{12\cdot2221\\1}, \substack{1122\cdot11\\1}, \substack{1221111\\1}, \substack{012\cdot111\\1}, \substack{0\cdot\cdot\cdot111\\0} \},$$

$$X_{29}^{300} = \{ \substack{246543\cdot\\3}, \substack{2454321\\\cdot}, \substack{2354321\\2}, \substack{1354321\\3}, \substack{\cdot34\cdot\cdot\cdot1\\2}, \substack{1243\cdot\cdot1\\2}, \substack{1233221\\2}, \substack{1233321\\1}, \substack{1233211\\\cdot}, \substack{1232211\\2}, \substack{1232111\\1}, \substack{1221111\\1}, \substack{1111111\\1}, \substack{0111111\\0}, \substack{000\cdot\cdot11\\0} \},$$

$$X_{29}^{301} = \{ \substack{24654\cdot\cdot\\3}, \substack{1354321\\\cdot}, \substack{\cdot\cdot4\cdot\cdot\cdot1\\2}, \substack{1233221\\2}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{1232221\\1}, \substack{1232111\\\cdot}, \substack{0\cdot11111\\\cdot}, \substack{000\cdot111\\0} \},$$

$$X_{29}^{302} = \{ \substack{2465432\\3}, \substack{2\cdot\cdot\cdot321\\\cdot}, \substack{1344321\\2}, \substack{1\cdot43221\\2}, \substack{1243321\\2}, \substack{123\cdot\cdot11\\\cdot}, \substack{1\cdot22221\\1}, \substack{01222\cdot1\\1}, \substack{1221111\\1}, \substack{1122111\\1}, \substack{1111111\\\cdot}, \substack{00000\cdot1\\0} \},$$

$$X_{29}^{303} = \{ \substack{24654\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\2}, \substack{1\cdot4\cdot321\\2}, \substack{1343211\\2}, \substack{1243221\\2}, \substack{1233211\\2}, \substack{1232221\\2}, \substack{1233221\\1}, \substack{1232111\\\cdot}, \substack{1232211\\1}, \substack{1222221\\1}, \substack{1122211\\1}, \substack{1111111\\1}, \substack{\cdot\cdot\cdot\cdot111\\0} \},$$

$$X_{29}^{304}=\{\substack{24654\cdot\cdot\\3},\substack{2354321\\2},\substack{1354321\\3},\substack{\cdot\cdot4\cdot\cdot\cdot1\\2},\substack{1233221\\2},\substack{1232211\\2},\substack{1233211\\1},\substack{1232221\\1},\substack{1232111\\\cdot},\substack{1111111\\1},\substack{0111111\\0},\substack{00\cdot\cdot111\\\cdot}\},$$

$$X_{29}^{305}=\{\substack{24654\cdot\cdot\\3},\substack{2\cdot\cdot\cdot\cdot21\\2},\substack{13\cdot\cdot321\\2},\substack{1343211\\2},\substack{12432\cdot1\\2},\substack{1233321\\2},\substack{1233211\\2},\substack{1232221\\2},\substack{1233221\\1},\substack{1232111\\2},\substack{1232211\\1},\substack{122\cdot111\\1},\substack{1122211\\1},\substack{0122221\\1},\substack{\cdot\cdot\cdot\cdot111\\0}\},$$

$$X_{29}^{306}=\{\substack{2465432\\3},\substack{\cdot\cdot54321\\\cdot},\substack{234\cdot\cdot\cdot1\\2},\substack{1343321\\2},\substack{1244321\\2},\substack{1\cdot432\cdot1\\2},\substack{1233321\\\cdot},\substack{1233211\\2},\substack{1232221\\2},\substack{1233221\\1},\substack{1232211\\1},\substack{1222111\\1},\substack{1121111\\1},\substack{000\cdot\cdot\cdot1\\0}\},$$

$$X_{29}^{307}=\{\substack{2465432\\3},\substack{2454321\\2},\substack{2354321\\3},\substack{1354321\\\cdot},\substack{234\cdot\cdot21\\2},\substack{1343321\\2},\substack{1244321\\2},\substack{\cdot\cdot\cdot3211\\2},\substack{1\cdot43221\\2},\substack{1233321\\2},\substack{1233\cdot21\\1},\substack{1232221\\2},\substack{1232211\\1},\substack{1222111\\1},\substack{1121111\\1},\substack{0111111\\1},\substack{00\cdot\cdot\cdot\cdot1\\0}\},$$

$$X_{29}^{308}=\{\substack{2465432\\3},\substack{\cdot3\cdot\cdot\cdot\cdot1\\\cdot},\substack{12432\cdot1\\2},\substack{1233321\\\cdot},\substack{1233211\\2},\substack{1232221\\2},\substack{1233221\\1},\substack{1232211\\1},\substack{122\cdot111\\1},\substack{00\cdot\cdot\cdot\cdot1\\\cdot}\},$$

$$X_{29}^{309}=\{\substack{2465\cdot\cdot\cdot\\3},\substack{\cdot\cdot54321\\\cdot},\substack{\cdot\cdot4\cdot\cdot\cdot1\\2},\substack{123\cdot\cdot21\\\cdot},\substack{0001111\\0}\},$$

$$X_{29}^{310}=\{\substack{2465\cdot\cdot\cdot\\3},\substack{\cdot\cdot54321\\\cdot},\substack{\cdot\cdot4\cdot\cdot\cdot1\\2},\substack{1233321\\\cdot},\substack{1233211\\2},\substack{1232221\\2},\substack{1233221\\1},\substack{1232211\\1},\substack{0001111\\0}\},$$

$$X_{29}^{311}=\{\substack{2\cdot\cdot\cdot\cdot\cdot\cdot\\3},\substack{\cdot\cdot54321\\2},\substack{\cdot\cdot\cdot3\cdot\cdot1\\2},\substack{1\cdot44321\\2},\substack{1233\cdot21\\1},\substack{1232221\\2},\substack{0121111\\1},\substack{1111111\\0}\},$$

$$X_{29}^{312}=\{\substack{2\cdot\cdot\cdot\cdot\cdot\cdot\\3},\substack{\cdot\cdot54321\\2},\substack{\cdot\cdot4\cdot\cdot21\\2},\substack{1\cdot\cdot\cdot211\\2},\substack{123\cdot221\\2},\substack{1233321\\1},\substack{1232111\\1},\substack{0122221\\1},\substack{1111111\\0}\},$$

$$X_{29}^{313}=\{\substack{24654\cdot\cdot\\3},\substack{\cdot\cdot\cdot4321\\\cdot},\substack{\cdot\cdot\cdot3\cdot\cdot1\\\cdot},\substack{0000111\\0}\},$$

$$X_{29}^{314}=\{\substack{2465\cdot\cdot\cdot\\3},\substack{\cdot\cdot54321\\\cdot},\substack{\cdot\cdot4\cdot\cdot\cdot1\\2},\substack{1233221\\2},\substack{1233321\\1},\substack{1232221\\\cdot},\substack{1232211\\2},\substack{1232111\\1},\substack{0001111\\0}\},$$

$$X_{29}^{315}=\{\substack{2\cdot\cdot\cdot\cdot\cdot\cdot\\3},\substack{2454321\\2},\substack{\cdot\cdot4\cdot\cdot\cdot1\\2},\substack{1354321\\2},\substack{1233\cdot21\\2},\substack{1232\cdot11\\2},\substack{1233211\\1},\substack{1232221\\1},\substack{1111111\\0},\substack{0111111\\1}\},$$

$$X_{29}^{316}=\{\substack{2\cdot\cdot\cdot\cdot\cdot\cdot\\3},\substack{\cdot\cdot54321\\2},\substack{1\cdot4\cdot321\\2},\substack{\cdot\cdot\cdot\cdot2\cdot1\\2},\substack{1233321\\\cdot},\substack{012\cdot111\\1},\substack{1111111\\0}\},$$

$$X_{29}^{317}=\{\substack{2465\cdot\cdot\cdot\\3},\substack{\cdot\cdot54321\\\cdot},\substack{\cdot\cdot4\cdot\cdot\cdot1\\2},\substack{1232\cdot\cdot1\\\cdot},\substack{0001111\\0}\},$$

$$X_{29}^{318}=\{\substack{24\cdot\cdot\cdot\cdot\cdot\\\cdot},\substack{\cdot3\cdot\cdot\cdot\cdot1\\2},\substack{123\cdot\cdot\cdot1\\2},\substack{122\cdot\cdot\cdot1\\1},\substack{\cdot111111\\0}\},$$

$$X_{29}^{319}=\{\substack{2465432\\3},\substack{\cdot\cdot\cdot\cdot321\\2},\substack{2343211\\2},\substack{1343221\\2},\substack{1233321\\1},\substack{12\cdot\cdot2\cdot1\\1},\substack{1122221\\1},\substack{0122211\\1},\substack{\cdot\cdot\cdot\cdot\cdot\cdot1\\0}\},$$

$$X_{29}^{320}=\{\substack{\cdot\cdot\cdot\cdot\cdot\cdot\cdot\\3},\substack{\cdot\cdot\cdot\cdot321\\2},\substack{1343211\\2},\substack{12\cdot3221\\2},\substack{1232\cdot11\\2},\substack{1233211\\1},\substack{12\cdot2221\\1},\substack{1122211\\1},\substack{01222\cdot1\\1}\},$$

$$X_{29}^{321}=\{\substack{24\cdot\cdot\cdot\cdot\cdot\\3},\substack{\cdot354321\\2},\substack{\cdot\cdot4\cdot\cdot\cdot1\\2},\substack{123\cdot\cdot21\\\cdot},\substack{\cdot111111\\0},\substack{0011111\\1}\},$$

$$X_{29}^{322}=\{\substack{246543\cdot\\3},\substack{24\cdot4321\\\cdot},\substack{\cdot3\cdot\cdot\cdot\cdot1\\\cdot},\substack{12\cdot\cdot\cdot21\\\cdot},\substack{0000\cdot11\\0}\},$$

$$X_{29}^{323}=\{\substack{24654\cdot\cdot\\3},\substack{2\cdot\cdot\cdot\cdot\cdot1\\2},\substack{13\cdot4321\\2},\substack{12\cdot3321\\\cdot},\substack{1343211\\2},\substack{1243221\\2},\substack{123\cdot2\cdot1\\1},\substack{1222221\\1},\substack{1122\cdot11\\1},\substack{1221111\\1},\substack{\cdot\cdot\cdot\cdot111\\0}\},$$

$$X_{29}^{324}=\{\substack{24654\cdot\cdot\\3},\substack{\cdot\cdot54321\\\cdot},\substack{\cdot34\cdot\cdot\cdot1\\2},\substack{12\cdot\cdot321\\\cdot},\substack{12\cdot3221\\2},\substack{1232211\\2},\substack{1233211\\1},\substack{12\cdot2221\\1},\substack{000\cdot111\\0}\},$$

$$X_{29}^{325}=\{\substack{246543\cdot\\3},\substack{\cdot\cdot54321\\\cdot},\substack{\cdot\cdot44321\\2},\substack{\cdot\cdot\cdot3\cdot21\\\cdot},\substack{2343211\\2},\substack{1233211\\\cdot},\substack{1\cdot22221\\1},\substack{000\cdot\cdot11\\0}\},$$

$$X_{29}^{326}=\{\substack{24\cdot\cdot\cdot\cdot\cdot\\\cdot},\substack{23\cdot\cdot\cdot\cdot1\\2},\substack{1354321\\2},\substack{1\cdot\cdot3\cdot21\\\cdot},\substack{1244321\\2},\substack{12\cdot3211\\2},\substack{1232221\\2},\substack{1122221\\1},\substack{1121111\\1},\substack{\cdot111111\\0}\},$$

$$X_{29}^{327}=\{\substack{246543\cdot\\3},\substack{\cdot\cdot54321\\\cdot},\substack{234\cdot\cdot\cdot1\\2},\substack{1\cdot4\cdot\cdot21\\2},\substack{1233321\\\cdot},\substack{1233221\\2},\substack{1232211\\2},\substack{1233211\\1},\substack{1\cdot\cdot2221\\1},\substack{000\cdot\cdot11\\0}\},$$

$$X_{29}^{328}=\{\substack{24\cdot\cdot\cdot\cdot\cdot\\3},\substack{\cdot3\cdot4321\\2},\substack{\cdot\cdot\cdot3\cdot21\\\cdot},\substack{1244321\\2},\substack{2343211\\2},\substack{12\cdot3211\\2},\substack{1232221\\2},\substack{1122221\\1},\substack{\cdot111111\\0},\substack{0011111\\1}\},$$

$$X_{29}^{329} = \{ \substack{24\cdot\cdot\cdot\cdot\cdot\\ \cdot}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\ 2}, \substack{124\cdot\cdot\cdot1\\ 2}, \substack{1233\cdot21\\ 1}, \substack{1233211\\ 2}, \substack{1232221\\ 2}, \substack{1232\cdot11\\ 1}, \substack{\cdot111111\\ 0} \},$$

$$X_{29}^{330} = \{ \substack{24\cdot\cdot\cdot\cdot\cdot\\ \cdot}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\ 2}, \substack{1244321\\ 2}, \substack{1243221\\ 2}, \substack{1233321\\ 2}, \substack{12\cdot3211\\ 2}, \substack{1233\cdot21\\ 1}, \substack{1232221\\ 2}, \substack{1232211\\ 1}, \substack{1222111\\ 1}, \substack{\cdot111111\\ 0} \},$$

$$X_{29}^{331} = \{ \substack{24\cdot\cdot\cdot\cdot\cdot\\ \cdot}, \substack{23\cdot\cdot\cdot\cdot1\\ 2}, \substack{1\cdot4\cdot\cdot\cdot1\\ 2}, \substack{1233321\\ \cdot}, \substack{1233211\\ 2}, \substack{1232221\\ 2}, \substack{1233221\\ 1}, \substack{1232211\\ 1}, \substack{1111111\\ 1}, \substack{\cdot111111\\ 0} \},$$

$$X_{29}^{332} = \{ \substack{24\cdot\cdot\cdot\cdot\cdot\\ 3}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\ 2}, \substack{124\cdot\cdot\cdot1\\ 2}, \substack{1233321\\ \cdot}, \substack{1233211\\ 2}, \substack{1232221\\ 2}, \substack{1233221\\ 1}, \substack{1232211\\ 1}, \substack{\cdot111111\\ 0}, \substack{0011111\\ 1} \},$$

$$X_{29}^{333} = \{ \substack{24\cdot\cdot\cdot\cdot\cdot\\ 3}, \substack{\cdot\cdot\cdot4321\\ 2}, \substack{2343221\\ 2}, \substack{1343321\\ 2}, \substack{\cdot\cdot\cdot3211\\ 2}, \substack{12\cdot3\cdot21\\ \cdot}, \substack{1232221\\ 2}, \substack{1122211\\ 1}, \substack{0122111\\ 1}, \substack{\cdot111111\\ 0} \},$$

$$X_{29}^{334} = \{ \substack{2465\cdot\cdot\cdot\\ 3}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\ \cdot}, \substack{124\cdot\cdot\cdot1\\ 2}, \substack{1233321\\ \cdot}, \substack{1233211\\ 2}, \substack{1232221\\ 2}, \substack{1233221\\ 1}, \substack{1232211\\ 1}, \substack{00\cdot1111\\ \cdot} \},$$

$$X_{29}^{335} = \{ \substack{24\cdot\cdot\cdot\cdot\cdot\\ 3}, \substack{23\cdot\cdot\cdot\cdot1\\ 2}, \substack{1\cdot\cdot\cdot321\\ 2}, \substack{1\cdot\cdot3211\\ 2}, \substack{12\cdot\cdot221\\ 2}, \substack{1233\cdot21\\ 1}, \substack{1122211\\ 1}, \substack{\cdot111111\\ 0}, \substack{0011111\\ 1} \},$$

$$X_{29}^{336} = \{ \substack{2465432\\ 3}, \substack{24\cdot\cdot321\\ \cdot}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\ \cdot}, \substack{124\cdot\cdot21\\ 2}, \substack{1233221\\ 2}, \substack{1232211\\ 2}, \substack{1233211\\ 1}, \substack{12\cdot2221\\ 1}, \substack{1232111\\ \cdot}, \substack{00000\cdot1\\ 0} \},$$

$$X_{29}^{337} = \{ \substack{24\cdot\cdot\cdot\cdot\cdot\\ \cdot}, \substack{\cdot3\cdot\cdot321\\ 2}, \substack{23432\cdot1\\ 2}, \substack{12\cdot\cdot321\\ 2}, \substack{12432\cdot1\\ 2}, \substack{1233221\\ 2}, \substack{1232\cdot11\\ 2}, \substack{1233211\\ 1}, \substack{1232221\\ 1}, \substack{11222\cdot1\\ 1}, \substack{\cdot111111\\ 0} \},$$

$$X_{29}^{338} = \{ \substack{24\cdot\cdot\cdot\cdot\cdot\\ 3}, \substack{\cdot\cdot\cdot\cdot321\\ 2}, \substack{2343211\\ 2}, \substack{1343221\\ 2}, \substack{12432\cdot1\\ 2}, \substack{1233221\\ 2}, \substack{1232\cdot11\\ 2}, \substack{1233211\\ 1}, \substack{1232221\\ 1}, \substack{1122221\\ 1}, \substack{0122211\\ 1}, \substack{\cdot111111\\ 0} \},$$

$$X_{29}^{339} = \{ \substack{2465\cdot\cdot\cdot\\ 3}, \substack{2\cdot54321\\ \cdot}, \substack{\cdot\cdot4\cdot\cdot\cdot1\\ 2}, \substack{1233221\\ 2}, \substack{1233321\\ 1}, \substack{1232221\\ \cdot}, \substack{1232211\\ 2}, \substack{1232111\\ 1}, \substack{1111111\\ \cdot}, \substack{0001111\\ 0} \},$$

$$X_{29}^{340} = \{ \substack{24\cdot\cdot\cdot\cdot\cdot\\ \cdot}, \substack{234\cdot\cdot\cdot1\\ 2}, \substack{13\cdot\cdot\cdot21\\ 2}, \substack{124\cdot\cdot\cdot1\\ 2}, \substack{123\cdot2\cdot1\\ 2}, \substack{1233321\\ 1}, \substack{1232111\\ 1}, \substack{1122221\\ 1}, \substack{\cdot111111\\ 0}, \substack{0111111\\ 1} \},$$

$$X_{29}^{341} = \{ \substack{2465\cdot\cdot\cdot\\ 3}, \substack{2454321\\ \cdot}, \substack{2354321\\ 3}, \substack{\cdot\cdot4\cdot\cdot\cdot1\\ 2}, \substack{1354321\\ 2}, \substack{1233221\\ 2}, \substack{1233321\\ 1}, \substack{1232221\\ \cdot}, \substack{1232211\\ 2}, \substack{1232111\\ 1}, \substack{1111111\\ 0}, \substack{0111111\\ 1}, \substack{0001111\\ 0} \},$$

$$X_{29}^{342} = \{ \substack{24\cdot\cdot\cdot\cdot\cdot\\ 3}, \substack{\cdot\cdot54321\\ 2}, \substack{134\cdot\cdot21\\ 2}, \substack{23432\cdot1\\ 2}, \substack{12\cdot\cdot321\\ \cdot}, \substack{12\cdot\cdot2\cdot1\\ 2}, \substack{1122221\\ 1}, \substack{012\cdot111\\ 1}, \substack{\cdot111111\\ 0} \},$$

$$X_{29}^{343} = \{ \substack{24\cdot\cdot\cdot\cdot\cdot\\ \cdot}, \substack{\cdot\cdot4\cdot\cdot\cdot1\\ 2}, \substack{1233\cdot21\\ 2}, \substack{1232\cdot11\\ 2}, \substack{1233211\\ 1}, \substack{1232221\\ 1}, \substack{\cdot111111\\ \cdot} \},$$

$$X_{29}^{344} = \{ \substack{24\cdot\cdot\cdot\cdot\cdot\\ 3}, \substack{2354321\\ 2}, \substack{\cdot\cdot4\cdot\cdot\cdot1\\ 2}, \substack{1233\cdot21\\ 2}, \substack{1232\cdot11\\ 2}, \substack{1233211\\ 1}, \substack{1232221\\ 1}, \substack{1111111\\ 1}, \substack{\cdot111111\\ 0}, \substack{0011111\\ 1} \},$$

$$X_{29}^{345} = \{ \substack{24\cdot\cdot\cdot\cdot\cdot\\ \cdot}, \substack{234\cdot321\\ 2}, \substack{1354321\\ 2}, \substack{\cdot\cdot\cdot\cdot2\cdot1\\ 2}, \substack{12\cdot\cdot321\\ \cdot}, \substack{112\cdot111\\ 1}, \substack{\cdot111111\\ 0}, \substack{0111111\\ 1} \},$$

$$X_{29}^{346} = \{ \substack{24\cdot\cdot\cdot\cdot\cdot\\ \cdot}, \substack{\cdot3\cdot4321\\ 2}, \substack{2343\cdot21\\ 2}, \substack{12\cdot\cdot\cdot\cdot1\\ 2}, \substack{1122\cdot\cdot1\\ 1}, \substack{0122221\\ 1}, \substack{\cdot111111\\ 0} \},$$

$$X_{29}^{347} = \{ \substack{24\cdot\cdot\cdot\cdot\cdot\\ \cdot}, \substack{\cdot3\cdot4321\\ 2}, \substack{2343221\\ 2}, \substack{1343321\\ 2}, \substack{12\cdot\cdot\cdot\cdot1\\ 2}, \substack{\cdot122221\\ 1}, \substack{1122211\\ 1}, \substack{0122111\\ 1}, \substack{\cdot111111\\ 0} \},$$

$$X_{29}^{348} = \{ \substack{24\cdot\cdot\cdot\cdot\cdot\\ 3}, \substack{\cdot\cdot54321\\ 2}, \substack{234\cdot\cdot\cdot1\\ 2}, \substack{12\cdot\cdot\cdot\cdot1\\ 2}, \substack{112\cdot\cdot\cdot1\\ 1}, \substack{\cdot111111\\ 0} \},$$

$$X_{29}^{349} = \{ \substack{2465432\\ 3}, \substack{24\cdot\cdot\cdot21\\ \cdot}, \substack{2354321\\ 3}, \substack{\cdot\cdot4\cdot\cdot\cdot1\\ 2}, \substack{1354321\\ 2}, \substack{123\cdot\cdot21\\ \cdot}, \substack{1111111\\ 0}, \substack{0111111\\ 1}, \substack{0000001\\ 0} \},$$

$$X_{29}^{350} = \{ \substack{2465432\\ 3}, \substack{\cdot\cdot\cdot\cdot321\\ \cdot}, \substack{\cdot3432\cdot1\\ 2}, \substack{12\cdot\cdot221\\ \cdot}, \substack{00000\cdot1\\ 0} \},$$

$$X_{29}^{351} = \{ \substack{24\cdot\cdot\cdot\cdot\cdot\\ \cdot}, \substack{\cdot354321\\ 2}, \substack{2344321\\ 2}, \substack{1343\cdot21\\ 2}, \substack{12\cdot\cdot\cdot\cdot1\\ 2}, \substack{2343211\\ 2}, \substack{1122221\\ 1}, \substack{0122\cdot11\\ 1}, \substack{1121111\\ 1}, \substack{\cdot111111\\ 0} \},$$

$$X_{29}^{352} = \{\substack{2465432\\3}, \substack{\cdot\cdot\cdot\cdot 321\\\cdot}, \substack{\cdot 3432\cdot 1\\2}, \substack{12\cdot 3221\\2}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{12\cdot 2221\\1}, \substack{00000\cdot 1\\0}\},$$

$$X_{29}^{353} = \{\substack{246\cdot\cdot\cdot\cdot\\3}, \substack{2454321\\2}, \substack{\cdot 3\cdot\cdot\cdot\cdot 1\\2}, \substack{12\cdot\cdot 321\\\cdot}, \substack{1233211\\2}, \substack{1232221\\2}, \substack{1233221\\1}, \substack{12\cdot 2211\\1}, \substack{1222221\\1}, \substack{\cdot\cdot 11111\\0}\},$$

$$X_{29}^{354} = \{\substack{246\cdot\cdot\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot\cdot 1\\2}, \substack{1\cdot\cdot\cdot 321\\2}, \substack{1343211\\2}, \substack{1243221\\2}, \substack{1233\cdot 21\\1}, \substack{1233211\\2}, \substack{1232221\\2}, \substack{1232211\\1}, \substack{1222221\\1}, \substack{1122211\\1}, \substack{\cdot\cdot 11111\\0}\},$$

$$X_{29}^{355} = \{\substack{24654\cdot\cdot\\3}, \substack{23\cdot\cdot\cdot 21\\\cdot}, \substack{1\cdot\cdot 4321\\\cdot}, \substack{1\cdot\cdot 3\cdot\cdot 1\\\cdot}, \substack{0122221\\1}, \substack{00\cdot\cdot 111\\\cdot}\},$$

$$X_{29}^{356} = \{\substack{246\cdot\cdot\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot 321\\2}, \substack{13\cdot\cdot 321\\\cdot}, \substack{124\cdot\cdot 21\\2}, \substack{1343211\\2}, \substack{1233\cdot 21\\2}, \substack{1232\cdot 11\\2}, \substack{1233211\\1}, \substack{12\cdot 2221\\1}, \substack{1122211\\1}, \substack{01222\cdot 1\\1}, \substack{0\cdot 11111\\0}\},$$

$$X_{29}^{357} = \{\substack{246543\cdot\\3}, \substack{24\cdot 4321\\\cdot}, \substack{2354321\\2}, \substack{1354321\\3}, \substack{\cdot 34\cdot\cdot\cdot 1\\2}, \substack{124\cdot\cdot 21\\2}, \substack{1233221\\2}, \substack{1233321\\1}, \substack{12\cdot 2221\\\cdot}, \substack{1232211\\2}, \substack{1232111\\1}, \substack{1111111\\1}, \substack{0111111\\0}, \substack{0000\cdot 11\\0}\},$$

$$X_{29}^{358} = \{\substack{2465432\\3}, \substack{2\cdot\cdot\cdot 321\\\cdot}, \substack{1354321\\3}, \substack{1344321\\2}, \substack{23432\cdot 1\\2}, \substack{1243321\\2}, \substack{1\cdot 43211\\2}, \substack{123\cdot 221\\\cdot}, \substack{1232111\\\cdot}, \substack{1\cdot 22211\\1}, \substack{1221111\\1}, \substack{1122111\\1}, \substack{1111111\\1}, \substack{00000\cdot 1\\0}\},$$

$$X_{29}^{359} = \{\substack{2465\cdot\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot 21\\2}, \substack{13\cdot\cdot 321\\2}, \substack{12\cdot 3321\\\cdot}, \substack{1343211\\2}, \substack{1243221\\2}, \substack{1233211\\2}, \substack{12322\cdot 1\\2}, \substack{1232211\\1}, \substack{1222221\\1}, \substack{1122211\\1}, \substack{0122221\\1}, \substack{1221111\\1}, \substack{\cdot\cdot\cdot 1111\\0}\},$$

$$X_{29}^{360} = \{\substack{24654\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot\cdot 1\\2}, \substack{13\cdot 4321\\2}, \substack{1\cdot 43221\\2}, \substack{12\cdot 3321\\\cdot}, \substack{123\cdot 211\\\cdot}, \substack{1\cdot 22221\\1}, \substack{1221111\\1}, \substack{1122111\\1}, \substack{\cdot\cdot\cdot\cdot 111\\0}\},$$

$$X_{29}^{361} = \{\substack{24654\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot 21\\2}, \substack{13\cdot\cdot 321\\2}, \substack{1\cdot\cdot 3211\\2}, \substack{1243221\\2}, \substack{1233321\\\cdot}, \substack{1232221\\2}, \substack{1233221\\1}, \substack{1232211\\1}, \substack{122\cdot 111\\1}, \substack{1122211\\1}, \substack{0122221\\1}, \substack{\cdot\cdot\cdot\cdot 111\\0}\},$$

$$X_{29}^{362} = \{\substack{246\cdot\cdot\cdot\cdot\\3}, \substack{2\cdot 54321\\2}, \substack{\cdot\cdot 4\cdot\cdot\cdot 1\\2}, \substack{1233\cdot 21\\1}, \substack{1233211\\2}, \substack{1232221\\2}, \substack{1232\cdot 11\\1}, \substack{1111111\\1}, \substack{\cdot\cdot 11111\\0}\},$$

$$X_{29}^{363} = \{\substack{246\cdot\cdot\cdot\cdot\\3}, \substack{\cdot\cdot 54321\\2}, \substack{234\cdot\cdot\cdot 1\\2}, \substack{1343321\\2}, \substack{1244321\\2}, \substack{1\cdot 432\cdot 1\\2}, \substack{1233321\\\cdot}, \substack{1233211\\2}, \substack{1232221\\2}, \substack{1233221\\1}, \substack{1232211\\1}, \substack{1222111\\1}, \substack{1121111\\1}, \substack{\cdot\cdot 11111\\0}\},$$

$$X_{29}^{364} = \{\substack{2465432\\3}, \substack{2\cdot\cdot\cdot 321\\\cdot}, \substack{13\cdot 4321\\2}, \substack{23432\cdot 1\\2}, \substack{12\cdot 3321\\\cdot}, \substack{1\cdot 43211\\2}, \substack{123\cdot 221\\\cdot}, \substack{1\cdot 22211\\1}, \substack{1221111\\1}, \substack{1122111\\1}, \substack{1111111\\0}, \substack{00000\cdot 1\\0}\},$$

$$X_{29}^{365} = \{\substack{246\cdot\cdot\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot\cdot 1\\2}, \substack{1\cdot 44321\\2}, \substack{13432\cdot 1\\2}, \substack{1243321\\2}, \substack{1233\cdot 21\\\cdot}, \substack{12\cdot 3211\\2}, \substack{1232221\\2}, \substack{1222211\\1}, \substack{1122111\\1}, \substack{1111111\\1}, \substack{\cdot\cdot 11111\\0}\},$$

$$X_{29}^{366} = \{\substack{246\cdot\cdot\cdot\cdot\\3}, \substack{2454321\\2}, \substack{\cdot 3\cdot\cdot\cdot\cdot 1\\2}, \substack{1233321\\\cdot}, \substack{123\cdot 2\cdot 1\\2}, \substack{122\cdot\cdot\cdot 1\\1}, \substack{\cdot\cdot 11111\\0}\},$$

$$X_{29}^{367} = \{\substack{24654\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot\cdot 1\\2}, \substack{13\cdot\cdot\cdot 21\\2}, \substack{124\cdot 321\\2}, \substack{1243211\\2}, \substack{123\cdot 2\cdot 1\\1}, \substack{1232111\\\cdot}, \substack{1222211\\1}, \substack{1122221\\1}, \substack{\cdot\cdot\cdot\cdot 111\\0}\},$$

$$X_{29}^{368} = \{\substack{2465432\\3}, \substack{24\cdot 4321\\\cdot}, \substack{\cdot 3\cdot\cdot\cdot\cdot 1\\\cdot}, \substack{12\cdot 3\cdot 21\\\cdot}, \substack{12\cdot 2221\\\cdot}, \substack{1221111\\1}, \substack{0000\cdot\cdot 1\\0}\},$$

$$X_{29}^{369} = \{\substack{24654\cdot\cdot\\3}, \substack{2\cdot 54321\\\cdot}, \substack{\cdot\cdot 44321\\2}, \substack{\cdot\cdot 432\cdot 1\\2}, \substack{1\cdot\cdot 3321\\\cdot}, \substack{1233221\\2}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{1232221\\1}, \substack{0122111\\1}, \substack{1111111\\\cdot}, \substack{000\cdot 111\\0}\},$$

$$X_{29}^{370} = \{\substack{2465432\\3}, \substack{24\cdot\cdot 321\\\cdot}, \substack{2354321\\2}, \substack{1354321\\3}, \substack{\cdot 34\cdot\cdot\cdot 1\\2}, \substack{1244321\\2}, \substack{12432\cdot 1\\2}, \substack{123\cdot\cdot 21\\\cdot}, \substack{1222111\\1}, \substack{1111111\\1}, \substack{0111111\\0}, \substack{00000\cdot 1\\0}\},$$

$$X_{29}^{371} = \{\substack{246\cdot\cdot\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot\cdot 1\\2}, \substack{134\cdot\cdot\cdot 1\\2}, \substack{12\cdot\cdot 321\\2}, \substack{1233221\\2}, \substack{1232\cdot 11\\2}, \substack{1233211\\1}, \substack{1232221\\1}, \substack{12222\cdot 1\\1}, \substack{1111111\\1}, \substack{\cdot\cdot 11111\\0}\},$$

$$X_{29}^{372} = \{ \substack{246\cdot\cdot\cdot\cdot\\3}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\2}, \substack{12\cdot\cdot321\\2}, \substack{1233221\\2}, \substack{1232\cdot11\\2}, \substack{1233211\\1}, \substack{1232221\\1}, \substack{12222\cdot1\\1}, \substack{\cdot111111\\0}, \substack{0011111\\\cdot} \},$$

$$X_{29}^{373} = \{ \substack{246\cdot\cdot\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\2}, \substack{1\cdot4\cdot321\\2}, \substack{1343221\\2}, \substack{1233\cdot21\\2}, \substack{1243211\\2}, \substack{1232\cdot11\\2}, \substack{1233211\\1}, \substack{1232221\\1}, \substack{1222211\\1}, \substack{1122221\\1}, \substack{1111111\\\cdot}, \substack{0\cdot11111\\0} \},$$

$$X_{29}^{374} = \{ \substack{2465432\\3}, \substack{2\cdot\cdot\cdot321\\\cdot}, \substack{1\cdot\cdot\cdot321\\2}, \substack{1343211\\2}, \substack{12\cdot3221\\2}, \substack{1233321\\1}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{12\cdot2221\\1}, \substack{1122211\\1}, \substack{01222\cdot1\\1}, \substack{1111111\\0}, \substack{00000\cdot1\\0} \},$$

$$X_{29}^{375} = \{ \substack{246\cdot\cdot\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot321\\2}, \substack{1\cdot4\cdot321\\2}, \substack{1343211\\2}, \substack{1243221\\2}, \substack{1233\cdot21\\2}, \substack{1232\cdot11\\2}, \substack{1233211\\1}, \substack{12\cdot2221\\1}, \substack{1122211\\1}, \substack{01222\cdot1\\1}, \substack{1111111\\\cdot}, \substack{0\cdot11111\\0} \},$$

$$X_{29}^{376} = \{ \substack{2465\cdot\cdot\cdot\\3}, \substack{2454321\\3}, \substack{23\cdot\cdot\cdot\cdot1\\2}, \substack{1\cdot4\cdot\cdot\cdot1\\2}, \substack{1233221\\2}, \substack{1233321\\1}, \substack{1232221\\\cdot}, \substack{1232211\\2}, \substack{1232111\\1}, \substack{1111111\\\cdot}, \substack{0111111\\0}, \substack{0011111\\1}, \substack{0001111\\0} \},$$

$$X_{29}^{377} = \{ \substack{2465432\\3}, \substack{24\cdot4321\\\cdot}, \substack{2354321\\2}, \substack{1354321\\3}, \substack{\cdot34\cdot\cdot\cdot1\\2}, \substack{124\cdot321\\2}, \substack{12\cdot22\cdot1\\\cdot}, \substack{1232111\\\cdot}, \substack{1111111\\1}, \substack{0111111\\0}, \substack{0000\cdot\cdot1\\0} \},$$

$$X_{29}^{378} = \{ \substack{246\cdot\cdot\cdot\cdot\\3}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\2}, \substack{123\cdot\cdot\cdot1\\2}, \substack{122\cdot\cdot\cdot1\\1}, \substack{\cdot111111\\0}, \substack{0011111\\\cdot} \},$$

$$X_{29}^{379} = \{ \substack{246\cdot\cdot\cdot\cdot\\3}, \substack{2454321\\2}, \substack{2344321\\2}, \substack{1354321\\2}, \substack{\cdot343\cdot\cdot1\\2}, \substack{1244321\\2}, \substack{123\cdot\cdot\cdot1\\2}, \substack{1222\cdot\cdot1\\1}, \substack{1121111\\1}, \substack{\cdot\cdot11111\\0}, \substack{0111111\\1} \},$$

$$X_{29}^{380} = \{ \substack{2465\cdot\cdot\cdot\\3}, \substack{2454321\\3}, \substack{23\cdot\cdot\cdot\cdot1\\2}, \substack{1\cdot4\cdot\cdot\cdot1\\2}, \substack{1232\cdot\cdot1\\\cdot}, \substack{1111111\\\cdot}, \substack{0111111\\0}, \substack{0011111\\1}, \substack{0001111\\0} \},$$

$$X_{29}^{381} = \{ \substack{2465432\\3}, \substack{\cdot344321\\2}, \substack{2343321\\2}, \substack{\cdot\cdot43211\\2}, \substack{1343221\\2}, \substack{1243\cdot21\\2}, \substack{123\cdot\cdot11\\\cdot}, \substack{1221111\\1}, \substack{1122111\\1}, \substack{0122211\\1}, \substack{\cdot\cdot11111\\\cdot}, \substack{000\cdot\cdot\cdot1\\0} \},$$

$$X_{29}^{382} = \{ \substack{2465\cdot\cdot\cdot\\3}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\2}, \substack{1244321\\2}, \substack{1243221\\2}, \substack{1233321\\1}, \substack{1233211\\2}, \substack{12322\cdot1\\2}, \substack{1232211\\1}, \substack{1222221\\1}, \substack{12\cdot2111\\1}, \substack{\cdot\cdot\cdot1111\\0}, \substack{0011111\\1} \},$$

$$X_{29}^{383} = \{ \substack{246543\cdot\\3}, \substack{2\cdot54321\\\cdot}, \substack{234\cdot\cdot\cdot1\\2}, \substack{1344321\\2}, \substack{1\cdot432\cdot1\\2}, \substack{12\cdot3321\\\cdot}, \substack{1233211\\2}, \substack{1232221\\2}, \substack{1233221\\1}, \substack{1232211\\1}, \substack{1221111\\1}, \substack{1122111\\1}, \substack{1111111\\\cdot}, \substack{000\cdot\cdot11\\0} \},$$

$$X_{29}^{384} = \{ \substack{2465432\\3}, \substack{\cdot\cdot54321\\\cdot}, \substack{\cdot\cdot44321\\2}, \substack{2343221\\2}, \substack{1\cdot43321\\2}, \substack{\cdot343211\\2}, \substack{1243221\\2}, \substack{1233\cdot\cdot1\\\cdot}, \substack{1222221\\1}, \substack{1122211\\1}, \substack{0122111\\1}, \substack{000\cdot\cdot\cdot1\\0} \},$$

$$X_{29}^{385} = \{ \substack{2465432\\3}, \substack{246\cdot\cdot21\\3}, \substack{2\cdot\cdot\cdot321\\2}, \substack{13\cdot4321\\2}, \substack{\cdot\cdot43211\\2}, \substack{1343221\\2}, \substack{1243\cdot21\\2}, \substack{123\cdot\cdot21\\\cdot}, \substack{1221111\\1}, \substack{1122111\\1}, \substack{0122211\\1}, \substack{\cdot\cdot11111\\0}, \substack{0000001\\0} \},$$

$$X_{29}^{386} = \{ \substack{2465\cdot\cdot\cdot\\3}, \substack{\cdot\cdot\cdot4321\\2}, \substack{2343321\\2}, \substack{\cdot343211\\2}, \substack{1343221\\2}, \substack{1243321\\2}, \substack{123\cdot221\\2}, \substack{1232\cdot11\\2}, \substack{12\cdot2221\\1}, \substack{1232111\\1}, \substack{1222211\\1}, \substack{1122111\\1}, \substack{0122211\\1}, \substack{\cdot\cdot\cdot1111\\0} \},$$

$$X_{29}^{387} = \{ \substack{24654\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\2}, \substack{13\cdot4321\\2}, \substack{1343221\\2}, \substack{1243321\\2}, \substack{1243211\\2}, \substack{1233221\\2}, \substack{1232\cdot11\\2}, \substack{1233211\\1}, \substack{1232221\\1}, \substack{1232111\\1}, \substack{1222211\\1}, \substack{1122221\\1}, \substack{1221111\\1}, \substack{1122111\\1}, \substack{\cdot\cdot\cdot\cdot111\\0} \},$$

$$X_{29}^{388} = \{ \substack{246543\cdot\\3}, \substack{2454321\\3}, \substack{23\cdot\cdot\cdot\cdot1\\2}, \substack{1343\cdot\cdot1\\2}, \substack{1244321\\2}, \substack{12432\cdot1\\2}, \substack{1233221\\2}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{1232221\\1}, \substack{12\cdot2111\\\cdot}, \substack{11\cdot1111\\\cdot}, \substack{0111111\\0}, \substack{0011111\\1}, \substack{000\cdot\cdot11\\0} \},$$

$$X_{29}^{389} = \{ \substack{2465432\\3}, \substack{2\cdot\cdot\cdot321\\\cdot}, \substack{13\cdot4321\\2}, \substack{12\cdot3321\\\cdot}, \substack{1\cdot43211\\2}, \substack{123\cdot221\\\cdot}, \substack{1\cdot22211\\1}, \substack{01222\cdot1\\1}, \substack{1221111\\1}, \substack{1122111\\1}, \substack{1111111\\0}, \substack{00000\cdot1\\0} \},$$

$$X_{29}^{390} = \{\genfrac{}{}{0pt}{}{2465\cdot\cdot\cdot}{3}, \genfrac{}{}{0pt}{}{2454321}{2}, \genfrac{}{}{0pt}{}{2344321}{2}, \genfrac{}{}{0pt}{}{1354321}{2}, \genfrac{}{}{0pt}{}{\cdot\cdot\cdot3321}{2}, \genfrac{}{}{0pt}{}{1244321}{2}, \genfrac{}{}{0pt}{}{\cdot343211}{2}, \genfrac{}{}{0pt}{}{12\cdot3221}{2}, \genfrac{}{}{0pt}{}{1232\cdot\cdot1}{2}, \genfrac{}{}{0pt}{}{12\cdot2221}{1}, \genfrac{}{}{0pt}{}{\cdot122211}{1}, \genfrac{}{}{0pt}{}{1121111}{1}, \genfrac{}{}{0pt}{}{\cdot\cdot\cdot1111}{0}, \genfrac{}{}{0pt}{}{0111111}{1}\},$$

$$X_{29}^{391} = \{\genfrac{}{}{0pt}{}{246543\cdot}{3}, \genfrac{}{}{0pt}{}{2454321}{2}, \genfrac{}{}{0pt}{}{2354321}{3}, \genfrac{}{}{0pt}{}{1354321}{\cdot}, \genfrac{}{}{0pt}{}{234\cdot\cdot21}{2}, \genfrac{}{}{0pt}{}{1343321}{2}, \genfrac{}{}{0pt}{}{1244321}{2}, \genfrac{}{}{0pt}{}{\cdot\cdot\cdot3211}{2}, \genfrac{}{}{0pt}{}{1\cdot43221}{2}, \genfrac{}{}{0pt}{}{1233321}{2}, \genfrac{}{}{0pt}{}{1233\cdot21}{1}, \genfrac{}{}{0pt}{}{1232221}{2}, \genfrac{}{}{0pt}{}{1232211}{1}, \genfrac{}{}{0pt}{}{1222111}{1}, \genfrac{}{}{0pt}{}{1121111}{1}, \genfrac{}{}{0pt}{}{0111111}{1}, \genfrac{}{}{0pt}{}{00\cdot\cdot\cdot11}{0}\},$$

$$X_{29}^{392} = \{\genfrac{}{}{0pt}{}{246543\cdot}{3}, \genfrac{}{}{0pt}{}{2454321}{2}, \genfrac{}{}{0pt}{}{2354321}{3}, \genfrac{}{}{0pt}{}{\cdot\cdot44321}{2}, \genfrac{}{}{0pt}{}{1\cdot43\cdot21}{2}, \genfrac{}{}{0pt}{}{\cdot\cdot43211}{2}, \genfrac{}{}{0pt}{}{1233221}{2}, \genfrac{}{}{0pt}{}{1233321}{1}, \genfrac{}{}{0pt}{}{1232221}{\cdot}, \genfrac{}{}{0pt}{}{1232211}{2}, \genfrac{}{}{0pt}{}{1232111}{1}, \genfrac{}{}{0pt}{}{0122\cdot11}{1}, \genfrac{}{}{0pt}{}{1111111}{\cdot}, \genfrac{}{}{0pt}{}{0111111}{1}, \genfrac{}{}{0pt}{}{00\cdot\cdot\cdot11}{0}\},$$

$$X_{29}^{393} = \{\genfrac{}{}{0pt}{}{246543\cdot}{3}, \genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot321}{\cdot}, \genfrac{}{}{0pt}{}{\cdot3432\cdot1}{2}, \genfrac{}{}{0pt}{}{12\cdot\cdot221}{\cdot}, \genfrac{}{}{0pt}{}{0000011}{0}\},$$

$$X_{29}^{394} = \{\genfrac{}{}{0pt}{}{2465432}{3}, \genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot321}{2}, \genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot2\cdot1}{1}, \genfrac{}{}{0pt}{}{1232111}{2}, \genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot\cdot\cdot1}{0}\},$$

$$X_{29}^{395} = \{\genfrac{}{}{0pt}{}{2465432}{3}, \genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot321}{\cdot}, \genfrac{}{}{0pt}{}{23432\cdot1}{2}, \genfrac{}{}{0pt}{}{1\cdot\cdot\cdot221}{\cdot}, \genfrac{}{}{0pt}{}{00000\cdot1}{0}\},$$

$$X_{29}^{396} = \{\genfrac{}{}{0pt}{}{2\cdot\cdot\cdot\cdot\cdot\cdot}{3}, \genfrac{}{}{0pt}{}{2\cdot\cdot\cdot\cdot21}{2}, \genfrac{}{}{0pt}{}{13\cdot4321}{2}, \genfrac{}{}{0pt}{}{1\cdot\cdot\cdot2\cdot1}{2}, \genfrac{}{}{0pt}{}{12\cdot3321}{\cdot}, \genfrac{}{}{0pt}{}{0122221}{1}, \genfrac{}{}{0pt}{}{1221111}{1}, \genfrac{}{}{0pt}{}{1122111}{1}, \genfrac{}{}{0pt}{}{1111111}{0}\},$$

$$X_{29}^{397} = \{\genfrac{}{}{0pt}{}{24\cdot\cdot\cdot\cdot\cdot}{\cdot}, \genfrac{}{}{0pt}{}{2354321}{3}, \genfrac{}{}{0pt}{}{\cdot\cdot4\cdot\cdot\cdot1}{2}, \genfrac{}{}{0pt}{}{1354321}{2}, \genfrac{}{}{0pt}{}{123\cdot\cdot21}{\cdot}, \genfrac{}{}{0pt}{}{1111111}{0}, \genfrac{}{}{0pt}{}{0111111}{1}\},$$

$$X_{29}^{398} = \{\genfrac{}{}{0pt}{}{24\cdot\cdot\cdot\cdot\cdot}{\cdot}, \genfrac{}{}{0pt}{}{2354321}{3}, \genfrac{}{}{0pt}{}{\cdot\cdot4\cdot\cdot\cdot1}{2}, \genfrac{}{}{0pt}{}{1354321}{2}, \genfrac{}{}{0pt}{}{1233321}{\cdot}, \genfrac{}{}{0pt}{}{1233221}{2}, \genfrac{}{}{0pt}{}{1232211}{2}, \genfrac{}{}{0pt}{}{1233211}{1}, \genfrac{}{}{0pt}{}{1232221}{1}, \genfrac{}{}{0pt}{}{1111111}{0}, \genfrac{}{}{0pt}{}{0111111}{1}\},$$

$$X_{29}^{399} = \{\genfrac{}{}{0pt}{}{2465432}{3}, \genfrac{}{}{0pt}{}{2465421}{3}, \genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot321}{\cdot}, \genfrac{}{}{0pt}{}{\cdot3432\cdot1}{2}, \genfrac{}{}{0pt}{}{1243211}{2}, \genfrac{}{}{0pt}{}{123\cdot221}{\cdot}, \genfrac{}{}{0pt}{}{1222211}{1}, \genfrac{}{}{0pt}{}{0000001}{0}\},$$

$$X_{29}^{400} = \{\genfrac{}{}{0pt}{}{246543\cdot}{3}, \genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot321}{\cdot}, \genfrac{}{}{0pt}{}{\cdot3432\cdot1}{2}, \genfrac{}{}{0pt}{}{1243211}{2}, \genfrac{}{}{0pt}{}{123\cdot221}{\cdot}, \genfrac{}{}{0pt}{}{1222211}{1}, \genfrac{}{}{0pt}{}{0000011}{0}\},$$

$$X_{29}^{401} = \{\genfrac{}{}{0pt}{}{2\cdot\cdot\cdot\cdot\cdot\cdot}{3}, \genfrac{}{}{0pt}{}{\cdot\cdot\cdot4321}{2}, \genfrac{}{}{0pt}{}{\cdot\cdot\cdot3321}{\cdot}, \genfrac{}{}{0pt}{}{\cdot\cdot43211}{2}, \genfrac{}{}{0pt}{}{1\cdot43221}{2}, \genfrac{}{}{0pt}{}{123\cdot221}{\cdot}, \genfrac{}{}{0pt}{}{0122211}{1}, \genfrac{}{}{0pt}{}{1111111}{0}\},$$

$$X_{29}^{402} = \{\genfrac{}{}{0pt}{}{2465\cdot\cdot\cdot}{3}, \genfrac{}{}{0pt}{}{\cdot\cdot54321}{\cdot}, \genfrac{}{}{0pt}{}{\cdot\cdot4\cdot\cdot21}{2}, \genfrac{}{}{0pt}{}{\cdot343211}{2}, \genfrac{}{}{0pt}{}{1233221}{2}, \genfrac{}{}{0pt}{}{1233321}{1}, \genfrac{}{}{0pt}{}{1233211}{\cdot}, \genfrac{}{}{0pt}{}{1232211}{2}, \genfrac{}{}{0pt}{}{1222221}{1}, \genfrac{}{}{0pt}{}{1232111}{1}, \genfrac{}{}{0pt}{}{0001111}{0}\},$$

$$X_{29}^{403} = \{\genfrac{}{}{0pt}{}{2465\cdot\cdot\cdot}{3}, \genfrac{}{}{0pt}{}{\cdot\cdot54321}{\cdot}, \genfrac{}{}{0pt}{}{\cdot34\cdot\cdot\cdot1}{2}, \genfrac{}{}{0pt}{}{1243\cdot\cdot1}{2}, \genfrac{}{}{0pt}{}{1233321}{\cdot}, \genfrac{}{}{0pt}{}{1233221}{2}, \genfrac{}{}{0pt}{}{1232211}{2}, \genfrac{}{}{0pt}{}{1233211}{1}, \genfrac{}{}{0pt}{}{1232221}{1}, \genfrac{}{}{0pt}{}{1221111}{1}, \genfrac{}{}{0pt}{}{0001111}{0}\},$$

$$X_{29}^{404} = \{\genfrac{}{}{0pt}{}{2465432}{3}, \genfrac{}{}{0pt}{}{2\cdot\cdot\cdot\cdot21}{3}, \genfrac{}{}{0pt}{}{\cdot3\cdot\cdot\cdot\cdot1}{2}, \genfrac{}{}{0pt}{}{12\cdot\cdot\cdot21}{\cdot}, \genfrac{}{}{0pt}{}{1111111}{0}, \genfrac{}{}{0pt}{}{0011111}{1}, \genfrac{}{}{0pt}{}{0000001}{0}\},$$

$$X_{29}^{405} = \{\genfrac{}{}{0pt}{}{24654\cdot\cdot}{3}, \genfrac{}{}{0pt}{}{2\cdot\cdot4321}{\cdot}, \genfrac{}{}{0pt}{}{1\cdot\cdot\cdot321}{\cdot}, \genfrac{}{}{0pt}{}{2343221}{2}, \genfrac{}{}{0pt}{}{\cdot343211}{2}, \genfrac{}{}{0pt}{}{12\cdot\cdot221}{\cdot}, \genfrac{}{}{0pt}{}{1122211}{1}, \genfrac{}{}{0pt}{}{0122111}{1}, \genfrac{}{}{0pt}{}{0000111}{0}\},$$

$$X_{29}^{406} = \{\genfrac{}{}{0pt}{}{2465432}{3}, \genfrac{}{}{0pt}{}{2\cdot\cdot\cdot321}{\cdot}, \genfrac{}{}{0pt}{}{13\cdot4321}{\cdot}, \genfrac{}{}{0pt}{}{\cdot\cdot432\cdot1}{2}, \genfrac{}{}{0pt}{}{1243321}{2}, \genfrac{}{}{0pt}{}{1233221}{2}, \genfrac{}{}{0pt}{}{1232211}{2}, \genfrac{}{}{0pt}{}{1233211}{1}, \genfrac{}{}{0pt}{}{1232221}{1}, \genfrac{}{}{0pt}{}{1232111}{\cdot}, \genfrac{}{}{0pt}{}{1221111}{1}, \genfrac{}{}{0pt}{}{1122111}{1}, \genfrac{}{}{0pt}{}{00000\cdot1}{0}\},$$

$$X_{29}^{407} = \{\genfrac{}{}{0pt}{}{24\cdot\cdot\cdot\cdot\cdot}{3}, \genfrac{}{}{0pt}{}{\cdot354321}{2}, \genfrac{}{}{0pt}{}{\cdot\cdot4\cdot\cdot\cdot1}{2}, \genfrac{}{}{0pt}{}{1233\cdot21}{1}, \genfrac{}{}{0pt}{}{1233211}{2}, \genfrac{}{}{0pt}{}{1232221}{2}, \genfrac{}{}{0pt}{}{1232\cdot11}{1}, \genfrac{}{}{0pt}{}{\cdot111111}{0}, \genfrac{}{}{0pt}{}{0011111}{1}\},$$

$$X_{29}^{408} = \{\genfrac{}{}{0pt}{}{24\cdot\cdot\cdot\cdot\cdot}{\cdot}, \genfrac{}{}{0pt}{}{\cdot3\cdot\cdot\cdot\cdot1}{2}, \genfrac{}{}{0pt}{}{1243221}{2}, \genfrac{}{}{0pt}{}{1233321}{2}, \genfrac{}{}{0pt}{}{12\cdot3211}{2}, \genfrac{}{}{0pt}{}{1233\cdot21}{1}, \genfrac{}{}{0pt}{}{1232221}{2}, \genfrac{}{}{0pt}{}{1232211}{1}, \genfrac{}{}{0pt}{}{122\cdot111}{1}, \genfrac{}{}{0pt}{}{\cdot111111}{0}\},$$

$$X_{29}^{409} = \{\genfrac{}{}{0pt}{}{2\cdot\cdot\cdot\cdot\cdot\cdot}{3}, \genfrac{}{}{0pt}{}{\cdot3\cdot\cdot\cdot\cdot1}{2}, \genfrac{}{}{0pt}{}{1243221}{2}, \genfrac{}{}{0pt}{}{1233321}{2}, \genfrac{}{}{0pt}{}{12\cdot3211}{2}, \genfrac{}{}{0pt}{}{1233\cdot21}{1}, \genfrac{}{}{0pt}{}{1232221}{2}, \genfrac{}{}{0pt}{}{1232211}{1}, \genfrac{}{}{0pt}{}{122\cdot111}{1}, \genfrac{}{}{0pt}{}{1111111}{0}, \genfrac{}{}{0pt}{}{0011111}{1}\},$$

$$X_{29}^{410} = \{\genfrac{}{}{0pt}{}{2465\cdot\cdot\cdot}{3}, \genfrac{}{}{0pt}{}{2454321}{\cdot}, \genfrac{}{}{0pt}{}{2354321}{2}, \genfrac{}{}{0pt}{}{1354321}{3}, \genfrac{}{}{0pt}{}{\cdot34\cdot\cdot\cdot1}{2}, \genfrac{}{}{0pt}{}{124\cdot\cdot21}{2}, \genfrac{}{}{0pt}{}{1233221}{2}, \genfrac{}{}{0pt}{}{1233321}{1}, \genfrac{}{}{0pt}{}{12\cdot2221}{\cdot}, \genfrac{}{}{0pt}{}{1232211}{2}, \genfrac{}{}{0pt}{}{1232111}{1}, \genfrac{}{}{0pt}{}{1111111}{1}, \genfrac{}{}{0pt}{}{0111111}{0}, \genfrac{}{}{0pt}{}{0001111}{0}\},$$

$$X_{29}^{411} = \{\genfrac{}{}{0pt}{}{24654\cdot\cdot}{3}, \genfrac{}{}{0pt}{}{\cdot\cdot54321}{\cdot}, \genfrac{}{}{0pt}{}{234\cdot\cdot\cdot1}{2}, \genfrac{}{}{0pt}{}{1344321}{2}, \genfrac{}{}{0pt}{}{1\cdot432\cdot1}{2}, \genfrac{}{}{0pt}{}{12\cdot3321}{\cdot}, \genfrac{}{}{0pt}{}{1233221}{2}, \genfrac{}{}{0pt}{}{1232211}{2}, \genfrac{}{}{0pt}{}{1233211}{1}, \genfrac{}{}{0pt}{}{1232221}{1}, \genfrac{}{}{0pt}{}{1221111}{1}, \genfrac{}{}{0pt}{}{1122111}{1}, \genfrac{}{}{0pt}{}{000\cdot111}{0}\},$$

$$X_{29}^{412} = \{\genfrac{}{}{0pt}{}{2465\cdot\cdot\cdot}{3}, \genfrac{}{}{0pt}{}{2454321}{\cdot}, \genfrac{}{}{0pt}{}{2354321}{2}, \genfrac{}{}{0pt}{}{1354321}{3}, \genfrac{}{}{0pt}{}{\cdot34\cdot\cdot\cdot1}{2}, \genfrac{}{}{0pt}{}{124\cdot\cdot21}{2}, \genfrac{}{}{0pt}{}{1233221}{2}, \genfrac{}{}{0pt}{}{1233321}{1}, \genfrac{}{}{0pt}{}{1233211}{\cdot}, \genfrac{}{}{0pt}{}{1232211}{2}, \genfrac{}{}{0pt}{}{1222221}{1}, \genfrac{}{}{0pt}{}{1232111}{1}, \genfrac{}{}{0pt}{}{1111111}{1}, \genfrac{}{}{0pt}{}{0111111}{0}, \genfrac{}{}{0pt}{}{0001111}{0}\},$$

$$X_{29}^{413} = \{ \substack{246\cdot\cdot\cdot\cdot\\3}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\2}, \substack{12\cdot\cdot321\\\cdot}, \substack{1233211\\2}, \substack{1232221\\2}, \substack{1233221\\1}, \substack{1232211\\1}, \substack{12222\cdot1\\1}, \substack{\cdot111111\\0}, \substack{0011111\\\cdot} \},$$

$$X_{29}^{414} = \{ \substack{246\cdot\cdot\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot\cdot\cdot1\\2}, \substack{134\cdot\cdot\cdot1\\2}, \substack{12\cdot\cdot321\\\cdot}, \substack{1233221\\2}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{1232221\\1}, \substack{12222\cdot1\\1}, \substack{1111111\\1}, \substack{\cdot\cdot11111\\0} \},$$

$$X_{29}^{415} = \{ \substack{24654\cdot\cdot\\3}, \substack{\cdot\cdot54321\\\cdot}, \substack{\cdot34\cdot\cdot\cdot1\\2}, \substack{124\cdot321\\2}, \substack{1233221\\2}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{1232221\\1}, \substack{1232111\\\cdot}, \substack{12222\cdot1\\1}, \substack{000\cdot111\\0} \},$$

$$X_{29}^{416} = \{ \substack{246\cdot\cdot\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot321\\2}, \substack{13\cdot4321\\2}, \substack{\cdot\cdot43211\\2}, \substack{1343221\\2}, \substack{1243\cdot21\\2}, \substack{123\cdot\cdot21\\\cdot}, \substack{1221111\\1}, \substack{1122111\\1}, \substack{0122211\\1}, \substack{\cdot\cdot11111\\0} \},$$

$$X_{29}^{417} = \{ \substack{2465432\\3}, \substack{2\cdot\cdot\cdot321\\\cdot}, \substack{1\cdot\cdot\cdot321\\\cdot}, \substack{23432\cdot1\\2}, \substack{1343211\\2}, \substack{12\cdot3221\\2}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{12\cdot2221\\1}, \substack{1122211\\1}, \substack{00000\cdot1\\0} \},$$

$$X_{29}^{418} = \{ \substack{2465432\\3}, \substack{2465\cdot21\\3}, \substack{\cdot3\cdot\cdot\cdot21\\\cdot}, \substack{1244321\\2}, \substack{12\cdot3\cdot\cdot1\\\cdot}, \substack{\cdot122221\\1}, \substack{00\cdot1111\\\cdot}, \substack{0000001\\0} \},$$

$$X_{29}^{419} = \{ \substack{2465432\\3}, \substack{2\cdot\cdot\cdot321\\\cdot}, \substack{1\cdot4\cdot321\\2}, \substack{23432\cdot1\\2}, \substack{1343211\\2}, \substack{12\cdot3221\\2}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{12\cdot2221\\1}, \substack{1232111\\\cdot}, \substack{1122211\\1}, \substack{1111111\\\cdot}, \substack{00000\cdot1\\0} \},$$

$$X_{29}^{420} = \{ \substack{2465432\\3}, \substack{\cdot\cdot\cdot4321\\\cdot}, \substack{2343221\\2}, \substack{1343321\\2}, \substack{\cdot343211\\2}, \substack{12\cdot3321\\\cdot}, \substack{12\cdot\cdot221\\\cdot}, \substack{1122211\\1}, \substack{0122111\\1}, \substack{0000\cdot\cdot1\\0} \},$$

$$X_{29}^{421} = \{ \substack{2465432\\3}, \substack{24\cdot4321\\\cdot}, \substack{\cdot34\cdot321\\2}, \substack{2343221\\2}, \substack{1243\cdot\cdot1\\2}, \substack{1343211\\2}, \substack{1233211\\2}, \substack{1232221\\2}, \substack{1233221\\1}, \substack{1232111\\\cdot}, \substack{1232211\\1}, \substack{1122211\\1}, \substack{0122221\\1}, \substack{1221111\\1}, \substack{\cdot111111\\\cdot}, \substack{0000\cdot\cdot1\\0} \},$$

$$X_{29}^{422} = \{ \substack{2465432\\3}, \substack{24\cdot4321\\\cdot}, \substack{23\cdot\cdot321\\2}, \substack{1354321\\3}, \substack{1\cdot44321\\2}, \substack{\cdot343211\\2}, \substack{1\cdot\cdot3221\\2}, \substack{1243321\\2}, \substack{1233321\\1}, \substack{12\cdot2221\\\cdot}, \substack{1232211\\2}, \substack{1232111\\1}, \substack{1122111\\1}, \substack{0122211\\1}, \substack{1111111\\1}, \substack{0111111\\0}, \substack{0000\cdot\cdot1\\0} \},$$

$$X_{29}^{423} = \{ \substack{2465432\\3}, \substack{2\cdot\cdot\cdot\cdot21\\\cdot}, \substack{1\cdot\cdot\cdot\cdot21\\\cdot}, \substack{2343211\\2}, \substack{0000001\\0} \},$$

$$X_{29}^{424} = \{ \substack{246543\cdot\\3}, \substack{\cdot\cdot\cdot\cdot321\\\cdot}, \substack{23432\cdot1\\2}, \substack{1\cdot\cdot\cdot221\\\cdot}, \substack{0000011\\0} \},$$

$$X_{29}^{425} = \{ \substack{2\cdot\cdot\cdot\cdot\cdot\cdot\\3}, \substack{\cdot3\cdot\cdot\cdot\cdot1\\2}, \substack{12\cdot\cdot\cdot21\\\cdot}, \substack{1111111\\0}, \substack{0011111\\1} \},$$

$$X_{29}^{426} = \{ \substack{2465432\\3}, \substack{\cdot\cdot\cdot\cdot321\\\cdot}, \substack{2343211\\2}, \substack{1\cdot\cdot\cdot221\\\cdot}, \substack{0122211\\1}, \substack{00000\cdot1\\0} \},$$

$$X_{29}^{427} = \{ \substack{246\cdot\cdot\cdot\cdot\\3}, \substack{2\cdot\cdot\cdot321\\2}, \substack{1354321\\\cdot}, \substack{1343321\\2}, \substack{1244321\\2}, \substack{\cdot\cdot432\cdot1\\2}, \substack{1233321\\2}, \substack{123\cdot2\cdot1\\1}, \substack{1232111\\2}, \substack{1222111\\1}, \substack{1121111\\1}, \substack{0\cdot11111\\0} \},$$

$$X_{29}^{428} = \{ \substack{2465432\\3}, \substack{\cdot\cdot\cdot\cdot\cdot21\\\cdot}, \substack{0000001\\0} \},$$

$$X_{29}^{429} = \{ \substack{246543\cdot\\3}, \substack{\cdot\cdot\cdot\cdot321\\\cdot}, \substack{2343211\\2}, \substack{1\cdot\cdot\cdot221\\\cdot}, \substack{0122211\\1}, \substack{0000011\\0} \},$$

$$X_{29}^{430} = \{ \substack{246543\cdot\\3}, \substack{\cdot\cdot\cdot\cdot\cdot21\\\cdot} \}.$$

8.2. Near-radical maximal abelian sets

As explained in section 3.1, here it will also be useful to consider subsets X of Φ^+ satisfying $|X \setminus \Omega| = 1$ or 2; we call these near-radical sets.

We begin with the case of maximal abelian sets X satisfying $|X \setminus \Omega| = 1$. Suppose X consists of $\substack{2465432\\3}$, one root of the form $\substack{\cdot\cdot\cdot\cdot\cdot\cdot0\\\cdot}$ and various roots of the form $\substack{\cdot\cdot\cdot\cdot\cdot\cdot1\\\cdot}$. Using the group $\langle w_1, w_2, w_3, w_4, w_5, w_6, w_7 \rangle$ we may assume the second root is $\substack{2343210\\2}$; this excludes $\substack{0\cdot\cdot\cdot\cdot\cdot1\\\cdot}$, giving $\substack{2\cdot\cdot\cdot\cdot\cdot1\\\cdot} \in X$ by default. We therefore have $\substack{2\cdot\cdot\cdot\cdot\cdot\cdot\\\cdot} \in X$, and the remaining roots of X are to be chosen one from each of 16 pairs summing to ρ, of the form $\{\substack{1\cdot\cdot\cdot\cdot11\\\cdot}, \substack{1\cdot\cdot\cdot\cdot21\\\cdot}\}$.

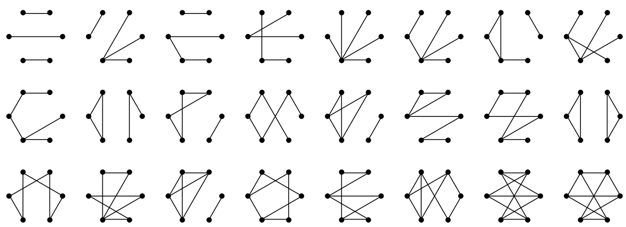
In terms of the identification with unordered pairs introduced in section 8.1, the pairs in which a choice must be made are ij where $i, j \in \{1, 2, 3, 4, 5, 6\}$, together with 78. Indeed, using the group $\langle w_2, w_3, w_4, w_5, w_6, w_7\rangle$ we may assume $\begin{smallmatrix}1354321\\3\end{smallmatrix} \in X$, which excludes $\begin{smallmatrix}1111111\\0\end{smallmatrix}$, leaving just the 15 pairs of the form $\{\begin{smallmatrix}1\cdots\cdots1\\1\end{smallmatrix}, \begin{smallmatrix}1\cdots\cdots1\\2\end{smallmatrix}\}$ corresponding to ij with $i, j \leq 6$. As in section 8.1 we may represent such a set by a graph, this time on vertices $1, 2, 3, 4, 5, 6$, where the presence or absence of an edge means that the α_2-coefficient of the corresponding root is odd or even respectively. We know that $\langle w_3, w_4, w_5, w_6, w_7\rangle$ acts as S_6 permuting the vertices; provided the set contains $\begin{smallmatrix}1354321\\2\end{smallmatrix}$ so that the edge 56 is absent, w_2 acts in such a way that the presence or absence of the edge ij gives rise to the absence or presence of the edge kl, where $\{i, j, k, l\} = \{1, 2, 3, 4\}$.

We are therefore in the same position as that which we faced when determining the radical sets in the E_7 root system in section 7.1; accordingly, we obtain the same list of 37 graphs requiring consideration as in Figure 7.2. Before going further it is worth considering the reason for the similarity. Set $\gamma_1 = \begin{smallmatrix}0111111\\0\end{smallmatrix}$, $\gamma_2 = \begin{smallmatrix}0011111\\1\end{smallmatrix}$, and write $w^* = w_{\gamma_1} w_{\gamma_2}$. Then w^* is an involution which sends $\begin{smallmatrix}2343210\\2\end{smallmatrix}$ to $\begin{smallmatrix}2465432\\3\end{smallmatrix}$, and therefore sends pairs of roots summing to $\begin{smallmatrix}2343210\\2\end{smallmatrix}$ to pairs of roots summing to $\begin{smallmatrix}2465432\\3\end{smallmatrix}$. Indeed, given any root β of the form $\begin{smallmatrix}1\cdots\cdots0\\\cdot\end{smallmatrix}$, we may write $\beta = \sum_{i=1}^{8} n_i\alpha_i$ with $n_1 = 1$, $n_8 = 0$, and $n_3 \in \{n_2, n_2 + 1\}$; the inner products with γ_1 and γ_2 are $n_3 - n_2 - 1$ and $n_2 - n_3$ respectively, so that $w^*(\beta)$ is $\beta + \gamma_1$ or $\beta + \gamma_2$ according as $n_3 = n_2$ or $n_3 = n_2 + 1$, and in either case we have $w^*(\beta) = \sum_{i=1}^{8} m_i\alpha_i$ with $m_1 = 1$, $m_8 = 1$, $m_2 = n_3$ and $m_3 = n_2 + 1$. Therefore w^* sends each pair $\{\begin{smallmatrix}11\cdots\cdots0\\\cdot\end{smallmatrix}, \begin{smallmatrix}12\cdots\cdots0\\\cdot\end{smallmatrix}\}$ to a pair $\{\begin{smallmatrix}1\cdots\cdots1\\1\end{smallmatrix}, \begin{smallmatrix}1\cdots\cdots1\\2\end{smallmatrix}\}$; indeed, given $i, j \leq 6$, if $\{\beta, \beta'\}$ is the pair corresponding to ij in the notation of section 7.1, then $\{w^*(\beta), w^*(\beta')\}$ is the pair corresponding to ij in the notation of section 8.1 (while $\{w^*(\begin{smallmatrix}1000000\\0\end{smallmatrix}), w^*(\begin{smallmatrix}1343210\\2\end{smallmatrix})\} = \{\begin{smallmatrix}1111111\\0\end{smallmatrix}, \begin{smallmatrix}1354321\\3\end{smallmatrix}\}$). Moreover, conjugation by w^* interchanges w_2 and w_3 while fixing w_4, w_5, w_6 and w_7, which explains the correspondence between the actions of these elements in the E_7 and E_8 situations. (We can now see that in section 7.1 we should really have identified the pair $\{\begin{smallmatrix}100000\\0\end{smallmatrix}, \begin{smallmatrix}134321\\2\end{smallmatrix}\}$ with the unordered pair 78, and represented radical sets by pairs of graphs, with vertex sets $\{1, 2, 3, 4, 5, 6\}$ and $\{7, 8\}$; then the effect of w_3 there would have been as stated on edges ij with $i, j \leq 4$, but also the presence or absence of 56 would have led to the absence or presence of 78, and vice versa.)

We digress briefly to observe that the notations used for radical sets in E_6 and E_7 are linked in a similar way. Write $w^\dagger = w_3w_4w_1w_2w_3w_4w_5w_6w_7w_6w_5w_6$. Then $w^\dagger$ (although not an involution) sends $\begin{smallmatrix}123210\\2\end{smallmatrix}$ to $\begin{smallmatrix}234321\\2\end{smallmatrix}$, and therefore sends pairs of roots summing to $\begin{smallmatrix}123210\\2\end{smallmatrix}$ to pairs of roots summing to $\begin{smallmatrix}234321\\2\end{smallmatrix}$; indeed, given $i \in \{1, 2, 3\}$ and $j \in \{4, 5, 6\}$, if $\{\beta, \beta'\}$ is the pair corresponding to ij in the notation of section 6.1, then $\{w^\dagger(\beta), w^\dagger(\beta')\}$ is the pair corresponding to ij in the notation of section 7.1 (while $\{w^\dagger(\begin{smallmatrix}000000\\1\end{smallmatrix}), w^\dagger(\begin{smallmatrix}123210\\1\end{smallmatrix})\} = \{\begin{smallmatrix}100000\\0\end{smallmatrix}, \begin{smallmatrix}134321\\2\end{smallmatrix}\}$). Moreover, conjugation by $w^\dagger$ (on the left) sends w_1, w_3, w_5, w_6 to w_4, w_2, w_7, w_6 respectively and w_4 to $w_7w_6w_5w_4w_3w_4w_5w_6w_7$, which explains the correspondence between the actions of these elements in the E_6 and E_7 situations.

Returning to the matter in hand, we must now consider which of the graphs in Figure 7.2 give rise to maximal abelian sets here; we shall see that the condition for maximality is actually different from that found in section 7.1. The presence of $\begin{smallmatrix}2\cdots\cdots\\\cdot\end{smallmatrix}$ in X excludes all negative roots along with all roots $\begin{smallmatrix}0\cdots\cdots\\\cdot\end{smallmatrix}$, while $\begin{smallmatrix}1354321\\3\end{smallmatrix}$

excludes the roots $\substack{1\cdot\cdot\cdot\cdot\cdot\cdot\\0}$; so the roots outside Ω which remain to be excluded are those of the form $\substack{1\cdot\cdot\cdot\cdot\cdot0\\1}$ or $\substack{1\cdot\cdot\cdot\cdot\cdot0\\2}$. To exclude $\substack{1110000\\1}$ we must have some root $\substack{1233\cdot\cdot1\\1}$ or some root $\substack{1\cdot\cdot4321\\2}$ present; using the group $\langle w_3, w_4, w_5, w_6, w_7\rangle$ we see that to exclude the roots $\substack{1\cdot\cdot\cdot\cdot\cdot0\\1}$ the graph cannot contain a triangle such that there are no edges between the other three vertices, which disposes of the sixth graph in the first row. To exclude $\substack{1232100\\2}$ we must have some root $\substack{1\cdot\cdot\cdot\cdot21\\1}$ present; using the group $\langle w_3, w_4, w_5, w_6, w_7\rangle$ we see that to exclude the roots $\substack{1\cdot\cdot\cdot\cdot\cdot0\\2}$ the graph cannot contain a vertex joined to none of the others, which disposes of all remaining graphs in the first row except for the ninth therein, the first, third and fourth in the second row and the fourth in the third row. We are therefore left with the following 24 graphs representing maximal abelian sets.



We take the vertices $1, 2, 3, 4, 5, 6$ to be arranged as in section 7.1. In fact in three of the 24 cases we shall choose to take not the set given by the graph above, but instead its image under w_2; the graphs concerned are the fifth in the second row and the third and sixth in the third row. As in section 7.1 we make this choice because it will in due course lead to a more convenient form for the stabilizer. We therefore set

$$X_{30}^{1} = \{\substack{2\cdot\cdot\cdot\cdot\cdot\cdot\\\cdot}, \substack{13\cdot\cdot\cdot21\\\cdot}, \substack{124\cdot321\\2}, \substack{123\cdot221\\2}, \substack{12\cdot\cdot\cdot11\\2}, \substack{1232111\\1}, \substack{1222211\\1}, \substack{1122221\\1}\},$$

$$X_{30}^{2} = \{\substack{2\cdot\cdot\cdot\cdot\cdot\cdot\\\cdot}, \substack{1\cdot\cdot4321\\\cdot}, \substack{1\cdot\cdot3\cdot21\\2}, \substack{1\cdot\cdot2221\\1}, \substack{1232\cdot11\\2}, \substack{1233211\\1}\},$$

$$X_{30}^{3} = \{\substack{2\cdot\cdot\cdot\cdot\cdot\cdot\\\cdot}, \substack{13\cdot\cdot\cdot21\\\cdot}, \substack{124\cdot321\\2}, \substack{1243211\\2}, \substack{1233321\\1}, \substack{123\cdot2\cdot1\\2}, \substack{1232111\\1}, \substack{1222211\\1}, \substack{1122221\\1}\},$$

$$X_{30}^{4} = \{\substack{2\cdot\cdot\cdot\cdot\cdot\cdot\\\cdot}, \substack{13\cdot\cdot\cdot21\\\cdot}, \substack{12\cdot\cdot321\\2}, \substack{12\cdot3211\\2}, \substack{1232221\\2}, \substack{1233221\\1}, \substack{1232111\\2}, \substack{12\cdot2211\\1}, \substack{1122221\\1}\},$$

$$X_{30}^{5} = \{\substack{2\cdot\cdot\cdot\cdot\cdot\cdot\\\cdot}, \substack{1\cdot\cdot\cdot\cdot21\\\cdot}\},$$

$$X_{30}^{6} = \{\substack{2\cdot\cdot\cdot\cdot\cdot\cdot\\\cdot}, \substack{1\cdot\cdot\cdot321\\\cdot}, \substack{1\cdot\cdot3221\\2}, \substack{1\cdot\cdot2221\\1}, \substack{1232211\\2}, \substack{1233211\\1}\},$$

$$X_{30}^{7} = \{\substack{2\cdot\cdot\cdot\cdot\cdot\cdot\\\cdot}, \substack{13\cdot\cdot\cdot21\\\cdot}, \substack{12\cdot3\cdot\cdot1\\\cdot}, \substack{1122221\\1}, \substack{1221111\\1}\},$$

$$X_{30}^{8} = \{\substack{2\cdot\cdot\cdot\cdot\cdot\cdot\\\cdot}, \substack{1\cdot\cdot\cdot321\\\cdot}, \substack{1343211\\2}, \substack{12\cdot3221\\2}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{12\cdot2221\\1}, \substack{1122211\\1}\},$$

$$X_{30}^{9} = \{\substack{2\cdot\cdot\cdot\cdot\cdot\cdot\\\cdot}, \substack{1\cdot\cdot4321\\\cdot}, \substack{1\cdot43\cdot21\\2}, \substack{1233221\\2}, \substack{1233321\\1}, \substack{1233211\\\cdot}, \substack{1232211\\2}, \substack{1\cdot22221\\1}, \substack{1232111\\1}\},$$

$$X_{30}^{10} = \{\substack{2\cdot\cdot\cdot\cdot\cdot\cdot\\\cdot}, \substack{1354321\\\cdot}, \substack{1\cdot\cdot3\cdot\cdot1\\\cdot}, \substack{1\cdot21111\\1}\},$$

$$X_{30}^{11} = \{\substack{2\cdot\cdot\cdot\cdot\cdot\cdot\\\cdot}, \substack{1354321\\3}, \substack{1\cdot4\cdot\cdot\cdot1\\2}, \substack{1233\cdot21\\1}, \substack{1233211\\2}, \substack{1232221\\2}, \substack{1232\cdot11\\1}, \substack{1111111\\1}\},$$

$$X_{30}^{12} = \{\substack{2\cdot\cdot\cdot\cdot\cdot\cdot\\\cdot}, \substack{13\cdot4321\\\cdot}, \substack{1\cdot432\cdot1\\2}, \substack{12\cdot3321\\\cdot}, \substack{1233221\\2}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{1232221\\1}, \substack{1221111\\1},$$
$$\substack{1122111\\1}\},$$

$$X_{30}^{13} = \{\substack{2\cdot\cdot\cdot\cdot\cdot\cdot\\\cdot}, \substack{1\cdot\cdot\cdot321\\2}, \substack{1\cdot\cdot\cdot2\cdot1\\2}, \substack{1233321\\1}, \substack{1111111\\0}\},$$

$$X_{30}^{14} = \{\substack{2\cdot\cdot\cdot\cdot\cdot\cdot\\\cdot}, \substack{13\cdot\cdot\cdot21\\\cdot}, \substack{124\cdot321\\2}, \substack{123\cdot\cdot11\\\cdot}, \substack{12222\cdot1\\1}, \substack{1122221\\1}\},$$

$X_{30}^{15} = \{\substack{2\cdot\cdot\cdot\cdot\cdot\cdot\\ \cdot}, \substack{13\cdot\cdot\cdot21\\ \cdot}, \substack{124\cdot321\\ 2}, \substack{1233221\\ 2}, \substack{1232\cdot11\\ 2}, \substack{1233211\\ 1}, \substack{1\cdot\cdot2221\\ 1}, \substack{1232111\\ 1}, \substack{1222211\\ 1}\}$,

$X_{30}^{16} = \{\substack{2\cdot\cdot\cdot\cdot\cdot\cdot\\ \cdot}, \substack{1354321\\ 3}, \substack{1\cdot\cdot3\cdot\cdot1\\ \cdot}, \substack{1\cdot\cdot1111\\ 1}\}$

$X_{30}^{17} = \{\substack{2\cdot\cdot\cdot\cdot\cdot\cdot\\ \cdot}, \substack{1354321\\ 3}, \substack{1343\cdot\cdot1\\ 2}, \substack{1244321\\ 2}, \substack{1243221\\ 2}, \substack{1233321\\ 2}, \substack{12\cdot3211\\ 2}, \substack{1233\cdot21\\ 1}, \substack{1232221\\ 2}, \substack{1232211\\ 1}, \substack{1222111\\ 1}, \substack{11\cdot1111\\ 1}\}$,

$X_{30}^{18} = \{\substack{2\cdot\cdot\cdot\cdot\cdot\cdot\\ \cdot}, \substack{1354321\\ \cdot}, \substack{1\cdot4\cdot321\\ 2}, \substack{123\cdot221\\ \cdot}, \substack{1232111\\ \cdot}, \substack{1\cdot222\cdot1\\ 1}\}$,

$X_{30}^{19} = \{\substack{2\cdot\cdot\cdot\cdot\cdot\cdot\\ \cdot}, \substack{1\cdot\cdot\cdot\cdot\cdot1\\ 2}, \substack{1111111\\ 0}\}$,

$X_{30}^{20} = \{\substack{2\cdot\cdot\cdot\cdot\cdot\cdot\\ \cdot}, \substack{1354321\\ \cdot}, \substack{1343321\\ 2}, \substack{1244321\\ 2}, \substack{1\cdot43221\\ 2}, \substack{1233321\\ \cdot}, \substack{123\cdot211\\ \cdot}, \substack{1\cdot22221\\ 1}, \substack{1222111\\ 1}, \substack{1121111\\ 1}\}$,

$X_{30}^{21} = \{\substack{2\cdot\cdot\cdot\cdot\cdot\cdot\\ \cdot}, \substack{1354321\\ \cdot}, \substack{1\cdot4\cdot321\\ 2}, \substack{1233211\\ 2}, \substack{1232221\\ 2}, \substack{1233221\\ 1}, \substack{1232111\\ \cdot}, \substack{1232211\\ 1}, \substack{1\cdot222\cdot1\\ 1}\}$,

$X_{30}^{22} = \{\substack{2\cdot\cdot\cdot\cdot\cdot\cdot\\ \cdot}, \substack{13\cdot4321\\ 2}, \substack{1\cdot\cdot\cdot2\cdot1\\ 2}, \substack{12\cdot3321\\ \cdot}, \substack{1221111\\ 1}, \substack{1122111\\ 1}, \substack{1111111\\ 0}\}$,

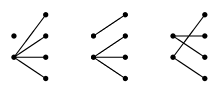
$X_{30}^{23} = \{\substack{2\cdot\cdot\cdot\cdot\cdot\cdot\\ \cdot}, \substack{1\cdot\cdot4321\\ \cdot}, \substack{1\cdot\cdot2\cdot\cdot1\\ \cdot}\}$,

$X_{30}^{24} = \{\substack{2\cdot\cdot\cdot\cdot\cdot\cdot\\ \cdot}, \substack{13\cdot4321\\ \cdot}, \substack{1243321\\ 2}, \substack{1233221\\ 2}, \substack{1232\cdot11\\ 2}, \substack{1233211\\ 1}, \substack{1\cdot\cdot2221\\ 1}, \substack{1232111\\ 1}, \substack{1222211\\ 1}, \substack{1122\cdot11\\ 1}, \substack{1221111\\ 1}\}$.

Now consider maximal abelian sets X with $|X \setminus \Omega| = 2$. Here we divide into two cases according as the two roots of $X \setminus \Omega$ are mutually orthogonal or not.

First suppose the two roots of $X \setminus \Omega$ are mutually orthogonal; using the group $\langle w_1, w_2, w_3, w_4, w_5, w_6, w_7\rangle$ we may assume one is $\substack{2343210\\ 2}$, which forces the other (which may now be negative) to be of the form $\substack{0\cdot\cdot\cdot\cdot\cdot0\\ \cdot}$, so using the group $\langle w_2, w_3, w_4, w_5, w_6, w_7\rangle$ we may assume it is $\substack{0122210\\ 1}$. This excludes $\substack{0\cdot\cdot\cdot\cdot\cdot1\\ \cdot}$, $\substack{1\cdot\cdot\cdot111\\ \cdot}$, $\substack{23432\cdot1\\ 2}$, giving $\substack{\cdot\cdot\cdot\cdot321\\ \cdot}$, $\substack{24654\cdot1\\ 3} \in X$ by default. Note that in the two pairs of the form $\{\substack{01222\cdot1\\ 1}, \substack{23432\cdot1\\ 2}\}$ both roots are excluded, so that X will consist of only 26 of the roots $\substack{\cdot\cdot\cdot\cdot\cdot\cdot1\\ \cdot}$ together with $\substack{2465432\\ 3}$, $\substack{2343210\\ 2}$, $\substack{0122210\\ 1}$, giving $|X| = 29$. The remaining roots of X are to be chosen one from each of 8 pairs of the form $\{\substack{1\cdot\cdot\cdot2\cdot1\\ 1}, \substack{1\cdot\cdot\cdot2\cdot1\\ 2}\}$ summing to $\substack{2465432\\ 3}$; the pairs concerned are those identified with unordered pairs ij with $i \in \{1, 2\}$ and $j \in \{3, 4, 5, 6\}$. As in section 7.2 we may represent these sets by bipartite graphs, this time with $\langle w_3, w_4, w_5, w_7\rangle$ acting as $S_2 \times S_4$ on the vertices.

The roots so far chosen exclude all roots outside Ω except those of the form $\substack{\cdot\cdot\cdot\cdot210\\ \cdot}$. To exclude $\substack{1343210\\ 2}$ we must have some root $\substack{11222\cdot1\\ 1}$ present, while to exclude $\substack{1122210\\ 1}$ we must have some root $\substack{13432\cdot1\\ 2}$ present; using the group $\langle w_3, w_4, w_5\rangle$ we see that to exclude the roots $\substack{1\cdot\cdot\cdot210\\ \cdot}$ the graph must have precisely one edge $1j$ or $2j$ for each $j \in \{3, 4, 5, 6\}$. Accordingly there are just the following 3 possible graphs.

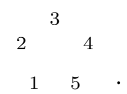


Again, we number the vertices as in section 7.2. In the first of these, applying w_8 gives a set lying in Ω, so only the other two require consideration. We choose to apply $w_5w_4w_3w_1$ to both; we therefore set

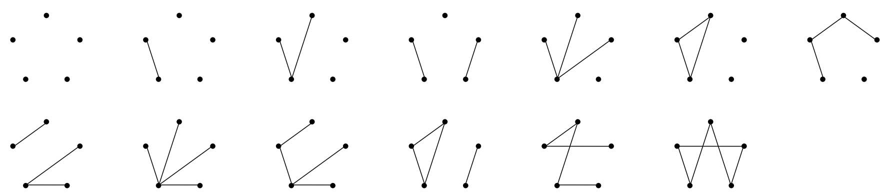
$X_{29}^{431} = \{\substack{24654\cdot\cdot\\ 3}, \substack{\cdot\cdot\cdot\cdot321\\ \cdot}, \substack{\cdot\cdot43221\\ 2}, \substack{1233211\\ 2}, \substack{1232210\\ 2}, \substack{1233210\\ 1}, \substack{1232211\\ 1}, \substack{\cdot\cdot22221\\ 1}\}$,

$X_{29}^{432} = \{\substack{24654\cdot\cdot\\ 3}, \substack{\cdot\cdot\cdot\cdot321\\ \cdot}, \substack{2343221\\ 2}, \substack{1\cdot43211\\ 2}, \substack{1233221\\ 2}, \substack{1232221\\ 1}, \substack{1232210\\ 2}, \substack{1233210\\ 1}, \substack{1\cdot22211\\ 1}, \substack{0122221\\ 1}\}$.

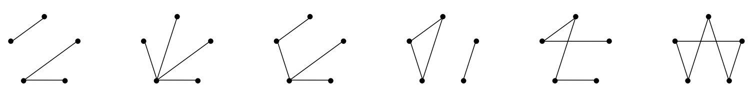
Now suppose the two roots of $X \setminus \Omega$ are not mutually orthogonal; using the group $\langle w_1, w_2, w_3, w_4, w_5, w_6, w_7 \rangle$ we may assume one is $\substack{2343210\\2}$, which forces the other to be of the form $\substack{1\cdot\cdot\cdot\cdot\cdot 0\\ \cdot}$, so using the group $\langle w_2, w_3, w_4, w_5, w_6, w_7 \rangle$ we may assume it is $\substack{1343210\\2}$. This excludes $\substack{0\cdot\cdot\cdot\cdot\cdot 1\\ \cdot}$, $\substack{11\cdot\cdot\cdot\cdot 1\\ \cdot}$, giving $\substack{2\cdot\cdot\cdot\cdot\cdot 1\\ \cdot}$, $\substack{13\cdot\cdot\cdot\cdot 1\\ \cdot} \in X$ by default. The remaining roots of X are to be chosen one from each of 10 pairs of the form $\{\substack{12\cdot\cdot\cdot\cdot 1\\1}, \substack{12\cdot\cdot\cdot\cdot 1\\2}\}$ summing to $\substack{2465432\\3}$; the pairs concerned are those identified with unordered pairs ij with $i, j \leq 5$. We may therefore represent these sets by graphs with vertex set $\{1, 2, 3, 4, 5\}$, where we arrange the vertices in a regular pentagon as follows:



As before $\langle w_4, w_5, w_6, w_7 \rangle \cong S_5$ acts by permuting vertices, so that it suffices to consider graphs up to isomorphism; we may obtain the list of 34 graphs from Figure 7.1 by simply deleting one isolated vertex from those which contain one. Again, w_2 acts as the bifid map on edges among the vertices $\{1, 2, 3, 4\}$, so as before we have equivalence classes of graphs. This reduces the number of graphs to be considered to 13; we may obtain the following list of graphs by again deleting an isolated vertex from those in Figure 7.2 which contain one.



We must now consider which of these graphs give maximal abelian sets. The roots so far chosen exclude all roots outside Ω except those of the form $\substack{12\cdot\cdot\cdot\cdot 0\\ \cdot}$. To exclude $\substack{1233210\\1}$ we must have $\substack{1221111\\1}$ or some root $\substack{1232\cdot\cdot 1\\2}$ present; using the group $\langle w_4, w_5, w_6, w_7 \rangle$ we see that to exclude the roots $\substack{12\cdot\cdot\cdot\cdot 0\\1}$ the graph cannot contain a triangle with the remaining two vertices not joined, which disposes of the sixth graph in the first row above. To exclude $\substack{1243210\\2}$ we must have some root $\substack{122\cdot\cdot\cdot 1\\1}$ present; using the group $\langle w_4, w_5, w_6, w_7 \rangle$ we see that to exclude the roots $\substack{12\cdot\cdot\cdot\cdot 0\\2}$ the graph cannot contain a vertex joined to none of the others, which disposes of the remaining graphs in the first row above. We are thus left with the following 6 graphs representing maximal abelian sets.



We therefore set

$X_{31}^1 = \{\substack{24\cdot\cdot\cdot\cdot\cdot\\ \cdot}, \substack{\cdot 3\cdot\cdot\cdot\cdot\cdot\\ \cdot}, \substack{1244321\\2}, \substack{12\cdot 3\cdot 21\\2}, \substack{1232\cdot 11\\2}, \substack{1233211\\1}, \substack{12\cdot 2221\\1}\}$,

$X_{31}^2 = \{\substack{24\cdot\cdot\cdot\cdot\cdot\\ \cdot}, \substack{\cdot 3\cdot\cdot\cdot\cdot\cdot\\ \cdot}, \substack{12\cdot\cdot\cdot 21\\ \cdot}\}$,

$X_{31}^3 = \{\substack{24\cdot\cdot\cdot\cdot\cdot\\ \cdot}, \substack{\cdot 3\cdot\cdot\cdot\cdot\cdot\\ \cdot}, \substack{12\cdot\cdot 321\\ \cdot}, \substack{12\cdot 3221\\2}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{12\cdot 2221\\1}\}$,

$X_{31}^4 = \{\substack{24\cdot\cdot\cdot\cdot\cdot\\ \cdot}, \substack{\cdot 3\cdot\cdot\cdot\cdot\cdot\\ \cdot}, \substack{12\cdot 3\cdot\cdot 1\\ \cdot}, \substack{1221111\\1}\}$,

$X_{31}^5 = \{\substack{24\cdot\cdot\cdot\cdot\cdot\\ \cdot}, \substack{\cdot 3\cdot\cdot\cdot\cdot\cdot\\ \cdot}, \substack{12\cdot\cdot 321\\2}, \substack{1243221\\2}, \substack{1233211\\ \cdot}, \substack{1233221\\1}, \substack{1232111\\2}, \substack{1232211\\1}, \substack{1222221\\1}\}$,

$X_{31}^6 = \{\substack{24\cdot\cdot\cdot\cdot\cdot\\ \cdot}, \substack{\cdot 3\cdot\cdot\cdot\cdot\cdot\\ \cdot}, \substack{1243221\\2}, \substack{1233321\\2}, \substack{12\cdot 3211\\2}, \substack{1233\cdot 21\\1}, \substack{1232221\\2}, \substack{1232211\\1}, \substack{122\cdot 111\\1}\}$.

8.3. Determination of maximal abelian sets

We begin by giving some maximal abelian sets which are neither radical nor near-radical; we set

$$X_{22} = \{\genfrac{}{}{0pt}{}{2\cdot\cdot\cdot\cdot\cdot\cdot}{\cdot}, \genfrac{}{}{0pt}{}{1343321}{2}, \genfrac{}{}{0pt}{}{1243221}{2}, \genfrac{}{}{0pt}{}{1233211}{2}, \genfrac{}{}{0pt}{}{1233210}{1}, \genfrac{}{}{0pt}{}{1221000}{1}, \genfrac{}{}{0pt}{}{1121100}{1}, \genfrac{}{}{0pt}{}{1111110}{1}, \genfrac{}{}{0pt}{}{1111111}{0}\},$$
$$X_{28} = \{\genfrac{}{}{0pt}{}{24\cdot\cdot\cdot\cdot\cdot}{\cdot}, \genfrac{}{}{0pt}{}{\cdot 3\cdot\cdot\cdot\cdot\cdot}{\cdot}, \genfrac{}{}{0pt}{}{1244321}{2}, \genfrac{}{}{0pt}{}{1232221}{2}, \genfrac{}{}{0pt}{}{1232211}{1}, \genfrac{}{}{0pt}{}{1232100}{2}, \genfrac{}{}{0pt}{}{1232110}{1}, \genfrac{}{}{0pt}{}{1222210}{1}, \genfrac{}{}{0pt}{}{1222111}{1}\},$$
$$X_{30}^{25} = \{\genfrac{}{}{0pt}{}{2\cdot\cdot\cdot\cdot\cdot\cdot}{\cdot}, \genfrac{}{}{0pt}{}{1\cdot\cdot\cdot 321}{\cdot}, \genfrac{}{}{0pt}{}{1343211}{2}, \genfrac{}{}{0pt}{}{123\cdot 221}{\cdot}, \genfrac{}{}{0pt}{}{1243210}{2}, \genfrac{}{}{0pt}{}{1222210}{1}, \genfrac{}{}{0pt}{}{1122211}{1}\},$$
$$X_{31}^{7} = \{\genfrac{}{}{0pt}{}{24\cdot\cdot\cdot\cdot\cdot}{\cdot}, \genfrac{}{}{0pt}{}{\cdot 3\cdot\cdot\cdot\cdot\cdot}{\cdot}, \genfrac{}{}{0pt}{}{12\cdot\cdot 321}{\cdot}, \genfrac{}{}{0pt}{}{1243221}{2}, \genfrac{}{}{0pt}{}{1233211}{2}, \genfrac{}{}{0pt}{}{1232210}{2}, \genfrac{}{}{0pt}{}{1233210}{1}, \genfrac{}{}{0pt}{}{1232211}{1}, \genfrac{}{}{0pt}{}{1222221}{1}\},$$
$$X_{32}^{1} = \{\genfrac{}{}{0pt}{}{246\cdot\cdot\cdot\cdot}{3}, \genfrac{}{}{0pt}{}{\cdot\cdot 54321}{\cdot}, \genfrac{}{}{0pt}{}{\cdot\cdot 4\cdot\cdot\cdot\cdot}{2}, \genfrac{}{}{0pt}{}{1233\cdot 21}{2}, \genfrac{}{}{0pt}{}{1232\cdot 11}{2}, \genfrac{}{}{0pt}{}{1233211}{1}, \genfrac{}{}{0pt}{}{1232221}{1}\},$$
$$X_{32}^{2} = \{\genfrac{}{}{0pt}{}{246\cdot\cdot\cdot\cdot}{3}, \genfrac{}{}{0pt}{}{\cdot\cdot 54321}{\cdot}, \genfrac{}{}{0pt}{}{\cdot\cdot 4\cdot\cdot\cdot\cdot}{2}, \genfrac{}{}{0pt}{}{1233321}{\cdot}, \genfrac{}{}{0pt}{}{1233221}{2}, \genfrac{}{}{0pt}{}{1232211}{2}, \genfrac{}{}{0pt}{}{1233211}{1}, \genfrac{}{}{0pt}{}{1232221}{1}\},$$
$$X_{32}^{3} = \{\genfrac{}{}{0pt}{}{246\cdot\cdot\cdot\cdot}{3}, \genfrac{}{}{0pt}{}{\cdot\cdot 54321}{\cdot}, \genfrac{}{}{0pt}{}{\cdot\cdot 4\cdot\cdot\cdot\cdot}{2}, \genfrac{}{}{0pt}{}{123\cdot\cdot 21}{\cdot}\},$$
$$X_{33} = \{\genfrac{}{}{0pt}{}{2465\cdot\cdot\cdot}{3}, \genfrac{}{}{0pt}{}{\cdot\cdot\cdot 4321}{\cdot}, \genfrac{}{}{0pt}{}{\cdot\cdot\cdot 3\cdot\cdot\cdot}{2}, \genfrac{}{}{0pt}{}{1233\cdot 21}{1}, \genfrac{}{}{0pt}{}{1232221}{2}\},$$
$$X_{34}^{1} = \{\genfrac{}{}{0pt}{}{24654\cdot\cdot}{3}, \genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot 321}{\cdot}, \genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot 2\cdot\cdot}{2}\},$$
$$X_{34}^{2} = \{\genfrac{}{}{0pt}{}{2465\cdot\cdot\cdot}{3}, \genfrac{}{}{0pt}{}{\cdot\cdot\cdot 4321}{\cdot}, \genfrac{}{}{0pt}{}{\cdot\cdot\cdot 3\cdot\cdot\cdot}{\cdot}\},$$
$$X_{36} = \{\genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot\cdot\cdot\cdot}{3}, \genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot\cdot\cdot\cdot}{2}\}.$$

We then set

$$\mathcal{S}(E_8) = \{X_{22}, X_{28}, X_{29}^1, \dots, X_{29}^{430}, X_{29}^{431}, X_{29}^{432}, X_{30}^1, \dots, X_{30}^{24}, X_{30}^{25},$$
$$X_{31}^1, \dots, X_{31}^6, X_{31}^7, X_{32}^1, X_{32}^2, X_{32}^3, X_{33}, X_{34}^1, X_{34}^2, X_{36}\}.$$

As in section 3.2, we let X be any maximal abelian set consisting of positive roots and containing a simple root α; we seek to show that X is known, i.e., a W-translate of a set in $\mathcal{S}(E_8)$. Here we note that if at some point the union of the sets of chosen and available roots is a W-translate of a set with at most two roots outside Ω, there will be no need to continue the line of investigation since we have determined the radical and near-radical maximal abelian sets. As with the analysis for E_7, at some points we shall write $X = X_c \cup X_a$, where X_c is the set of roots which have been chosen by then (including those known to be in X by default), and X_a is a subset (to be determined) of the available roots.

We work through the possibilities for the simple root α contained in X. In the first of these we take $\alpha = \genfrac{}{}{0pt}{}{0000001}{0}$. Much as in the E_7 analysis, since the radical sets have been treated we may also assume that X contains some root $\genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot\cdot\cdot 0}{\cdot}$; as α excludes the roots $\genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot\cdot 10}{\cdot}$, we may assume that X contains some root of the form $\genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot\cdot 00}{\cdot}$, and hence some simple root α' of this form. For convenience we shall subdivide this first step of the analysis according to the possibilities for α'. Indeed, we shall subdivide still further the first of these possibilities, where $\alpha' = \genfrac{}{}{0pt}{}{0000100}{0}$.

LEMMA 8.14. *If* $\genfrac{}{}{0pt}{}{0000001}{0}, \genfrac{}{}{0pt}{}{0000100}{0} \in X$ *and* X *meets* $\{\genfrac{}{}{0pt}{}{\cdot\cdot\cdot 0000}{\cdot}\}$ *then* X *is known.*

PROOF. Assume $\genfrac{}{}{0pt}{}{0000001}{0}, \genfrac{}{}{0pt}{}{0000100}{0} \in X$; this excludes $\genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot\cdot 10}{\cdot}, \genfrac{}{}{0pt}{}{0000011}{0}, \genfrac{}{}{0pt}{}{\cdot\cdot\cdot 1000}{\cdot}, \genfrac{}{}{0pt}{}{\cdot\cdot\cdot 2111}{\cdot}, \genfrac{}{}{0pt}{}{\cdot\cdot\cdot 3221}{\cdot}, \genfrac{}{}{0pt}{}{2465321}{3}, \genfrac{}{}{0pt}{}{2465431}{3}$, giving $\genfrac{}{}{0pt}{}{1232211}{2}, \genfrac{}{}{0pt}{}{2465421}{3} \in X$ by default. If in addition X meets $\{\genfrac{}{}{0pt}{}{\cdot\cdot\cdot 0000}{\cdot}\}$, by Lemma 3.1 (with $Y = \{\genfrac{}{}{0pt}{}{0000001}{0}, \genfrac{}{}{0pt}{}{0000100}{0}\}$), we may assume that one of the following holds: (a) $\genfrac{}{}{0pt}{}{0000000}{1} \in X$; (b) $\genfrac{}{}{0pt}{}{0000000}{1} \notin X$, $\genfrac{}{}{0pt}{}{0010000}{\cdot} \in X$; (c) $\genfrac{}{}{0pt}{}{00\cdot 0000}{\cdot} \notin X$, $\genfrac{}{}{0pt}{}{01\cdot 0000}{\cdot} \in X$; (d) $\genfrac{}{}{0pt}{}{0\cdot\cdot 0000}{\cdot} \notin X$, $\genfrac{}{}{0pt}{}{1\cdot\cdot 0000}{\cdot} \in X$.

We show that we need only treat (a). If (b) holds, this excludes $\genfrac{}{}{0pt}{}{\cdot 100000}{0}, \genfrac{}{}{0pt}{}{0001100}{0}, \genfrac{}{}{0pt}{}{0001111}{0}, \genfrac{}{}{0pt}{}{\cdot 111100}{\cdot}, \genfrac{}{}{0pt}{}{\cdot 111111}{\cdot}, \genfrac{}{}{0pt}{}{1222100}{1}, \genfrac{}{}{0pt}{}{12222\cdot 1}{1}, \genfrac{}{}{0pt}{}{1233211}{\cdot}, \genfrac{}{}{0pt}{}{1233321}{\cdot}, \genfrac{}{}{0pt}{}{\cdot 344321}{2}, \genfrac{}{}{0pt}{}{2454321}{\cdot}$; but now

$-\genfrac{}{}{0pt}{}{1222110}{1}$ cannot be excluded. Likewise if (c) holds, this excludes $\genfrac{}{}{0pt}{}{1000000}{0}$, $\genfrac{}{}{0pt}{}{00\cdot1100}{\cdot}$, $\genfrac{}{}{0pt}{}{00\cdot1111}{\cdot}$, $\genfrac{}{}{0pt}{}{11\cdot\cdot100}{\cdot}$, $\genfrac{}{}{0pt}{}{11\cdot1111}{\cdot}$, $\genfrac{}{}{0pt}{}{11222\cdot1}{1}$, $\genfrac{}{}{0pt}{}{12\cdot3211}{\cdot}$, $\genfrac{}{}{0pt}{}{12\cdot\cdot321}{\cdot}$, $\genfrac{}{}{0pt}{}{23\cdot4321}{\cdot}$; but now $-\genfrac{}{}{0pt}{}{1122110}{1}$ cannot be excluded. Similarly if (d) holds, this excludes $\genfrac{}{}{0pt}{}{0\cdot\cdot1100}{\cdot}$, $\genfrac{}{}{0pt}{}{0\cdot\cdot1111}{\cdot}$, $\genfrac{}{}{0pt}{}{0122100}{1}$, $\genfrac{}{}{0pt}{}{01222\cdot1}{1}$, $\genfrac{}{}{0pt}{}{1\cdot\cdot3211}{\cdot}$, $\genfrac{}{}{0pt}{}{1\cdot\cdot\cdot321}{\cdot}$; but now $-\genfrac{}{}{0pt}{}{0122110}{1}$ cannot be excluded. Thus (b), (c) and (d) give rise to no sets requiring consideration.

Thus we may assume (a) holds; this excludes $\genfrac{}{}{0pt}{}{\cdot\cdot10000}{0}$, $\genfrac{}{}{0pt}{}{\cdot\cdot11100}{0}$, $\genfrac{}{}{0pt}{}{\cdot\cdot11111}{0}$, $\genfrac{}{}{0pt}{}{1232100}{1}$, $\genfrac{}{}{0pt}{}{12322\cdot1}{1}$, $\genfrac{}{}{0pt}{}{1233211}{1}$, $\genfrac{}{}{0pt}{}{1233321}{1}$, $\genfrac{}{}{0pt}{}{\cdot\cdot54321}{2}$, giving $\genfrac{}{}{0pt}{}{1233211}{2}$, $\genfrac{}{}{0pt}{}{1233321}{2}$, $\genfrac{}{}{0pt}{}{2454321}{3}$ $\in X$ by default. Using Lemma 3.1 (with $Y = \{\genfrac{}{}{0pt}{}{0000001}{0}, \genfrac{}{}{0pt}{}{0000100}{0}, \genfrac{}{}{0pt}{}{0000000}{1}\}$) we may assume that one of the following holds: (i) $\genfrac{}{}{0pt}{}{0100000}{0} \in X$; (ii) $\genfrac{}{}{0pt}{}{0100000}{0} \notin X$, $\genfrac{}{}{0pt}{}{1\cdot00000}{0} \in X$; (iii) $\genfrac{}{}{0pt}{}{\cdot\cdot00000}{0} \notin X$.

Assume (i) holds; this excludes $\genfrac{}{}{0pt}{}{1000000}{0}$, $\genfrac{}{}{0pt}{}{0010000}{1}$, $\genfrac{}{}{0pt}{}{0011100}{1}$, $\genfrac{}{}{0pt}{}{0011111}{1}$, $\genfrac{}{}{0pt}{}{112\cdot100}{1}$, $\genfrac{}{}{0pt}{}{1121111}{1}$, $\genfrac{}{}{0pt}{}{11222\cdot1}{1}$, $\genfrac{}{}{0pt}{}{1243211}{2}$, $\genfrac{}{}{0pt}{}{124\cdot321}{2}$, $\genfrac{}{}{0pt}{}{2354321}{3}$, giving $\genfrac{}{}{0pt}{}{0111100}{1}$, $\genfrac{}{}{0pt}{}{0111111}{1}$, $\genfrac{}{}{0pt}{}{1222211}{1}$, $\genfrac{}{}{0pt}{}{1343211}{2}$, $\genfrac{}{}{0pt}{}{1343321}{2}$ $\in X$ by default. To exclude the negative root $-\genfrac{}{}{0pt}{}{0011000}{0}$ we must then have $\genfrac{}{}{0pt}{}{0122100}{1} \in X$, which excludes $\genfrac{}{}{0pt}{}{1100000}{0}$, $\genfrac{}{}{0pt}{}{1110000}{1}$, $\genfrac{}{}{0pt}{}{0000111}{0}$, $\genfrac{}{}{0pt}{}{1111111}{1}$, $\genfrac{}{}{0pt}{}{1221111}{1}$, $\genfrac{}{}{0pt}{}{1222221}{1}$, $\genfrac{}{}{0pt}{}{1232221}{2}$, $\genfrac{}{}{0pt}{}{2343321}{2}$, giving $\genfrac{}{}{0pt}{}{0122211}{1}$, $\genfrac{}{}{0pt}{}{1344321}{2}$, $\genfrac{}{}{0pt}{}{1354321}{3}$ $\in X$ by default; likewise to exclude $-\genfrac{}{}{0pt}{}{0011110}{0}$ we must have $\genfrac{}{}{0pt}{}{0122221}{1} \in X$, which excludes $\genfrac{}{}{0pt}{}{1111100}{1}$, $\genfrac{}{}{0pt}{}{122\cdot100}{1}$, $\genfrac{}{}{0pt}{}{1232100}{2}$, $\genfrac{}{}{0pt}{}{2343211}{2}$; to exclude $-\genfrac{}{}{0pt}{}{1000000}{0}$ we must have $\genfrac{}{}{0pt}{}{2344321}{2} \in X$, which excludes $\genfrac{}{}{0pt}{}{0110000}{1}$, $\genfrac{}{}{0pt}{}{0121100}{1}$, $\genfrac{}{}{0pt}{}{0121111}{1}$; to exclude $-\genfrac{}{}{0pt}{}{0010000}{0}$ we must have $\genfrac{}{}{0pt}{}{2464321}{3} \in X$, which excludes $\genfrac{}{}{0pt}{}{0001100}{0}$, $\genfrac{}{}{0pt}{}{0001111}{0}$. Thus

$$\begin{aligned} X = \{ & \genfrac{}{}{0pt}{}{2465432}{3}, \genfrac{}{}{0pt}{}{2465421}{3}, \genfrac{}{}{0pt}{}{24\cdot4321}{3}, \genfrac{}{}{0pt}{}{1354321}{3}, \genfrac{}{}{0pt}{}{\cdot344321}{2}, \genfrac{}{}{0pt}{}{1343321}{2}, \genfrac{}{}{0pt}{}{1343211}{2}, \genfrac{}{}{0pt}{}{1233321}{2}, \genfrac{}{}{0pt}{}{123\cdot211}{2}, \\ & \genfrac{}{}{0pt}{}{1222211}{1}, \genfrac{}{}{0pt}{}{01222\cdot1}{1}, \genfrac{}{}{0pt}{}{0122100}{1}, \genfrac{}{}{0pt}{}{0111111}{1}, \genfrac{}{}{0pt}{}{0111100}{1}, \genfrac{}{}{0pt}{}{0100000}{0}, \genfrac{}{}{0pt}{}{0000000}{1}, \genfrac{}{}{0pt}{}{0000100}{0}, \genfrac{}{}{0pt}{}{0000001}{0}\} \\ = {} & w_7w_4w_5w_6w_1w_3w_4w_5w_2w_4w_3w_1w_6w_5(X_{22}). \end{aligned}$$

Assume instead (ii) holds; this excludes $\genfrac{}{}{0pt}{}{0\cdot10000}{1}$, $\genfrac{}{}{0pt}{}{0\cdot\cdot\cdot100}{1}$, $\genfrac{}{}{0pt}{}{0\cdot\cdot1111}{1}$, $\genfrac{}{}{0pt}{}{01222\cdot1}{1}$, $\genfrac{}{}{0pt}{}{1\cdot43211}{2}$, $\genfrac{}{}{0pt}{}{1\cdot4\cdot321}{2}$, $\genfrac{}{}{0pt}{}{1354321}{3}$, giving $\genfrac{}{}{0pt}{}{1111100}{1}$, $\genfrac{}{}{0pt}{}{1111111}{1}$, $\genfrac{}{}{0pt}{}{1\cdot22211}{1}$, $\genfrac{}{}{0pt}{}{2343211}{2}$, $\genfrac{}{}{0pt}{}{234\cdot321}{2}$, $\genfrac{}{}{0pt}{}{2354321}{3}$ $\in X$ by default. To exclude $-\genfrac{}{}{0pt}{}{0011000}{0}$ we must have $\genfrac{}{}{0pt}{}{1122100}{1} \in X$, which excludes $\genfrac{}{}{0pt}{}{0000111}{0}$, $\genfrac{}{}{0pt}{}{1221111}{1}$, $\genfrac{}{}{0pt}{}{1222221}{1}$, $\genfrac{}{}{0pt}{}{1232221}{2}$; to exclude $-\genfrac{}{}{0pt}{}{0000010}{0}$ we must have $\genfrac{}{}{0pt}{}{1122221}{1} \in X$, which excludes $\genfrac{}{}{0pt}{}{122\cdot100}{1}$, $\genfrac{}{}{0pt}{}{1232100}{2}$. However, now $-\genfrac{}{}{0pt}{}{0111000}{0}$ cannot be excluded; so no sets require consideration.

Thus we may assume for the remainder of this proof that (iii) holds; this gives $\genfrac{}{}{0pt}{}{\cdot354321}{3} \in X$ by default. To exclude $-\genfrac{}{}{0pt}{}{1232110}{1}$ we must have $\genfrac{}{}{0pt}{}{2464321}{3} \in X$, which excludes $\genfrac{}{}{0pt}{}{0001100}{0}$, $\genfrac{}{}{0pt}{}{0001111}{0}$. Suppose X contains some root $\genfrac{}{}{0pt}{}{\cdot\cdot22100}{1}$; using $\langle w_1, w_3\rangle$ we may assume $\genfrac{}{}{0pt}{}{0122100}{1} \in X$. This excludes $\genfrac{}{}{0pt}{}{0000111}{0}$, $\genfrac{}{}{0pt}{}{1110000}{1}$, $\genfrac{}{}{0pt}{}{1\cdot\cdot1111}{1}$, $\genfrac{}{}{0pt}{}{1\cdot22221}{1}$, $\genfrac{}{}{0pt}{}{1232221}{2}$, $\genfrac{}{}{0pt}{}{2343321}{2}$, giving $\genfrac{}{}{0pt}{}{0\cdot11100}{1}$, $\genfrac{}{}{0pt}{}{0122211}{1}$, $\genfrac{}{}{0pt}{}{1\cdot43211}{2}$ $\in X$ by default; to exclude $-\genfrac{}{}{0pt}{}{0000010}{0}$ we must then have $\genfrac{}{}{0pt}{}{0122221}{1} \in X$, which excludes $\genfrac{}{}{0pt}{}{1\cdot\cdot1100}{1}$, $\genfrac{}{}{0pt}{}{1\cdot22100}{1}$, $\genfrac{}{}{0pt}{}{1232100}{2}$, $\genfrac{}{}{0pt}{}{2343211}{2}$, giving $\genfrac{}{}{0pt}{}{0\cdot11111}{1}$, $\genfrac{}{}{0pt}{}{1\cdot43321}{2}$ $\in X$ by default; to exclude $-\genfrac{}{}{0pt}{}{1111000}{0}$ we must have $\genfrac{}{}{0pt}{}{2344321}{2} \in X$, which excludes $\genfrac{}{}{0pt}{}{0\cdot10000}{1}$, $\genfrac{}{}{0pt}{}{0121100}{1}$, $\genfrac{}{}{0pt}{}{0121111}{1}$, giving $\genfrac{}{}{0pt}{}{1\cdot22211}{1}$, $\genfrac{}{}{0pt}{}{1\cdot44321}{2}$ $\in X$ by default; so

$$\begin{aligned} X = \{ & \genfrac{}{}{0pt}{}{2465432}{3}, \genfrac{}{}{0pt}{}{2465421}{3}, \genfrac{}{}{0pt}{}{\cdot\cdot\cdot4321}{3}, \genfrac{}{}{0pt}{}{\cdot\cdot44321}{2}, \genfrac{}{}{0pt}{}{1\cdot\cdot3321}{2}, \genfrac{}{}{0pt}{}{1\cdot\cdot\cdot211}{2}, \genfrac{}{}{0pt}{}{1\cdot22211}{1}, \genfrac{}{}{0pt}{}{01222\cdot1}{1}, \genfrac{}{}{0pt}{}{0122100}{1}, \\ & \genfrac{}{}{0pt}{}{0\cdot11111}{1}, \genfrac{}{}{0pt}{}{0\cdot11100}{1}, \genfrac{}{}{0pt}{}{0000000}{1}, \genfrac{}{}{0pt}{}{0000100}{0}, \genfrac{}{}{0pt}{}{0000001}{0}\} \\ = {} & w_7w_4w_5w_6w_3w_4w_5w_1w_3w_4w_5w_2w_4w_3w_1w_6w_5w_4w_3w_4w_5(X_{28}). \end{aligned}$$

Thus we may assume $\genfrac{}{}{0pt}{}{\cdot\cdot22100}{1} \notin X$, giving $\genfrac{}{}{0pt}{}{1232221}{2}$, $\genfrac{}{}{0pt}{}{\cdot\cdot43321}{2}$ $\in X$ by default. Suppose X contains some root $\genfrac{}{}{0pt}{}{\cdot\cdot22221}{1}$; using $\langle w_1, w_3\rangle$ we may assume $\genfrac{}{}{0pt}{}{0122221}{1} \in X$. This excludes $\genfrac{}{}{0pt}{}{1110000}{1}$, $\genfrac{}{}{0pt}{}{1\cdot\cdot1100}{1}$, $\genfrac{}{}{0pt}{}{1232100}{2}$, $\genfrac{}{}{0pt}{}{2343211}{2}$, giving $\genfrac{}{}{0pt}{}{0000111}{0}$, $\genfrac{}{}{0pt}{}{0\cdot11111}{1}$, $\genfrac{}{}{0pt}{}{0122211}{1}$ $\in X$ by default; to exclude $-\genfrac{}{}{0pt}{}{1111000}{0}$ we must then have $\genfrac{}{}{0pt}{}{2344321}{2} \in X$, which excludes $\genfrac{}{}{0pt}{}{0\cdot10000}{1}$, $\genfrac{}{}{0pt}{}{0121100}{1}$, $\genfrac{}{}{0pt}{}{0121111}{1}$, giving $\genfrac{}{}{0pt}{}{1111111}{1}$, $\genfrac{}{}{0pt}{}{1\cdot22211}{1}$ $\in X$ by default; to exclude $-\genfrac{}{}{0pt}{}{0011000}{0}$

we must have $\substack{1243211\\2}$ or $\substack{1244321\\2}$ present, and similarly to exclude $-\substack{0111000\\0}$ we must have $\substack{1343211\\2}$ or $\substack{1344321\\2}$ present. Thus $X = X_c \cup X_a$ where

$$X_c = \{\substack{2465432\\3}, \substack{2465421\\3}, \substack{\cdot\cdot\cdot 4321\\3}, \substack{2344321\\2}, \substack{\cdot\cdot\cdot 3321\\2}, \substack{1232221\\2}, \substack{123\cdot 211\\2}, \substack{1\cdot 22211\\1}, \substack{01222\cdot 1\\1}, \substack{\cdot\cdot 11111\\1}, \substack{0000111\\0}, \substack{0000000\\1}, \substack{0000100\\0}, \substack{0000001\\0}\},$$

$$X_a \subset \{\substack{1\cdot 44321\\2}, \substack{1\cdot 43211\\2}, \substack{1\cdot 22221\\1}, \substack{1\cdot 21111\\1}, \substack{0\cdot 11100\\1}\},$$

and X_a must contain at least one of $\substack{1243211\\2}$ and $\substack{1244321\\2}$, and at least one of $\substack{1343211\\2}$ and $\substack{1344321\\2}$; set

$$w = w_6w_5w_4w_3w_2w_1w_4w_7w_6w_5w_4w_3w_2w_1w_4w_3w_6w_5w_4w_7,$$

then we have

$$w(X_c) = \{\substack{24654\cdot\cdot\\3}, \substack{\cdot\cdot\cdot\cdot 321\\\cdot}, \substack{2343211\\2}, \substack{1343221\\2}, \substack{1232210\\2}, \substack{1233210\\1}, \substack{1122221\\1}, \substack{0122211\\1}\},$$

$$w(X_a) \subset \{\substack{12\cdot 32\cdot\cdot\\2}, \substack{12\cdot 22\cdot 1\\1}\},$$

and $w(X_a)$ must contain both some root $\substack{12432\cdot 1\\2}$ and some root $\substack{12332\cdot 1\\2}$. To avoid $|w(X)\setminus\Omega| = 2$ we must have some root $\substack{12\cdot 3210\\2}$ present in $w(X)$; using $\langle w_4\rangle$ we may assume $\substack{1243210\\2} \in w(X)$, which excludes $\substack{12222\cdot 1\\1}$, giving $\substack{12432\cdot 1\\2} \in w(X)$ by default. If also $\substack{1233210\\2} \in w(X)$ this excludes $\substack{12322\cdot 1\\1}$, giving $\substack{12332\cdot 1\\2} \in w(X)$ by default, so

$$\begin{aligned} w(X) &= \{\substack{24654\cdot\cdot\\3}, \substack{\cdot\cdot\cdot\cdot 321\\\cdot}, \substack{2343211\\2}, \substack{1343221\\2}, \substack{12\cdot 32\cdot\cdot\\2}, \substack{1232210\\2}, \substack{1233210\\1}, \substack{1122221\\1}, \substack{0122211\\1}\} \\ &= w_3w_1w_4w_3w_4(X^7_{31}). \end{aligned}$$

So assume $\substack{1233210\\2} \notin w(X)$; to exclude this we must have some root $\substack{12322\cdot 1\\1}$ present. Using $\langle w_1w_7\rangle$ (which preserves $w(X_c)$ and $\{\substack{12432\cdot\cdot\\2}\}$) we may assume $\substack{1232221\\1} \in w(X)$, which excludes $\substack{1233211\\2}$; since $w(X_a)$ must contain some root $\substack{12332\cdot 1\\2}$ we must have $\substack{1233221\\2} \in w(X)$, which excludes $\substack{1232211\\1}$, so

$$\begin{aligned} w(X) &= \{\substack{24654\cdot\cdot\\3}, \substack{\cdot\cdot\cdot\cdot 321\\\cdot}, \substack{2343211\\2}, \substack{1\cdot\cdot 3221\\2}, \substack{124321\cdot\\2}, \substack{1232221\\1}, \substack{1232210\\2}, \substack{1233210\\1}, \substack{1122221\\1}, \\ &\qquad \substack{0122211\\1}\} \\ &= w_3w_1w_5w_4(X^{25}_{30}). \end{aligned}$$

Thus we may assume $\substack{\cdot\cdot 22221\\1} \notin X$, giving $\substack{\cdot\cdot 43211\\2} \in X$ by default. Thus $X = X_c \cup X_a$ where

$$X_c = \{\substack{2465432\\3}, \substack{2465421\\3}, \substack{\cdot\cdot\cdot 4321\\3}, \substack{\cdot\cdot\cdot 3321\\2}, \substack{\cdot\cdot\cdot\cdot 211\\2}, \substack{1232221\\2}, \substack{0000000\\1}, \substack{0000100\\0}, \substack{0000001\\0}\},$$

$$X_a \subset \{\substack{\cdot\cdot 44321\\2}, \substack{\cdot\cdot 22211\\1}, \substack{1232100\\2}, \substack{\cdot\cdot\cdot 1111\\1}, \substack{\cdot\cdot\cdot 1100\\1}, \substack{\cdot\cdot 10000\\1}, \substack{0000111\\0}\};$$

set

$$w = w_4w_3w_5w_4w_1w_2w_3w_4w_5w_8w_7w_6w_5w_4w_3w_2w_1w_4w_3w_5w_4w_6w_5w_7,$$

then we have

$$w(X_c) = \{\substack{\cdot\cdot\cdot\cdot\cdot\cdot\cdot\\3}, \substack{\cdot\cdot 4\cdot 321\\2}, \substack{1233\cdot 21\\2}, \substack{1232221\\1}, \substack{1233211\\1}, \substack{1232111\\2}\},$$

$$w(X_a) \subset \{\substack{\cdot\cdot 54321\\2}, \substack{\cdot\cdot 432\cdot\cdot\\2}, \substack{1233221\\1}, \substack{1232211\\2}, \substack{\cdot\cdot 222\cdot 1\\1}, \substack{\cdot\cdot 11111\\1}\}.$$

To avoid $|w(X)\setminus\Omega| \leq 2$ we must have $\substack{\cdot\cdot 43210\\2} \in w(X)$, which excludes $\substack{\cdot\cdot 11111\\1}$, $\substack{\cdot\cdot 222\cdot 1\\1}$, giving $\substack{\cdot\cdot 432\cdot 1\\2}$, $\substack{\cdot\cdot 54321\\2} \in w(X)$ by default. If $\substack{1232211\\2} \in w(X)$ this excludes $\substack{1233221\\1}$, so

$$w(X) = \{\substack{246\cdot\cdot\cdot\cdot\\3}, \substack{\cdot\cdot 54321\\\cdot}, \substack{\cdot\cdot 4\cdot\cdot\cdot\cdot\\2}, \substack{1233\cdot 21\\2}, \substack{1232\cdot 11\\2}, \substack{1233211\\1}, \substack{1232221\\1}\} = X^1_{32};$$

if instead ${}^{1232211}_{2} \notin w(X)$ we have ${}^{1233221}_{1} \in w(X)$ by default, so

$$w(X) = \{ {}^{246\cdot\cdot\cdot\cdot}_{3}, {}^{\cdot\cdot54321}_{\cdot}, {}^{\cdot\cdot4\cdot\cdot\cdot\cdot}_{2}, {}^{1233\cdot21}_{2}, {}^{12332\cdot1}_{1}, {}^{1232221}_{1}, {}^{1232111}_{2} \} = w_6(X^2_{32}).$$

This proves the lemma. ∎

LEMMA 8.15. *If* ${}^{0000001}_{0}, {}^{0000100}_{0} \in X$ *then* X *is known.*

PROOF. As before, assume ${}^{0000001}_{0}, {}^{0000100}_{0} \in X$; this excludes ${}^{\cdot\cdot\cdot\cdot\cdot10}_{\cdot}$, ${}^{0000011}_{0}$, ${}^{\cdot\cdot\cdot1000}_{\cdot}$, ${}^{\cdot\cdot\cdot2111}_{\cdot}$, ${}^{\cdot\cdot\cdot3221}_{\cdot}$, ${}^{2465321}_{3}$, ${}^{2465431}_{3}$, giving ${}^{1232211}_{2}, {}^{2465421}_{3} \in X$ by default. By Lemma 8.14 we may assume ${}^{\cdot\cdot\cdot0000}_{\cdot} \notin X$; this gives ${}^{\cdot\cdot\cdot2211}_{1} \in X$ by default. To exclude $-{}^{0000010}_{0}$ we must have some root ${}^{\cdot\cdot\cdot2221}_{\cdot}$ present; using $\langle w_1, w_2, w_3, w_4 \rangle$ we may assume ${}^{0122221}_{1} \in X$, which excludes ${}^{1\cdot\cdot\cdot100}_{\cdot}$, ${}^{2343211}_{2}$, giving ${}^{1\cdot\cdot3321}_{\cdot} \in X$ by default.

First suppose ${}^{0122100}_{1} \in X$; this excludes ${}^{0000111}_{0}$, ${}^{1\cdot\cdot1111}_{\cdot}$, ${}^{1\cdot\cdot2221}_{\cdot}$, ${}^{2343321}_{2}$, giving ${}^{1\cdot\cdot3211}_{\cdot}, {}^{1\cdot\cdot4321}_{\cdot} \in X$ by default. Thus $X = X_c \cup X_a$ where

$$X_c = \{ {}^{2465432}_{3}, {}^{2465421}_{3}, {}^{1\cdot\cdot\cdot321}_{\cdot}, {}^{1\cdot\cdot\cdot211}_{\cdot}, {}^{01222\cdot1}_{1}, {}^{0122100}_{1}, {}^{0000100}_{0}, {}^{0000001}_{0} \},$$
$$X_a \subset \{ {}^{2\cdot\cdot4321}_{\cdot}, {}^{0\cdot\cdot1111}_{\cdot}, {}^{0\cdot\cdot1100}_{\cdot} \};$$

set

$$w = w_8w_7w_6w_5w_4w_3w_2w_1w_4w_3w_5w_4w_2w_6w_5w_4w_3w_1w_7,$$

then we have

$$w(X_c) = \{ {}^{24654\cdot\cdot}_{3}, {}^{\cdot\cdot\cdot\cdot321}_{\cdot}, {}^{2343221}_{2}, {}^{1343211}_{2}, {}^{1122211}_{1}, {}^{0122221}_{1} \},$$
$$w(X_a) \subset \{ {}^{12\cdot\cdot2\cdot\cdot}_{\cdot} \}.$$

To avoid $|w(X) \setminus \Omega| \le 2$ we must have at least three of the six roots ${}^{12\cdot\cdot210}_{\cdot}$ present in $w(X)$; on the other hand to avoid $|w_8w(X) \setminus \Omega| \le 2$ or $|w_8w_7w(X) \setminus \Omega| \le 2$ we must have some root ${}^{12\cdot\cdot211}_{\cdot}$ and some root ${}^{12\cdot\cdot221}_{\cdot}$ present, which will exclude one of the roots ${}^{12\cdot\cdot210}_{\cdot}$. Thus $|w(X) \cap \{ {}^{12\cdot\cdot210}_{\cdot} \}| = 3$, 4 or 5; using $\langle w_2, w_4, w_5 \rangle$ we may assume $w(X) \cap \{ {}^{12\cdot\cdot210}_{\cdot} \}$ is one of the following: (i) $\{ {}^{12\cdot\cdot210}_{2} \}$; (ii) $\{ {}^{12\cdot3210}_{\cdot} \}$; (iii) $\{ {}^{12\cdot3210}_{2}, {}^{1232210}_{1} \}$; (iv) $\{ {}^{12\cdot\cdot210}_{2}, {}^{1233210}_{1} \}$; (v) $\{ {}^{123\cdot210}_{\cdot} \}$; (vi) $\{ {}^{1243210}_{2}, {}^{123\cdot210}_{\cdot} \}$.

If (i) holds this excludes ${}^{12\cdot\cdot2\cdot1}_{1}$, giving ${}^{12\cdot\cdot2\cdot1}_{2} \in w(X)$ by default; so

$$w(X) = \{ {}^{24654\cdot\cdot}_{3}, {}^{\cdot\cdot\cdot\cdot321}_{\cdot}, {}^{2343221}_{2}, {}^{1343211}_{2}, {}^{12\cdot\cdot2\cdot\cdot}_{2}, {}^{1122211}_{1}, {}^{0122221}_{1} \}$$
$$= w_3w_1w_4w_3w_5w_4(X^2_{32}).$$

If (ii) holds this excludes ${}^{12\cdot22\cdot1}_{\cdot}$, giving ${}^{12\cdot32\cdot1}_{\cdot} \in w(X)$ by default; so

$$w(X) = \{ {}^{24654\cdot\cdot}_{3}, {}^{\cdot\cdot\cdot\cdot321}_{\cdot}, {}^{2343221}_{2}, {}^{1343211}_{2}, {}^{12\cdot32\cdot\cdot}_{\cdot}, {}^{1122211}_{1}, {}^{0122221}_{1} \}$$
$$= w_3w_1w_4w_3w_2w_4(X^2_{32}).$$

If (iii) holds this excludes ${}^{12\cdot22\cdot1}_{1}$, ${}^{12332\cdot1}_{2}$, giving ${}^{12432\cdot1}_{2} \in w(X)$ by default; to exclude ${}^{1232210}_{2}$ we must have some root ${}^{12332\cdot1}_{1}$, and using $\langle w_1w_7 \rangle$ (which preserves $w(X_c)$ and $\{ {}^{12\cdot3210}_{2}, {}^{1232210}_{1}, {}^{12432\cdot1}_{2} \}$) we may assume ${}^{1233221}_{1} \in w(X)$, which excludes ${}^{1232211}_{2}$; to exclude ${}^{1233210}_{1}$ we must have ${}^{1232221}_{2} \in w(X)$, which excludes ${}^{1233211}_{1}$; so

$$w(X) = \{ {}^{24654\cdot\cdot}_{3}, {}^{\cdot\cdot\cdot\cdot321}_{\cdot}, {}^{2343221}_{2}, {}^{1343211}_{2}, {}^{12432\cdot\cdot}_{2}, {}^{1232221}_{2}, {}^{1233221}_{1}, {}^{1233210}_{2}, {}^{1232210}_{1}, {}^{1122211}_{1}, {}^{0122221}_{1} \}$$
$$= w_3w_1w_3w_4(X^{25}_{30}).$$

If (iv) holds this excludes $\substack{12\cdot22\cdot1\\ \cdot}$, $\substack{12332\cdot1\\ 1}$, giving $\substack{12\cdot32\cdot1\\ 2}$ $\in w(X)$ by default; so

$$\begin{aligned} w(X) &= \{\substack{24654\cdot\cdot\\ 3}, \substack{\cdot\cdot\cdot\cdot321\\ \cdot}, \substack{2343221\\ 2}, \substack{1343211\\ 2}, \substack{12\cdot32\cdot\cdot\\ 2}, \substack{1232210\\ 2}, \substack{1233210\\ 1}, \substack{1122211\\ 1}, \substack{0122221\\ 1}\} \\ &= w_3w_1w_4w_3(X^7_{31}). \end{aligned}$$

If (v) holds this excludes $\substack{123\cdot2\cdot1\\ \cdot}$; to exclude $\substack{1243210\\ 2}$ we must have some root $\substack{12222\cdot1\\ 1}$, and using $\langle w_1w_7\rangle$ (which preserves $w(X_c)$ and $\{\substack{123\cdot210\\ \cdot}\}$) we may assume $\substack{1222221\\ 1}$ $\in w(X)$, which also excludes $\substack{1243211\\ 2}$; since we must have some root $\substack{12\cdot\cdot211\\ \cdot}$ we must have $\substack{1222211\\ 1}$ $\in w(X)$, which excludes $\substack{1243221\\ 2}$; but now there is no root to exclude $\substack{1222210\\ 1}$, so no set arises from this possibility. Finally if (vi) holds this excludes $\substack{123\cdot2\cdot1\\ \cdot}$, $\substack{12222\cdot1\\ 1}$, giving $\substack{12432\cdot1\\ 2}$ $\in w(X)$ by default; so

$$\begin{aligned} w(X) &= \{\substack{24654\cdot\cdot\\ 3}, \substack{\cdot\cdot\cdot\cdot321\\ \cdot}, \substack{2343221\\ 2}, \substack{1343211\\ 2}, \substack{12432\cdot\cdot\\ 2}, \substack{123\cdot210\\ \cdot}, \substack{1122211\\ 1}, \substack{0122221\\ 1}\} \\ &= w_3w_1w_8w_7(X^{25}_{30}). \end{aligned}$$

Thus we may assume $\substack{0122100\\ 1}$ $\notin X$, giving $\substack{0000111\\ 0}$, $\substack{2343321\\ 2}$ $\in X$ by default. To avoid $|X \setminus \Omega| = 1$ we must have some root $\substack{0\cdot\cdot1100\\ \cdot}$ present; using $\langle w_2, w_3, w_4\rangle$ we may assume $\substack{0001100\\ 0}$ $\in X$, which excludes $\substack{\cdot\cdot21111\\ 1}$, $\substack{1232221\\ \cdot}$, $\substack{2464321\\ 3}$, giving $\substack{1233211\\ \cdot}$, $\substack{1\cdot44321\\ 2}$ $\in X$ by default. Suppose $\substack{0121100\\ 1}$ $\in X$; this excludes $\substack{0001111\\ 0}$, $\substack{1111111\\ \cdot}$, $\substack{1\cdot22221\\ 1}$, $\substack{2344321\\ 2}$, giving $\substack{1\cdot43211\\ 2}$, $\substack{1354321\\ \cdot}$ $\in X$ by default. Thus $X = X_c \cup X_a$ where

$$\begin{aligned} X_c &= \{\substack{2465432\\ 3}, \substack{2465421\\ 3}, \substack{1\cdot\cdot4321\\ \cdot}, \substack{\cdot\cdot\cdot3321\\ \cdot}, \substack{1\cdot\cdot\cdot211\\ \cdot}, \substack{01222\cdot1\\ 1}, \substack{0121100\\ 1}, \substack{0000111\\ 0}, \substack{000\cdot100\\ 0}, \\ &\qquad \substack{0000001\\ 0}\}, \\ X_a &\subset \{\substack{2\cdot54321\\ \cdot}, \substack{0\cdot11111\\ \cdot}, \substack{0\cdot11100\\ \cdot}\}; \end{aligned}$$

set

$$w = w_4w_2w_5w_4w_3w_8w_7w_6w_5w_4w_3w_2w_1w_4w_3w_5w_4w_2w_6w_5w_4w_3w_1w_7,$$

then we have

$$\begin{aligned} w(X_c) &= \{\substack{24654\cdot\cdot\\ 3}, \substack{\cdot\cdot\cdot\cdot321\\ \cdot}, \substack{2343221\\ 2}, \substack{1343211\\ 2}, \substack{1122211\\ 1}, \substack{0122221\\ 1}, \substack{12432\cdot\cdot\\ 2}\}, \\ w(X_a) &\subset \{\substack{123\cdot2\cdot\cdot\\ \cdot}\}, \end{aligned}$$

which is covered by the situation which arose two paragraphs above.

Thus we may assume $\substack{0121100\\ 1}$ $\notin X$, giving $\substack{0001111\\ 0}$, $\substack{2344321\\ 2}$ $\in X$ by default. To avoid $|X \setminus \Omega| = 2$ we must have some root $\substack{0\cdot11100\\ \cdot}$ present; using $\langle w_2, w_3\rangle$ we may assume $\substack{0011100\\ 0}$ $\in X$, which excludes $\substack{\cdot111111\\ 1}$, $\substack{1222221\\ 1}$, $\substack{2454321\\ 3}$, giving $\substack{1243211\\ 2}$, $\substack{1354321\\ 2}$ $\in X$ by default. We may assume $\substack{0111100\\ 1}$ $\notin X$ (else we could apply w_4 to reduce to the case just considered), giving $\substack{0011111\\ 0}$, $\substack{2354321\\ 2}$ $\in X$ by default. To exclude $-\substack{1221000\\ 1}$ we must have $\substack{1343211\\ 2}$ or $\substack{2454321\\ 2}$ present; likewise to exclude $-\substack{1232110\\ 2}$ we must have some root $\substack{\cdot354321\\ 3}$ present. Thus $X = X_c \cup X_a$ where

$$\begin{aligned} X_c &= \{\substack{2465432\\ 3}, \substack{2465421\\ 3}, \substack{\cdot354321\\ 2}, \substack{\cdot\cdot44321\\ 2}, \substack{\cdot\cdot\cdot3321\\ \cdot}, \substack{12\cdot\cdot211\\ \cdot}, \substack{\cdot122211\\ 1}, \substack{0122221\\ 1}, \substack{00\cdot\cdot111\\ 0}, \\ &\qquad \substack{00\cdot\cdot100\\ 0}, \substack{0000001\\ 0}\}, \\ X_a &\subset \{\substack{2454321\\ 2}, \substack{\cdot354321\\ 3}, \substack{1343211\\ 2}, \substack{1122221\\ 1}, \substack{\cdot111111\\ 0}, \substack{0011111\\ 1}, \substack{0111100\\ 0}, \substack{0011100\\ 1}\}, \end{aligned}$$

and X_a must contain at least one of $\substack{1343211\\ 2}$ and $\substack{2454321\\ 2}$, and some root $\substack{\cdot354321\\ 3}$; set

$$w = w_5w_4w_2w_5w_4w_6w_3w_4w_5w_6w_2w_4w_5w_3w_1w_3w_4w_2w_5w_4w_3w_1w_6w_5w_4w_3w_2w_7,$$

then we have

$$w(X_c)=\{\substack{2\cdot\cdot\cdot\cdot\cdot\cdot\\ \cdot},\ \substack{13\cdot\cdot\cdot\cdot\cdot\\ \cdot},\ \substack{124\cdot\cdot\cdot\cdot\\ 2},\ \substack{1233211\\ 2},\ \substack{1232111\\ 1}\},$$
$$w(X_a)\subset\{\substack{123\cdot221\\ 2},\ \substack{1233\cdot21\\ 1},\ \substack{1232211\\ \cdot},\ \substack{123321\cdot\\ 1},\ \substack{123211\cdot\\ 2}\},$$

and $w(X_a)$ must contain both some root $\substack{123\cdot221\\ 2}$ and some root $\substack{1233\cdot21\\ 1}$.

Let $n=|w(X)\cap\{\substack{123\cdot221\\ 2},\ \substack{1233\cdot21\\ 1}\}|$; then $2\le n\le 4$. First suppose $n=4$; then $\substack{123\cdot221\\ 2},\ \substack{1233\cdot21\\ 1}\in w(X)$, which excludes $\substack{1232211\\ \cdot},\ \substack{123321\cdot\\ 1},\ \substack{123211\cdot\\ 2}$; so

$$w(X)=\{\substack{2\cdot\cdot\cdot\cdot\cdot\cdot\\ \cdot},\ \substack{13\cdot\cdot\cdot\cdot\cdot\\ \cdot},\ \substack{124\cdot\cdot\cdot\cdot\\ 2},\ \substack{1233211\\ 2},\ \substack{123\cdot221\\ 2},\ \substack{1233\cdot21\\ 1},\ \substack{1232111\\ 1}\}=w_6w_7(X_{32}^2).$$

Next suppose $n=3$; then using $\langle w_2w_6w_5w_6\rangle$ (which preserves $w(X_c)$) we may assume $\substack{1233\cdot21\\ 1}\in w(X)$, which excludes $\substack{1232\cdot11\\ 2},\ \substack{1232110\\ 2}$. If $\substack{1233221\\ 2}\in w(X)$, $\substack{1232221\\ 2}\notin w(X)$ this excludes $\substack{1232211\\ 1}$ and gives $\substack{123321\cdot\\ 1}\in w(X)$ by default; so

$$w(X)=\{\substack{2\cdot\cdot\cdot\cdot\cdot\cdot\\ \cdot},\ \substack{13\cdot\cdot\cdot\cdot\cdot\\ \cdot},\ \substack{124\cdot\cdot\cdot\cdot\\ 2},\ \substack{12332\cdot1\\ 2},\ \substack{1233\cdot\cdot\cdot\\ 1},\ \substack{1232111\\ 1}\}=w_6w_7w_2(X_{33}).$$

On the other hand if $\substack{1232221\\ 2}\in w(X)$, $\substack{1233221\\ 2}\notin w(X)$ this excludes $\substack{123321\cdot\\ 1}$ and gives $\substack{1232211\\ 1}\in w(X)$ by default; so

$$w(X)=\{\substack{2\cdot\cdot\cdot\cdot\cdot\cdot\\ \cdot},\ \substack{13\cdot\cdot\cdot\cdot\cdot\\ \cdot},\ \substack{124\cdot\cdot\cdot\cdot\\ 2},\ \substack{1233211\\ 2},\ \substack{1233\cdot21\\ 1},\ \substack{1233221\\ 2},\ \substack{1232\cdot11\\ 1}\}=w_2(X_{32}^1).$$

Thus we may assume $n=2$. If $\substack{1232221\\ 2},\ \substack{1233321\\ 1}\in w(X)$, $\substack{1233221\\ \cdot}\notin w(X)$ this excludes $\substack{123321\cdot\\ 1},\ \substack{123211\cdot\\ 2}$, giving $\substack{1232211\\ \cdot}\in w(X)$ by default; so

$$w(X)=\{\substack{2\cdot\cdot\cdot\cdot\cdot\cdot\\ \cdot},\ \substack{13\cdot\cdot\cdot\cdot\cdot\\ \cdot},\ \substack{124\cdot\cdot\cdot\cdot\\ 2},\ \substack{1233211\\ 2},\ \substack{1232221\\ 2},\ \substack{1233321\\ 1},\ \substack{1232211\\ \cdot},\ \substack{1232111\\ 1}\}$$
$$=w_2w_5w_7w_6(X_{32}^2).$$

If $\substack{1233221\\ 2},\ \substack{1233321\\ 1}\in w(X)$, $\substack{1232221\\ 2},\ \substack{1233221\\ 1}\notin w(X)$ this excludes $\substack{1232211\\ 1},\ \substack{123211\cdot\\ 2}$, giving $\substack{1233211\\ 1}\in w(X)$ by default; to exclude $\substack{1233221\\ 1}$ we must have $\substack{1232211\\ 2}\in w(X)$, which excludes $\substack{1233210\\ 1}$; so

$$w(X)=\{\substack{2\cdot\cdot\cdot\cdot\cdot\cdot\\ \cdot},\ \substack{13\cdot\cdot\cdot\cdot\cdot\\ \cdot},\ \substack{124\cdot\cdot\cdot\cdot\\ 2},\ \substack{12332\cdot1\\ 2},\ \substack{1233321\\ 1},\ \substack{1232211\\ 2},\ \substack{1233211\\ 1},\ \substack{1232111\\ 1}\}$$
$$=w_7w_6w_5(X_{32}^2).$$

If $\substack{1232221\\ 2},\ \substack{1233221\\ 1}\in w(X)$, $\substack{1233221\\ 2},\ \substack{1233321\\ 1}\notin w(X)$ we may apply $w_2w_6w_5w_6$ to reduce to the previous case. Thus we may assume $\substack{1233221\\ \cdot}\in w(X)$, $\substack{1232221\\ 2},\ \substack{1233321\\ 1}\notin w(X)$, which excludes $\substack{1232211\\ \cdot}$. If $\substack{1232111\\ 2}\in w(X)$ this excludes $\substack{1233210\\ 1}$; to exclude $\substack{1232221\\ 2}$ we must have $\substack{1233211\\ 1}$, which excludes $\substack{1232110\\ 2}$; so

$$w(X)=\{\substack{2\cdot\cdot\cdot\cdot\cdot\cdot\\ \cdot},\ \substack{13\cdot\cdot\cdot\cdot\cdot\\ \cdot},\ \substack{124\cdot\cdot\cdot\cdot\\ 2},\ \substack{12332\cdot1\\ \cdot},\ \substack{1232111\\ \cdot}\}=w_6w_7(X_{32}^3).$$

If instead $\substack{1232111\\ 2}\notin w(X)$ this gives $\substack{1233210\\ 1}\in w(X)$ by default; to exclude $\substack{1233321\\ 1}$ we must have $\substack{1232110\\ 2}\in w(X)$, which excludes $\substack{1233211\\ 1}$; so

$$w(X)=\{\substack{2\cdot\cdot\cdot\cdot\cdot\cdot\\ \cdot},\ \substack{13\cdot\cdot\cdot\cdot\cdot\\ \cdot},\ \substack{124\cdot\cdot\cdot\cdot\\ 2},\ \substack{12332\cdot1\\ 2},\ \substack{1233221\\ 1},\ \substack{1233210\\ 1},\ \substack{1232110\\ 2},\ \substack{1232111\\ 1}\}$$
$$=w_6w_7w_8(X_{32}^2).$$

This proves the lemma. ☐

This completes the treatment of the sets with $\alpha=\substack{0000001\\ 0}$ for which $\alpha'=\substack{0000100\\ 0}$. We therefore move on to consider the other possibilities for α'.

Lemma 8.16. *If* $\substack{0000001\\ 0},\ \substack{0001000\\ 0}\in X$ *then* X *is known.*

PROOF. By Lemma 3.1 and the previous results we may assume ${}^{0000100}_{0} \notin X$, ${}^{0000001}_{0}$, ${}^{0001\cdot00}_{0} \in X$; this excludes ${}^{\cdot\cdot\cdot\cdot\cdot10}_{\cdot}$, ${}^{\cdot\cdot10000}_{\cdot}$, ${}^{0000\cdot11}_{0}$, ${}^{\cdot\cdot21\cdot00}_{1}$, ${}^{\cdot\cdot21111}_{1}$, ${}^{1232\cdot\cdot1}_{\cdot}$, ${}^{\cdot\cdot43\cdot21}_{2}$, ${}^{2464321}_{3}$, ${}^{2465431}_{3}$, giving ${}^{1233211}_{2}$, ${}^{2344321}_{2}$, ${}^{2465\cdot21}_{3} \in X$ b default.

First suppose X meets $\{{}^{\cdot\cdot00000}_{0}\}$; then by Lemma 3.1 we may assume one of the following holds: (a) ${}^{0100000}_{0} \in X$; (b) ${}^{0100000}_{0} \notin X$, ${}^{1\cdot00000}_{0} \in X$. If (a) holds this excludes ${}^{1000000}_{0}$, ${}^{0011\cdot00}_{\cdot}$, ${}^{0011111}_{\cdot}$, ${}^{1122100}_{1}$, ${}^{1122\cdot\cdot1}_{1}$, ${}^{1243211}_{2}$, ${}^{1244321}_{2}$, ${}^{2354321}_{\cdot}$; however, now $-{}^{1121110}_{1}$ cannot be excluded. Similarly if (b) holds this excludes ${}^{0\cdot11\cdot00}_{\cdot}$, ${}^{0\cdot11111}_{\cdot}$, ${}^{0122100}_{1}$, ${}^{0122\cdot\cdot1}_{1}$, ${}^{1\cdot43211}_{2}$, ${}^{1\cdot\cdot4321}_{\cdot}$; however, now $-{}^{0121110}_{1}$ cannot be excluded. Thus ${}^{\cdot\cdot00000}_{0} \notin X$, which gives ${}^{1\cdot44321}_{2} \in X$ by default.

Next suppose ${}^{0000000}_{1} \in X$; this then excludes ${}^{\cdot\cdot11\cdot00}_{0}$, ${}^{\cdot\cdot11111}_{0}$, ${}^{1232100}_{1}$, ${}^{1233\cdot\cdot1}_{1}$, ${}^{\cdot\cdot54321}_{2}$, giving ${}^{1233\cdot21}_{2}$, ${}^{\cdot\cdot54321}_{3} \in X$ by default. To exclude $-{}^{\cdot\cdot10000}_{0}$ we must have ${}^{\cdot\cdot43211}_{2} \in X$, which excludes ${}^{\cdot\cdot22221}_{1}$; so

$$\begin{aligned} X \subset \{ & {}^{2465432}_{3}, {}^{2465\cdot21}_{3}, {}^{\cdot\cdot54321}_{3}, {}^{\cdot\cdot44321}_{2}, {}^{\cdot\cdot43211}_{2}, {}^{1233\cdot\cdot1}_{2}, {}^{\cdot\cdot22\cdot11}_{1}, {}^{1232100}_{2}, {}^{\cdot\cdot22100}_{1}, \\ & {}^{\cdot\cdot11111}_{1}, {}^{\cdot\cdot11\cdot00}_{1}, {}^{0001111}_{0}, {}^{0001\cdot00}_{0}, {}^{0000000}_{1}, {}^{0000001}_{0}\} \\ \subset\ & w_4w_3w_1w_7w_6w_5w_4w_3w_2w_4w_5w_6w_7w_8(\Omega). \end{aligned}$$

Thus we may assume ${}^{0000000}_{1} \notin X$, which gives ${}^{1233211}_{1} \in X$ by default. By Corollary 3.2 (with $Y = \{{}^{0000001}_{0}\}$) and the previous lemmas in this section, we may assume $X \cap \{{}^{\cdot\cdot11\cdot00}_{\cdot}\}$ is stable under $\langle w_6 \rangle$.

Suppose ${}^{\cdot\cdot11\cdot00}_{0} \in X$; this excludes ${}^{\cdot\cdot11\cdot00}_{1}$, ${}^{\cdot\cdot11111}_{1}$, ${}^{\cdot\cdot22\cdot\cdot1}_{1}$, ${}^{1233\cdot21}_{2}$, ${}^{\cdot\cdot54321}_{3}$, giving ${}^{\cdot\cdot43211}_{2}$, ${}^{\cdot\cdot54321}_{2} \in X$ by default. To exclude $-{}^{0000\cdot10}_{0}$ we must have ${}^{1233\cdot21}_{1} \in X$, which excludes ${}^{1232100}_{2}$; so

$$\begin{aligned} X \subset \{ & {}^{2465432}_{3}, {}^{2465\cdot21}_{3}, {}^{\cdot\cdot\cdot4321}_{2}, {}^{\cdot\cdot\cdot3211}_{\cdot}, {}^{1233\cdot21}_{1}, {}^{\cdot\cdot\cdot2100}_{1}, {}^{\cdot\cdot\cdot1111}_{0}, {}^{\cdot\cdot\cdot1\cdot00}_{0}, {}^{0000001}_{0}\} \\ \subset\ & w_2w_4w_3w_1w_7w_6w_5w_4w_3w_2w_4w_5w_6w_7w_8(\Omega). \end{aligned}$$

Thus we may assume some root ${}^{\cdot\cdot11\cdot00}_{0}$ is absent; using $\langle w_1, w_3 \rangle$ we may assume ${}^{1111\cdot00}_{0} \notin X$.

Suppose ${}^{0\cdot11\cdot00}_{0} \in X$; this excludes ${}^{\cdot\cdot11\cdot00}_{1}$, ${}^{\cdot\cdot11111}_{1}$, ${}^{1\cdot22\cdot\cdot1}_{1}$, ${}^{1233\cdot21}_{2}$, ${}^{2\cdot54321}_{3}$, giving ${}^{0122\cdot11}_{1}$, ${}^{1\cdot43211}_{2}$, ${}^{\cdot\cdot54321}_{2} \in X$ by default. To exclude $-{}^{1110000}_{1}$ we must have ${}^{1232100}_{2}$ or ${}^{2343211}_{2}$ present, either of which excludes ${}^{0122221}_{1}$, giving ${}^{2343211}_{2} \in X$ by default. To exclude $-{}^{0000\cdot10}_{0}$ we must have ${}^{1233\cdot21}_{1} \in X$, which excludes ${}^{1232100}_{2}$; so

$$\begin{aligned} X \subset \{ & {}^{2465432}_{3}, {}^{2465\cdot21}_{3}, {}^{\cdot\cdot\cdot4321}_{2}, {}^{1354321}_{3}, {}^{\cdot\cdot\cdot3211}_{\cdot}, {}^{1233\cdot21}_{1}, {}^{0122\cdot11}_{1}, {}^{\cdot\cdot\cdot2100}_{1}, {}^{\cdot\cdot\cdot1111}_{0}, \\ & {}^{0\cdot\cdot1\cdot00}_{0}, {}^{0000001}_{0}\} \\ \subset\ & w_2w_4w_3w_1w_7w_6w_5w_4w_3w_2w_4w_5w_6w_7w_8(\Omega \cup \{{}^{1233210}_{1}\}). \end{aligned}$$

Thus we may assume some root ${}^{0\cdot11\cdot00}_{0}$ is absent; using $\langle w_3 \rangle$ we may assume ${}^{0111\cdot00}_{0} \notin X$.

Suppose ${}^{0011\cdot00}_{0} \in X$; this excludes ${}^{\cdot111\cdot00}_{1}$, ${}^{\cdot111111}_{1}$, ${}^{1222\cdot\cdot1}_{1}$, ${}^{1233\cdot21}_{2}$, ${}^{2454321}_{3}$, giving ${}^{\cdot122\cdot11}_{1}$, ${}^{1243211}_{2}$, ${}^{\cdot354321}_{2} \in X$ by default. To exclude $-{}^{1221110}_{1}$ we must have ${}^{2454321}_{2} \in X$, which excludes ${}^{0011\cdot00}_{1}$, ${}^{0011111}_{1}$; to exclude $-{}^{0000010}_{0}$ we must have ${}^{1233221}_{1}$ or some root ${}^{\cdot122221}_{1}$ present, any of which excludes ${}^{1232100}_{2}$, giving ${}^{1233\cdot21}_{1} \in X$ by default. To exclude $-{}^{0110000}_{1}$ we must have ${}^{1343211}_{2}$ or ${}^{1354321}_{3}$ present; likewise to exclude $-{}^{1110000}_{1}$ we must have ${}^{2343211}_{2}$ or ${}^{2354321}_{3}$ present. Thus

$X = X_c \cup X_a$ where

$$X_c = \{ \substack{2465432\\3}, \substack{2465\cdot21\\3}, \substack{\cdot\cdot\cdot4321\\2}, \substack{12\cdot3211\\\cdot}, \substack{1233\cdot21\\1}, \substack{\cdot122\cdot11\\1}, \substack{00\cdot1\cdot00\\0}, \substack{0000001\\0} \},$$
$$X_a \subset \{ \substack{\cdot354321\\3}, \substack{\cdot343211\\2}, \substack{\cdot122221\\1}, \substack{\cdot\cdot\cdot2100\\1}, \substack{\cdot\cdot\cdot1111\\0} \},$$

and X_a must contain at least one of $\substack{1343211\\2}$ and $\substack{1354321\\3}$, and at least one of $\substack{2343211\\2}$ and $\substack{2354321\\3}$; set

$$w = w_7w_6w_5w_4w_3w_2w_4w_8w_7w_6w_5w_4w_3w_2w_1w_4w_3w_5w_4w_2w_6w_7,$$

then we have

$$w(X_c) = \{ \substack{\cdot\cdot\cdot\cdot\cdot\cdot\cdot\\3}, \substack{\cdot\cdot\cdot4321\\2}, \substack{\cdot\cdot43321\\2}, \substack{2343211\\2}, \substack{123\cdot221\\\cdot}, \substack{0122211\\1} \},$$
$$w(X_a) \subset \{ \substack{1\cdot432\cdot\cdot\\2}, \substack{1233321\\\cdot}, \substack{1232111\\\cdot}, \substack{1\cdot2221\cdot\\1} \},$$

and $w(X_a)$ must contain both some root $\substack{134321\cdot\\2}$ and some root $\substack{124321\cdot\\2}$. To avoid $|w(X) \setminus \Omega| \le 2$ we must have at least three of $\substack{1\cdot43210\\2}, \substack{1\cdot22210\\1}$ present in $w(X)$; either of the latter pair excludes $\substack{1232111\\\cdot}$, giving $\substack{1233321\\\cdot} \in w(X)$ by default. If $\substack{1\cdot22210\\1} \in w(X)$ this excludes $\substack{1\cdot432\cdot1\\2}$, so we must have $\substack{1\cdot43210\\2} \in w(X)$, which excludes $\substack{1\cdot22211\\1}$; but now $|w_8w(X) \setminus \Omega| = 2$. Thus using $\langle w_3 \rangle$ we may assume $\substack{1\cdot43210\\2}, \substack{1222210\\1} \in w(X)$, $\substack{1122210\\1} \notin w(X)$, which excludes $\substack{12432\cdot1\\2}, \substack{1\cdot22211\\1}$, giving $\substack{13432\cdot1\\2} \in w(X)$ by default; so

$$\begin{aligned} w(X) &= \{ \substack{24654\cdot\cdot\\3}, \substack{\cdot\cdot\cdot\cdot321\\\cdot}, \substack{2343211\\2}, \substack{13432\cdot1\\2}, \substack{123\cdot221\\\cdot}, \substack{1\cdot43210\\2}, \substack{1222210\\1}, \substack{0122211\\1} \} \\ &= w_1(X^{25}_{30}). \end{aligned}$$

Thus we may assume $\substack{0011\cdot00\\0} \notin X$. If we had some root $\substack{\cdot\cdot11\cdot00\\1}$ present we could apply w_2 to reduce to the case just considered in the preceding three paragraphs; so we may assume $\substack{\cdot\cdot11\cdot00\\1} \notin X$, which gives $\substack{\cdot\cdot22\cdot11\\1} \in X$ by default.

Suppose X contains some root $\substack{1232100\\\cdot}$; using $\langle w_2 \rangle$ we may assume $\substack{1232100\\1} \in X$, which excludes $\substack{0001111\\0}, \substack{\cdot\cdot11111\\1}, \substack{\cdot\cdot22221\\1}, \substack{1233\cdot21\\2}$, giving $\substack{\cdot\cdot43211\\2}, \substack{\cdot\cdot54321\\2} \in X$ by default. To exclude $-\substack{0000\cdot10\\0}$ we must have $\substack{1233\cdot21\\1} \in X$, which excludes $\substack{1232100\\2}$. Thus $X = X_c \cup X_a$ where

$$X_c = \{ \substack{2465432\\3}, \substack{2465\cdot21\\3}, \substack{\cdot\cdot\cdot4321\\2}, \substack{\cdot\cdot\cdot3211\\\cdot}, \substack{1233\cdot21\\1}, \substack{\cdot\cdot22\cdot11\\1}, \substack{1232100\\1}, \substack{0001\cdot00\\0}, \substack{0000001\\0} \},$$
$$X_a \subset \{ \substack{\cdot\cdot54321\\3}, \substack{\cdot\cdot22100\\1}, \substack{\cdot\cdot11111\\0} \};$$

set

$$w = w_8w_7w_6w_5w_4w_3w_2w_1w_4w_3w_5w_4w_2w_6w_7,$$

then we have

$$w(X_c) = \{ \substack{\cdot\cdot\cdot\cdot\cdot\cdot\cdot\\3}, \substack{\cdot\cdot\cdot4321\\2}, \substack{2343\cdot21\\2}, \substack{12\cdot3\cdot21\\\cdot}, \substack{1343211\\2}, \substack{1122\cdot11\\1}, \substack{0122221\\1} \},$$
$$w(X_a) \subset \{ \substack{12\cdot321\cdot\\\cdot}, \substack{12\cdot2221\\\cdot} \}.$$

To avoid $|w(X) \setminus \Omega| \le 2$ we must have $\substack{12\cdot3210\\\cdot} \in X$, which excludes $\substack{12\cdot2221\\\cdot}$, giving $\substack{12\cdot3211\\\cdot} \in w(X)$ by default; so

$$\begin{aligned} w(X) &= \{ \substack{\cdot\cdot\cdot\cdot\cdot\cdot\cdot\\3}, \substack{\cdot\cdot\cdot4321\\2}, \substack{2343\cdot21\\2}, \substack{12\cdot3\cdot\cdot\cdot\\\cdot}, \substack{1343211\\2}, \substack{1122\cdot11\\1}, \substack{0122221\\1} \} \\ &= w_3w_1w_4w_3w_2w_4(X^1_{32}). \end{aligned}$$

Thus we may assume $\substack{1232100\\\cdot} \notin X$, which gives $\substack{0001111\\0}, \substack{1233\cdot21\\\cdot} \in X$ by default. To avoid $|X \setminus \Omega| \le 2$ we must have some root $\substack{\cdot\cdot22100\\1}$ present; using $\langle w_1, w_3 \rangle$ we may assume $\substack{0122100\\1} \in X$, which excludes $\substack{1111111\\\cdot}, \substack{1\cdot22221\\1}$, giving $\substack{1\cdot43211\\2}, \substack{1354321\\\cdot} \in X$

by default. To exclude $-\substack{1221110\\1}$, $-\substack{1121110\\1}$, $-\substack{1110000\\1}$ and $-\substack{1110000\\0}$ we must have respectively some root $\substack{2454321\\\cdot}$, some root $\substack{2354321\\\cdot}$, some root $\substack{2\cdot54321\\3}$ or $\substack{2343211\\2}$, and some root $\substack{2\cdot54321\\2}$ or $\substack{2343211\\2}$ present. Thus $X = X_c \cup X_a$ where

$$X_c = \{\substack{2465432\\3}, \substack{2465\cdot21\\3}, \substack{1354321\\\cdot}, \substack{\cdot\cdot44321\\2}, \substack{1\cdot\cdot3211\\\cdot}, \substack{1233\cdot21\\\cdot}, \substack{\cdot\cdot22\cdot11\\1}, \substack{0122100\\1}, \substack{0001111\\0}, \substack{0001\cdot00\\0}, \substack{0000001\\0}\},$$
$$X_a \subset \{\substack{2\cdot54321\\\cdot}, \substack{2343211\\2}, \substack{0122221\\1}, \substack{1\cdot22100\\1}, \substack{0\cdot11111\\\cdot}\},$$

and X_a must contain some root $\substack{2454321\\\cdot}$ and some root $\substack{2354321\\\cdot}$, and if it does not contain $\substack{2343211\\2}$ it must also contain some root $\substack{2\cdot54321\\3}$ and some root $\substack{2\cdot54321\\2}$. Set

$$w = w_6w_5w_7w_6w_4w_5w_6w_7w_3w_4w_5w_6w_2w_4w_3w_5w_4w_1w_3w_4w_2w_5w_4w_3w_1w_6w_7;$$

then we have

$$w(X_c) = \{\substack{246\cdot\cdot\cdot\cdot\\3}, \substack{\cdot\cdot54321\\\cdot}, \substack{\cdot\cdot4\cdot\cdot\cdot\cdot\\2}, \substack{1233321\\1}\},$$
$$w(X_a) \subset \{\substack{1233321\\2}, \substack{123\cdot2\cdot1\\\cdot}, \substack{123\cdot210\\2}, \substack{1232111\\1}\},$$

and $w(X_a)$ must contain some root $\substack{12332\cdot1\\2}$ and some root $\substack{12322\cdot1\\2}$, and if it does not contain $\substack{1233321\\2}$ it must also contain some root $\substack{123\cdot221\\2}$ and some root $\substack{123\cdot211\\2}$.

First suppose $\substack{1232111\\1} \in w(X)$; this excludes $\substack{1233321\\2}$, $\substack{123\cdot210\\2}$. If $\substack{123\cdot2\cdot1\\2} \in w(X)$ this excludes $\substack{123\cdot2\cdot1\\1}$; so

$$w(X) = \{\substack{246\cdot\cdot\cdot\cdot\\3}, \substack{\cdot\cdot54321\\\cdot}, \substack{\cdot\cdot4\cdot\cdot\cdot\cdot\\2}, \substack{1233321\\1}, \substack{123\cdot2\cdot1\\2}, \substack{1232111\\1}\} = w_6w_7(X^1_{32}).$$

If $w(X)$ contains exactly three of the roots $\substack{123\cdot2\cdot1\\2}$, using $\langle w_5, w_7\rangle$ (which preserves $w(X_c)$ and $\substack{1232111\\1}$) we may assume $\substack{12332\cdot1\\2}$, $\substack{1232221\\2} \in w(X)$, $\substack{1232211\\2} \notin w(X)$, which excludes $\substack{12322\cdot1\\1}$, $\substack{1233211\\1}$ and gives $\substack{1233221\\1} \in w(X)$ by default; so

$$w(X) = \{\substack{246\cdot\cdot\cdot\cdot\\3}, \substack{\cdot\cdot54321\\\cdot}, \substack{\cdot\cdot4\cdot\cdot\cdot\cdot\\2}, \substack{1233\cdot21\\1}, \substack{12332\cdot1\\2}, \substack{1232221\\2}, \substack{1232111\\1}\} = w_6w_5(X^2_{32}).$$

If on the other hand $w(X)$ contains just two of the roots $\substack{123\cdot2\cdot1\\2}$, using $\langle w_5, w_7\rangle$ we may assume $\substack{1233221\\2}$, $\substack{1232211\\2} \in w(X)$, $\substack{1233211\\2}$, $\substack{1232221\\2} \notin w(X)$, which excludes $\substack{1232211\\1}$, $\substack{1233221\\1}$ and gives $\substack{1233211\\1}$, $\substack{1232221\\1} \in w(X)$ by default; so

$$\begin{aligned} w(X) &= \{\substack{246\cdot\cdot\cdot\cdot\\3}, \substack{\cdot\cdot54321\\\cdot}, \substack{\cdot\cdot4\cdot\cdot\cdot\cdot\\2}, \substack{1233321\\1}, \substack{1233221\\2}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{1232221\\1}, \substack{1232111\\1}\} \\ &= w_2w_5(X^1_{32}). \end{aligned}$$

So we may assume $\substack{1232111\\1} \notin w(X)$, which gives $\substack{1233321\\2} \in w(X)$ by default. If $\substack{123\cdot210\\2} \in w(X)$ this excludes $\substack{123\cdot2\cdot1\\1}$, giving $\substack{123\cdot2\cdot1\\2} \in w(X)$ by default; so

$$w(X) = \{\substack{24654\cdot\cdot\\3}, \substack{\cdot\cdot\cdot\cdot321\\\cdot}, \substack{\cdot\cdot\cdot\cdot2\cdot\cdot\\2}\} = X^1_{34}.$$

If $w(X)$ contains just one of the roots $\substack{123\cdot210\\2}$, using $\langle w_5\rangle$ (which preserves $w(X_c)$ and $\substack{1233321\\2}$) we may assume $\substack{1233210\\2} \in w(X)$, $\substack{1232210\\2} \notin w(X)$, which excludes $\substack{12322\cdot1\\1}$, giving $\substack{12332\cdot1\\2} \in w(X)$ by default; using $\langle w_7\rangle$ we may assume $\substack{1232221\\2} \in w(X)$, which excludes $\substack{1233211\\1}$, and then to exclude $\substack{1232210\\2}$ we must have $\substack{1233221\\1} \in w(X)$, which excludes $\substack{1232211\\2}$; so

$$w(X) = \{\substack{2465\cdot\cdot\cdot\\3}, \substack{\cdot\cdot\cdot4321\\\cdot}, \substack{\cdot\cdot\cdot3\cdot\cdot\cdot\\2}, \substack{1233\cdot21\\1}, \substack{1232221\\2}\} = X_{33}.$$

Thus we may assume $\substack{123\cdot210\\2} \notin w(X)$. Using $\langle w_7\rangle$ we may assume $\substack{1233221\\2} \in w(X)$, which excludes $\substack{1232211\\1}$; to exclude $\substack{1233210\\2}$ we must have $\substack{1232221\\1} \in w(X)$, which excludes ${}^{1233211}_{2}$. Finally if ${}^{1232221}_{2} \in w(X)$ this excludes ${}^{1233211}_{1}$, and then to exclude ${}^{1232210}_{2}$ we must have ${}^{1233221}_{1} \in w(X)$, which excludes ${}^{1232211}_{2}$; so

$$w(X) = \{ {}^{246\cdot\cdot\cdot\cdot}_{3}, {}^{\cdot\cdot54321}_{\cdot}, {}^{\cdot\cdot4\cdot\cdot\cdot\cdot}_{2}, {}^{123\cdot\cdot21}_{\cdot} \} = X^3_{32}.$$

If on the other hand ${}^{1232221}_{2} \notin w(X)$, giving ${}^{1233211}_{1} \in w(X)$ by default, then we must have ${}^{1232211}_{2} \in w(X)$, which excludes ${}^{1233221}_{1}$; so

$$w(X) = \{ {}^{246\cdot\cdot\cdot\cdot}_{3}, {}^{\cdot\cdot54321}_{\cdot}, {}^{\cdot\cdot4\cdot\cdot\cdot\cdot}_{2}, {}^{1233321}_{\cdot}, {}^{1233221}_{2}, {}^{1232211}_{2}, {}^{1233211}_{1}, {}^{1232221}_{1} \} = X^2_{32}.$$

This proves the lemma. □

LEMMA 8.17. *If* ${}^{0000001}_{0}, {}^{0010000}_{0} \in X$ *then* X *is known.*

PROOF. By Lemma 3.1 and the previous results we may assume ${}^{000\cdot\cdot00}_{0} \notin X$, ${}^{0000001}_{0}, {}^{001\cdot\cdot00}_{0} \in X$; this excludes ${}^{\cdot\cdot\cdot\cdot\cdot10}_{\cdot}, {}^{\cdot100000}_{0}, {}^{0000000}_{1}, {}^{000\cdot\cdot11}_{0}, {}^{\cdot11\cdot\cdot00}_{1}, {}^{\cdot111111}_{1}, {}^{122\cdot\cdot00}_{1}, {}^{122\cdot\cdot\cdot1}_{1}, {}^{123\cdot\cdot\cdot1}_{2}, {}^{\cdot34\cdot\cdot21}_{2}, {}^{2454321}_{3}, {}^{2465431}_{3}$, giving ${}^{1243211}_{2}, {}^{2354321}_{2}, {}^{246\cdot\cdot21}_{3} \in X$ by default. To exclude $-{}^{0111110}_{1}$ we must have ${}^{0122221}_{1}$ or ${}^{1354321}_{3}$ present, either of which excludes ${}^{1000000}_{0}, {}^{111\cdot\cdot00}_{0}$, giving ${}^{1354321}_{2} \in X$ by default. By Corollary 3.2 (with $Y = \{ {}^{0000001}_{0} \}$) and the previous lemmas in this section, we may assume $X \cap \{ {}^{001\cdot\cdot00}_{1} \}$ is stable under $\langle w_5, w_6 \rangle$. Suppose ${}^{001\cdot\cdot00}_{1} \in X$; this excludes ${}^{011\cdot\cdot00}_{0}, {}^{\cdot111111}_{0}, {}^{123\cdot\cdot\cdot1}_{1}, {}^{2454321}_{2}$, giving ${}^{124\cdot\cdot21}_{2}, {}^{\cdot354321}_{3} \in X$ by default; to exclude $-{}^{\cdot100000}_{0}$ we must have ${}^{\cdot343211}_{2} \in X$, which excludes ${}^{\cdot122221}_{1}$; so

$$\begin{aligned} X \subset \{ & {}^{2465432}_{3}, {}^{246\cdot\cdot21}_{3}, {}^{\cdot354321}_{\cdot}, {}^{124\cdot\cdot21}_{2}, {}^{\cdot\cdot43211}_{2}, {}^{1232100}_{\cdot}, {}^{\cdot12\cdot\cdot11}_{1}, {}^{\cdot12\cdot\cdot00}_{1}, {}^{0011111}_{\cdot}, \\ & {}^{001\cdot\cdot00}_{\cdot}, {}^{0000001}_{0} \} \\ \subset\ & w_3 w_1 w_7 w_6 w_5 w_4 w_3 w_2 w_4 w_5 w_6 w_7 w_8(\Omega). \end{aligned}$$

So we may assume ${}^{001\cdot\cdot00}_{1} \notin X$, which gives ${}^{123\cdot\cdot11}_{1} \in X$ by default. To exclude $-{}^{1111110}_{1}$ we must have ${}^{1122221}_{1}$ or ${}^{2354321}_{3}$ present, either of which excludes ${}^{011\cdot\cdot00}_{0}$, giving ${}^{124\cdot\cdot21}_{2} \in X$ by default. If some root ${}^{\cdot12\cdot\cdot00}_{1}$ were absent, by using $\langle w_1, w_5, w_6 \rangle$ we could assume ${}^{0121000}_{1} \notin X$, and then applying $w_2 w_3 w_4 w_3 w_2$ would produce a positive set meeting $\{ {}^{00\cdot\cdot\cdot00}_{0} \}$ in a proper non-empty subset of $\{ {}^{001\cdot\cdot00}_{0} \}$. Thus we may assume ${}^{\cdot12\cdot\cdot00}_{1} \in X$, which excludes ${}^{\cdot111111}_{0}, {}^{\cdot12\cdot\cdot\cdot1}_{1}, {}^{123\cdot\cdot21}_{1}$, giving ${}^{\cdot343211}_{2}, {}^{\cdot354321}_{3} \in X$ by default; so

$$\begin{aligned} X \subset \{ & {}^{2465432}_{3}, {}^{246\cdot\cdot21}_{3}, {}^{\cdot354321}_{\cdot}, {}^{2454321}_{2}, {}^{124\cdot\cdot21}_{2}, {}^{\cdot\cdot43211}_{2}, {}^{123\cdot\cdot11}_{1}, {}^{1232100}_{\cdot}, {}^{\cdot12\cdot\cdot00}_{1}, \\ & {}^{0011111}_{\cdot}, {}^{001\cdot\cdot00}_{0}, {}^{0000001}_{0} \} \\ \subset\ & w_7 w_6 w_5 w_3 w_4 w_1 w_2 w_3 w_4 w_5 w_6 w_7 w_8(\Omega \cup \{ {}^{1243210}_{2} \}). \end{aligned}$$

This proves the lemma. □

LEMMA 8.18. *If* ${}^{0000001}_{0}, {}^{0000000}_{1} \in X$ *then* X *is known.*

PROOF. By Lemma 3.1 and the previous results we may assume ${}^{00\cdot\cdot\cdot00}_{0} \notin X$, ${}^{0000001}_{0}, {}^{00\cdot\cdot\cdot00}_{1} \in X$; this excludes ${}^{\cdot\cdot\cdot\cdot\cdot10}_{\cdot}, {}^{\cdot1\cdot\cdot\cdot00}_{0}, {}^{\cdot\cdot\cdot\cdot\cdot11}_{0}, {}^{12\cdot\cdot\cdot00}_{1}, {}^{12\cdot\cdot\cdot\cdot1}_{1}, {}^{\cdot3\cdot\cdot\cdot21}_{2}, {}^{2454321}_{2}, {}^{2465431}_{3}$, giving ${}^{0011111}_{1}, {}^{12\cdot\cdot\cdot11}_{2}, {}^{2\cdot\cdot\cdot\cdot21}_{3} \in X$ by default. If some root ${}^{011\cdot\cdot00}_{1}$ were absent, applying w_3 would produce a positive set meeting $\{ {}^{00\cdot\cdot\cdot00}_{\cdot} \}$ in a proper non-empty subset of $\{ {}^{00\cdot\cdot\cdot00}_{1} \}$, whence by Lemma 3.1 and the previous lemmas in this section X would be known; so we may assume ${}^{011\cdot\cdot00}_{1} \in X$, which excludes ${}^{1000000}_{0}, {}^{112\cdot\cdot00}_{1}, {}^{112\cdot\cdot\cdot1}_{1}, {}^{124\cdot\cdot21}_{2}$, giving ${}^{0111111}_{1}, {}^{1343211}_{2}, {}^{1354321}_{3} \in X$ by default; but now $-\substack{1111110\\0}$ cannot be excluded, so no sets require consideration. This proves the lemma. □

Lemma 8.19. *If* $\substack{0000001\\0}, \substack{0100000\\0} \in X$ *then* X *is known.*

Proof. By Lemma 3.1 and the previous results we may assume $\substack{00\cdot\cdot\cdot00\\\cdot} \notin X$, $\substack{0000001\\0}, \substack{01\cdot\cdot\cdot00\\\cdot} \in X$; this excludes $\substack{\cdot\cdot\cdot\cdot\cdot10\\\cdot}, \substack{1000000\\0}, \substack{11\cdot\cdot\cdot00\\\cdot}, \substack{00\cdot\cdot\cdot11\\\cdot}, \substack{11\cdot\cdot\cdot\cdot1\\\cdot}, \substack{12\cdot\cdot\cdot\cdot1\\\cdot}, \substack{23\cdot\cdot\cdot21\\\cdot}, \substack{2465431\\3}$, giving $\substack{13\cdot\cdot\cdot\cdot1\\\cdot}, \substack{24\cdot\cdot\cdot21\\\cdot} \in X$ by default. To exclude $-\substack{1000000\\0}$ we must have $\substack{2343211\\2} \in X$, which excludes $\substack{0122221\\1}$; so

$$\begin{aligned} X &\subset \{\substack{2465432\\3}, \substack{24\cdot\cdot\cdot21\\\cdot}, \substack{13\cdot\cdot\cdot\cdot1\\\cdot}, \substack{2343211\\2}, \substack{12\cdot\cdot\cdot00\\\cdot}, \substack{01\cdot\cdot\cdot11\\\cdot}, \substack{01\cdot\cdot\cdot00\\\cdot}, \substack{0000001\\0}\} \\ &\subset w_7w_6w_5w_4w_1w_2w_3w_4w_5w_6w_7w_8(\Omega). \end{aligned}$$

This proves the lemma. □

Lemma 8.20. *If* $\substack{0000001\\0}, \substack{1000000\\0} \in X$ *then* X *is known.*

Proof. By Lemma 3.1 and the previous results we may assume $\substack{0\cdot\cdot\cdot\cdot00\\\cdot} \notin X$, $\substack{0000001\\0}, \substack{1\cdot\cdot\cdot\cdot00\\\cdot} \in X$; this excludes $\substack{\cdot\cdot\cdot\cdot\cdot10\\\cdot}, \substack{0\cdot\cdot\cdot\cdot11\\\cdot}, \substack{0122221\\1}, \substack{1\cdot\cdot\cdot\cdot\cdot1\\\cdot}, \substack{2465431\\3}$, giving $\substack{2343211\\2}, \substack{2\cdot\cdot\cdot\cdot21\\\cdot} \in X$ by default; so

$$\begin{aligned} X &= \{\substack{2465432\\3}, \substack{2\cdot\cdot\cdot\cdot21\\\cdot}, \substack{2343211\\2}, \substack{1\cdot\cdot\cdot\cdot00\\\cdot}, \substack{0000001\\0}\} \\ &\subset w_7w_6w_5w_4w_2w_3w_4w_5w_6w_7w_8(\Omega). \end{aligned}$$

This proves the lemma. □

This completes the treatment of the sets with $\alpha = \substack{0000001\\0}$. We therefore move on to consider the other possibilities for α. As in the E_7 analysis, if at some stage we are able to find $w \in W$ which sends one of the chosen roots to $\substack{0000001\\0}$ while preserving the positivity of the union of the sets of chosen and available roots, there will be no need to pursue the line of reasoning further.

Lemma 8.21. *If* $\substack{0000010\\0} \in X$ *then* X *is known.*

Proof. We assume $\substack{0000001\\0} \notin X$, $\substack{000001\cdot\\0} \in X$; this excludes $\substack{\cdot\cdot\cdot\cdot100\\\cdot}, \substack{\cdot\cdot\cdot\cdot21\cdot\\\cdot}, \substack{2465421\\3}$, giving $\substack{2343221\\2}, \substack{2465431\\3} \in X$ by default. Using Lemma 3.1 we may assume one of the following holds: (i) $\substack{0001000\\0} \in X$; (ii) $\substack{0001000\\0} \notin X$, $\substack{001\cdot000\\0} \in X$; (iii) $\substack{00\cdot\cdot000\\0} \notin X$, $\substack{00\cdot\cdot000\\1} \in X$; (iv) $\substack{00\cdot\cdot000\\\cdot} \notin X$, $\substack{01\cdot\cdot000\\\cdot} \in X$; (v) $\substack{0\cdot\cdot\cdot000\\\cdot} \notin X$, $\substack{1\cdot\cdot\cdot000\\\cdot} \in X$; (vi) $\substack{\cdot\cdot\cdot\cdot000\\\cdot} \notin X$.

Assume (i) holds; this excludes $\substack{000011\cdot\\0}, \substack{\cdot\cdot10000\\\cdot}, \substack{\cdot\cdot2111\cdot\\1}, \substack{1232221\\\cdot}, \substack{\cdot\cdot43321\\2}, \substack{2464321\\3}$, giving $\substack{1233221\\2}, \substack{2465321\\3} \in X$ by default. To exclude $-\substack{0121100\\1}$ we must have $\substack{0121000\\1}$ or some root $\substack{1354321\\\cdot}$ present, any of which excludes $\substack{1\cdot00000\\0}$, giving $\substack{1343221\\2} \in X$ by default. Likewise to exclude $-\substack{1121100\\1}$ we must have $\substack{1121000\\1}$ or some root $\substack{2354321\\\cdot}$ present, any of which excludes $\substack{0100000\\0}$, giving $\substack{1243221\\2} \in X$ by default. Suppose $\substack{0000000\\1} \in X$; this excludes $\substack{\cdot\cdot11000\\0}, \substack{\cdot\cdot1111\cdot\\0}, \substack{123211\cdot\\1}, \substack{1233\cdot21\\1}, \substack{\cdot\cdot54321\\2}$, giving $\substack{\cdot\cdot54321\\3} \in X$ by default; to exclude $-\substack{0000100\\0}$ we must have $\substack{1233321\\2} \in X$, which excludes $\substack{\cdot\cdot21000\\1}$, giving $\substack{\cdot\cdot44321\\2} \in X$ by default; so

$$\begin{aligned} X \subset \{&\substack{246543\cdot\\3}, \substack{2465321\\3}, \substack{\cdot\cdot54321\\3}, \substack{\cdot\cdot44321\\2}, \substack{\cdot\cdot\cdot3221\\2}, \substack{1233321\\2}, \substack{123211\cdot\\2}, \substack{\cdot\cdot22221\\1}, \substack{\cdot\cdot2211\cdot\\1}, \\ &\substack{\cdot\cdot1111\cdot\\1}, \substack{\cdot\cdot11000\\1}, \substack{000111\cdot\\0}, \substack{000001\cdot\\0}, \substack{0000000\\1}, \substack{0001000\\0}\} \\ \subset\ & w_4w_3w_1w_6w_5w_4w_2w_3w_4w_5w_6w_7w_8(\Omega). \end{aligned}$$

Thus we may assume $\substack{0000000\\1} \notin X$, which gives $\substack{1233221\\1} \in X$ by default. To exclude $-\substack{0000100\\0}$ we must have some root $\substack{1233321\\\cdot}$ present, so using $\langle w_2 \rangle$ we may assume $\substack{1233321\\1} \in X$, which excludes $\substack{\cdot\cdot\cdot1000\\1}$, $\substack{123211\cdot\\2}$, giving $\substack{\cdot\cdot44321\\2} \in X$ by default. Thus $X = X_c \cup X_a$ where

$$X_c = \{ \substack{246543\cdot\\3}, \substack{2465321\\3}, \substack{\cdot\cdot44321\\2}, \substack{\cdot\cdot\cdot3221\\\cdot}, \substack{1233321\\1}, \substack{000001\cdot\\0}, \substack{0001000\\0} \},$$
$$X_a \subset \{ \substack{\cdot\cdot54321\\\cdot}, \substack{1233321\\2}, \substack{\cdot\cdot22221\\1}, \substack{\cdot\cdot\cdot211\cdot\\1}, \substack{\cdot\cdot1111\cdot\\\cdot}, \substack{000111\cdot\\0}, \substack{\cdot\cdot11000\\0} \};$$

set

$$w = w_7w_6w_5w_4w_2w_3w_4w_5w_8w_7w_6w_5w_4w_3w_2w_1w_4w_3w_5w_4w_2w_4w_5w_6,$$

then we have $w(X) \subset \Phi^+$, and $\substack{0000001\\0} = w(\substack{0001000\\0}) \in w(X_c)$.

Assume (ii) holds; this then excludes $\substack{0000000\\1}$, $\substack{\cdot100000\\0}$, $\substack{000\cdot11\cdot\\0}$, $\substack{\cdot11\cdot000\\1}$, $\substack{\cdot11111\cdot\\1}$, $\substack{122\cdot11\cdot\\1}$, $\substack{1222221\\1}$, $\substack{123\cdot\cdot21\\2}$, $\substack{\cdot34\cdot321\\2}$, $\substack{2454321\\3}$, giving $\substack{1243221\\2}$, $\substack{2354321\\2}$, $\substack{246\cdot321\\3} \in X$ by default. To exclude $-\substack{0111100\\1}$ we must have $\substack{1354321\\3} \in X$, which excludes $\substack{1000000\\0}$, $\substack{111\cdot000\\0}$, $\substack{111111\cdot\\0}$; to exclude $-\substack{1111100\\1}$ we must have $\substack{2354321\\3} \in X$, which excludes $\substack{011\cdot000\\0}$, $\substack{011111\cdot\\0}$, giving $\substack{1343221\\2}$, $\substack{1354321\\2} \in X$ by default; to exclude $-\substack{0000100\\0}$ we must have $\substack{1243321\\2}$ or $\substack{1233321\\1}$ present, either of which excludes $\substack{1221000\\1}$, giving $\substack{124\cdot321\\2} \in X$ by default. Thus $X = X_c \cup X_a$ where

$$X_c = \{ \substack{246543\cdot\\3}, \substack{246\cdot321\\3}, \substack{\cdot354321\\\cdot}, \substack{\cdot\cdot43221\\2}, \substack{124\cdot321\\2}, \substack{001\cdot000\\0}, \substack{000001\cdot\\0} \},$$
$$X_a \subset \{ \substack{2454321\\2}, \substack{123\cdot\cdot21\\1}, \substack{123211\cdot\\\cdot}, \substack{\cdot122221\\1}, \substack{\cdot12\cdot11\cdot\\1}, \substack{\cdot121000\\1}, \substack{001111\cdot\\\cdot}, \substack{001\cdot000\\1} \};$$

set

$$w = w_7w_6w_5w_4w_2w_3w_4w_8w_7w_6w_5w_4w_3w_2w_1w_4w_3w_4w_5w_6,$$

then we have $w(X) \subset \Phi^+$, and $\substack{0000001\\0} = w(\substack{0010000\\0}) \in w(X_c)$.

Assume (iii) holds; this then excludes $\substack{\cdot1\cdot\cdot000\\0}$, $\substack{\cdot\cdot\cdot\cdot11\cdot\\0}$, $\substack{12\cdot\cdot11\cdot\\1}$, $\substack{12\cdot\cdot\cdot21\\1}$, $\substack{\cdot3\cdot\cdot321\\2}$, $\substack{2454321\\2}$, giving $\substack{123211\cdot\\2}$, $\substack{12\cdot\cdot221\\2}$, $\substack{2\cdot\cdot\cdot321\\3} \in X$ by default; but now $-\substack{0111100\\0}$ cannot be excluded, so no sets require consideration.

Assume (iv) holds; this then excludes $\substack{1000000\\0}$, $\substack{11\cdot\cdot000\\\cdot}$, $\substack{00\cdot\cdot11\cdot\\\cdot}$, $\substack{11\cdot\cdot11\cdot\\\cdot}$, $\substack{1122221\\1}$, $\substack{12\cdot\cdot\cdot21\\\cdot}$, $\substack{23\cdot\cdot321\\\cdot}$, giving $\substack{13\cdot\cdot\cdot21\\\cdot}$, $\substack{24\cdot\cdot321\\\cdot} \in X$ by default; so

$$\begin{aligned} X \subset \{ & \substack{246543\cdot\\3}, \substack{24\cdot\cdot321\\\cdot}, \substack{13\cdot\cdot\cdot21\\\cdot}, \substack{2343221\\2}, \substack{12\cdot\cdot11\cdot\\\cdot}, \substack{0122221\\1}, \substack{01\cdot\cdot11\cdot\\\cdot}, \substack{1221000\\1}, \substack{01\cdot\cdot000\\\cdot}, \\ & \substack{000001\cdot\\0} \} \\ \subset\ & w_6w_5w_4w_1w_2w_3w_4w_5w_6w_7w_8(\Omega). \end{aligned}$$

Assume (v) holds; this excludes $\substack{0\cdot\cdot\cdot11\cdot\\\cdot}$, $\substack{0122221\\1}$, $\substack{1\cdot\cdot\cdot\cdot21\\\cdot}$, giving $\substack{2\cdot\cdot\cdot321\\\cdot} \in X$ by default; so

$$\begin{aligned} X &\subset \{ \substack{246543\cdot\\3}, \substack{2\cdot\cdot\cdot321\\\cdot}, \substack{2343221\\2}, \substack{1\cdot\cdot\cdot11\cdot\\\cdot}, \substack{1\cdot\cdot\cdot000\\\cdot}, \substack{000001\cdot\\0} \} \\ &\subset w_6w_5w_4w_2w_3w_4w_5w_6w_7w_8(\Omega). \end{aligned}$$

Thus we may assume (vi) holds; this gives $\substack{0122221\\1}$, $\substack{1\cdot\cdot\cdot221\\\cdot} \in X$ by default. By Corollary 3.2 and the previous lemmas in this section, we may assume $X \cap \{ \substack{\cdot\cdot\cdot\cdot11\cdot\\\cdot} \}$ is stable under $\langle w_8 \rangle$. To avoid $|X \setminus \Omega| \leq 2$ we must have at least two roots $\substack{\cdot\cdot\cdot\cdot110\\\cdot}$ present; using $\langle w_1, w_2, w_3, w_4, w_5 \rangle$ we may first assume $\substack{000011\cdot\\0} \in X$, which excludes $\substack{\cdot\cdot\cdot211\cdot\\\cdot}$, $\substack{2465321\\3}$, giving $\substack{\cdot\cdot\cdot3321\\\cdot} \in X$ by default; using $\langle w_1, w_2, w_3, w_4 \rangle$ we may then assume $\substack{000111\cdot\\0} \in X$, which excludes $\substack{\cdot\cdot2111\cdot\\1}$, $\substack{2464321\\3}$, giving $\substack{\cdot\cdot44321\\2} \in X$ by default.

To exclude $-\underset{\cdot}{1232100}$ we must have some root $\underset{3}{\cdot\cdot54321}$ and some root $\underset{2}{\cdot\cdot54321}$ present. Thus $X = X_c \cup X_a$ where

$$X_c = \{\underset{3}{246543\cdot}, \underset{2}{\cdot\cdot44321}, \underset{\cdot}{\cdot\cdot\cdot3321}, \underset{\cdot}{\cdot\cdot\cdot\cdot221}, \underset{0}{000\cdot\cdot1\cdot}\},$$
$$X_a \subset \{\underset{\cdot}{\cdot\cdot54321}, \underset{\cdot}{\cdot\cdot1111\cdot}\},$$

and X_a must be stable under $\langle w_8\rangle$, and must contain some root $\underset{3}{\cdot\cdot54321}$ and some root $\underset{2}{\cdot\cdot54321}$; set

$$w = (w_4w_5w_6w_3w_4w_5w_1w_3w_4w_2)^3,$$

then we have

$$w(X_c) = \{\underset{3}{246\cdot\cdot\cdot\cdot}, \underset{\cdot}{\cdot\cdot54321}, \underset{2}{\cdot\cdot4\cdot\cdot\cdot\cdot}\},$$
$$w(X_a) \subset \{\underset{\cdot}{123\cdot\cdot21}, \underset{\cdot}{123\cdot\cdot1\cdot}\},$$

and $w(X_a)$ must be stable under $\langle w_8\rangle$, and must contain some root $\underset{1}{123\cdot\cdot21}$ and some root $\underset{2}{123\cdot\cdot21}$.

If $\underset{\cdot}{123\cdot\cdot21} \in w(X)$, this excludes $\underset{\cdot}{123\cdot\cdot1\cdot}$; so

$$w(X) = \{\underset{3}{246\cdot\cdot\cdot\cdot}, \underset{\cdot}{\cdot\cdot54321}, \underset{2}{\cdot\cdot4\cdot\cdot\cdot\cdot}, \underset{\cdot}{123\cdot\cdot21}\} = X_{32}^3.$$

Thus we may assume some root $\underset{\cdot}{123\cdot\cdot21}$ is absent; using $\langle w_2, w_5, w_6\rangle$ we may assume $\underset{1}{1232221} \notin w(X)$; to exclude it we must have $\underset{2}{123321\cdot} \in w(X)$, which excludes $\underset{1}{1232\cdot1\cdot}$, giving $\underset{2}{1233\cdot21} \in w(X)$ by default. Since some root $\underset{1}{1233\cdot21}$ must be present, using $\langle w_6\rangle$ we may assume $\underset{1}{1233321} \in w(X)$, which excludes $\underset{2}{123211\cdot}$. Suppose $\underset{1}{1233221} \notin w(X)$; to exclude it we must have $\underset{2}{123221\cdot} \in w(X)$, which excludes $\underset{1}{123321\cdot}$, giving $\underset{2}{1232221} \in w(X)$ by default; so

$$w(X) = \{\underset{3}{24654\cdot\cdot}, \underset{\cdot}{\cdot\cdot\cdot\cdot321}, \underset{2}{\cdot\cdot\cdot\cdot2\cdot\cdot}\} = X_{34}^1.$$

So we may assume $\underset{1}{1233221} \in w(X)$, which excludes $\underset{2}{123221\cdot}$. If $\underset{2}{1232221} \in w(X)$ this excludes $\underset{1}{123321\cdot}$, and

$$w(X) = \{\underset{3}{2465\cdot\cdot\cdot}, \underset{\cdot}{\cdot\cdot\cdot4321}, \underset{2}{\cdot\cdot\cdot3\cdot\cdot\cdot}, \underset{1}{1233\cdot21}, \underset{2}{1232221}\} = X_{33};$$

if on the other hand $\underset{2}{1232221} \notin w(X)$ this gives $\underset{1}{123321\cdot} \in w(X)$ by default, and

$$w(X) = \{\underset{3}{2465\cdot\cdot\cdot}, \underset{\cdot}{\cdot\cdot\cdot4321}, \underset{\cdot}{\cdot\cdot\cdot3\cdot\cdot\cdot}\} = X_{34}^2.$$

This proves the lemma. ∎

LEMMA 8.22. *If $\underset{0}{0000100} \in X$ then X is known.*

PROOF. We assume $\underset{0}{00000\cdot\cdot} \notin X$, $\underset{0}{00001\cdot\cdot} \in X$; this excludes $\underset{\cdot}{\cdot\cdot\cdot1000}$, $\underset{\cdot}{\cdot\cdot\cdot21\cdot\cdot}$, $\underset{\cdot}{\cdot\cdot\cdot32\cdot\cdot}$, $\underset{3}{2465321}$, giving $\underset{2}{2343321}$, $\underset{3}{24654\cdot1} \in X$ by default. Using Lemma 3.1 we may assume one of the following holds: (i) $\underset{1}{0000000} \in X$; (ii) $\underset{1}{0000000} \notin X$, $\underset{\cdot}{0010000} \in X$; (iii) $\underset{\cdot}{00\cdot0000} \notin X$, $\underset{\cdot}{01\cdot0000} \in X$; (iv) $\underset{\cdot}{0\cdot\cdot0000} \notin X$, $\underset{\cdot}{1\cdot\cdot0000} \in X$; (v) $\underset{\cdot}{\cdot\cdot\cdot0000} \notin X$.

Assume (i) holds; this excludes $\underset{0}{\cdot\cdot10000}$, $\underset{0}{\cdot\cdot111\cdot\cdot}$, $\underset{1}{12322\cdot\cdot}$, $\underset{1}{1233321}$, $\underset{2}{\cdot\cdot54321}$, giving $\underset{2}{1233321}$, $\underset{3}{2454321} \in X$ by default. To exclude $-\underset{0}{0011000}$ we must have $\underset{2}{1244321} \in X$, which excludes $\underset{0}{\cdot100000}$, $\underset{1}{\cdot110000}$, $\underset{1}{12211\cdot\cdot}$, giving $\underset{2}{1243321}$, $\underset{3}{2354321} \in X$ by default; to exclude $-\underset{0}{0111000}$ we must have $\underset{2}{1344321} \in X$, which excludes $\underset{0}{1000000}$, $\underset{1}{0010000}$, $\underset{1}{11211\cdot\cdot}$, giving $\begin{smallmatrix}1343321\\2\end{smallmatrix}$, $\begin{smallmatrix}1354321\\3\end{smallmatrix} \in X$ by default; to exclude $-\begin{smallmatrix}1111000\\0\end{smallmatrix}$ we must have $\begin{smallmatrix}2344321\\2\end{smallmatrix} \in X$, which excludes $\begin{smallmatrix}01211\cdot\cdot\\1\end{smallmatrix}$; so

$$X \subset \{\begin{smallmatrix}24654\cdot\cdot\\3\end{smallmatrix}, \begin{smallmatrix}\cdot\cdot\cdot4321\\3\end{smallmatrix}, \begin{smallmatrix}\cdot\cdot44321\\2\end{smallmatrix}, \begin{smallmatrix}\cdot\cdot\cdot3321\\2\end{smallmatrix}, \begin{smallmatrix}12322\cdot\cdot\\2\end{smallmatrix}, \begin{smallmatrix}\cdot\cdot222\cdot\cdot\\1\end{smallmatrix}, \begin{smallmatrix}\cdot\cdot111\cdot\cdot\\1\end{smallmatrix}, \begin{smallmatrix}000\cdot1\cdot\cdot\\0\end{smallmatrix}, \begin{smallmatrix}0000000\\1\end{smallmatrix}\},$$
$$\subset w_4w_3w_5w_4w_1w_2w_3w_4w_5w_6w_7w_8(\Omega \cup \{\begin{smallmatrix}1343210\\2\end{smallmatrix}\}).$$

Assume (ii) holds; this then excludes $\begin{smallmatrix}\cdot100000\\0\end{smallmatrix}$, $\begin{smallmatrix}00011\cdot\cdot\\0\end{smallmatrix}$, $\begin{smallmatrix}\cdot1111\cdot\cdot\\\cdot\end{smallmatrix}$, $\begin{smallmatrix}12222\cdot\cdot\\1\end{smallmatrix}$, $\begin{smallmatrix}1233321\\\cdot\end{smallmatrix}$, $\begin{smallmatrix}\cdot344321\\2\end{smallmatrix}$, $\begin{smallmatrix}2454321\\\cdot\end{smallmatrix}$, giving $\begin{smallmatrix}1243321\\2\end{smallmatrix}$, $\begin{smallmatrix}2464321\\3\end{smallmatrix} \in X$ by default. To exclude $-\begin{smallmatrix}0001000\\0\end{smallmatrix}$ we must have $\begin{smallmatrix}1244321\\2\end{smallmatrix} \in X$, which excludes $\begin{smallmatrix}\cdot110000\\\cdot\end{smallmatrix}$, $\begin{smallmatrix}12211\cdot\cdot\\1\end{smallmatrix}$, giving $\begin{smallmatrix}2354321\\\cdot\end{smallmatrix} \in X$ by default; to exclude $-\begin{smallmatrix}0111000\\\cdot\end{smallmatrix}$ we must have $\begin{smallmatrix}1354321\\\cdot\end{smallmatrix} \in X$, which excludes $\begin{smallmatrix}1000000\\0\end{smallmatrix}$, giving $\begin{smallmatrix}1343321\\2\end{smallmatrix} \in X$ by default; so

$$X \subset \{\begin{smallmatrix}24654\cdot\cdot\\3\end{smallmatrix}, \begin{smallmatrix}2464321\\3\end{smallmatrix}, \begin{smallmatrix}\cdot354321\\\cdot\end{smallmatrix}, \begin{smallmatrix}\cdot\cdot43321\\2\end{smallmatrix}, \begin{smallmatrix}1244321\\2\end{smallmatrix}, \begin{smallmatrix}12322\cdot\cdot\\\cdot\end{smallmatrix}, \begin{smallmatrix}\cdot1222\cdot\cdot\\1\end{smallmatrix}, \begin{smallmatrix}\cdot1211\cdot\cdot\\1\end{smallmatrix}, \begin{smallmatrix}00111\cdot\cdot\\\cdot\end{smallmatrix},$$
$$\begin{smallmatrix}00001\cdot\cdot\\0\end{smallmatrix}, \begin{smallmatrix}0010000\\\cdot\end{smallmatrix}\}$$
$$\subset w_5w_3w_4w_1w_2w_3w_4w_5w_6w_7w_8(\Omega).$$

Assume (iii) holds; this then excludes $\begin{smallmatrix}1000000\\0\end{smallmatrix}$, $\begin{smallmatrix}00\cdot11\cdot\cdot\\\cdot\end{smallmatrix}$, $\begin{smallmatrix}11\cdot11\cdot\cdot\\\cdot\end{smallmatrix}$, $\begin{smallmatrix}11222\cdot\cdot\\1\end{smallmatrix}$, $\begin{smallmatrix}12\cdot\cdot321\\\cdot\end{smallmatrix}$, $\begin{smallmatrix}23\cdot4321\\\cdot\end{smallmatrix}$, giving $\begin{smallmatrix}1343321\\2\end{smallmatrix}$, $\begin{smallmatrix}24\cdot4321\\\cdot\end{smallmatrix} \in X$ by default. To exclude $-\begin{smallmatrix}00\cdot1000\\\cdot\end{smallmatrix}$ we must have $\begin{smallmatrix}13\cdot4321\\\cdot\end{smallmatrix} \in X$, which excludes $\begin{smallmatrix}11\cdot0000\\\cdot\end{smallmatrix}$; so

$$X \subset \{\begin{smallmatrix}24654\cdot\cdot\\3\end{smallmatrix}, \begin{smallmatrix}24\cdot4321\\\cdot\end{smallmatrix}, \begin{smallmatrix}13\cdot4321\\\cdot\end{smallmatrix}, \begin{smallmatrix}\cdot343321\\2\end{smallmatrix}, \begin{smallmatrix}12\cdot22\cdot\cdot\\\cdot\end{smallmatrix}, \begin{smallmatrix}01222\cdot\cdot\\1\end{smallmatrix}, \begin{smallmatrix}12211\cdot\cdot\\1\end{smallmatrix}, \begin{smallmatrix}01\cdot11\cdot\cdot\\\cdot\end{smallmatrix}, \begin{smallmatrix}01\cdot0000\\\cdot\end{smallmatrix},$$
$$\begin{smallmatrix}00001\cdot\cdot\\0\end{smallmatrix}\}$$
$$\subset w_5w_4w_1w_2w_3w_4w_5w_6w_7w_8(\Omega).$$

Assume (iv) holds; this excludes $\begin{smallmatrix}0\cdot\cdot11\cdot\cdot\\\cdot\end{smallmatrix}$, $\begin{smallmatrix}01222\cdot\cdot\\1\end{smallmatrix}$, $\begin{smallmatrix}1\cdot\cdot\cdot321\\\cdot\end{smallmatrix}$, giving $\begin{smallmatrix}2\cdot\cdot4321\\\cdot\end{smallmatrix} \in X$ by default; so

$$X \subset \{\begin{smallmatrix}24654\cdot\cdot\\3\end{smallmatrix}, \begin{smallmatrix}2\cdot\cdot4321\\\cdot\end{smallmatrix}, \begin{smallmatrix}2343321\\2\end{smallmatrix}, \begin{smallmatrix}1\cdot\cdot22\cdot\cdot\\\cdot\end{smallmatrix}, \begin{smallmatrix}1\cdot\cdot11\cdot\cdot\\\cdot\end{smallmatrix}, \begin{smallmatrix}1\cdot\cdot0000\\\cdot\end{smallmatrix}, \begin{smallmatrix}00001\cdot\cdot\\0\end{smallmatrix}\}$$
$$\subset w_5w_4w_2w_3w_4w_5w_6w_7w_8(\Omega).$$

Thus we may assume (v) holds; this gives $\begin{smallmatrix}1\cdot\cdot3321\\\cdot\end{smallmatrix} \in X$ by default. By Corollary 3.2 and the previous lemmas in this section, we may assume $X \cap \{\begin{smallmatrix}\cdot\cdot\cdot11\cdot\cdot\\\cdot\end{smallmatrix}\}$ is stable under $\langle w_7, w_8\rangle$. Suppose $\begin{smallmatrix}\cdot\cdot\cdot11\cdot\cdot\\\cdot\end{smallmatrix} \notin X$; this gives $\begin{smallmatrix}\cdot\cdot\cdot22\cdot\cdot\\\cdot\end{smallmatrix}$, $\begin{smallmatrix}\cdot\cdot\cdot4321\\\cdot\end{smallmatrix} \in X$ by default, so

$$X = \{\begin{smallmatrix}24654\cdot\cdot\\3\end{smallmatrix}, \begin{smallmatrix}\cdot\cdot\cdot4321\\\cdot\end{smallmatrix}, \begin{smallmatrix}\cdot\cdot\cdot3321\\\cdot\end{smallmatrix}, \begin{smallmatrix}\cdot\cdot\cdot22\cdot\cdot\\\cdot\end{smallmatrix}, \begin{smallmatrix}00001\cdot\cdot\\0\end{smallmatrix}\} = w_5w_4w_3w_1w_2w_4w_3w_5w_4w_2(X_{36}).$$

So we may assume some root $\begin{smallmatrix}\cdot\cdot\cdot11\cdot\cdot\\\cdot\end{smallmatrix}$ is present; using $\langle w_1, w_2, w_3, w_4\rangle$ we may assume $\begin{smallmatrix}00011\cdot\cdot\\0\end{smallmatrix} \in X$, which excludes $\begin{smallmatrix}\cdot\cdot211\cdot\cdot\\1\end{smallmatrix}$, $\begin{smallmatrix}12322\cdot\cdot\\\cdot\end{smallmatrix}$, $\begin{smallmatrix}2464321\\3\end{smallmatrix}$, giving $\begin{smallmatrix}\cdot\cdot44321\\2\end{smallmatrix} \in X$ by default. Suppose $\begin{smallmatrix}\cdot\cdot111\cdot\cdot\\\cdot\end{smallmatrix} \notin X$; this gives $\begin{smallmatrix}\cdot\cdot222\cdot\cdot\\1\end{smallmatrix}$, $\begin{smallmatrix}\cdot\cdot54321\\\cdot\end{smallmatrix} \in X$ by default, so

$$X = \{\begin{smallmatrix}24654\cdot\cdot\\3\end{smallmatrix}, \begin{smallmatrix}\cdot\cdot54321\\\cdot\end{smallmatrix}, \begin{smallmatrix}\cdot\cdot44321\\2\end{smallmatrix}, \begin{smallmatrix}\cdot\cdot\cdot3321\\\cdot\end{smallmatrix}, \begin{smallmatrix}\cdot\cdot222\cdot\cdot\\1\end{smallmatrix}, \begin{smallmatrix}000\cdot1\cdot\cdot\\0\end{smallmatrix}\}$$
$$= w_4w_3w_5w_4w_1w_2w_3w_4w_5w_3w_1w_4w_3w_2w_4w_6w_7w_8w_5w_6w_7(X^3_{32}).$$

So we may assume some root $\begin{smallmatrix}\cdot\cdot111\cdot\cdot\\\cdot\end{smallmatrix}$ is present; using $\langle w_1, w_2, w_3\rangle$ we may assume $\begin{smallmatrix}00111\cdot\cdot\\0\end{smallmatrix} \in X$, which excludes $\begin{smallmatrix}\cdot1111\cdot\cdot\\1\end{smallmatrix}$, $\begin{smallmatrix}12222\cdot\cdot\\1\end{smallmatrix}$, $\begin{smallmatrix}2454321\\3\end{smallmatrix}$, giving $\begin{smallmatrix}\cdot354321\\2\end{smallmatrix} \in X$ by default. To exclude $-\begin{smallmatrix}1221000\\1\end{smallmatrix}$ we must have $\begin{smallmatrix}2454321\\2\end{smallmatrix} \in X$, which excludes $\begin{smallmatrix}00111\cdot\cdot\\1\end{smallmatrix}$; so

$$X \subset \{\begin{smallmatrix}24654\cdot\cdot\\3\end{smallmatrix}, \begin{smallmatrix}\cdot354321\\3\end{smallmatrix}, \begin{smallmatrix}\cdot\cdot\cdot4321\\2\end{smallmatrix}, \begin{smallmatrix}\cdot\cdot\cdot3321\\\cdot\end{smallmatrix}, \begin{smallmatrix}\cdot1222\cdot\cdot\\1\end{smallmatrix}, \begin{smallmatrix}\cdot\cdot\cdot\cdot1\cdot\cdot\\0\end{smallmatrix}\}$$
$$\subset w_2w_4w_5w_3w_4w_1w_2w_3w_4w_5w_6w_7w_8(\Omega \cup \{\begin{smallmatrix}1233210\\\cdot\end{smallmatrix}\}).$$

This proves the lemma. $\square$

LEMMA 8.23. *If* $\substack{0001000\\0} \in X$ *then* X *is known.*

PROOF. We assume $\substack{0000\cdots\\0} \notin X$, $\substack{0001\cdots\\0} \in X$; this excludes $\substack{\cdot\cdot10000\\\cdot}$, $\substack{\cdot\cdot21\cdots\\1}$, $\substack{1232\cdots\\\cdot}$, $\substack{\cdot\cdot43\cdots\\2}$, $\substack{2464321\\3}$, giving $\substack{2344321\\2}$, $\substack{2465\cdot\cdot1\\3} \in X$ by default. To exclude $-\substack{0010000\\0}$ we must have some root $\substack{0011\cdots\\0}$ or some root $\substack{\cdot354321\\2}$ present, any of which excludes $\substack{0000000\\1}$. Using Lemma 3.1 we may assume one of the following holds: (i) $\substack{0100000\\0} \in X$; (ii) $\substack{0100000\\0} \notin X$, $\substack{1\cdot00000\\0} \in X$; (iii) $\substack{\cdot\cdot00000\\0} \notin X$.

Assume (i) holds; this excludes $\substack{1000000\\0}$, $\substack{0011\cdots\\\cdot}$, $\substack{1122\cdots\\1}$, $\substack{1244321\\2}$, $\substack{2354321\\\cdot}$, giving $\substack{1344321\\2}$, $\substack{2454321\\\cdot} \in X$ by default. To exclude $-\substack{0010000\\\cdot}$ we must have $\substack{1354321\\\cdot} \in X$, which excludes $\substack{1100000\\0}$, $\substack{1111\cdots\\\cdot}$; so

$$\begin{aligned} X \subset \{ & \substack{2465\cdots\\3}, \substack{2454321\\\cdot}, \substack{1354321\\\cdot}, \substack{\cdot344321\\2}, \substack{1233\cdots\\\cdot}, \substack{1222\cdots\\1}, \substack{0122\cdots\\1}, \substack{0111\cdots\\\cdot}, \substack{0001\cdots\\0}, \\ & \substack{0100000\\0} \} \\ \subset\ & w_4w_1w_2w_3w_4w_5w_6w_7w_8(\Omega). \end{aligned}$$

Assume (ii) holds; this excludes $\substack{0\cdot11\cdots\\\cdot}$, $\substack{0122\cdots\\1}$, $\substack{1\cdot\cdot4321\\\cdot}$, giving $\substack{2\cdot54321\\\cdot} \in X$ by default; so

$$\begin{aligned} X \subset \{ & \substack{2465\cdots\\3}, \substack{2\cdot54321\\\cdot}, \substack{2344321\\2}, \substack{1233\cdots\\\cdot}, \substack{1\cdot22\cdots\\1}, \substack{1111\cdots\\\cdot}, \substack{0001\cdots\\0}, \substack{1\cdot00000\\0} \} \\ \subset\ & w_4w_2w_3w_4w_5w_6w_7w_8(\Omega). \end{aligned}$$

Thus we may assume (iii) holds; this gives $\substack{1\cdot44321\\2} \in X$ by default. By Corollary 3.2 and the previous lemmas in this section, we may assume $X \cap \{\substack{\cdot\cdot11\cdots\\\cdot}\}$ is stable under $\langle w_6, w_7, w_8\rangle$. To exclude $-\substack{0000000\\1}$ we must have some root $\substack{\cdot\cdot11\cdots\\1}$, some root $\substack{1233\cdots\\2}$ or some root $\substack{\cdot\cdot54321\\3}$ present, any of which excludes some root $\substack{\cdot\cdot11\cdots\\0}$; using $\langle w_1, w_3\rangle$ we may assume $\substack{1111\cdots\\0} \notin X$, which gives $\substack{1354321\\3} \in X$ by default. Likewise to exclude $-\substack{1110000\\1}$ we must have some root $\substack{1111\cdots\\1}$ or some root $\substack{2\cdot54321\\3}$ present, any of which excludes some root $\substack{0\cdot11\cdots\\0}$; using $\langle w_3\rangle$ we may assume $\substack{0111\cdots\\0} \notin X$, which gives $\substack{2354321\\3} \in X$ by default. If $\substack{0011\cdots\\0} \in X$ this excludes $\substack{\cdot111\cdots\\1}$, $\substack{1222\cdots\\1}$, $\substack{1233\cdots\\2}$, $\substack{2454321\\3}$, giving $\substack{\cdot354321\\2} \in X$ by default; so

$$\begin{aligned} X \subset \{ & \substack{2465\cdots\\3}, \substack{2454321\\2}, \substack{\cdot354321\\\cdot}, \substack{\cdot\cdot44321\\2}, \substack{1233\cdots\\1}, \substack{\cdot122\cdots\\1}, \substack{00\cdot1\cdots\\\cdot} \} \\ \subset\ & w_3w_4w_1w_2w_3w_4w_5w_6w_7w_8(\Omega \cup \{\substack{1243210\\2}\}). \end{aligned}$$

So we may assume $\substack{0011\cdots\\0} \notin X$, which gives $\substack{1233\cdots\\2}$, $\substack{2454321\\3} \in X$ by default. If we had some root $\substack{\cdot\cdot11\cdots\\1}$ present we could apply w_2 to reduce to the case just considered; so we may assume $\substack{\cdot\cdot11\cdots\\1} \notin X$, which gives $\substack{1233\cdots\\1}$, $\substack{\cdot\cdot54321\\2} \in X$ by default. We cannot have $\substack{\cdot\cdot22\cdots\\1} \in X$, so using $\langle w_1, w_3, w_6, w_7, w_8\rangle$ we may assume $\substack{0122100\\1} \notin X$. Thus $X = X_c \cup X_a$ where

$$\begin{aligned} X_c &= \{\substack{2465\cdots\\3}, \substack{\cdot\cdot54321\\\cdot}, \substack{\cdot\cdot44321\\2}, \substack{1233\cdots\\\cdot}, \substack{0001\cdots\\0}\}, \\ X_a &\subset \{\substack{1\cdot22\cdots\\1}, \substack{0122\cdot\cdot1\\1}, \substack{0122\cdot10\\1}\}; \end{aligned}$$

set

$$w = w_5w_4w_2w_3w_4w_6w_5w_4w_3w_2w_4,$$

then we have $w(X) \subset \Phi^+$, and $\substack{0000100\\0} = w(\substack{0001000\\0}) \in w(X_c)$. This proves the lemma. $\square$

LEMMA 8.24. *If* $\substack{0010000\\0} \in X$ *then* X *is known.*

Proof. We assume $\substack{000\cdot\cdot\cdot\cdot\\0} \notin X$, $\substack{001\cdot\cdot\cdot\cdot\\0} \in X$; this excludes $\substack{0000000\\1}$, $\substack{\cdot100000\\0}$, $\substack{\cdot11\cdot\cdot\cdot\cdot\\1}$, $\substack{122\cdot\cdot\cdot\cdot\\1}$, $\substack{123\cdot\cdot\cdot\cdot\\2}$, $\substack{\cdot34\cdot\cdot\cdot\cdot\\2}$, $\substack{2454321\\3}$, giving $\substack{2354321\\2}$, $\substack{246\cdot\cdot\cdot1\\3} \in X$ by default. Suppose $\substack{1000000\\0} \in X$; this excludes $\substack{011\cdot\cdot\cdot\cdot\\0}$, $\substack{012\cdot\cdot\cdot\cdot\\1}$, $\substack{1354321\\\cdot}$, giving $\substack{2354321\\3} \in X$ by default; to exclude $-\substack{0100000\\0}$ we must have $\substack{2454321\\2} \in X$, which excludes $\substack{001\cdot\cdot\cdot\cdot\\1}$; so

$$\begin{aligned} X &\subset \{\substack{246\cdot\cdot\cdot\cdot\\3}, \substack{2454321\\2}, \substack{2354321\\\cdot}, \substack{124\cdot\cdot\cdot\cdot\\2}, \substack{123\cdot\cdot\cdot\cdot\\1}, \substack{112\cdot\cdot\cdot\cdot\\1}, \substack{111\cdot\cdot\cdot\cdot\\0}, \substack{001\cdot\cdot\cdot\cdot\\0}, \substack{1000000\\0}\} \\ &\subset w_2w_3w_4w_5w_6w_7w_8(\Omega). \end{aligned}$$

So we may assume $\substack{1000000\\0} \notin X$, which gives $\substack{1354321\\2} \in X$ by default. By Corollary 3.2 and the previous lemmas in this section, we may assume $X \cap \{\substack{\cdot11\cdot\cdot\cdot\cdot\\0}\}$ is stable under $\langle w_5, w_6, w_7, w_8\rangle$. To exclude $-\substack{0000000\\1}$ we must have some root $\substack{001\cdot\cdot\cdot\cdot\\1}$ or some root $\substack{\cdot354321\\3}$ present, any of which excludes some root $\substack{\cdot11\cdot\cdot\cdot\cdot\\0}$; using $\langle w_1\rangle$ we may assume $\substack{111\cdot\cdot\cdot\cdot\\0} \notin X$, which gives $\substack{1354321\\3} \in X$ by default. To exclude $-\substack{1100000\\0}$ we must have $\substack{2454321\\2} \in X$, which excludes $\substack{001\cdot\cdot\cdot\cdot\\1}$. If some root $\substack{112\cdot\cdot\cdot\cdot\\1}$ were absent, by using $\langle w_5, w_6, w_7, w_8\rangle$ we could assume $\substack{1121000\\1} \notin X$, and then applying $w_1w_2w_3w_4w_3w_2w_1$ would produce a positive set meeting $\{\substack{00\cdot\cdot\cdot\cdot\cdot\\0}\}$ in a proper non-empty subset of $\{\substack{001\cdot\cdot\cdot\cdot\\0}\}$. Thus we may assume $\substack{112\cdot\cdot\cdot\cdot\\1} \in X$, which excludes $\substack{011\cdot\cdot\cdot\cdot\\0}$, $\substack{012\cdot\cdot\cdot\cdot\\1}$, $\substack{123\cdot\cdot\cdot\cdot\\1}$, giving $\substack{124\cdot\cdot\cdot\cdot\\2}$, $\substack{2354321\\3} \in X$ by default; so

$$\begin{aligned} X &= \{\substack{246\cdot\cdot\cdot\cdot\\3}, \substack{2454321\\2}, \substack{\cdot354321\\\cdot}, \substack{124\cdot\cdot\cdot\cdot\\2}, \substack{112\cdot\cdot\cdot\cdot\\1}, \substack{001\cdot\cdot\cdot\cdot\\0}\} \\ &= w_2w_3w_4w_5w_6w_7w_8w_1w_2w_4w_3w_5w_4w_2w_6w_5w_4w_3w_7w_6w_5w_4w_2(X^{13}_{30}). \end{aligned}$$

This proves the lemma. □

Lemma 8.25. *If* $\substack{0000000\\1} \in X$ *then* X *is known.*

Proof. We assume $\substack{00\cdot\cdot\cdot\cdot\cdot\\0} \notin X$, $\substack{00\cdot\cdot\cdot\cdot\cdot\\1} \in X$; this excludes $\substack{\cdot1\cdot\cdot\cdot\cdot\cdot\\0}$, $\substack{12\cdot\cdot\cdot\cdot\cdot\\1}$, $\substack{\cdot3\cdot\cdot\cdot\cdot\cdot\\2}$, $\substack{2454321\\2}$, giving $\substack{2\cdot\cdot\cdot\cdot\cdot1\\3} \in X$ by default. Suppose $\substack{1000000\\0} \in X$; this excludes $\substack{01\cdot\cdot\cdot\cdot\cdot\\1}$, $\substack{1354321\\3}$, so

$$\begin{aligned} X &\subset \{\substack{2\cdot\cdot\cdot\cdot\cdot\cdot\\3}, \substack{12\cdot\cdot\cdot\cdot\cdot\\2}, \substack{11\cdot\cdot\cdot\cdot\cdot\\1}, \substack{00\cdot\cdot\cdot\cdot\cdot\\1}, \substack{1000000\\0}\} \\ &\subset w_3w_4w_5w_6w_7w_8(\Omega). \end{aligned}$$

So we may assume $\substack{1000000\\0} \notin X$, which gives $\substack{1354321\\3} \in X$ by default. We cannot have $\substack{\cdot1\cdot\cdot\cdot\cdot\cdot\\1} \in X$; using $\langle w_1, w_4, w_5, w_6, w_7, w_8\rangle$ we may assume $\substack{0110000\\1} \notin X$. Thus $X = X_c \cup X_a$ where

$$\begin{aligned} X_c &= \{\substack{\cdot\cdot\cdot\cdot\cdot\cdot\cdot\\3}, \substack{00\cdot\cdot\cdot\cdot\cdot\\1}\}, \\ X_a &\subset \{\substack{12\cdot\cdot\cdot\cdot\cdot\\2}, \substack{\cdot12\cdot\cdot\cdot\cdot\\1}, \substack{111\cdot\cdot\cdot\cdot\\1}, \substack{0111\cdot\cdot\cdot\\1}\}; \end{aligned}$$

set

$$w = w_2w_4w_3,$$

then we have $w(X) \subset \Phi^+$, and $\substack{0010000\\0} = w(\substack{0000000\\1}) \in w(X_c)$. This proves the lemma. □

Lemma 8.26. *If* $\substack{0100000\\0} \in X$ *then* X *is known.*

Proof. We assume $\substack{00\cdot\cdot\cdot\cdot\cdot\\\cdot} \notin X$, $\substack{01\cdot\cdot\cdot\cdot\cdot\\\cdot} \in X$; this excludes $\substack{1000000\\0}$, $\substack{11\cdot\cdot\cdot\cdot\cdot\\\cdot}$, $\substack{12\cdot\cdot\cdot\cdot\cdot\\\cdot}$, $\substack{23\cdot\cdot\cdot\cdot\cdot\\\cdot}$, giving $\substack{13\cdot\cdot\cdot\cdot\cdot\\\cdot}$, $\substack{24\cdot\cdot\cdot\cdot1\\\cdot} \in X$ by default. However, now $-\substack{1000000\\0}$ cannot be excluded; so no sets require consideration. This proves the lemma. □

LEMMA 8.27. *If* $\begin{smallmatrix}1000000\\0\end{smallmatrix} \in X$ *then* X *is known.*

PROOF. We cannot have $\begin{smallmatrix}1\cdots\cdots\\\cdot\end{smallmatrix} \in X$, so no sets require consideration. This proves the lemma. □

Combining the various lemmas in this section we have proved the following.

THEOREM 8.28. *If* X *is a maximal abelian set in a root system of type* E_8, *then a* W*-translate of* X *lies in* $\mathcal{S}(E_8)$.

8.4. Stabilizers and structure of maximal abelian sets

For each set $X \in \mathcal{S}(E_8)$ we shall determine its stabilizer W_X in W, and find the W_X-orbits on X. Recall that for $\beta \in X$ the orthogonality count $o(\beta)$ is simply the number of roots in X which are orthogonal to β. If in fact X is radical, we may read off the orthogonality counts from the graph Γ_X. Here, just as we saw in the E_7 case, two roots represented by (black or white) edges in Γ_X are orthogonal if and only if the edges either meet at a vertex and are of different colours, or do not meet and are of the same colour. Suppose Γ_X has e (black) edges. We have $o(\begin{smallmatrix}2465432\\3\end{smallmatrix}) = 0$; for any $\beta \in X$ represented by a (black) edge in Γ_X which meets t others, we have $o(\beta) = (12 - t) + (e - t - 1) = 11 + e - 2t$; for any $\beta \in X$ represented by an absent (white) edge in Γ_X which meets t (black) edges, we have $o(\beta) = (15 - (e - t)) + t = 15 - e + 2t$. (Again we observe that, as given by Lemma 3.3, $o(\beta)$ therefore has the same parity for all $\beta \in X \setminus \{\rho\}$.)

As in the E_7 case we also have the near-radical sets to consider. The seven sets X with $|X \setminus \Omega| = 2$ may be handled by inspection. If X is a set with $X \setminus \Omega = \{\begin{smallmatrix}2343210\\2\end{smallmatrix}\}$, we have $o(\begin{smallmatrix}2465432\\3\end{smallmatrix}) = 1$ and $o(\begin{smallmatrix}2343210\\2\end{smallmatrix}) = 17$, since $\begin{smallmatrix}2343210\\2\end{smallmatrix}$ is orthogonal to $\begin{smallmatrix}2465432\\3\end{smallmatrix}$ and the 16 roots of the form $\begin{smallmatrix}1\cdots\cdot 1\\\cdot\end{smallmatrix}$ in X. For the other roots the orthogonality count is most simply obtained by taking the graph on vertices $\{1, 2, 3, 4, 5, 6\}$ representing X, adding vertices 7 and 8 and edges $i8$ for $i < 8$, and treating the resulting graph as above, except that for roots represented by (black or white) edges ij with $i, j < 7$ the orthogonality count must be increased by one because of the presence of $\begin{smallmatrix}2343210\\2\end{smallmatrix}$.

Again we use the method employed in the E_6 and E_7 analyses. In the vast majority of cases W_X must fix $\begin{smallmatrix}2465432\\3\end{smallmatrix}$, and thus must lie in $\langle w_1, w_2, w_3, w_4, w_5, w_6, w_7\rangle$. We begin with the sets other than the radical sets $X_{29}^1, \ldots, X_{29}^{430}$; here we work through the possibilities for X in turn. In a few cases we shall use the notation τ for the high root $\begin{smallmatrix}1232100\\2\end{smallmatrix}$ of the E_6 subsystem.

If $X = X_{22}$ we must fix $\{\begin{smallmatrix}2\cdots\cdots\\\cdot\end{smallmatrix}\}, \{\begin{smallmatrix}1343321\\2\end{smallmatrix}, \begin{smallmatrix}1243221\\2\end{smallmatrix}, \begin{smallmatrix}1233211\\2\end{smallmatrix}, \begin{smallmatrix}1233210\\1\end{smallmatrix}, \begin{smallmatrix}1221000\\1\end{smallmatrix}, \begin{smallmatrix}1121100\\1\end{smallmatrix}, \begin{smallmatrix}1111110\\1\end{smallmatrix}, \begin{smallmatrix}1111111\\0\end{smallmatrix}\}$ ($o(\beta) = 5, 14$ respectively). Here we first note that each of the roots in the first set is orthogonal to four in the second, in such a way that we may identify the two sets with the affine hyperplanes and vectors of a 3-dimensional space over the field of two elements; consequently, W_X is isomorphic to a subgroup of $AGL_3(2)$. We set

$$G = \langle w_2w_8, w_3w_6, w_4w_7, w_4{w_7}^{w_3w_6w_5}\rangle;$$

then G is 3-transitive on the set of roots with $o(\beta) = 14$, and the stabilizer of $\begin{smallmatrix}1343321\\2\end{smallmatrix}$, $\begin{smallmatrix}1243221\\2\end{smallmatrix}$ and $\begin{smallmatrix}1233211\\2\end{smallmatrix}$ contains $\langle w_4{w_7}^{w_3w_6w_5}, w_4{w_7}^{w_5w_6w_5}\rangle$ (note that $w_4{w_7}^{w_5w_6w_5} = (w_4{w_7}^{w_5w_6w_5})^{w_3w_6} \in G$), so that $|G| \geq 8.7.6.4 = 1344 = |AGL_3(2)|$. Thus $W_X = G$.

If $X = X_{28}$ we must fix $\{\substack{24\cdot\cdot\cdot\cdot\cdot\\\cdot}\}, \{\substack{\cdot3\cdot\cdot\cdot\cdot\cdot\\\cdot}\}, \{\substack{1244321\\2}, \substack{1232221\\2}, \substack{1232211\\1}, \substack{1232100\\2}, \substack{1232110\\1}, \substack{1222210\\1}, \substack{1222111\\1}\}$ ($o(\beta) = 5, 11, 17$ respectively). Here we first note that each of the roots in the first set is orthogonal to three in the third, in such a way that we may identify the two sets with the points and lines of the Fano plane; moreover, each root in the first set is orthogonal to just two in the second, the two being interchanged by w_1, which fixes each root in the first and third sets. Consequently W_X is the direct product of $\langle w_1 \rangle$ and a subgroup of $L_3(2)$. We set

$$G = \langle w_1, w_2w_7, w_6w_8, w_4{w_6}^{w_5} \rangle;$$

this is 2-transitive on $\{\substack{24\cdot\cdot\cdot\cdot\cdot\\\cdot}\}$ and the stabilizer of $\substack{2465432\\3}$ and $\substack{2465431\\3}$ contains $\langle w_1, w_4w_6, w_4{w_6}^{w_5} \rangle$ (note that $w_4w_6 = [w_6w_8, w_4{w_6}^{w_5}] \in G$), so that $|G| \geq 7.6.8 = 336 = 2|L_3(2)|$. Thus $W_X = G$.

If $X = X_{29}^{431}$ we must fix each of $\{\substack{246543\cdot\\3}\}, \substack{2465421\\3}, \{\substack{\cdot\cdot\cdot\cdot321\\\cdot}\}, \{\substack{\cdot\cdot43221\\2}, \substack{\cdot\cdot22221\\1}\}, \{\substack{1233211\\2}, \substack{1232210\\2}, \substack{1233210\\1}, \substack{1232211\\1}\}$ ($o(\beta) = 2, 6, 10, 14, 18$ respectively). We set

$$G = \langle w_1, w_3, w_2w_8, w_2{w_5}^{w_4} \rangle;$$

then G is transitive on $\{\substack{\cdot\cdot\cdot\cdot321\\\cdot}\}$, so we may fix $\beta_1 = \substack{2465321\\3}$. We must then fix $\{\substack{\cdot\cdot\cdot3321\\\cdot}\}, \{\substack{\cdot\cdot22221\\1}\}, \{\substack{1232210\\2}, \substack{1232211\\1}\}$ (by orthogonality to β_1) and hence $\{\substack{\cdot\cdot\cdot4321\\\cdot}\}, \{\substack{\cdot\cdot43221\\2}\}, \{\substack{1233211\\2}, \substack{1233210\\1}\}$; we must then fix $\{\substack{\cdot\cdot44321\\2}\}$ (by orthogonality to both of $\{\substack{1232210\\2}, \substack{1232211\\1}\}$), $\{\substack{\cdot\cdot54321\\\cdot}\}, \{\substack{1233321\\\cdot}\}$ (by orthogonality to one of $\{\substack{1232210\\2}, \substack{1232211\\1}\}$) and hence $\substack{2464321\\3}, \{\substack{\cdot\cdot43321\\2}\}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_1, w_3, w_2w_8 \rangle$ giving transitivity on $\{\substack{1233321\\\cdot}\}$, so we may fix $\beta_2 = \substack{1233321\\2}$. We must then fix $\{\substack{\cdot\cdot54321\\2}\}, \substack{1233210\\1}, \substack{1232211\\1}$ (by orthogonality to β_2) and hence $\{\substack{\cdot\cdot54321\\3}\}, \substack{1233211\\2}, \substack{1232210\\2}$; we must then fix $\substack{2465432\\3}$ (by orthogonality to $\substack{1232210\\2}$) and hence $\substack{2465431\\3}$. Inside $\mathrm{stab}_G(\beta_1, \beta_2)$ we have $\langle w_1, w_3 \rangle$, which acts as S_3 on $\{\substack{\cdot\cdot54321\\3}\}$, so we may fix all of these roots. We then have fixed all of the roots $\substack{\cdot\cdot\cdot\cdot\cdot\cdot\cdot\\3}$, which span $\mathbb{R}\Phi$; so $W_X = G$.

If $X = X_{29}^{432}$ we must fix $\substack{2465432\\3}, \{\substack{24654\cdot1\\3}\}, \{\substack{\cdot\cdot\cdot\cdot321\\\cdot}\}, \{\substack{2343221\\2}, \substack{1\cdot43211\\2}, \substack{1233221\\2}, \substack{1232221\\1}, \substack{1\cdot22211\\1}, \substack{0122221\\1}\}, \{\substack{1232210\\2}, \substack{1233210\\1}\}$ ($o(\beta) = 2, 4, 10, 16, 18$ respectively). We set

$$G = \langle w_3, w_2w_5, w_1w_4w_7, w_2{w_5}^{w_4} \rangle;$$

then G is transitive on $\{\substack{\cdot\cdot\cdot\cdot321\\\cdot}\}$, so we may fix $\beta_1 = \substack{2465321\\3}$. We must then fix $\{\substack{\cdot\cdot\cdot3321\\\cdot}\}, \{\substack{1232221\\1}, \substack{1\cdot22211\\1}, \substack{0122221\\1}\}, \substack{1232210\\2}$ (by orthogonality to β_1) and hence $\{\substack{\cdot\cdot\cdot4321\\\cdot}\}, \{\substack{2343221\\2}, \substack{1\cdot43211\\2}, \substack{1233221\\2}\}, \substack{1233210\\1}$; we must then fix $\{\substack{\cdot\cdot\cdot4321\\3}\}, \{\substack{\cdot\cdot\cdot3321\\2}\}$ (by orthogonality to $\substack{1233210\\1}$) and hence $\{\substack{\cdot\cdot\cdot4321\\2}\}, \substack{1233321\\1}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_3, w_1w_4w_7 \rangle$ giving transitivity on $\{\substack{\cdot\cdot\cdot4321\\3}\}$, so we may fix $\beta_2 = \substack{2464321\\3}$. We must then fix $\{\substack{\cdot\cdot44321\\2}\}, \substack{1233321\\2}, \substack{1233221\\2}, \{\substack{1\cdot22211\\1}, \substack{0122221\\1}\}$ (by orthogonality to β_2) and hence $\{\substack{\cdot\cdot54321\\2}\}, \{\substack{\cdot\cdot43321\\2}\}, \{\substack{2343221\\2}, \substack{1\cdot43211\\2}\}, \substack{1232221\\1}$; we must then fix $\substack{2465421\\3}, \{\substack{1\cdot43211\\2}\}, \{\substack{1\cdot22211\\1}\}$ (by orthogonality to $\substack{1233221\\2}$) and hence $\substack{2465431\\3}, \substack{2343221\\2}, \substack{0122221\\1}$; we must then fix $\substack{1354321\\3}, \substack{1354321\\2}, \{\substack{1\cdot44321\\2}\}, \{\substack{1\cdot43321\\2}\}$ (by orthogonality to $\substack{2343221\\2}$) and hence $\{\substack{2\cdot54321\\3}\}, \{\substack{2\cdot54321\\2}\}, \substack{2344321\\2}, \substack{2343321\\2}$. Inside $\mathrm{stab}_G(\beta_1, \beta_2)$ we have $\langle w_3 \rangle$, which acts as S_2 on $\{\substack{2\cdot54321\\3}\}$, so we may fix both of these roots. We then have fixed all of the roots $\substack{\cdot\cdot\cdot\cdot\cdot\cdot\cdot\\3}$, which span $\mathbb{R}\Phi$; so $W_X = G$.

If $X = X^1_{30}$ we must fix $\substack{2465432\\3}$, $\{\substack{2\cdot\cdot\cdot\cdot\cdot1\\3}\}$, $\substack{1354321\\3}$, $\{\substack{2\cdot\cdot\cdot\cdot\cdot1\\2}\}$, $\{\substack{13\cdot\cdot\cdot21\\2}, \substack{124\cdot321\\2}, \substack{123\cdot221\\2}, \substack{12\cdot\cdot\cdot11\\2}\}$, $\substack{2343210\\2}$, $\{\substack{1232111\\1}, \substack{1222211\\1}, \substack{1122221\\1}\}$ ($o(\beta) = 1, 7, 10, 11, 14, 17, 18$ respectively). We set

$$G = \langle w_5, w_4w_6, w_3w_7\rangle;$$

then G is transitive on $\{\substack{13\cdot\cdot\cdot21\\2}, \substack{124\cdot321\\2}, \substack{123\cdot221\\2}, \substack{12\cdot\cdot\cdot11\\2}\}$, so we may fix $\beta_1 = \substack{1354321\\2}$. We must then fix $\{\substack{2\cdot54321\\3}\}$, $\{\substack{234\cdot\cdot\cdot1\\2}\}$, $\{\substack{123\cdot2\cdot1\\2}, \substack{1232111\\2}\}$, $\{\substack{1222211\\1}, \substack{1122221\\1}\}$ (by orthogonality to β_1) and hence $\{\substack{246\cdot\cdot\cdot1\\3}\}$, $\{\substack{2\cdot54321\\2}\}$, $\{\substack{1\cdot4\cdot321\\2}, \substack{1343221\\2}, \substack{1243211\\2}\}$, $\substack{1232111\\1}$; we must then fix $\{\substack{24654\cdot1\\3}\}$, $\{\substack{234\cdot321\\2}\}$, $\{\substack{1\cdot4\cdot321\\2}\}$, $\{\substack{123\cdot2\cdot1\\2}\}$ (by orthogonality to $\substack{1232111\\1}$) and hence $\{\substack{246\cdot321\\3}\}$, $\{\substack{23432\cdot1\\2}\}$, $\{\substack{1343221\\2}, \substack{1243211\\2}\}$, $\substack{1232111\\2}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_5, w_3w_7\rangle$ giving transitivity on $\{\substack{1\cdot4\cdot321\\2}\}$, so we may fix $\beta_2 = \substack{1344321\\2}$. We must then fix each of $\substack{2464321\\3}$, $\substack{2354321\\3}$, $\substack{2354321\\2}$, $\substack{2343321\\2}$, $\substack{1243321\\2}$, $\substack{1243211\\2}$, $\{\substack{12322\cdot1\\2}\}$ and $\substack{1122221\\1}$ (by orthogonality to β_2) and hence $\substack{2465321\\3}$, $\substack{2454321\\3}$, $\substack{2454321\\2}$, $\substack{2344321\\2}$, $\{\substack{1343321\\2}, \substack{1244321\\2}\}$, $\substack{1343221\\2}$, $\{\substack{12332\cdot1\\2}\}$ and $\substack{1222211\\1}$; we must then fix $\substack{2465431\\3}$, $\substack{2343221\\2}$, $\substack{1244321\\2}$, $\substack{1233221\\2}$, $\substack{1232221\\2}$ (by orthogonality to $\substack{1222211\\1}$), by which point all roots are fixed; so $W_X = G$.

If $X = X^2_{30}$ we must fix $\substack{2465432\\3}$, $\substack{2465431\\3}$, $\substack{1354321\\3}$, $\substack{2343211\\2}$, $\{\substack{1\cdot\cdot2221\\1}\}$ and $\substack{1233211\\1}$ ($o(\beta) = 1, 4, 11, 14, 15, 19$ respectively), $\{\substack{2\cdot\cdot4321\\3}\}$, $\{\substack{2343\cdot21\\2}\}$, $\{\substack{1\cdot\cdot3\cdot21\\2}\}$ ($o(\beta) = 8, 10, 13$ respectively, orthogonal to $\substack{1233211\\1}$), $\{\substack{2465\cdot21\\3}\}$, $\{\substack{2\cdot\cdot4321\\2}\}$, $\{\substack{1\cdot\cdot4321\\2}\}$ ($o(\beta) = 8, 10, 13$ respectively, not orthogonal to $\substack{1233211\\1}$), $\substack{2343210\\2}$ ($o(\beta) = 17$, orthogonal to $\substack{2465432\\3}$), $\{\substack{1232\cdot11\\2}\}$ ($o(\beta) = 17$, not orthogonal to $\substack{2465432\\3}$). We set

$$G = \langle w_3, w_4, w_6\rangle;$$

then G acts as S_3 on $\{\substack{2\cdot\cdot4321\\3}\}$ and independently as S_2 on $\{\substack{2465\cdot21\\3}\}$, so we may fix all of these roots. We then have fixed all of the roots $\substack{\cdot\cdot\cdot\cdot\cdot\cdot\cdot\\3}$, which span $\mathbb{R}\Phi$; so $W_X = G$.

If $X = X^3_{30}$ we must fix $\substack{2465432\\3}$, $\{\substack{24654\cdot1\\3}\}$, $\substack{1354321\\3}$, $\{\substack{23432\cdot1\\2}\}$, $\substack{1232111\\1}$ ($o(\beta) = 1, 6, 11, 12, 19$ respectively), $\{\substack{2\cdot54321\\3}\}$, $\{\substack{234\cdot321\\2}\}$, $\{\substack{1\cdot4\cdot321\\2}\}$ ($o(\beta) = 8, 10, 13$ respectively, orthogonal to $\substack{1232111\\1}$), $\{\substack{246\cdot321\\3}\}$, $\{\substack{2\cdot54321\\2}\}$, $\substack{1354321\\2}$ ($o(\beta) = 8, 10, 13$ respectively, not orthogonal to $\substack{1232111\\1}$), $\{\substack{1343221\\2}, \substack{1243211\\2}\}$ ($o(\beta) = 15$, orthogonal to both of $\{\substack{234\cdot321\\2}\}$), $\{\substack{123\cdot2\cdot1\\2}\}$ ($o(\beta) = 15$, orthogonal to one of $\{\substack{234\cdot321\\2}\}$), $\substack{1233321\\1}$ ($o(\beta) = 15$, orthogonal to neither of $\{\substack{234\cdot321\\2}\}$), $\substack{2343210\\2}$ ($o(\beta) = 17$, orthogonal to $\substack{1233321\\1}$), $\{\substack{1222211\\1}, \substack{1122221\\1}\}$ ($o(\beta) = 17$, not orthogonal to $\substack{1233321\\1}$). We set

$$G = \langle w_5, w_3w_7\rangle;$$

then G is transitive on $\{\substack{123\cdot2\cdot1\\2}\}$, so we may fix $\beta_1 = \substack{1233221\\2}$. We must then fix $\substack{2465421\\3}$, $\substack{2464321\\3}$, $\substack{2343321\\2}$, $\substack{2343211\\2}$, $\{\substack{1\cdot43321\\2}\}$, $\substack{1243211\\2}$, $\substack{1232211\\2}$, $\substack{1222211\\1}$ (by orthogonality to β_1) and hence $\substack{2465431\\3}$, $\substack{2465321\\3}$, $\substack{2344321\\2}$, $\substack{2343221\\2}$, $\{\substack{1\cdot44321\\2}\}$, $\substack{1343221\\2}$, $\{\substack{1233211\\2}, \substack{1232221\\2}\}$, $\substack{1122221\\1}$; we must then fix $\substack{2454321\\3}$, $\substack{2454321\\2}$, $\substack{1344321\\2}$, $\substack{1343321\\2}$, $\substack{1233211\\2}$ (by orthogonality to $\substack{1122221\\1}$), by which point all roots are fixed; so $W_X = G$.

If $X = X^4_{30}$ we must fix each of $\substack{2465432\\3}$, $\{\substack{24654\cdot1\\3}\}$, $\{\substack{2\cdot\cdot\cdot321\\3}\}$, $\{\substack{2\cdot\cdot\cdot321\\2}\}$, $\substack{1354321\\3}$, $\{\substack{23432\cdot1\\2}\}$, $\{\substack{1\cdot\cdot\cdot321\\2}\}$, $\{\substack{1343221\\2}, \substack{12\cdot3211\\2}, \substack{1232221\\2}\}$ ($o(\beta) = 1, 6, 8, 10, 11, 12, 13, 15$ respectively), $\substack{1232111\\2}$ ($o(\beta) = 17$, orthogonal to both of $\{\substack{24654\cdot1\\3}\}$), $\{\substack{1233221\\1}, \substack{12\cdot2211\\1}, \substack{1122221\\1}\}$ ($o(\beta) = 17$, orthogonal to one of $\{\substack{24654\cdot1\\3}\}$), $\substack{2343210\\2}$ ($o(\beta) = 17$, orthogonal to neither of $\{\substack{24654\cdot1\\3}\}$). We set

$$G = \langle w_4, w_3w_5w_7\rangle;$$

then G is transitive on $\{{1343221 \atop 2}, {12\cdot3211 \atop 2}, {1232221 \atop 2}\}$, so we may fix $\beta_1 = {1232221 \atop 2}$. We must then fix ${2465421 \atop 3}, {2465321 \atop 3}, \{{2\cdot\cdot4321 \atop 2}\}, {2343211 \atop 2}, \{{1\cdot\cdot4321 \atop 2}\}, \{{12\cdot3211 \atop 2}\}, \{{1233221 \atop 1}, {12\cdot2211 \atop 1}\}$ (by orthogonality to β_1) and hence each of ${2465431 \atop 3}, \{{2\cdot\cdot4321 \atop 3}\}, {2343321 \atop 2}, {2343221 \atop 2}, \{{1\cdot\cdot3321 \atop 2}\}, {1343221 \atop 2}, {1122221 \atop 1}$; we must then fix $\{{24\cdot4321 \atop 3}\}, {2454321 \atop 2}, \{{13\cdot4321 \atop 2}\}, {1343321 \atop 2}, \{{12\cdot2211 \atop 1}\}$ (by orthogonality to ${1122221 \atop 1}$) and hence ${2354321 \atop 3}, \{{23\cdot4321 \atop 2}\}, {1244321 \atop 2}, \{{12\cdot3321 \atop 2}\}, {1233221 \atop 1}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_4\rangle$ giving transitivity on $\{{24\cdot4321 \atop 3}\}$, so we may fix $\beta_2 = {2464321 \atop 3}$. We must then fix ${2344321 \atop 2}, {1344321 \atop 2}, {1233321 \atop 2}, {1233211 \atop 2}, {1222211 \atop 1}$ (by orthogonality to β_2), by which point all roots are fixed; so $W_X = G$. (In this case we note also that the set of roots with $o(\beta) = 13$ is in fact a union of the two W_X-orbits $\{{13\cdot4321 \atop 2}, {12\cdot3321 \atop 2}\}$ and $\{{1343321 \atop 2}, {1244321 \atop 2}\}$.)

If $X = X_{30}^5$ we must fix $\{{246543\cdot \atop 3}\}, \{{2\cdot\cdot\cdot\cdot21 \atop \cdot}\}, \{{1\cdot\cdot\cdot\cdot21 \atop \cdot}\}, \{{234321\cdot \atop 2}\}$ ($o(\beta) = 1, 9, 12, 17$ respectively). We set

$$G = \langle w_2, w_3, w_4, w_5, w_6, w_8\rangle;$$

then G is transitive on $\{{1\cdot\cdot\cdot\cdot21 \atop \cdot}\}$, so we may fix $\beta_1 = {1354321 \atop 3}$. We must then fix $\{{2\cdot\cdot\cdot\cdot21 \atop 2}\}, \{{1\cdot\cdot\cdot\cdot21 \atop 1}\}$ (by orthogonality to β_1) and hence $\{{2\cdot\cdot\cdot\cdot21 \atop 3}\}, \{{1\cdot\cdot\cdot\cdot21 \atop 2}\}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_3, w_4, w_5, w_6, w_8\rangle$, which acts as S_5 on $\{{2\cdot\cdot\cdot\cdot21 \atop 3}\}$ and independently as S_2 on $\{{246543\cdot \atop 3}\}$, so we may fix all of these roots. We then have fixed all of the roots ${\cdot\cdot\cdot\cdot\cdot\cdot\cdot \atop 3}$, which span $\mathbb{R}\Phi$, so $W_X = G$.

If $X = X_{30}^6$ we must fix ${2465432 \atop 3}, {2465431 \atop 3}, {2465421 \atop 3}, \{{2\cdot\cdot\cdot321 \atop \cdot}\}, {2343221 \atop 2}, \{{1\cdot\cdot\cdot321 \atop \cdot}\}, \{{1\cdot\cdot3221 \atop 2}, {1\cdot\cdot2221 \atop 1}\}, {2343211 \atop 2}, {2343210 \atop 2}, \{{1232211 \atop 2}, {1233211 \atop 1}\}$ ($o(\beta) = 1, 3, 7, 9, 11, 12, 14, 15, 17, 18$ respectively). We set

$$G = \langle w_3, w_4, w_2w_5\rangle;$$

then G is transitive on $\{{1\cdot\cdot\cdot321 \atop \cdot}\}$, so we may fix $\beta_1 = {1354321 \atop 3}$. We must then fix $\{{2\cdot\cdot\cdot321 \atop 2}\}, {1233321 \atop 1}, \{{1\cdot\cdot2221 \atop 1}\}, {1233211 \atop 1}$ (by orthogonality to β_1) and hence $\{{2\cdot\cdot\cdot321 \atop 3}\}, \{{1\cdot\cdot\cdot321 \atop 2}\}, \{{1\cdot\cdot3221 \atop 2}\}, {1232211 \atop 2}$; we must then fix ${2465321 \atop 3}, \{{2\cdot\cdot4321 \atop 2}\}, \{{1\cdot\cdot4321 \atop 2}\}$ (by orthogonality to ${1232211 \atop 2}$) and hence $\{{2\cdot\cdot4321 \atop 3}\}, {2343321 \atop 2}, \{{1\cdot\cdot3321 \atop 2}\}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_3, w_4\rangle$, which acts as S_3 on $\{{2\cdot\cdot4321 \atop 3}\}$, so we may fix all of these roots. We then have fixed all of the roots ${\cdot\cdot\cdot\cdot\cdot\cdot\cdot \atop 3}$, which span $\mathbb{R}\Phi$, so $W_X = G$.

If $X = X_{30}^7$ we must fix ${2465432 \atop 3}, {2465431 \atop 3}, \{{2465\cdot21 \atop 3}\}, \{{2343\cdot21 \atop 2}\}, \{{13\cdot4321 \atop \cdot}\}, {2343211 \atop 2}, {2343210 \atop 2}, {1221111 \atop 1}$ ($o(\beta) = 1, 5, 7, 11, 12, 13, 17, 20$ respectively), $\{{23\cdot4321 \atop \cdot}\}, \{{12\cdot3\cdot21 \atop \cdot}\}$ ($o(\beta) = 9, 14$ respectively, orthogonal to ${1221111 \atop 1}$), $\{{24\cdot4321 \atop \cdot}\}, \{{1343\cdot21 \atop 2}\}$ ($o(\beta) = 9, 14$ respectively, not orthogonal to ${1221111 \atop 1}$), $\{{12\cdot3211 \atop \cdot}\}$ ($o(\beta) = 16$, orthogonal to ${2465431 \atop 3}$), ${1122221 \atop 1}$ ($o(\beta) = 16$, not orthogonal to ${2465431 \atop 3}$). We set

$$G = \langle w_2, w_4, w_6\rangle;$$

then G acts as S_3 on $\{{24\cdot4321 \atop \cdot}\}$ and independently as S_2 on $\{{2465\cdot21 \atop 3}\}$, so we may fix all of these roots. We then have fixed all of the roots ${24\cdot\cdot\cdot\cdot\cdot \atop \cdot}, {1122221 \atop 1}$, which span $\mathbb{R}\Phi$, so $W_X = G$.

If $X = X_{30}^8$ we must fix all of ${2465432 \atop 3}, \{{24654\cdot1 \atop 3}\}, \{{2\cdot\cdot\cdot321 \atop \cdot}\}, \{{1\cdot\cdot\cdot321 \atop \cdot}\}, \{{23432\cdot1 \atop 2}\}, \{{1343211 \atop 2}, {12\cdot3221 \atop 2}, {1232211 \atop 2}, {1233211 \atop 1}, {12\cdot2221 \atop 1}, {1122211 \atop 1}\}, {2343210 \atop 2}$ ($o(\beta) = 1, 5, 9, 12, 13, 16, 17$ respectively). We set

$$G = \langle w_4, w_2w_5, w_3w_5w_7\rangle;$$

then G is transitive on $\{{1343211 \atop 2}, {12\cdot3221 \atop 2}, {1232211 \atop 2}, {1233211 \atop 1}, {12\cdot2221 \atop 1}, {1122211 \atop 1}\}$, so we may fix $\beta_1 = {1343211 \atop 2}$. We must then fix each of ${2465431 \atop 3}, \{{23\cdot\cdot321 \atop \cdot}\}, {2343221 \atop 2}, \{{12\cdot\cdot321 \atop \cdot}\}$

and $\{\substack{12\cdot3221\\2}, \substack{12\cdot2221\\1}, \substack{1122211\\1}\}$ (by orthogonality to β_1) and hence $\substack{2465421\\3}, \{\substack{24\cdot\cdot321\\\cdot}\}$, $\substack{2343211\\2}, \{\substack{13\cdot\cdot321\\\cdot}\}, \{\substack{1232211\\2}, \substack{1233211\\1}\}$; we must then fix $\substack{1122211\\1}$ (by orthogonality to $\substack{2465431\\3}$) and hence $\{\substack{12\cdot3221\\2}, \substack{12\cdot2221\\1}\}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_4, w_2w_5\rangle$ giving transitivity on $\{\substack{12\cdot3221\\2}, \substack{12\cdot2221\\1}\}$, so we may fix $\beta_2 = \substack{1243221\\2}$. We must then fix $\{\substack{2454321\\\cdot}\}, \{\substack{234\cdot321\\2}\}, \{\substack{134\cdot321\\2}\}, \{\substack{1233321\\\cdot}\}, \substack{1222221\\1}$ (by orthogonality to β_2) and hence $\{\substack{246\cdot321\\3}\}, \{\substack{2354321\\\cdot}\}, \{\substack{1354321\\\cdot}\}, \{\substack{124\cdot321\\2}\}, \{\substack{1233221\\2}, \substack{1232221\\1}\}$. Inside $\mathrm{stab}_G(\beta_1, \beta_2)$ we have $\langle w_2w_5\rangle$ giving transitivity on $\{\substack{1233221\\2}, \substack{1232221\\1}\}$, so we may fix $\beta_3 = \substack{1233221\\2}$. We must then fix $\substack{2464321\\3}, \substack{2454321\\2}, \substack{2354321\\2}, \substack{2343321\\2}, \substack{1354321\\2}, \substack{1343321\\2}, \substack{1243321\\2}, \substack{1233321\\1}$ (by orthogonality to β_3) and hence $\substack{2465321\\3}, \substack{2454321\\3}, \substack{2354321\\3}, \substack{2344321\\2}, \substack{1354321\\3}, \substack{1344321\\2}$, $\substack{1244321\\2}, \substack{1233321\\2}$; we must then fix $\substack{1232211\\2}$ (by orthogonality to $\substack{2465321\\3}$), by which point all roots are fixed; so $W_X = G$.

If $X = X_{30}^{9}$ we must fix $\substack{2465432\\3}, \substack{2465431\\3}, \{\substack{2465\cdot21\\3}\}, \{\substack{2343\cdot21\\2}\}, \substack{2343211\\2}, \substack{2343210\\2}$ and $\{\substack{1232211\\2}, \substack{1232111\\1}\}$ ($o(\beta) = 1, 5, 7, 11, 13, 17, 18$ respectively), $\substack{2344321\\2}, \{\substack{1\cdot44321\\2}\}$, $\{\substack{1233221\\2}, \substack{1233321\\1}\}, \{\substack{1\cdot22221\\1}\}$ ($o(\beta) = 9, 12, 14, 16$ respectively, orthogonal to both of $\{\substack{1232211\\2}, \substack{1232111\\1}\}$), $\{\substack{2\cdot54321\\\cdot}\}, \{\substack{1354321\\\cdot}\}, \{\substack{1\cdot43\cdot21\\2}\}, \{\substack{1233211\\\cdot}\}$ ($o(\beta) = 9, 12, 14, 16$ respectively, orthogonal to one of $\{\substack{1232211\\2}, \substack{1232111\\1}\}$), $\substack{2464321\\3}$ ($o(\beta) = 9$, orthogonal to neither of $\{\substack{1232211\\2}, \substack{1232111\\1}\}$). We set

$$G = \langle w_3, w_2w_6\rangle;$$

then G is transitive on $\{\substack{1\cdot43\cdot21\\2}\}$, so we may fix $\beta_1 = \substack{1343321\\2}$. We must then fix $\substack{2465321\\3}, \{\substack{2354321\\\cdot}\}, \substack{2343221\\2}, \substack{1244321\\2}, \substack{1243221\\2}, \substack{1233221\\2}, \substack{1122221\\1}$ and $\substack{1232111\\1}$ (by orthogonality to β_1) and hence $\substack{2465421\\3}, \{\substack{2454321\\\cdot}\}, \substack{2343321\\2}, \substack{1344321\\2}, \{\substack{1343221\\2}, \substack{1243321\\2}\}$, $\substack{1233321\\1}, \substack{1222221\\1}, \substack{1232211\\2}$; we must then fix $\substack{2454321\\3}, \substack{2354321\\3}, \substack{1354321\\3}, \substack{1243321\\2}, \substack{1233211\\2}$ (by orthogonality to $\substack{1232111\\1}$), by which point all roots are fixed; so $W_X = G$.

If $X = X_{30}^{10}$ we must fix all of $\substack{2465432\\3}, \{\substack{2\cdot54321\\\cdot}\}, \{\substack{1354321\\\cdot}\}, \{\substack{1\cdot43\cdot\cdot1\\2}\}, \{\substack{1233\cdot\cdot1\\\cdot}\}$, $\substack{2343210\\2}, \{\substack{1\cdot21111\\1}\}$ ($o(\beta) = 1, 9, 12, 14, 16, 17, 18$ respectively), $\{\substack{2465\cdot\cdot1\\3}\}$ and $\substack{2344321\\2}$ ($o(\beta) = 7, 11$ respectively, orthogonal to both of $\{\substack{1\cdot21111\\1}\}$), $\substack{2464321\\3}, \{\substack{2343\cdot\cdot1\\2}\}$ ($o(\beta) = 7, 11$ respectively, orthogonal to neither of $\{\substack{1\cdot21111\\1}\}$). We set

$$G = \langle w_2, w_3, w_6, w_7\rangle;$$

then G acts as S_3 on $\{\substack{2465\cdot\cdot1\\3}\}$ and independently as S_2 on each of $\{\substack{1354321\\\cdot}\}$ and $\{\substack{1\cdot21111\\1}\}$, so we may fix all of these roots. We then have fixed all of the roots $\substack{2465\cdot\cdot\cdot\\3}, \substack{1354321\\\cdot}, \substack{1\cdot21111\\1}$, which span $\mathbb{R}\Phi$; so $W_X = G$.

If $X = X_{30}^{11}$ we must fix $\substack{2465432\\3}, \{\substack{246\cdot\cdot\cdot1\\3}\}, \{\substack{234\cdot\cdot\cdot1\\2}\}, \substack{1354321\\3}, \{\substack{1\cdot4\cdot\cdot\cdot1\\2}\}, \substack{2343210\\2}$, $\substack{1111111\\1}$ ($o(\beta) = 1, 7, 11, 12, 14, 17, 20$ respectively), $\{\substack{2\cdot54321\\2}\}$ and $\{\substack{1233\cdot21\\1}, \substack{1232\cdot11\\1}\}$ ($o(\beta) = 9, 16$ respectively, orthogonal to $\substack{1111111\\1}$), $\{\substack{2\cdot54321\\3}\}$ and $\{\substack{1233211\\2}, \substack{1232221\\2}\}$ ($o(\beta) = 9, 16$ respectively, not orthogonal to $\substack{1111111\\1}$). We set

$$G = \langle w_3, w_6, w_5w_7\rangle;$$

then G is transitive on $\{\substack{1\cdot4\cdot\cdot\cdot1\\2}\}$, so we may fix $\beta_1 = \substack{1344321\\2}$. We must then fix $\substack{2464321\\3}, \substack{2354321\\3}, \substack{2354321\\2}, \{\substack{2343\cdot\cdot1\\2}\}, \{\substack{1243\cdot\cdot1\\2}\}, \substack{1232221\\2}, \{\substack{1232\cdot11\\1}\}$ (by orthogonality to β_1) and hence $\{\substack{2465\cdot\cdot1\\3}\}, \substack{2454321\\3}, \substack{2454321\\2}, \substack{2344321\\2}, \{\substack{1244321\\2}, \substack{1343\cdot\cdot1\\2}\}, \substack{1233211\\2}$ and $\{\substack{1233\cdot21\\1}\}$; we must then fix $\{\substack{2465\cdot21\\3}\}, \substack{2343211\\2}, \{\substack{1244321\\2}, \substack{1343211\\2}\}, \substack{1243211\\2}$ (by orthogonality to $\substack{1232221\\2}$) and hence $\substack{2465431\\3}, \{\substack{2343\cdot21\\2}\}, \{\substack{1343\cdot21\\2}\}, \{\substack{1243\cdot21\\2}\}$; we must then fix $\substack{1244321\\2}$ (by orthogonality to $\substack{2343211\\2}$) and hence $\substack{1343211\\2}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_6\rangle$, which acts as S_2 on $\{\substack{2465\cdot 21\\3}\}$, so we may fix both of these roots. We then have fixed all of the roots $\substack{\cdot\cdot\cdot\cdot\cdot\cdot\cdot\\3}$, which span $\mathbb{R}\Phi$; so $W_X = G$.

If $X = X_{30}^{12}$ we must fix $\substack{2465432\\3}$, $\{\substack{2\cdot 54321\\\cdot}\}$, $\{\substack{1354321\\\cdot}\}$, $\substack{2343210\\2}$, $\{\substack{1221111\\1}, \substack{1122111\\1}\}$ ($o(\beta) = 1, 9, 12, 17, 18$ respectively), $\{\substack{24654\cdot 1\\3}\}$, $\{\substack{1344321\\2}, \substack{1243321\\2}\}$, $\{\substack{1233321\\\cdot}\}$ ($o(\beta) = 7, 14, 16$ respectively, orthogonal to both of $\{\substack{1221111\\1}, \substack{1122111\\1}\}$), $\{\substack{246\cdot 321\\3}\}$, $\{\substack{234\cdot 321\\2}\}$, $\{\substack{1\cdot 432\cdot 1\\2}\}$, $\{\substack{1233221\\2}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{1232221\\1}\}$ ($o(\beta) = 7, 11, 14, 16$ respectively, orthogonal to one of $\{\substack{1221111\\1}, \substack{1122111\\1}\}$), $\{\substack{23432\cdot 1\\2}\}$ ($o(\beta) = 11$, orthogonal to neither of $\{\substack{1221111\\1}, \substack{1122111\\1}\}$). We set

$$G = \langle w_2w_7, w_3w_5w_7\rangle;$$

then G is transitive on $\{\substack{1\cdot 432\cdot 1\\2}\}$, so we may fix $\beta_1 = \substack{1343221\\2}$. We must then fix $\substack{2465421\\3}$, $\{\substack{2354321\\\cdot}\}$, $\substack{2343211\\2}$, $\substack{1243321\\2}$, $\substack{1243211\\2}$, $\{\substack{1232211\\2}, \substack{1233211\\1}\}$ and $\substack{1122111\\1}$ (by orthogonality to β_1) and hence $\substack{2465431\\3}$, $\{\substack{2454321\\\cdot}\}$, $\substack{2343221\\2}$, $\substack{1344321\\2}$, $\{\substack{1343211\\2}, \substack{1243221\\2}\}$, $\{\substack{1233221\\2}, \substack{1232221\\1}\}$, $\substack{1221111\\1}$; we must then fix $\substack{2465321\\3}$, $\substack{2344321\\2}$, $\substack{1243221\\2}$, $\substack{1233221\\2}$, $\substack{1233211\\1}$ (by orthogonality to $\substack{1221111\\1}$) and hence $\substack{2464321\\3}$, $\substack{2343321\\2}$, $\substack{1343211\\2}$, $\substack{1232221\\1}$, $\substack{1232211\\2}$; we must then fix $\substack{2454321\\2}$, $\substack{2354321\\2}$, $\substack{1354321\\2}$, $\substack{1233321\\1}$ (by orthogonality to $\substack{1232211\\2}$), by which point all roots are fixed; so $W_X = G$.

If $X = X_{30}^{13}$ we must fix all of $\substack{2465432\\3}$, $\{\substack{24654\cdot 1\\3}\}$, $\{\substack{2\cdot\cdot\cdot 321\\3}\}$, $\{\substack{2\cdot\cdot\cdot 321\\2}\}$, $\{\substack{23432\cdot 1\\2}\}$, $\{\substack{1\cdot\cdot\cdot 321\\2}\}$, $\substack{2343210\\2}$, $\substack{1111111\\0}$ ($o(\beta) = 1, 6, 8, 10, 12, 13, 17, 21$ respectively), $\{\substack{1\cdot\cdot\cdot 2\cdot 1\\2}\}$ ($o(\beta) = 15$, orthogonal to $\substack{1111111\\0}$), $\substack{1233321\\1}$ ($o(\beta) = 15$, not orthogonal to $\substack{1111111\\0}$). We set

$$G = \langle w_3, w_4, w_5, w_7\rangle;$$

then G acts as S_4 on $\{\substack{2\cdot\cdot\cdot 321\\3}\}$ and independently as S_2 on $\{\substack{24654\cdot 1\\3}\}$, so we may fix all of these roots. We then have fixed all of the roots $\substack{2\cdot\cdot\cdot\cdot\cdot\cdot\\3}$, $\substack{1111111\\0}$, which span $\mathbb{R}\Phi$; so $W_X = G$.

If $X = X_{30}^{14}$ we must fix $\substack{2465432\\3}$, $\substack{2465421\\3}$, $\substack{2343221\\2}$, $\substack{1122221\\1}$ ($o(\beta) = 1, 6, 12, 19$ respectively), $\substack{2343210\\2}$ ($o(\beta) = 17$, orthogonal to $\substack{2465432\\3}$), $\{\substack{1232111\\\cdot}\}$ ($o(\beta) = 17$, orthogonal to $\substack{1122221\\1}$ but not $\substack{2465432\\3}$), $\substack{1222221\\1}$ ($o(\beta) = 17$, orthogonal to neither $\substack{1122221\\1}$ nor $\substack{2465432\\3}$), $\substack{2465431\\3}$, $\{\substack{234\cdot 321\\2}\}$, $\{\substack{134\cdot 321\\2}\}$ ($o(\beta) = 8, 10, 13$ respectively, orthogonal to both of $\{\substack{1232111\\\cdot}\}$), $\{\substack{2454321\\\cdot}\}$, $\{\substack{2354321\\\cdot}\}$, $\{\substack{1354321\\\cdot}\}$ ($o(\beta) = 8, 10, 13$ respectively, orthogonal to one of $\{\substack{1232111\\\cdot}\}$), $\{\substack{246\cdot 321\\3}\}$, $\substack{2343211\\2}$ ($o(\beta) = 8, 10$ respectively, orthogonal to neither of $\{\substack{1232111\\\cdot}\}$), $\substack{1222211\\1}$ ($o(\beta) = 15$, orthogonal to both of $\{\substack{1354321\\\cdot}\}$), $\{\substack{123\cdot 211\\\cdot}\}$ ($o(\beta) = 15$, orthogonal to one of $\{\substack{1354321\\\cdot}\}$), $\{\substack{124\cdot 321\\2}\}$ ($o(\beta) = 15$, not orthogonal to $\substack{1122221\\1}$), $\substack{1343221\\2}$ ($o(\beta) = 15$, orthogonal to $\substack{2465421\\3}$). We set

$$G = \langle w_2, w_5\rangle;$$

then G acts as S_2 on each of $\{\substack{246\cdot 321\\3}\}$ and $\{\substack{2454321\\\cdot}\}$ independently, so we may fix all of these roots. We then have fixed all of the roots $\substack{24\cdot\cdot\cdot\cdot\cdot\\\cdot}$, $\substack{1122221\\1}$, which span $\mathbb{R}\Phi$; so $W_X = G$.

If $X = X_{30}^{15}$ we must fix $\substack{2465432\\3}$, $\substack{2465431\\3}$, $\{\substack{24\cdot\cdot\cdot 21\\\cdot}\}$, $\{\substack{23\cdot\cdot\cdot 21\\\cdot}\}$, $\substack{2343211\\2}$, $\{\substack{13\cdot\cdot\cdot 21\\\cdot}\}$, $\{\substack{124\cdot 321\\2}, \substack{1233221\\2}, \substack{12\cdot 2221\\1}\}$ ($o(\beta) = 1, 6, 8, 10, 12, 13, 15$ respectively), $\substack{1122221\\1}$ ($o(\beta) = 17$, orthogonal to all of $\{\substack{24\cdot\cdot\cdot 21\\\cdot}\}$), $\{\substack{1232\cdot 11\\2}, \substack{1233211\\1}, \substack{1232111\\1}, \substack{1222211\\1}\}$ ($o(\beta) = 17$, orthogonal to two of $\{\substack{24\cdot\cdot\cdot 21\\\cdot}\}$), $\substack{2343210\\2}$ ($o(\beta) = 17$, orthogonal to none of $\{\substack{24\cdot\cdot\cdot 21\\\cdot}\}$). We set

$$G = \langle w_2w_5, w_4w_6\rangle;$$

then G is transitive on $\{\substack{124\cdot321\\2}, \substack{1233221\\2}, \substack{12\cdot2221\\1}\}$, so we may fix $\beta_1 = \substack{1244321\\2}$. We must then fix $\{\substack{24\cdot4321\\\cdot}\}, \{\substack{2343\cdot21\\2}\}, \{\substack{1343\cdot21\\2}\}, \{\substack{12\cdot2221\\1}\}$ and $\{\substack{1232\cdot11\\2}, \substack{1232111\\1}, \substack{1222211\\1}\}$ (by orthogonality to β_1) and hence $\{\substack{2465\cdot21\\3}\}, \{\substack{23\cdot4321\\\cdot}\}, \{\substack{13\cdot4321\\\cdot}\}, \{\substack{1243321\\2}, \substack{1233221\\2}\}, \substack{1233211\\1}$; we must then fix $\{\substack{24\cdot4321\\3}\}, \substack{2354321\\3}, \substack{1354321\\3}, \{\substack{1232\cdot11\\2}\}$ (by orthogonality to $\substack{1233211\\1}$) and hence $\substack{2454321\\2}, \{\substack{23\cdot4321\\2}\}, \{\substack{13\cdot4321\\2}\}, \{\substack{1232111\\1}, \substack{1222211\\1}\}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_4w_6\rangle$ giving transitivity on $\{\substack{12\cdot2221\\1}\}$, so we may fix $\beta_2 = \substack{1232221\\1}$. We must then fix $\substack{2454321\\3}, \substack{2344321\\2}, \substack{1344321\\2}, \substack{1233221\\2}, \substack{1222211\\1}$ (by orthogonality to β_2) and hence $\substack{2464321\\3}, \substack{2354321\\2}, \substack{1354321\\2}, \substack{1243321\\2}, \substack{1232111\\1}$; we must then fix $\substack{2465321\\3}, \substack{2343221\\2}, \substack{1343221\\2}, \substack{1232111\\2}$ (by orthogonality to $\substack{1222211\\1}$), by which point all roots are fixed; so $W_X = G$.

If $X = X_{30}^{16}$ we must fix $\substack{2465432\\3}, \{\substack{2\cdot\cdot\cdot\cdot\cdot1\\3}\}, \{\substack{2\cdot\cdot\cdot\cdot\cdot1\\2}\}, \substack{1354321\\3}, \{\substack{1\cdot\cdot3\cdot\cdot1\\2}\}$ ($o(\beta) = 1, 8, 10, 13, 15$ respectively), $\substack{2343210\\2}$ ($o(\beta) = 17$, orthogonal to $\substack{2465432\\3}$) and $\{\substack{1233\cdot\cdot1\\1}, \substack{1\cdot\cdot1111\\1}\}$ ($o(\beta) = 17$, not orthogonal to $\substack{2465432\\3}$). We set

$$G = \langle w_3, w_4, w_3w_5{w_7}^{w_4w_6w_5}\rangle;$$

then G is transitive on $\{\substack{1\cdot\cdot3\cdot\cdot1\\2}\}$, so we may fix $\beta_1 = \substack{1343321\\2}$. We must then fix each of $\{\substack{2465321\\3}, \substack{2354321\\3}\}, \{\substack{23\cdot4321\\2}, \substack{23432\cdot1\\2}\}, \{\substack{12\cdot32\cdot1\\2}\}$ and $\{\substack{12332\cdot1\\1}, \substack{11\cdot1111\\1}\}$ (by orthogonality to β_1) and hence each of $\{\substack{24654\cdot1\\3}, \substack{24\cdot4321\\3}\}, \{\substack{2454321\\2}, \substack{2343321\\2}\}, \{\substack{13432\cdot1\\2}, \substack{12\cdot3321\\2}\}, \{\substack{1233321\\1}, \substack{1221111\\1}\}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_4, w_3w_5{w_7}^{w_4w_6w_5}\rangle$ giving transitivity on $\{\substack{13432\cdot1\\2}, \substack{12\cdot3321\\2}\}$, so we may fix $\beta_2 = \substack{1343221\\2}$. We must then fix each of $\substack{2465421\\3}, \substack{2354321\\3}, \substack{2343321\\2}, \{\substack{23\cdot4321\\2}, \substack{2343211\\2}\}, \{\substack{12\cdot3321\\2}\}, \{\substack{12\cdot3211\\2}\}, \substack{1233321\\1}$ and $\{\substack{1233211\\1}, \substack{11\cdot1111\\1}\}$ (by orthogonality to β_2) and hence $\{\substack{2465431\\3}, \substack{24\cdot4321\\3}\}, \substack{2465321\\3}, \substack{2454321\\2}, \substack{2343221\\2}, \substack{1343211\\2}, \{\substack{12\cdot3221\\2}\}, \substack{1221111\\1}, \substack{1233221\\1}$; we must then fix each of $\substack{2465431\\3}$, $\{\substack{23\cdot4321\\2}\}$ and $\substack{1233211\\1}$ (by orthogonality to $\substack{1221111\\1}$) and hence $\{\substack{24\cdot4321\\3}\}, \substack{2343211\\2}, \{\substack{11\cdot1111\\1}\}$. Inside $\mathrm{stab}_G(\beta_1, \beta_2)$ we have $\langle w_4\rangle$ giving transitivity on $\{\substack{11\cdot1111\\1}\}$, so we may fix $\beta_3 = \substack{1111111\\1}$. We must then fix $\substack{2464321\\3}, \substack{2354321\\2}, \substack{1243321\\2}, \substack{1243221\\2}, \substack{1243211\\2}$ (by orthogonality to β_3), by which point all roots are fixed; so $W_X = G$.

If $X = X_{30}^{17}$ we must fix $\substack{2465432\\3}, \{\substack{2\cdot\cdot\cdot\cdot\cdot1\\3}\}, \{\substack{2\cdot\cdot\cdot\cdot\cdot1\\2}\}, \substack{1354321\\3}, \{\substack{1343\cdot\cdot1\\2}, \substack{1244321\\2}, \substack{1243221\\2}, \substack{1233321\\2}, \substack{12\cdot3211\\2}, \substack{1232221\\2}\}$ ($o(\beta) = 1, 8, 10, 13$ and 15 respectively), $\substack{2343210\\2}$ ($o(\beta) = 17$, orthogonal to $\substack{2465432\\3}$), $\{\substack{1233\cdot21\\1}, \substack{1232211\\1}, \substack{1222111\\1}, \substack{11\cdot1111\\1}\}$ ($o(\beta) = 17$, not orthogonal to $\substack{2465432\\3}$). We set

$$G = \langle w_4w_6, w_3w_5w_7\rangle;$$

then G is transitive on $\{\substack{1233\cdot21\\1}, \substack{1232211\\1}, \substack{1222111\\1}, \substack{11\cdot1111\\1}\}$, so we may fix $\beta_1 = \substack{1233321\\1}$. We must then fix $\{\substack{2\cdot\cdot\cdot321\\3}\}, \{\substack{23432\cdot1\\2}\}, \{\substack{1\cdot432\cdot1\\2}, \substack{1233211\\2}, \substack{1232221\\2}\}, \{\substack{1222111\\1}, \substack{11\cdot1111\\1}\}$ (by orthogonality to β_1) and hence $\{\substack{24654\cdot1\\3}\}, \{\substack{2\cdot\cdot\cdot321\\2}\}, \{\substack{1343321\\2}, \substack{1244321\\2}, \substack{1233321\\2}\}, \{\substack{1233221\\1}, \substack{1232211\\1}\}$; we must then fix $\{\substack{2\cdot54321\\3}\}, \substack{1233321\\2}, \{\substack{1233211\\2}, \substack{1232221\\2}\}$, $\substack{1111111\\1}$ (by orthogonality to both of $\{\substack{1233221\\1}, \substack{1232211\\1}\}$), $\{\substack{2\cdot54321\\2}\}$ (by orthogonality to neither of $\{\substack{1233221\\1}, \substack{1232211\\1}\}$) and hence $\{\substack{246\cdot321\\3}\}, \{\substack{234\cdot321\\2}\}, \{\substack{1343321\\2}, \substack{1244321\\2}\}, \{\substack{1\cdot432\cdot1\\2}\}, \{\substack{1222111\\1}, \substack{1121111\\1}\}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_3w_5w_7\rangle$ giving transitivity on $\{\substack{1233221\\1}, \substack{1232211\\1}\}$, so we may fix $\beta_2 = \substack{1233221\\1}$. We must then fix $\substack{2465421\\3}, \substack{2464321\\3}, \substack{2343321\\2}, \substack{2343211\\2}, \substack{1343321\\2}, \{\substack{1\cdot43211\\2}\}$ and $\substack{1121111\\1}$ (by orthogonality to β_2) and hence $\substack{2465431\\3}, \substack{2465321\\3}, \substack{2344321\\2}, \substack{2343221\\2}, \substack{1244321\\2}, \{\substack{1\cdot43221\\2}\}, \substack{1222111\\1}$; we must then fix $\substack{2354321\\3}, \substack{2354321\\2}, \substack{1243221\\2}, \substack{1243211\\2}, \substack{1232221\\2}$ (by orthogonality to $\substack{1222111\\1}$), by which point all roots are fixed; so $W_X = G$. (In this case we note also that the set of roots with

$o(\beta) = 15$ is in fact a union of the two W_X-orbits $\{\substack{1343\cdot21\\2}, \substack{1244321\\2}, \substack{12\cdot3211\\2}, \substack{1232221\\2}\}$ and $\{\substack{1343211\\2}, \substack{1243221\\2}, \substack{1233321\\2}\}$.)

If $X = X_{30}^{18}$ we must fix $\substack{2465432\\3}$, $\substack{2465431\\3}$, $\{\substack{2\cdot\cdot\cdot\cdot21\\\cdot}\}$, $\substack{2343211\\2}$, $\{\substack{1354321\\\cdot}, \substack{1\cdot4\cdot321\\2}, \substack{123\cdot221\\\cdot}, \substack{1\cdot22221\\1}\}$, $\substack{2343210\\2}$, $\{\substack{1232111\\\cdot}, \substack{1\cdot22211\\1}\}$ ($o(\beta) = 1, 5, 9, 13, 14, 17$ and 18 respectively). We set

$$G = \langle w_3, w_5, w_2w_3{w_6}^{w_4}\rangle;$$

then G is transitive on $\{\substack{1354321\\\cdot}, \substack{1\cdot4\cdot321\\2}, \substack{123\cdot221\\\cdot}, \substack{1\cdot22221\\1}\}$, so we may fix $\beta_1 = \substack{1354321\\3}$. We must then fix $\{\substack{2\cdot\cdot\cdot\cdot21\\2}\}$, $\{\substack{1\cdot\cdot\cdot221\\1}\}$, $\{\substack{1232111\\1}, \substack{1\cdot22211\\1}\}$ (by orthogonality to β_1) and hence $\{\substack{2\cdot\cdot\cdot\cdot21\\3}\}$, $\{\substack{1354321\\2}, \substack{1\cdot4\cdot321\\2}, \substack{123\cdot221\\2}\}$, $\substack{1232111\\2}$; we must then fix $\substack{2465421\\3}$, $\{\substack{2\cdot\cdot\cdot321\\2}\}$, $\{\substack{1354321\\2}, \substack{1\cdot4\cdot321\\2}\}$, $\{\substack{1\cdot22211\\1}\}$ (by orthogonality to $\substack{1232111\\2}$) and hence $\{\substack{2\cdot\cdot\cdot321\\3}\}$, $\substack{2343221\\2}$, $\{\substack{123\cdot221\\2}\}$, $\substack{1232111\\1}$; we must then fix $\{\substack{2\cdot54321\\3}\}$, $\{\substack{234\cdot321\\2}\}$, $\{\substack{1\cdot4\cdot321\\2}\}$, $\{\substack{1\cdot22221\\1}\}$ (by orthogonality to $\substack{1232111\\1}$) and hence $\{\substack{246\cdot321\\3}\}$, $\{\substack{2\cdot54321\\2}\}$, $\substack{1354321\\2}$, $\{\substack{123\cdot221\\1}\}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_3, w_5\rangle$, which acts as S_2 on each of $\{\substack{246\cdot321\\3}\}$ and $\{\substack{2\cdot54321\\3}\}$ independently, so we may fix all of these roots. We then have fixed all of the roots $\substack{\cdot\cdot\cdot\cdot\cdot\cdot\cdot\\3}$, which span $\mathbb{R}\Phi$; so $W_X = G$. (In this case we note also that the set of roots with $o(\beta) = 9$ is in fact a union of the two W_X-orbits $\{\substack{246\cdot\cdot21\\3}, \substack{234\cdot\cdot21\\2}\}$ and $\{\substack{2\cdot54321\\\cdot}\}$.)

If $X = X_{30}^{19}$ we must fix $\substack{2465432\\3}$, $\{\substack{2\cdot\cdot\cdot\cdot\cdot1\\3}\}$, $\{\substack{2\cdot\cdot\cdot\cdot\cdot1\\2}\}$, $\{\substack{1\cdot\cdot\cdot\cdot\cdot1\\2}\}$, $\substack{2343210\\2}$, $\substack{1111111\\0}$ ($o(\beta) = 1, 7, 11, 14, 17, 22$ respectively). We set

$$G = \langle w_3, w_4, w_5, w_6, w_7\rangle;$$

then G acts as S_6 on $\{\substack{2\cdot\cdot\cdot\cdot\cdot1\\3}\}$, so we may fix all of these roots. We then have fixed all of the roots $\substack{2\cdot\cdot\cdot\cdot\cdot\cdot\\3}$, $\substack{1111111\\0}$, which span $\mathbb{R}\Phi$; so $W_X = G$.

If $X = X_{30}^{20}$ we must fix each of $\substack{2465432\\3}$, $\{\substack{24654\cdot1\\3}\}$, $\{\substack{2\cdot\cdot\cdot321\\\cdot}\}$, $\{\substack{23432\cdot1\\2}\}$, $\{\substack{1354321\\\cdot}, \substack{1343321\\2}, \substack{1244321\\2}, \substack{1233321\\\cdot}\}$, $\{\substack{1\cdot43221\\2}, \substack{123\cdot211\\\cdot}, \substack{1\cdot22221\\1}\}$, $\substack{2343210\\2}$, $\{\substack{1222111\\1}, \substack{1121111\\1}\}$ ($o(\beta) = 1, 7, 9, 11, 14, 16, 17, 18$ respectively). We set

$$G = \langle w_2, w_3w_5, w_3w_5{w_7}^{w_4}\rangle;$$

then G is transitive on $\{\substack{1\cdot43221\\2}, \substack{123\cdot211\\\cdot}, \substack{1\cdot22221\\1}\}$, so we may fix $\beta_1 = \substack{1343221\\2}$. We must then fix each of $\substack{2465421\\3}$, $\{\substack{23\cdot\cdot321\\\cdot}\}$, $\substack{2343211\\2}$, $\{\substack{1244321\\2}, \substack{1233321\\\cdot}\}$, $\{\substack{123\cdot211\\\cdot}, \substack{1122221\\1}\}$ and $\substack{1121111\\1}$ (by orthogonality to β_1) and hence $\substack{2465431\\3}$, $\{\substack{24\cdot\cdot321\\\cdot}\}$, $\substack{2343221\\2}$, $\{\substack{1354321\\\cdot}, \substack{1343321\\2}\}$, $\{\substack{1243221\\2}, \substack{1222221\\1}\}$ and $\substack{1222111\\1}$; we must then fix $\{\substack{2465321\\3}, \substack{2454321\\\cdot}\}$, $\substack{2344321\\2}$, $\substack{1222221\\1}$, $\{\substack{1233211\\\cdot}\}$ (by orthogonality to $\substack{1121111\\1}$) and hence $\substack{2464321\\3}$, $\{\substack{2354321\\\cdot}, \substack{2343321\\2}\}$, $\substack{1243221\\2}$, $\{\substack{1232211\\\cdot}, \substack{1122221\\1}\}$; we must then fix each of $\substack{2465321\\3}$, $\{\substack{2354321\\\cdot}\}$, $\{\substack{1354321\\\cdot}\}$, $\substack{1244321\\2}$, $\{\substack{1232211\\\cdot}\}$ (by orthogonality to $\substack{1222221\\1}$) and hence $\{\substack{2454321\\\cdot}\}$, $\substack{2343321\\2}$, $\substack{1343321\\2}$, $\{\substack{1233321\\\cdot}\}$, $\substack{1122221\\1}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_2\rangle$ giving transitivity on $\{\substack{1232211\\\cdot}\}$, so we may fix $\beta_2 = \substack{1232211\\1}$. We must then fix $\substack{2454321\\3}$, $\substack{2354321\\3}$, $\substack{1354321\\3}$, $\substack{1233321\\2}$, $\substack{1233211\\2}$ (by orthogonality to β_2), by which point all roots are fixed; so $W_X = G$. (In this case we note also that the set of roots with $o(\beta) = 14$ is in fact a union of the two W_X-orbits $\{\substack{1354321\\\cdot}, \substack{1233321\\\cdot}\}$ and $\{\substack{1343321\\2}, \substack{1244321\\2}\}$.)

If $X = X_{30}^{21}$ we must fix each of $\substack{2465432\\3}$, $\{\substack{24654\cdot1\\3}\}$, $\{\substack{23432\cdot1\\2}\}$, $\substack{2343210\\2}$, $\{\substack{1232111\\\cdot}\}$ ($o(\beta) = 1, 7, 11, 17, 18$ respectively), $\{\substack{234\cdot321\\2}\}$, $\{\substack{1\cdot4\cdot321\\2}\}$, $\{\substack{1\cdot222\cdot1\\1}\}$ ($o(\beta) = 9, 14, 16$ respectively, orthogonal to both of $\{\substack{1232111\\\cdot}\}$), $\{\substack{2\cdot54321\\\cdot}\}$, $\{\substack{1354321\\\cdot}\}$, $\{\substack{1233211\\2}, \substack{1232221\\2}, \substack{1233221\\1}, \substack{1232211\\1}\}$ ($o(\beta) = 9, 14, 16$ respectively, orthogonal to one of $\{\substack{1232111\\\cdot}\}$),

$\{\substack{246\cdot321\\3}\}$ ($o(\beta)=9$, orthogonal to neither of $\{\substack{1232111\\\cdot}\}$). We set

$$G=\langle w_3, w_2w_7, w_5w_7\rangle;$$

then G is transitive on $\{\substack{1\cdot4\cdot321\\2}\}$, so we may fix $\beta_1=\substack{1344321\\2}$. We must then fix $\substack{2464321\\3}, \{\substack{2354321\\\cdot}\}, \substack{2343321\\2}, \substack{1243321\\2}, \{\substack{1232221\\2}, \substack{1232211\\1}\}, \{\substack{11222\cdot1\\1}\}$ (by orthogonality to β_1) and hence $\substack{2465321\\3}, \{\substack{2454321\\\cdot}\}, \substack{2344321\\2}, \{\substack{1343321\\2}, \substack{1244321\\2}\}, \{\substack{1233211\\2}, \substack{1233221\\1}\}, \{\substack{12222\cdot1\\1}\}$; we must then fix $\substack{1244321\\2}$ (by orthogonality to $\substack{2464321\\3}$) and hence $\substack{1343321\\2}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_2w_7\rangle$ giving transitivity on $\{\substack{1233211\\2}, \substack{1233221\\1}\}$, so we may fix $\beta_2=\substack{1233221\\1}$. We must then fix $\substack{2465421\\3}, \substack{2454321\\3}, \substack{2354321\\3}, \substack{2343211\\2}, \substack{1354321\\3}, \substack{1222211\\1}, \substack{1122211\\1}, \substack{1232111\\2}$ (by orthogonality to β_2) and hence $\substack{2465431\\3}, \substack{2454321\\2}, \substack{2354321\\2}, \substack{2343221\\2}, \substack{1354321\\2}, \substack{1222221\\1}, \substack{1122221\\1}$ and $\substack{1232111\\1}$; we must then fix $\substack{1232221\\2}$ (by orthogonality to $\substack{1354321\\2}$), by which point all roots are fixed; so $W_X=G$.

If $X=X_{30}^{22}$ we must fix $\substack{2465432\\3}, \{\substack{2\cdot\cdot\cdot\cdot\cdot1\\3}\}, \{\substack{2\cdot\cdot\cdot\cdot\cdot1\\2}\}, \{\substack{13\cdot4321\\2}, \substack{1\cdot\cdot\cdot2\cdot1\\2}, \substack{12\cdot3321\\2}\}$, $\substack{1111111\\0}$ ($o(\beta)=1,8,10,15,19$ respectively), $\substack{2343210\\2}$ ($o(\beta)=17$, orthogonal to $\substack{2465432\\3}$), $\{\substack{1233321\\1}, \substack{1221111\\1}, \substack{1122111\\1}\}$ ($o(\beta)=17$, not orthogonal to $\substack{2465432\\3}$). We set

$$G=\langle w_7, w_3w_5, w_4{w_7}^{w_6w_5w_6}\rangle;$$

then G is transitive on $\{\substack{13\cdot4321\\2}, \substack{1\cdot\cdot\cdot2\cdot1\\2}, \substack{12\cdot3321\\2}\}$, so we may fix $\beta_1=\substack{1354321\\2}$. We must then fix $\{\substack{2\cdot54321\\3}\}, \{\substack{234\cdot\cdot\cdot1\\2}\}, \{\substack{1233321\\2}, \substack{123\cdot2\cdot1\\2}\}, \{\substack{1221111\\1}, \substack{1122111\\1}\}$ (by orthogonality to β_1) and hence $\{\substack{246\cdot\cdot\cdot1\\3}\}, \{\substack{2\cdot54321\\2}\}, \{\substack{1344321\\2}, \substack{1\cdot432\cdot1\\2}, \substack{1243321\\2}\}, \substack{1233321\\1}$; we must then fix $\{\substack{246\cdot321\\3}\}, \{\substack{23432\cdot1\\2}\}, \{\substack{1\cdot432\cdot1\\2}\}, \{\substack{123\cdot2\cdot1\\2}\}$ (by orthogonality to $\substack{1233321\\1}$) and hence $\{\substack{24654\cdot1\\3}\}, \{\substack{234\cdot321\\2}\}, \{\substack{1344321\\2}, \substack{1243321\\2}\}, \substack{1233321\\2}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_7, w_3w_5\rangle$ giving transitivity on $\{\substack{1\cdot432\cdot1\\2}\}$, so we may fix $\beta_2=\substack{1343221\\2}$. We must then fix $\substack{2465421\\3}, \substack{2354321\\3}, \substack{2354321\\2}, \substack{2343211\\2}, \substack{1243321\\2}, \substack{1243211\\2}, \{\substack{123\cdot211\\2}\}, \substack{1122111\\1}$ (by orthogonality to β_2) and hence $\substack{2465431\\3}, \substack{2454321\\3}, \substack{2454321\\2}, \substack{2343221\\2}, \substack{1344321\\2}, \{\substack{1343211\\2}, \substack{1243221\\2}\}, \{\substack{123\cdot221\\2}\}, \substack{1221111\\1}$; we must then fix all of $\substack{2464321\\3}, \substack{2343321\\2}, \substack{1343211\\2}, \substack{1232221\\2}, \substack{1232211\\2}$ (by orthogonality to $\substack{1122111\\1}$), by which point all roots are fixed; so $W_X=G$.

If $X=X_{30}^{23}$ we must fix $\substack{2465432\\3}, \{\substack{2\cdot\cdot\cdot\cdot\cdot1\\\cdot}\}, \{\substack{1\cdot\cdot4321\\\cdot}, \substack{1\cdot\cdot2\cdot\cdot1\\\cdot}\}, \substack{2343210\\2}$ ($o(\beta)=1,9,16,17$ respectively). We set

$$G=\langle w_2, w_3, w_4, w_3w_5{w_7}^{w_4w_6w_5}\rangle;$$

then G is transitive on $\{\substack{1\cdot\cdot4321\\\cdot}, \substack{1\cdot\cdot2\cdot\cdot1\\\cdot}\}$, so we may fix $\beta_1=\substack{1354321\\3}$. We must then fix $\{\substack{2\cdot\cdot\cdot\cdot\cdot1\\2}\}, \{\substack{1\cdot\cdot2\cdot\cdot1\\1}\}$ (by orthogonality to β_1) and hence $\{\substack{2\cdot\cdot\cdot\cdot\cdot1\\3}\}, \{\substack{1\cdot\cdot4321\\2}, \substack{1232\cdot\cdot1\\2}\}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_3, w_4, w_3w_5{w_7}^{w_4w_6w_5}\rangle$ giving transitivity on $\{\substack{1\cdot\cdot2\cdot\cdot1\\1}\}$, so we may fix $\beta_2=\substack{1122111\\1}$. We must then fix each of $\{\substack{24654\cdot1\\3}, \substack{24\cdot4321\\3}\}$, $\{\substack{2454321\\2}, \substack{2343321\\2}\}, \{\substack{13\cdot4321\\2}, \substack{12322\cdot1\\2}\}$ and $\{\substack{12\cdot22\cdot1\\1}\}$ (by orthogonality to β_2) and hence $\{\substack{2465321\\3}, \substack{2354321\\3}\}, \{\substack{23\cdot4321\\2}, \substack{23432\cdot1\\2}\}, \{\substack{1244321\\2}, \substack{1232111\\2}\}$ and $\{\substack{12\cdot2111\\1}, \substack{11222\cdot1\\1}\}$. Inside $\mathrm{stab}_G(\beta_1,\beta_2)$ we have $\langle w_4, w_3w_5{w_7}^{w_4w_6w_5}\rangle$ giving transitivity on $\{\substack{12\cdot2111\\1}, \substack{11222\cdot1\\1}\}$, so we may fix $\beta_3=\substack{1122211\\1}$. We must then fix $\{\substack{2465431\\3}, \substack{24\cdot4321\\3}\}, \substack{2465321\\3}, \substack{2454321\\2}, \substack{2343221\\2}, \{\substack{13\cdot4321\\2}, \substack{1232221\\2}\}, \substack{1232111\\2}, \{\substack{12\cdot2221\\1}\}, \{\substack{12\cdot2111\\1}\}$ (by orthogonality to β_3) and hence $\substack{2465421\\3}, \substack{2354321\\3}, \substack{2343321\\2}, \{\substack{23\cdot4321\\2}, \substack{2343211\\2}\}, \substack{1232211\\2}, \substack{1244321\\2}, \{\substack{12\cdot2211\\1}\}$, $\substack{1122221\\1}$; we must then fix $\{\substack{24\cdot4321\\3}\}, \substack{2343211\\2}, \{\substack{13\cdot4321\\2}\}$ (by orthogonality to $\substack{1122221\\1}$) and hence $\substack{2465431\\3}, \{\substack{23\cdot4321\\2}\}, \substack{1232221\\2}$. Inside $\mathrm{stab}_G(\beta_1,\beta_2,\beta_3)$ we have $\langle w_4\rangle$ giving transitivity on $\{\substack{12\cdot2111\\1}\}$, so we may fix $\beta_4=\substack{1222111\\1}$. We must then fix

$\substack{2464321\\3}, \substack{2354321\\2}, \substack{1354321\\2}, \substack{1232221\\1}, \substack{1232211\\1}$ (by orthogonality to β_4), by which point all roots are fixed; so $W_X = G$.

If $X = X_{30}^{24}$ we must fix $\substack{2465432\\3}, \{\substack{2\cdot\cdot\cdot\cdot\cdot1\\\cdot}\}, \{\substack{13\cdot4321\\\cdot}, \substack{1243321\\2}, \substack{1233221\\2}, \substack{1232\cdot11\\2}, \substack{1233211\\1}, \substack{1\cdot\cdot2221\\1}, \substack{1232111\\1}, \substack{1222211\\1}, \substack{1122\cdot11\\1}, \substack{1221111\\1}\}, \substack{2343210\\2}$ ($o(\beta) = 1, 9, 16, 17$ respectively). We set

$$G = \langle w_2w_7, w_4w_6, w_3w_5w_7\rangle;$$

then G is transitive on $\{\substack{13\cdot4321\\\cdot}, \substack{1243321\\2}, \substack{1233221\\2}, \substack{1232\cdot11\\2}, \substack{1233211\\1}, \substack{1\cdot\cdot2221\\1}, \substack{1232111\\1}, \substack{1222211\\1}, \substack{1122\cdot11\\1}, \substack{1221111\\1}\}$, so we may fix $\beta_1 = \substack{1354321\\3}$. We must then fix $\{\substack{2\cdot\cdot\cdot\cdot\cdot1\\2}\}$ and $\{\substack{1233211\\1}, \substack{1\cdot\cdot2221\\1}, \substack{1232111\\1}, \substack{1222211\\1}, \substack{1122\cdot11\\1}, \substack{1221111\\1}\}$ (by orthogonality to β_1) and hence $\{\substack{2\cdot\cdot\cdot\cdot\cdot1\\3}\}$ and $\{\substack{13\cdot4321\\2}, \substack{1243321\\2}, \substack{1233221\\2}, \substack{1232\cdot11\\2}\}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_4w_6, w_3w_5w_7\rangle$ giving transitivity on $\{\substack{13\cdot4321\\2}, \substack{1243321\\2}, \substack{1233221\\2}, \substack{1232\cdot11\\2}\}$, so we may fix $\beta_2 = \substack{1354321\\2}$. We must then fix $\{\substack{2\cdot54321\\3}\}, \{\substack{234\cdot\cdot\cdot1\\2}\}, \{\substack{1233221\\2}, \substack{1232\cdot11\\2}\}, \{\substack{1\cdot222\cdot1\\1}, \substack{1221111\\1}, \substack{1122111\\1}\}$ (by orthogonality to β_2) and hence $\{\substack{246\cdot\cdot\cdot1\\3}\}, \{\substack{2\cdot54321\\2}\}, \{\substack{1344321\\2}, \substack{1243321\\2}\}, \{\substack{1233211\\1}, \substack{1232221\\1}, \substack{1232111\\1}\}$; we must then fix $\{\substack{23432\cdot1\\2}\}, \substack{1232111\\2}, \substack{1232111\\1}$ and $\{\substack{1221111\\1}, \substack{1122111\\1}\}$ (by orthogonality to both of $\{\substack{1344321\\2}, \substack{1243321\\2}\}$) and hence each of $\{\substack{234\cdot321\\2}\}, \{\substack{1233221\\2}, \substack{1232211\\2}\}, \{\substack{1233211\\1}, \substack{1232221\\1}\}$ and $\{\substack{1\cdot222\cdot1\\1}\}$; we must then fix $\{\substack{24654\cdot1\\3}\}$ (by orthogonality to $\substack{1232111\\1}$) and hence $\{\substack{246\cdot321\\3}\}$. Inside $\mathrm{stab}_G(\beta_1, \beta_2)$ we have $\langle w_3w_5w_7\rangle$ giving transitivity on $\{\substack{1221111\\1}, \substack{1122111\\1}\}$, so we may fix $\beta_3 = \substack{1122111\\1}$. We must then fix $\substack{2464321\\3}, \substack{2454321\\3}, \substack{2454321\\2}, \substack{2343321\\2}, \substack{1232211\\2}, \substack{1232221\\1}$ and $\{\substack{12222\cdot1\\1}\}$ (by orthogonality to β_3) and hence $\substack{2465321\\3}, \substack{2354321\\3}, \substack{2354321\\2}, \substack{2344321\\2}, \substack{1233221\\2}, \substack{1233211\\1}$ and $\{\substack{11222\cdot1\\1}\}$; we must then fix $\substack{2465421\\3}, \substack{2343211\\2}, \substack{1243321\\2}, \substack{1222211\\1}, \substack{1122211\\1}$ (by orthogonality to $\substack{1233221\\2}$), by which point all roots are fixed; so $W_X = G$.

If $X = X_{30}^{25}$ we must fix each of $\{\substack{246543\cdot\\3}\}, \substack{2465421\\3}, \{\substack{2\cdot\cdot321\\\cdot}\}, \{\substack{1\cdot\cdot321\\\cdot}\}, \substack{2343221\\2}, \{\substack{234321\cdot\\2}\}, \{\substack{123\cdot221\\\cdot}\}, \{\substack{1343211\\2}, \substack{1243210\\2}, \substack{1222210\\1}, \substack{1122211\\1}\}$ ($o(\beta) = 3, 5, 9, 12, 13, 15, 16, 18$ respectively). We set

$$G = \langle w_2, w_3w_8, w_2{w_5}^{w_4}\rangle;$$

then G is transitive on $\{\substack{2\cdot\cdot321\\\cdot}\}$, so we may fix $\beta_1 = \substack{2465321\\3}$. We must then fix $\substack{2343321\\2}, \{\substack{1\cdot\cdot3321\\\cdot}\}, \{\substack{1232221\\\cdot}\}, \{\substack{1222210\\1}, \substack{1122211\\1}\}$ (by orthogonality to β_1) and hence $\{\substack{2\cdot\cdot4321\\\cdot}\}, \{\substack{1\cdot\cdot4321\\\cdot}\}, \{\substack{1233221\\\cdot}\}$ and $\{\substack{1343211\\2}, \substack{1243210\\2}\}$; we must then fix $\substack{2344321\\2}$ and $\{\substack{1\cdot44321\\2}\}$ (by orthogonality to both of $\{\substack{1232221\\\cdot}\}$), $\{\substack{2\cdot54321\\\cdot}\}, \{\substack{1354321\\\cdot}\}, \{\substack{1233321\\\cdot}\}$ (by orthogonality to one of $\{\substack{1232221\\\cdot}\}$) and hence $\substack{2464321\\3}, \{\substack{1\cdot43321\\2}\}$ (by orthogonality to neither of $\{\substack{1232221\\\cdot}\}$). Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_2, w_3w_8\rangle$ giving transitivity on $\{\substack{2\cdot54321\\\cdot}\}$, so we may fix $\beta_2 = \substack{2454321\\3}$. We must then fix each of $\substack{2354321\\2}, \substack{1354321\\2}, \substack{1244321\\2}, \substack{1243321\\2}, \substack{1233321\\1}, \substack{1233221\\1}, \substack{1232221\\1}, \substack{1243210\\2}$ and $\substack{1122211\\1}$ (by orthogonality to β_2) and hence each of $\{\substack{2454321\\2}, \substack{2354321\\3}\}, \substack{1354321\\3}, \substack{1344321\\2}, \substack{1343321\\2}, \substack{1233321\\2}, \substack{1233221\\2}, \substack{1232221\\2}, \substack{1343211\\2}, \substack{1222210\\1}$; we must then fix $\substack{2465431\\3}, \substack{2354321\\3}, \substack{2343210\\2}$ (by orthogonality to $\substack{1343211\\2}$), by which point all roots are fixed; so $W_X = G$.

If $X = X_{31}^{1}$ we must fix $\substack{2465432\\3}, \substack{2465431\\3}, \substack{2454321\\2}, \{\substack{\cdot354321\\3}\}, \{\substack{\cdot343211\\2}\}, \substack{1233211\\1}$ ($o(\beta) = 2, 5, 9, 10, 14, 19$ respectively), $\{\substack{24\cdot4321\\3}\}, \{\substack{\cdot343\cdot21\\2}\}\{\substack{12\cdot3\cdot21\\2}\}$ ($o(\beta) = 7, 12, 15$ respectively, orthogonal to $\substack{1233211\\1}$), $\{\substack{2465\cdot21\\3}\}, \{\substack{\cdot3\cdot4321\\2}\}, \substack{1244321\\2}$ ($o(\beta) = 7, 12, 15$ respectively, not orthogonal to $\substack{1233211\\1}$), $\{\substack{12\cdot2221\\1}\}$ ($o(\beta) = 17$ respectively, orthogonal to both of $\{\substack{\cdot354321\\3}\}$), $\{\substack{\cdot343210\\2}\}$ ($o(\beta) = 17$, orthogonal to one of $\{\substack{\cdot354321\\3}\}$), $\{\substack{1232\cdot11\\2}\}$ ($o(\beta) = 17$, orthogonal to neither of $\{\substack{\cdot354321\\3}\}$). We set

$$G = \langle w_1, w_4, w_6\rangle;$$

then G acts as S_2 on each of $\{\substack{\cdot354321\\3}\}$, $\{\substack{24\cdot4321\\3}\}$ and $\{\substack{2465\cdot21\\3}\}$ independently, so we may fix all of these roots. We then have fixed all of the roots $\substack{\cdot\cdot\cdot\cdot\cdot\cdot\cdot\\3}$, which span $\mathbb{R}\Phi$; so $W_X = G$.

If $X = X_{31}^2$ we must fix $\{\substack{246543\cdot\\3}\}, \{\substack{24\cdot\cdot\cdot21\\\cdot}\}, \{\substack{\cdot3\cdot\cdot\cdot21\\\cdot}\}, \{\substack{12\cdot\cdot\cdot21\\\cdot}\}, \{\substack{\cdot34321\cdot\\2}\}$ ($o(\beta) = 2, 8, 11, 14, 17$ respectively). We set

$$G = \langle w_1, w_2, w_4, w_5, w_6, w_8\rangle;$$

then G is transitive on $\{\substack{\cdot34321\cdot\\2}\}$, so we may fix $\beta_1 = \substack{2343211\\2}$. We must then fix $\substack{2465431\\3}, \{\substack{13\cdot\cdot\cdot21\\\cdot}\}, \substack{1343210\\2}$ (by orthogonality to β_1) and hence $\substack{2465432\\3}, \{\substack{23\cdot\cdot\cdot21\\\cdot}\}$, $\{\substack{2343210\\2}, \substack{1343211\\2}\}$; we must then fix $\substack{2343210\\2}$ (by orthogonality to $\substack{2465432\\3}$) and hence $\substack{1343211\\2}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_2, w_4, w_5, w_6\rangle$, which acts as S_5 on $\{\substack{24\cdot\cdot\cdot21\\\cdot}\}$, so we may fix all of these roots. We then have fixed all of the roots $\substack{24\cdot\cdot\cdot\cdot\cdot\\\cdot}, \substack{2343210\\2}$, which span $\mathbb{R}\Phi$; so $W_X = G$.

If $X = X_{31}^3$ we must fix $\substack{2465432\\3}, \substack{2465431\\3}, \substack{2465421\\3}, \{\substack{24\cdot\cdot321\\\cdot}\}, \{\substack{\cdot3\cdot\cdot321\\\cdot}\}, \{\substack{\cdot343221\\2}\}$, $\{\substack{12\cdot\cdot321\\\cdot}\}, \{\substack{\cdot343211\\2}\}, \{\substack{12\cdot3221\\2}, \substack{12\cdot2221\\1}\}, \{\substack{\cdot343210\\2}\}, \{\substack{1232211\\2}, \substack{1233211\\1}\}$ ($o(\beta) = 2, 4, 6, 8, 11, 13, 14, 15, 16, 17, 18$ respectively). We set

$$G = \langle w_1, w_4, w_2w_5\rangle;$$

then G is transitive on $\{\substack{\cdot3\cdot\cdot321\\\cdot}\}$, so we may fix $\beta_1 = \substack{2354321\\3}$. We must then fix $\substack{2454321\\2}, \{\substack{13\cdot\cdot321\\2}\}, \substack{1343221\\2}, \substack{1343211\\2}, \substack{1343210\\2}, \substack{1233321\\1}, \{\substack{12\cdot2221\\1}\}$ and $\substack{1233211\\1}$ (by orthogonality to β_1) and hence $\{\substack{24\cdot\cdot321\\3}\}, \{\substack{1354321\\3}, \substack{23\cdot\cdot321\\2}\}, \substack{2343221\\2}, \substack{2343211\\2}, \substack{2343210\\2}$, $\{\substack{12\cdot\cdot321\\2}\}, \{\substack{12\cdot3221\\2}\}, \substack{1232211\\2}$; we must then fix $\{\substack{24\cdot4321\\3}\}, \{\substack{1354321\\3}, \substack{2343321\\2}\}, \substack{1343321\\2}$, $\{\substack{12\cdot3321\\2}\}$ (by orthogonality to $\substack{1233211\\1}$) and hence $\substack{2465321\\3}, \{\substack{23\cdot4321\\2}\}, \{\substack{13\cdot4321\\2}\}$ and $\substack{1244321\\2}$; we must then fix $\substack{1354321\\3}$ (by orthogonality to $\substack{2454321\\2}$) and hence $\substack{2343321\\2}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_4\rangle$ giving transitivity on $\{\substack{12\cdot2221\\1}\}$, so we may fix $\beta_2 = \substack{1222221\\1}$. We must then fix $\substack{2464321\\3}, \substack{2354321\\2}, \substack{1354321\\2}, \substack{1243321\\2}, \substack{1243221\\2}$ (by orthogonality to β_2), by which point all roots are fixed; so $W_X = G$.

If $X = X_{31}^4$ we must fix all of $\substack{2465432\\3}, \{\substack{2465\cdot\cdot1\\3}\}, \{\substack{24\cdot4321\\\cdot}\}, \{\substack{\cdot3\cdot4321\\\cdot}\}, \{\substack{\cdot343\cdot\cdot1\\2}\}$, $\{\substack{12\cdot3\cdot\cdot1\\\cdot}\}, \{\substack{\cdot343210\\2}\}, \substack{1221111\\1}$ ($o(\beta) = 2, 6, 8, 11, 13, 16, 17, 20$ respectively). We set

$$G = \langle w_1, w_2, w_4, w_6, w_7\rangle;$$

then G acts as S_2 on $\{\substack{\cdot343210\\2}\}$ and independently as S_3 on each of $\{\substack{2465\cdot\cdot1\\3}\}$ and $\{\substack{24\cdot4321\\\cdot}\}$, so we may fix all of these roots, which span $\mathbb{R}\Phi$, so $W_X = G$.

If $X = X_{31}^5$ we must fix all of $\substack{2465432\\3}, \{\substack{2465\cdot\cdot1\\3}\}, \{\substack{24\cdot4321\\\cdot}\}, \{\substack{\cdot3\cdot4321\\\cdot}\}, \{\substack{\cdot343\cdot\cdot1\\2}\}$, $\substack{1244321\\2}, \{\substack{12\cdot3321\\2}, \substack{1243221\\2}, \substack{1233211\\\cdot}, \substack{1233221\\1}\}, \{\substack{\cdot343210\\2}\}$ and $\{\substack{1232111\\2}, \substack{1232211\\1}, \substack{1222221\\1}\}$ ($o(\beta) = 2, 6, 8, 11, 13, 14, 16, 17, 18$ respectively). We set

$$G = \langle w_1, w_2w_6, w_4w_7\rangle;$$

then G is transitive on $\{\substack{\cdot3\cdot4321\\\cdot}\}$, so we may fix $\beta_1 = \substack{2354321\\3}$. We must then fix $\substack{2454321\\2}, \{\substack{13\cdot4321\\2}\}, \{\substack{1343\cdot\cdot1\\2}\}, \substack{1343210\\2}, \{\substack{12332\cdot1\\1}\}, \{\substack{1232211\\1}, \substack{1222221\\1}\}$ (by orthogonality to β_1) and hence $\{\substack{24\cdot4321\\3}\}, \{\substack{1354321\\3}, \substack{23\cdot4321\\2}\}, \{\substack{2343\cdot\cdot1\\2}\}, \substack{2343210\\2}, \{\substack{12\cdot3321\\2}, \substack{1243221\\2}$, $\substack{1233211\\2}\}, \substack{1232111\\2}$; we must then fix $\{\substack{24654\cdot1\\3}\}, \{\substack{23\cdot4321\\2}\}, \substack{2343321\\2}, \substack{1343321\\2}, \{\substack{12\cdot3321\\2}\}$ (by orthogonality to $\substack{1232111\\2}$) and hence $\substack{2465321\\3}, \substack{1354321\\3}, \{\substack{23432\cdot1\\2}\}, \{\substack{13432\cdot1\\2}\}$ and $\{\substack{1243221\\2}, \substack{1233211\\2}\}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_4w_7\rangle$ giving transitivity on $\{\substack{1232211\\1}$, $\substack{1222221\\1}\}$, so we may fix $\beta_2 = \substack{1222221\\1}$. We must then fix $\substack{2465421\\3}, \substack{2464321\\3}, \substack{2354321\\2}$, $\substack{1354321\\2}, \substack{2343211\\2}, \substack{1343211\\2}, \substack{1243321\\2}, \substack{1233211\\1}$ (by orthogonality to β_2) and hence $\substack{2465431\\3}$,

$\substack{2454321\\3}, \substack{2344321\\2}, \substack{1344321\\2}, \substack{2343221\\2}, \substack{1343221\\2}, \substack{1233321\\2}, \substack{1233221\\1}$; we must then fix $\substack{1233211\\2}$ (by orthogonality to $\substack{2465431\\3}$), by which point all roots are fixed; so $W_X = G$.

If $X = X_{31}^6$ we must fix $\substack{2465432\\3}, \{\substack{24\cdot\cdot\cdot\cdot 1\\\cdot}\}, \{\substack{\cdot 3\cdot\cdot\cdot\cdot 1\\\cdot}\}$ ($o(\beta) = 2, 7, 12$ respectively), $\{\substack{\cdot 343210\\2}\}$ ($o(\beta) = 17$, orthogonal to $\substack{2465432\\3}$), $\{\substack{1243221\\2}, \substack{1233321\\2}, \substack{12\cdot 3211\\2}, \substack{1233\cdot 21\\1}, \substack{1232221\\2}, \substack{1232211\\1}, \substack{122\cdot 111\\1}\}$ ($o(\beta) = 17$, not orthogonal to $\substack{2465432\\3}$). We set

$$G = \langle w_1, w_2w_5, w_4w_6, w_5w_7\rangle;$$

then G is transitive on $\{\substack{\cdot 3\cdot\cdot\cdot\cdot 1\\\cdot}\}$, so we may fix $\beta_1 = \substack{2354321\\3}$. We must then fix each of $\substack{2454321\\2}, \{\substack{13\cdot\cdot\cdot\cdot 1\\2}\}, \substack{1343210\\2}$ and $\{\substack{1233\cdot 21\\1}, \substack{1232211\\1}, \substack{122\cdot 111\\1}\}$ (by orthogonality to β_1) and hence each of $\{\substack{24\cdot\cdot\cdot\cdot 1\\3}\}, \{\substack{1354321\\3}, \substack{23\cdot\cdot\cdot\cdot 1\\2}\}, \substack{2343210\\2}$ and $\{\substack{1243221\\2}, \substack{1233321\\2}, \substack{12\cdot 3211\\2}, \substack{1232221\\2}\}$; we must then fix $\substack{1354321\\3}$ (by orthogonality to $\substack{2454321\\2}$) and hence $\{\substack{23\cdot\cdot\cdot\cdot 1\\2}\}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_4w_6, w_5w_7\rangle$ giving transitivity on $\{\substack{23\cdot\cdot\cdot\cdot 1\\2}\}$, so we may fix $\beta_2 = \substack{2354321\\2}$. We must then fix $\substack{2454321\\3}, \{\substack{134\cdot\cdot\cdot 1\\2}\}, \{\substack{1233321\\2}, \substack{1233211\\2}, \substack{1232221\\2}\}, \{\substack{122\cdot 111\\1}\}$ (by orthogonality to β_2) and hence $\{\substack{246\cdot\cdot\cdot 1\\3}\}, \substack{1354321\\2}, \{\substack{12432\cdot 1\\2}\}, \{\substack{1233\cdot 21\\1}, \substack{1232211\\1}\}$. Inside $\mathrm{stab}_G(\beta_1, \beta_2)$ we have $\langle w_5w_7\rangle$ giving transitivity on $\{\substack{122\cdot 111\\1}\}$, so we may fix $\beta_3 = \substack{1221111\\1}$. We must then fix $\{\substack{2465\cdot\cdot 1\\3}\}, \substack{2344321\\2}, \substack{1344321\\2}, \{\substack{1233321\\2}, \substack{1233211\\2}\}$ and $\{\substack{1233\cdot 21\\1}\}$ (by orthogonality to β_3) and hence all of $\substack{2464321\\3}, \{\substack{2343\cdot\cdot 1\\2}\}, \{\substack{1343\cdot\cdot 1\\2}\}, \substack{1232221\\2}, \substack{1232211\\1}$; we must then fix $\{\substack{2465\cdot 21\\3}\}, \substack{2343211\\2}, \substack{1343211\\2}, \substack{1243211\\2}, \substack{1233211\\2}$ (by orthogonality to $\substack{1232221\\2}$) and hence $\substack{2465431\\3}, \{\substack{2343\cdot 21\\2}\}, \{\substack{1343\cdot 21\\2}\}, \substack{1243221\\2}, \substack{1233321\\2}$; we must then fix $\substack{2465321\\3}, \substack{2343221\\2}, \substack{1343221\\2}, \substack{1233221\\1}$ (by orthogonality to $\substack{1232211\\1}$), by which point all roots are fixed; so $W_X = G$.

If $X = X_{31}^7$ we must fix $\{\substack{24654\cdot\cdot\\3}\}, \{\substack{24\cdot\cdot 321\\\cdot}\}, \{\substack{\cdot 3\cdot\cdot 321\\\cdot}\}, \{\substack{12\cdot\cdot 321\\\cdot}\}, \{\substack{\cdot 3432\cdot\cdot\\2}\}$, $\{\substack{1243221\\2}, \substack{1233211\\2}, \substack{1232210\\2}, \substack{1233210\\1}, \substack{1232211\\1}, \substack{1222221\\1}\}$ ($o(\beta) = 4, 8, 11, 14, 15, 18$ respectively). We set

$$G = \langle w_1, w_2w_5, w_4w_7, w_5w_8\rangle;$$

then G is transitive on $\{\substack{24654\cdot\cdot\\3}\}$, so we may fix $\beta_1 = \substack{2465432\\3}$. We must then fix $\{\substack{\cdot 343210\\2}\}, \{\substack{1232210\\2}, \substack{1233210\\1}\}$ (by orthogonality to β_1) and hence $\{\substack{\cdot 3432\cdot 1\\2}\}, \{\substack{1243221\\2}, \substack{1233211\\2}, \substack{1232211\\1}, \substack{1222221\\1}\}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_1, w_2w_5, w_4w_7\rangle$ giving transitivity on $\{\substack{24\cdot\cdot 321\\\cdot}\}$, so we may fix $\beta_2 = \substack{2465321\\3}$. We must then fix $\{\substack{\cdot 343321\\2}\}$, $\{\substack{12\cdot 3321\\\cdot}\}, \substack{1232210\\2}, \{\substack{1232211\\1}, \substack{1222221\\1}\}$ (by orthogonality to β_2) and hence $\{\substack{\cdot 3\cdot 4321\\\cdot}\}$, $\substack{1244321\\2}, \substack{1233210\\1}, \{\substack{1243221\\2}, \substack{1233211\\2}\}$; we must then fix $\substack{2454321\\2}, \{\substack{\cdot 3\cdot 4321\\2}\}$ and $\substack{1233321\\1}$ (by orthogonality to $\substack{1232210\\2}$) and hence $\{\substack{24\cdot 4321\\3}\}, \{\substack{\cdot 354321\\3}\}$ and $\{\substack{12\cdot 3321\\2}\}$. Inside $\mathrm{stab}_G(\beta_1, \beta_2)$ we have $\langle w_1, w_4w_7\rangle$ giving transitivity on $\{\substack{\cdot 3\cdot 4321\\2}\}$, so we may fix $\beta_3 = \substack{2354321\\2}$. We must then fix $\substack{2454321\\3}, \substack{1354321\\3}, \substack{1344321\\2}, \substack{1343321\\2}, \{\substack{13432\cdot 1\\2}\}, \substack{1343210\\2}$, $\substack{1233321\\2}, \substack{1233211\\2}, \substack{1222221\\1}$ (by orthogonality to β_3) and hence $\substack{2464321\\3}, \substack{2354321\\3}, \{\substack{2344321\\2}, \substack{1354321\\2}\}, \substack{2343321\\2}, \{\substack{23432\cdot 1\\2}\}, \substack{2343210\\2}, \substack{1243321\\2}, \substack{1243221\\2}$ and $\substack{1232211\\1}$; we must then fix $\substack{2465431\\3}, \substack{2344321\\2}, \substack{2343221\\2}, \substack{1343221\\2}$ (by orthogonality to $\substack{1232211\\1}$), by which point all roots are fixed; so $W_X = G$.

If $X = X_{32}^1$ we must fix $\substack{2465432\\3}, \{\substack{246\cdot\cdot\cdot 1\\3}\}, \{\substack{\cdot\cdot 54321\\3}\}, \{\substack{\cdot\cdot 54321\\2}\}, \{\substack{\cdot\cdot 4\cdot\cdot\cdot 1\\2}\}$ and $\{\substack{1233211\\1}, \substack{1232221\\1}\}$ ($o(\beta) = 3, 6, 9, 11, 14, 19$ respectively), $\{\substack{\cdot\cdot 43210\\2}\}$ ($o(\beta) = 17$, orthogonal to $\substack{2465432\\3}$), $\{\substack{1233\cdot 21\\2}, \substack{1232\cdot 11\\2}\}$ ($o(\beta) = 17$, not orthogonal to $\substack{2465432\\3}$). We set

$$G = \langle w_1, w_3, w_6, w_5w_7\rangle;$$

then G is transitive on $\{\substack{\cdot\cdot 4\cdot\cdot\cdot 1\\2}\}$, so we may fix $\beta_1 = \substack{2344321\\2}$. We must then fix $\substack{2464321\\3}, \substack{1354321\\3}, \substack{1354321\\2}, \{\substack{1\cdot 43\cdot\cdot 1\\2}\}, \{\substack{1\cdot 43210\\2}\}, \{\substack{1232\cdot 11\\2}\}, \substack{1232221\\1}$ (by orthogonality to

β_1) and hence $\{\genfrac{}{}{0pt}{}{2465\cdot\cdot1}{3}\}, \{\genfrac{}{}{0pt}{}{2\cdot54321}{3}\}, \{\genfrac{}{}{0pt}{}{2\cdot54321}{2}\}, \{\genfrac{}{}{0pt}{}{1\cdot44321}{2}, \genfrac{}{}{0pt}{}{2343\cdot\cdot1}{2}\}, \genfrac{}{}{0pt}{}{2343210}{2}, \{\genfrac{}{}{0pt}{}{1233\cdot21}{2}\}, \genfrac{}{}{0pt}{}{1233211}{1}$; we must then fix $\{\genfrac{}{}{0pt}{}{2465\cdot21}{3}\}, \{\genfrac{}{}{0pt}{}{1\cdot44321}{2}, \genfrac{}{}{0pt}{}{2343211}{2}\}, \{\genfrac{}{}{0pt}{}{1\cdot43211}{2}\}$ (by orthogonality to $\genfrac{}{}{0pt}{}{1232221}{1}$) and hence $\genfrac{}{}{0pt}{}{2465431}{3}, \{\genfrac{}{}{0pt}{}{2343\cdot21}{2}\}, \{\genfrac{}{}{0pt}{}{1\cdot43\cdot21}{2}\}$; we must then fix $\genfrac{}{}{0pt}{}{2343211}{2}$ (by orthogonality to $\genfrac{}{}{0pt}{}{2465431}{3}$) and hence $\{\genfrac{}{}{0pt}{}{1\cdot44321}{2}\}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_3, w_6\rangle$, which acts as S_2 on each of $\{\genfrac{}{}{0pt}{}{2\cdot54321}{3}\}$ and $\{\genfrac{}{}{0pt}{}{2465\cdot21}{3}\}$ independently, so we may fix all of these roots. We then have fixed all of $\genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot\cdot\cdot\cdot}{3}$, which span $\mathbb{R}\Phi$; so $W_X = G$.

If $X = X_{32}^2$ we must fix all of $\genfrac{}{}{0pt}{}{2465432}{3}, \{\genfrac{}{}{0pt}{}{24654\cdot1}{3}\}, \{\genfrac{}{}{0pt}{}{246\cdot321}{3}\}, \{\genfrac{}{}{0pt}{}{\cdot\cdot54321}{\cdot}\}, \{\genfrac{}{}{0pt}{}{\cdot\cdot4\cdot321}{2}\}, \{\genfrac{}{}{0pt}{}{\cdot\cdot432\cdot1}{2}\}, \{\genfrac{}{}{0pt}{}{1233321}{\cdot}\}, \{\genfrac{}{}{0pt}{}{\cdot\cdot43210}{2}\}, \{\genfrac{}{}{0pt}{}{1233221}{2}, \genfrac{}{}{0pt}{}{1232211}{2}, \genfrac{}{}{0pt}{}{1233211}{1}, \genfrac{}{}{0pt}{}{1232221}{1}\}$ ($o(\beta) = 3, 5, 7, 10, 13, 15, 16, 17, 18$ respectively). We set

$$G = \langle w_1, w_3, w_2w_5, w_2w_7\rangle;$$

then G is transitive on $\{\genfrac{}{}{0pt}{}{\cdot\cdot432\cdot1}{2}\}$, so we may fix $\beta_1 = \genfrac{}{}{0pt}{}{2343221}{2}$. We must then fix $\genfrac{}{}{0pt}{}{2465421}{3}, \{\genfrac{}{}{0pt}{}{1354321}{\cdot}\}, \{\genfrac{}{}{0pt}{}{1\cdot4\cdot321}{2}\}, \{\genfrac{}{}{0pt}{}{1\cdot43211}{2}\}, \{\genfrac{}{}{0pt}{}{1\cdot43210}{2}\}, \{\genfrac{}{}{0pt}{}{1232211}{2}, \genfrac{}{}{0pt}{}{1233211}{1}\}$ (by orthogonality to β_1) and hence $\genfrac{}{}{0pt}{}{2465431}{3}, \{\genfrac{}{}{0pt}{}{2\cdot54321}{\cdot}\}, \{\genfrac{}{}{0pt}{}{234\cdot321}{2}\}, \{\genfrac{}{}{0pt}{}{2343211}{2}, \genfrac{}{}{0pt}{}{1\cdot43221}{2}\}, \genfrac{}{}{0pt}{}{2343210}{2}$ and $\{\genfrac{}{}{0pt}{}{1233221}{2}, \genfrac{}{}{0pt}{}{1232221}{1}\}$; we must then fix $\{\genfrac{}{}{0pt}{}{1\cdot43221}{2}\}$ (by orthogonality to $\genfrac{}{}{0pt}{}{2343210}{2}$) and hence $\genfrac{}{}{0pt}{}{2343211}{2}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_3, w_2w_5\rangle$ giving transitivity on $\{\genfrac{}{}{0pt}{}{1\cdot4\cdot321}{2}\}$, so we may fix $\beta_2 = \genfrac{}{}{0pt}{}{1344321}{2}$. We must then fix each of $\genfrac{}{}{0pt}{}{2464321}{3}, \{\genfrac{}{}{0pt}{}{2354321}{\cdot}\}, \genfrac{}{}{0pt}{}{2343321}{2}, \genfrac{}{}{0pt}{}{1243321}{2}, \genfrac{}{}{0pt}{}{1243221}{2}, \genfrac{}{}{0pt}{}{1243211}{2}, \genfrac{}{}{0pt}{}{1243210}{2}, \genfrac{}{}{0pt}{}{1232221}{1}, \genfrac{}{}{0pt}{}{1232211}{2}$ (by orthogonality to β_2) and hence each of $\genfrac{}{}{0pt}{}{2465321}{3}, \{\genfrac{}{}{0pt}{}{2454321}{\cdot}\}, \genfrac{}{}{0pt}{}{2344321}{2}, \{\genfrac{}{}{0pt}{}{1343321}{2}, \genfrac{}{}{0pt}{}{1244321}{2}\}, \genfrac{}{}{0pt}{}{1343221}{2}, \genfrac{}{}{0pt}{}{1343211}{2}, \genfrac{}{}{0pt}{}{1343210}{2}, \genfrac{}{}{0pt}{}{1233221}{2}$ and $\genfrac{}{}{0pt}{}{1233211}{1}$; we must then fix $\genfrac{}{}{0pt}{}{2454321}{2}, \genfrac{}{}{0pt}{}{2354321}{2}, \genfrac{}{}{0pt}{}{1354321}{2}, \genfrac{}{}{0pt}{}{1244321}{2}, \genfrac{}{}{0pt}{}{1233321}{1}$ (by orthogonality to $\genfrac{}{}{0pt}{}{1232211}{2}$), by which point all roots are fixed; so $W_X = G$.

If $X = X_{32}^3$ we must fix $\{\genfrac{}{}{0pt}{}{246543\cdot}{3}\}, \{\genfrac{}{}{0pt}{}{246\cdot\cdot21}{3}\}, \{\genfrac{}{}{0pt}{}{\cdot\cdot54321}{\cdot}\}, \{\genfrac{}{}{0pt}{}{\cdot\cdot4\cdot\cdot21}{2}\}, \{\genfrac{}{}{0pt}{}{123\cdot\cdot21}{\cdot}\}$ and $\{\genfrac{}{}{0pt}{}{\cdot\cdot4321\cdot}{2}\}$ ($o(\beta) = 3, 7, 10, 13, 16, 17$ respectively). We set

$$G = \langle w_1, w_2, w_3, w_5, w_6, w_8\rangle;$$

then G is transitive on $\{\genfrac{}{}{0pt}{}{\cdot\cdot54321}{\cdot}\}$, so we may fix $\beta_1 = \genfrac{}{}{0pt}{}{2454321}{3}$. We must then fix $\{\genfrac{}{}{0pt}{}{\cdot354321}{2}\}, \{\genfrac{}{}{0pt}{}{124\cdot\cdot21}{2}\}, \{\genfrac{}{}{0pt}{}{124321\cdot}{2}\}, \{\genfrac{}{}{0pt}{}{123\cdot\cdot21}{1}\}$ (by orthogonality to β_1) and hence $\{\genfrac{}{}{0pt}{}{2454321}{2}, \genfrac{}{}{0pt}{}{\cdot354321}{3}\}, \{\genfrac{}{}{0pt}{}{\cdot34\cdot\cdot21}{2}\}, \{\genfrac{}{}{0pt}{}{\cdot34321\cdot}{2}\}, \{\genfrac{}{}{0pt}{}{123\cdot\cdot21}{2}\}$; we must then fix $\genfrac{}{}{0pt}{}{2454321}{2}$ (by orthogonality to both of $\{\genfrac{}{}{0pt}{}{124321\cdot}{2}\}$) and hence $\{\genfrac{}{}{0pt}{}{\cdot354321}{3}\}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_1, w_5, w_6, w_8\rangle$, which acts as S_3 on $\{\genfrac{}{}{0pt}{}{246\cdot\cdot21}{3}\}$ and independently as S_2 on each of $\{\genfrac{}{}{0pt}{}{246543\cdot}{3}\}$ and $\{\genfrac{}{}{0pt}{}{\cdot354321}{3}\}$, so we may fix all of these roots. We then have fixed all of $\genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot\cdot\cdot\cdot}{3}$, which span $\mathbb{R}\Phi$; so $W_X = G$.

If $X = X_{33}$ we must fix $\{\genfrac{}{}{0pt}{}{246543\cdot}{3}\}, \{\genfrac{}{}{0pt}{}{2465\cdot21}{3}\}, \{\genfrac{}{}{0pt}{}{\cdot\cdot\cdot4321}{3}\}, \{\genfrac{}{}{0pt}{}{\cdot\cdot\cdot4321}{2}\}, \{\genfrac{}{}{0pt}{}{\cdot\cdot\cdot3\cdot21}{2}\}$ and $\{\genfrac{}{}{0pt}{}{\cdot\cdot\cdot321\cdot}{2}\}$ ($o(\beta) = 4, 6, 9, 12, 15, 17$ respectively), $\genfrac{}{}{0pt}{}{1232221}{2}$ ($o(\beta) = 18$, orthogonal to both of $\{\genfrac{}{}{0pt}{}{2465\cdot21}{3}\}$), $\{\genfrac{}{}{0pt}{}{1233\cdot21}{1}\}$ ($o(\beta) = 18$, orthogonal to one of $\{\genfrac{}{}{0pt}{}{2465\cdot21}{3}\}$). We set

$$G = \langle w_1, w_3, w_4, w_6, w_8\rangle;$$

then G acts as S_4 on $\{\genfrac{}{}{0pt}{}{\cdot\cdot\cdot4321}{3}\}$ and independently as S_2 on each of $\{\genfrac{}{}{0pt}{}{2465\cdot21}{3}\}$ and $\{\genfrac{}{}{0pt}{}{246543\cdot}{3}\}$, so we may fix all of these roots, which span $\mathbb{R}\Phi$, so $W_X = G$.

If $X = X_{34}^1$ we must fix $\{\genfrac{}{}{0pt}{}{24654\cdot\cdot}{3}\}, \{\genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot321}{3}\}, \{\genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot321}{2}\}, \{\genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot2\cdot\cdot}{2}\}$ and $\genfrac{}{}{0pt}{}{1233321}{1}$ ($o(\beta) = 5, 8, 14, 17, 20$ respectively). We set

$$G = \langle w_1, w_3, w_4, w_5, w_7, w_8\rangle;$$

then G acts as S_5 on $\{\genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot321}{3}\}$ and independently as S_3 on $\{\genfrac{}{}{0pt}{}{24654\cdot\cdot}{3}\}$, so we may fix all of these roots, which span $\mathbb{R}\Phi$, so $W_X = G$.

If $X = X_{34}^2$ we must fix $\{\genfrac{}{}{0pt}{1}{2465\cdot\cdot\cdot}{3}\}, \{\genfrac{}{}{0pt}{1}{\cdot\cdot\cdot 4321}{\cdot}\}, \{\genfrac{}{}{0pt}{1}{\cdot\cdot\cdot 3\cdot\cdot\cdot}{\cdot}\}$ ($o(\beta) = 5, 11, 17$ respectively). We set

$$G = \langle w_1, w_2, w_3, w_4, w_6, w_7, w_8 \rangle;$$

then G is transitive on $\{\genfrac{}{}{0pt}{1}{\cdot\cdot\cdot 4321}{\cdot}\}$, so we may fix $\beta_1 = \genfrac{}{}{0pt}{1}{2464321}{3}$. We must then fix $\{\genfrac{}{}{0pt}{1}{\cdot\cdot 44321}{2}\}, \{\genfrac{}{}{0pt}{1}{1233\cdot\cdot\cdot}{\cdot}\}$ (by orthogonality to β_1) and hence $\{\genfrac{}{}{0pt}{1}{\cdot\cdot 54321}{\cdot}\}, \{\genfrac{}{}{0pt}{1}{\cdot\cdot 43\cdot\cdot\cdot}{2}\}$. Inside $\mathrm{stab}_G(\beta_1)$ we have $\langle w_1, w_2, w_3, w_6, w_7, w_8 \rangle$ giving transitivity on $\{\genfrac{}{}{0pt}{1}{\cdot\cdot 54321}{\cdot}\}$, so we may fix $\beta_2 = \genfrac{}{}{0pt}{1}{2454321}{3}$. We must then fix $\{\genfrac{}{}{0pt}{1}{\cdot 354321}{2}\}, \genfrac{}{}{0pt}{1}{1244321}{2}, \{\genfrac{}{}{0pt}{1}{1243\cdot\cdot\cdot}{2}\}, \{\genfrac{}{}{0pt}{1}{1233\cdot\cdot\cdot}{1}\}$ (by orthogonality to β_2) and hence $\{\genfrac{}{}{0pt}{1}{2454321}{2}, \genfrac{}{}{0pt}{1}{\cdot 354321}{3}\}, \{\genfrac{}{}{0pt}{1}{\cdot 344321}{2}\}, \{\genfrac{}{}{0pt}{1}{\cdot 343\cdot\cdot\cdot}{2}\}, \{\genfrac{}{}{0pt}{1}{1233\cdot\cdot\cdot}{2}\}$; we must then fix $\genfrac{}{}{0pt}{1}{2454321}{2}$ (by orthogonality to $\genfrac{}{}{0pt}{1}{1244321}{2}$) and hence $\{\genfrac{}{}{0pt}{1}{\cdot 354321}{3}\}$. Inside $\mathrm{stab}_G(\beta_1, \beta_2)$ we have $\langle w_1, w_6, w_7, w_8 \rangle$, which acts as S_4 on $\{\genfrac{}{}{0pt}{1}{2465\cdot\cdot\cdot}{3}\}$ and independently as S_2 on $\{\genfrac{}{}{0pt}{1}{\cdot 354321}{3}\}$, so we may fix all of these roots. We then have fixed all of the roots $\genfrac{}{}{0pt}{1}{\cdot\cdot\cdot\cdot\cdot\cdot\cdot}{3}$, which span $\mathbb{R}\Phi$; so $W_X = G$.

If $X = X_{36}$ we must fix $\{\genfrac{}{}{0pt}{1}{\cdot\cdot\cdot\cdot\cdot\cdot\cdot}{3}\}, \{\genfrac{}{}{0pt}{1}{\cdot\cdot\cdot\cdot\cdot\cdot\cdot}{2}\}$ ($o(\beta) = 7, 17$ respectively). We set

$$G = \langle w_1, w_3, w_4, w_5, w_6, w_7, w_8 \rangle;$$

then G acts as S_8 on $\{\genfrac{}{}{0pt}{1}{\cdot\cdot\cdot\cdot\cdot\cdot\cdot}{3}\}$, so we may fix all of these roots, which span $\mathbb{R}\Phi$, so $W_X = G$.

This completes the treatment of the sets which are not radical. We therefore now turn to the radical sets $X_{29}^1, \ldots, X_{29}^{430}$. In 414 of these 430 cases we find that arguments entirely similar to those above suffice to determine the stabilizer and its orbits; moreover, in all cases except that of X_{29}^{430} the stabilizer W_X must lie in $\langle w_1, w_2, w_3, w_4, w_5, w_6, w_7 \rangle$, and thus can be read off from the graph representing the set. In the remaining 16 cases it is necessary to argue a little more carefully to produce sufficient sets which must be fixed by W_X on which the group G acts transitively. In the interests of space we shall restrict ourselves to the consideration of these 16 cases; in addition, as $\genfrac{}{}{0pt}{1}{2465432}{3}$ is the only root with orthogonality count 0 in each of these cases, we shall not bother to list it.

If $X = X_{29}^{20}$ all roots (other than $\genfrac{}{}{0pt}{1}{2465432}{3}$) have $o(\beta) = 15$. We set

$$G = \langle w_4, w_3 w_5, w_1 w_6, w_7 w_\sigma \rangle.$$

Here we shall temporarily call a set of six mutually non-orthogonal roots a *clique*; note that $\genfrac{}{}{0pt}{1}{0000011}{0}$ lies in the clique $\{\genfrac{}{}{0pt}{1}{\cdot\cdot\cdot\cdot\cdot 11}{0}\}$. However, there is no clique containing $\genfrac{}{}{0pt}{1}{2465431}{3}$: the roots not orthogonal to it are $\{\genfrac{}{}{0pt}{1}{2\cdot\cdot\cdot 321}{2}, \genfrac{}{}{0pt}{1}{13\cdot 4321}{2}, \genfrac{}{}{0pt}{1}{1\cdot\cdot\cdot 221}{2}, \genfrac{}{}{0pt}{1}{12\cdot 3321}{2}\}$, and $\langle w_4, w_3 w_5, w_1 w_6 \rangle$ is transitive on this set while fixing $\genfrac{}{}{0pt}{1}{2465431}{3}$, so if such a clique existed there would be one also containing $\genfrac{}{}{0pt}{1}{2454321}{2}$; since this root is orthogonal to $\genfrac{}{}{0pt}{1}{12\cdot 3321}{2}, \genfrac{}{}{0pt}{1}{12\cdot\cdot 221}{2}$, the four remaining would have to be chosen from the six roots $\{\genfrac{}{}{0pt}{1}{23\cdot\cdot 321}{2}, \genfrac{}{}{0pt}{1}{13\cdot 4321}{2}, \genfrac{}{}{0pt}{1}{1343221}{2}\}$; however, of these each of $\genfrac{}{}{0pt}{1}{2343321}{2}$ and $\genfrac{}{}{0pt}{1}{1343221}{2}$ is orthogonal to three of the others, which would leave too few roots, while the four remaining roots $\genfrac{}{}{0pt}{1}{\cdot 3\cdot 4321}{2}$ are not mutually non-orthogonal. Thus we must fix $\{\genfrac{}{}{0pt}{1}{2\cdot\cdot\cdot 321}{2}, \genfrac{}{}{0pt}{1}{13\cdot 4321}{2}, \genfrac{}{}{0pt}{1}{\cdot\cdot\cdot\cdot\cdot 11}{2}, \genfrac{}{}{0pt}{1}{1\cdot\cdot\cdot 221}{2}, \genfrac{}{}{0pt}{1}{12\cdot 3321}{2}, \genfrac{}{}{0pt}{1}{\cdot\cdot\cdot\cdot\cdot 11}{0}\}$ (contained in a clique) and $\{\genfrac{}{}{0pt}{1}{2465431}{3}, \genfrac{}{}{0pt}{1}{1221111}{1}, \genfrac{}{}{0pt}{1}{1122111}{1}, \genfrac{}{}{0pt}{1}{0122211}{1}\}$ (not contained in a clique). Now G is transitive on each of these sets and we may argue as before to see that $W_X = G$.

If $X = X_{29}^{37}$ we must fix $\{\genfrac{}{}{0pt}{1}{1233\cdot\cdot 1}{1}, \genfrac{}{}{0pt}{1}{\cdot\cdot\cdot 1111}{0}\}, \{\genfrac{}{}{0pt}{1}{2\cdot 54321}{2}, \genfrac{}{}{0pt}{1}{1\cdot 44321}{2}, \genfrac{}{}{0pt}{1}{0000\cdot\cdot 1}{0}\}, \{\genfrac{}{}{0pt}{1}{\cdot\cdot\cdot 3\cdot\cdot 1}{2}, \genfrac{}{}{0pt}{1}{1111111}{1}, \genfrac{}{}{0pt}{1}{0121111}{1}\}$ ($o(\beta) = 12, 14, 16$ respectively). We set

$$G = \langle w_3, w_6, w_7, w_1 w_4 \rangle.$$

Here we note that among the roots with $o(\beta) = 14$, each is non-orthogonal to just two of the others; however, ${2454321 \atop 2}$ is non-orthogonal to $\{{2354321 \atop 2}, {1344321 \atop 2}\}$, which are themselves orthogonal, while ${0000001 \atop 0}$ is non-orthogonal to $\{{0000\cdot 11 \atop 0}\}$, which are themselves non-orthogonal. Thus we must fix $\{{2\cdot 54321 \atop 2}, {1\cdot 44321 \atop 2}\}$ (by orthogonality among the two non-orthogonal roots with $o(\beta) = 14$) and hence $\{{0000\cdot\cdot 1 \atop 0}\}$; we must then fix $\{{1233\cdot\cdot 1 \atop 1}\}, \{{\cdot\cdot\cdot 3\cdot\cdot 1 \atop 2}\}$ (by orthogonality to two of $\{{0000\cdot\cdot 1 \atop 0}\}$) and hence $\{{\cdot\cdot\cdot 1111 \atop 0}\}, \{{1111111 \atop 1}, {0121111 \atop 1}\}$. Now G is transitive on each of these sets and we may argue as before to see that $W_X = G$.

If $X = X^{43}_{29}$ we must fix all of ${2465431 \atop 3}$, $\{{2465421 \atop 3}, {2\cdot 54321 \atop 2}, {1\cdot 44321 \atop 2}, {\cdot\cdot\cdot 3\cdot 21 \atop 2}, {1232221 \atop 2}, {1233221 \atop 1}\}, \{{\cdot\cdot\cdot 3211 \atop 2}, {1232111 \atop 2}, {1111111 \atop 1}, {0121111 \atop 1}, {\cdot\cdot\cdot\cdot 111 \atop 0}\}$ ($o(\beta) = 12, 14, 16$ respectively). We set

$$G = \langle w_3, w_1 w_4, w_6 w_\tau \rangle.$$

Here we note that among the roots with $o(\beta) = 16$, each is orthogonal to just five of the others; however, ${0000111 \atop 0}$ is orthogonal to $\{{\cdot\cdot\cdot 3211 \atop 2}, {1232111 \atop 2}\}$, which are mutually non-orthogonal, while ${2343211 \atop 2}$ is orthogonal to $\{{0121111 \atop 1}, {0\cdot\cdot\cdot 111 \atop 0}\}$, of which ${0121111 \atop 1}$ is orthogonal to ${0001111 \atop 0}$. Thus we must fix $\{{1232111 \atop 2}, {0000111 \atop 0}\}$ (by non-orthogonality among the five orthogonal roots with $o(\beta) = 16$) and hence $\{{\cdot\cdot\cdot 3211 \atop 2}, {1111111 \atop 1}, {0121111 \atop 1}, {\cdot\cdot\cdot 1111 \atop 0}\}$; we must then fix $\{{2465421 \atop 3}, {2\cdot 54321 \atop 2}, {1\cdot 44321 \atop 2}, {1233221 \atop 1}\}$ (by orthogonality to both of $\{{1232111 \atop 2}, {0000111 \atop 0}\}$), $\{{\cdot\cdot\cdot 3\cdot 21 \atop 2}\}, \{{\cdot\cdot\cdot 3211 \atop 2}, {\cdot\cdot\cdot 1111 \atop 0}\}$ (by orthogonality to one of $\{{1232111 \atop 2}, {0000111 \atop 0}\}$) and hence ${1232221 \atop 2}$, $\{{1111111 \atop 1}, {0121111 \atop 1}\}$. Now G is transitive on the sets of roots with $o(\beta) = 16$, and we may argue as before to see that $W_X = G$; finally we note that the set of roots with $o(\beta) = 14$ which are orthogonal to both of $\{{1232111 \atop 2}, {0000111 \atop 0}\}$ is in fact a union of the two W_X-orbits $\{{2\cdot 54321 \atop 2}, {1\cdot 44321 \atop 2}\}$ and $\{{2465421 \atop 3}, {1233221 \atop 1}\}$.

If $X = X^{44}_{29}$ we must fix $\{{2465421 \atop 3}, {2454321 \atop 3}, {\cdot 3\cdot\cdot\cdot 21 \atop 2}, {12\cdot\cdot\cdot 21 \atop 2}, {123\cdot 221 \atop 1}\}, \{{\cdot\cdot 43211 \atop 2}, {1232111 \atop 2}, {\cdot 111111 \atop 0}, {0011111 \atop 1}, {000\cdot 111 \atop 0}\}$, ${0000001 \atop 0}$ ($o(\beta) = 14, 16, 18$ respectively). We set

$$G = \langle w_1, w_5, w_4 w_6 w_\tau \rangle.$$

Here we note that among the roots with $o(\beta) = 16$, each is orthogonal to just four of the others; however, ${1243211 \atop 2}$ is orthogonal to $\{{\cdot 111111 \atop 0}, {000\cdot 111 \atop 0}\}$, which are also orthogonal to ${1232111 \atop 2}$, while ${0011111 \atop 1}$ is orthogonal to $\{{\cdot 343211 \atop 2}, {\cdot 111111 \atop 0}\}$, and none of the other roots is orthogonal to this set. Thus we must fix $\{{1243211 \atop 2}, {1232111 \atop 2}, {000\cdot 111 \atop 0}\}$ (by non-uniqueness of the set of orthogonal roots with $o(\beta) = 16$) and hence $\{{\cdot 343211 \atop 2}, {\cdot 111111 \atop 0}, {0011111 \atop 1}\}$; we must then fix $\{{12\cdot\cdot\cdot 21 \atop 2}\}, \{{\cdot 343211 \atop 2}, {\cdot 111111 \atop 0}\}$ (by orthogonality to two of $\{{1243211 \atop 2}, {1232111 \atop 2}, {000\cdot 111 \atop 0}\}$) and hence $\{{2465421 \atop 3}, {2454321 \atop 3}, {\cdot 3\cdot\cdot\cdot 21 \atop 2}, {123\cdot 221 \atop 1}\}$, ${0011111 \atop 1}$. Next we note that each root in $\{{2465421 \atop 3}, {2454321 \atop 3}, {\cdot 3\cdot\cdot\cdot 21 \atop 2}, {123\cdot 221 \atop 1}\}$ is orthogonal to just three in $\{{12\cdot\cdot\cdot 21 \atop 2}\}$; however, ${2354321 \atop 2}$ is orthogonal to $\{{123\cdot\cdot 21 \atop 2}\}$, which are also orthogonal to ${1354321 \atop 2}$, while ${2465421 \atop 3}$ is orthogonal to $\{{12\cdot\cdot 221 \atop 2}\}$, and none of the other roots is orthogonal to this set. Thus we must fix $\{{\cdot 3\cdot\cdot\cdot 21 \atop 2}\}$ (by non-uniqueness of the set of orthogonal roots among $\{{12\cdot\cdot\cdot 21 \atop 2}\}$) and hence $\{{2465421 \atop 3}, {2454321 \atop 3}, {123\cdot 221 \atop 1}\}$. Now G is transitive on $\{{\cdot 3\cdot\cdot\cdot 21 \atop 2}\}$, and we may argue as before to see that $W_X = G$; finally we note that the set of roots with $o(\beta) = 14$ orthogonal to two of $\{{1243211 \atop 2}, {1232111 \atop 2}, {000\cdot 111 \atop 0}\}$ is in fact a union of the two W_X-orbits $\{{124\cdot 321 \atop 2}, {123\cdot 221 \atop 2}\}$ and $\{{1243221 \atop 2}, {1233321 \atop 2}\}$.

If $X = X_{29}^{51}$ we must fix $\{\genfrac{}{}{0pt}{}{246\cdot\cdot\cdot1}{3}, \genfrac{}{}{0pt}{}{\cdot\cdot54321}{2}\}, \{\genfrac{}{}{0pt}{}{\cdot\cdot4\cdot\cdot\cdot1}{2}, \genfrac{}{}{0pt}{}{1233211}{1}, \genfrac{}{}{0pt}{}{1232221}{1}\}, \{\genfrac{}{}{0pt}{}{1233\cdot21}{2}, \genfrac{}{}{0pt}{}{1232\cdot11}{2}, \genfrac{}{}{0pt}{}{\cdot\cdot11111}{0}\}$ ($o(\beta) = 9, 13, 17$ respectively). We set

$$G = \langle w_1, w_3, w_6, w_5w_7 \rangle.$$

Here we note that among the roots with $o(\beta) = 17$, each is orthogonal to just four others; however, $\genfrac{}{}{0pt}{}{0011111}{0}$ is orthogonal to $\{\genfrac{}{}{0pt}{}{1233\cdot21}{2}, \genfrac{}{}{0pt}{}{1232\cdot11}{2}\}$, which are also orthogonal to both $\genfrac{}{}{0pt}{}{1111111}{0}$ and $\genfrac{}{}{0pt}{}{0111111}{0}$, while $\genfrac{}{}{0pt}{}{1233321}{2}$ is orthogonal to $\{\genfrac{}{}{0pt}{}{1232111}{2}, \genfrac{}{}{0pt}{}{\cdot\cdot11111}{0}\}$, and none of the other roots is orthogonal to this set. Thus we must fix $\{\genfrac{}{}{0pt}{}{\cdot\cdot11111}{0}\}$ (by non-uniqueness of the set of orthogonal roots with $o(\beta) = 17$) and hence $\{\genfrac{}{}{0pt}{}{1233\cdot21}{2}, \genfrac{}{}{0pt}{}{1232\cdot11}{2}\}$; we must then fix $\{\genfrac{}{}{0pt}{}{246\cdot\cdot\cdot1}{3}\}$ (by orthogonality to all of $\{\genfrac{}{}{0pt}{}{\cdot\cdot11111}{0}\}$), $\{\genfrac{}{}{0pt}{}{\cdot\cdot4\cdot\cdot\cdot1}{2}\}$ (by orthogonality to two of $\{\genfrac{}{}{0pt}{}{\cdot\cdot11111}{0}\}$) and hence $\{\genfrac{}{}{0pt}{}{\cdot\cdot54321}{2}\}, \{\genfrac{}{}{0pt}{}{1233211}{1}, \genfrac{}{}{0pt}{}{1232221}{1}\}$. Now G is transitive on each of these sets and we may argue as before to see that $W_X = G$.

If $X = X_{29}^{72}$ we must fix $\genfrac{}{}{0pt}{}{2465431}{3}, \{\genfrac{}{}{0pt}{}{2465421}{3}, \genfrac{}{}{0pt}{}{2454321}{3}, \genfrac{}{}{0pt}{}{\cdot3\cdot\cdot\cdot21}{2}, \genfrac{}{}{0pt}{}{12\cdot\cdot\cdot21}{2}, \genfrac{}{}{0pt}{}{123\cdot221}{1}\}$ and $\{\genfrac{}{}{0pt}{}{\cdot\cdot43211}{2}, \genfrac{}{}{0pt}{}{1232111}{2}, \genfrac{}{}{0pt}{}{\cdot111111}{0}, \genfrac{}{}{0pt}{}{0011111}{1}, \genfrac{}{}{0pt}{}{000\cdot111}{0}\}$ ($o(\beta) = 9, 13, 17$ respectively). Replacing the fixed root $\genfrac{}{}{0pt}{}{2465431}{3}$ by $\genfrac{}{}{0pt}{}{0000001}{0}$ would give the sets seen in treating X_{29}^{44}; consequently the argument here is identical to that two paragraphs earlier.

If $X = X_{29}^{77}$ we must fix $\{\genfrac{}{}{0pt}{}{2465\cdot\cdot1}{3}, \genfrac{}{}{0pt}{}{2\cdot\cdot4321}{2}\}, \{\genfrac{}{}{0pt}{}{1\cdot\cdot4321}{2}, \genfrac{}{}{0pt}{}{2343\cdot\cdot1}{2}\}, \{\genfrac{}{}{0pt}{}{1\cdot43321}{2}, \genfrac{}{}{0pt}{}{1343221}{2}, \genfrac{}{}{0pt}{}{1243211}{2}, \genfrac{}{}{0pt}{}{12332\cdot1}{2}, \genfrac{}{}{0pt}{}{1232111}{1}, \genfrac{}{}{0pt}{}{1222211}{1}, \genfrac{}{}{0pt}{}{1122221}{1}, \genfrac{}{}{0pt}{}{1111111}{0}\}, \{\genfrac{}{}{0pt}{}{1232\cdot\cdot1}{2}, \genfrac{}{}{0pt}{}{0\cdot\cdot1111}{0}\}$ ($o(\beta) = 11, 13, 15, 17$ respectively). We set

$$G = \langle w_3w_7, w_4w_6 \rangle.$$

Here we note that each root with $o(\beta) = 15$ is orthogonal to just three roots with $o(\beta) = 17$; however, $\genfrac{}{}{0pt}{}{1111111}{0}$ is orthogonal to $\{\genfrac{}{}{0pt}{}{1232\cdot\cdot1}{2}\}$, which are mutually non-orthogonal, while $\genfrac{}{}{0pt}{}{1343321}{2}$ is orthogonal to $\{\genfrac{}{}{0pt}{}{1232111}{2}, \genfrac{}{}{0pt}{}{00\cdot1111}{0}\}$, of which $\genfrac{}{}{0pt}{}{1232111}{2}$ is orthogonal to the other two roots. Thus we must fix $\genfrac{}{}{0pt}{}{1111111}{0}$ (by mutual non-orthogonality among the three orthogonal roots with $o(\beta) = 17$) and hence $\{\genfrac{}{}{0pt}{}{1\cdot43321}{2}, \genfrac{}{}{0pt}{}{1343221}{2}, \genfrac{}{}{0pt}{}{1243211}{2}, \genfrac{}{}{0pt}{}{12332\cdot1}{2}, \genfrac{}{}{0pt}{}{1232111}{1}, \genfrac{}{}{0pt}{}{1222211}{1}, \genfrac{}{}{0pt}{}{1122221}{1}\}$; we must then fix $\{\genfrac{}{}{0pt}{}{2465\cdot\cdot1}{3}\}, \{\genfrac{}{}{0pt}{}{1\cdot\cdot4321}{2}\}, \{\genfrac{}{}{0pt}{}{1\cdot43321}{2}, \genfrac{}{}{0pt}{}{1343221}{2}, \genfrac{}{}{0pt}{}{1243211}{2}, \genfrac{}{}{0pt}{}{12332\cdot1}{2}\}, \{\genfrac{}{}{0pt}{}{1232\cdot\cdot1}{2}\}$ (by orthogonality to $\genfrac{}{}{0pt}{}{1111111}{0}$) and hence $\{\genfrac{}{}{0pt}{}{2\cdot\cdot4321}{2}\}, \{\genfrac{}{}{0pt}{}{2343\cdot\cdot1}{2}\}, \{\genfrac{}{}{0pt}{}{1232111}{1}, \genfrac{}{}{0pt}{}{1222211}{1}, \genfrac{}{}{0pt}{}{1122221}{1}\}$ and $\{\genfrac{}{}{0pt}{}{0\cdot\cdot1111}{0}\}$. Now G is transitive on each of these sets and we may argue as before to see that $W_X = G$.

If $X = X_{29}^{88}$ we must fix all of $\genfrac{}{}{0pt}{}{2454321}{2}, \{\genfrac{}{}{0pt}{}{24654\cdot1}{3}, \genfrac{}{}{0pt}{}{\cdot3\cdot\cdot\cdot\cdot1}{2}, \genfrac{}{}{0pt}{}{1233221}{1}, \genfrac{}{}{0pt}{}{1232211}{1}, \genfrac{}{}{0pt}{}{122\cdot111}{1}, \genfrac{}{}{0pt}{}{\cdot111111}{0}\}, \{\genfrac{}{}{0pt}{}{12432\cdot1}{2}, \genfrac{}{}{0pt}{}{1233321}{2}, \genfrac{}{}{0pt}{}{1233211}{2}, \genfrac{}{}{0pt}{}{1232221}{2}, \genfrac{}{}{0pt}{}{1232111}{2}, \genfrac{}{}{0pt}{}{00\cdot\cdot111}{0}\}$ ($o(\beta) = 9, 13, 17$ respectively). We set

$$G = \langle w_1, w_5w_7, w_4w_6w_\tau \rangle.$$

Here we note that among the roots with $o(\beta) = 17$, each is orthogonal to just four others; however, $\genfrac{}{}{0pt}{}{1233321}{2}$ is orthogonal to $\{\genfrac{}{}{0pt}{}{12432\cdot1}{2}, \genfrac{}{}{0pt}{}{1232111}{2}, \genfrac{}{}{0pt}{}{0011111}{0}\}$, of which both $\genfrac{}{}{0pt}{}{1243221}{2}$ and $\genfrac{}{}{0pt}{}{1243211}{2}$ are non-orthogonal to each of the others, while $\genfrac{}{}{0pt}{}{0011111}{0}$ is orthogonal to $\{\genfrac{}{}{0pt}{}{1233321}{2}, \genfrac{}{}{0pt}{}{1233211}{2}, \genfrac{}{}{0pt}{}{1232221}{2}, \genfrac{}{}{0pt}{}{1232111}{2}\}$, each of which is orthogonal to one of the others. Thus we must fix $\{\genfrac{}{}{0pt}{}{12432\cdot1}{2}, \genfrac{}{}{0pt}{}{1233321}{2}, \genfrac{}{}{0pt}{}{1232111}{2}, \genfrac{}{}{0pt}{}{000\cdot111}{0}\}, \{\genfrac{}{}{0pt}{}{1233211}{2}, \genfrac{}{}{0pt}{}{1232221}{2}, \genfrac{}{}{0pt}{}{0011111}{0}\}$ (by the degree of orthogonality among the four orthogonal roots with $o(\beta) = 17$). Now G is transitive on these sets, and we may argue as before to see that $W_X = G$; finally we note that the set of roots with $o(\beta) = 13$ is in fact a union of the two W_X-orbits $\{\genfrac{}{}{0pt}{}{\cdot3\cdot\cdot\cdot\cdot1}{2}, \genfrac{}{}{0pt}{}{\cdot111111}{0}\}$ and $\{\genfrac{}{}{0pt}{}{24654\cdot1}{3}, \genfrac{}{}{0pt}{}{1233221}{1}, \genfrac{}{}{0pt}{}{1232211}{1}, \genfrac{}{}{0pt}{}{122\cdot111}{1}\}$.

If $X = X_{29}^{97}$ we must fix $\{\genfrac{}{}{0pt}{}{2454321}{\cdot}, \genfrac{}{}{0pt}{}{\cdot 34\cdot 321}{2}, \genfrac{}{}{0pt}{}{1233321}{\cdot}\}$, $\{\genfrac{}{}{0pt}{}{2354321}{2}, \genfrac{}{}{0pt}{}{1354321}{3}, \genfrac{}{}{0pt}{}{\cdot 3432\cdot 1}{2}, \genfrac{}{}{0pt}{}{124\cdot 321}{2}, \genfrac{}{}{0pt}{}{1233221}{2}, \genfrac{}{}{0pt}{}{1232211}{2}, \genfrac{}{}{0pt}{}{1233211}{1}, \genfrac{}{}{0pt}{}{1232221}{1}, \genfrac{}{}{0pt}{}{1111111}{1}, \genfrac{}{}{0pt}{}{0111111}{0}, \genfrac{}{}{0pt}{}{000\cdot 111}{0}\}$, $\{\genfrac{}{}{0pt}{}{12432\cdot 1}{2}, \genfrac{}{}{0pt}{}{00000\cdot 1}{0}\}$ ($o(\beta) = 13, 15, 17$ respectively). We set

$$G = \langle w_5w_7, w_1w_2w_5, {w_1w_4w_6w_\tau}^{w_3w_5w_4}\rangle.$$

Here we note that each root with $o(\beta) = 15$ is orthogonal to just two roots with $o(\beta) = 17$, these two being non-orthogonal; however, the roots with $o(\beta) = 15$ orthogonal to $\{\genfrac{}{}{0pt}{}{00000\cdot 1}{0}\}$ are $\{\genfrac{}{}{0pt}{}{2354321}{2}, \genfrac{}{}{0pt}{}{1354321}{3}, \genfrac{}{}{0pt}{}{124\cdot 321}{2}\}$, and the only orthogonality among these four roots is that between $\genfrac{}{}{0pt}{}{2354321}{2}$ and $\genfrac{}{}{0pt}{}{1354321}{3}$. Thus we must fix each of $\{\genfrac{}{}{0pt}{}{2354321}{2}, \genfrac{}{}{0pt}{}{1354321}{3}, \genfrac{}{}{0pt}{}{1233221}{2}, \genfrac{}{}{0pt}{}{1232211}{2}, \genfrac{}{}{0pt}{}{1233211}{1}, \genfrac{}{}{0pt}{}{1232221}{1}, \genfrac{}{}{0pt}{}{1111111}{1}, \genfrac{}{}{0pt}{}{0111111}{0}\}$, $\{\genfrac{}{}{0pt}{}{\cdot 3432\cdot 1}{2}, \genfrac{}{}{0pt}{}{124\cdot 321}{2}, \genfrac{}{}{0pt}{}{000\cdot 111}{0}\}$ (by the degree of orthogonality among the roots with $o(\beta) = 15$ orthogonal to the same pair of roots with $o(\beta) = 17$). Now G is transitive on these sets and we may argue as before to see that $W_X = G$; finally we note that the set of roots with $o(\beta) = 13$ is in fact a union of the two W_X-orbits $\{\genfrac{}{}{0pt}{}{2454321}{\cdot}, \genfrac{}{}{0pt}{}{1233321}{\cdot}\}$ and $\{\genfrac{}{}{0pt}{}{\cdot 34\cdot 321}{2}\}$.

If $X = X_{29}^{169}$ we must fix $\genfrac{}{}{0pt}{}{2465431}{3}$, $\{\genfrac{}{}{0pt}{}{1\cdot\cdot\cdot 321}{\cdot}\}$, $\{\genfrac{}{}{0pt}{}{2454321}{3}, \genfrac{}{}{0pt}{}{23\cdot\cdot 321}{2}, \genfrac{}{}{0pt}{}{1\cdot\cdot 3221}{2}, \genfrac{}{}{0pt}{}{1232221}{1}\}$, $\{\genfrac{}{}{0pt}{}{2343211}{2}, \genfrac{}{}{0pt}{}{0000011}{0}\}$ ($o(\beta) = 10, 12, 14, 18$ respectively), $\genfrac{}{}{0pt}{}{2343221}{2}$ ($o(\beta) = 16$, not orthogonal to $\genfrac{}{}{0pt}{}{2465431}{3}$), $\{\genfrac{}{}{0pt}{}{1\cdot 43211}{2}, \genfrac{}{}{0pt}{}{1232211}{2}, \genfrac{}{}{0pt}{}{1233211}{1}, \genfrac{}{}{0pt}{}{0111111}{0}, \genfrac{}{}{0pt}{}{0011111}{1}, \genfrac{}{}{0pt}{}{000\cdot 111}{0}\}$ ($o(\beta) = 16$, orthogonal to $\genfrac{}{}{0pt}{}{2465431}{3}$). We set

$$G = \langle w_2w_3w_5, {w_1w_4w_6w_\tau}^{w_3w_4w_5w_4w_3w_1}\rangle.$$

Here we note that each of the roots with $o(\beta) = 14$ is orthogonal to just one root with $o(\beta) = 18$; however, the roots with $o(\beta) = 14$ orthogonal to $\genfrac{}{}{0pt}{}{0000011}{0}$ are $\{\genfrac{}{}{0pt}{}{2454321}{3}, \genfrac{}{}{0pt}{}{23\cdot\cdot 321}{2}\}$, and the only orthogonality among these four roots is that between $\genfrac{}{}{0pt}{}{2454321}{3}$ and $\genfrac{}{}{0pt}{}{2354321}{2}$. Thus we must fix $\{\genfrac{}{}{0pt}{}{2454321}{3}, \genfrac{}{}{0pt}{}{2354321}{2}, \genfrac{}{}{0pt}{}{1233221}{2}, \genfrac{}{}{0pt}{}{1232221}{1}\}$, $\{\genfrac{}{}{0pt}{}{234\cdot 321}{2}, \genfrac{}{}{0pt}{}{1\cdot 43221}{2}\}$ (by the degree of orthogonality among the roots with $o(\beta) = 14$ orthogonal to the same root with $o(\beta) = 18$). Now G is transitive on these sets and we may argue as before to see that $W_X = G$; finally we note that the set of roots with $o(\beta) = 12$ is in fact a union of the three W_X-orbits $\{\genfrac{}{}{0pt}{}{1354321}{\cdot}, \genfrac{}{}{0pt}{}{1233321}{\cdot}\}$, $\{\genfrac{}{}{0pt}{}{1344321}{2}, \genfrac{}{}{0pt}{}{1243321}{2}\}$ and $\{\genfrac{}{}{0pt}{}{1343321}{2}, \genfrac{}{}{0pt}{}{1244321}{2}\}$, while the set of roots with $o(\beta) = 16$ orthogonal to $\genfrac{}{}{0pt}{}{2465431}{3}$ is in fact a union of the two W_X-orbits $\{\genfrac{}{}{0pt}{}{1\cdot 43211}{2}, \genfrac{}{}{0pt}{}{000\cdot 111}{0}\}$ and $\{\genfrac{}{}{0pt}{}{1232211}{2}, \genfrac{}{}{0pt}{}{1233211}{1}, \genfrac{}{}{0pt}{}{0111111}{0}, \genfrac{}{}{0pt}{}{0011111}{1}\}$.

If $X = X_{29}^{188}$ we must fix $\{\genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot\cdot 11}{2}, \genfrac{}{}{0pt}{}{123\cdot 211}{1}, \genfrac{}{}{0pt}{}{1222111}{1}, \genfrac{}{}{0pt}{}{1121111}{1}, \genfrac{}{}{0pt}{}{0\cdot 11111}{1}, \genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot\cdot 11}{0}\}$, $\{\genfrac{}{}{0pt}{}{2344321}{2}, \genfrac{}{}{0pt}{}{1354321}{2}, \genfrac{}{}{0pt}{}{\cdot 343\cdot 21}{2}, \genfrac{}{}{0pt}{}{1244321}{2}, \genfrac{}{}{0pt}{}{1243221}{2}, \genfrac{}{}{0pt}{}{1233321}{2}\}$, $\genfrac{}{}{0pt}{}{2465431}{3}$ ($o(\beta) = 14, 16, 18$ respectively). We set

$$G = \langle w_\tau, w_3w_5, w_1w_4w_6\rangle.$$

Here we note that each root with $o(\beta) = 14$ is non-orthogonal to just three roots with $o(\beta) = 16$; however, $\genfrac{}{}{0pt}{}{0000011}{0}$ is non-orthogonal to $\{\genfrac{}{}{0pt}{}{\cdot\cdot 43221}{2}\}$, which are mutually non-orthogonal, while $\genfrac{}{}{0pt}{}{1232211}{1}$ is non-orthogonal to $\{\genfrac{}{}{0pt}{}{1354321}{2}, \genfrac{}{}{0pt}{}{\cdot 343321}{2}\}$, of which $\genfrac{}{}{0pt}{}{1354321}{2}$ is orthogonal to $\genfrac{}{}{0pt}{}{2343321}{2}$. Thus we must fix $\{\genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot\cdot 11}{2}, \genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot\cdot 11}{0}\}$ (by mutual non-orthogonality among the three non-orthogonal roots with $o(\beta) = 16$) and hence $\{\genfrac{}{}{0pt}{}{123\cdot 211}{1}, \genfrac{}{}{0pt}{}{1222111}{1}, \genfrac{}{}{0pt}{}{1121111}{1}, \genfrac{}{}{0pt}{}{0\cdot 11111}{1}\}$. Now G is transitive on these sets and we may argue as before to see that $W_X = G$; finally we note that the set of roots with $o(\beta) = 16$ is in fact a union of the two W_X-orbits $\{\genfrac{}{}{0pt}{}{234\cdot 321}{2}, \genfrac{}{}{0pt}{}{1354321}{2}, \genfrac{}{}{0pt}{}{1\cdot 43221}{2}, \genfrac{}{}{0pt}{}{1233321}{2}\}$ and $\{\genfrac{}{}{0pt}{}{2343221}{2}, \genfrac{}{}{0pt}{}{1343321}{2}, \genfrac{}{}{0pt}{}{1244321}{2}\}$.

If $X = X_{29}^{191}$ we must fix all of $\substack{1243221\\2}$, $\{\substack{2465431\\3}, \substack{2354321\\2}, \substack{1354321\\3}, \substack{\cdot343221\\2}, \substack{124\cdot321\\2}, \substack{1243211\\2}, \substack{1233221\\2}, \substack{1232221\\1}, \substack{1232111\\\cdot}, \substack{0011111\\\cdot}, \substack{0000011\\0}\}$, $\{\substack{\cdot34\cdot321\\2}, \substack{\cdot343211\\2}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{1111111\\1}, \substack{0111111\\0}, \substack{000\cdot111\\0}\}$ ($o(\beta) = 12, 14, 16$ respectively). We set

$$G = \langle w_1w_2w_5, w_1w_4w_6{w_\tau}^{w_3w_5w_4}\rangle.$$

Here we note that among the roots with $o(\beta) = 14$, each is orthogonal to just six of the others; however, $\substack{2465431\\3}$ is orthogonal to $\{\substack{1243211\\2}, \substack{1232111\\\cdot}, \substack{0011111\\\cdot}, \substack{0000011\\0}\}$, each of which is orthogonal to one of the others, $\substack{2354321\\2}$ is orthogonal to $\{\substack{1354321\\3}, \substack{1343221\\2}, \substack{1233221\\2}, \substack{1232111\\2}, \substack{0011111\\1}, \substack{0000011\\0}\}$, of which two ($\substack{1233221\\2}$ and $\substack{1232111\\2}$) are orthogonal to none of the others, and $\substack{2343221\\2}$ is orthogonal to $\{\substack{1354321\\3}, \substack{124\cdot321\\2}, \substack{1243211\\2}, \substack{0011111\\\cdot}\}$, among which the only instance of orthogonality is between $\substack{1354321\\3}$ and $\substack{0011111\\0}$. Thus we must fix $\substack{2465431\\3}$, $\{\substack{2354321\\2}, \substack{1354321\\3}, \substack{1243211\\2}, \substack{1233221\\2}, \substack{1232221\\1}, \substack{1232111\\\cdot}, \substack{0011111\\\cdot}, \substack{0000011\\0}\}$, $\{\substack{\cdot343221\\2}, \substack{124\cdot321\\2}\}$ (by the degree of non-orthogonality among the six orthogonal roots with $o(\beta) = 16$); we must then fix $\{\substack{1243211\\2}, \substack{1232111\\\cdot}, \substack{0011111\\\cdot}, \substack{0000011\\0}\}$ and $\{\substack{\cdot343211\\2}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{1111111\\1}, \substack{0111111\\0}, \substack{000\cdot111\\0}\}$ (by orthogonality to $\substack{2465431\\3}$) and hence $\{\substack{2354321\\2}, \substack{1354321\\3}, \substack{1233221\\2}, \substack{1232221\\1}\}$, $\{\substack{\cdot34\cdot321\\2}\}$. Now G is transitive on the set $\{\substack{\cdot343221\\2}, \substack{124\cdot321\\2}\}$ and we may argue as before to see that $W_X = G$; finally we note that the set of roots with $o(\beta) = 14$ orthogonal to $\substack{2465431\\3}$ is in fact a union of the two W_X-orbits $\{\substack{1232111\\\cdot}, \substack{0011111\\\cdot}\}$ and $\{\substack{1243211\\2}, \substack{0000011\\0}\}$, the set of roots with $o(\beta) = 16$ orthogonal to $\substack{2465431\\3}$ is in fact a union of the two W_X-orbits $\{\substack{\cdot343211\\2}, \substack{000\cdot111\\0}\}$ and $\{\substack{1232211\\2}, \substack{1233211\\1}, \substack{1111111\\1}, \substack{0111111\\0}\}$, while the set of roots with $o(\beta) = 16$ not orthogonal to $\substack{2465431\\3}$ is in fact a union of the two W_X-orbits $\{\substack{2344321\\2}, \substack{1343321\\2}\}$ and $\{\substack{2343321\\2}, \substack{1344321\\2}\}$.

If $X = X_{29}^{192}$ we must fix $\substack{1232111\\2}$, $\{\substack{\cdot\cdot\cdot\cdot2\cdot1\\2}, \substack{12\cdot2111\\1}, \substack{1121111\\1}, \substack{0\cdot11111\\1}\}$, $\{\substack{24654\cdot1\\3}, \substack{2344321\\2}, \substack{1354321\\2}, \substack{\cdot343321\\2}, \substack{1244321\\2}, \substack{\cdot\cdot\cdot\cdot111\\0}\}$ ($o(\beta) = 12, 14, 16$ respectively). We set

$$G = \langle w_7, w_1w_4, w_3w_5\rangle.$$

Here we note that among the roots with $o(\beta) = 16$, each is orthogonal to just five of the others; however, $\substack{2465431\\3}$ is orthogonal to $\{\substack{\cdot\cdot\cdot\cdot111\\0}\}$, which are mutually non-orthogonal, while $\substack{0011111\\0}$ is orthogonal to $\{\substack{24654\cdot1\\3}, \substack{2344321\\2}, \substack{\cdot343321\\2}\}$, among which the single instance of orthogonality is between $\substack{2344321\\2}$ and $\substack{1343321\\2}$, and indeed $\substack{1354321\\2}$ is orthogonal to $\{\substack{234\cdot321\\2}, \substack{1111111\\0}, \substack{000\cdot111\\0}\}$, of which the only root orthogonal to none of the others is $\substack{1111111\\0}$. Thus we must fix $\{\substack{24654\cdot1\\3}\}$, $\{\substack{\cdot\cdot\cdot\cdot111\\0}\}$, $\{\substack{2344321\\2}, \substack{1354321\\2}, \substack{\cdot343321\\2}, \substack{1244321\\2}\}$ (by the degree of orthogonality among the five orthogonal roots with $o(\beta) = 16$); we must then fix $\{\substack{12\cdot2111\\1}, \substack{1121111\\1}, \substack{0\cdot11111\\1}\}$ (by orthogonality to both of $\{\substack{24654\cdot1\\3}\}$) and hence $\{\substack{\cdot\cdot\cdot\cdot2\cdot1\\2}\}$. Now G is transitive on each of these sets and we may argue as before to see that $W_X = G$.

If $X = X_{29}^{193}$ we must fix $\substack{1244321\\2}$, $\{\substack{\cdot354321\\3}, \substack{2344321\\2}, \substack{1354321\\2}, \substack{12\cdot3\cdot\cdot1\\\cdot}, \substack{00\cdot1111\\0}\}$ and $\{\substack{2454321\\2}, \substack{\cdot343\cdot\cdot1\\2}, \substack{1121111\\1}, \substack{0111111\\1}, \substack{0000\cdot\cdot1\\0}\}$ ($o(\beta) = 12, 14, 16$ respectively). We set

$$G = \langle w_6, w_7, w_1w_4, w_2w_\sigma\rangle.$$

Here we note that among the roots with $o(\beta) = 16$, each is orthogonal to just five of the others; however, $\substack{2454321\\2}$ is orthogonal to $\{\substack{1121111\\1}, \substack{0111111\\1}, \substack{0000\cdot\cdot1\\0}\}$, among which the single instance of orthogonality is between $\substack{1121111\\1}$ and $\substack{0111111\\1}$, while $\substack{2343321\\2}$ is orthogonal to $\{\substack{13432\cdot1\\2}, \substack{0111111\\1}, \substack{00000\cdot1\\0}\}$, of which only $\substack{0111111\\1}$ is orthogonal to none of the others. Thus we must fix $\{\substack{2454321\\2}, \substack{1121111\\1}, \substack{0111111\\1}\}$, $\{\substack{\cdot343\cdot\cdot1\\2},$

$\genfrac{}{}{0pt}{}{0000\cdot\cdot1}{0}\}$ (by the degree of orthogonality among the five orthogonal roots with $o(\beta) = 16$). Now G is transitive on these sets and we may argue as before to see that $W_X = G$; finally we note that the set of roots with $o(\beta) = 14$ is in fact a union of the two W_X-orbits $\{\genfrac{}{}{0pt}{}{12\cdot3\cdot\cdot1}{\cdot}\}$ and $\{\genfrac{}{}{0pt}{}{\cdot354321}{3}, \genfrac{}{}{0pt}{}{2344321}{2}, \genfrac{}{}{0pt}{}{1354321}{2}, \genfrac{}{}{0pt}{}{00\cdot1111}{0}\}$.

If $X = X_{29}^{307}$ all roots (other than $\genfrac{}{}{0pt}{}{2465432}{3}$) have $o(\beta) = 15$. We set

$$G = \langle w_1w_4w_6, w_2w_3w_5, w_2w_7w_\sigma\rangle.$$

Here we note that the 12 roots not orthogonal to $\genfrac{}{}{0pt}{}{0000001}{0}$ are $\{\genfrac{}{}{0pt}{}{\cdot\cdot\cdot3211}{2}, \genfrac{}{}{0pt}{}{1232211}{1}, \genfrac{}{}{0pt}{}{1222111}{1}, \genfrac{}{}{0pt}{}{1121111}{1}, \genfrac{}{}{0pt}{}{0111111}{1}, \genfrac{}{}{0pt}{}{00\cdot\cdot\cdot11}{0}\}$, and $\langle w_1w_4w_6, w_2w_3w_5\rangle$ acts transitively on this set while fixing $\genfrac{}{}{0pt}{}{0000001}{0}$; each of these 12 roots is orthogonal to just five of the others; of the five roots orthogonal to $\genfrac{}{}{0pt}{}{2343211}{2}$, there are four which are also orthogonal to $\genfrac{}{}{0pt}{}{1343211}{2}$, and four which are also orthogonal to $\genfrac{}{}{0pt}{}{1243211}{2}$. Similarly, the 12 roots not orthogonal to $\genfrac{}{}{0pt}{}{0000011}{0}$ are $\{\genfrac{}{}{0pt}{}{\cdot\cdot43221}{2}, \genfrac{}{}{0pt}{}{1232221}{2}, \genfrac{}{}{0pt}{}{1233221}{1}, \genfrac{}{}{0pt}{}{1222111}{1}, \genfrac{}{}{0pt}{}{1121111}{1}, \genfrac{}{}{0pt}{}{0111111}{1}, \genfrac{}{}{0pt}{}{00\cdot\cdot111}{0}, \genfrac{}{}{0pt}{}{0000001}{0}\}$; again each of these 12 roots is orthogonal to just five of the others; but this time, of the five roots orthogonal to $\genfrac{}{}{0pt}{}{1343221}{2}$, there are four which are also orthogonal to $\genfrac{}{}{0pt}{}{2343221}{2}$, but no other root among the 12 is orthogonal to four of the five. Thus we must fix $\{\genfrac{}{}{0pt}{}{2343221}{2}, \genfrac{}{}{0pt}{}{1343321}{2}, \genfrac{}{}{0pt}{}{1244321}{2}, \genfrac{}{}{0pt}{}{0000001}{0}\}$ and its complement in $X \setminus \{\genfrac{}{}{0pt}{}{2465432}{3}\}$ (by this configuration among the non-orthogonal roots). Now G is transitive on each of these sets and we may argue as before to see that $W_X = G$.

If $X = X_{29}^{308}$ all roots (other than $\genfrac{}{}{0pt}{}{2465432}{3}$) have $o(\beta) = 15$. We set

$$G = \langle w_1, w_\sigma, w_2w_5, w_4w_6, w_5w_7\rangle.$$

As with the treatment of the set X_{29}^{20}, we shall call a set of six mutually non-orthogonal roots a *clique*; note that $\genfrac{}{}{0pt}{}{0000001}{0}$ lies in the clique $\{\genfrac{}{}{0pt}{}{00\cdot\cdot\cdot\cdot1}{\cdot}\}$. However, there is no clique containing $\genfrac{}{}{0pt}{}{1233321}{2}$: the roots not orthogonal to it are $\{\genfrac{}{}{0pt}{}{\cdot354321}{3}, \genfrac{}{}{0pt}{}{\cdot34\cdot321}{2}, \genfrac{}{}{0pt}{}{1233321}{1}, \genfrac{}{}{0pt}{}{1233211}{2}, \genfrac{}{}{0pt}{}{1232221}{2}, \genfrac{}{}{0pt}{}{0011111}{1}, \genfrac{}{}{0pt}{}{000\cdot111}{0}\}$, and $\langle w_1, w_\sigma, w_5w_7\rangle$ breaks this set into the orbits $\{\genfrac{}{}{0pt}{}{\cdot354321}{3}, \genfrac{}{}{0pt}{}{0011111}{1}\}$, $\{\genfrac{}{}{0pt}{}{\cdot34\cdot321}{2}, \genfrac{}{}{0pt}{}{000\cdot111}{0}\}$, $\{\genfrac{}{}{0pt}{}{1233211}{2}, \genfrac{}{}{0pt}{}{1232221}{2}\}$ and $\{\genfrac{}{}{0pt}{}{1233321}{1}\}$ while fixing $\genfrac{}{}{0pt}{}{1233321}{2}$. If there were a clique meeting the first of these orbits we could assume it contained $\genfrac{}{}{0pt}{}{2354321}{3}$, which is orthogonal to $\genfrac{}{}{0pt}{}{134\cdot321}{2}, \genfrac{}{}{0pt}{}{1233321}{1}, \genfrac{}{}{0pt}{}{000\cdot111}{0}$, so the remaining four roots would have to be chosen from the six roots $\{\genfrac{}{}{0pt}{}{1354321}{3}, \genfrac{}{}{0pt}{}{234\cdot321}{2}, \genfrac{}{}{0pt}{}{1233211}{2}, \genfrac{}{}{0pt}{}{1232221}{2}, \genfrac{}{}{0pt}{}{0011111}{1}\}$; but of these each of $\genfrac{}{}{0pt}{}{2344321}{2}$ and $\genfrac{}{}{0pt}{}{2343321}{2}$ is orthogonal to three of the others, which would leave too few roots, while the four remaining roots are not mutually non-orthogonal. Thus the clique could not meet the first orbit, and so would have to meet the second; thus we could assume it contained $\genfrac{}{}{0pt}{}{2344321}{2}$, which is orthogonal to $\genfrac{}{}{0pt}{}{1343321}{2}, \genfrac{}{}{0pt}{}{1232221}{2}, \genfrac{}{}{0pt}{}{0000111}{0}$, so the remaining four roots would have to be chosen from the five roots $\{\genfrac{}{}{0pt}{}{2343321}{2}, \genfrac{}{}{0pt}{}{1344321}{2}, \genfrac{}{}{0pt}{}{1233321}{1}, \genfrac{}{}{0pt}{}{1233211}{2}, \genfrac{}{}{0pt}{}{0001111}{0}\}$; but of these $\genfrac{}{}{0pt}{}{2343321}{2}$ is orthogonal to $\genfrac{}{}{0pt}{}{1344321}{2}$ while $\genfrac{}{}{0pt}{}{1233321}{1}$ is orthogonal to $\genfrac{}{}{0pt}{}{1233211}{2}$. Thus we must fix $\{\genfrac{}{}{0pt}{}{\cdot3\cdot\cdot\cdot\cdot1}{\cdot}, \genfrac{}{}{0pt}{}{00\cdot\cdot\cdot\cdot1}{\cdot}\}$ (contained in a clique) and $\{\genfrac{}{}{0pt}{}{12432\cdot1}{2}, \genfrac{}{}{0pt}{}{1233321}{\cdot}, \genfrac{}{}{0pt}{}{1233211}{2}, \genfrac{}{}{0pt}{}{1232221}{2}, \genfrac{}{}{0pt}{}{1233221}{1}, \genfrac{}{}{0pt}{}{1232211}{1}, \genfrac{}{}{0pt}{}{122\cdot111}{1}\}$ (not contained in a clique). Now G is transitive on each of these sets and we may argue as before to see that $W_X = G$.

This completes the treatment of the 16 cases which required careful consideration.

The results found here are presented in tabular form in the final chapter of this work. Again, in most instances we may immediately see that the sets represent different W-orbits, because they are uniquely identified by their signatures. Here there are 25 cases of pairs of sets sharing the same signature; one such pair is X_{30}^{23}

and X_{30}^{24}, which may be distinguished by the orders of the stabilizers. In each of the remainder the two sets are radical, and the sets concerned are X_{29}^{i} and X_{29}^{j} where the pair $\{i, j\}$ is one of the following.

$$\begin{array}{cccccc} \{9,391\} & \{20,307\} & \{31,158\} & \{40,159\} & \{60,419\} & \{63,243\} \\ \{65,70\} & \{66,247\} & \{78,258\} & \{82,278\} & \{83,84\} & \{90,291\} \\ \{93,422\} & \{153,383\} & \{154,387\} & \{180,185\} & \{183,184\} & \{200,223\} \\ \{216,219\} & \{237,246\} & \{255,256\} & \{279,298\} & \{338,363\} & \{376,386\} \end{array}$$

Of these the first, second, fourth and fifth in the first row and the first in each of the second and third rows may again be distinguished by the orders of the stabilizers. In the remaining eighteen cases it is necessary to consider the geometry of the two sets a little more closely; to do so we introduce a little more notation. Given a set X, if there is a unique W_X-orbit of size l consisting of roots β with $o(\beta) = k$, we denote it by $O_k^l(X)$; if there are m such W_X-orbits, we write them as $O_k^{l,1}(X), \ldots, O_k^{l,m}(X)$.

We begin with three cases where orthogonality within a single W_X-orbit suffices to distinguish the two sets: if $(i, j) = (153, 383)$, $(363, 338)$ or $(376, 386)$, the two roots in $O_{18}^2(X)$ are orthogonal to each other if $X = X_{29}^i$ but not if $X = X_{29}^j$. In the remaining fifteen cases we consider orthogonality between W_X-orbits. In fourteen of the cases the following table gives one way of distinguishing between the two sets.

(i,j)	first orbit(s)	second orbit(s)	orthogonality
$(31,158)$	$O_{18}^1(X)$	$O_{14}^4(X)$	$0:4$
$(63,243)$	$O_{19}^1(X)$	$O_{17}^{1,1}(X),\ldots,O_{17}^{1,4}(X)$	$2:1$
$(66,247)$	$O_{17}^1(X)$	$O_{17}^{2,1}(X),O_{17}^{2,2}(X)$	$4:2$
$(78,258)$	$O_{17}^1(X)$	$O_{17}^2(X)$	$0:2$
$(82,278)$	$O_{17}^{1,1}(X),O_{17}^{1,2}(X)$	$O_{15}^{2,1}(X),\ldots,O_{15}^{2,3}(X)$	$2,6:4,4$
$(83,84)$	$O_{11}^1(X)$	$O_{15}^{1,1}(X),O_{15}^{1,2}(X)$	$2:0$
$(90,291)$	$O_{13}^{1,1}(X),O_{13}^{1,2}(X)$	$O_{17}^{1,1}(X),O_{17}^{1,2}(X)$	$2,2:1,1$
$(180,185)$	$O_{18}^2(X)$	$O_{16}^{2,1}(X),O_{16}^{2,2}(X)$	$2:4$
$(183,184)$	$O_{16}^{2,1}(X),\ldots,O_{16}^{2,3}(X)$	$O_{16}^{1,1}(X),O_{16}^{1,2}(X)$	$0,2,4:2,2,2$
$(200,223)$	$O_{19}^1(X)$	$O_{17}^{1,1}(X),O_{17}^{1,2}(X)$	$1:2$
$(216,219)$	$O_{21}^1(X)$	$O_{17}^1(X)$	$1:0$
$(237,246)$	$O_{19}^1(X)$	$O_{17}^{1,1}(X),O_{17}^{1,2}(X)$	$0:1$
$(255,256)$	$O_{17}^{1,1}(X),\ldots,O_{17}^{1,5}(X)$	$O_{15}^{1,1}(X),\ldots,O_{15}^{1,6}(X)$	$1,3,3,5,5:3,3,3,3,5$
$(279,298)$	$O_{17}^2(X)$	$O_{19}^2(X)$	$0:2$

In this table the final column counts the instances of orthogonality between roots in the orbits listed in the second and third columns, with a colon separating the values for the sets X_{29}^i and X_{29}^j. For example, the first row indicates that if $(i, j) = (31, 158)$, the number of roots in $O_{14}^4(X)$ orthogonal to the root in $O_{18}^1(X)$ is 0 if $X = X_{29}^i$ and 4 if $X = X_{29}^j$. If the column headed 'second orbit(s)' contains more than just a single orbit, the orthogonality numbers are aggregated: thus for example the second row indicates that if $(i, j) = (63, 243)$, the number of roots in $O_{17}^{1,1}(X) \cup \cdots \cup O_{17}^{1,4}(X)$ orthogonal to the root in $O_{19}^1(X)$ is 2 if $X = X_{29}^i$ and 1 if $X = X_{29}^j$. If however the column headed 'first orbit(s)' contains more than just a single orbit, the final column gives the orthogonality numbers for each separately: thus for example the fifth row indicates that if $(i, j) = (82, 278)$, there are 8 instances of orthogonality between the roots in $O_{17}^{1,1}(X) \cup O_{17}^{1,2}(X)$ and those in $O_{15}^{2,1}(X) \cup \cdots \cup O_{15}^{2,3}(X)$, but if $X = X_{29}^{i}$ one of the former roots is orthogonal to 2 of the latter and the other to all 6, whereas if $X = X_{29}^{j}$ each of the former is orthogonal to 4 of the latter.

The remaining case is that where $(i, j) = (154, 387)$. Here the two sets X cannot be distinguished in this fashion, as the patterns of orthogonality numbers between W_X-orbits are in fact identical; it is therefore necessary to descend to the level of individual roots within W_X-orbits. For $X = X_{29}^{i}$ or X_{29}^{j} we have unique sets $O_{10}^{1}(X)$, $O_{10}^{2}(X)$, $O_{18}^{1}(X)$ and $O_{18}^{2}(X)$. We also have three sets $O_{16}^{2,1}(X), O_{16}^{2,2}(X), O_{16}^{2,3}(X)$, which we may distinguish as follows: firstly $O_{16}^{2,1}(X)$ consists of roots orthogonal to those in both $O_{10}^{1}(X)$ and $O_{18}^{1}(X)$; next $O_{16}^{2,2}(X)$ consists of roots orthogonal to that in $O_{10}^{1}(X)$ but not that in $O_{18}^{1}(X)$; finally $O_{16}^{2,3}(X)$ consists of roots orthogonal to that in $O_{18}^{1}(X)$ but not that in $O_{10}^{1}(X)$. We shall consider orthogonality among the sets $O_{10}^{2}(X)$, $O_{18}^{2}(X)$, $O_{16}^{2,1}(X)$ and $O_{16}^{2,2}(X)$; each root in $O_{10}^{2}(X) \cup O_{18}^{2}(X)$ is orthogonal to one root in each of $O_{16}^{2,1}(X)$ and $O_{16}^{2,2}(X)$. Choose a root $\beta \in O_{10}^{2}(X)$, and let the roots orthogonal to β in $O_{16}^{2,1}(X)$ and $O_{16}^{2,2}(X)$ be γ_1 and γ_2 respectively. If $X = X_{29}^{i}$, the roots γ_1 and γ_2 are orthogonal to different roots in $O_{18}^{2}(X)$; if however $X = X_{29}^{j}$ they are orthogonal to the same root in $O_{18}^{2}(X)$.

We have thus shown the following.

THEOREM 8.29. *The* 473 *sets in* $\mathcal{S}(E_8)$ *represent different* W*-orbits.*

CHAPTER 9

Tables of maximal abelian sets

In this final chapter we provide tables listing maximal abelian sets for each of the root systems of exceptional type. As we have proved in the previous five chapters, in each case the sets listed form a complete set of representatives for the W-orbits of maximal abelian sets.

Each table has four columns. The first simply contains the notation used for the set X. The second gives the stabilizer W_X in terms of a generating set of involutions; almost always the isomorphism type of W_X is given, although in a small number of cases where W_X is a fairly large soluble group we use the notation '$[n]$' for a group of order n. We denote the dihedral group of order $2n$ by Dih_{2n}, and the alternating group of degree n by Alt_n; we also denote e.g. the Weyl group of type B_4 by $W(B_4)$. In the table for the E_8 root system, when dealing with radical sets we sometimes use additional notation for convenience. Thus we write w_σ and w_τ for the reflections in the roots $\sigma = {}^{2343210}_{\ \ 2}$ and $\tau = {}^{1232100}_{\ \ 2}$; in terms of the action on vertices of the corresponding graph we have $w_\sigma = (7\ 8)$ (as given in section 8.1) and $w_\tau = (1\ 8)$. Also, we occasionally write $w^\diamond$ for $w_0 w_\rho$, where w_0 is the longest element of $W(E_8)$ and $\rho = {}^{2465432}_{\ \ 3}$; recall from section 8.1 that $w_0 w_\rho$ is the longest element of the subgroup $W(E_7)$, and acts on the corresponding graph as complementation.

The third column contains the signature $\mathrm{Sig}(X)$: this gives the sequence of orthogonality counts of roots in X, grouped into W_X-orbits. Each component of $\mathrm{Sig}(X)$ is an orthogonality count, with a superscript if it applies to more than one root; the superscript consists of the sizes of the W_X-orbits, separated by plus signs, with multiple orbits of the same size indicated by a multiplier. Thus for example if the roots with $o(\beta) = 5$ comprised four W_X-orbits of sizes 2, 2, 2 and 1, this would be denoted $5^{3.2+1}$. If there are two root lengths, as explained in sections 4.3 and 5.3 each orthogonality count distinguishes between long and short roots; in the signature a separating semi-colon is used, with orthogonality counts of long roots preceding it and those of short roots following it.

Finally the fourth column lists the W_X-orbits on X, taken in the order given by the signature.

Table 1: Maximal abelian sets in the G_2 root system

X	W_X	Sig(X)	W_X-orbits on X
X_3^1	$\langle w_2\rangle \cong S_2$	00^2; 00	$\{3\cdot\}, \{21\}$
X_3^2	1	00 01; 10	$\{32\}, \{31\}, \{11\}$

Table 2: Maximal abelian sets in the F_4 root system

X	W_X	Sig(X)	W_X-orbits on X
X_8^1	$\langle w_1, w_3, w_4\rangle \cong S_3 \times S_2$	00^2 21^3; 10^3	$\{\cdot 342\}, \{1242, 1222, 1220\}, \{123\cdot, 1221\}$
X_8^2	$\langle w_4\rangle \cong S_2$	00 20 01 22^2; 10^2 30	$\{2342\}, \{1342\}, \{1242\}, \{1122, 1120\}, \{123\cdot\}, \{1221\}$
X_8^3	$\langle w_3, w_4\rangle \cong S_3$	00 30 32^3; 20^3	$\{2342\}, \{1342\}, \{1122, 1120, 1100\}, \{123\cdot, 1221\}$
X_8^4	$\langle w_3, w_4\rangle \cong S_3$	00 31^3 33; 20^3	$\{2342\}, \{1242, 1222, 1220\}, \{1000\}, \{123\cdot, 1221\}$
X_8^5	$\langle w_3\rangle \cong S_2$	00 21^2 02 23; 10 30^2	$\{2342\}, \{1242, 1222\}, \{1122\}, \{1000\}, \{1232\}, \{12\cdot 1\}$
X_8^6	$\langle w_4\rangle \cong S_2$	00 11 12^2 13; 20^2 40	$\{2342\}, \{1242\}, \{1122, 1120\}, \{1000\}, \{123\cdot\}, \{1221\}$
X_8^7	$\langle w_3, w_4\rangle \cong S_3$	00 22^3 03; 30^3	$\{2342\}, \{1122, 1120, 1100\}, \{1000\}, \{123\cdot, 1221\}$
X_9	$\langle w_1, w_3\rangle \cong {S_2}^2$	10^2 11^2 12^2; 00 30^2	$\{\cdot 342\}, \{1242, 1222\}, \{\cdot 122\}, \{1232\}, \{12\cdot 1\}$

Table 3: Maximal abelian sets in the E_6 root system

X	W_X	Sig(X)	W_X-orbits on X
X_{11}^1	$\langle w_1, w_2, w_3, w_5, w_6\rangle \cong {S_3}^2 \times S_2$	$0^2\ 4^9$	$\{\genfrac{}{}{0pt}{}{12321}{\cdot}\}, \{\genfrac{}{}{0pt}{}{\cdot\cdot2\cdot\cdot}{1}\}$
X_{11}^2	$\langle w_1w_4, w_4w_6\rangle \cong {S_2}^2$	$0\ 2^2\ 4^{2.2}\ 6^4$	$\{\genfrac{}{}{0pt}{}{12321}{2}\}, \{\genfrac{}{}{0pt}{}{12\cdot21}{1}\}, \{\genfrac{}{}{0pt}{}{1221\cdot}{1}\}, \{\genfrac{}{}{0pt}{}{\cdot1221}{1}\}, \{\genfrac{}{}{0pt}{}{11211}{1}, \genfrac{}{}{0pt}{}{01210}{1}, \genfrac{}{}{0pt}{}{11110}{1}, \genfrac{}{}{0pt}{}{01111}{1}\}$
X_{11}^3	$\langle w_5, w_4w_6\rangle \cong Dih_8$	$0\ 3^4\ 5^4\ 7^2$	$\{\genfrac{}{}{0pt}{}{12321}{2}\}, \{\genfrac{}{}{0pt}{}{12\cdot\cdot\cdot}{1}\}, \{\genfrac{}{}{0pt}{}{112\cdot1}{1}, \genfrac{}{}{0pt}{}{111\cdot0}{1}\}, \{\genfrac{}{}{0pt}{}{01210}{1}, \genfrac{}{}{0pt}{}{01111}{1}\}$
X_{11}^4	$\langle w_3, w_1w_4\rangle \cong Dih_8$	$0\ 3^4\ 5^4\ 7^2$	$\{\genfrac{}{}{0pt}{}{12321}{2}\}, \{\genfrac{}{}{0pt}{}{\cdot\cdot\cdot21}{1}\}, \{\genfrac{}{}{0pt}{}{1\cdot211}{1}, \genfrac{}{}{0pt}{}{0\cdot111}{1}\}, \{\genfrac{}{}{0pt}{}{01210}{1}, \genfrac{}{}{0pt}{}{11110}{1}\}$
X_{11}^5	$\langle w_1w_5, w_3w_6\rangle \cong S_3$	$0\ 3\ 5^6\ 7^3$	$\{\genfrac{}{}{0pt}{}{12321}{2}\}, \{\genfrac{}{}{0pt}{}{12321}{1}\}, \{\genfrac{}{}{0pt}{}{122\cdot1}{1}, \genfrac{}{}{0pt}{}{11221}{1}, \genfrac{}{}{0pt}{}{11210}{1}, \genfrac{}{}{0pt}{}{0121\cdot}{1}\}, \{\genfrac{}{}{0pt}{}{01110}{1}, \genfrac{}{}{0pt}{}{11100}{1}, \genfrac{}{}{0pt}{}{00111}{1}\}$
X_{11}^6	$\langle w_3, w_5\rangle \cong {S_2}^2$	$0\ 4^{4+1}\ 6^{2.2}\ 8$	$\{\genfrac{}{}{0pt}{}{12321}{2}\}, \{\genfrac{}{}{0pt}{}{1\cdot2\cdot1}{1}\}, \{\genfrac{}{}{0pt}{}{12321}{1}\}, \{\genfrac{}{}{0pt}{}{111\cdot0}{1}\}, \{\genfrac{}{}{0pt}{}{0\cdot111}{1}\}, \{\genfrac{}{}{0pt}{}{01210}{1}\}$
X_{11}^7	$\langle w_1w_4, w_3w_5, w_4w_6\rangle \cong Alt_5$	$0\ 6^{10}$	$\{\genfrac{}{}{0pt}{}{12321}{2}\}, \{\genfrac{}{}{0pt}{}{12\cdot21}{1}, \genfrac{}{}{0pt}{}{11211}{1}, \genfrac{}{}{0pt}{}{01210}{1}, \genfrac{}{}{0pt}{}{111\cdot0}{1}, \genfrac{}{}{0pt}{}{0\cdot111}{1}, \genfrac{}{}{0pt}{}{01100}{1}, \genfrac{}{}{0pt}{}{00110}{1}\}$
X_{11}^8	$\langle w_1, w_3, w_5, w_6\rangle \cong {S_3}^2$	$0\ 5^9\ 9$	$\{\genfrac{}{}{0pt}{}{12321}{2}\}, \{\genfrac{}{}{0pt}{}{\cdot\cdot1\cdot\cdot}{1}\}, \{\genfrac{}{}{0pt}{}{12321}{1}\}$
X_{12}	$\langle w_2, w_3, w_5\rangle \cong {S_2}^3$	$1^2\ 3^4\ 5^2\ 6^{2.2}$	$\{\genfrac{}{}{0pt}{}{12321}{\cdot}\}, \{\genfrac{}{}{0pt}{}{1\cdot2\cdot1}{1}\}, \{\genfrac{}{}{0pt}{}{11111}{\cdot}\}, \{\genfrac{}{}{0pt}{}{1\cdot210}{1}\}, \{\genfrac{}{}{0pt}{}{012\cdot1}{1}\}$
X_{13}^1	$\langle w_1, w_2, w_4, w_5\rangle \cong S_4 \times S_2$	$2^4\ 5^8\ 8$	$\{\genfrac{}{}{0pt}{}{12\cdot\cdot1}{\cdot}\}, \{\genfrac{}{}{0pt}{}{\cdot1\cdot\cdot1}{\cdot}\}, \{\genfrac{}{}{0pt}{}{12210}{1}\}$
X_{13}^2	$\langle w_2, w_3, w_4, w_6\rangle \cong S_4 \times S_2$	$2^4\ 5^8\ 8$	$\{\genfrac{}{}{0pt}{}{1\cdot\cdot21}{\cdot}\}, \{\genfrac{}{}{0pt}{}{1\cdot\cdot1\cdot}{\cdot}\}, \{\genfrac{}{}{0pt}{}{01221}{1}\}$
X_{16}^1	$\langle w_2, w_3, w_4, w_5, w_6\rangle \cong W(D_5)$	5^{16}	$\{\genfrac{}{}{0pt}{}{1\cdot\cdot\cdot\cdot}{\cdot}\}$
X_{16}^2	$\langle w_1, w_2, w_3, w_4, w_5\rangle \cong W(D_5)$	5^{16}	$\{\genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot1}{\cdot}\}$

Table 4: Maximal abelian sets

X	W_X	$\mathrm{Sig}(X)$
X_{14}	$\langle w_1w_4, w_3w_7, w_4{w_6}^{w_5}\rangle \cong L_3(2)$	$3^7\ 9^7$
X_{17}^1	$\langle w_1, w_2, w_4, w_5, w_6, w_7\rangle \cong S_6 \times S_2$	$0^2\ 6^{15}$
X_{17}^2	$\langle w_2, w_6, w_5w_7\rangle \cong S_2 \times Dih_8$	$0\ 2\ 4\ 6^8\ 8^4\ 10^2$
X_{17}^3	$\langle w_4, w_6\rangle \cong {S_2}^2$	$0\ 3\ 5^{2.2}\ 7^{4+2.1}\ 9^{2.2}\ 11$
X_{17}^4	$\langle w_5, w_4w_6, w_2w_7\rangle \cong W(C_3)$	$0\ 3\ 7^{12}\ 11^3$
X_{17}^5	$\langle w_2, w_4, w_6\rangle \cong S_3 \times S_2$	$0\ 4\ 6^{6+3}\ 8^3\ 10^2\ 12$
X_{17}^6	$\langle w_3, w_4, w_6, w_7\rangle \cong {S_3}^2$	$0\ 4^3\ 6^3\ 8^9\ 12$
X_{17}^7	$\langle w_3w_6, w_4w_7\rangle \cong S_3$	$0\ 4^3\ 6^{3+1}\ 8^6\ 10^3$
X_{17}^8	$\langle w_5, w_2w_7\rangle \cong {S_2}^2$	$0\ 4\ 6^{4+1}\ 8^{4+2+1}\ 10^2\ 12$
X_{17}^9	$\langle w_4, w_2w_5w_7\rangle \cong Dih_8$	$0\ 4\ 6^{4+2}\ 8^4\ 10^{4+1}$
X_{17}^{10}	$\langle w_3, w_4, w_6\rangle \cong S_3 \times S_2$	$0\ 5^3\ 7^{6+2}\ 9^{3+1}\ 13$
X_{17}^{11}	$\langle w_2, w_3w_6\rangle \cong {S_2}^2$	$0\ 5^{2.2}\ 7^{4+2}\ 9^{2.2}\ 11^2$
X_{17}^{12}	$\langle w_2, w_3, w_6, w_7\rangle \cong S_3 \times {S_2}^2$	$0\ 5^2\ 7^6\ 9^6\ 11^2$
X_{17}^{13}	$\langle w_2, w_6, w_5w_7\rangle \cong S_2 \times Dih_8$	$0\ 5\ 7^8\ 9^{4+2}\ 13$
X_{17}^{14}	$\langle w_3w_5, w_4w_6, w_5w_7\rangle \cong Alt_5$	$0\ 5^6\ 9^{10}$
X_{17}^{15}	$\langle w_3w_7, w_2w_5w_7\rangle \cong {S_2}^2$	$0\ 5^2\ 7^{4+2}\ 9^{4+2}\ 11^2$
X_{17}^{16}	$\langle w_2, w_4, w_5, w_7\rangle \cong S_4 \times S_2$	$0\ 6^6\ 8^{8+1}\ 14$
X_{17}^{17}	$\langle w_3, w_5\rangle \cong {S_2}^2$	$0\ 6^{2.2}\ 8^{4+2+2.1}\ 10^{2+1}\ 12$
X_{17}^{18}	$\langle w_3w_5, w_4w_6\rangle \cong Dih_{10}$	$0\ 6^5\ 8^5\ 10^{5+1}$
X_{17}^{19}	$\langle w_2, w_4, w_2w_5{w_7}^{w_4w_6w_5}\rangle \cong S_3 \wr S_2$	$0\ 6\ 8^9\ 10^6$
X_{17}^{20}	$\langle w_4w_6, w_2w_5w_7\rangle \cong Dih_{12}$	$0\ 6\ 8^{6+3}\ 10^6$
X_{17}^{21}	$\langle w_2, w_5, w_2w_3{w_6}^{w_4}\rangle = [2^43]$	$0\ 7^{12}\ 11^4$
X_{17}^{22}	$\langle w_2, w_4, w_5, w_6, w_7\rangle \cong S_6$	$0\ 7^{15}\ 15$
X_{17}^{23}	$\langle w_3, w_2w_5, w_2w_5{w_7}^{w_4}\rangle \cong S_2 \times Dih_8$	$0\ 7^{4+2}\ 9^8\ 11^2$
X_{17}^{24}	$\langle w_2, w_3w_7, w_5w_7\rangle \cong {S_2}^3$	$0\ 7^{4+2}\ 9^{2.4}\ 11^2$
X_{17}^{25}	$\langle w_7, w_2w_5, w_4{w_7}^{w_6w_5w_6}\rangle = [2^43]$	$0\ 8^{12}\ 10^3\ 12$
X_{17}^{26}	$\langle w_2, w_3, w_4, w_2w_5{w_7}^{w_4w_6w_5}\rangle = [2^73^2]$	$0\ 9^{16}$
X_{17}^{27}	$\langle w_3w_7, w_4w_6, w_2w_5w_7\rangle = [2^63]$	$0\ 9^{16}$
X_{17}^{28}	$\langle w_2, w_1w_5, w_3w_6, w_5w_7\rangle \cong S_4 \times S_2$	$2^3\ 6^8\ 10^6$

in the E_7 root system
W_X-orbits on X
$\{\substack{\cdot\cdot\cdot\cdot\cdot\cdot\\2}\},\{\substack{123321\\1},\substack{122111\\1},\substack{112110\\1},\substack{012100\\1},\substack{111100\\1},\substack{011110\\1},\substack{001111\\1}\}$
$\{\substack{\cdot34321\\2}\},\{\substack{12\cdot\cdot\cdot\cdot\\\cdot}\}$
$\{\substack{234321\\2}\},\{\substack{134321\\2}\},\{\substack{124321\\2}\},\{\substack{123\cdot\cdot\cdot\\\cdot}\},\{\substack{1222\cdot1\\1},\substack{1221\cdot0\\1}\},\{\substack{112210\\1},\substack{112111\\1}\}$
$\{\substack{234321\\2}\},\{\substack{134321\\2}\},\{\substack{12\cdot321\\2}\},\{\substack{1232\cdot1\\2}\},\{\substack{12\cdot2\cdot1\\1}\},\{\substack{123321\\1}\},\{\substack{123210\\2}\},\{\substack{1221\cdot0\\1}\},\{\substack{11\cdot111\\1}\},\{\substack{112210\\1}\}$
$\{\substack{234321\\2}\},\{\substack{134321\\2}\},\{\substack{12\cdot\cdot\cdot1\\2},\substack{123\cdot21\\1},\substack{122\cdot11\\1},\substack{12\cdot\cdot\cdot0\\1}\},\{\substack{112100\\1},\substack{111110\\1},\substack{111111\\0}\}$
$\{\substack{234321\\2}\},\{\substack{134321\\2}\},\{\substack{12\cdot2\cdot1\\\cdot}\},\{\substack{12\cdot321\\\cdot}\},\{\substack{11\cdot111\\\cdot}\},\{\substack{1221\cdot0\\1}\},\{\substack{112210\\1}\}$
$\{\substack{234321\\2}\},\{\substack{1\cdot\cdot321\\2}\},\{\substack{1232\cdot\cdot\\2}\},\{\substack{1\cdot\cdot2\cdot\cdot\\1}\},\{\substack{111000\\1}\}$
$\{\substack{234321\\2}\},\{\substack{1\cdot\cdot321\\2}\},\{\substack{1232\cdot\cdot\\2}\},\{\substack{123321\\1}\},\{\substack{1232\cdot1\\1},\substack{122221\\1},\substack{122210\\1},\substack{11221\cdot\\1}\},\{\substack{122100\\1},\substack{112110\\1},\substack{111111\\1}\}$
$\{\substack{234321\\2}\},\{\substack{134321\\2}\},\{\substack{123\cdot21\\\cdot}\},\{\substack{124321\\2}\},\{\substack{122\cdot1\cdot\\1}\},\{\substack{123211\\2},\substack{123210\\1}\},\{\substack{112221\\1}\},\{\substack{111110\\1},\substack{111111\\0}\},\{\substack{112100\\1}\}$
$\{\substack{234321\\2}\},\{\substack{134321\\2}\},\{\substack{12\cdot321\\2},\substack{12\cdot221\\1}\},\{\substack{123321\\1},\substack{123221\\2}\},\{\substack{123211\\2},\substack{12\cdot210\\1},\substack{122111\\1}\},\{\substack{112211\\1},\substack{11\cdot110\\1},\substack{111111\\0}\},\{\substack{122100\\1}\}$
$\{\substack{234321\\2}\},\{\substack{1\cdot\cdot321\\2}\},\{\substack{1\cdot\cdot2\cdot1\\1}\},\{\substack{1232\cdot1\\2}\},\{\substack{1\cdot\cdot210\\1}\},\{\substack{111111\\0}\},\{\substack{111000\\1}\}$
$\{\substack{234321\\2}\},\{\substack{1\cdot4321\\2}\},\{\substack{123321\\\cdot}\},\{\substack{1232\cdot1\\\cdot}\},\{\substack{122211\\1},\substack{112221\\1}\},\{\substack{1\cdot2210\\1}\},\{\substack{111111\\\cdot}\},\{\substack{122110\\1},\substack{112100\\1}\}$
$\{\substack{234321\\2}\},\{\substack{1\cdot4321\\2}\},\{\substack{1232\cdot\cdot\\\cdot}\},\{\substack{1\cdot22\cdot\cdot\\1}\},\{\substack{111000\\\cdot}\}$
$\{\substack{234321\\2}\},\{\substack{134321\\2}\},\{\substack{123\cdot\cdot\cdot\\\cdot}\},\{\substack{1122\cdot1\\1},\substack{1121\cdot0\\1}\},\{\substack{122210\\1},\substack{122111\\1}\},\{\substack{110000\\0}\}$
$\{\substack{234321\\2}\},\{\substack{1\cdot\cdot\cdot\cdot\cdot\\2}\},\{\substack{12321\cdot\\1},\substack{122221\\1},\substack{122111\\1},\substack{122210\\1},\substack{1122\cdot1\\1},\substack{112110\\1},\substack{111\cdot00\\1}\}$
$\{\substack{234321\\2}\},\{\substack{1\cdot4321\\2}\},\{\substack{12321\cdot\\\cdot}\},\{\substack{123321\\2},\substack{123221\\1}\},\{\substack{122211\\1},\substack{122110\\1},\substack{112210\\1},\substack{112111\\1}\},\{\substack{1\cdot2221\\1}\},\{\substack{111000\\1},\substack{111100\\0}\}$
$\{\substack{234321\\2}\},\{\substack{12\cdot\cdot21\\\cdot}\},\{\substack{12\cdot\cdot1\cdot\\\cdot}\},\{\substack{112221\\1}\},\{\substack{100000\\0}\}$
$\{\substack{234321\\2}\},\{\substack{1\cdot4321\\2}\},\{\substack{123\cdot21\\2}\},\{\substack{1\cdot2\cdot10\\1}\},\{\substack{123\cdot21\\1}\},\{\substack{123211\\2}\},\{\substack{111110\\1}\},\{\substack{1\cdot2100\\1}\},\{\substack{111111\\1}\},\{\substack{111111\\0}\}$
$\{\substack{234321\\2}\},\{\substack{1\cdot\cdot\cdot\cdot1\\2}\},\{\substack{123\cdot21\\1},\substack{122211\\1},\substack{11\cdot111\\1}\},\{\substack{112210\\1},\substack{122110\\1},\substack{1\cdot2100\\1},\substack{111110\\1}\},\{\substack{111111\\0}\}$
$\{\substack{234321\\2}\},\{\substack{134321\\2}\},\{\substack{12\cdot2\cdot\cdot\\\cdot}\},\{\substack{1122\cdot\cdot\\1},\substack{11\cdot000\\\cdot}\}$
$\{\substack{234321\\2}\},\{\substack{134321\\2}\},\{\substack{123321\\1},\substack{1232\cdot1\\2},\substack{12\cdot210\\1},\substack{122111\\1}\},\{\substack{122221\\1},\substack{123211\\1},\substack{123210\\2}\},\{\substack{1122\cdot1\\1},\substack{112110\\1},\substack{111100\\1},\substack{11\cdot000\\0}\}$
$\{\substack{234321\\2}\},\{\substack{1\cdot4321\\2},\substack{123\cdot21\\\cdot},\substack{1\cdot2\cdot11\\1},\substack{111111\\\cdot}\},\{\substack{1\cdot2100\\1},\substack{111110\\\cdot}\}$
$\{\substack{234321\\2}\},\{\substack{12\cdot\cdot\cdot\cdot\\\cdot}\},\{\substack{100000\\0}\}$
$\{\substack{234321\\2}\},\{\substack{1\cdot4321\\2},\substack{1\cdot2221\\1}\},\{\substack{123321\\1},\substack{123221\\2}\},\{\substack{123211\\\cdot},\substack{1\cdot2\cdot10\\1},\substack{111111\\\cdot}\},\{\substack{111100\\1},\substack{111000\\0}\}$
$\{\substack{234321\\2}\},\{\substack{123\cdot21\\\cdot}\},\{\substack{1\cdot4321\\2}\},\{\substack{122210\\1},\substack{122111\\1},\substack{112211\\1},\substack{112110\\1}\},\{\substack{11111\cdot\\\cdot}\},\{\substack{1\cdot2100\\1}\}$
$\{\substack{234321\\2}\},\{\substack{12\cdot321\\2},\substack{12\cdot221\\1},\substack{12\cdot\cdot1\cdot\\\cdot}\},\{\substack{112221\\1},\substack{111000\\1},\substack{111100\\0}\},\{\substack{100000\\0}\}$
$\{\substack{234321\\2}\},\{\substack{1\cdot\cdot321\\\cdot},\substack{1\cdot\cdot1\cdot\cdot\\\cdot}\}$
$\{\substack{234321\\2}\},\{\substack{1\cdot\cdot321\\2},\substack{123221\\1},\substack{122211\\1},\substack{1221\cdot0\\1},\substack{11\cdot111\\\cdot},\substack{112210\\1},\substack{112100\\1},\substack{111110\\\cdot},\substack{111100\\0},\substack{111000\\1}\}$
$\{\substack{\cdot\cdot4321\\2}\},\{\substack{123\cdot\cdot\cdot\\\cdot}\},\{\substack{122221\\1},\substack{112211\\1},\substack{122100\\1},\substack{112110\\1},\substack{012210\\1},\substack{012111\\1}\}$

Table 4: Maximal abelian sets

X	W_X	Sig(X)
X_{18}^1	$\langle w_1, w_2, w_4, w_5, w_7\rangle \cong S_4 \times {S_2}^2$	$1^2\ 5^6\ 8^8\ 9^2$
X_{18}^2	$\langle w_2, w_3w_7, w_5w_7\rangle \cong {S_2}^3$	$1\ 3^2\ 5^4\ 7^2\ 8^4\ 9\ 10^4$
X_{18}^3	$\langle w_4, w_3w_5\rangle \cong Dih_8$	$1\ 4^4\ 6^4\ 7\ 9^{4+2.1}\ 11^2$
X_{18}^4	$\langle w_2, w_4, w_3w_5\rangle \cong W(B_3)$	$1\ 5^8\ 8^6\ 9\ 12^2$
X_{18}^5	$\langle w_4, w_3w_5, w_2w_5w_7\rangle = [2^6]$	$1\ 5^8\ 9\ 10^8$
X_{19}	$\langle w_1, w_2, w_4, w_6\rangle \cong S_3 \times {S_2}^2$	$2^2\ 4^3\ 7^6\ 9^4\ 10^{3+1}$
X_{20}^1	$\langle w_1, w_2, w_3, w_5, w_6\rangle \cong {S_3}^2 \times S_2$	$3^3\ 6^6\ 9^9\ 12^2$
X_{20}^2	$\langle w_1, w_2, w_3, w_4, w_6, w_7\rangle \cong S_5 \times S_3$	$3^5\ 9^{15}$
X_{22}	$\langle w_1, w_3, w_4, w_5, w_6\rangle \cong S_6$	$5^6\ 9^{15}\ 15$
X_{27}	$\langle w_1, w_2, w_3, w_4, w_5, w_6\rangle \cong W(E_6)$	10^{27}

in the E_7 root system (continued)
W_X-orbits on X
$\{\substack{\cdot 34321\\ 2}\}, \{\substack{12\cdot\cdot 21\\ \cdot}\}, \{\substack{12\cdot\cdot 1\cdot\\ \cdot}\}, \{\substack{\cdot 12221\\ 1}\}$
$\{\substack{234321\\ 2}\}, \{\substack{1\cdot 4321\\ 2}\}, \{\substack{123\cdot 21\\ \cdot}\}, \{\substack{1\cdot 2221\\ 1}\}, \{\substack{12321\cdot\\ \cdot}\}, \{\substack{012221\\ 1}\}, \{\substack{122211\\ 1}, \substack{112111\\ 1}, \substack{112210\\ 1}, \substack{122110\\ 1}\}$
$\{\substack{234321\\ 2}\}, \{\substack{1\cdot\cdot\cdot 21\\ 2}\}, \{\substack{1\cdot\cdot\cdot 21\\ 1}\}, \{\substack{123211\\ 2}\}, \{\substack{12\cdot 211\\ 1}, \substack{11\cdot 111\\ 1}\}, \{\substack{123210\\ 2}\}, \{\substack{012221\\ 1}\}, \{\substack{112210\\ 1}, \substack{122110\\ 1}\}$
$\{\substack{234321\\ 2}\}, \{\substack{1\cdot\cdot\cdot 21\\ \cdot}\}, \{\substack{12\cdot 211\\ \cdot}, \substack{11\cdot 111\\ \cdot}\}, \{\substack{012221\\ 1}\}, \{\substack{112210\\ 1}, \substack{122110\\ 1}\}$
$\{\substack{234321\\ 2}\}, \{\substack{1\cdot\cdot\cdot 21\\ \cdot}\}, \{\substack{012221\\ 1}\}, \{\substack{123211\\ 2}, \substack{12\cdot 210\\ 1}, \substack{112211\\ 1}, \substack{122111\\ 1}, \substack{11\cdot 110\\ 1}, \substack{111111\\ 0}\}$
$\{\substack{\cdot 34321\\ 2}\}, \{\substack{12\cdot 321\\ \cdot}\}, \{\substack{12\cdot 2\cdot 1\\ \cdot}\}, \{\substack{\cdot 122\cdot 1\\ 1}\}, \{\substack{12\cdot 210\\ \cdot}\}, \{\substack{122111\\ 1}\}$
$\{\substack{\cdot\cdot 4321\\ 2}\}, \{\substack{123\cdot\cdot 1\\ \cdot}\}, \{\substack{\cdot\cdot 2\cdot\cdot 1\\ 1}\}, \{\substack{123210\\ \cdot}\}$
$\{\substack{\cdot\cdot\cdot 321\\ \cdot}\}, \{\substack{\cdot\cdot\cdot 2\cdot\cdot\\ \cdot}\}$
$\{\substack{\cdot\cdot\cdot\cdot\cdot 1\\ 2}\}, \{\substack{\cdot\cdot\cdot\cdot\cdot 1\\ 1}\}, \{\substack{123210\\ 2}\}$
$\{\substack{\cdot\cdot\cdot\cdot\cdot 1\\ \cdot}\}$

Table 5: Maximal abelian sets

X	W_X	Sig(X)
X_{22}	$\langle w_2w_8, w_3w_6, w_4w_7, {w_4w_7}^{w_3w_6w_5}\rangle \cong AGL_3(2)$	$5^{14}\ 14^8$
X_{28}	$\langle w_1, w_2w_7, w_6w_8, {w_4w_6}^{w_5}\rangle \cong S_2 \times L_3(2)$	$5^7\ 11^{14}\ 17^7$
X_{29}^1	$\langle w_1, w_3, w_4, w_5, w_6, w_7, w_\sigma\rangle \cong S_8$	$0\ 15^{28}$
X_{29}^2	$\langle w_1, w_3, w_4, w_5, w_6, w_\tau\rangle \cong S_6 \times S_2$	$0\ 12\ 14^{15}\ 16^{12}$
X_{29}^3	$\langle w_1, w_3, w_4, w_5, w_7\rangle \cong S_5 \times S_2$	$0\ 11^2\ 13^{10}\ 15^{10}\ 17^{5+1}$
X_{29}^4	$\langle w_1, w_3, w_4, w_6, {w_6w_\tau}^{w_7}\rangle \cong S_4 \times Dih_8$	$0\ 13^{6+2}\ 15^{16}\ 17^4$
X_{29}^5	$\langle w_1, w_3, w_4, w_6, w_7\rangle \cong S_4 \times S_3$	$0\ 10^3\ 12^6\ 14^{12}\ 16^3\ 18^4$
X_{29}^6	$\langle w_1, w_3, w_4, w_5, w_7, w_\tau\rangle \cong S_5 \times S_3$	$0\ 10^3\ 12^{10}\ 16^{15}$
X_{29}^7	$\langle w_1, w_3, w_4, w_6w_\tau\rangle \cong S_4 \times S_2$	$0\ 10\ 12^{6+2}\ 14^8\ 16^{8+1}\ 18^2$
X_{29}^8	$\langle w_1, w_3, w_5, w_7\rangle \cong S_3 \times {S_2}^2$	$0\ 12^{3+2}\ 14^{2.6+1}\ 16^{4+3+1}\ 18^2$
X_{29}^9	$\langle w_1, w_6, {w_4w_6}^{w_5}, {w_6w_\tau}^{w_7}\rangle \cong S_2 \times (S_2 \wr S_3)$	$0\ 12\ 14^{12+3}\ 16^{12}$
X_{29}^{10}	$\langle w_1, w_3, w_5, w_6, w_7\rangle \cong S_4 \times S_3$	$0\ 9^4\ 11^3\ 13^{12}\ 15^6\ 19^3$
X_{29}^{11}	$\langle w_1, w_3, w_4, w_7\rangle \cong S_4 \times S_2$	$0\ 9^2\ 11^{6+2.1}\ 13^4\ 15^8\ 17^{4+2}$
X_{29}^{12}	$\langle w_1, w_3, w_6\rangle \cong S_3 \times S_2$	$0\ 9\ 11^{3+2}\ 13^{6+3+1}\ 15^{3+2+1}\ 17^{3+2}\ 19$
X_{29}^{13}	$\langle w_1, w_4, w_6, w_7\rangle \cong S_3 \times {S_2}^2$	$0\ 11^{3+1}\ 13^{6+4}\ 15^{6+3+1}\ 17^2\ 19^2$
X_{29}^{14}	$\langle w_1, w_3, w_5, w_7, w_\tau\rangle \cong {S_3}^2 \times S_2$	$0\ 11^{2.3}\ 13^6\ 15^{9+1}\ 17^6$
X_{29}^{15}	$\langle w_1, w_3, w_4, w_7, w_6w_\tau\rangle \cong S_4 \times Dih_8$	$0\ 11^{6+4}\ 15^{16}\ 19^2$
X_{29}^{16}	$\langle w_1, w_3, w_5w_7\rangle \cong S_3 \times S_2$	$0\ 11^{3+2}\ 13^{6+2}\ 15^{6+3+1}\ 17^{2.2}\ 19$
X_{29}^{17}	$\langle w_1, w_4, w_6w_\tau\rangle \cong {S_2}^3$	$0\ 11^{2.1}\ 13^{2.4+2}\ 15^{2.4+2.1}\ 17^{4+2}$
X_{29}^{18}	$\langle w_1, w_4, {w_4w_6w_\tau}^{w_5}\rangle \cong S_2 \times Dih_8$	$0\ 11\ 13^{8+4}\ 15^{2.4+2}\ 17^4\ 19$
X_{29}^{19}	$\langle w_4, w_7, w_3w_5\rangle \cong S_2 \times Dih_8$	$0\ 13^{4+2.2}\ 15^{8+4+2+2.1}\ 17^4$
X_{29}^{20}	$\langle w_4, w_3w_5, w_1w_6, w_7w_\sigma\rangle \cong W(C_4)$	$0\ 15^{24+4}$
X_{29}^{21}	$\langle w_1, w_4, w_5, w_6, w_7\rangle \cong S_5 \times S_2$	$0\ 8^5\ 10\ 12^{10}\ 14^{10}\ 20^2$
X_{29}^{22}	$\langle w_1, w_3, w_5, w_7\rangle \cong S_3 \times {S_2}^2$	$0\ 8^2\ 10^{3+2}\ 12^{6+1}\ 14^{6+1}\ 16^4\ 18^3$
X_{29}^{23}	$\langle w_1, w_5, w_6\rangle \cong S_3 \times S_2$	$0\ 8\ 10^{3+1}\ 12^{6+2}\ 14^{2.3+2+1}\ 16^3\ 18^2\ 20$

in the E_8 root system
W_X-orbits on X
$\{\genfrac{}{}{0pt}{}{2\cdot\cdot\cdot\cdot\cdot\cdot}{\cdot}\},\{\genfrac{}{}{0pt}{}{1343321}{2},\genfrac{}{}{0pt}{}{1243221}{2},\genfrac{}{}{0pt}{}{1233211}{2},\genfrac{}{}{0pt}{}{1233210}{1},\genfrac{}{}{0pt}{}{1221000}{1},\genfrac{}{}{0pt}{}{1121100}{1},\genfrac{}{}{0pt}{}{1111110}{1},\genfrac{}{}{0pt}{}{1111111}{0}\}$
$\{\genfrac{}{}{0pt}{}{24\cdot\cdot\cdot\cdot\cdot}{\cdot}\},\{\genfrac{}{}{0pt}{}{\cdot3\cdot\cdot\cdot\cdot\cdot}{\cdot}\},\{\genfrac{}{}{0pt}{}{1244321}{2},\genfrac{}{}{0pt}{}{1232221}{2},\genfrac{}{}{0pt}{}{1232211}{1},\genfrac{}{}{0pt}{}{1232100}{2},\genfrac{}{}{0pt}{}{1232110}{1},\genfrac{}{}{0pt}{}{1222210}{1},\genfrac{}{}{0pt}{}{1222111}{1}\}$
$\{\genfrac{}{}{0pt}{}{2465432}{3}\},\{\genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot\cdot\cdot1}{2},\genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot\cdot\cdot1}{0}\}$
$\{\genfrac{}{}{0pt}{}{2465432}{3}\},\{\genfrac{}{}{0pt}{}{2465431}{3}\},\{\genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot\cdot21}{2}\},\{\genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot\cdot11}{2},\genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot\cdot11}{0}\}$
$\{\genfrac{}{}{0pt}{}{2465432}{3}\},\{\genfrac{}{}{0pt}{}{24654\cdot1}{3}\},\{\genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot321}{2}\},\{\genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot2\cdot1}{2}\},\{\genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot111}{0}\},\{\genfrac{}{}{0pt}{}{1232111}{2}\}$
$\{\genfrac{}{}{0pt}{}{2465432}{3}\},\{\genfrac{}{}{0pt}{}{\cdot\cdot\cdot4321}{2}\},\{\genfrac{}{}{0pt}{}{2465431}{3},\genfrac{}{}{0pt}{}{1233211}{1}\},\{\genfrac{}{}{0pt}{}{\cdot\cdot\cdot3\cdot\cdot1}{2},\genfrac{}{}{0pt}{}{\cdot\cdot\cdot1111}{0}\},\{\genfrac{}{}{0pt}{}{1232\cdot11}{2},\genfrac{}{}{0pt}{}{0000\cdot11}{0}\}$
$\{\genfrac{}{}{0pt}{}{2465432}{3}\},\{\genfrac{}{}{0pt}{}{2465\cdot\cdot1}{3}\},\{\genfrac{}{}{0pt}{}{\cdot\cdot\cdot4321}{2}\},\{\genfrac{}{}{0pt}{}{\cdot\cdot\cdot3\cdot\cdot1}{2}\},\{\genfrac{}{}{0pt}{}{1232\cdot\cdot1}{2}\},\{\genfrac{}{}{0pt}{}{\cdot\cdot\cdot1111}{0}\}$
$\{\genfrac{}{}{0pt}{}{2465432}{3}\},\{\genfrac{}{}{0pt}{}{24654\cdot1}{3},\genfrac{}{}{0pt}{}{1233321}{1}\},\{\genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot321}{2}\},\{\genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot2\cdot1}{2},\genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot111}{0}\}$
$\{\genfrac{}{}{0pt}{}{2465432}{3}\},\{\genfrac{}{}{0pt}{}{2465431}{3}\},\{\genfrac{}{}{0pt}{}{\cdot\cdot\cdot4321}{2}\},\{\genfrac{}{}{0pt}{}{2465421}{3},\genfrac{}{}{0pt}{}{1233221}{1}\},\{\genfrac{}{}{0pt}{}{\cdot\cdot\cdot3\cdot21}{2}\},\{\genfrac{}{}{0pt}{}{\cdot\cdot\cdot3211}{2},\genfrac{}{}{0pt}{}{\cdot\cdot\cdot1111}{0}\},$ $\{\genfrac{}{}{0pt}{}{1232221}{2}\},\{\genfrac{}{}{0pt}{}{1232111}{2},\genfrac{}{}{0pt}{}{0000111}{0}\}$
$\{\genfrac{}{}{0pt}{}{2465432}{3}\},\{\genfrac{}{}{0pt}{}{\cdot\cdot54321}{2}\},\{\genfrac{}{}{0pt}{}{24654\cdot1}{3}\},\{\genfrac{}{}{0pt}{}{\cdot\cdot4\cdot321}{2}\},\{\genfrac{}{}{0pt}{}{\cdot\cdot432\cdot1}{2}\},\{\genfrac{}{}{0pt}{}{1232111}{1}\},\{\genfrac{}{}{0pt}{}{123\cdot2\cdot1}{2}\},$ $\{\genfrac{}{}{0pt}{}{\cdot\cdot11111}{0}\},\{\genfrac{}{}{0pt}{}{1232111}{2}\},\{\genfrac{}{}{0pt}{}{000\cdot111}{0}\}$
$\{\genfrac{}{}{0pt}{}{2465432}{3}\},\{\genfrac{}{}{0pt}{}{2454321}{2}\},\{\genfrac{}{}{0pt}{}{\cdot3\cdot\cdot\cdot\cdot1}{2},\genfrac{}{}{0pt}{}{\cdot111111}{0}\},\{\genfrac{}{}{0pt}{}{2465431}{3},\genfrac{}{}{0pt}{}{1233211}{1},\genfrac{}{}{0pt}{}{1221111}{1}\},\{\genfrac{}{}{0pt}{}{12\cdot3\cdot21}{2},$ $\genfrac{}{}{0pt}{}{12\cdot\cdot\cdot11}{2},\genfrac{}{}{0pt}{}{00\cdot\cdot\cdot11}{0}\}$
$\{\genfrac{}{}{0pt}{}{2465432}{3}\},\{\genfrac{}{}{0pt}{}{246\cdot\cdot\cdot1}{3}\},\{\genfrac{}{}{0pt}{}{\cdot\cdot54321}{2}\},\{\genfrac{}{}{0pt}{}{\cdot\cdot4\cdot\cdot\cdot1}{2}\},\{\genfrac{}{}{0pt}{}{123\cdot\cdot\cdot1}{2}\},\{\genfrac{}{}{0pt}{}{\cdot\cdot11111}{0}\}$
$\{\genfrac{}{}{0pt}{}{2465432}{3}\},\{\genfrac{}{}{0pt}{}{24654\cdot1}{3}\},\{\genfrac{}{}{0pt}{}{\cdot\cdot\cdot4321}{2}\},\{\genfrac{}{}{0pt}{}{2465321}{3}\},\{\genfrac{}{}{0pt}{}{1233321}{1}\},\{\genfrac{}{}{0pt}{}{\cdot\cdot\cdot3321}{2}\},\{\genfrac{}{}{0pt}{}{\cdot\cdot\cdot32\cdot1}{2}\},$ $\{\genfrac{}{}{0pt}{}{\cdot\cdot\cdot1111}{0}\},\{\genfrac{}{}{0pt}{}{12322\cdot1}{2}\}$
$\{\genfrac{}{}{0pt}{}{2465432}{3}\},\{\genfrac{}{}{0pt}{}{2465431}{3}\},\{\genfrac{}{}{0pt}{}{\cdot\cdot54321}{2}\},\{\genfrac{}{}{0pt}{}{2465\cdot21}{3}\},\{\genfrac{}{}{0pt}{}{\cdot\cdot43\cdot21}{2}\},\{\genfrac{}{}{0pt}{}{\cdot\cdot44321}{2}\},\{\genfrac{}{}{0pt}{}{1232221}{1}\},$ $\{\genfrac{}{}{0pt}{}{\cdot\cdot43211}{2}\},\{\genfrac{}{}{0pt}{}{1233\cdot21}{2}\},\{\genfrac{}{}{0pt}{}{1232221}{2}\},\{\genfrac{}{}{0pt}{}{\cdot\cdot11111}{0}\},\{\genfrac{}{}{0pt}{}{1232\cdot11}{2}\},\{\genfrac{}{}{0pt}{}{0001111}{0}\}$
$\{\genfrac{}{}{0pt}{}{2465432}{3}\},\{\genfrac{}{}{0pt}{}{2465\cdot\cdot1}{3}\},\{\genfrac{}{}{0pt}{}{2454321}{2}\},\{\genfrac{}{}{0pt}{}{\cdot343\cdot\cdot1}{2}\},\{\genfrac{}{}{0pt}{}{\cdot3\cdot4321}{2}\},\{\genfrac{}{}{0pt}{}{12\cdot3\cdot\cdot1}{2}\},\{\genfrac{}{}{0pt}{}{1232\cdot\cdot1}{2}\},$ $\{\genfrac{}{}{0pt}{}{1221111}{1}\},\{\genfrac{}{}{0pt}{}{\cdot111111}{0}\},\{\genfrac{}{}{0pt}{}{00\cdot1111}{0}\}$
$\{\genfrac{}{}{0pt}{}{2465432}{3}\},\{\genfrac{}{}{0pt}{}{24654\cdot1}{3},\genfrac{}{}{0pt}{}{1233321}{1}\},\{\genfrac{}{}{0pt}{}{\cdot\cdot54321}{2}\},\{\genfrac{}{}{0pt}{}{\cdot\cdot4\cdot321}{2}\},\{\genfrac{}{}{0pt}{}{\cdot\cdot432\cdot1}{2},\genfrac{}{}{0pt}{}{\cdot\cdot11111}{0}\},\{\genfrac{}{}{0pt}{}{1232111}{1}\},$ $\{\genfrac{}{}{0pt}{}{123\cdot2\cdot1}{2},\genfrac{}{}{0pt}{}{000\cdot111}{0}\}$
$\{\genfrac{}{}{0pt}{}{2465432}{3}\},\{\genfrac{}{}{0pt}{}{\cdot\cdot\cdot4321}{2}\},\{\genfrac{}{}{0pt}{}{24654\cdot1}{3},\genfrac{}{}{0pt}{}{12332\cdot1}{1}\},\{\genfrac{}{}{0pt}{}{\cdot\cdot\cdot3\cdot\cdot1}{2},\genfrac{}{}{0pt}{}{\cdot\cdot\cdot1111}{0}\},\{\genfrac{}{}{0pt}{}{1232111}{2},\genfrac{}{}{0pt}{}{0000111}{0}\}$
$\{\genfrac{}{}{0pt}{}{2465432}{3}\},\{\genfrac{}{}{0pt}{}{\cdot\cdot54321}{2}\},\{\genfrac{}{}{0pt}{}{24654\cdot1}{3}\},\{\genfrac{}{}{0pt}{}{\cdot\cdot4\cdot321}{2}\},\{\genfrac{}{}{0pt}{}{1233221}{1},\genfrac{}{}{0pt}{}{1232211}{1}\},\{\genfrac{}{}{0pt}{}{\cdot\cdot432\cdot1}{2}\},\{\genfrac{}{}{0pt}{}{\cdot\cdot11111}{0}\},$ $\{\genfrac{}{}{0pt}{}{1233321}{2}\},\{\genfrac{}{}{0pt}{}{1233211}{2},\genfrac{}{}{0pt}{}{1232221}{2}\},\{\genfrac{}{}{0pt}{}{000\cdot111}{0}\},\{\genfrac{}{}{0pt}{}{1232111}{2}\}$
$\{\genfrac{}{}{0pt}{}{2465432}{3}\},\{\genfrac{}{}{0pt}{}{2465431}{3}\},\{\genfrac{}{}{0pt}{}{2454321}{2}\},\{\genfrac{}{}{0pt}{}{\cdot3\cdot4321}{2}\},\{\genfrac{}{}{0pt}{}{\cdot343\cdot21}{2}\},\{\genfrac{}{}{0pt}{}{2465421}{3},\genfrac{}{}{0pt}{}{1233221}{1}\},\{\genfrac{}{}{0pt}{}{\cdot343211}{2},$ $\genfrac{}{}{0pt}{}{\cdot111111}{0}\},\{\genfrac{}{}{0pt}{}{12\cdot3\cdot21}{2}\},\{\genfrac{}{}{0pt}{}{1232221}{2}\},\{\genfrac{}{}{0pt}{}{1221111}{1}\},\{\genfrac{}{}{0pt}{}{12\cdot3211}{2},\genfrac{}{}{0pt}{}{00\cdot1111}{0}\},\{\genfrac{}{}{0pt}{}{1232111}{2},\genfrac{}{}{0pt}{}{0000111}{0}\}$
$\{\genfrac{}{}{0pt}{}{2465432}{3}\},\{\genfrac{}{}{0pt}{}{2454321}{2}\},\{\genfrac{}{}{0pt}{}{\cdot3\cdot\cdot\cdot21}{2}\},\{\genfrac{}{}{0pt}{}{2465\cdot21}{3},\genfrac{}{}{0pt}{}{12\cdot2221}{1}\},\{\genfrac{}{}{0pt}{}{12\cdot3\cdot21}{2}\},\{\genfrac{}{}{0pt}{}{\cdot343211}{2},\genfrac{}{}{0pt}{}{\cdot111111}{0}\},$ $\{\genfrac{}{}{0pt}{}{1244321}{2},\genfrac{}{}{0pt}{}{1232221}{2}\},\{\genfrac{}{}{0pt}{}{1232\cdot11}{2},\genfrac{}{}{0pt}{}{00\cdot1111}{0}\},\{\genfrac{}{}{0pt}{}{0000001}{0}\}$
$\{\genfrac{}{}{0pt}{}{2465432}{3}\},\{\genfrac{}{}{0pt}{}{2\cdot\cdot\cdot321}{2}\},\{\genfrac{}{}{0pt}{}{24654\cdot1}{3}\},\{\genfrac{}{}{0pt}{}{23432\cdot1}{2}\},\{\genfrac{}{}{0pt}{}{1\cdot\cdot\cdot2\cdot1}{2}\},\{\genfrac{}{}{0pt}{}{13\cdot4321}{2},\genfrac{}{}{0pt}{}{12\cdot3321}{2}\},\{\genfrac{}{}{0pt}{}{1221111}{1},$ $\genfrac{}{}{0pt}{}{1122111}{1}\},\{\genfrac{}{}{0pt}{}{1232111}{2}\},\{\genfrac{}{}{0pt}{}{1111111}{0}\},\{\genfrac{}{}{0pt}{}{0\cdot\cdot\cdot111}{0}\}$
$\{\genfrac{}{}{0pt}{}{2465432}{3}\},\{\genfrac{}{}{0pt}{}{2\cdot\cdot\cdot321}{2},\genfrac{}{}{0pt}{}{13\cdot4321}{2},\genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot\cdot11}{2},\genfrac{}{}{0pt}{}{1\cdot\cdot\cdot221}{2},\genfrac{}{}{0pt}{}{12\cdot3321}{2},\genfrac{}{}{0pt}{}{\cdot\cdot\cdot\cdot\cdot11}{0}\},\{\genfrac{}{}{0pt}{}{2465431}{3},\genfrac{}{}{0pt}{}{1221111}{1},$ $\genfrac{}{}{0pt}{}{1122111}{1},\genfrac{}{}{0pt}{}{0122211}{1}\}$
$\{\genfrac{}{}{0pt}{}{2465432}{3}\},\{\genfrac{}{}{0pt}{}{24\cdot\cdot\cdot\cdot1}{3}\},\{\genfrac{}{}{0pt}{}{2454321}{2}\},\{\genfrac{}{}{0pt}{}{\cdot3\cdot\cdot\cdot\cdot1}{2}\},\{\genfrac{}{}{0pt}{}{12\cdot\cdot\cdot\cdot1}{2}\},\{\genfrac{}{}{0pt}{}{\cdot111111}{0}\}$
$\{\genfrac{}{}{0pt}{}{2465432}{3}\},\{\genfrac{}{}{0pt}{}{24654\cdot1}{3}\},\{\genfrac{}{}{0pt}{}{\cdot\cdot54321}{2}\},\{\genfrac{}{}{0pt}{}{246\cdot321}{3}\},\{\genfrac{}{}{0pt}{}{\cdot\cdot4\cdot321}{2}\},\{\genfrac{}{}{0pt}{}{1233321}{1}\},\{\genfrac{}{}{0pt}{}{\cdot\cdot432\cdot1}{2}\},$ $\{\genfrac{}{}{0pt}{}{1233321}{2}\},\{\genfrac{}{}{0pt}{}{123\cdot2\cdot1}{2}\},\{\genfrac{}{}{0pt}{}{\cdot\cdot11111}{0}\}$
$\{\genfrac{}{}{0pt}{}{2465432}{3}\},\{\genfrac{}{}{0pt}{}{2465431}{3}\},\{\genfrac{}{}{0pt}{}{246\cdot\cdot21}{3}\},\{\genfrac{}{}{0pt}{}{2454321}{2}\},\{\genfrac{}{}{0pt}{}{\cdot34\cdot\cdot21}{2}\},\{\genfrac{}{}{0pt}{}{\cdot354321}{2}\},\{\genfrac{}{}{0pt}{}{124\cdot\cdot21}{2}\},$ $\{\genfrac{}{}{0pt}{}{123\cdot\cdot21}{2}\},\{\genfrac{}{}{0pt}{}{\cdot343211}{2}\},\{\genfrac{}{}{0pt}{}{1222221}{1}\},\{\genfrac{}{}{0pt}{}{123\cdot\cdot11}{2}\},\{\genfrac{}{}{0pt}{}{\cdot111111}{0}\},\{\genfrac{}{}{0pt}{}{0011111}{0}\}$

Table 5: Maximal abelian sets

X	W_X	Sig(X)
X_{29}^{24}	$\langle w_3, w_5, w_6, w_7\rangle \cong S_4 \times S_2$	0 10^4 12^{4+2} 14^{8+6} 16 18 20^2
X_{29}^{25}	$\langle w_1, w_3, w_4, w_6, w_\tau\rangle \cong S_4 \times {S_2}^2$	0 8 10^{6+4} 14^8 16^8 18
X_{29}^{26}	$\langle w_1, w_3, w_5w_7\rangle \cong S_3 \times S_2$	0 8 10^{3+2} 12^{6+2} 14^{3+1} 16^{6+2} 18^2
X_{29}^{27}	$\langle w_1, w_4, w_4w_6{w_\tau}^{w_5}\rangle \cong S_2 \times Dih_8$	0 8 10 12^{8+4} 14^{4+2} 16^4 18^4
X_{29}^{28}	$\langle w_1, w_3, w_7\rangle \cong S_3 \times S_2$	0 10^{3+2+1} 12^{3+1} 14^{6+3+1} 16^{3+2} 18^{2+1}
X_{29}^{29}	$\langle w_1, w_4, w_7\rangle \cong {S_2}^3$	0 10^{2+1} $12^{4+2+2.1}$ 14^{4+2} $16^{4+2.2+1}$ 18^2
X_{29}^{30}	$\langle w_1, w_3, w_7\rangle \cong S_3 \times S_2$	0 10^{3+2} 12^{3+2+1} 14^{6+3} 16^{3+2+1} 18 20
X_{29}^{31}	$\langle w_1, w_4w_7\rangle \cong {S_2}^2$	0 10^{2+1} 12^{4+2+1} $14^{4+2.2+1}$ $16^{3.2}$ 18^{2+1}
X_{29}^{32}	$\langle w_1, w_6\rangle \cong {S_2}^2$	0 $10^{2.1}$ $12^{4+2.2+1}$ $14^{3.2+2.1}$ $16^{3.2+1}$ 18 20
X_{29}^{33}	$\langle w_3, w_6\rangle \cong {S_2}^2$	0 10 $12^{3.2+1}$ $14^{4+2.2+3.1}$ $16^{2.2+2.1}$ 18^{2+1}
X_{29}^{34}	$\langle w_3, w_6, w_7\rangle \cong S_3 \times S_2$	0 $12^{2.3+2}$ $14^{6+3+2+2.1}$ 16^{3+1} 18^2 20
X_{29}^{35}	$\langle w_3, w_6, w_7, w_1w_4\rangle \cong S_3 \times Dih_8$	0 12^3 14^{12+4+3} 16^2 18^4
X_{29}^{36}	$\langle w_1, w_4, w_5, w_7\rangle \cong S_3 \times {S_2}^2$	0 10 12^{4+3} $14^{6+2.2+1}$ 16^6 18^3
X_{29}^{37}	$\langle w_3, w_6, w_7, w_1w_4\rangle \cong S_3 \times Dih_8$	0 12^{4+3} 14^{4+3} 16^{12+2}
X_{29}^{38}	$\langle w_1, w_4, w_7, w_6w_\tau\rangle \cong {S_2}^2 \times Dih_8$	0 10 $12^{2.4}$ 14^8 16^{8+1} 18^2
X_{29}^{39}	$\langle w_1, w_3, w_5w_7, w_6w_\tau\rangle \cong S_3 \times Dih_{10}$	0 10^3 12^5 14^{15} 18^5
X_{29}^{40}	$\langle w_1, w_4w_6w_\tau\rangle \cong {S_2}^2$	0 10 12^{4+2+1} $14^{2.4+2+1}$ $16^{3.2}$ 18^{2+1}

in the E_8 root system (continued)
W_X-orbits on X
$\{\substack{2465432\\3}\},\{\substack{246\cdot\cdot\cdot1\\3}\},\{\substack{234\cdot\cdot\cdot1\\2}\},\{\substack{2\cdot54321\\2}\},\{\substack{1\cdot4\cdot\cdot\cdot1\\2}\},\{\substack{123\cdot\cdot\cdot1\\2}\},\{\substack{1111111\\1}\},\{\substack{1111111\\0}\},\{\substack{0\cdot11111\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{2465431\\3}\},\{\substack{\cdot\cdot\cdot4321\\2}\},\{\substack{2465\cdot21\\3},\substack{1233\cdot21\\1}\},\{\substack{\cdot\cdot\cdot3\cdot21\\2}\},\{\substack{\cdot\cdot\cdot3211\\2},\substack{\cdot\cdot\cdot1111\\0}\},\{\substack{1232221\\2}\}$
$\{\substack{2465432\\3}\},\{\substack{1233321\\1}\},\{\substack{\cdot\cdot54321\\2}\},\{\substack{24654\cdot1\\3}\},\{\substack{\cdot\cdot4\cdot321\\2}\},\{\substack{1233211\\1},\substack{1232221\\1}\},\{\substack{\cdot\cdot11111\\0}\},\{\substack{1233321\\2}\},\{\substack{\cdot\cdot432\cdot1\\2}\},\{\substack{000\cdot111\\0}\},\{\substack{1233221\\2},\substack{1232211\\2}\}$
$\{\substack{2465432\\3}\},\{\substack{2465431\\3}\},\{\substack{2454321\\2}\},\{\substack{\cdot3\cdot\cdot\cdot21\\2}\},\{\substack{2465\cdot21\\3},\substack{12\cdot2221\\1}\},\{\substack{12\cdot3\cdot21\\2}\},\{\substack{1244321\\2},\substack{1232221\\2}\},\{\substack{\cdot343211\\2},\substack{\cdot111111\\0}\},\{\substack{1232\cdot11\\2},\substack{00\cdot1111\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{\cdot\cdot54321\\2}\},\{\substack{24654\cdot1\\3}\},\{\substack{2465321\\3}\},\{\substack{\cdot\cdot44321\\2}\},\{\substack{1233321\\1}\},\{\substack{\cdot\cdot432\cdot1\\2}\},\{\substack{\cdot\cdot43321\\2}\},\{\substack{1232111\\1}\},\{\substack{\cdot\cdot11111\\0}\},\{\substack{12332\cdot1\\2}\},\{\substack{12322\cdot1\\2}\},\{\substack{0001111\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{24654\cdot1\\3}\},\{\substack{2454321\\2}\},\{\substack{\cdot3\cdot4321\\2}\},\{\substack{\cdot343321\\2}\},\{\substack{2465321\\3}\},\{\substack{1233321\\1}\},\{\substack{\cdot3432\cdot1\\2}\},\{\substack{12\cdot3321\\2}\},\{\substack{12\cdot32\cdot1\\2}\},\{\substack{12322\cdot1\\2}\},\{\substack{\cdot111111\\0}\},\{\substack{1221111\\1}\},\{\substack{00\cdot1111\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{\cdot\cdot54321\\2}\},\{\substack{24654\cdot1\\3}\},\{\substack{\cdot\cdot43321\\2}\},\{\substack{12322\cdot1\\1}\},\{\substack{2465321\\3}\},\{\substack{\cdot\cdot432\cdot1\\2}\},\{\substack{\cdot\cdot44321\\2}\},\{\substack{\cdot\cdot11111\\0}\},\{\substack{12322\cdot1\\2}\},\{\substack{1233321\\2}\},\{\substack{1232111\\2}\},\{\substack{0001111\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{24654\cdot1\\3}\},\{\substack{2454321\\2}\},\{\substack{\cdot3\cdot4321\\2}\},\{\substack{\cdot343321\\2}\},\{\substack{2465321\\3}\},\{\substack{\cdot3432\cdot1\\2}\},\{\substack{12\cdot3321\\2}\},\{\substack{1232211\\1},\substack{1222221\\1}\},\{\substack{1244321\\2}\},\{\substack{1243221\\2},\substack{1233211\\2}\},\{\substack{12322\cdot1\\2}\},\{\substack{\cdot111111\\0}\},\{\substack{00\cdot1111\\0}\},\{\substack{1232111\\2}\}$
$\{\substack{2465432\\3}\},\{\substack{2465431\\3}\},\{\substack{2454321\\2}\},\{\substack{\cdot343\cdot21\\2}\},\{\substack{2465\cdot21\\3}\},\{\substack{\cdot354321\\2}\},\{\substack{1232221\\1}\},\{\substack{\cdot344321\\2}\},\{\substack{\cdot343211\\2}\},\{\substack{1243\cdot21\\2}\},\{\substack{1232221\\2}\},\{\substack{1221111\\1}\},\{\substack{1233\cdot21\\2}\},\{\substack{1232\cdot11\\2}\},\{\substack{\cdot111111\\0}\},\{\substack{1243211\\2}\},\{\substack{0011111\\0}\},\{\substack{0001111\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{2465431\\3}\},\{\substack{2465\cdot21\\3}\},\{\substack{2\cdot54321\\2}\},\{\substack{2343\cdot21\\2}\},\{\substack{2344321\\2}\},\{\substack{1\cdot43\cdot21\\2}\},\{\substack{1\cdot44321\\2}\},\{\substack{1233\cdot21\\2}\},\{\substack{2343211\\2}\},\{\substack{1232221\\2}\},\{\substack{1232221\\1}\},\{\substack{1\cdot43211\\2}\},\{\substack{1232\cdot11\\2}\},\{\substack{1111111\\1}\},\{\substack{1111111\\0}\},\{\substack{0\cdot11111\\0}\},\{\substack{0001111\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{2465\cdot\cdot1\\3}\},\{\substack{2343\cdot\cdot1\\2}\},\{\substack{2\cdot54321\\2}\},\{\substack{1\cdot43\cdot\cdot1\\2}\},\{\substack{1232\cdot\cdot1\\2}\},\{\substack{1\cdot21111\\1}\},\{\substack{2344321\\2}\},\{\substack{1354321\\2}\},\{\substack{1233\cdot\cdot1\\2}\},\{\substack{1111111\\0}\},\{\substack{0\cdot11111\\0}\},\{\substack{0001111\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{2465\cdot\cdot1\\3}\},\{\substack{\cdot\cdot\cdot3\cdot\cdot1\\2}\},\{\substack{2\cdot54321\\2},\substack{1\cdot44321\\2}\},\{\substack{1232\cdot\cdot1\\2}\},\{\substack{1111111\\1},\substack{0121111\\1}\},\{\substack{\cdot\cdot\cdot1111\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{2454321\\2}\},\{\substack{\cdot3432\cdot1\\2}\},\{\substack{12\cdot\cdot111\\1}\},\{\substack{\cdot3\cdot\cdot321\\2}\},\{\substack{24654\cdot1\\3}\},\{\substack{\cdot111111\\0}\},\{\substack{1232111\\2}\},\{\substack{12\cdot\cdot2\cdot1\\2}\},\{\substack{00\cdot\cdot111\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{\cdot\cdot\cdot1111\\0}\},\{\substack{1233\cdot\cdot1\\1}\},\{\substack{2\cdot54321\\2},\substack{1\cdot44321\\2}\},\{\substack{0000\cdot\cdot1\\0}\},\{\substack{\cdot\cdot\cdot3\cdot\cdot1\\2}\},\{\substack{1111111\\1},\substack{0121111\\1}\}$
$\{\substack{2465432\\3}\},\{\substack{2454321\\2}\},\{\substack{24654\cdot1\\3},\substack{12332\cdot1\\1}\},\{\substack{\cdot3\cdot4321\\2}\},\{\substack{\cdot343\cdot\cdot1\\2},\substack{\cdot111111\\0}\},\{\substack{12\cdot3\cdot\cdot1\\2},\substack{00\cdot1111\\0}\},\{\substack{1221111\\1}\},\{\substack{1232111\\2},\substack{0000111\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{\cdot\cdot54321\\2}\},\{\substack{24654\cdot1\\3},\substack{1233221\\1},\substack{1232\cdot11\\1}\},\{\substack{\cdot\cdot4\cdot\cdot\cdot1\\2},\substack{\cdot\cdot11111\\0}\},\{\substack{1233211\\2},\substack{1232221\\2},\substack{1232111\\2},\substack{000\cdot111\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{2454321\\2}\},\{\substack{\cdot3\cdot4321\\2}\},\{\substack{2465421\\3},\substack{1233221\\1}\},\{\substack{2465431\\3}\},\{\substack{\cdot343\cdot21\\2}\},\{\substack{\cdot343211\\2},\substack{\cdot111111\\0}\},\{\substack{1232211\\1},\substack{1222111\\1}\},\{\substack{1244321\\2}\},\{\substack{1243221\\2},\substack{1233321\\2}\},\{\substack{1243211\\2},\substack{0001111\\0}\},\{\substack{1233211\\2},\substack{0011111\\0}\},\{\substack{1232111\\2},\substack{0000111\\0}\},\{\substack{1232221\\2}\}$

Table 5: Maximal abelian sets

X	W_X	Sig(X)
X_{29}^{41}	$\langle w_3, w_5w_7\rangle \cong {S_2}^2$	0 $12^{3.2}$ $14^{4+2.2+2.1}$ $16^{4+3.2+1}$ 18
X_{29}^{42}	$\langle w_7, w_3w_5\rangle \cong {S_2}^2$	0 $12^{2.2+1}$ $14^{4+3.2+3.1}$ $16^{4+2.2}$ 18^2
X_{29}^{43}	$\langle w_3, w_1w_4, w_6w_\tau\rangle \cong S_2 \times Dih_8$	0 12 $14^{8+4+2+1}$ $16^{8+2.2}$
X_{29}^{44}	$\langle w_1, w_5, w_4w_6w_\tau\rangle \cong S_2 \times Dih_8$	0 $14^{8+2.4+2}$ $16^{2.4+1}$ 18
X_{29}^{45}	$\langle w_1, w_4, w_5, w_7\rangle \cong S_3 \times {S_2}^2$	0 7^2 9^{3+1} 11^6 13^{4+3+1} 15^6 19^2
X_{29}^{46}	$\langle w_4, w_5, w_6\rangle \cong S_4$	0 7 9^4 11^{4+1} 13^{6+4+1} 15^{4+1} 19 21
X_{29}^{47}	$\langle w_1, w_4, w_5, w_6, w_7\rangle \cong S_5 \times S_2$	0 9^5 $13^{2.10}$ 17 21^2
X_{29}^{48}	$\langle w_1, w_3, w_6\rangle \cong S_3 \times S_2$	0 7 9^{3+2} 11^{3+2+1} 13^6 15^{3+2} $17^{3+2.1}$
X_{29}^{49}	$\langle w_1, w_5\rangle \cong {S_2}^2$	0 7 $9^{2.1}$ $11^{4+2.2+1}$ $13^{2.2+2.1}$ $15^{2.2+1}$ $17^{2.2}$ 19
X_{29}^{50}	$\langle w_3, w_5, w_6\rangle \cong S_3 \times S_2$	0 7 $11^{2.3+2}$ $13^{6+3+2+1}$ 15 17^{3+1} 19^2
X_{29}^{51}	$\langle w_1, w_3, w_6, w_5w_7\rangle \cong S_3 \times Dih_8$	0 9^{4+3} 13^{12+2} 17^{4+3}
X_{29}^{52}	$\langle w_1, w_7\rangle \cong {S_2}^2$	0 $9^{2+2.1}$ $11^{2.2+1}$ $13^{4+2+2.1}$ $15^{2.2+2.1}$ $17^{2.2}$ 19
X_{29}^{53}	$\langle w_3, w_5, w_7\rangle \cong {S_2}^3$	0 9^2 $11^{3.2}$ $13^{4+2+2.1}$ $15^{2.4}$ $17^{2.1}$ 19^2
X_{29}^{54}	$\langle w_1, w_5, w_7\rangle \cong {S_2}^3$	0 9^{2+1} 11^{4+2} $13^{4+2.2+1}$ 15^{4+2} 17^{2+1} 21
X_{29}^{55}	$\langle w_5, w_3w_7\rangle \cong {S_2}^2$	0 9^2 $11^{3.2}$ $13^{4+2+2.1}$ $15^{4+2.2}$ $17^{2.1}$ 19^2
X_{29}^{56}	$\langle w_5, w_6\rangle \cong S_3$	0 9 $11^{2.3+1}$ $13^{2.3+3.1}$ $15^{2.3+2.1}$ 17 19 21
X_{29}^{57}	$\langle w_1, w_5, w_6\rangle \cong S_3 \times S_2$	0 9 11^3 $13^{6+2.3+2}$ 15^{3+2+1} 17 19^{2+1}

in the E_8 root system (continued)
W_X-orbits on X
$\{\substack{2465432\\3}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{2\cdot54321\\2}\}, \{\substack{234\cdot321\\2}\}, \{\substack{1\cdot4\cdot321\\2}\}, \{\substack{23432\cdot1\\2}\}, \{\substack{1233221\\1}, \substack{1232211\\1}\}, \{\substack{1233321\\2}\}, \{\substack{1111111\\0}\}, \{\substack{1\cdot432\cdot1\\2}\}, \{\substack{1233211\\2}, \substack{1232221\\2}\}, \{\substack{0\cdot11111\\0}\}, \{\substack{000\cdot111\\0}\}, \{\substack{1111111\\1}\}, \{\substack{1232111\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{2\cdot54321\\2}\}, \{\substack{23432\cdot1\\2}\}, \{\substack{1232111\\1}\}, \{\substack{1\cdot432\cdot1\\2}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{234\cdot321\\2}\}, \{\substack{1221111\\1}, \substack{1122111\\1}\}, \{\substack{1354321\\2}\}, \{\substack{1232111\\2}\}, \{\substack{1111111\\0}\}, \{\substack{123\cdot2\cdot1\\2}\}, \{\substack{1344321\\2}, \substack{1243321\\2}\}, \{\substack{0\cdot11111\\0}\}, \{\substack{000\cdot111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{\cdot\cdot\cdot3\cdot21\\2}\}, \{\substack{2\cdot54321\\2}, \substack{1\cdot44321\\2}\}, \{\substack{2465421\\3}, \substack{1233221\\1}\}, \{\substack{1232221\\2}\}, \{\substack{\cdot\cdot\cdot3211\\2}, \substack{\cdot\cdot\cdot1111\\0}\}, \{\substack{1232111\\2}, \substack{0000111\\0}\}, \{\substack{1111111\\1}, \substack{0121111\\1}\}$
$\{\substack{2465432\\3}\}, \{\substack{\cdot3\cdot\cdot\cdot21\\2}\}, \{\substack{2465421\\3}, \substack{2454321\\3}, \substack{123\cdot221\\1}\}, \{\substack{124\cdot321\\2}, \substack{123\cdot221\\2}\}, \{\substack{1243221\\2}, \substack{1233321\\2}\}, \{\substack{\cdot343211\\2}, \substack{\cdot111111\\0}\}, \{\substack{1243211\\2}, \substack{1232111\\2}, \substack{000\cdot111\\0}\}, \{\substack{0011111\\1}\}, \{\substack{0000001\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{24\cdot\cdot321\\3}\}, \{\substack{2454321\\2}\}, \{\substack{\cdot3\cdot\cdot321\\2}\}, \{\substack{\cdot3432\cdot1\\2}\}, \{\substack{12\cdot\cdot321\\2}\}, \{\substack{1233321\\1}\}, \{\substack{12\cdot\cdot2\cdot1\\2}\}, \{\substack{\cdot111111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{24\cdot\cdot\cdot21\\3}\}, \{\substack{23\cdot\cdot\cdot21\\2}\}, \{\substack{2454321\\2}\}, \{\substack{12\cdot\cdot\cdot21\\2}\}, \{\substack{13\cdot\cdot\cdot21\\2}\}, \{\substack{2343211\\2}\}, \{\substack{12\cdot\cdot\cdot11\\2}\}, \{\substack{1122221\\1}\}, \{\substack{1111111\\0}\}, \{\substack{0111111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{24\cdot\cdot\cdot\cdot1\\3}\}, \{\substack{\cdot3\cdot\cdot\cdot\cdot1\\2}\}, \{\substack{12\cdot\cdot\cdot\cdot1\\2}\}, \{\substack{0011111\\1}\}, \{\substack{\cdot111111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{\cdot\cdot54321\\2}\}, \{\substack{2465\cdot21\\3}\}, \{\substack{\cdot\cdot44321\\2}\}, \{\substack{1233\cdot21\\1}\}, \{\substack{2464321\\3}\}, \{\substack{\cdot\cdot43\cdot21\\2}\}, \{\substack{\cdot\cdot43211\\2}\}, \{\substack{1233\cdot21\\2}\}, \{\substack{\cdot\cdot11111\\0}\}, \{\substack{1233211\\2}\}, \{\substack{1232221\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{2465421\\3}\}, \{\substack{2454321\\2}\}, \{\substack{\cdot34\cdot321\\2}\}, \{\substack{246\cdot321\\3}\}, \{\substack{\cdot354321\\2}\}, \{\substack{1233321\\1}\}, \{\substack{\cdot343221\\2}\}, \{\substack{124\cdot321\\2}\}, \{\substack{1233321\\2}\}, \{\substack{1222221\\1}\}, \{\substack{\cdot343211\\2}\}, \{\substack{123\cdot221\\2}\}, \{\substack{1243221\\2}\}, \{\substack{123\cdot211\\2}\}, \{\substack{\cdot111111\\0}\}, \{\substack{0011111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{246\cdot\cdot21\\3}\}, \{\substack{234\cdot\cdot21\\2}\}, \{\substack{2\cdot54321\\2}\}, \{\substack{1\cdot4\cdot\cdot21\\2}\}, \{\substack{123\cdot\cdot21\\2}\}, \{\substack{1\cdot22221\\1}\}, \{\substack{1354321\\2}\}, \{\substack{2343211\\2}\}, \{\substack{123\cdot\cdot11\\2}\}, \{\substack{1111111\\0}\}, \{\substack{0\cdot11111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{246\cdot\cdot\cdot1\\3}\}, \{\substack{\cdot\cdot54321\\2}\}, \{\substack{\cdot\cdot4\cdot\cdot\cdot1\\2}\}, \{\substack{1233211\\1}, \substack{1232221\\1}\}, \{\substack{1233\cdot21\\2}, \substack{1232\cdot11\\2}\}, \{\substack{\cdot\cdot11111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{2465321\\3}\}, \{\substack{2454321\\2}\}, \{\substack{\cdot354321\\2}\}, \{\substack{\cdot344321\\2}\}, \{\substack{2464321\\3}\}, \{\substack{\cdot3432\cdot1\\2}\}, \{\substack{\cdot343321\\2}\}, \{\substack{1244321\\2}\}, \{\substack{1233321\\1}\}, \{\substack{12432\cdot1\\2}\}, \{\substack{12332\cdot1\\2}\}, \{\substack{1233321\\2}\}, \{\substack{1222111\\1}\}, \{\substack{12322\cdot1\\2}\}, \{\substack{\cdot111111\\0}\}, \{\substack{0011111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{246\cdot321\\3}\}, \{\substack{2\cdot54321\\2}\}, \{\substack{234\cdot321\\2}\}, \{\substack{1\cdot4\cdot321\\2}\}, \{\substack{23432\cdot1\\2}\}, \{\substack{1233321\\2}\}, \{\substack{1233321\\1}\}, \{\substack{1\cdot432\cdot1\\2}\}, \{\substack{123\cdot2\cdot1\\2}\}, \{\substack{1111111\\1}\}, \{\substack{1111111\\0}\}, \{\substack{0\cdot11111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{2454321\\2}\}, \{\substack{\cdot34\cdot321\\2}\}, \{\substack{246\cdot321\\3}\}, \{\substack{\cdot3432\cdot1\\2}\}, \{\substack{\cdot354321\\2}\}, \{\substack{12222\cdot1\\1}\}, \{\substack{1233321\\2}\}, \{\substack{123\cdot2\cdot1\\2}\}, \{\substack{124\cdot321\\2}\}, \{\substack{\cdot111111\\0}\}, \{\substack{1232111\\2}\}, \{\substack{0011111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{246\cdot321\\3}\}, \{\substack{2\cdot54321\\2}\}, \{\substack{234\cdot321\\2}\}, \{\substack{1\cdot4\cdot321\\2}\}, \{\substack{23432\cdot1\\2}\}, \{\substack{1354321\\2}\}, \{\substack{1233321\\2}\}, \{\substack{123\cdot2\cdot1\\2}\}, \{\substack{1343211\\2}, \substack{1243221\\2}\}, \{\substack{1222221\\1}, \substack{1122211\\1}\}, \{\substack{1232111\\2}\}, \{\substack{1111111\\0}\}, \{\substack{0\cdot11111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{246\cdot\cdot21\\3}\}, \{\substack{234\cdot\cdot21\\2}\}, \{\substack{2454321\\2}\}, \{\substack{134\cdot\cdot21\\2}\}, \{\substack{123\cdot\cdot21\\2}\}, \{\substack{2354321\\2}\}, \{\substack{2343211\\2}\}, \{\substack{1222221\\1}\}, \{\substack{124\cdot\cdot21\\2}\}, \{\substack{123\cdot\cdot11\\2}\}, \{\substack{1343211\\2}\}, \{\substack{1111111\\1}\}, \{\substack{1111111\\0}\}, \{\substack{0111111\\0}\}, \{\substack{0011111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{246\cdot\cdot21\\3}\}, \{\substack{\cdot34\cdot\cdot21\\2}\}, \{\substack{124\cdot\cdot21\\2}\}, \{\substack{123\cdot\cdot21\\2}\}, \{\substack{\cdot354321\\2}\}, \{\substack{123\cdot\cdot11\\2}\}, \{\substack{\cdot343211\\2}\}, \{\substack{1222221\\1}\}, \{\substack{0011111\\1}\}, \{\substack{\cdot111111\\0}\}, \{\substack{0011111\\0}\}$

Table 5: Maximal abelian sets

X	W_X	Sig(X)
X_{29}^{58}	$\langle w_1, w_5, w_6, w_7\rangle \cong S_4 \times S_2$	0 11^4 13^{8+6+1} 15^{4+2} 19^2 21
X_{29}^{59}	$\langle w_1, w_3, w_7\rangle \cong S_3 \times S_2$	0 9^{3+2+1} 11^{3+2} 13^{3+1} 15^{6+3+1} 17^2 19
X_{29}^{60}	$\langle w_3, w_5, w_2w_3w_7{}^{w_4}\rangle \cong {S_2}^2 \wr S_2$	0 9^2 11^8 13^4 15^8 17^{4+2}
X_{29}^{61}	$\langle w_1, w_4w_7, w_5w_\tau\rangle \cong S_3 \times S_2$	0 9^{3+1} 11^6 $13^{2.3}$ 15^6 17^6
X_{29}^{62}	$\langle w_1, w_3, w_5w_7\rangle \cong S_3 \times S_2$	0 9^{3+1} $11^{2.2}$ 13^{6+3+1} 15^6 17^2 19^2
X_{29}^{63}	$\langle w_1\rangle \cong S_2$	0 $9^{2.1}$ $11^{2.2+3.1}$ $13^{2.2+2.1}$ $15^{2.2+4.1}$ $17^{4.1}$ 19
X_{29}^{64}	$\langle w_3, w_5w_7\rangle \cong {S_2}^2$	0 9 $11^{3.2}$ $13^{4+2+2.1}$ $15^{3.2}$ 17^{4+2+1}
X_{29}^{65}	$\langle w_1, w_4, w_7\rangle \cong {S_2}^3$	0 $9^{2.1}$ 11^{4+2} $13^{4+2+2.1}$ $15^{4+2.2}$ 17^2 19^2
X_{29}^{66}	$\langle w_1, w_7\rangle \cong {S_2}^2$	0 9 $11^{2.2+1}$ $13^{4+2.2+2.1}$ $15^{2.2+2.1}$ $17^{2.2+1}$ 19
X_{29}^{67}	$\langle w_3, w_7\rangle \cong {S_2}^2$	0 $11^{2.2+2.1}$ $13^{2.2+2.1}$ $15^{4+2.2+2.1}$ $17^{2.2+2.1}$
X_{29}^{68}	$\langle w_4, w_3w_5w_7\rangle \cong Dih_8$	0 11^{4+1} $13^{4+2.2}$ $15^{2.4+2}$ 17^4 19
X_{29}^{69}	$\langle w_3, w_7, w_1w_4\rangle \cong S_2 \times Dih_8$	0 11^2 $13^{2.4+2.1}$ 15^{8+2} 17^{4+2}
X_{29}^{70}	$\langle w_1, w_4w_6w_\tau\rangle \cong {S_2}^2$	0 $9^{2.1}$ 11^{4+2} $13^{4+2+2.1}$ $15^{4+2.2}$ 17^2 19^2
X_{29}^{71}	$\langle w_4\rangle \cong S_2$	0 9 $11^{2+3.1}$ $13^{3.2+4.1}$ $15^{2+4.1}$ $17^{2+3.1}$ 19
X_{29}^{72}	$\langle w_1, w_5, w_4w_6w_\tau\rangle \cong S_2 \times Dih_8$	0 9 $13^{8+2.4+2}$ $17^{2.4+1}$

in the E_8 root system (continued)
W_X-orbits on X
$\{\substack{2465432\\3}\},\{\substack{246\cdot\cdot\cdot1\\3}\},\{\substack{\cdot34\cdot\cdot\cdot1\\2}\},\{\substack{123\cdot\cdot\cdot1\\2}\},\{\substack{2454321\\2}\},\{\substack{124\cdot\cdot\cdot1\\2}\},\{\substack{\cdot111111\\1}\},\{\substack{\cdot111111\\0}\},\{\substack{0011111\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{\cdot\cdot54321\\2}\},\{\substack{24654\cdot1\\3}\},\{\substack{1233321\\1}\},\{\substack{\cdot\cdot43321\\2}\},\{\substack{12322\cdot1\\1}\},\{\substack{\cdot\cdot44321\\2}\},\{\substack{2465321\\3}\},\{\substack{\cdot\cdot432\cdot1\\2}\},\{\substack{\cdot\cdot11111\\0}\},\{\substack{1233321\\2}\},\{\substack{12322\cdot1\\2}\},\{\substack{0001111\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{24654\cdot1\\3}\},\{\substack{1\cdot\cdot\cdot321\\\cdot}\},\{\substack{2\cdot54321\\\cdot}\},\{\substack{1\cdot43211\\2},\substack{123\cdot221\\\cdot},\substack{1\cdot22211\\1}\},\{\substack{012\cdot111\\1},\substack{000\cdot111\\0}\},\{\substack{23432\cdot1\\2}\}$
$\{\substack{2465432\\3}\},\{\substack{24654\cdot1\\3},\substack{1233321\\1}\},\{\substack{2454321\\2}\},\{\substack{\cdot3\cdot\cdot321\\2}\},\{\substack{2465321\\3},\substack{1232221\\1},\substack{1222211\\1}\},\{\substack{12\cdot\cdot321\\2}\},\{\substack{\cdot3432\cdot1\\2},\substack{\cdot111111\\0}\},\{\substack{1243211\\2},\substack{1233221\\2},\substack{12322\cdot1\\2},\substack{00\cdot1111\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{\cdot\cdot54321\\2}\},\{\substack{1233321\\1}\},\{\substack{24654\cdot1\\3}\},\{\substack{1233211\\1},\substack{1232221\\1}\},\{\substack{\cdot\cdot4\cdot321\\2}\},\{\substack{\cdot\cdot11111\\0}\},\{\substack{1232111\\1}\},\{\substack{\cdot\cdot432\cdot1\\2}\},\{\substack{000\cdot111\\0}\},\{\substack{1233221\\2},\substack{1232211\\2}\}$
$\{\substack{2465432\\3}\},\{\substack{2465431\\3}\},\{\substack{2454321\\2}\},\{\substack{\cdot354321\\2}\},\{\substack{\cdot344321\\2}\},\{\substack{2465421\\3}\},\{\substack{2465321\\3}\},\{\substack{1233321\\1}\},\{\substack{\cdot343321\\2}\},\{\substack{\cdot343221\\2}\},\{\substack{1244321\\2}\},\{\substack{1232221\\1}\},\{\substack{\cdot343211\\2}\},\{\substack{\cdot111111\\0}\},\{\substack{1243221\\2}\},\{\substack{1233321\\2}\},\{\substack{1233221\\2}\},\{\substack{1222111\\1}\},\{\substack{1243211\\2}\},\{\substack{1232221\\2}\},\{\substack{0011111\\0}\},\{\substack{0001111\\0}\},\{\substack{1232211\\2}\}$
$\{\substack{2465432\\3}\},\{\substack{1233321\\1}\},\{\substack{24654\cdot1\\3}\},\{\substack{2\cdot54321\\2}\},\{\substack{234\cdot321\\2}\},\{\substack{1\cdot4\cdot321\\2}\},\{\substack{1233211\\1},\substack{1232221\\1}\},\{\substack{1233321\\2}\},\{\substack{1111111\\0}\},\{\substack{23432\cdot1\\2}\},\{\substack{0\cdot11111\\0}\},\{\substack{000\cdot111\\0}\},\{\substack{1\cdot432\cdot1\\2}\},\{\substack{1233221\\2},\substack{1232211\\2}\},\{\substack{1111111\\1}\}$
$\{\substack{2465432\\3}\},\{\substack{2465321\\3}\},\{\substack{2454321\\2}\},\{\substack{\cdot3\cdot4321\\2}\},\{\substack{24654\cdot1\\3}\},\{\substack{\cdot3432\cdot1\\2}\},\{\substack{12\cdot2111\\1}\},\{\substack{1244321\\2}\},\{\substack{1233321\\1}\},\{\substack{12\cdot32\cdot1\\2}\},\{\substack{\cdot343321\\2}\},\{\substack{\cdot111111\\0}\},\{\substack{00\cdot1111\\0}\},\{\substack{12322\cdot1\\2}\}$
$\{\substack{2465432\\3}\},\{\substack{2454321\\2}\},\{\substack{24654\cdot1\\3}\},\{\substack{\cdot354321\\2}\},\{\substack{2465321\\3}\},\{\substack{\cdot3432\cdot1\\2}\},\{\substack{\cdot344321\\2}\},\{\substack{\cdot343321\\2}\},\{\substack{1233321\\1}\},\{\substack{1232111\\1}\},\{\substack{12432\cdot1\\2}\},\{\substack{\cdot111111\\0}\},\{\substack{1243321\\2}\},\{\substack{1221111\\1}\},\{\substack{12332\cdot1\\2}\},\{\substack{12322\cdot1\\2}\},\{\substack{0011111\\0}\},\{\substack{0001111\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{24654\cdot1\\3}\},\{\substack{2\cdot54321\\2}\},\{\substack{2465321\\3}\},\{\substack{2344321\\2}\},\{\substack{1\cdot44321\\2}\},\{\substack{23432\cdot1\\2}\},\{\substack{2343321\\2}\},\{\substack{1233321\\1}\},\{\substack{1\cdot432\cdot1\\2}\},\{\substack{1\cdot43321\\2}\},\{\substack{12332\cdot1\\2}\},\{\substack{1232111\\1}\},\{\substack{1111111\\0}\},\{\substack{12322\cdot1\\2}\},\{\substack{0\cdot11111\\0}\},\{\substack{1111111\\1}\},\{\substack{0001111\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{2\cdot\cdot\cdot321\\2}\},\{\substack{1111111\\0}\},\{\substack{1233211\\1},\substack{12\cdot2221\\1},\substack{1122211\\1}\},\{\substack{24654\cdot1\\3}\},\{\substack{23432\cdot1\\2}\},\{\substack{13\cdot4321\\2},\substack{12\cdot3321\\2}\},\{\substack{0\cdot\cdot\cdot111\\0}\},\{\substack{1221111\\1},\substack{1122111\\1}\},\{\substack{1343211\\2},\substack{12\cdot3221\\2},\substack{1232211\\2}\},\{\substack{1232111\\2}\}$
$\{\substack{2465432\\3}\},\{\substack{24654\cdot1\\3}\},\{\substack{2\cdot54321\\2},\substack{1\cdot44321\\2}\},\{\substack{\cdot\cdot\cdot3321\\2}\},\{\substack{2465321\\3}\},\{\substack{1233321\\1}\},\{\substack{\cdot\cdot\cdot32\cdot1\\2}\},\{\substack{12322\cdot1\\2}\},\{\substack{\cdot\cdot\cdot1111\\0}\},\{\substack{1111111\\1},\substack{0121111\\1}\}$
$\{\substack{2465432\\3}\},\{\substack{2465431\\3}\},\{\substack{2454321\\2}\},\{\substack{\cdot3\cdot4321\\2}\},\{\substack{2465421\\3},\substack{1233221\\1}\},\{\substack{\cdot343\cdot21\\2}\},\{\substack{2464321\\3},\substack{1222221\\1}\},\{\substack{1244321\\2}\},\{\substack{1233211\\1}\},\{\substack{\cdot343211\\2},\substack{\cdot111111\\0}\},\{\substack{1243321\\2},\substack{1233221\\2}\},\{\substack{1243221\\2},\substack{1233321\\2}\},\{\substack{1233211\\2},\substack{0011111\\0}\},\{\substack{1232111\\2},\substack{0000111\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{2465431\\3}\},\{\substack{23\cdot4321\\2}\},\{\substack{2465421\\3}\},\{\substack{2454321\\2}\},\{\substack{2343321\\2}\},\{\substack{13\cdot4321\\2}\},\{\substack{12\cdot3321\\2}\},\{\substack{12\cdot2221\\1}\},\{\substack{2465321\\3}\},\{\substack{2343221\\2}\},\{\substack{1343321\\2}\},\{\substack{1244321\\2}\},\{\substack{12\cdot3221\\2}\},\{\substack{2343211\\2}\},\{\substack{1232221\\2}\},\{\substack{1122211\\1}\},\{\substack{1111111\\0}\},\{\substack{00\cdot1111\\0}\},\{\substack{1343211\\2}\},\{\substack{1232211\\2}\},\{\substack{0111111\\0}\},\{\substack{1232111\\2}\}$
$\{\substack{2465432\\3}\},\{\substack{2465431\\3}\},\{\substack{\cdot3\cdot\cdot\cdot21\\2}\},\{\substack{2465421\\3},\substack{2454321\\3},\substack{123\cdot221\\1}\},\{\substack{124\cdot321\\2},\substack{123\cdot221\\2}\},\{\substack{1243221\\2},\substack{1233321\\2}\},\{\substack{\cdot343211\\2},\substack{\cdot111111\\0}\},\{\substack{1243211\\2},\substack{1232111\\2},\substack{000\cdot111\\0}\},\{\substack{0011111\\1}\}$

Table 5: Maximal abelian sets

X	W_X	Sig(X)
X_{29}^{73}	$\langle w_1, w_3, w_5, w_6, w_\tau\rangle \cong {S_3}^2 \times S_2$	0 9^3 11^6 13^9 15^6 17^3 21
X_{29}^{74}	$\langle w_1, w_7\rangle \cong {S_2}^2$	0 9 $11^{2.2+1}$ $13^{4+3.2}$ $15^{2.2+2.1}$ $17^{2+3.1}$ 19
X_{29}^{75}	$\langle w_1, w_6, w_4w_\tau\rangle \cong {S_2}^3$	0 9 $11^{2.4}$ 13^{4+2+1} $15^{2.4}$ 17^{2+1} 21
X_{29}^{76}	$\langle w_3, w_7\rangle \cong {S_2}^2$	0 $11^{2.2+1}$ $13^{3.2+2.1}$ $15^{4+2.2+2.1}$ $17^{2+2.1}$ 19
X_{29}^{77}	$\langle w_3w_7, w_4w_6\rangle \cong S_3$	0 $11^{2.3}$ $13^{2.3}$ 15^{6+3+1} $17^{2.3}$
X_{29}^{78}	$\langle w_1, w_4w_7\rangle \cong {S_2}^2$	0 9 $11^{2.2}$ $13^{2.4+2+2.1}$ $15^{3.2}$ 17^{2+1} 19^2
X_{29}^{79}	1	0 $11^{5.1}$ $13^{8.1}$ $15^{10.1}$ $17^{4.1}$ 19
X_{29}^{80}	$\langle w_1, w_4w_7\rangle \cong {S_2}^2$	0 11^2 $13^{4+2.2+2.1}$ $15^{4+3.2}$ $17^{2.2+2.1}$
X_{29}^{81}	$\langle w_4, w_3w_5w_7\rangle \cong Dih_8$	0 11^{4+2} $13^{2.4+2+1}$ 15^{4+2} 17^4 21
X_{29}^{82}	$\langle w_6\rangle \cong S_2$	0 $11^{2+2.1}$ $13^{2.2+6.1}$ $15^{3.2+4.1}$ $17^{2.1}$ $19^{2.1}$
X_{29}^{83}	$\langle w_1, w_6\rangle \cong {S_2}^2$	0 11 $13^{4+3.2+2.1}$ $15^{4.2+2.1}$ $17^{2+2.1}$ 19
X_{29}^{84}	$\langle w_1, w_6\rangle \cong {S_2}^2$	0 11 $13^{4+3.2+2.1}$ $15^{4.2+2.1}$ $17^{2+2.1}$ 19
X_{29}^{85}	$\langle w_1, w_2, w_3, w_1w_4w_6{w_\tau}^{w_3w_5w_4}\rangle \cong S_2 \times (S_3 \wr S_2)$	0 13^{12+9} 17^6 21
X_{29}^{86}	$\langle w_1, w_2, w_3, w_6, w_7\rangle \cong {S_3}^2 \times S_2$	0 11^3 $13^{2.6}$ 15^{9+1} 19^3
X_{29}^{87}	$\langle w_1, w_3, w_2w_5, w_5w_7\rangle \cong S_3 \times {S_2}^2$	0 $13^{2.6+2}$ 15^{6+4} 17^2 19^2

in the E_8 root system (continued)
W_X-orbits on X
$\{\substack{2465432\\3}\}, \{\substack{\cdot\cdot54321\\2}\}, \{\substack{246\cdot\cdot21\\3}, \substack{123\cdot\cdot21\\1}\}, \{\substack{\cdot\cdot4\cdot\cdot21\\2}\}, \{\substack{\cdot\cdot43211\\2}, \substack{\cdot\cdot11111\\0}\}, \{\substack{123\cdot\cdot21\\2}\}, \{\substack{0000001\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2454321\\2}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{\cdot354321\\2}\}, \{\substack{2465321\\3}\}, \{\substack{\cdot3432\cdot1\\2}\}, \{\substack{\cdot344321\\2}\}, \{\substack{\cdot343321\\2}\}, \{\substack{12322\cdot1\\1}\}, \{\substack{12432\cdot1\\2}\}, \{\substack{\cdot111111\\0}\}, \{\substack{1244321\\2}\}, \{\substack{1222111\\1}\}, \{\substack{12322\cdot1\\2}\}, \{\substack{1233321\\2}\}, \{\substack{1232111\\2}\}, \{\substack{0011111\\0}\}, \{\substack{0001111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2454321\\2}\}, \{\substack{2465\cdot21\\3}, \substack{1233\cdot21\\1}\}, \{\substack{\cdot3\cdot4321\\2}\}, \{\substack{\cdot343\cdot21\\2}\}, \{\substack{2454321\\3}, \substack{1232221\\1}\}, \{\substack{1244321\\2}\}, \{\substack{\cdot343211\\2}, \substack{\cdot111111\\0}\}, \{\substack{12\cdot3\cdot21\\2}\}, \{\substack{1243211\\2}, \substack{0001111\\0}\}, \{\substack{1232221\\2}\}, \{\substack{0000001\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{2\cdot54321\\2}\}, \{\substack{2343321\\2}\}, \{\substack{23432\cdot1\\2}\}, \{\substack{1\cdot43321\\2}\}, \{\substack{12322\cdot1\\1}\}, \{\substack{2465321\\3}\}, \{\substack{2344321\\2}\}, \{\substack{1\cdot432\cdot1\\2}\}, \{\substack{1\cdot44321\\2}\}, \{\substack{12322\cdot1\\2}\}, \{\substack{1233321\\2}\}, \{\substack{1111111\\0}\}, \{\substack{0\cdot11111\\0}\}, \{\substack{1232111\\2}\}, \{\substack{1111111\\1}\}, \{\substack{0001111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465\cdot\cdot1\\3}\}, \{\substack{2\cdot\cdot4321\\2}\}, \{\substack{1\cdot\cdot4321\\2}\}, \{\substack{2343\cdot\cdot1\\2}\}, \{\substack{1\cdot43321\\2}, \substack{1343221\\2}, \substack{1243211\\2}, \substack{12332\cdot1\\2}\}, \{\substack{1232111\\1}, \substack{1222211\\1}, \substack{1122221\\1}\}, \{\substack{1111111\\0}\}, \{\substack{1232\cdot\cdot1\\2}\}, \{\substack{0\cdot\cdot1111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2454321\\2}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{\cdot343321\\2}\}, \{\substack{\cdot3\cdot4321\\2}\}, \{\substack{\cdot3432\cdot1\\2}\}, \{\substack{1232211\\1}, \substack{1222221\\1}\}, \{\substack{2465321\\3}\}, \{\substack{1221111\\1}\}, \{\substack{12\cdot3321\\2}\}, \{\substack{12322\cdot1\\2}\}, \{\substack{\cdot111111\\0}\}, \{\substack{1243221\\2}, \substack{1233211\\2}\}, \{\substack{1232111\\2}\}, \{\substack{00\cdot1111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{2465421\\3}\}, \{\substack{2454321\\2}\}, \{\substack{2354321\\2}\}, \{\substack{2343321\\2}\}, \{\substack{2465321\\3}\}, \{\substack{2344321\\2}\}, \{\substack{1354321\\2}\}, \{\substack{2343221\\2}\}, \{\substack{1343321\\2}\}, \{\substack{2343211\\2}\}, \{\substack{1243321\\2}\}, \{\substack{1232221\\1}\}, \{\substack{1344321\\2}\}, \{\substack{1343211\\2}\}, \{\substack{1243221\\2}\}, \{\substack{1233321\\2}\}, \{\substack{1243211\\2}\}, \{\substack{1232221\\2}\}, \{\substack{1232211\\2}\}, \{\substack{1122211\\1}\}, \{\substack{1221111\\1}\}, \{\substack{1111111\\0}\}, \{\substack{1233221\\2}\}, \{\substack{1232111\\2}\}, \{\substack{0111111\\0}\}, \{\substack{0011111\\0}\}, \{\substack{0001111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{\cdot3\cdot4321\\2}\}, \{\substack{\cdot343321\\2}\}, \{\substack{12\cdot3321\\2}\}, \{\substack{2465321\\3}\}, \{\substack{1244321\\2}\}, \{\substack{\cdot3432\cdot1\\2}\}, \{\substack{1243221\\2}, \substack{1233211\\2}\}, \{\substack{12322\cdot1\\2}\}, \{\substack{1232211\\1}, \substack{1222221\\1}\}, \{\substack{\cdot111111\\0}\}, \{\substack{00\cdot1111\\0}\}, \{\substack{1232111\\2}\}, \{\substack{0011111\\1}\}$
$\{\substack{2465432\\3}\}, \{\substack{2\cdot\cdot\cdot321\\2}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{13\cdot4321\\2}, \substack{12\cdot3321\\2}\}, \{\substack{1233211\\1}, \substack{12\cdot2221\\1}, \substack{1122211\\1}\}, \{\substack{1343321\\2}, \substack{1244321\\2}\}, \{\substack{1111111\\0}\}, \{\substack{0\cdot\cdot\cdot111\\0}\}, \{\substack{23432\cdot1\\2}\}, \{\substack{1343211\\2}, \substack{12\cdot3221\\2}, \substack{1232211\\2}\}, \{\substack{1232111\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{2343\cdot21\\2}\}, \{\substack{2465431\\3}\}, \{\substack{2454321\\2}\}, \{\substack{2465\cdot21\\3}\}, \{\substack{1343\cdot21\\2}\}, \{\substack{2354321\\2}\}, \{\substack{2344321\\2}\}, \{\substack{2343211\\2}\}, \{\substack{1232221\\2}\}, \{\substack{1232221\\1}\}, \{\substack{1221111\\1}\}, \{\substack{1243\cdot21\\2}\}, \{\substack{1233\cdot21\\2}\}, \{\substack{1232\cdot11\\2}\}, \{\substack{1344321\\2}\}, \{\substack{1343211\\2}\}, \{\substack{1111111\\1}\}, \{\substack{1111111\\0}\}, \{\substack{1243211\\2}\}, \{\substack{0111111\\0}\}, \{\substack{0011111\\0}\}, \{\substack{0001111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{\cdot343\cdot21\\2}\}, \{\substack{2465\cdot21\\3}\}, \{\substack{\cdot354321\\2}\}, \{\substack{1243\cdot21\\2}\}, \{\substack{1232221\\2}\}, \{\substack{1232221\\1}\}, \{\substack{\cdot344321\\2}\}, \{\substack{\cdot343211\\2}\}, \{\substack{1233\cdot21\\2}\}, \{\substack{1232\cdot11\\2}\}, \{\substack{1243211\\2}\}, \{\substack{1221111\\1}\}, \{\substack{\cdot111111\\0}\}, \{\substack{0011111\\1}\}, \{\substack{0011111\\0}\}, \{\substack{0001111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{\cdot343\cdot21\\2}\}, \{\substack{2465\cdot21\\3}\}, \{\substack{\cdot344321\\2}\}, \{\substack{1233\cdot21\\2}\}, \{\substack{2454321\\2}\}, \{\substack{1232221\\2}\}, \{\substack{\cdot343211\\2}\}, \{\substack{1243\cdot21\\2}\}, \{\substack{1232\cdot11\\2}\}, \{\substack{\cdot111111\\1}\}, \{\substack{1244321\\2}\}, \{\substack{1232221\\1}\}, \{\substack{\cdot111111\\0}\}, \{\substack{1243211\\2}\}, \{\substack{0001111\\0}\}, \{\substack{0011111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{\cdot\cdot54321\\\cdot}, \substack{123\cdot\cdot21\\\cdot}\}, \{\substack{\cdot\cdot4\cdot\cdot21\\2}\}, \{\substack{\cdot\cdot43211\\2}, \substack{000\cdot\cdot11\\0}\}, \{\substack{0000001\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{\cdot\cdot44321\\2}\}, \{\substack{\cdot\cdot54321\\\cdot}\}, \{\substack{1233\cdot\cdot1\\\cdot}\}, \{\substack{\cdot\cdot43\cdot\cdot1\\2}\}, \{\substack{0001111\\0}\}, \{\substack{0000\cdot\cdot1\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{\cdot\cdot54321\\\cdot}\}, \{\substack{\cdot\cdot4\cdot321\\2}\}, \{\substack{1233321\\\cdot}\}, \{\substack{\cdot\cdot432\cdot1\\2}\}, \{\substack{1233221\\2}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{1232221\\1}\}, \{\substack{000\cdot111\\0}\}, \{\substack{00000\cdot1\\0}\}$

Table 5: Maximal abelian sets

X	W_X	Sig(X)
X_{29}^{88}	$\langle w_1, w_5w_7, w_4w_6w_\tau\rangle \cong S_2 \times Dih_{12}$	0 9 13^{12+6} 17^{6+3}
X_{29}^{89}	$\langle w_1, w_2, w_3, w_\tau, w_1w_4w_6{}^{w_3w_5w_4}\rangle \cong S_3 \times (S_3 \wr S_2)$	0 9 13^{18} 17^{9}
X_{29}^{90}	$\langle w_3w_5w_7\rangle \cong S_2$	0 11^2 $13^{4.2+2.1}$ $15^{5.2}$ $17^{2.2+2.1}$
X_{29}^{91}	$\langle w_1, w_4w_6, w_5w_7\rangle \cong S_2 \times Dih_{10}$	0 11^2 $13^{2.5}$ 15^{10} 17^{5+1}
X_{29}^{92}	$\langle w_1, w_3, w_2w_7, w_5w_7\rangle \cong S_3 \times {S_2}^2$	0 11^2 $13^{6+2.2}$ 15^{6+4} 17^{6}
X_{29}^{93}	$\langle w_6, w_5w_7, w_1w_6w_\tau{}^{w_3w_5w_4w_2w_3w_5w_4w_5}\rangle \cong Dih_8 \wr S_2$	0 13^8 15^{16} 17^4
X_{29}^{94}	$\langle w_1, w_4, w_5, w_7\rangle \cong S_3 \times {S_2}^2$	0 13^{4+3+1} $15^{2.6+2.2}$ 17^{3+1}
X_{29}^{95}	$\langle w_1, w_4w_6w_\tau\rangle \cong {S_2}^2$	0 $13^{4+2+2.1}$ $15^{2.4+4.2}$ $17^{2+2.1}$
X_{29}^{96}	$\langle w_1, w_5w_7\rangle \cong {S_2}^2$	0 $13^{4+2+2.1}$ $15^{4+6.2}$ $17^{2+2.1}$
X_{29}^{97}	$\langle w_5w_7, w_1w_2w_5, w_1w_4w_6w_\tau{}^{w_3w_5w_4}\rangle \cong S_2 \times Dih_8$	0 $13^{2.4}$ $15^{2.8}$ 17^4
X_{29}^{98}	$\langle w_3, w_4, w_5, w_6\rangle \cong S_5$	0 6 8^5 12^{10+5} 14^5 16 22
X_{29}^{99}	$\langle w_1, w_4, w_6\rangle \cong {S_2}^3$	0 6 8^{2+1} 10^{4+2} 12^{4+2+1} 14^{4+2} 16^{2+1} 18^2
X_{29}^{100}	$\langle w_4, w_5\rangle \cong S_3$	0 6 8 $10^{2.3+1}$ $12^{2.3+2.1}$ $14^{3+3.1}$ 16^3 18 20
X_{29}^{101}	$\langle w_1, w_6, w_5w_7\rangle \cong S_2 \times Dih_8$	0 8^{4+1} 10^{2+1} 12^8 14^{4+2} 16^4 18^2
X_{29}^{102}	$\langle w_4, w_7\rangle \cong {S_2}^2$	0 8^{2+1} $10^{2.2+1}$ $12^{2.2+2.1}$ $14^{4+2.2+1}$ 16^{2+1} 18 20
X_{29}^{103}	$\langle w_1, w_4, w_5, w_7\rangle \cong S_3 \times {S_2}^2$	0 8^2 10^3 12^{6+3} 14^{6+4+1} 18 20^2
X_{29}^{104}	$\langle w_1, w_4, w_5, w_6\rangle \cong S_4 \times S_2$	0 6 10^4 12^{8+6+1} 14^2 16^4 20^2
X_{29}^{105}	$\langle w_4, w_5, w_7\rangle \cong S_3 \times S_2$	0 8^2 $10^{2.3}$ 12^{3+2+1} 14^{6+3+2} 16 18 22

in the E_8 root system (continued)
W_X-orbits on X
$\{\substack{2465432\\3}\}, \{\substack{2454321\\2}\}, \{\substack{\cdot3\cdot\cdot\cdot\cdot1\\2}, \substack{\cdot111111\\0}\}, \{\substack{24654\cdot1\\3}, \substack{1233221\\1}, \substack{1232211\\1}, \substack{122\cdot111\\1}\}, \{\substack{12432\cdot1\\2}, \substack{1233321\\2}, \substack{1232111\\2}, \substack{000\cdot111\\0}\}, \{\substack{1233211\\2}, \substack{1232221\\2}, \substack{0011111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{0000001\\0}\}, \{\substack{\cdot\cdot43211\\2}, \substack{123\cdot\cdot11\\\cdot}, \substack{\cdot\cdot11111\\\cdot}, \substack{000\cdot\cdot11\\0}\}, \{\substack{\cdot\cdot4\cdot\cdot21\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{2\cdot54321\\2}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{234\cdot321\\2}\}, \{\substack{23432\cdot1\\2}\}, \{\substack{1233221\\1}, \substack{1232211\\1}\}, \{\substack{1354321\\2}\}, \{\substack{1111111\\0}\}, \{\substack{1343321\\2}, \substack{1244321\\2}\}, \{\substack{1343221\\2}, \substack{1243211\\2}\}, \{\substack{1343211\\2}, \substack{1243221\\2}\}, \{\substack{1222111\\1}, \substack{1121111\\1}\}, \{\substack{0\cdot11111\\0}\}, \{\substack{1233211\\2}, \substack{1232221\\2}\}, \{\substack{000\cdot111\\0}\}, \{\substack{1233321\\2}\}, \{\substack{1232111\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{\cdot111111\\0}\}, \{\substack{1233\cdot21\\1}, \substack{1232211\\1}, \substack{122\cdot111\\1}\}, \{\substack{00\cdot\cdot\cdot\cdot1\\0}\}, \{\substack{\cdot3\cdot\cdot\cdot\cdot1\\2}\}, \{\substack{12432\cdot1\\2}, \substack{1233321\\2}, \substack{1233211\\2}, \substack{1232221\\2}\}, \{\substack{0011111\\1}\}$
$\{\substack{2465432\\3}\}, \{\substack{000\cdot111\\0}\}, \{\substack{\cdot\cdot11111\\\cdot}\}, \{\substack{1233321\\\cdot}\}, \{\substack{00000\cdot1\\0}\}, \{\substack{\cdot\cdot4\cdot321\\2}\}, \{\substack{1233221\\2}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{1232221\\1}\}, \{\substack{\cdot\cdot432\cdot1\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{2\cdot54321\\2}, \substack{1233\cdot21\\1}, \substack{1232\cdot11\\1}, \substack{0\cdot11111\\0}\}, \{\substack{\cdot\cdot4\cdot\cdot\cdot1\\2}, \substack{000\cdot\cdot\cdot1\\0}\}, \{\substack{1354321\\3}, \substack{1233211\\2}, \substack{1232221\\2}, \substack{1111111\\1}\}$
$\{\substack{2465432\\3}\}, \{\substack{\cdot3432\cdot1\\2}\}, \{\substack{12\cdot\cdot111\\1}\}, \{\substack{1232111\\2}\}, \{\substack{\cdot3\cdot\cdot321\\2}\}, \{\substack{12\cdot\cdot2\cdot1\\2}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{\cdot111111\\0}\}, \{\substack{00\cdot\cdot111\\0}\}, \{\substack{0011111\\1}\}$
$\{\substack{2465432\\3}\}, \{\substack{\cdot3\cdot4321\\2}\}, \{\substack{2465421\\3}, \substack{1233221\\1}\}, \{\substack{2465431\\3}\}, \{\substack{1244321\\2}\}, \{\substack{\cdot343\cdot21\\2}\}, \{\substack{\cdot343211\\2}, \substack{\cdot111111\\0}\}, \{\substack{1243321\\2}, \substack{1233221\\2}\}, \{\substack{1243211\\2}, \substack{0001111\\0}\}, \{\substack{1233211\\2}, \substack{0011111\\0}\}, \{\substack{1232111\\1}, \substack{1222211\\1}\}, \{\substack{1232111\\2}, \substack{0000111\\0}\}, \{\substack{1232221\\2}\}, \{\substack{0011111\\1}\}$
$\{\substack{2465432\\3}\}, \{\substack{\cdot34\cdot321\\2}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{2454321\\2}\}, \{\substack{1233321\\2}\}, \{\substack{\cdot3432\cdot1\\2}\}, \{\substack{124\cdot321\\2}\}, \{\substack{1233221\\2}, \substack{1232211\\2}\}, \{\substack{1233211\\1}, \substack{1232221\\1}\}, \{\substack{\cdot111111\\1}\}, \{\substack{\cdot111111\\0}\}, \{\substack{000\cdot111\\0}\}, \{\substack{12432\cdot1\\2}\}, \{\substack{1232111\\2}\}, \{\substack{0011111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2454321\\\cdot}, \substack{1233321\\\cdot}\}, \{\substack{\cdot34\cdot321\\2}\}, \{\substack{2354321\\2}, \substack{1354321\\3}, \substack{1233221\\2}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{1232221\\1}, \substack{1111111\\1}, \substack{0111111\\0}\}, \{\substack{\cdot3432\cdot1\\2}, \substack{124\cdot321\\2}, \substack{000\cdot111\\0}\}, \{\substack{12432\cdot1\\2}, \substack{00000\cdot1\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{2\cdot\cdot\cdot\cdot21\\3}\}, \{\substack{1\cdot\cdot\cdot\cdot21\\2}\}, \{\substack{2\cdot\cdot\cdot\cdot21\\2}\}, \{\substack{1\cdot\cdot\cdot\cdot11\\2}\}, \{\substack{0122221\\1}\}, \{\substack{1111111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{2465\cdot21\\3}\}, \{\substack{2454321\\2}\}, \{\substack{\cdot3\cdot4321\\2}\}, \{\substack{24\cdot4321\\3}\}, \{\substack{\cdot343\cdot21\\2}\}, \{\substack{1233\cdot21\\1}\}, \{\substack{1244321\\2}\}, \{\substack{12\cdot3\cdot21\\2}\}, \{\substack{\cdot343211\\2}\}, \{\substack{12\cdot3211\\2}\}, \{\substack{1232221\\2}\}, \{\substack{\cdot111111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{2465421\\3}\}, \{\substack{24\cdot\cdot321\\3}\}, \{\substack{23\cdot\cdot321\\2}\}, \{\substack{2454321\\2}\}, \{\substack{13\cdot\cdot321\\2}\}, \{\substack{12\cdot\cdot321\\2}\}, \{\substack{2343221\\2}\}, \{\substack{1233321\\1}\}, \{\substack{12\cdot\cdot221\\2}\}, \{\substack{2343211\\2}\}, \{\substack{1343221\\2}\}, \{\substack{1122221\\1}\}, \{\substack{12\cdot\cdot211\\2}\}, \{\substack{1111111\\0}\}, \{\substack{0111111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{246\cdot\cdot\cdot1\\3}\}, \{\substack{2454321\\2}\}, \{\substack{\cdot354321\\2}\}, \{\substack{2454321\\3}\}, \{\substack{\cdot34\cdot\cdot\cdot1\\2}\}, \{\substack{124\cdot\cdot\cdot1\\2}\}, \{\substack{1233211\\1}, \substack{1232221\\1}\}, \{\substack{1233\cdot21\\2}, \substack{1232\cdot11\\2}\}, \{\substack{\cdot111111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{2465321\\3}\}, \{\substack{24\cdot4321\\3}\}, \{\substack{23\cdot4321\\2}\}, \{\substack{2454321\\2}\}, \{\substack{13\cdot4321\\2}\}, \{\substack{23432\cdot1\\2}\}, \{\substack{2343321\\2}\}, \{\substack{1244321\\2}\}, \{\substack{12\cdot32\cdot1\\2}\}, \{\substack{13432\cdot1\\2}\}, \{\substack{12\cdot3321\\2}\}, \{\substack{1233321\\1}\}, \{\substack{12322\cdot1\\2}\}, \{\substack{1122111\\1}\}, \{\substack{1111111\\0}\}, \{\substack{0111111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{24\cdot\cdot321\\3}\}, \{\substack{\cdot3\cdot\cdot321\\2}\}, \{\substack{12\cdot\cdot321\\2}\}, \{\substack{12\cdot\cdot2\cdot1\\2}\}, \{\substack{\cdot3432\cdot1\\2}\}, \{\substack{1233321\\1}\}, \{\substack{0011111\\1}\}, \{\substack{\cdot111111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{24\cdot\cdot\cdot21\\3}\}, \{\substack{\cdot3\cdot\cdot\cdot21\\2}\}, \{\substack{12\cdot\cdot\cdot21\\2}\}, \{\substack{2454321\\2}\}, \{\substack{\cdot122221\\1}\}, \{\substack{12\cdot\cdot\cdot11\\2}\}, \{\substack{\cdot111111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{24\cdot\cdot321\\3}\}, \{\substack{23\cdot\cdot321\\2}\}, \{\substack{12\cdot\cdot321\\2}\}, \{\substack{23432\cdot1\\2}\}, \{\substack{2454321\\2}\}, \{\substack{12\cdot\cdot2\cdot1\\2}\}, \{\substack{13\cdot\cdot321\\2}\}, \{\substack{11222\cdot1\\1}\}, \{\substack{1232111\\2}\}, \{\substack{1111111\\0}\}, \{\substack{0111111\\0}\}$

Table 5: Maximal abelian sets

X	W_X	Sig(X)
X_{29}^{106}	$\langle w_4, w_5, w_1w_7\rangle \cong S_3 \times S_2$	0 8^2 10^3 12^{6+3+1} 14^{6+2} 16^{2+1} 20^2
X_{29}^{107}	$\langle w_4, w_5, w_6\rangle \cong S_4$	0 8 10^4 12^{6+4} $14^{2.4+2.1}$ 16 20 22
X_{29}^{108}	$\langle w_1, w_3, w_5, w_6, w_\tau\rangle \cong {S_3}^2 \times S_2$	0 6 8^3 10^6 12^9 16^{6+3}
X_{29}^{109}	$\langle w_1, w_6, w_4w_\tau\rangle \cong {S_2}^3$	0 6 8 $10^{2.4}$ 12^{4+2+1} 14^4 16^{4+1} 18^2
X_{29}^{110}	$\langle w_4, w_3w_5w_7\rangle \cong Dih_8$	0 6 10^{4+2} $12^{2.4+2+1}$ 14^4 16^2 18^4
X_{29}^{111}	$\langle w_1, w_2, w_3, w_1w_4w_6{w_\tau}^{w_3w_5w_4}\rangle \cong S_2 \times (S_3 \wr S_2)$	0 6 12^{12+9} 18^6
X_{29}^{112}	$\langle w_1, w_3, w_5w_7\rangle \cong S_3 \times S_2$	0 8^{3+2} 10^{2+1} 12^{6+2} 14^6 16^{3+1} 18^2
X_{29}^{113}	$\langle w_1\rangle \cong S_2$	0 $8^{3.1}$ $10^{2.2+2.1}$ $12^{2+3.1}$ $14^{2.2+3.1}$ $16^{2+3.1}$ $18^{2.1}$
X_{29}^{114}	$\langle w_6, w_3w_5w_7\rangle \cong Dih_8$	0 8^2 10^{4+1} $12^{2.4}$ 14^4 16^{4+2+1} 18^2
X_{29}^{115}	$\langle w_3, w_6\rangle \cong {S_2}^2$	0 8 $10^{2.2+1}$ $12^{3.2+1}$ 14^{4+2+1} $16^{2+3.1}$ 18^{2+1}
X_{29}^{116}	$\langle w_1, w_5, w_7\rangle \cong {S_2}^3$	0 8^{2+1} 10^{4+1} $12^{3.2+1}$ 14^{4+2} 16^{4+2} 20
X_{29}^{117}	$\langle w_5, w_3w_7\rangle \cong {S_2}^2$	0 8^2 $10^{2.2+1}$ $12^{4+2+2.1}$ $14^{2.2}$ 16^{4+2+1} 18^2
X_{29}^{118}	$\langle w_1, w_5\rangle \cong {S_2}^2$	0 $8^{2.1}$ $10^{2.2+1}$ 12^{4+2+1} $14^{2.2+3.1}$ $16^{2.2}$ 18^{2+1}
X_{29}^{119}	$\langle w_1\rangle \cong S_2$	0 $8^{2.1}$ $10^{2+2.1}$ $12^{3.2+3.1}$ $14^{2+4.1}$ $16^{2+3.1}$ 18 20

in the E_8 root system (continued)
W_X-orbits on X
$\{\substack{2465432\\3}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{24\cdot\cdot321\\3}\}, \{\substack{\cdot3\cdot\cdot321\\2}\}, \{\substack{12\cdot\cdot321\\2}\}, \{\substack{2454321\\2}\}, \{\substack{12\cdot\cdot2\cdot1\\2}\}, \{\substack{2343221\\2}, \substack{1343211\\2}\}, \{\substack{1122211\\1}, \substack{0122221\\1}\}, \{\substack{1232111\\2}\}, \{\substack{\cdot111111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{24\cdot\cdot\cdot21\\3}\}, \{\substack{12\cdot\cdot\cdot21\\2}\}, \{\substack{23\cdot\cdot\cdot21\\2}\}, \{\substack{13\cdot\cdot\cdot21\\2}\}, \{\substack{12\cdot\cdot\cdot11\\2}\}, \{\substack{2343211\\2}\}, \{\substack{1122221\\1}\}, \{\substack{0011111\\1}\}, \{\substack{1111111\\0}\}, \{\substack{0111111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{\cdot\cdot54321\\2}\}, \{\substack{246\cdot\cdot21\\3}, \substack{123\cdot\cdot21\\1}\}, \{\substack{\cdot\cdot4\cdot\cdot21\\2}\}, \{\substack{\cdot\cdot43211\\2}, \substack{\cdot\cdot11111\\0}\}, \{\substack{123\cdot\cdot21\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{2454321\\2}\}, \{\substack{2465\cdot21\\3}, \substack{1233\cdot21\\1}\}, \{\substack{\cdot3\cdot4321\\2}\}, \{\substack{\cdot343\cdot21\\2}\}, \{\substack{2464321\\3}, \substack{1222221\\1}\}, \{\substack{1244321\\2}\}, \{\substack{12\cdot3\cdot21\\2}\}, \{\substack{\cdot343211\\2}, \substack{\cdot111111\\0}\}, \{\substack{1232221\\2}\}, \{\substack{1233211\\2}, \substack{0011111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{1233321\\1}\}, \{\substack{2\cdot\cdot\cdot321\\2}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{13\cdot4321\\2}, \substack{12\cdot3321\\2}\}, \{\substack{1233221\\1}, \substack{12\cdot2211\\1}, \substack{1122221\\1}\}, \{\substack{1343321\\2}, \substack{1244321\\2}\}, \{\substack{1111111\\0}\}, \{\substack{0\cdot\cdot\cdot111\\0}\}, \{\substack{23432\cdot1\\2}\}, \{\substack{1343221\\2}, \substack{12\cdot3211\\2}, \substack{1232221\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{\cdot\cdot54321\\\cdot}, \substack{123\cdot\cdot21\\\cdot}\}, \{\substack{\cdot\cdot4\cdot\cdot21\\2}\}, \{\substack{\cdot\cdot43211\\2}, \substack{000\cdot\cdot11\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{\cdot\cdot54321\\2}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{246\cdot321\\3}\}, \{\substack{1233321\\1}\}, \{\substack{\cdot\cdot4\cdot321\\2}\}, \{\substack{1233221\\1}, \substack{1232211\\1}\}, \{\substack{\cdot\cdot432\cdot1\\2}\}, \{\substack{\cdot\cdot11111\\0}\}, \{\substack{1233321\\2}\}, \{\substack{1233211\\2}, \substack{1232221\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{2465421\\3}\}, \{\substack{2454321\\2}\}, \{\substack{\cdot354321\\2}\}, \{\substack{\cdot344321\\2}\}, \{\substack{2465321\\3}\}, \{\substack{1233321\\1}\}, \{\substack{\cdot343321\\2}\}, \{\substack{2464321\\3}\}, \{\substack{1244321\\2}\}, \{\substack{1233221\\1}\}, \{\substack{\cdot343221\\2}\}, \{\substack{\cdot343211\\2}\}, \{\substack{1243321\\2}\}, \{\substack{1233321\\2}\}, \{\substack{1222211\\1}\}, \{\substack{\cdot111111\\0}\}, \{\substack{1243211\\2}\}, \{\substack{1233221\\2}\}, \{\substack{1233211\\2}\}, \{\substack{1232221\\2}\}, \{\substack{0011111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2\cdot54321\\2}\}, \{\substack{234\cdot\cdot\cdot1\\2}\}, \{\substack{1111111\\0}\}, \{\substack{246\cdot\cdot\cdot1\\3}\}, \{\substack{1233\cdot21\\1}, \substack{1232\cdot11\\1}\}, \{\substack{1222221\\1}, \substack{1122\cdot11\\1}, \substack{1221111\\1}\}, \{\substack{1344321\\2}, \substack{1243\cdot21\\2}, \substack{1343211\\2}\}, \{\substack{0\cdot11111\\0}\}, \{\substack{1111111\\1}\}, \{\substack{1233211\\2}, \substack{1232221\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{2465\cdot21\\3}\}, \{\substack{2\cdot54321\\2}\}, \{\substack{2344321\\2}\}, \{\substack{2343\cdot21\\2}\}, \{\substack{1\cdot44321\\2}\}, \{\substack{1233\cdot21\\1}\}, \{\substack{2464321\\3}\}, \{\substack{1\cdot43\cdot21\\2}\}, \{\substack{1233\cdot21\\2}\}, \{\substack{2343211\\2}\}, \{\substack{1\cdot43211\\2}\}, \{\substack{1233211\\2}\}, \{\substack{1232221\\2}\}, \{\substack{1111111\\0}\}, \{\substack{0\cdot11111\\0}\}, \{\substack{1111111\\1}\}$
$\{\substack{2465432\\3}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{2454321\\2}\}, \{\substack{\cdot34\cdot321\\2}\}, \{\substack{1233321\\1}\}, \{\substack{246\cdot321\\3}\}, \{\substack{\cdot354321\\2}\}, \{\substack{12222\cdot1\\1}\}, \{\substack{1233321\\2}\}, \{\substack{\cdot3432\cdot1\\2}\}, \{\substack{124\cdot321\\2}\}, \{\substack{123\cdot2\cdot1\\2}\}, \{\substack{\cdot111111\\0}\}, \{\substack{0011111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{2\cdot54321\\2}\}, \{\substack{234\cdot321\\2}\}, \{\substack{1233321\\1}\}, \{\substack{1\cdot4\cdot321\\2}\}, \{\substack{246\cdot321\\3}\}, \{\substack{1354321\\2}\}, \{\substack{1233321\\2}\}, \{\substack{23432\cdot1\\2}\}, \{\substack{1222221\\1}, \substack{1122211\\1}\}, \{\substack{123\cdot2\cdot1\\2}\}, \{\substack{1343211\\2}, \substack{1243221\\2}\}, \{\substack{1111111\\0}\}, \{\substack{0\cdot11111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{2454321\\2}\}, \{\substack{246\cdot321\\3}\}, \{\substack{\cdot354321\\2}\}, \{\substack{2465421\\3}\}, \{\substack{\cdot34\cdot321\\2}\}, \{\substack{\cdot343221\\2}\}, \{\substack{1233321\\1}\}, \{\substack{124\cdot321\\2}\}, \{\substack{\cdot343211\\2}\}, \{\substack{1243221\\2}\}, \{\substack{1222221\\1}\}, \{\substack{1232111\\1}\}, \{\substack{123\cdot221\\2}\}, \{\substack{\cdot111111\\0}\}, \{\substack{123\cdot211\\2}\}, \{\substack{0011111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{2454321\\2}\}, \{\substack{\cdot344321\\2}\}, \{\substack{2465421\\3}\}, \{\substack{2465321\\3}\}, \{\substack{\cdot354321\\2}\}, \{\substack{\cdot343321\\2}\}, \{\substack{\cdot343221\\2}\}, \{\substack{2464321\\3}\}, \{\substack{1233321\\1}\}, \{\substack{1222221\\1}\}, \{\substack{\cdot343211\\2}\}, \{\substack{1244321\\2}\}, \{\substack{1233321\\2}\}, \{\substack{1233221\\2}\}, \{\substack{1222111\\1}\}, \{\substack{\cdot111111\\0}\}, \{\substack{1243221\\2}\}, \{\substack{1233211\\2}\}, \{\substack{1232221\\2}\}, \{\substack{1232211\\2}\}, \{\substack{0011111\\0}\}$

Table 5: Maximal abelian sets

X	W_X	Sig(X)
X_{29}^{120}	1	$0\ 8\ 10^{5.1}\ 12^{7.1}\ 14^{7.1}\ 16^{5.1}\ 18^{3.1}$
X_{29}^{121}	$\langle w_5\rangle \cong S_2$	$0\ 8\ 10^{2+2.1}\ 12^{2.2+5.1}\ 14^{2.2+2.1}\ 16^{2+4.1}\ 18\ 20$
X_{29}^{122}	$\langle w_1, w_5\rangle \cong {S_2}^2$	$0\ 8\ 10\ 12^{4+3.2+2.1}\ 14^{2.2+2.1}\ 16^{2.2}\ 18^{2+2.1}$
X_{29}^{123}	$\langle w_3, w_6, w_5w_7\rangle \cong S_2 \times Dih_8$	$0\ 10^{4+2}\ 12^{4}\ 14^{8+2}\ 16^{4+1}\ 18^{2+1}$
X_{29}^{124}	$\langle w_1, w_5, w_7\rangle \cong {S_2}^3$	$0\ 8\ 10^{2.2}\ 12^{2.4+2}\ 14^{2+1}\ 16^{4+2.2+1}\ 20$
X_{29}^{125}	$\langle w_4, w_3w_5w_7\rangle \cong Dih_8$	$0\ 8\ 10^{4+1}\ 12^{4+2}\ 14^{2.4+2}\ 16^{2}\ 18^{4}$
X_{29}^{126}	$\langle w_7, w_3w_5\rangle \cong {S_2}^2$	$0\ 10^{3.2}\ 12^{2.2+1}\ 14^{4+2+1}\ 16^{4+2+2.1}\ 18^{2}$
X_{29}^{127}	$\langle w_7\rangle \cong S_2$	$0\ 10^{2+3.1}\ 12^{2+4.1}\ 14^{2.2+5.1}\ 16^{2.2+2.1}\ 18\ 20$
X_{29}^{128}	$\langle w_1, w_7\rangle \cong {S_2}^2$	$0\ 10^{2+1}\ 12^{2.2+2.1}\ 14^{4+3.2+2.1}\ 16^{2+1}\ 18^{2+2.1}$
X_{29}^{129}	$\langle w_1, w_5, w_7\rangle \cong {S_2}^3$	$0\ 10^{2}\ 12^{4+2+2.1}\ 14^{2.4+2+1}\ 16^{2.2}\ 18^{2}\ 20$
X_{29}^{130}	$\langle w_5\rangle \cong S_2$	$0\ 8\ 10^{2+2.1}\ 12^{2.2+4.1}\ 14^{2.2+5.1}\ 16^{2+1}\ 18^{2.1}\ 20$
X_{29}^{131}	$\langle w_3, w_5\rangle \cong {S_2}^2$	$0\ 8\ 10\ 12^{4+3.2+2.1}\ 14^{3.2}\ 16^{2+2.1}\ 18^{2+2.1}$
X_{29}^{132}	$\langle w_1, w_6, w_7\rangle \cong S_3 \times S_2$	$0\ 8\ 10^{3+2}\ 12^{6+3+1}\ 14^{3+2}\ 16^{3+2+1}\ 22$

in the E_8 root system (continued)
W_X-orbits on X
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{2465421\\3}\}, \{\substack{2465321\\3}\}, \{\substack{2454321\\2}\}, \{\substack{2354321\\2}\}, \{\substack{2344321\\2}\}, \{\substack{2464321\\3}\}, \{\substack{1354321\\2}\}, \{\substack{2343321\\2}\}, \{\substack{1344321\\2}\}, \{\substack{2343221\\2}\}, \{\substack{1244321\\2}\}, \{\substack{1233321\\1}\}, \{\substack{2343211\\2}\}, \{\substack{1343221\\2}\}, \{\substack{1243321\\2}\}, \{\substack{1243221\\2}\}, \{\substack{1233321\\2}\}, \{\substack{1233221\\2}\}, \{\substack{1222221\\1}\}, \{\substack{1343211\\2}\}, \{\substack{1233211\\2}\}, \{\substack{1232221\\2}\}, \{\substack{1122111\\1}\}, \{\substack{1111111\\0}\}, \{\substack{1232211\\2}\}, \{\substack{0111111\\0}\}, \{\substack{0011111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{234\cdot321\\2}\}, \{\substack{2465421\\3}\}, \{\substack{2454321\\2}\}, \{\substack{246\cdot321\\3}\}, \{\substack{134\cdot321\\2}\}, \{\substack{2354321\\2}\}, \{\substack{2343221\\2}\}, \{\substack{1233321\\2}\}, \{\substack{1233321\\1}\}, \{\substack{1222221\\1}\}, \{\substack{124\cdot321\\2}\}, \{\substack{123\cdot221\\2}\}, \{\substack{2343211\\2}\}, \{\substack{1343221\\2}\}, \{\substack{123\cdot211\\2}\}, \{\substack{1343211\\2}\}, \{\substack{1243221\\2}\}, \{\substack{1111111\\1}\}, \{\substack{1111111\\0}\}, \{\substack{0111111\\0}\}, \{\substack{0011111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{2465421\\3}\}, \{\substack{\cdot34\cdot321\\2}\}, \{\substack{246\cdot321\\3}\}, \{\substack{\cdot354321\\2}\}, \{\substack{124\cdot321\\2}\}, \{\substack{1233321\\2}\}, \{\substack{1233321\\1}\}, \{\substack{\cdot343221\\2}\}, \{\substack{123\cdot221\\2}\}, \{\substack{1243221\\2}\}, \{\substack{1222221\\1}\}, \{\substack{\cdot343211\\2}\}, \{\substack{123\cdot211\\2}\}, \{\substack{\cdot111111\\0}\}, \{\substack{0011111\\1}\}, \{\substack{0011111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{246\cdot\cdot\cdot1\\3}\}, \{\substack{2\cdot54321\\2}\}, \{\substack{234\cdot\cdot\cdot1\\2}\}, \{\substack{1\cdot4\cdot\cdot\cdot1\\2}\}, \{\substack{1233211\\1}, \substack{1232221\\1}\}, \{\substack{1233\cdot21\\2}, \substack{1232\cdot11\\2}\}, \{\substack{1111111\\0}\}, \{\substack{0\cdot11111\\0}\}, \{\substack{1111111\\1}\}$
$\{\substack{2465432\\3}\}, \{\substack{2454321\\2}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{246\cdot321\\3}\}, \{\substack{\cdot34\cdot321\\2}\}, \{\substack{\cdot3432\cdot1\\2}\}, \{\substack{\cdot354321\\2}\}, \{\substack{122\cdot111\\1}\}, \{\substack{1233321\\1}\}, \{\substack{123\cdot2\cdot1\\2}\}, \{\substack{12432\cdot1\\2}\}, \{\substack{\cdot111111\\0}\}, \{\substack{1233321\\2}\}, \{\substack{0011111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{1233321\\1}\}, \{\substack{2\cdot\cdot\cdot321\\2}\}, \{\substack{1111111\\0}\}, \{\substack{1233211\\1}, \substack{12\cdot2221\\1}, \substack{1122211\\1}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{13\cdot4321\\2}, \substack{12\cdot3321\\2}\}, \{\substack{0\cdot\cdot\cdot111\\0}\}, \{\substack{23432\cdot1\\2}\}, \{\substack{1221111\\1}, \substack{1122111\\1}\}, \{\substack{1343211\\2}, \substack{12\cdot3221\\2}, \substack{1232211\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{246\cdot321\\3}\}, \{\substack{2\cdot54321\\2}\}, \{\substack{234\cdot321\\2}\}, \{\substack{23432\cdot1\\2}\}, \{\substack{1354321\\2}\}, \{\substack{1\cdot432\cdot1\\2}\}, \{\substack{1344321\\2}, \substack{1243321\\2}\}, \{\substack{1233321\\1}\}, \{\substack{123\cdot2\cdot1\\2}\}, \{\substack{1221111\\1}, \substack{1122111\\1}\}, \{\substack{1233321\\2}\}, \{\substack{1111111\\0}\}, \{\substack{0\cdot11111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{2465321\\3}\}, \{\substack{2454321\\2}\}, \{\substack{2344321\\2}\}, \{\substack{23432\cdot1\\2}\}, \{\substack{2464321\\3}\}, \{\substack{2354321\\2}\}, \{\substack{2343321\\2}\}, \{\substack{1344321\\2}\}, \{\substack{13432\cdot1\\2}\}, \{\substack{12332\cdot1\\2}\}, \{\substack{1343321\\2}\}, \{\substack{1244321\\2}\}, \{\substack{1233321\\2}\}, \{\substack{1233321\\1}\}, \{\substack{1222111\\1}\}, \{\substack{12432\cdot1\\2}\}, \{\substack{12322\cdot1\\2}\}, \{\substack{1111111\\1}\}, \{\substack{1111111\\0}\}, \{\substack{0111111\\0}\}, \{\substack{0011111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{2465321\\3}\}, \{\substack{\cdot354321\\2}\}, \{\substack{\cdot344321\\2}\}, \{\substack{2464321\\3}\}, \{\substack{1244321\\2}\}, \{\substack{\cdot3432\cdot1\\2}\}, \{\substack{\cdot343321\\2}\}, \{\substack{12432\cdot1\\2}\}, \{\substack{12332\cdot1\\2}\}, \{\substack{1233321\\2}\}, \{\substack{1233321\\1}\}, \{\substack{12322\cdot1\\2}\}, \{\substack{1222111\\1}\}, \{\substack{\cdot111111\\0}\}, \{\substack{0011111\\1}\}, \{\substack{0011111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{\cdot34\cdot321\\2}\}, \{\substack{246\cdot321\\3}\}, \{\substack{2454321\\2}\}, \{\substack{1233321\\2}\}, \{\substack{\cdot3432\cdot1\\2}\}, \{\substack{123\cdot2\cdot1\\2}\}, \{\substack{124\cdot321\\2}\}, \{\substack{1233321\\1}\}, \{\substack{12432\cdot1\\2}\}, \{\substack{\cdot111111\\1}\}, \{\substack{\cdot111111\\0}\}, \{\substack{0011111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{234\cdot321\\2}\}, \{\substack{2465421\\3}\}, \{\substack{2454321\\2}\}, \{\substack{246\cdot321\\3}\}, \{\substack{134\cdot321\\2}\}, \{\substack{2354321\\2}\}, \{\substack{2343221\\2}\}, \{\substack{1233321\\2}\}, \{\substack{1222221\\1}\}, \{\substack{124\cdot321\\2}\}, \{\substack{123\cdot221\\2}\}, \{\substack{1354321\\2}\}, \{\substack{2343211\\2}\}, \{\substack{1343221\\2}\}, \{\substack{1222211\\1}\}, \{\substack{1122221\\1}\}, \{\substack{123\cdot211\\2}\}, \{\substack{1111111\\0}\}, \{\substack{1232111\\2}\}, \{\substack{0111111\\0}\}, \{\substack{0011111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{2465421\\3}\}, \{\substack{1\cdot4\cdot321\\2}\}, \{\substack{246\cdot321\\3}\}, \{\substack{2\cdot54321\\2}\}, \{\substack{234\cdot321\\2}\}, \{\substack{1354321\\2}\}, \{\substack{1233321\\2}\}, \{\substack{1\cdot43221\\2}\}, \{\substack{123\cdot221\\2}\}, \{\substack{1\cdot22221\\1}\}, \{\substack{123\cdot211\\2}\}, \{\substack{2343211\\2}\}, \{\substack{0122211\\1}\}, \{\substack{0\cdot11111\\0}\}, \{\substack{1232111\\2}\}, \{\substack{1111111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2454321\\2}\}, \{\substack{2465\cdot\cdot1\\3}\}, \{\substack{\cdot344321\\2}\}, \{\substack{\cdot343\cdot\cdot1\\2}\}, \{\substack{1222\cdot\cdot1\\1}\}, \{\substack{2464321\\3}\}, \{\substack{1233\cdot\cdot1\\2}\}, \{\substack{\cdot354321\\2}\}, \{\substack{1232\cdot\cdot1\\2}\}, \{\substack{\cdot111111\\0}\}, \{\substack{1244321\\2}\}, \{\substack{0011111\\0}\}$

Table 5: Maximal abelian sets

X	W_X	Sig(X)
X_{29}^{133}	$\langle w_7\rangle \cong S_2$	0 $10^{2+3.1}$ $12^{2+4.1}$ $14^{3.2+3.1}$ $16^{2+4.1}$ 18 20
X_{29}^{134}	$\langle w_1w_7, w_3w_6\rangle \cong S_3$	0 10^{3} $12^{2.3+1}$ 14^{6+3} $16^{2.3}$ 18^{3}
X_{29}^{135}	$\langle w_1, w_2, w_3, w_6\rangle \cong S_3 \times {S_2}^2$	0 8 10^{3} 12^{6+4} 14^{6+2} 16^{3+1} 20^{2}
X_{29}^{136}	$\langle w_1, w_3, w_2w_5\rangle \cong S_3 \times S_2$	0 8 $12^{2.6+2}$ 14^{3+2} 16^{3+2} 18^{2} 20
X_{29}^{137}	$\langle w_5, w_7\rangle \cong {S_2}^2$	0 $10^{2.2+1}$ $12^{4.2+1}$ $14^{4+2+2.1}$ $16^{2+2.1}$ 18 22
X_{29}^{138}	$\langle w_1, w_5, w_7\rangle \cong {S_2}^3$	0 10^{2} 12^{4+2+1} $14^{2.4+3.2}$ 16 18^{2+1} 20
X_{29}^{139}	$\langle w_5, w_3w_7\rangle \cong {S_2}^2$	0 $10^{2.2}$ $12^{3.2+1}$ $14^{2.4+2+1}$ $16^{2+2.1}$ 20^{2}
X_{29}^{140}	$\langle w_5\rangle \cong S_2$	0 $10^{2.1}$ $12^{3.2+2.1}$ $14^{3.2+5.1}$ $16^{4.1}$ $18^{2.1}$ 20
X_{29}^{141}	$\langle w_1, w_2, w_3, w_5\rangle \cong S_3 \times {S_2}^2$	0 10 $12^{2.6+2}$ 14^{4+3} 16^{3} 18^{2} 22
X_{29}^{142}	$\langle w_1, w_3, w_2w_6\rangle \cong S_3 \times S_2$	0 10 12^{6+3} $14^{6+3+2.2}$ 16^{2} 18 20^{2}
X_{29}^{143}	$\langle w_1, w_2, w_3, w_4, w_6, w_7\rangle \cong S_5 \times S_3$	0 12^{10} 14^{15} 20^{3}
X_{29}^{144}	$\langle w_1, w_3, w_5, w_7\rangle \cong S_3 \times {S_2}^2$	0 8^{3} 10^{4+1} 12^{3+2} $14^{2.6}$ 18^{2} 20
X_{29}^{145}	$\langle w_1, w_\tau, w_4w_6\rangle \cong {S_2}^3$	0 8 $10^{2.4+1}$ 12 $14^{2.4+2}$ 16^{4+2} 20
X_{29}^{146}	$\langle w_6, w_3w_5w_7\rangle \cong Dih_8$	0 10^{4+2} 12^{4+1} 14^{4+2+1} $16^{2.4}$ 18^{2}
X_{29}^{147}	$\langle w_1, w_4w_7\rangle \cong {S_2}^2$	0 8 $10^{2.2+1}$ 12^{4+2} $14^{4+2.2+2.1}$ 16^{2} $18^{2.2}$

in the E_8 root system (continued)
W_X-orbits on X
$\{{2465432 \atop 3}\}, \{{24654\cdot 1 \atop 3}\}, \{{2465321 \atop 3}\}, \{{2454321 \atop 2}\}, \{{2344321 \atop 2}\}, \{{23432\cdot 1 \atop 2}\}, \{{2464321 \atop 3}\}, \{{2354321 \atop 2}\}, \{{2343321 \atop 2}\}, \{{1344321 \atop 2}\}, \{{13432\cdot 1 \atop 2}\}, \{{12332\cdot 1 \atop 2}\}, \{{12222\cdot 1 \atop 1}\}, \{{1354321 \atop 2}\}, \{{1244321 \atop 2}\}, \{{1233321 \atop 2}\}, \{{12322\cdot 1 \atop 2}\}, \{{1243321 \atop 2}\}, \{{1232111 \atop 2}\}, \{{1122111 \atop 1}\}, \{{1111111 \atop 0}\}, \{{0111111 \atop 0}\}, \{{0011111 \atop 0}\}$
$\{{2465432 \atop 3}\}, \{{2465\cdot\cdot 1 \atop 3}\}, \{{\cdot\cdot 54321 \atop 2}\}, \{{\cdot\cdot 44321 \atop 2}\}, \{{2464321 \atop 3}\}, \{{\cdot 343321 \atop 2}, {2343221 \atop 2}, {1343211 \atop 2}, {12432\cdot 1 \atop 2}\}, \{{1233\cdot\cdot 1 \atop 2}\}, \{{1232\cdot\cdot 1 \atop 2}\}, \{{1222111 \atop 1}, {1122211 \atop 1}, {0122221 \atop 1}\}, \{{\cdot\cdot 11111 \atop 0}\}$
$\{{2465432 \atop 3}\}, \{{2465431 \atop 3}\}, \{{\cdot\cdot 44321 \atop 2}\}, \{{\cdot\cdot 54321 \atop \cdot}\}, \{{1233\cdot 21 \atop \cdot}\}, \{{\cdot\cdot 43\cdot 21 \atop 2}\}, \{{1233211 \atop \cdot}\}, \{{\cdot\cdot 43211 \atop 2}\}, \{{0001111 \atop 0}\}, \{{0000\cdot 11 \atop 0}\}$
$\{{2465432 \atop 3}\}, \{{2465431 \atop 3}\}, \{{\cdot\cdot 54321 \atop \cdot}\}, \{{\cdot\cdot 4\cdot 321 \atop 2}\}, \{{1233321 \atop \cdot}\}, \{{\cdot\cdot 43221 \atop 2}\}, \{{1233221 \atop 2}, {1232221 \atop 1}\}, \{{\cdot\cdot 43211 \atop 2}\}, \{{1232211 \atop 2}, {1233211 \atop 1}\}, \{{000\cdot 111 \atop 0}\}, \{{0000011 \atop 0}\}$
$\{{2465432 \atop 3}\}, \{{24654\cdot 1 \atop 3}\}, \{{234\cdot 321 \atop 2}\}, \{{2454321 \atop 2}\}, \{{246\cdot 321 \atop 3}\}, \{{134\cdot 321 \atop 2}\}, \{{23432\cdot 1 \atop 2}\}, \{{12222\cdot 1 \atop 1}\}, \{{1233321 \atop 2}\}, \{{123\cdot 2\cdot 1 \atop 2}\}, \{{13432\cdot 1 \atop 2}\}, \{{2354321 \atop 2}\}, \{{1111111 \atop 1}\}, \{{124\cdot 321 \atop 2}\}, \{{1232111 \atop 2}\}, \{{1111111 \atop 0}\}, \{{0111111 \atop 0}\}, \{{0011111 \atop 0}\}$
$\{{2465432 \atop 3}\}, \{{24654\cdot 1 \atop 3}\}, \{{\cdot 34\cdot 321 \atop 2}\}, \{{246\cdot 321 \atop 3}\}, \{{1233321 \atop 2}\}, \{{\cdot 3432\cdot 1 \atop 2}\}, \{{123\cdot 2\cdot 1 \atop 2}\}, \{{\cdot 354321 \atop 2}\}, \{{124\cdot 321 \atop 2}\}, \{{12222\cdot 1 \atop 1}\}, \{{1232111 \atop 2}\}, \{{\cdot 111111 \atop 0}\}, \{{0011111 \atop 1}\}, \{{0011111 \atop 0}\}$
$\{{2465432 \atop 3}\}, \{{24654\cdot 1 \atop 3}\}, \{{234\cdot 321 \atop 2}\}, \{{246\cdot 321 \atop 3}\}, \{{2\cdot 54321 \atop 2}\}, \{{23432\cdot 1 \atop 2}\}, \{{1233321 \atop 2}\}, \{{1\cdot 4\cdot 321 \atop 2}\}, \{{123\cdot 2\cdot 1 \atop 2}\}, \{{1222211 \atop 1}, {1122221 \atop 1}\}, \{{1111111 \atop 1}\}, \{{1343221 \atop 2}, {1243211 \atop 2}\}, \{{1232111 \atop 2}\}, \{{1111111 \atop 0}\}, \{{0\cdot 11111 \atop 0}\}$
$\{{2465432 \atop 3}\}, \{{2465431 \atop 3}\}, \{{2465421 \atop 3}\}, \{{246\cdot 321 \atop 3}\}, \{{234\cdot 321 \atop 2}\}, \{{134\cdot 321 \atop 2}\}, \{{2454321 \atop 2}\}, \{{1233321 \atop 2}\}, \{{124\cdot 321 \atop 2}\}, \{{123\cdot 221 \atop 2}\}, \{{123\cdot 211 \atop 2}\}, \{{1354321 \atop 2}\}, \{{2343221 \atop 2}\}, \{{2343211 \atop 2}\}, \{{1343211 \atop 2}\}, \{{1222221 \atop 1}\}, \{{1243221 \atop 2}\}, \{{1232111 \atop 2}\}, \{{1122211 \atop 1}\}, \{{0111111 \atop 1}\}, \{{1111111 \atop 0}\}, \{{0111111 \atop 0}\}, \{{0011111 \atop 0}\}$
$\{{2465432 \atop 3}\}, \{{2465431 \atop 3}\}, \{{\cdot\cdot 54321 \atop \cdot}\}, \{{\cdot\cdot 4\cdot 321 \atop 2}\}, \{{1233321 \atop \cdot}\}, \{{123\cdot 211 \atop \cdot}\}, \{{\cdot\cdot 43211 \atop 2}\}, \{{\cdot\cdot 43221 \atop 2}\}, \{{000\cdot 111 \atop 0}\}, \{{0000011 \atop 0}\}$
$\{{2465432 \atop 3}\}, \{{2465431 \atop 3}\}, \{{\cdot\cdot 54321 \atop \cdot}\}, \{{\cdot\cdot 44321 \atop 2}\}, \{{\cdot\cdot 43\cdot 21 \atop 2}\}, \{{\cdot\cdot 43211 \atop 2}\}, \{{1233221 \atop 2}, {1233321 \atop 1}\}, \{{1233211 \atop \cdot}\}, \{{1232211 \atop 2}, {1232111 \atop 1}\}, \{{0001111 \atop 0}\}, \{{0000\cdot 11 \atop 0}\}$
$\{{2465432 \atop 3}\}, \{{\cdot\cdot\cdot 4321 \atop \cdot}\}, \{{\cdot\cdot\cdot 3\cdot\cdot 1 \atop \cdot}\}, \{{0000\cdot\cdot 1 \atop 0}\}$
$\{{2465432 \atop 3}\}, \{{\cdot\cdot 54321 \atop 2}\}, \{{123\cdot 2\cdot 1 \atop 1}\}, \{{1232111 \atop 1}\}, \{{\cdot\cdot 11111 \atop 0}\}, \{{24654\cdot 1 \atop 3}\}, \{{\cdot\cdot 4\cdot 321 \atop 2}\}, \{{\cdot\cdot 432\cdot 1 \atop 2}\}, \{{000\cdot 111 \atop 0}\}, \{{1232111 \atop 2}\}$
$\{{2465432 \atop 3}\}, \{{2454321 \atop 2}\}, \{{2465\cdot 21 \atop 3}, {1233\cdot 21 \atop 1}\}, \{{\cdot 3\cdot 4321 \atop 2}\}, \{{2465431 \atop 3}\}, \{{1244321 \atop 2}\}, \{{\cdot 343\cdot 21 \atop 2}\}, \{{\cdot 343211 \atop 2}, {\cdot 111111 \atop 0}\}, \{{1232111 \atop 1}, {1222211 \atop 1}\}, \{{12\cdot 3211 \atop 2}, {00\cdot 1111 \atop 0}\}, \{{1243321 \atop 2}, {1233221 \atop 2}\}, \{{1232221 \atop 2}\}$
$\{{2465432 \atop 3}\}, \{{246\cdot\cdot\cdot 1 \atop 3}\}, \{{2\cdot 54321 \atop 2}\}, \{{234\cdot\cdot\cdot 1 \atop 2}\}, \{{1354321 \atop 2}\}, \{{1344321 \atop 2}, {1243\cdot 21 \atop 2}, {1343211 \atop 2}\}, \{{1233211 \atop 1}, {1232221 \atop 1}\}, \{{1354321 \atop 3}\}, \{{1233\cdot 21 \atop 2}, {1232\cdot 11 \atop 2}\}, \{{1222221 \atop 1}, {1122\cdot 11 \atop 1}, {1221111 \atop 1}\}, \{{0\cdot 11111 \atop 0}\}$
$\{{2465432 \atop 3}\}, \{{2454321 \atop 2}\}, \{{24654\cdot 1 \atop 3}\}, \{{\cdot 343321 \atop 2}\}, \{{1233321 \atop 1}\}, \{{\cdot 3\cdot 4321 \atop 2}\}, \{{1232221 \atop 1}, {1222211 \atop 1}\}, \{{\cdot 3432\cdot 1 \atop 2}\}, \{{12\cdot 3321 \atop 2}\}, \{{\cdot 111111 \atop 0}\}, \{{2465321 \atop 3}\}, \{{1221111 \atop 1}\}, \{{12322\cdot 1 \atop 2}\}, \{{1243211 \atop 2}, {1233221 \atop 2}\}, \{{00\cdot 1111 \atop 0}\}$

Table 5: Maximal abelian sets

X	W_X	Sig(X)
X_{29}^{148}	$\langle w_4w_7\rangle \cong S_2$	0 $10^{2.2+2.1}$ $12^{2+3.1}$ $14^{3.2+1}$ $16^{3.2+2.1}$ 18^2
X_{29}^{149}	$\langle w_1, w_4w_7, w_5w_\tau\rangle \cong S_3 \times S_2$	0 10^3 12^{6+3} 14^3 $16^{2.6}$ 18
X_{29}^{150}	$\langle w_1\rangle \cong S_2$	0 8 $10^{2+2.1}$ $12^{2.2+4.1}$ $14^{3.2+3.1}$ $16^{3.1}$ $18^{2.1}$ 20
X_{29}^{151}	$\langle w_3, w_5w_7\rangle \cong {S_2}^2$	0 10^{2+1} $12^{3.2+1}$ $14^{4+2.2+1}$ 16^{4+2} 18^{2+1}
X_{29}^{152}	$\langle w_1, w_5w_7\rangle \cong {S_2}^2$	0 8 10 $12^{4+4.2}$ 14^{4+2} $16^{2+2.1}$ $18^{2.2}$
X_{29}^{153}	$\langle w_3w_5w_7\rangle \cong S_2$	0 10^{2+1} $12^{3.2+2.1}$ $14^{3.2}$ $16^{4.2+1}$ 18^2
X_{29}^{154}	$\langle w_3w_5w_7\rangle \cong S_2$	0 10^{2+1} $12^{3.2+1}$ $14^{4.2+1}$ $16^{3.2}$ 18^{2+1}
X_{29}^{155}	$\langle w_1\rangle \cong S_2$	0 10 $12^{2.2+4.1}$ $14^{2.2+4.1}$ $16^{2.2+5.1}$ $18^{2.1}$
X_{29}^{156}	$\langle w_1, w_5w_7\rangle \cong {S_2}^2$	0 10 $12^{4+2+2.1}$ $14^{4.2}$ $16^{4+2.2+1}$ 18^2
X_{29}^{157}	$\langle w_1, w_2, w_4, w_1w_6{w_\tau}^{w_3w_4w_5w_4w_3}\rangle \cong S_3 \times Dih_8$	0 8 10 12^{12} 14^6 16^4 18^4
X_{29}^{158}	$\langle w_1, w_2w_5\rangle \cong {S_2}^2$	0 10^{2+1} 12^{4+2+1} $14^{4+2.2+1}$ $16^{3.2}$ 18^{2+1}
X_{29}^{159}	$\langle w_1, w_4, w_7\rangle \cong {S_2}^3$	0 10 12^{4+2+1} $14^{2.4+2+1}$ $16^{3.2}$ 18^{2+1}
X_{29}^{160}	$\langle w_1, w_7\rangle \cong {S_2}^2$	0 $12^{2.2+1}$ $14^{4+3.2+3.1}$ $16^{3.2+2.1}$ $18^{2.1}$

in the E_8 root system (continued)
W_X-orbits on X
$\{\substack{2465432\\3}\}, \{\substack{24654\cdot 1\\3}\}, \{\substack{23\cdot 4321\\2}\}, \{\substack{2454321\\2}\}, \{\substack{1233321\\1}\}, \{\substack{13\cdot 4321\\2}\}, \{\substack{2465321\\3}\}, \{\substack{2343321\\2}\}, \{\substack{1244321\\2}\}, \{\substack{23432\cdot 1\\2}\}, \{\substack{12\cdot 3321\\2}\}, \{\substack{1232221\\1}, \substack{1222211\\1}\}, \{\substack{1111111\\0}\}, \{\substack{13432\cdot 1\\2}\}, \{\substack{1243211\\2}, \substack{1233221\\2}\}, \{\substack{00\cdot 1111\\0}\}, \{\substack{1122111\\1}\}, \{\substack{0111111\\0}\}, \{\substack{12322\cdot 1\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{24654\cdot 1\\3}, \substack{1233321\\1}\}, \{\substack{\cdot 3\cdot\cdot 321\\2}\}, \{\substack{12\cdot\cdot 321\\2}\}, \{\substack{2465321\\3}, \substack{1232221\\1}, \substack{1222211\\1}\}, \{\substack{\cdot 3432\cdot 1\\2}, \substack{\cdot 111111\\0}\}, \{\substack{1243211\\2}, \substack{123\cdot 221\\2}, \substack{1232211\\2}, \substack{00\cdot 1111\\0}\}, \{\substack{0011111\\1}\}$
$\{\substack{2465432\\3}\}, \{\substack{2454321\\2}\}, \{\substack{\cdot 354321\\2}\}, \{\substack{2465431\\3}\}, \{\substack{2465321\\3}\}, \{\substack{\cdot 344321\\2}\}, \{\substack{\cdot 343221\\2}\}, \{\substack{2465421\\3}\}, \{\substack{1233321\\1}\}, \{\substack{1232221\\1}\}, \{\substack{1232111\\1}\}, \{\substack{\cdot 343321\\2}\}, \{\substack{\cdot 343211\\2}\}, \{\substack{\cdot 111111\\0}\}, \{\substack{1244321\\2}\}, \{\substack{1243221\\2}\}, \{\substack{1222111\\1}\}, \{\substack{1243211\\2}\}, \{\substack{1233221\\2}\}, \{\substack{0011111\\0}\}, \{\substack{1232221\\2}\}, \{\substack{0001111\\0}\}, \{\substack{1232211\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{2\cdot 54321\\2}\}, \{\substack{1233321\\1}\}, \{\substack{24654\cdot 1\\3}\}, \{\substack{234\cdot 321\\2}\}, \{\substack{1233211\\1}, \substack{1232221\\1}\}, \{\substack{1111111\\0}\}, \{\substack{1\cdot 4\cdot 321\\2}\}, \{\substack{23432\cdot 1\\2}\}, \{\substack{0\cdot 11111\\0}\}, \{\substack{1232111\\1}\}, \{\substack{1\cdot 432\cdot 1\\2}\}, \{\substack{000\cdot 111\\0}\}, \{\substack{1233221\\2}, \substack{1232211\\2}\}, \{\substack{1111111\\1}\}$
$\{\substack{2465432\\3}\}, \{\substack{2454321\\2}\}, \{\substack{1233321\\1}\}, \{\substack{\cdot 34\cdot 321\\2}\}, \{\substack{24654\cdot 1\\3}\}, \{\substack{\cdot 354321\\2}\}, \{\substack{1233221\\1}, \substack{1232211\\1}\}, \{\substack{\cdot 111111\\0}\}, \{\substack{\cdot 3432\cdot 1\\2}\}, \{\substack{122\cdot 111\\1}\}, \{\substack{000\cdot 111\\0}\}, \{\substack{1233321\\2}\}, \{\substack{0011111\\0}\}, \{\substack{12432\cdot 1\\2}\}, \{\substack{1233211\\2}, \substack{1232221\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{2\cdot 54321\\2}\}, \{\substack{1233321\\1}\}, \{\substack{24654\cdot 1\\3}\}, \{\substack{234\cdot 321\\2}\}, \{\substack{1233221\\1}, \substack{1232211\\1}\}, \{\substack{1354321\\2}\}, \{\substack{1111111\\0}\}, \{\substack{1344321\\2}, \substack{1243321\\2}\}, \{\substack{23432\cdot 1\\2}\}, \{\substack{0\cdot 11111\\0}\}, \{\substack{1343221\\2}, \substack{1243211\\2}\}, \{\substack{1343211\\2}, \substack{1243221\\2}\}, \{\substack{1221111\\1}, \substack{1122111\\1}\}, \{\substack{000\cdot 111\\0}\}, \{\substack{1233321\\2}\}, \{\substack{1233211\\2}, \substack{1232221\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{2\cdot 54321\\2}\}, \{\substack{1233321\\1}\}, \{\substack{24654\cdot 1\\3}\}, \{\substack{234\cdot 321\\2}\}, \{\substack{1233221\\1}, \substack{1232211\\1}\}, \{\substack{1354321\\2}\}, \{\substack{1344321\\2}, \substack{1243321\\2}\}, \{\substack{23432\cdot 1\\2}\}, \{\substack{1222221\\1}, \substack{1122211\\1}\}, \{\substack{0\cdot 11111\\0}\}, \{\substack{1232111\\1}\}, \{\substack{1343211\\2}, \substack{1243221\\2}\}, \{\substack{1221111\\1}, \substack{1122111\\1}\}, \{\substack{000\cdot 111\\0}\}, \{\substack{1233211\\2}, \substack{1232221\\2}\}, \{\substack{1354321\\3}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{\cdot 354321\\2}\}, \{\substack{\cdot 344321\\2}\}, \{\substack{2465421\\3}\}, \{\substack{2465321\\3}\}, \{\substack{1244321\\2}\}, \{\substack{1233321\\1}\}, \{\substack{\cdot 343321\\2}\}, \{\substack{\cdot 343221\\2}\}, \{\substack{1243221\\2}\}, \{\substack{1233321\\2}\}, \{\substack{1233221\\2}\}, \{\substack{1232221\\1}\}, \{\substack{\cdot 343211\\2}\}, \{\substack{\cdot 111111\\0}\}, \{\substack{1243211\\2}\}, \{\substack{1232221\\2}\}, \{\substack{1222111\\1}\}, \{\substack{0011111\\0}\}, \{\substack{0001111\\0}\}, \{\substack{1232211\\2}\}, \{\substack{0011111\\1}\}$
$\{\substack{2465432\\3}\}, \{\substack{1233321\\1}\}, \{\substack{\cdot 34\cdot 321\\2}\}, \{\substack{24654\cdot 1\\3}\}, \{\substack{2454321\\2}\}, \{\substack{1233321\\2}\}, \{\substack{124\cdot 321\\2}\}, \{\substack{1233211\\1}, \substack{1232221\\1}\}, \{\substack{\cdot 111111\\0}\}, \{\substack{000\cdot 111\\0}\}, \{\substack{\cdot 3432\cdot 1\\2}\}, \{\substack{1233221\\2}, \substack{1232211\\2}\}, \{\substack{\cdot 111111\\1}\}, \{\substack{0011111\\0}\}, \{\substack{12432\cdot 1\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{1244321\\2}\}, \{\substack{2465431\\3}\}, \{\substack{\cdot 3\cdot 4321\\\cdot}, \substack{12\cdot 3\cdot 21\\\cdot}\}, \{\substack{12\cdot 3211\\\cdot}, \substack{00\cdot 1111\\\cdot}\}, \{\substack{\cdot 343\cdot 21\\2}\}, \{\substack{\cdot 343211\\2}, \substack{0000\cdot 11\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{124\cdot 321\\2}\}, \{\substack{2465431\\3}\}, \{\substack{\cdot 354321\\\cdot}\}, \{\substack{1233321\\\cdot}\}, \{\substack{1243221\\2}\}, \{\substack{\cdot 34\cdot 321\\2}\}, \{\substack{1233221\\2}, \substack{1232221\\1}\}, \{\substack{0011111\\\cdot}\}, \{\substack{1243211\\2}\}, \{\substack{\cdot 343221\\2}\}, \{\substack{1232211\\2}, \substack{1233211\\1}\}, \{\substack{000\cdot 111\\0}\}, \{\substack{\cdot 343211\\2}\}, \{\substack{0000011\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465321\\3}\}, \{\substack{\cdot 3\cdot 4321\\2}\}, \{\substack{24654\cdot 1\\3}\}, \{\substack{1244321\\2}\}, \{\substack{\cdot 3432\cdot 1\\2}\}, \{\substack{12\cdot 32\cdot 1\\2}\}, \{\substack{12\cdot 2111\\1}\}, \{\substack{1233321\\1}\}, \{\substack{\cdot 343321\\2}\}, \{\substack{\cdot 111111\\0}\}, \{\substack{00\cdot 1111\\0}\}, \{\substack{12322\cdot 1\\2}\}, \{\substack{0011111\\1}\}$
$\{\substack{2465432\\3}\}, \{\substack{24654\cdot 1\\3}\}, \{\substack{\cdot 354321\\2}\}, \{\substack{2465321\\3}\}, \{\substack{\cdot 3432\cdot 1\\2}\}, \{\substack{\cdot 344321\\2}\}, \{\substack{\cdot 343321\\2}\}, \{\substack{12432\cdot 1\\2}\}, \{\substack{1243321\\2}\}, \{\substack{1233321\\1}\}, \{\substack{1232111\\1}\}, \{\substack{12332\cdot 1\\2}\}, \{\substack{12322\cdot 1\\2}\}, \{\substack{\cdot 111111\\0}\}, \{\substack{1221111\\1}\}, \{\substack{0011111\\0}\}, \{\substack{0011111\\1}\}, \{\substack{0001111\\0}\}$

Table 5: Maximal abelian sets

X	W_X	Sig(X)
X_{29}^{161}	$\langle w_4, w_3w_5w_7\rangle \cong Dih_8$	0 12^{4+2} $14^{2.4+2}$ $16^{2.4+2+1}$ 18
X_{29}^{162}	$\langle w_1, w_5, w_7\rangle \cong {S_2}^3$	0 8 $10^{2.2}$ $12^{2.4}$ $14^{4+2.2+1}$ 16^{2+1} 18^2 20
X_{29}^{163}	$\langle w_3w_7\rangle \cong S_2$	0 $10^{2.2+1}$ $12^{2.2+2.1}$ $14^{3.2+3.1}$ $16^{3.2}$ 18 20
X_{29}^{164}	$\langle w_4, w_1w_6, w_1w_6{w_\tau}^{w_3w_4w_5w_4w_3}\rangle \cong {S_2}^3$	0 8 10 12^{8+4} 14^{4+2} $16^{2.2}$ 18^4
X_{29}^{165}	1	0 $10^{3.1}$ $12^{7.1}$ $14^{9.1}$ $16^{6.1}$ $18^{3.1}$
X_{29}^{166}	$\langle w_1, w_4w_6w_\tau\rangle \cong {S_2}^2$	0 10 12^{4+2+1} $14^{4+3.2+1}$ 16^{4+2} 18^{2+1}
X_{29}^{167}	$\langle w_3, w_1w_4w_7\rangle \cong Dih_8$	0 10^2 $12^{2.4+2}$ 14^{4+1} $16^{2.4+2}$ 20
X_{29}^{168}	$\langle w_3w_5w_7\rangle \cong S_2$	0 10^{2+1} $12^{2.2+2.1}$ $14^{6.2}$ 16^{2+1} $18^{2.2}$
X_{29}^{169}	$\langle w_2w_3w_5, w_1w_4w_6{w_\tau}^{w_3w_4w_5w_4w_3w_1}\rangle \cong {S_2}^2$	0 10 $12^{4+2.2}$ $14^{2.4}$ $16^{2.4+1}$ 18^2
X_{29}^{170}	$\langle w_1, w_4w_6\rangle \cong {S_2}^2$	0 10 $12^{4+2+2.1}$ $14^{4+2.2}$ $16^{4.2+1}$ 18^2
X_{29}^{171}	$\langle w_6\rangle \cong S_2$	0 10 $12^{2.2+3.1}$ $14^{3.2+5.1}$ $16^{2+4.1}$ $18^{3.1}$
X_{29}^{172}	$\langle w_1, w_4, w_5, w_7\rangle \cong S_3 \times {S_2}^2$	0 10^2 12^{6+3} 14^{6+2} 16^{4+3} 18 20
X_{29}^{173}	$\langle w_1, w_6, w_4w_\tau\rangle \cong {S_2}^3$	0 8 $12^{3.4+2}$ 14^{4+1} 16^{4+1} 18^2 20

in the E_8 root system (continued)
W_X-orbits on X
$\{{2465432 \atop 3}\}, \{{2\cdot\cdot\cdot321 \atop 2}\}, \{{23432\cdot1 \atop 2}\}, \{{1233221 \atop 1}, {12\cdot2211 \atop 1}, {1122221 \atop 1}\}, \{{12\cdot2111 \atop 1}, {11\cdot1111 \atop 1}\}, \{{24654\cdot1 \atop 3}\}, \{{1343221 \atop 2}, {12\cdot3211 \atop 2}, {1232221 \atop 2}\}, \{{0\cdot\cdot\cdot111 \atop 0}\}, \{{1343321 \atop 2}, {1244321 \atop 2}\}, \{{1232111 \atop 2}\}, \{{1354321 \atop 3}\}$
$\{{2465432 \atop 3}\}, \{{2454321 \atop 2}\}, \{{24654\cdot1 \atop 3}\}, \{{\cdot354321 \atop 2}\}, \{{\cdot34\cdot321 \atop 2}\}, \{{123\cdot2\cdot1 \atop 1}\}, \{{\cdot3432\cdot1 \atop 2}\}, \{{124\cdot321 \atop 2}\}, \{{\cdot111111 \atop 0}\}, \{{2454321 \atop 3}\}, \{{12432\cdot1 \atop 2}\}, \{{1233321 \atop 2}\}, \{{000\cdot111 \atop 0}\}, \{{1232111 \atop 2}\}$
$\{{2465432 \atop 3}\}, \{{24654\cdot1 \atop 3}\}, \{{2\cdot54321 \atop 2}\}, \{{2344321 \atop 2}\}, \{{1\cdot44321 \atop 2}\}, \{{12332\cdot1 \atop 1}\}, \{{1354321 \atop 2}\}, \{{2343321 \atop 2}\}, \{{23432\cdot1 \atop 2}\}, \{{1\cdot43321 \atop 2}\}, \{{1222211 \atop 1}, {1122221 \atop 1}\}, \{{2464321 \atop 3}\}, \{{1233321 \atop 2}\}, \{{1111111 \atop 0}\}, \{{1343221 \atop 2}, {1243211 \atop 2}\}, \{{12332\cdot1 \atop 2}\}, \{{0\cdot11111 \atop 0}\}, \{{0000111 \atop 0}\}, \{{1232111 \atop 2}\}$
$\{{2465432 \atop 3}\}, \{{1244321 \atop 2}\}, \{{2465431 \atop 3}\}, \{{\cdot3\cdot4321 \atop 2}, {12\cdot3\cdot21 \atop 2}\}, \{{\cdot354321 \atop 3}, {1233\cdot21 \atop 1}\}, \{{12\cdot3211 \atop 2}, {00\cdot1111 \atop 0}\}, \{{1122211 \atop 1}, {0122111 \atop 1}\}, \{{2454321 \atop 2}, {1232221 \atop 2}\}, \{{2343221 \atop 2}, {1343321 \atop 2}\}, \{{\cdot343211 \atop 2}, {0000\cdot11 \atop 0}\}$
$\{{2465432 \atop 3}\}, \{{2465421 \atop 3}\}, \{{2454321 \atop 2}\}, \{{2354321 \atop 2}\}, \{{2465431 \atop 3}\}, \{{2465321 \atop 3}\}, \{{2344321 \atop 2}\}, \{{1354321 \atop 2}\}, \{{2343321 \atop 2}\}, \{{2343211 \atop 2}\}, \{{1232211 \atop 1}\}, \{{1344321 \atop 2}\}, \{{2343221 \atop 2}\}, \{{1343321 \atop 2}\}, \{{1244321 \atop 2}\}, \{{1343211 \atop 2}\}, \{{1243211 \atop 2}\}, \{{1232221 \atop 1}\}, \{{1122211 \atop 1}\}, \{{1111111 \atop 0}\}, \{{1243221 \atop 2}\}, \{{1233321 \atop 2}\}, \{{1232211 \atop 2}\}, \{{1222111 \atop 1}\}, \{{0111111 \atop 0}\}, \{{0011111 \atop 0}\}, \{{1232221 \atop 2}\}, \{{1232111 \atop 2}\}, \{{0001111 \atop 0}\}$
$\{{2465432 \atop 3}\}, \{{2465431 \atop 3}\}, \{{\cdot3\cdot4321 \atop 2}\}, \{{2465321 \atop 3}, {1233321 \atop 1}\}, \{{1244321 \atop 2}\}, \{{\cdot343\cdot21 \atop 2}\}, \{{2454321 \atop 3}, {1232221 \atop 1}\}, \{{1243321 \atop 2}, {1233221 \atop 2}\}, \{{1243221 \atop 2}, {1233321 \atop 2}\}, \{{1233211 \atop 1}\}, \{{\cdot343211 \atop 2}, {\cdot111111 \atop 0}\}, \{{1243211 \atop 2}, {0001111 \atop 0}\}, \{{1232211 \atop 2}, {0000011 \atop 0}\}, \{{0011111 \atop 1}\}$
$\{{2465432 \atop 3}\}, \{{24654\cdot1 \atop 3}\}, \{{2\cdot54321 \atop 2}, {1\cdot44321 \atop 2}\}, \{{\cdot\cdot\cdot3321 \atop 2}\}, \{{2344321 \atop 2}, {1354321 \atop 2}\}, \{{1232221 \atop 1}, {1\cdot22211 \atop 1}, {0122221 \atop 1}\}, \{{2465321 \atop 3}\}, \{{2343221 \atop 2}, {1\cdot43211 \atop 2}, {1233221 \atop 2}\}, \{{\cdot\cdot\cdot1111 \atop 0}\}, \{{12322\cdot1 \atop 2}\}, \{{1232111 \atop 2}\}$
$\{{2465432 \atop 3}\}, \{{2\cdot54321 \atop 2}\}, \{{1232111 \atop 1}\}, \{{23432\cdot1 \atop 2}\}, \{{1233221 \atop 1}, {1232211 \atop 1}\}, \{{1354321 \atop 2}\}, \{{1111111 \atop 0}\}, \{{24654\cdot1 \atop 3}\}, \{{234\cdot321 \atop 2}\}, \{{1343221 \atop 2}, {1243211 \atop 2}\}, \{{1343211 \atop 2}, {1243221 \atop 2}\}, \{{1221111 \atop 1}, {1122111 \atop 1}\}, \{{0\cdot11111 \atop 0}\}, \{{1344321 \atop 2}, {1243321 \atop 2}\}, \{{1232111 \atop 2}\}, \{{1233211 \atop 2}, {1232221 \atop 2}\}, \{{000\cdot111 \atop 0}\}$
$\{{2465432 \atop 3}\}, \{{2465431 \atop 3}\}, \{{1354321 \atop \cdot}, {1233321 \atop \cdot}\}, \{{1344321 \atop 2}, {1243321 \atop 2}\}, \{{1343321 \atop 2}, {1244321 \atop 2}\}, \{{2454321 \atop 3}, {2354321 \atop 2}, {1233221 \atop 2}, {1232221 \atop 1}\}, \{{234\cdot321 \atop 2}, {1\cdot43221 \atop 2}\}, \{{1\cdot43211 \atop 2}, {000\cdot111 \atop 0}\}, \{{1232211 \atop 2}, {1233211 \atop 1}, {0111111 \atop 0}, {0011111 \atop 1}\}, \{{2343221 \atop 2}\}, \{{2343211 \atop 2}, {0000011 \atop 0}\}$
$\{{2465432 \atop 3}\}, \{{2465431 \atop 3}\}, \{{\cdot3\cdot4321 \atop 2}\}, \{{2465\cdot21 \atop 3}\}, \{{2454321 \atop 2}\}, \{{1244321 \atop 2}\}, \{{\cdot343\cdot21 \atop 2}\}, \{{1243321 \atop 2}, {1233221 \atop 2}\}, \{{\cdot122221 \atop 1}\}, \{{12\cdot3211 \atop 2}\}, \{{1232111 \atop 1}, {1222211 \atop 1}\}, \{{\cdot111111 \atop 0}\}, \{{00\cdot1111 \atop 0}\}, \{{1232221 \atop 2}\}, \{{1232\cdot11 \atop 2}\}$
$\{{2465432 \atop 3}\}, \{{2465431 \atop 3}\}, \{{2343\cdot21 \atop 2}\}, \{{1233\cdot21 \atop 2}\}, \{{2344321 \atop 2}\}, \{{1232221 \atop 2}\}, \{{1222221 \atop 1}\}, \{{2465\cdot21 \atop 3}\}, \{{1343\cdot21 \atop 2}\}, \{{1243\cdot21 \atop 2}\}, \{{2354321 \atop 2}\}, \{{1344321 \atop 2}\}, \{{1244321 \atop 2}\}, \{{1232221 \atop 1}\}, \{{1111111 \atop 1}\}, \{{1232\cdot11 \atop 2}\}, \{{2343211 \atop 2}\}, \{{1111111 \atop 0}\}, \{{0011111 \atop 1}\}, \{{0001111 \atop 0}\}, \{{1343211 \atop 2}\}, \{{0111111 \atop 0}\}, \{{0011111 \atop 0}\}$
$\{{2465432 \atop 3}\}, \{{\cdot111111 \atop 0}\}, \{{12\cdot\cdot2\cdot1 \atop 1}\}, \{{00\cdot\cdot111 \atop 0}\}, \{{\cdot3\cdot\cdot321 \atop 2}\}, \{{00000\cdot1 \atop 0}\}, \{{\cdot3432\cdot1 \atop 2}\}, \{{12\cdot\cdot321 \atop 2}\}, \{{0011111 \atop 1}\}, \{{1232111 \atop 2}\}$
$\{{2465432 \atop 3}\}, \{{2454321 \atop 2}\}, \{{2465\cdot21 \atop 3}, {1233\cdot21 \atop 1}\}, \{{\cdot3\cdot4321 \atop 2}\}, \{{\cdot343\cdot21 \atop 2}\}, \{{2464321 \atop 3}, {1222221 \atop 1}\}, \{{\cdot343211 \atop 2}, {\cdot111111 \atop 0}\}, \{{1221111 \atop 1}\}, \{{12\cdot3\cdot21 \atop 2}\}, \{{1232221 \atop 2}\}, \{{1233211 \atop 2}, {0011111 \atop 0}\}, \{{0000001 \atop 0}\}$

Table 5: Maximal abelian sets

X	W_X	Sig(X)
X_{29}^{174}	$\langle w_7 \rangle \cong S_2$	0 $10^{2.1}$ $12^{3.2+3.1}$ $14^{2.2+4.1}$ $16^{2+5.1}$ 18 20
X_{29}^{175}	$\langle w_6, w_3w_5w_7 \rangle \cong Dih_8$	0 12^{4+1} $14^{3.4+1}$ $16^{4+2.2}$ 18^2
X_{29}^{176}	$\langle w_1, w_6, w_4w_\tau \rangle \cong {S_2}^3$	0 $12^{2.4+1}$ $14^{2.4+2}$ 16^{4+2+1} 18 20
X_{29}^{177}	$\langle w_4w_7 \rangle \cong S_2$	0 10 $12^{3.2+2.1}$ $14^{3.2+2.1}$ $16^{3.2+3.1}$ 18^2
X_{29}^{178}	$\langle w_4w_7 \rangle \cong S_2$	0 $12^{2.2+2.1}$ $14^{4.2+2.1}$ $16^{4.2+3.1}$ 18
X_{29}^{179}	$\langle w_3w_5w_7 \rangle \cong S_2$	0 10^2 $12^{3.2+2.1}$ $14^{5.2+1}$ $16^{2.2}$ 18^2 20
X_{29}^{180}	$\langle w_1, w_4w_7 \rangle \cong {S_2}^2$	0 $12^{2.2}$ $14^{2.4+3.2+2.1}$ $16^{2.2+1}$ 18^{2+1}
X_{29}^{181}	$\langle w_1, w_2w_5 \rangle \cong {S_2}^2$	0 $12^{4+2.2+1}$ $14^{4+3.2}$ $16^{3.2+1}$ 18 20
X_{29}^{182}	$\langle w_4w_7 \rangle \cong S_2$	0 10 $12^{3.2+1}$ $14^{4.2+3.1}$ $16^{2.2+2.1}$ 18^{2+1}
X_{29}^{183}	$\langle w_1w_4w_6 \rangle \cong S_2$	0 $12^{2.2+1}$ $14^{6.2+1}$ $16^{3.2+2.1}$ 18^2
X_{29}^{184}	$\langle w_1w_2w_6 \rangle \cong S_2$	0 $12^{2.2+1}$ $14^{6.2+1}$ $16^{3.2+2.1}$ 18^2
X_{29}^{185}	$\langle w_1, w_2w_6 \rangle \cong {S_2}^2$	0 $12^{2.2}$ $14^{2.4+3.2+2.1}$ $16^{2.2+1}$ 18^{2+1}
X_{29}^{186}	$\langle w_2, w_3, w_1w_4w_6{w_\tau}^{w_3w_5w_4w_3w_1} \rangle \cong S_2 \times Dih_8$	0 $12^{2.4}$ 14^{8+4+1} 16^4 18^2 20
X_{29}^{187}	$\langle w_2, w_6, w_7, w_2{w_3}^{w_4} \rangle \cong S_3 \times Dih_8$	0 12^4 14^{12+4} 16^{3+2} 18^3

in the E_8 root system (continued)
W_X-orbits on X
$\{\substack{2465432\\3}\}, \{\substack{2454321\\2}\}, \{\substack{2354321\\2}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{23432\cdot1\\2}\}, \{\substack{12322\cdot1\\1}\}, \{\substack{2465321\\3}\}, \{\substack{1354321\\2}\}, \{\substack{2343321\\2}\}, \{\substack{13432\cdot1\\2}\}, \{\substack{12432\cdot1\\2}\}, \{\substack{2344321\\2}\}, \{\substack{1243321\\2}\}, \{\substack{1221111\\1}\}, \{\substack{1111111\\0}\}, \{\substack{12322\cdot1\\2}\}, \{\substack{1344321\\2}\}, \{\substack{1232111\\2}\}, \{\substack{1122111\\1}\}, \{\substack{0111111\\0}\}, \{\substack{0011111\\0}\}, \{\substack{1233321\\2}\}, \{\substack{0001111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{234\cdot\cdot\cdot1\\2}\}, \{\substack{1111111\\1}\}, \{\substack{246\cdot\cdot\cdot1\\3}\}, \{\substack{1233\cdot21\\2}, \substack{1232\cdot11\\2}\}, \{\substack{1222221\\1}, \substack{1122\cdot11\\1}, \substack{1221111\\1}\}, \{\substack{1111111\\0}\}, \{\substack{1344321\\2}, \substack{1243\cdot21\\2}, \substack{1343211\\2}\}, \{\substack{1233211\\1}, \substack{1232221\\1}\}, \{\substack{0\cdot11111\\1}\}, \{\substack{0\cdot11111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465\cdot21\\3}, \substack{1233\cdot21\\1}\}, \{\substack{\cdot3\cdot4321\\2}\}, \{\substack{1244321\\2}\}, \{\substack{\cdot343\cdot21\\2}\}, \{\substack{12\cdot3\cdot21\\2}\}, \{\substack{2454321\\3}, \substack{1232221\\1}\}, \{\substack{\cdot343211\\2}, \substack{\cdot111111\\0}\}, \{\substack{1243211\\2}, \substack{0001111\\0}\}, \{\substack{1232221\\2}\}, \{\substack{0011111\\1}\}, \{\substack{0000001\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2454321\\2}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{23\cdot4321\\2}\}, \{\substack{23432\cdot1\\2}\}, \{\substack{2465321\\3}\}, \{\substack{2343321\\2}\}, \{\substack{13\cdot4321\\2}\}, \{\substack{13432\cdot1\\2}\}, \{\substack{1232211\\1}, \substack{1222221\\1}\}, \{\substack{1221111\\1}\}, \{\substack{1111111\\0}\}, \{\substack{12\cdot3321\\2}\}, \{\substack{1243221\\2}, \substack{1233211\\2}\}, \{\substack{12322\cdot1\\2}\}, \{\substack{1232111\\2}\}, \{\substack{1122111\\1}\}, \{\substack{0111111\\0}\}, \{\substack{00\cdot1111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{23\cdot4321\\2}\}, \{\substack{2465321\\3}\}, \{\substack{1244321\\2}\}, \{\substack{13\cdot4321\\2}\}, \{\substack{23432\cdot1\\2}\}, \{\substack{12\cdot3321\\2}\}, \{\substack{1243221\\2}, \substack{1233211\\2}\}, \{\substack{2343321\\2}\}, \{\substack{1122111\\1}\}, \{\substack{13432\cdot1\\2}\}, \{\substack{12322\cdot1\\2}\}, \{\substack{1232211\\1}, \substack{1222221\\1}\}, \{\substack{00\cdot1111\\0}\}, \{\substack{1232111\\2}\}, \{\substack{1111111\\0}\}, \{\substack{0011111\\1}\}, \{\substack{0111111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2\cdot54321\\2}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{234\cdot321\\2}\}, \{\substack{1233221\\1}, \substack{1232211\\1}\}, \{\substack{1354321\\2}\}, \{\substack{1111111\\0}\}, \{\substack{1344321\\2}, \substack{1243321\\2}\}, \{\substack{23432\cdot1\\2}\}, \{\substack{1343321\\2}, \substack{1244321\\2}\}, \{\substack{1222211\\1}, \substack{1122221\\1}\}, \{\substack{0\cdot11111\\0}\}, \{\substack{1232111\\1}\}, \{\substack{1343221\\2}, \substack{1243211\\2}\}, \{\substack{000\cdot111\\0}\}, \{\substack{1233211\\2}, \substack{1232221\\2}\}, \{\substack{1232111\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{\cdot343321\\2}\}, \{\substack{\cdot3\cdot4321\\2}\}, \{\substack{\cdot3432\cdot1\\2}\}, \{\substack{12\cdot3321\\2}\}, \{\substack{12322\cdot1\\2}\}, \{\substack{1232211\\1}, \substack{1222221\\1}\}, \{\substack{2465321\\3}\}, \{\substack{1221111\\1}\}, \{\substack{1243221\\2}, \substack{1233211\\2}\}, \{\substack{\cdot111111\\0}\}, \{\substack{1232111\\2}\}, \{\substack{00\cdot1111\\0}\}, \{\substack{0011111\\1}\}$
$\{\substack{2465432\\3}\}, \{\substack{\cdot34\cdot321\\2}\}, \{\substack{2454321\\\cdot}\}, \{\substack{1233321\\\cdot}\}, \{\substack{1222221\\1}\}, \{\substack{\cdot354321\\\cdot}\}, \{\substack{\cdot343221\\2}\}, \{\substack{124\cdot321\\2}\}, \{\substack{1233221\\2}, \substack{1232221\\1}\}, \{\substack{\cdot343211\\2}\}, \{\substack{1232211\\2}, \substack{1233211\\1}\}, \{\substack{000\cdot111\\0}\}, \{\substack{1243221\\2}\}, \{\substack{0000011\\0}\}, \{\substack{0000001\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2343321\\2}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{23\cdot4321\\2}\}, \{\substack{23432\cdot1\\2}\}, \{\substack{2454321\\2}\}, \{\substack{12\cdot3321\\2}\}, \{\substack{12322\cdot1\\2}\}, \{\substack{1232221\\1}, \substack{1222211\\1}\}, \{\substack{11\cdot1111\\1}\}, \{\substack{2465321\\3}\}, \{\substack{1343321\\2}\}, \{\substack{1111111\\0}\}, \{\substack{13432\cdot1\\2}\}, \{\substack{1243211\\2}, \substack{1233221\\2}\}, \{\substack{1244321\\2}\}, \{\substack{1232111\\2}\}, \{\substack{00\cdot1111\\0}\}, \{\substack{0111111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465\cdot21\\3}\}, \{\substack{\cdot343211\\2}\}, \{\substack{2454321\\2}\}, \{\substack{2354321\\2}, \substack{1344321\\2}\}, \{\substack{2343321\\2}, \substack{1343221\\2}\}, \{\substack{2343221\\2}, \substack{1343321\\2}\}, \{\substack{12\cdot3211\\2}\}, \{\substack{1232\cdot11\\2}\}, \{\substack{1232111\\1}, \substack{1222211\\1}\}, \{\substack{2465431\\3}\}, \{\substack{1243321\\2}, \substack{1233221\\2}\}, \{\substack{1111111\\1}, \substack{0121111\\1}\}, \{\substack{\cdot111111\\0}\}, \{\substack{1244321\\2}\}, \{\substack{1232221\\2}\}, \{\substack{00\cdot1111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2454321\\\cdot}\}, \{\substack{\cdot344321\\2}\}, \{\substack{2465431\\3}\}, \{\substack{2354321\\2}, \substack{1354321\\3}\}, \{\substack{2343321\\2}, \substack{1343221\\2}\}, \{\substack{2343221\\2}, \substack{1343321\\2}\}, \{\substack{\cdot343211\\2}\}, \{\substack{1233221\\2}, \substack{1233321\\1}\}, \{\substack{1233211\\\cdot}\}, \{\substack{1244321\\2}\}, \{\substack{1243\cdot21\\2}\}, \{\substack{1232211\\2}, \substack{1232111\\1}\}, \{\substack{1111111\\1}, \substack{0111111\\0}\}, \{\substack{1243211\\2}\}, \{\substack{0001111\\0}\}, \{\substack{0000\cdot11\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2454321\\\cdot}\}, \{\substack{\cdot344321\\2}\}, \{\substack{\cdot354321\\\cdot}\}, \{\substack{\cdot343\cdot21\\2}\}, \{\substack{\cdot343211\\2}\}, \{\substack{1233221\\2}, \substack{1233321\\1}\}, \{\substack{1233211\\\cdot}\}, \{\substack{1244321\\2}\}, \{\substack{1222221\\1}\}, \{\substack{1243\cdot21\\2}\}, \{\substack{1232211\\2}, \substack{1232111\\1}\}, \{\substack{0001111\\0}\}, \{\substack{0000\cdot11\\0}\}, \{\substack{0000001\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{1354321\\\cdot}, \substack{1233321\\\cdot}\}, \{\substack{1\cdot4\cdot321\\2}\}, \{\substack{123\cdot211\\\cdot}, \substack{0\cdot11111\\\cdot}\}, \{\substack{1\cdot43211\\2}, \substack{000\cdot111\\0}\}, \{\substack{2465431\\3}\}, \{\substack{234\cdot321\\2}, \substack{1\cdot43221\\2}\}, \{\substack{2343211\\2}, \substack{0000011\\0}\}, \{\substack{2343221\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{1\cdot\cdot4321\\\cdot}\}, \{\substack{1\cdot\cdot3\cdot\cdot1\\\cdot}\}, \{\substack{2\cdot54321\\\cdot}\}, \{\substack{2343\cdot\cdot1\\2}\}, \{\substack{0121111\\1}, \substack{0001111\\0}\}, \{\substack{0000\cdot\cdot1\\0}\}$

Table 5: Maximal abelian sets

X	W_X	Sig(X)
X_{29}^{188}	$\langle w_\tau, w_3w_5, w_1w_4w_6\rangle \cong S_2 \times Dih_{12}$	0 14^{12+6} 16^{6+3} 18
X_{29}^{189}	$\langle w_1, w_2, w_3, w_\tau, {w_1w_4w_6}^{w_3w_5w_4}\rangle \cong S_3 \times (S_3 \wr S_2)$	0 14^{18} 16^9 18
X_{29}^{190}	$\langle w_1w_4w_6, w_3w_5w_7\rangle \cong Dih_{14}$	0 12^7 14^7 $16^{2.7}$
X_{29}^{191}	$\langle w_1w_2w_5, {w_1w_4w_6w_\tau}^{w_3w_5w_4}\rangle \cong {S_2}^2$	0 12 $14^{3.4+2+1}$ $16^{2.4+2.2}$
X_{29}^{192}	$\langle w_7, w_1w_4, w_3w_5\rangle \cong S_2 \times Dih_{10}$	0 12 14^{10+5} $16^{2.5+2}$
X_{29}^{193}	$\langle w_6, w_7, w_1w_4, w_2w_\sigma\rangle \cong {S_3}^2$	0 12 14^{9+6} 16^{9+3}
X_{29}^{194}	$\langle w_3, w_4, w_5\rangle \cong S_4$	0 5 7 9^4 11^{6+4} $13^{4+2.1}$ 15^{4+1} 21
X_{29}^{195}	$\langle w_3, w_4, w_7\rangle \cong S_3 \times S_2$	0 7^{2+1} 9^3 $11^{2.3}$ 13^{6+3+2} 15^{2+1} 17 21
X_{29}^{196}	$\langle w_3, w_4, w_5, w_7\rangle \cong S_4 \times S_2$	0 7^2 9^4 11^6 13^{8+4} 15^{2+1} 23
X_{29}^{197}	$\langle w_1, w_5, w_6\rangle \cong S_3 \times S_2$	0 5 7 9^{3+2} 11^{6+3+1} 13^3 15^{3+2} 17^{2+1}
X_{29}^{198}	$\langle w_4, w_6\rangle \cong {S_2}^2$	0 5 $9^{2.2+1}$ $11^{4.2+1}$ 13^{4+2+1} $15^{2.1}$ 17^{2+1} 19
X_{29}^{199}	$\langle w_1, w_5w_7\rangle \cong {S_2}^2$	0 7^{2+1} $9^{2.2}$ $11^{4+2.1}$ $13^{4+2.2}$ 15^{2+1} $17^{2.2}$
X_{29}^{200}	$\langle w_4\rangle \cong S_2$	0 $7^{2.1}$ $9^{2+2.1}$ $11^{2.2+3.1}$ $13^{2+4.1}$ $15^{2.2+2.1}$ $17^{2.1}$ 19
X_{29}^{201}	$\langle w_1, w_4, w_6\rangle \cong {S_2}^3$	0 $7^{2.1}$ $9^{2.2}$ $11^{2.4}$ $13^{2.2}$ $15^{4+2.1}$ $17^{2.2}$
X_{29}^{202}	$\langle w_6\rangle \cong S_2$	0 7 $9^{2+3.1}$ $11^{2+3.1}$ $13^{3.2+2.1}$ $15^{2+3.1}$ $17^{3.1}$ 19
X_{29}^{203}	$\langle w_1, w_4, w_6\rangle \cong {S_2}^3$	0 7 9^2 11^{4+2+1} $13^{2.4+2}$ $15^{2.2+1}$ 19^{2+1}
X_{29}^{204}	$\langle w_1, w_2, w_3, w_7\rangle \cong S_3 \times {S_2}^2$	0 5 9 $11^{2.6+2}$ 13^{4+3} 17^3 19^2

in the E_8 root system (continued)
W_X-orbits on X
$\{\substack{2465432\\3}\}, \{\substack{\cdot\cdot\cdot\cdot\cdot 11\\2}, \substack{\cdot\cdot\cdot\cdot\cdot 11\\0}\}, \{\substack{123\cdot 211\\1}, \substack{1222111\\1}, \substack{1121111\\1}, \substack{0\cdot 11111\\1}\}, \{\substack{234\cdot 321\\2}, \substack{1354321\\2}, \substack{1\cdot 43221\\2}, \substack{1233321\\2}\}, \{\substack{2343221\\2}, \substack{1343321\\2}, \substack{1244321\\2}\}, \{\substack{2465431\\3}\}$
$\{\substack{2465432\\3}\}, \{\substack{\cdot\cdot\cdot\cdot\cdot 11\\2}, \substack{123\cdot\cdot 11\\1}, \substack{\cdot\cdot 11111\\1}, \substack{\cdot\cdot\cdot\cdot\cdot 11\\0}\}, \{\substack{\cdot\cdot 4\cdot\cdot 21\\2}\}, \{\substack{2465431\\3}\}$
$\{\substack{2465432\\3}\}, \{\substack{\cdot\cdot\cdot\cdot\cdot\cdot 1\\0}\}, \{\substack{1233\cdot 21\\1}, \substack{1232211\\1}, \substack{1222111\\1}, \substack{1121111\\1}, \substack{0\cdot 11111\\1}\}, \{\substack{234\cdot 321\\2}, \substack{1354321\\2}, \substack{1343221\\2}, \substack{12\cdot 3211\\2}, \substack{1232221\\2}\}, \{\substack{23432\cdot 1\\2}, \substack{1343321\\2}, \substack{1244321\\2}, \substack{1343211\\2}, \substack{1243221\\2}, \substack{1233321\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{1243221\\2}\}, \{\substack{2354321\\2}, \substack{1354321\\3}, \substack{1233221\\2}, \substack{1232221\\1}\}, \{\substack{\cdot 343221\\2}, \substack{124\cdot 321\\2}\}, \{\substack{1232111\\\cdot}, \substack{0011111\\\cdot}\}, \{\substack{1243211\\2}, \substack{0000011\\0}\}, \{\substack{2465431\\3}\}, \{\substack{\cdot 343211\\2}, \substack{000\cdot 111\\0}\}, \{\substack{1232211\\2}, \substack{1233211\\1}, \substack{1111111\\1}, \substack{0111111\\0}\}, \{\substack{2344321\\2}, \substack{1343321\\2}\}, \{\substack{2343321\\2}, \substack{1344321\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{1232111\\2}\}, \{\substack{\cdot\cdot\cdot\cdot 2\cdot 1\\2}\}, \{\substack{12\cdot 2111\\1}, \substack{1121111\\1}, \substack{0\cdot 11111\\1}\}, \{\substack{2344321\\2}, \substack{1354321\\2}, \substack{\cdot 343321\\2}, \substack{1244321\\2}\}, \{\substack{\cdot\cdot\cdot\cdot 111\\0}\}, \{\substack{24654\cdot 1\\3}\}$
$\{\substack{2465432\\3}\}, \{\substack{1244321\\2}\}, \{\substack{12\cdot 3\cdot\cdot 1\\\cdot}\}, \{\substack{\cdot 354321\\3}, \substack{2344321\\2}, \substack{1354321\\2}, \substack{00\cdot 1111\\0}\}, \{\substack{\cdot 343\cdot\cdot 1\\2}, \substack{0000\cdot\cdot 1\\0}\}, \{\substack{2454321\\2}, \substack{1121111\\1}, \substack{0111111\\1}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{2465421\\3}\}, \{\substack{2\cdot\cdot\cdot 321\\3}\}, \{\substack{1\cdot\cdot\cdot 321\\2}\}, \{\substack{2\cdot\cdot\cdot 321\\2}\}, \{\substack{1\cdot\cdot\cdot 221\\2}\}, \{\substack{2343221\\2}\}, \{\substack{1233321\\1}\}, \{\substack{1\cdot\cdot\cdot 211\\2}\}, \{\substack{0122221\\1}\}, \{\substack{1111111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{24654\cdot 1\\3}\}, \{\substack{2465321\\3}\}, \{\substack{2\cdot\cdot 4321\\3}\}, \{\substack{2\cdot\cdot 4321\\2}\}, \{\substack{1\cdot\cdot 4321\\2}\}, \{\substack{1\cdot\cdot 32\cdot 1\\2}\}, \{\substack{1\cdot\cdot 3321\\2}\}, \{\substack{23432\cdot 1\\2}\}, \{\substack{12322\cdot 1\\2}\}, \{\substack{1233321\\1}\}, \{\substack{0122111\\1}\}, \{\substack{1111111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{24654\cdot 1\\3}\}, \{\substack{2\cdot\cdot\cdot 321\\3}\}, \{\substack{1\cdot\cdot\cdot 321\\2}\}, \{\substack{1\cdot\cdot\cdot 2\cdot 1\\2}\}, \{\substack{2\cdot\cdot\cdot 321\\2}\}, \{\substack{01222\cdot 1\\1}\}, \{\substack{1232111\\2}\}, \{\substack{1111111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{2454321\\2}\}, \{\substack{246\cdot\cdot 21\\3}\}, \{\substack{\cdot 354321\\2}\}, \{\substack{\cdot 34\cdot\cdot 21\\2}\}, \{\substack{123\cdot\cdot 21\\1}\}, \{\substack{2454321\\3}\}, \{\substack{124\cdot\cdot 21\\2}\}, \{\substack{123\cdot\cdot 21\\2}\}, \{\substack{\cdot 343211\\2}\}, \{\substack{\cdot 111111\\0}\}, \{\substack{1243211\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{2465\cdot 21\\3}\}, \{\substack{23\cdot 4321\\2}\}, \{\substack{2454321\\2}\}, \{\substack{24\cdot 4321\\3}\}, \{\substack{13\cdot 4321\\2}\}, \{\substack{2343\cdot 21\\2}\}, \{\substack{1233\cdot 21\\1}\}, \{\substack{1244321\\2}\}, \{\substack{12\cdot 3\cdot 21\\2}\}, \{\substack{1343\cdot 21\\2}\}, \{\substack{1122221\\1}\}, \{\substack{2343211\\2}\}, \{\substack{1232221\\2}\}, \{\substack{12\cdot 3211\\2}\}, \{\substack{1111111\\0}\}, \{\substack{0111111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{24654\cdot 1\\3}\}, \{\substack{2454321\\2}\}, \{\substack{246\cdot 321\\3}\}, \{\substack{\cdot 354321\\2}\}, \{\substack{\cdot 34\cdot 321\\2}\}, \{\substack{2454321\\3}\}, \{\substack{1233321\\1}\}, \{\substack{\cdot 3432\cdot 1\\2}\}, \{\substack{124\cdot 321\\2}\}, \{\substack{1233221\\1}, \substack{1232211\\1}\}, \{\substack{12432\cdot 1\\2}\}, \{\substack{1233321\\2}\}, \{\substack{1233211\\2}, \substack{1232221\\2}\}, \{\substack{\cdot 111111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{2465421\\3}\}, \{\substack{23\cdot 4321\\2}\}, \{\substack{2465321\\3}\}, \{\substack{2454321\\2}\}, \{\substack{24\cdot 4321\\3}\}, \{\substack{13\cdot 4321\\2}\}, \{\substack{2343321\\2}\}, \{\substack{1244321\\2}\}, \{\substack{1233321\\1}\}, \{\substack{12\cdot 3321\\2}\}, \{\substack{2343221\\2}\}, \{\substack{1343321\\2}\}, \{\substack{2343211\\2}\}, \{\substack{1233221\\1}\}, \{\substack{12\cdot 3221\\2}\}, \{\substack{12\cdot 3211\\2}\}, \{\substack{1343211\\2}\}, \{\substack{1122211\\1}\}, \{\substack{1232221\\2}\}, \{\substack{1111111\\0}\}, \{\substack{0111111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{2454321\\2}\}, \{\substack{2465\cdot 21\\3}\}, \{\substack{24\cdot 4321\\3}\}, \{\substack{\cdot 3\cdot 4321\\2}\}, \{\substack{\cdot 343\cdot 21\\2}\}, \{\substack{\cdot 343211\\2}\}, \{\substack{1233\cdot 21\\1}\}, \{\substack{12\cdot 3\cdot 21\\2}\}, \{\substack{1232221\\2}\}, \{\substack{1221111\\1}\}, \{\substack{12\cdot 3211\\2}\}, \{\substack{\cdot 111111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{2465\cdot 21\\3}\}, \{\substack{2464321\\3}\}, \{\substack{2454321\\2}\}, \{\substack{2354321\\2}\}, \{\substack{2343\cdot 21\\2}\}, \{\substack{2454321\\3}\}, \{\substack{2344321\\2}\}, \{\substack{1354321\\2}\}, \{\substack{1343\cdot 21\\2}\}, \{\substack{1243\cdot 21\\2}\}, \{\substack{1233\cdot 21\\1}\}, \{\substack{1244321\\2}\}, \{\substack{2343211\\2}\}, \{\substack{1233\cdot 21\\2}\}, \{\substack{1343211\\2}\}, \{\substack{1243211\\2}\}, \{\substack{1232221\\2}\}, \{\substack{1233211\\2}\}, \{\substack{1121111\\1}\}, \{\substack{1111111\\0}\}, \{\substack{0111111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{2465\cdot 21\\3}\}, \{\substack{\cdot 3\cdot 4321\\2}\}, \{\substack{24\cdot 4321\\3}\}, \{\substack{1244321\\2}\}, \{\substack{\cdot 343\cdot 21\\2}\}, \{\substack{12\cdot 3\cdot 21\\2}\}, \{\substack{1233\cdot 21\\1}\}, \{\substack{\cdot 343211\\2}\}, \{\substack{12\cdot 3211\\2}\}, \{\substack{1232221\\2}\}, \{\substack{\cdot 111111\\0}\}, \{\substack{0011111\\1}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465321\\3}\}, \{\substack{2464321\\3}\}, \{\substack{\cdot\cdot 54321\\\cdot}\}, \{\substack{\cdot\cdot 432\cdot 1\\2}\}, \{\substack{1232111\\\cdot}\}, \{\substack{12332\cdot 1\\\cdot}\}, \{\substack{\cdot\cdot 44321\\2}\}, \{\substack{\cdot\cdot 43321\\2}\}, \{\substack{00000\cdot 1\\0}\}$

Table 5: Maximal abelian sets

X	W_X	Sig(X)
X_{29}^{205}	$\langle w_1, w_2, w_3, w_5\rangle \cong S_3 \times {S_2}^2$	0 7^2 9^3 11^{6+2} 13^{4+3} 15^6 17 21
X_{29}^{206}	$\langle w_1, w_3, w_2w_5, w_5w_7\rangle \cong S_3 \times {S_2}^2$	0 7^2 $11^{2.6+2}$ 15^{6+4} 19^2
X_{29}^{207}	$\langle w_5\rangle \cong S_2$	0 7 $9^{2+3.1}$ $11^{2+3.1}$ $13^{2.2+4.1}$ $15^{2+3.1}$ 17^{2+1} 19
X_{29}^{208}	$\langle w_4\rangle \cong S_2$	0 7 $9^{2+2.1}$ $11^{2+4.1}$ $13^{3.2+3.1}$ $15^{2+3.1}$ $17^{2.1}$ 21
X_{29}^{209}	$\langle w_4\rangle \cong S_2$	0 7 $9^{2.1}$ $11^{3.2+2.1}$ $13^{2.2+4.1}$ $15^{2+3.1}$ $17^{2.1}$ $19^{2.1}$
X_{29}^{210}	$\langle w_4, w_5\rangle \cong S_3$	0 7 9 $11^{3.3}$ $13^{2.3+3.1}$ $15^{3+2.1}$ 17 19 21
X_{29}^{211}	$\langle w_6, w_5w_7\rangle \cong Dih_8$	0 $9^{4+2.1}$ 11^{4+1} 13^4 $15^{2.4+2}$ $17^{2.1}$ 19
X_{29}^{212}	$\langle w_1, w_6, w_5w_7\rangle \cong S_2 \times Dih_8$	0 9^4 11^{2+1} 13^{8+4} 15^{4+2} 19^{2+1}
X_{29}^{213}	$\langle w_1, w_4, w_7\rangle \cong {S_2}^3$	0 7 9^2 $11^{4+2+2.1}$ $13^{2.4}$ $15^{2.2+1}$ 17^2 19^2
X_{29}^{214}	$\langle w_5, w_7\rangle \cong {S_2}^2$	0 $9^{2.2+1}$ $11^{2.2+2.1}$ $13^{2.2+1}$ $15^{4+2.2+2.1}$ 17 21
X_{29}^{215}	$\langle w_7, w_1w_5\rangle \cong {S_2}^2$	0 $9^{2.2}$ $11^{2+2.1}$ $13^{4+3.2}$ $15^{4+2.1}$ 17^2 19^2
X_{29}^{216}	$\langle w_4, w_7\rangle \cong {S_2}^2$	0 9^{2+1} $11^{2.2+1}$ $13^{4+3.2+1}$ $15^{2.2+2.1}$ 17 19 21
X_{29}^{217}	$\langle w_1, w_2, w_3, w_4, w_7\rangle \cong S_5 \times S_2$	0 7 11^{10} 13^{10} 15^5 21^2
X_{29}^{218}	$\langle w_4, w_6, w_7\rangle \cong S_3 \times S_2$	0 9^{3+2} 11^{3+2+1} 13^{6+3+1} 15^{3+2} 17 23
X_{29}^{219}	$\langle w_4, w_7\rangle \cong {S_2}^2$	0 9^{2+1} $11^{2.2+1}$ $13^{4+3.2+1}$ $15^{2.2+2.1}$ 17 19 21

in the E_8 root system (continued)
W_X-orbits on X
$\{\substack{2465432\\3}\},\{\substack{246\cdot321\\3}\},\{\substack{\cdot\cdot43221\\2}\},\{\substack{\cdot\cdot54321\\\cdot}\},\{\substack{1232111\\\cdot}\},\{\substack{123\cdot221\\\cdot}\},\{\substack{\cdot\cdot43211\\2}\},\{\substack{\cdot\cdot4\cdot321\\2}\},\{\substack{0000011\\0}\},\{\substack{0000001\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{246\cdot321\\3}\},\{\substack{\cdot\cdot54321\\\cdot}\},\{\substack{\cdot\cdot432\cdot1\\2}\},\{\substack{1232111\\\cdot}\},\{\substack{\cdot\cdot4\cdot321\\2}\},\{\substack{1233221\\2},\substack{1232211\\2},\substack{1233211\\1},\substack{1232221\\1}\},\{\substack{00000\cdot1\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{2465431\\3}\},\{\substack{246\cdot321\\3}\},\{\substack{2465421\\3}\},\{\substack{2454321\\2}\},\{\substack{2354321\\2}\},\{\substack{234\cdot321\\2}\},\{\substack{2454321\\3}\},\{\substack{1354321\\2}\},\{\substack{2343221\\2}\},\{\substack{134\cdot321\\2}\},\{\substack{124\cdot321\\2}\},\{\substack{2343211\\2}\},\{\substack{1343221\\2}\},\{\substack{1243221\\2}\},\{\substack{1233321\\1}\},\{\substack{123\cdot221\\2}\},\{\substack{1243211\\2}\},\{\substack{1232111\\1}\},\{\substack{1122221\\1}\},\{\substack{123\cdot211\\2}\},\{\substack{1111111\\0}\},\{\substack{0111111\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{2465431\\3}\},\{\substack{23\cdot4321\\2}\},\{\substack{2465421\\3}\},\{\substack{2465321\\3}\},\{\substack{24\cdot4321\\3}\},\{\substack{2454321\\2}\},\{\substack{2343321\\2}\},\{\substack{2343221\\2}\},\{\substack{1244321\\2}\},\{\substack{13\cdot4321\\2}\},\{\substack{12\cdot3321\\2}\},\{\substack{12\cdot3221\\2}\},\{\substack{2343211\\2}\},\{\substack{1233321\\1}\},\{\substack{1122221\\1}\},\{\substack{12\cdot3211\\2}\},\{\substack{1343221\\2}\},\{\substack{1232221\\2}\},\{\substack{1122111\\1}\},\{\substack{1232211\\2}\},\{\substack{1111111\\0}\},\{\substack{0111111\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{2465431\\3}\},\{\substack{2465421\\3}\},\{\substack{2465321\\3}\},\{\substack{24\cdot4321\\3}\},\{\substack{23\cdot4321\\2}\},\{\substack{13\cdot4321\\2}\},\{\substack{2454321\\2}\},\{\substack{1244321\\2}\},\{\substack{12\cdot3321\\2}\},\{\substack{12\cdot3221\\2}\},\{\substack{2343221\\2}\},\{\substack{1343321\\2}\},\{\substack{1343221\\2}\},\{\substack{1233321\\1}\},\{\substack{12\cdot3211\\2}\},\{\substack{2343211\\2}\},\{\substack{1232221\\2}\},\{\substack{1122221\\1}\},\{\substack{1232211\\2}\},\{\substack{0122111\\1}\},\{\substack{1111111\\0}\},\{\substack{0111111\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{2465431\\3}\},\{\substack{2465421\\3}\},\{\substack{24\cdot\cdot321\\3}\},\{\substack{23\cdot\cdot321\\2}\},\{\substack{12\cdot\cdot321\\2}\},\{\substack{13\cdot\cdot321\\2}\},\{\substack{12\cdot\cdot221\\2}\},\{\substack{2343221\\2}\},\{\substack{1233321\\1}\},\{\substack{1122221\\1}\},\{\substack{12\cdot\cdot211\\2}\},\{\substack{2343211\\2}\},\{\substack{1343221\\2}\},\{\substack{0011111\\1}\},\{\substack{1111111\\0}\},\{\substack{0111111\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{246\cdot\cdot\cdot1\\3}\},\{\substack{2454321\\3}\},\{\substack{2454321\\2}\},\{\substack{234\cdot\cdot\cdot1\\2}\},\{\substack{2354321\\2}\},\{\substack{134\cdot\cdot\cdot1\\2}\},\{\substack{124\cdot\cdot\cdot1\\2}\},\{\substack{1233\cdot21\\2},\substack{1232\cdot11\\2}\},\{\substack{1233211\\1},\substack{1232221\\1}\},\{\substack{1111111\\1}\},\{\substack{1111111\\0}\},\{\substack{0111111\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{246\cdot\cdot\cdot1\\3}\},\{\substack{\cdot354321\\2}\},\{\substack{2454321\\3}\},\{\substack{\cdot34\cdot\cdot\cdot1\\2}\},\{\substack{124\cdot\cdot\cdot1\\2}\},\{\substack{1233\cdot21\\2},\substack{1232\cdot11\\2}\},\{\substack{1233211\\1},\substack{1232221\\1}\},\{\substack{\cdot111111\\0}\},\{\substack{0011111\\1}\}$
$\{\substack{2465432\\3}\},\{\substack{2465321\\3}\},\{\substack{24654\cdot1\\3}\},\{\substack{\cdot3\cdot4321\\2}\},\{\substack{24\cdot4321\\3}\},\{\substack{2454321\\2}\},\{\substack{1244321\\2}\},\{\substack{\cdot3432\cdot1\\2}\},\{\substack{12\cdot32\cdot1\\2}\},\{\substack{12\cdot3321\\2}\},\{\substack{\cdot122111\\1}\},\{\substack{1233321\\1}\},\{\substack{12322\cdot1\\2}\},\{\substack{\cdot111111\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{24654\cdot1\\3}\},\{\substack{246\cdot321\\3}\},\{\substack{2354321\\2}\},\{\substack{234\cdot321\\2}\},\{\substack{23432\cdot1\\2}\},\{\substack{2454321\\3}\},\{\substack{2454321\\2}\},\{\substack{124\cdot321\\2}\},\{\substack{12432\cdot1\\2}\},\{\substack{1354321\\2}\},\{\substack{123\cdot2\cdot1\\2}\},\{\substack{13432\cdot1\\2}\},\{\substack{112\cdot111\\1}\},\{\substack{1233321\\2}\},\{\substack{1233321\\1}\},\{\substack{1111111\\0}\},\{\substack{0111111\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{24654\cdot1\\3}\},\{\substack{246\cdot321\\3}\},\{\substack{\cdot354321\\2}\},\{\substack{2454321\\3}\},\{\substack{2454321\\2}\},\{\substack{\cdot3432\cdot1\\2}\},\{\substack{2343321\\2},\substack{1344321\\2}\},\{\substack{124\cdot321\\2}\},\{\substack{12432\cdot1\\2}\},\{\substack{123\cdot2\cdot1\\2}\},\{\substack{1233321\\2}\},\{\substack{1233321\\1}\},\{\substack{1122111\\1},\substack{0121111\\1}\},\{\substack{\cdot111111\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{24654\cdot1\\3}\},\{\substack{2465321\\3}\},\{\substack{24\cdot4321\\3}\},\{\substack{23\cdot4321\\2}\},\{\substack{1244321\\2}\},\{\substack{12\cdot32\cdot1\\2}\},\{\substack{13\cdot4321\\2}\},\{\substack{23432\cdot1\\2}\},\{\substack{12\cdot3321\\2}\},\{\substack{2343321\\2}\},\{\substack{13432\cdot1\\2}\},\{\substack{12322\cdot1\\2}\},\{\substack{1233321\\1}\},\{\substack{1122111\\1}\},\{\substack{0011111\\1}\},\{\substack{1111111\\0}\},\{\substack{0111111\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{2465321\\3}\},\{\substack{\cdot\cdot\cdot4321\\\cdot}\},\{\substack{\cdot\cdot\cdot32\cdot1\\\cdot}\},\{\substack{\cdot\cdot\cdot3321\\\cdot}\},\{\substack{00000\cdot1\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{2465\cdot\cdot1\\3}\},\{\substack{23\cdot4321\\2}\},\{\substack{2343\cdot\cdot1\\2}\},\{\substack{24\cdot4321\\3}\},\{\substack{1244321\\2}\},\{\substack{12\cdot3\cdot\cdot1\\2}\},\{\substack{1122\cdot\cdot1\\1}\},\{\substack{2454321\\2}\},\{\substack{1232\cdot\cdot1\\2}\},\{\substack{13\cdot4321\\2}\},\{\substack{1111111\\0}\},\{\substack{0111111\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{24654\cdot1\\3}\},\{\substack{2465321\\3}\},\{\substack{24\cdot4321\\3}\},\{\substack{23\cdot4321\\2}\},\{\substack{1244321\\2}\},\{\substack{12\cdot32\cdot1\\2}\},\{\substack{13\cdot4321\\2}\},\{\substack{23432\cdot1\\2}\},\{\substack{12\cdot3321\\2}\},\{\substack{2454321\\2}\},\{\substack{12322\cdot1\\2}\},\{\substack{11222\cdot1\\1}\},\{\substack{1343321\\2}\},\{\substack{1232111\\2}\},\{\substack{0122111\\1}\},\{\substack{1111111\\0}\},\{\substack{0111111\\0}\}$

Table 5: Maximal abelian sets

X	W_X	Sig(X)
X_{29}^{220}	$\langle w_1, w_2, w_3, w_5\rangle \cong S_3 \times {S_2}^2$	0 9^2 11^{6+3} 13^{6+4} 15^{3+2} 19 23
X_{29}^{221}	$\langle w_1, w_3, w_2w_5, w_5w_7\rangle \cong S_3 \times {S_2}^2$	0 9^2 11^6 $13^{2.6}$ 15^{4+2} 21^2
X_{29}^{222}	$\langle w_1, w_4, w_5, w_7\rangle \cong S_3 \times {S_2}^2$	0 7 9^2 11^{6+3} 13^6 15^{3+2} 17^4 19
X_{29}^{223}	$\langle w_1\rangle \cong S_2$	0 $7^{2.1}$ $9^{2+2.1}$ $11^{2.2+3.1}$ $13^{2+4.1}$ $15^{2.2+2.1}$ $17^{2.1}$ 19
X_{29}^{224}	$\langle w_3w_7\rangle \cong S_2$	0 7 $9^{2.2+1}$ $11^{2.2+2.1}$ $13^{2.2+2.1}$ $15^{2.2+1}$ $17^{2.2+1}$
X_{29}^{225}	$\langle w_2, w_3w_5, w_3w_5{w_7}^{w_4}\rangle \cong S_2 \times Dih_8$	0 7^2 11^{8+4+2} 15^{8+2} 19^2
X_{29}^{226}	$\langle w_6\rangle \cong S_2$	0 7 $9^{2.1}$ $11^{3.2+3.1}$ $13^{2.2+2.1}$ $15^{2+3.1}$ $17^{4.1}$ 19
X_{29}^{227}	$\langle w_1, w_6, w_4w_\tau\rangle \cong {S_2}^3$	0 7 $11^{2.4+1}$ $13^{2.4+2}$ 15 17^{4+2} 19
X_{29}^{228}	$\langle w_3, w_1w_4w_7\rangle \cong Dih_8$	0 7 9^2 $11^{2.4+2}$ 13^4 15^{4+1} 17^{4+2}
X_{29}^{229}	$\langle w_3w_5w_7\rangle \cong S_2$	0 7 9^2 $11^{3.2+2.1}$ $13^{4.2}$ $15^{2.2+1}$ 17^2 19^2
X_{29}^{230}	$\langle w_1, w_2w_5\rangle \cong {S_2}^2$	0 7 $11^{4+2.2+1}$ $13^{4+3.2}$ 15 $17^{3.2}$ 19
X_{29}^{231}	$\langle w_1, w_2, w_1w_4w_6{w_\tau}^{w_3w_4w_5w_4w_3}\rangle \cong S_2 \times Dih_8$	0 7 $11^{2.4}$ 13^{8+4} 15 17^4 19^2
X_{29}^{232}	$\langle w_1, w_3, w_6, w_5w_7\rangle \cong S_3 \times Dih_8$	0 7^3 9^4 11^4 13^{12} 15^3 19^2
X_{29}^{233}	$\langle w_3, w_5w_7\rangle \cong {S_2}^2$	0 $9^{2.2}$ $11^{2.2+1}$ $13^{4+2.2}$ $15^{4+2.1}$ $17^{2.2}$ 19
X_{29}^{234}	$\langle w_1, w_6\rangle \cong {S_2}^2$	0 7 $9^{2.2+1}$ $11^{2.2}$ $13^{4+2.2+2.1}$ $15^{2.2+1}$ 17 $19^{2.1}$

in the E_8 root system (continued)

W_X-orbits on X
$\{\substack{2465432\\3}\}, \{\substack{246\cdot321\\3}\}, \{\substack{\cdot\cdot54321\\\cdot}\}, \{\substack{\cdot\cdot43221\\2}\}, \{\substack{\cdot\cdot4\cdot321\\2}\}, \{\substack{123\cdot221\\\cdot}\}, \{\substack{\cdot\cdot43211\\2}\}, \{\substack{1233321\\\cdot}\}, \{\substack{0000011\\0}\}, \{\substack{0000001\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{246\cdot321\\3}\}, \{\substack{\cdot\cdot54321\\\cdot}\}, \{\substack{\cdot\cdot4\cdot321\\2}\}, \{\substack{\cdot\cdot432\cdot1\\2}\}, \{\substack{1233221\\2}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{1232221\\1}\}, \{\substack{1233321\\\cdot}\}, \{\substack{00000\cdot1\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{1233321\\1}\}, \{\substack{\cdot111111\\0}\}, \{\substack{12\cdot\cdot2\cdot1\\1}\}, \{\substack{00\cdot\cdot111\\0}\}, \{\substack{\cdot3\cdot\cdot321\\2}\}, \{\substack{12\cdot\cdot321\\2}\}, \{\substack{00000\cdot1\\0}\}, \{\substack{\cdot3432\cdot1\\2}\}, \{\substack{0011111\\1}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{2454321\\2}\}, \{\substack{\cdot354321\\2}\}, \{\substack{2465421\\3}\}, \{\substack{1233321\\1}\}, \{\substack{\cdot344321\\2}\}, \{\substack{\cdot343321\\2}\}, \{\substack{2465321\\3}\}, \{\substack{2464321\\3}\}, \{\substack{1233221\\1}\}, \{\substack{\cdot343221\\2}\}, \{\substack{1244321\\2}\}, \{\substack{1243321\\2}\}, \{\substack{1232211\\1}\}, \{\substack{1222221\\1}\}, \{\substack{\cdot343211\\2}\}, \{\substack{\cdot111111\\0}\}, \{\substack{1243221\\2}\}, \{\substack{1233321\\2}\}, \{\substack{1232221\\2}\}, \{\substack{0011111\\0}\}, \{\substack{1233211\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{1233321\\1}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{2\cdot54321\\2}\}, \{\substack{2343321\\2}\}, \{\substack{1\cdot43321\\2}\}, \{\substack{12322\cdot1\\1}\}, \{\substack{2344321\\2}\}, \{\substack{1354321\\2}\}, \{\substack{1\cdot44321\\2}\}, \{\substack{1222211\\1}, \substack{1122221\\1}\}, \{\substack{1233321\\2}\}, \{\substack{1111111\\0}\}, \{\substack{23432\cdot1\\2}\}, \{\substack{0\cdot11111\\0}\}, \{\substack{2465321\\3}\}, \{\substack{1343221\\2}, \substack{1243211\\2}\}, \{\substack{12322\cdot1\\2}\}, \{\substack{0001111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{23432\cdot1\\2}\}, \{\substack{2\cdot\cdot\cdot321\\\cdot}\}, \{\substack{1232111\\\cdot}, \substack{1111111\\\cdot}\}, \{\substack{1222111\\1}, \substack{1121111\\1}\}, \{\substack{1\cdot43221\\2}, \substack{123\cdot211\\\cdot}, \substack{1\cdot22221\\1}\}, \{\substack{00000\cdot1\\0}\}, \{\substack{1343321\\2}, \substack{1244321\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{2454321\\2}\}, \{\substack{2354321\\2}\}, \{\substack{2465\cdot21\\3}\}, \{\substack{2343\cdot21\\2}\}, \{\substack{1233\cdot21\\1}\}, \{\substack{2464321\\3}\}, \{\substack{2344321\\2}\}, \{\substack{1354321\\2}\}, \{\substack{1343\cdot21\\2}\}, \{\substack{1243\cdot21\\2}\}, \{\substack{1244321\\2}\}, \{\substack{1222221\\1}\}, \{\substack{1233\cdot21\\2}\}, \{\substack{2343211\\2}\}, \{\substack{1232221\\2}\}, \{\substack{1111111\\0}\}, \{\substack{1343211\\2}\}, \{\substack{1121111\\1}\}, \{\substack{0111111\\0}\}, \{\substack{0011111\\0}\}, \{\substack{1233211\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{2465\cdot21\\3}, \substack{1233\cdot21\\1}\}, \{\substack{\cdot3\cdot4321\\2}\}, \{\substack{1244321\\2}\}, \{\substack{\cdot343\cdot21\\2}\}, \{\substack{12\cdot3\cdot21\\2}\}, \{\substack{2464321\\3}, \substack{1222221\\1}\}, \{\substack{1232221\\2}\}, \{\substack{\cdot343211\\2}, \substack{\cdot111111\\0}\}, \{\substack{1233211\\2}, \substack{0011111\\0}\}, \{\substack{0011111\\1}\}$
$\{\substack{2465432\\3}\}, \{\substack{1233321\\1}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{2\cdot54321\\2}, \substack{1\cdot44321\\2}\}, \{\substack{\cdot\cdot\cdot3321\\2}\}, \{\substack{2344321\\2}, \substack{1354321\\2}\}, \{\substack{1232211\\1}, \substack{1\cdot22221\\1}, \substack{0122211\\1}\}, \{\substack{\cdot\cdot\cdot1111\\0}\}, \{\substack{2465321\\3}\}, \{\substack{2343211\\2}, \substack{1\cdot43221\\2}, \substack{1233211\\2}\}, \{\substack{12322\cdot1\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{1233321\\1}\}, \{\substack{2\cdot54321\\2}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{234\cdot321\\2}\}, \{\substack{1233221\\1}, \substack{1232211\\1}\}, \{\substack{1354321\\2}\}, \{\substack{1111111\\0}\}, \{\substack{1344321\\2}, \substack{1243321\\2}\}, \{\substack{1343321\\2}, \substack{1244321\\2}\}, \{\substack{1222211\\1}, \substack{1122221\\1}\}, \{\substack{0\cdot11111\\0}\}, \{\substack{23432\cdot1\\2}\}, \{\substack{000\cdot111\\0}\}, \{\substack{1232111\\1}\}, \{\substack{1343221\\2}, \substack{1243211\\2}\}, \{\substack{1233211\\2}, \substack{1232221\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{\cdot34\cdot321\\2}\}, \{\substack{2454321\\\cdot}\}, \{\substack{1233321\\\cdot}\}, \{\substack{1222221\\1}\}, \{\substack{\cdot354321\\\cdot}\}, \{\substack{\cdot343221\\2}\}, \{\substack{124\cdot321\\2}\}, \{\substack{1233221\\2}, \substack{1232221\\1}\}, \{\substack{1243221\\2}\}, \{\substack{\cdot343211\\2}\}, \{\substack{1232211\\2}, \substack{1233211\\1}\}, \{\substack{000\cdot111\\0}\}, \{\substack{0000011\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{2454321\\\cdot}, \substack{1232221\\\cdot}\}, \{\substack{\cdot343\cdot21\\2}\}, \{\substack{\cdot354321\\\cdot}, \substack{1233\cdot21\\\cdot}\}, \{\substack{\cdot344321\\2}, \substack{1243\cdot21\\2}\}, \{\substack{1221111\\1}\}, \{\substack{\cdot343211\\2}, \substack{0000\cdot11\\0}\}, \{\substack{1243211\\2}, \substack{0001111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{\cdot\cdot54321\\2}\}, \{\substack{246\cdot\cdot\cdot1\\3}\}, \{\substack{1233\cdot21\\1}, \substack{1232\cdot11\\1}\}, \{\substack{\cdot\cdot4\cdot\cdot\cdot1\\2}\}, \{\substack{\cdot\cdot11111\\0}\}, \{\substack{1233211\\2}, \substack{1232221\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{2\cdot54321\\2}\}, \{\substack{246\cdot321\\3}\}, \{\substack{234\cdot321\\2}\}, \{\substack{1233321\\1}\}, \{\substack{1\cdot4\cdot321\\2}\}, \{\substack{23432\cdot1\\2}\}, \{\substack{1233221\\1}, \substack{1232211\\1}\}, \{\substack{1\cdot432\cdot1\\2}\}, \{\substack{1233321\\2}\}, \{\substack{1111111\\0}\}, \{\substack{1233211\\2}, \substack{1232221\\2}\}, \{\substack{0\cdot11111\\0}\}, \{\substack{1111111\\1}\}$
$\{\substack{2465432\\3}\}, \{\substack{2454321\\2}\}, \{\substack{2465\cdot21\\3}\}, \{\substack{\cdot344321\\2}\}, \{\substack{2465431\\3}\}, \{\substack{\cdot354321\\2}\}, \{\substack{1233\cdot21\\1}\}, \{\substack{\cdot343\cdot21\\2}\}, \{\substack{\cdot343211\\2}\}, \{\substack{1222\cdot11\\1}\}, \{\substack{2464321\\3}\}, \{\substack{1244321\\2}\}, \{\substack{1233\cdot21\\2}\}, \{\substack{\cdot111111\\0}\}, \{\substack{1233211\\2}\}, \{\substack{1243211\\2}\}, \{\substack{1232221\\2}\}, \{\substack{0011111\\0}\}$

Table 5: Maximal abelian sets

X	W_X	Sig(X)
X_{29}^{235}	$\langle w_3w_6\rangle \cong S_2$	$0\ 9^{2.2+2.1}\ 11^{2.2+1}\ 13^{2+2.1}\ 15^{4.2+2.1}\ 17^2\ 19$
X_{29}^{236}	$\langle w_3w_5w_7\rangle \cong S_2$	$0\ 9^{2.2}\ 11^{2.2+2.1}\ 13^{3.2}\ 15^{2.2+2.1}\ 17^{3.2}$
X_{29}^{237}	$\langle w_1\rangle \cong S_2$	$0\ 9^{2.1}\ 11^{2.2+3.1}\ 13^{2+4.1}\ 15^{2.2+4.1}\ 17^{2+2.1}\ 19$
X_{29}^{238}	$\langle w_1, w_7\rangle \cong {S_2}^2$	$0\ 7\ 9^{2.2}\ 11^{2.2+2.1}\ 13^{4+2+3.1}\ 15^{2.2+1}\ 17^2\ 21$
X_{29}^{239}	$\langle w_1, w_5, w_7\rangle \cong {S_2}^3$	$0\ 9^2\ 11^{4+2.1}\ 13^{4.2}\ 15^{2.4}\ 17^2\ 19^{2.1}$
X_{29}^{240}	$\langle w_1, w_2w_5, w_5w_7\rangle \cong {S_2}^3$	$0\ 9^{2.2}\ 11^{4+2}\ 13^{4+2}\ 15^{4+2}\ 17^{4+2}$
X_{29}^{241}	1	$0\ 9^{4.1}\ 11^{5.1}\ 13^{8.1}\ 15^{6.1}\ 17^{4.1}\ 19$
X_{29}^{242}	$\langle w_5w_7, w_2w_3w_5\rangle \cong {S_2}^2$	$0\ 9^2\ 11^{4+2.2}\ 13^{2.2}\ 15^{2.4}\ 17^{3.2}$
X_{29}^{243}	$\langle w_5\rangle \cong S_2$	$0\ 9^{2.1}\ 11^{2.2+3.1}\ 13^{2.2+2.1}\ 15^{2.2+4.1}\ 17^{4.1}\ 19$
X_{29}^{244}	$\langle w_1, w_5w_7\rangle \cong {S_2}^2$	$0\ 7\ 9^2\ 11^{4+2.2}\ 13^{4+2.2}\ 15^{2.2+1}\ 17^2\ 19^{2.1}$
X_{29}^{245}	$\langle w_3w_5w_7\rangle \cong S_2$	$0\ 9^{2.2}\ 11^{2.2+1}\ 13^{4.2}\ 15^{2.2+2.1}\ 17^{2.2}\ 19$
X_{29}^{246}	$\langle w_5\rangle \cong S_2$	$0\ 9^{2.1}\ 11^{2.2+3.1}\ 13^{2+4.1}\ 15^{2.2+4.1}\ 17^{2+2.1}\ 19$

in the E_8 root system (continued)
W_X-orbits on X
$\{{2465432 \atop 3}\}, \{{2465\cdot 21 \atop 3}\}, \{{2\cdot 54321 \atop 2}\}, \{{2465431 \atop 3}\}, \{{2344321 \atop 2}\}, \{{1\cdot 44321 \atop 2}\}, \{{1233\cdot 21 \atop 1}\}, \{{1354321 \atop 2}\}, \{{2343\cdot 21 \atop 2}\}, \{{2464321 \atop 3}\}, \{{2343211 \atop 2}\}, \{{1343221 \atop 2}, {1243321 \atop 2}\}, \{{1\cdot 43211 \atop 2}\}, \{{1233\cdot 21 \atop 2}\}, \{{1222211 \atop 1}, {1122111 \atop 1}\}, \{{1233211 \atop 2}\}, \{{1111111 \atop 0}\}, \{{0\cdot 11111 \atop 0}\}, \{{1232221 \atop 2}\}$
$\{{2465432 \atop 3}\}, \{{24654\cdot 1 \atop 3}\}, \{{2\cdot 54321 \atop 2}\}, \{{246\cdot 321 \atop 3}\}, \{{234\cdot 321 \atop 2}\}, \{{1354321 \atop 2}\}, \{{1233321 \atop 1}\}, \{{1344321 \atop 2}, {1243321 \atop 2}\}, \{{23432\cdot 1 \atop 2}\}, \{{1233211 \atop 1}, {1232221 \atop 1}\}, \{{1343211 \atop 2}, {1243221 \atop 2}\}, \{{1222221 \atop 1}, {1122211 \atop 1}\}, \{{1354321 \atop 3}\}, \{{1233321 \atop 2}\}, \{{1233221 \atop 2}, {1232211 \atop 2}\}, \{{1221111 \atop 1}, {1122111 \atop 1}\}, \{{0\cdot 11111 \atop 0}\}$
$\{{2465432 \atop 3}\}, \{{2465431 \atop 3}\}, \{{2465421 \atop 3}\}, \{{\cdot 354321 \atop 2}\}, \{{\cdot 344321 \atop 2}\}, \{{2465321 \atop 3}\}, \{{1244321 \atop 2}\}, \{{1233321 \atop 1}\}, \{{\cdot 343321 \atop 2}\}, \{{2464321 \atop 3}\}, \{{1243321 \atop 2}\}, \{{1233321 \atop 2}\}, \{{1233221 \atop 1}\}, \{{\cdot 343221 \atop 2}\}, \{{\cdot 343211 \atop 2}\}, \{{1243211 \atop 2}\}, \{{1233221 \atop 2}\}, \{{1233211 \atop 2}\}, \{{1222211 \atop 1}\}, \{{\cdot 111111 \atop 0}\}, \{{1232221 \atop 2}\}, \{{0011111 \atop 0}\}, \{{0011111 \atop 1}\}$
$\{{2465432 \atop 3}\}, \{{2454321 \atop 2}\}, \{{24654\cdot 1 \atop 3}\}, \{{\cdot 344321 \atop 2}\}, \{{\cdot 343321 \atop 2}\}, \{{12222\cdot 1 \atop 1}\}, \{{2465321 \atop 3}\}, \{{1233321 \atop 1}\}, \{{\cdot 3432\cdot 1 \atop 2}\}, \{{\cdot 354321 \atop 2}\}, \{{2464321 \atop 3}\}, \{{1233321 \atop 2}\}, \{{1222111 \atop 1}\}, \{{12332\cdot 1 \atop 2}\}, \{{\cdot 111111 \atop 0}\}, \{{1244321 \atop 2}\}, \{{12322\cdot 1 \atop 2}\}, \{{0011111 \atop 0}\}$
$\{{2465432 \atop 3}\}, \{{24654\cdot 1 \atop 3}\}, \{{\cdot 34\cdot 321 \atop 2}\}, \{{1233321 \atop 2}\}, \{{1233321 \atop 1}\}, \{{246\cdot 321 \atop 3}\}, \{{\cdot 354321 \atop 2}\}, \{{124\cdot 321 \atop 2}\}, \{{12222\cdot 1 \atop 1}\}, \{{\cdot 3432\cdot 1 \atop 2}\}, \{{123\cdot 2\cdot 1 \atop 2}\}, \{{\cdot 111111 \atop 0}\}, \{{0011111 \atop 1}\}, \{{0011111 \atop 0}\}$
$\{{2465432 \atop 3}\}, \{{24654\cdot 1 \atop 3}\}, \{{124\cdot 321 \atop 2}\}, \{{\cdot 354321 \atop \cdot}\}, \{{1233321 \atop \cdot}\}, \{{\cdot 34\cdot 321 \atop 2}\}, \{{12432\cdot 1 \atop 2}\}, \{{1233221 \atop 2}, {1232211 \atop 2}, {1233211 \atop 1}, {1232221 \atop 1}\}, \{{0011111 \atop \cdot}\}, \{{\cdot 3432\cdot 1 \atop 2}\}, \{{000\cdot 111 \atop 0}\}$
$\{{2465432 \atop 3}\}, \{{2465431 \atop 3}\}, \{{2465421 \atop 3}\}, \{{2454321 \atop 2}\}, \{{2344321 \atop 2}\}, \{{2465321 \atop 3}\}, \{{2354321 \atop 2}\}, \{{2343321 \atop 2}\}, \{{1344321 \atop 2}\}, \{{1233321 \atop 1}\}, \{{2464321 \atop 3}\}, \{{1354321 \atop 2}\}, \{{2343221 \atop 2}\}, \{{1343321 \atop 2}\}, \{{1244321 \atop 2}\}, \{{2343211 \atop 2}\}, \{{1233321 \atop 2}\}, \{{1222221 \atop 1}\}, \{{1343211 \atop 2}\}, \{{1233221 \atop 2}\}, \{{1233211 \atop 2}\}, \{{1222111 \atop 1}\}, \{{1122211 \atop 1}\}, \{{1111111 \atop 0}\}, \{{1243221 \atop 2}\}, \{{1232221 \atop 2}\}, \{{1232211 \atop 2}\}, \{{0111111 \atop 0}\}, \{{0011111 \atop 0}\}$
$\{{2465432 \atop 3}\}, \{{24654\cdot 1 \atop 3}\}, \{{1\cdot 4\cdot 321 \atop 2}\}, \{{1354321 \atop \cdot}\}, \{{1233321 \atop \cdot}\}, \{{2454321 \atop 2}, {2354321 \atop 3}\}, \{{234\cdot 321 \atop 2}\}, \{{1\cdot 432\cdot 1 \atop 2}\}, \{{1233221 \atop 2}, {1232211 \atop 2}, {1233211 \atop 1}, {1232221 \atop 1}\}, \{{23432\cdot 1 \atop 2}\}, \{{0111111 \atop 1}, {0011111 \atop 0}\}, \{{000\cdot 111 \atop 0}\}$
$\{{2465432 \atop 3}\}, \{{2465431 \atop 3}\}, \{{2465421 \atop 3}\}, \{{234\cdot 321 \atop 2}\}, \{{134\cdot 321 \atop 2}\}, \{{2454321 \atop 2}\}, \{{1233321 \atop 2}\}, \{{1233321 \atop 1}\}, \{{246\cdot 321 \atop 3}\}, \{{124\cdot 321 \atop 2}\}, \{{1354321 \atop 2}\}, \{{1222221 \atop 1}\}, \{{123\cdot 221 \atop 2}\}, \{{123\cdot 211 \atop 2}\}, \{{2343221 \atop 2}\}, \{{2343211 \atop 2}\}, \{{1343211 \atop 2}\}, \{{1122211 \atop 1}\}, \{{1243221 \atop 2}\}, \{{1111111 \atop 0}\}, \{{0111111 \atop 1}\}, \{{0111111 \atop 0}\}, \{{0011111 \atop 0}\}$
$\{{2465432 \atop 3}\}, \{{2454321 \atop 2}\}, \{{24654\cdot 1 \atop 3}\}, \{{\cdot 34\cdot 321 \atop 2}\}, \{{246\cdot 321 \atop 3}\}, \{{\cdot 354321 \atop 2}\}, \{{\cdot 3432\cdot 1 \atop 2}\}, \{{1233211 \atop 1}, {1232221 \atop 1}\}, \{{12222\cdot 1 \atop 1}\}, \{{124\cdot 321 \atop 2}\}, \{{\cdot 111111 \atop 0}\}, \{{1233321 \atop 2}\}, \{{1233221 \atop 2}, {1232211 \atop 2}\}, \{{1232111 \atop 2}\}, \{{0011111 \atop 0}\}$
$\{{2465432 \atop 3}\}, \{{24654\cdot 1 \atop 3}\}, \{{2\cdot 54321 \atop 2}\}, \{{246\cdot 321 \atop 3}\}, \{{234\cdot 321 \atop 2}\}, \{{1354321 \atop 2}\}, \{{1344321 \atop 2}, {1243321 \atop 2}\}, \{{23432\cdot 1 \atop 2}\}, \{{1343321 \atop 2}, {1244321 \atop 2}\}, \{{1233211 \atop 1}, {1232221 \atop 1}\}, \{{1343221 \atop 2}, {1243211 \atop 2}\}, \{{1222211 \atop 1}, {1122221 \atop 1}\}, \{{1233321 \atop 2}\}, \{{1111111 \atop 0}\}, \{{1233221 \atop 2}, {1232211 \atop 2}\}, \{{0\cdot 11111 \atop 0}\}, \{{1232111 \atop 2}\}$
$\{{2465432 \atop 3}\}, \{{2465431 \atop 3}\}, \{{2454321 \atop 2}\}, \{{246\cdot 321 \atop 3}\}, \{{234\cdot 321 \atop 2}\}, \{{2465421 \atop 3}\}, \{{2354321 \atop 2}\}, \{{2343221 \atop 2}\}, \{{134\cdot 321 \atop 2}\}, \{{2343211 \atop 2}\}, \{{1343221 \atop 2}\}, \{{1233321 \atop 1}\}, \{{1222221 \atop 1}\}, \{{124\cdot 321 \atop 2}\}, \{{123\cdot 221 \atop 2}\}, \{{1343211 \atop 2}\}, \{{1243221 \atop 2}\}, \{{1232111 \atop 1}\}, \{{1111111 \atop 0}\}, \{{123\cdot 211 \atop 2}\}, \{{1111111 \atop 1}\}, \{{0111111 \atop 0}\}, \{{0011111 \atop 0}\}$

Table 5: Maximal abelian sets

X	W_X	Sig(X)
X_{29}^{247}	$\langle w_1, w_5\rangle \cong {S_2}^2$	$0\ 9\ 11^{2.2+1}\ 13^{4+2.2+2.1}\ 15^{2.2+2.1}\ 17^{2.2+1}\ 19$
X_{29}^{248}	$\langle w_3w_5w_7\rangle \cong S_2$	$0\ 9^{2+2.1}\ 11^{2.2}\ 13^{4.2+2.1}\ 15^{3.2}\ 17^{2}\ 19^{2}$
X_{29}^{249}	$\langle w_1, w_5\rangle \cong {S_2}^2$	$0\ 7\ 9\ 11^{4+2.2+2.1}\ 13^{3.2+1}\ 15^{2.2+1}\ 17^{2+1}\ 21$
X_{29}^{250}	1	$0\ 9^{2.1}\ 11^{7.1}\ 13^{6.1}\ 15^{8.1}\ 17^{4.1}\ 19$
X_{29}^{251}	$\langle w_3w_5w_7\rangle \cong S_2$	$0\ 9^{2+1}\ 11^{3.2}\ 13^{4.2+1}\ 15^{3.2}\ 17^{2+1}\ 21$
X_{29}^{252}	$\langle w_1\rangle \cong S_2$	$0\ 9\ 11^{2+2.1}\ 13^{3.2+6.1}\ 15^{2+4.1}\ 17^{2+1}\ 19^{2.1}$
X_{29}^{253}	$\langle w_1w_5\rangle \cong S_2$	$0\ 9\ 11^{2.2+2.1}\ 13^{3.2+2.1}\ 15^{2.2+2.1}\ 17^{3.2+1}$
X_{29}^{254}	1	$0\ 9^{2.1}\ 11^{6.1}\ 13^{8.1}\ 15^{8.1}\ 17^{2.1}\ 19^{2.1}$
X_{29}^{255}	1	$0\ 9\ 11^{5.1}\ 13^{10.1}\ 15^{6.1}\ 17^{5.1}\ 19$
X_{29}^{256}	1	$0\ 9\ 11^{5.1}\ 13^{10.1}\ 15^{6.1}\ 17^{5.1}\ 19$

in the E_8 root system (continued)
W_X-orbits on X
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{246\cdot321\\3}\}, \{\substack{\cdot354321\\2}\}, \{\substack{2465421\\3}\}, \{\substack{\cdot34\cdot321\\2}\}, \{\substack{\cdot343221\\2}\},$ $\{\substack{124\cdot321\\2}\}, \{\substack{1243221\\2}\}, \{\substack{1233321\\1}\}, \{\substack{\cdot343211\\2}\}, \{\substack{123\cdot221\\2}\}, \{\substack{1222221\\1}\}, \{\substack{1232111\\1}\},$ $\{\substack{123\cdot211\\2}\}, \{\substack{\cdot111111\\0}\}, \{\substack{0011111\\0}\}, \{\substack{0011111\\1}\}$
$\{\substack{2465432\\3}\}, \{\substack{2\cdot54321\\2}\}, \{\substack{1233321\\1}\}, \{\substack{1111111\\0}\}, \{\substack{234\cdot321\\2}\}, \{\substack{1233211\\1}, \substack{1232221\\1}\}, \{\substack{24654\cdot1\\3}\},$ $\{\substack{23432\cdot1\\2}\}, \{\substack{1222221\\1}, \substack{1122211\\1}\}, \{\substack{0\cdot11111\\0}\}, \{\substack{1354321\\2}\}, \{\substack{1232111\\1}\}, \{\substack{1344321\\2}, \substack{1243321\\2}\},$ $\{\substack{1221111\\1}, \substack{1122111\\1}\}, \{\substack{000\cdot111\\0}\}, \{\substack{1343211\\2}, \substack{1243221\\2}\}, \{\substack{1233221\\2}, \substack{1232211\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{2454321\\2}\}, \{\substack{2465431\\3}\}, \{\substack{\cdot34\cdot321\\2}\}, \{\substack{246\cdot321\\3}\}, \{\substack{\cdot343221\\2}\}, \{\substack{2465421\\3}\},$ $\{\substack{1222221\\1}\}, \{\substack{\cdot354321\\2}\}, \{\substack{\cdot343211\\2}\}, \{\substack{122\cdot111\\1}\}, \{\substack{1233321\\1}\}, \{\substack{123\cdot221\\2}\}, \{\substack{\cdot111111\\0}\},$ $\{\substack{1233321\\2}\}, \{\substack{123\cdot211\\2}\}, \{\substack{1243221\\2}\}, \{\substack{0011111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{2454321\\2}\}, \{\substack{2465421\\3}\}, \{\substack{2465321\\3}\}, \{\substack{2464321\\3}\}, \{\substack{2354321\\2}\},$ $\{\substack{2344321\\2}\}, \{\substack{2343321\\2}\}, \{\substack{2343221\\2}\}, \{\substack{1354321\\2}\}, \{\substack{1343321\\2}\}, \{\substack{2343211\\2}\}, \{\substack{1343221\\2}\},$ $\{\substack{1233321\\1}\}, \{\substack{1222221\\1}\}, \{\substack{1244321\\2}\}, \{\substack{1343211\\2}\}, \{\substack{1243221\\2}\}, \{\substack{1233321\\2}\}, \{\substack{1233221\\2}\},$ $\{\substack{1232221\\2}\}, \{\substack{1222111\\1}\}, \{\substack{1111111\\0}\}, \{\substack{1233211\\2}\}, \{\substack{1232211\\2}\}, \{\substack{1121111\\1}\}, \{\substack{0111111\\0}\},$ $\{\substack{0011111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2\cdot54321\\2}\}, \{\substack{1111111\\0}\}, \{\substack{234\cdot321\\2}\}, \{\substack{1233221\\1}, \substack{1232211\\1}\}, \{\substack{1233211\\1}, \substack{1232221\\1}\},$ $\{\substack{24654\cdot1\\3}\}, \{\substack{23432\cdot1\\2}\}, \{\substack{1222221\\1}, \substack{1122211\\1}\}, \{\substack{0\cdot11111\\0}\}, \{\substack{1354321\\2}\}, \{\substack{1344321\\2}, \substack{1243321\\2}\},$ $\{\substack{1221111\\1}, \substack{1122111\\1}\}, \{\substack{000\cdot111\\0}\}, \{\substack{1343211\\2}, \substack{1243221\\2}\}, \{\substack{1233321\\2}\}, \{\substack{1232111\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{\cdot344321\\2}\}, \{\substack{2465421\\3}\}, \{\substack{2465321\\3}\}, \{\substack{\cdot354321\\2}\}, \{\substack{\cdot343321\\2}\},$ $\{\substack{\cdot343221\\2}\}, \{\substack{2464321\\3}\}, \{\substack{1244321\\2}\}, \{\substack{1233321\\2}\}, \{\substack{1233221\\2}\}, \{\substack{1233321\\1}\}, \{\substack{1222221\\1}\},$ $\{\substack{\cdot343211\\2}\}, \{\substack{1243221\\2}\}, \{\substack{1233211\\2}\}, \{\substack{1232221\\2}\}, \{\substack{1222111\\1}\}, \{\substack{\cdot111111\\0}\}, \{\substack{1232211\\2}\},$ $\{\substack{0011111\\1}\}, \{\substack{0011111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{246\cdot321\\3}\}, \{\substack{\cdot354321\\2}\}, \{\substack{2465421\\3}\}, \{\substack{2454321\\2}\}, \{\substack{2343321\\2}, \substack{1344321\\2}\},$ $\{\substack{\cdot343221\\2}\}, \{\substack{124\cdot321\\2}\}, \{\substack{1243221\\2}\}, \{\substack{1233321\\1}\}, \{\substack{\cdot343211\\2}\}, \{\substack{123\cdot221\\2}\}, \{\substack{1233321\\2}\},$ $\{\substack{1222221\\1}\}, \{\substack{123\cdot211\\2}\}, \{\substack{1122111\\1}, \substack{0121111\\1}\}, \{\substack{\cdot111111\\0}\}, \{\substack{0011111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{2344321\\2}\}, \{\substack{2465421\\3}\}, \{\substack{2465321\\3}\}, \{\substack{2454321\\2}\}, \{\substack{2354321\\2}\},$ $\{\substack{2343321\\2}\}, \{\substack{2343221\\2}\}, \{\substack{2464321\\3}\}, \{\substack{1344321\\2}\}, \{\substack{1244321\\2}\}, \{\substack{2343211\\2}\}, \{\substack{1233321\\2}\},$ $\{\substack{1233221\\2}\}, \{\substack{1233321\\1}\}, \{\substack{1222221\\1}\}, \{\substack{1343221\\2}\}, \{\substack{1243321\\2}\}, \{\substack{1243221\\2}\}, \{\substack{1233211\\2}\},$ $\{\substack{1232221\\2}\}, \{\substack{1122111\\1}\}, \{\substack{1111111\\1}\}, \{\substack{1111111\\0}\}, \{\substack{1343211\\2}\}, \{\substack{1232211\\2}\}, \{\substack{0111111\\0}\},$ $\{\substack{0011111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{2465421\\3}\}, \{\substack{2465321\\3}\}, \{\substack{2454321\\2}\}, \{\substack{2344321\\2}\}, \{\substack{1344321\\2}\},$ $\{\substack{2464321\\3}\}, \{\substack{1354321\\2}\}, \{\substack{2343321\\2}\}, \{\substack{2343221\\2}\}, \{\substack{1244321\\2}\}, \{\substack{1343221\\2}\}, \{\substack{1233321\\2}\},$ $\{\substack{1233221\\2}\}, \{\substack{1233321\\1}\}, \{\substack{1222221\\1}\}, \{\substack{2343211\\2}\}, \{\substack{1243321\\2}\}, \{\substack{1343211\\2}\}, \{\substack{1243221\\2}\},$ $\{\substack{1233211\\2}\}, \{\substack{1232221\\2}\}, \{\substack{1232211\\2}\}, \{\substack{1122111\\1}\}, \{\substack{1111111\\0}\}, \{\substack{0111111\\1}\}, \{\substack{0111111\\0}\},$ $\{\substack{0011111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{2465421\\3}\}, \{\substack{2465321\\3}\}, \{\substack{2354321\\2}\}, \{\substack{2344321\\2}\}, \{\substack{1244321\\2}\},$ $\{\substack{2464321\\3}\}, \{\substack{1354321\\2}\}, \{\substack{2343321\\2}\}, \{\substack{1344321\\2}\}, \{\substack{2343221\\2}\}, \{\substack{1243321\\2}\}, \{\substack{1243221\\2}\},$ $\{\substack{1233321\\2}\}, \{\substack{1233221\\2}\}, \{\substack{1233321\\1}\}, \{\substack{2343211\\2}\}, \{\substack{1343221\\2}\}, \{\substack{1233211\\2}\}, \{\substack{1232221\\2}\},$ $\{\substack{1222221\\1}\}, \{\substack{1122111\\1}\}, \{\substack{1343211\\2}\}, \{\substack{1232211\\2}\}, \{\substack{1111111\\0}\}, \{\substack{0011111\\1}\}, \{\substack{0011111\\0}\},$ $\{\substack{0111111\\0}\}$

Table 5: Maximal abelian sets

X	W_X	Sig(X)
X_{29}^{257}	$\langle w_1, w_2, w_5\rangle \cong {S_2}^3$	0 9 $11^{4+2.2}$ 13^{4+2+1} $15^{4+2.2}$ 17^{2+1} 21
X_{29}^{258}	$\langle w_1, w_2w_6\rangle \cong {S_2}^2$	0 9 $11^{2.2}$ $13^{2.4+2+2.1}$ $15^{3.2}$ 17^{2+1} 19^2
X_{29}^{259}	$\langle w_1, w_6, w_5w_7\rangle \cong S_2 \times Dih_8$	0 11^{4+1} 13^8 $15^{2.4+2}$ $17^{2.2}$ 19
X_{29}^{260}	$\langle w_1, w_5, w_7\rangle \cong {S_2}^3$	0 $11^{2.2}$ $13^{2.4+2}$ $15^{4+2.2+2.1}$ 17^2 $19^{2.1}$
X_{29}^{261}	$\langle w_4, w_3w_5w_7\rangle \cong Dih_8$	0 9 11^{4+2} $13^{2.4}$ 15^{4+2} 17^{4+2+1}
X_{29}^{262}	$\langle w_1, w_2w_5, w_5w_7\rangle \cong {S_2}^3$	0 $11^{4+3.2}$ $15^{3.4+2.2}$ 19^2
X_{29}^{263}	$\langle w_7\rangle \cong S_2$	0 $11^{2+3.1}$ $13^{2.2+4.1}$ $15^{3.2+4.1}$ $17^{4.1}$ 19
X_{29}^{264}	$\langle w_5, w_1w_7\rangle \cong {S_2}^2$	0 9^2 $11^{4+2.1}$ $13^{4.2}$ $15^{4+2.2}$ 17^2 $19^{2.1}$
X_{29}^{265}	$\langle w_5, w^\diamond w_2w_3w_7w_\sigma\rangle \cong {S_2}^2$	0 9^{2+1} 11^{4+2} $13^{4+2.2+1}$ $15^{3.2}$ 17^{2+1} 21
X_{29}^{266}	$\langle w_3w_5, w^\diamond w_2w_3w_7w_\sigma\rangle \cong {S_2}^2$	0 9^2 11^{4+2} $13^{4+2.2}$ $15^{4+2.2}$ 17^2 19^2
X_{29}^{267}	$\langle w_1, w_2w_6\rangle \cong {S_2}^2$	0 9 $11^{4+2.2}$ 13^{4+2+1} $15^{4.2}$ 17^{2+1} 21
X_{29}^{268}	$\langle w_5\rangle \cong S_2$	0 9 $11^{2+2.1}$ $13^{4.2+4.1}$ $15^{2+4.1}$ $17^{3.1}$ $19^{2.1}$
X_{29}^{269}	$\langle w_1, w_4, w_2w_5\rangle \cong S_2 \times Dih_8$	0 9 11^4 13^{8+4} 15^{4+2} 17^{2+1} 19^2
X_{29}^{270}	$\langle w_1, w_5, w_6, w_7, w^\diamond w_2w_3w_\sigma\rangle \cong S_4 \times Dih_8$	0 7 11^{16} 15^{6+4} 23
X_{29}^{271}	$\langle w_6, w_7, w^\diamond w_2w_3w_\sigma\rangle \cong S_3 \times S_2$	0 9 11^{6+2} 13^{6+1} 15^{6+2} 17^{2+1} 21

in the E_8 root system (continued)
W_X-orbits on X
$\{\substack{2465432\\3}\},\{\substack{2465431\\3}\},\{\substack{\cdot34\cdot321\\2}\},\{\substack{2454321\\\cdot}\},\{\substack{1233321\\\cdot}\},\{\substack{\cdot354321\\\cdot}\},\{\substack{124\cdot321\\2}\},\{\substack{1222221\\1}\},\{\substack{123\cdot211\\\cdot}\},\{\substack{\cdot343221\\2}\},\{\substack{\cdot343211\\2}\},\{\substack{000\cdot111\\0}\},\{\substack{1243221\\2}\},\{\substack{0000011\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{2465431\\3}\},\{\substack{2454321\\\cdot}\},\{\substack{\cdot344321\\2}\},\{\substack{\cdot354321\\\cdot}\},\{\substack{\cdot343\cdot21\\2}\},\{\substack{1233221\\2},\substack{1233321\\1}\},\{\substack{1244321\\2}\},\{\substack{1222221\\1}\},\{\substack{\cdot343211\\2}\},\{\substack{1243\cdot21\\2}\},\{\substack{1233211\\\cdot}\},\{\substack{1232211\\2},\substack{1232111\\1}\},\{\substack{0001111\\0}\},\{\substack{0000\cdot11\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{246\cdot\cdot\cdot1\\3}\},\{\substack{2454321\\2}\},\{\substack{\cdot34\cdot\cdot\cdot1\\2}\},\{\substack{124\cdot\cdot\cdot1\\2}\},\{\substack{1233\cdot21\\2},\substack{1232\cdot11\\2}\},\{\substack{1233211\\1},\substack{1232221\\1}\},\{\substack{\cdot111111\\1}\},\{\substack{\cdot111111\\0}\},\{\substack{0011111\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{24654\cdot1\\3}\},\{\substack{246\cdot321\\3}\},\{\substack{\cdot34\cdot321\\2}\},\{\substack{\cdot3432\cdot1\\2}\},\{\substack{\cdot354321\\2}\},\{\substack{123\cdot2\cdot1\\2}\},\{\substack{12432\cdot1\\2}\},\{\substack{122\cdot111\\1}\},\{\substack{1233321\\2}\},\{\substack{1233321\\1}\},\{\substack{\cdot111111\\0}\},\{\substack{0011111\\1}\},\{\substack{0011111\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{1233321\\1}\},\{\substack{2\cdot\cdot\cdot321\\2}\},\{\substack{24654\cdot1\\3}\},\{\substack{13\cdot4321\\2},\substack{12\cdot3321\\2}\},\{\substack{1233211\\1},\substack{12\cdot2221\\1},\substack{1122211\\1}\},\{\substack{0\cdot\cdot\cdot111\\0}\},\{\substack{23432\cdot1\\2}\},\{\substack{1343211\\2},\substack{12\cdot3221\\2},\substack{1232211\\2}\},\{\substack{1221111\\1},\substack{1122111\\1}\},\{\substack{1354321\\3}\}$
$\{\substack{2465432\\3}\},\{\substack{\cdot354321\\\cdot}\},\{\substack{24654\cdot1\\3}\},\{\substack{124\cdot321\\2}\},\{\substack{12432\cdot1\\2}\},\{\substack{\cdot34\cdot321\\2}\},\{\substack{\cdot3432\cdot1\\2}\},\{\substack{1233221\\2},\substack{1232211\\2},\substack{1233211\\1},\substack{1232221\\1}\},\{\substack{1232111\\\cdot}\},\{\substack{0011111\\\cdot}\},\{\substack{000\cdot111\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{24654\cdot1\\3}\},\{\substack{2465321\\3}\},\{\substack{2464321\\3}\},\{\substack{2454321\\2}\},\{\substack{23432\cdot1\\2}\},\{\substack{13432\cdot1\\2}\},\{\substack{2344321\\2}\},\{\substack{1354321\\2}\},\{\substack{2343321\\2}\},\{\substack{1344321\\2}\},\{\substack{12432\cdot1\\2}\},\{\substack{12332\cdot1\\2}\},\{\substack{12322\cdot1\\2}\},\{\substack{1243321\\2}\},\{\substack{1233321\\2}\},\{\substack{1233321\\1}\},\{\substack{1221111\\1}\},\{\substack{1122111\\1}\},\{\substack{1111111\\0}\},\{\substack{0111111\\1}\},\{\substack{0111111\\0}\},\{\substack{0011111\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{24654\cdot1\\3}\},\{\substack{\cdot34\cdot321\\2}\},\{\substack{2454321\\2}\},\{\substack{1233321\\2}\},\{\substack{246\cdot321\\3}\},\{\substack{\cdot354321\\2}\},\{\substack{124\cdot321\\2}\},\{\substack{12222\cdot1\\1}\},\{\substack{123\cdot2\cdot1\\2}\},\{\substack{2343211\\2},\substack{1343221\\2}\},\{\substack{1122221\\1},\substack{0122211\\1}\},\{\substack{\cdot111111\\0}\},\{\substack{1232111\\2}\},\{\substack{0011111\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{2465431\\3},\substack{2343221\\2}\},\{\substack{2454321\\2}\},\{\substack{246\cdot321\\3},\substack{234\cdot321\\2}\},\{\substack{1343221\\2},\substack{1222221\\1}\},\{\substack{134\cdot321\\2},\substack{122\cdot111\\1}\},\{\substack{2465421\\3},\substack{2343211\\2}\},\{\substack{123\cdot221\\2}\},\{\substack{2354321\\2}\},\{\substack{1354321\\2},\substack{1111111\\0}\},\{\substack{1243221\\2},\substack{1122221\\1}\},\{\substack{1233321\\2},\substack{1232111\\2}\},\{\substack{123\cdot211\\2}\},\{\substack{0111111\\0}\},\{\substack{0011111\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{2465431\\3},\substack{2343221\\2}\},\{\substack{246\cdot321\\3},\substack{234\cdot321\\2}\},\{\substack{2\cdot54321\\2}\},\{\substack{1\cdot43221\\2},\substack{1\cdot22221\\1}\},\{\substack{2465421\\3},\substack{2343211\\2}\},\{\substack{123\cdot221\\2}\},\{\substack{1344321\\2},\substack{1243321\\2},\substack{1221111\\1},\substack{1122111\\1}\},\{\substack{1354321\\2},\substack{1111111\\0}\},\{\substack{1233321\\2},\substack{1232111\\2}\},\{\substack{123\cdot211\\2}\},\{\substack{0\cdot11111\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{2465431\\3}\},\{\substack{\cdot343\cdot21\\2}\},\{\substack{2454321\\\cdot}\},\{\substack{1232221\\\cdot}\},\{\substack{\cdot354321\\\cdot}\},\{\substack{1243\cdot21\\2}\},\{\substack{1221111\\1}\},\{\substack{\cdot344321\\2}\},\{\substack{\cdot343211\\2}\},\{\substack{1233221\\2},\substack{1233321\\1}\},\{\substack{1232211\\2},\substack{1232111\\1}\},\{\substack{0000\cdot11\\0}\},\{\substack{1243211\\2}\},\{\substack{0001111\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{2465431\\3}\},\{\substack{234\cdot321\\2}\},\{\substack{2465421\\3}\},\{\substack{1233321\\2}\},\{\substack{246\cdot321\\3}\},\{\substack{134\cdot321\\2}\},\{\substack{124\cdot321\\2}\},\{\substack{123\cdot221\\2}\},\{\substack{2354321\\2}\},\{\substack{2343221\\2}\},\{\substack{1222221\\1}\},\{\substack{1122221\\1}\},\{\substack{123\cdot211\\2}\},\{\substack{1354321\\2}\},\{\substack{2343211\\2}\},\{\substack{1343221\\2}\},\{\substack{1222211\\1}\},\{\substack{1232111\\2}\},\{\substack{1111111\\0}\},\{\substack{0011111\\1}\},\{\substack{0111111\\0}\},\{\substack{0011111\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{2465431\\3}\},\{\substack{12\cdot\cdot321\\\cdot}\},\{\substack{\cdot3\cdot\cdot321\\\cdot}\},\{\substack{12\cdot3221\\2},\substack{12\cdot2221\\1}\},\{\substack{00\cdot\cdot111\\\cdot}\},\{\substack{\cdot343221\\2}\},\{\substack{1232211\\2},\substack{1233211\\1}\},\{\substack{0000011\\0}\},\{\substack{\cdot343211\\2}\}$
$\{\substack{2465432\\3}\},\{\substack{2454321\\2}\},\{\substack{246\cdot\cdot\cdot1\\3},\substack{\cdot34\cdot\cdot\cdot1\\2},\substack{122\cdot\cdot\cdot1\\1}\},\{\substack{123\cdot\cdot\cdot1\\2}\},\{\substack{\cdot354321\\2},\substack{\cdot111111\\0}\},\{\substack{0011111\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{2454321\\2}\},\{\substack{2465\cdot\cdot1\\3},\substack{2343\cdot\cdot1\\2}\},\{\substack{2464321\\3},\substack{2344321\\2}\},\{\substack{1343\cdot\cdot1\\2},\substack{1222\cdot\cdot1\\1}\},\{\substack{2354321\\2}\},\{\substack{123\cdot\cdot\cdot1\\2}\},\{\substack{1354321\\2},\substack{1111111\\0}\},\{\substack{1244321\\2},\substack{1121111\\1}\},\{\substack{0111111\\0}\},\{\substack{0011111\\0}\}$

Table 5: Maximal abelian sets

X	W_X	Sig(X)
X_{29}^{272}	$\langle w_4, w_5, w_1w_7\rangle \cong S_3 \times S_2$	$0\ 9^2\ 11^{6+3+1}\ 13^{6+2}\ 15^{3+2}\ 17^2\ 23$
X_{29}^{273}	$\langle w_1, w_6, w_7\rangle \cong S_3 \times S_2$	$0\ 11^{3+2}\ 13^{6+2.3+1}\ 15^{3+2+1}\ 17^2\ 19\ 21$
X_{29}^{274}	$\langle w_6, w_3w_5w_7, w^{\diamond}w_2w_3w_{\sigma}\rangle \cong S_2 \times Dih_8$	$0\ 11^{8+2}\ 15^{8+4+2.2}\ 19^2$
X_{29}^{275}	$\langle w_7, w_1w_5\rangle \cong {S_2}^2$	$0\ 11^{2.2+1}\ 13^{4+2.2}\ 15^{4+2.2+2.1}\ 17^{2.2}\ 19$
X_{29}^{276}	$\langle w_7\rangle \cong S_2$	$0\ 9\ 11^{2.2+3.1}\ 13^{2.2+5.1}\ 15^{2.2+4.1}\ 17\ 19\ 21$
X_{29}^{277}	$\langle w_5, w_1w_2\rangle \cong {S_2}^2$	$0\ 11^{3.2}\ 13^{2.4+2+1}\ 15^{3.2}\ 17^{2+2.1}\ 21$
X_{29}^{278}	$\langle w_7\rangle \cong S_2$	$0\ 11^{2+2.1}\ 13^{2.2+6.1}\ 15^{3.2+4.1}\ 17^{2.1}\ 19^{2.1}$
X_{29}^{279}	$\langle w_5w_7, w_1w_2w_5\rangle \cong {S_2}^2$	$0\ 11^{2.2}\ 13^{2.4+2}\ 15^{4+3.2}\ 17^2\ 19^2$
X_{29}^{280}	$\langle w_1, w_2, w_4, w_1w_6{w_{\tau}}^{w_3w_4w_5w_4w_3}\rangle \cong S_3 \times Dih_8$	$0\ 11^{6+4}\ 13^{12}\ 15\ 17^4\ 23$
X_{29}^{281}	$\langle w_5, w_7\rangle \cong {S_2}^2$	$0\ 11^{2.2+1}\ 13^{4+4.2+1}\ 15^{2.2+2.1}\ 17^{2.1}\ 19\ 21$
X_{29}^{282}	$\langle w_1, w_2w_7, w_4w_6\rangle \cong S_3 \times S_2$	$0\ 11^3\ 13^{2.6}\ 15^{6+3+1}\ 19^3$
X_{29}^{283}	$\langle w_1, w_4w_7\rangle \cong {S_2}^2$	$0\ 7\ 11^{4+2+2.1}\ 13^{4+4.2}\ 15\ 17^{2.2}\ 19^2$
X_{29}^{284}	$\langle w_3w_5, w_3w_5{w_7}^{w_4}\rangle \cong {S_2}^2$	$0\ 9\ 11^{4+2}\ 13^{4+2.2}\ 15^{4+2}\ 17^{4+2+1}$
X_{29}^{285}	$\langle w_1w_4w_7\rangle \cong S_2$	$0\ 11^{2.2+2.1}\ 13^{3.2}\ 15^{4.2+2.1}\ 17^{3.2}$

in the E_8 root system (continued)
W_X-orbits on X
$\{{2465432 \atop 3}\}, \{{\cdot111111 \atop 0}\}, \{{12\cdot\cdot2\cdot1 \atop 1}\}, \{{00\cdot\cdot111 \atop 0}\}, \{{2454321 \atop 2}\}, \{{\cdot3\cdot\cdot321 \atop 2}\}, \{{1122221 \atop 1}, {0122211 \atop 1}\}, \{{12\cdot\cdot321 \atop 2}\}, \{{00000\cdot1 \atop 0}\}, \{{2343211 \atop 2}, {1343221 \atop 2}\}, \{{1232111 \atop 2}\}$
$\{{2465432 \atop 3}\}, \{{2465\cdot\cdot1 \atop 3}\}, \{{\cdot344321 \atop 2}\}, \{{\cdot343\cdot\cdot1 \atop 2}\}, \{{1233\cdot\cdot1 \atop 2}\}, \{{1222\cdot\cdot1 \atop 1}\}, \{{2464321 \atop 3}\}, \{{1232\cdot\cdot1 \atop 2}\}, \{{\cdot354321 \atop 2}\}, \{{1244321 \atop 2}\}, \{{\cdot111111 \atop 0}\}, \{{0011111 \atop 1}\}, \{{0011111 \atop 0}\}$
$\{{2465432 \atop 3}\}, \{{246\cdot\cdot\cdot1 \atop 3}, {234\cdot\cdot\cdot1 \atop 2}\}, \{{2\cdot54321 \atop 2}\}, \{{1344321 \atop 2}, {1243\cdot21 \atop 2}, {1343211 \atop 2}, {1222221 \atop 1}, {1122\cdot11 \atop 1}, {1221111 \atop 1}\}, \{{1233\cdot21 \atop 2}, {1232\cdot11 \atop 2}\}, \{{1354321 \atop 2}, {1111111 \atop 0}\}, \{{1233211 \atop 2}, {1232221 \atop 2}\}, \{{0\cdot11111 \atop 0}\}$
$\{{2465432 \atop 3}\}, \{{24654\cdot1 \atop 3}\}, \{{246\cdot321 \atop 3}\}, \{{2454321 \atop 2}\}, \{{\cdot3432\cdot1 \atop 2}\}, \{{\cdot354321 \atop 2}\}, \{{2343321 \atop 2}, {1344321 \atop 2}\}, \{{123\cdot2\cdot1 \atop 2}\}, \{{124\cdot321 \atop 2}\}, \{{12222\cdot1 \atop 1}\}, \{{1233321 \atop 2}\}, \{{1232111 \atop 2}\}, \{{1122111 \atop 1}, {0121111 \atop 1}\}, \{{\cdot111111 \atop 0}\}, \{{0011111 \atop 0}\}$
$\{{2465432 \atop 3}\}, \{{2344321 \atop 2}\}, \{{24654\cdot1 \atop 3}\}, \{{23432\cdot1 \atop 2}\}, \{{2465321 \atop 3}\}, \{{2454321 \atop 2}\}, \{{2343321 \atop 2}\}, \{{12332\cdot1 \atop 2}\}, \{{12222\cdot1 \atop 1}\}, \{{2464321 \atop 3}\}, \{{2354321 \atop 2}\}, \{{1344321 \atop 2}\}, \{{1233321 \atop 2}\}, \{{1111111 \atop 1}\}, \{{13432\cdot1 \atop 2}\}, \{{12322\cdot1 \atop 2}\}, \{{1244321 \atop 2}\}, \{{1232111 \atop 2}\}, \{{1122111 \atop 1}\}, \{{1111111 \atop 0}\}, \{{1243321 \atop 2}\}, \{{0111111 \atop 0}\}, \{{0011111 \atop 0}\}$
$\{{2465432 \atop 3}\}, \{{246\cdot321 \atop 3}\}, \{{2454321 \atop \cdot}\}, \{{\cdot343221 \atop 2}\}, \{{\cdot34\cdot321 \atop 2}\}, \{{123\cdot221 \atop \cdot}\}, \{{2354321 \atop 3}, {1354321 \atop 2}\}, \{{1243221 \atop 2}\}, \{{124\cdot321 \atop 2}\}, \{{\cdot343211 \atop 2}\}, \{{1233321 \atop \cdot}\}, \{{1111111 \atop 0}, {0111111 \atop 1}\}, \{{1243211 \atop 2}\}, \{{0000011 \atop 0}\}, \{{0000001 \atop 0}\}$
$\{{2465432 \atop 3}\}, \{{24654\cdot1 \atop 3}\}, \{{2465321 \atop 3}\}, \{{2344321 \atop 2}\}, \{{23432\cdot1 \atop 2}\}, \{{12332\cdot1 \atop 2}\}, \{{2464321 \atop 3}\}, \{{2354321 \atop 2}\}, \{{2343321 \atop 2}\}, \{{1344321 \atop 2}\}, \{{1244321 \atop 2}\}, \{{1233321 \atop 2}\}, \{{13432\cdot1 \atop 2}\}, \{{12322\cdot1 \atop 2}\}, \{{12222\cdot1 \atop 1}\}, \{{1354321 \atop 2}\}, \{{1243321 \atop 2}\}, \{{1232111 \atop 2}\}, \{{1122111 \atop 1}\}, \{{1111111 \atop 0}\}, \{{0011111 \atop 1}\}, \{{0111111 \atop 0}\}, \{{0011111 \atop 0}\}$
$\{{2465432 \atop 3}\}, \{{24654\cdot1 \atop 3}\}, \{{2454321 \atop \cdot}\}, \{{\cdot34\cdot321 \atop 2}\}, \{{\cdot3432\cdot1 \atop 2}\}, \{{2354321 \atop 2}, {1354321 \atop 3}\}, \{{1233211 \atop 2}, {1232221 \atop 2}, {1233221 \atop 1}, {1232211 \atop 1}\}, \{{124\cdot321 \atop 2}\}, \{{12432\cdot1 \atop 2}\}, \{{1232111 \atop \cdot}\}, \{{1111111 \atop 1}, {0111111 \atop 0}\}, \{{000\cdot111 \atop 0}\}$
$\{{2465432 \atop 3}\}, \{{24\cdot4321 \atop \cdot}, {12\cdot2221 \atop \cdot}\}, \{{\cdot343\cdot21 \atop 2}\}, \{{\cdot3\cdot4321 \atop \cdot}, {12\cdot3\cdot21 \atop \cdot}\}, \{{1244321 \atop 2}\}, \{{\cdot343211 \atop 2}, {0000\cdot11 \atop 0}\}, \{{0000001 \atop 0}\}$
$\{{2465432 \atop 3}\}, \{{24654\cdot1 \atop 3}\}, \{{234\cdot321 \atop 2}\}, \{{1233321 \atop 2}\}, \{{123\cdot2\cdot1 \atop 2}\}, \{{246\cdot321 \atop 3}\}, \{{134\cdot321 \atop 2}\}, \{{23432\cdot1 \atop 2}\}, \{{12222\cdot1 \atop 1}\}, \{{1111111 \atop 1}\}, \{{124\cdot321 \atop 2}\}, \{{13432\cdot1 \atop 2}\}, \{{2354321 \atop 2}\}, \{{1232111 \atop 2}\}, \{{1111111 \atop 0}\}, \{{0011111 \atop 1}\}, \{{0111111 \atop 0}\}, \{{0011111 \atop 0}\}$
$\{{2465432 \atop 3}\}, \{{24\cdot4321 \atop \cdot}\}, \{{\cdot3\cdot4321 \atop \cdot}\}, \{{\cdot343\cdot\cdot1 \atop 2}\}, \{{12432\cdot1 \atop 2}, {1233321 \atop \cdot}, {1233211 \atop 2}, {1233221 \atop 1}\}, \{{1232221 \atop 2}, {1232211 \atop 1}, {1222111 \atop 1}\}, \{{1244321 \atop 2}\}, \{{0000\cdot\cdot1 \atop 0}\}$
$\{{2465432 \atop 3}\}, \{{2454321 \atop 2}\}, \{{\cdot3\cdot4321 \atop 2}\}, \{{24654\cdot1 \atop 3}\}, \{{2465321 \atop 3}\}, \{{1233321 \atop 1}\}, \{{\cdot3432\cdot1 \atop 2}\}, \{{\cdot343321 \atop 2}\}, \{{1232221 \atop 1}, {1222211 \atop 1}\}, \{{12\cdot2111 \atop 1}\}, \{{\cdot111111 \atop 0}\}, \{{1244321 \atop 2}\}, \{{1243211 \atop 2}, {1233221 \atop 2}\}, \{{00\cdot1111 \atop 0}\}, \{{12322\cdot1 \atop 2}\}$
$\{{2465432 \atop 3}\}, \{{1233321 \atop 1}\}, \{{2\cdot\cdot\cdot321 \atop 2}\}, \{{24654\cdot1 \atop 3}\}, \{{123\cdot221 \atop 1}, {1\cdot22211 \atop 1}\}, \{{1354321 \atop 2}, {1233321 \atop 2}\}, \{{1344321 \atop 2}, {1243321 \atop 2}\}, \{{0\cdot\cdot\cdot111 \atop 0}\}, \{{23432\cdot1 \atop 2}\}, \{{1\cdot43211 \atop 2}, {123\cdot221 \atop 2}\}, \{{1221111 \atop 1}, {1122111 \atop 1}\}, \{{1354321 \atop 3}\}$
$\{{2465432 \atop 3}\}, \{{24654\cdot1 \atop 3}\}, \{{\cdot343321 \atop 2}\}, \{{2454321 \atop 2}\}, \{{1233321 \atop 1}\}, \{{2354321 \atop 2}, {1344321 \atop 2}\}, \{{12\cdot3321 \atop 2}\}, \{{1232221 \atop 1}, {1222211 \atop 1}\}, \{{2343221 \atop 2}, {1343211 \atop 2}\}, \{{2343211 \atop 2}, {1343221 \atop 2}\}, \{{12322\cdot1 \atop 2}\}, \{{\cdot111111 \atop 0}\}, \{{2465321 \atop 3}\}, \{{1244321 \atop 2}\}, \{{1243211 \atop 2}, {1233221 \atop 2}\}, \{{1111111 \atop 1}, {0121111 \atop 1}\}, \{{00\cdot1111 \atop 0}\}$

Table 5: Maximal abelian sets

X	W_X	Sig(X)
X_{29}^{286}	$\langle w_5w_7, w_1w_2w_5\rangle \cong {S_2}^2$	0 $11^{3.2}$ 13^{4+2} $15^{4+3.2}$ 17^{4+2}
X_{29}^{287}	$\langle w_2, w_\tau, w_3w_5, w_1w_4{w_6}^{w_3w_5w_4w_3w_1}\rangle \cong S_3 \times {S_2}^2$	0 7 11^{6+2} 13^{12} 15 17^4 19^2
X_{29}^{288}	$\langle w_2w_7, w_1w_5w_7\rangle \cong {S_2}^2$	0 9^2 $11^{3.2}$ $13^{4+2.2}$ $15^{2.4}$ 17^2 19^2
X_{29}^{289}	$\langle w_1, w_5w_7\rangle \cong {S_2}^2$	0 $11^{2.1}$ $13^{4+3.2}$ $15^{4+2.2+2.1}$ $17^{3.2}$
X_{29}^{290}	$\langle w_1, w_5w_7\rangle \cong {S_2}^2$	0 11 $13^{4+4.2}$ $15^{4+2.2+2.1}$ $17^{2.2}$ 19
X_{29}^{291}	$\langle w_3w_5w_7\rangle \cong S_2$	0 11^2 $13^{4.2+2.1}$ $15^{5.2}$ $17^{2.2+2.1}$
X_{29}^{292}	$\langle w_1, w_2, w_4, w_1w_6{w_\tau}^{w_3w_4w_5w_4w_3}\rangle \cong S_3 \times Dih_8$	0 11 13^{12} 15^{6+4} 17^4 19
X_{29}^{293}	$\langle w_1w_2w_6, w_3{w_5}^{w_4w_3}, w_3w_5w_7{w_\sigma}^{w_1w_3w_6w_7}\rangle = [2^5]$	0 11^{8+2} 15^{16} 19^2
X_{29}^{294}	$\langle w_3w_7\rangle \cong S_2$	0 $11^{2.2+1}$ $13^{3.2+2.1}$ $15^{4.2+2.1}$ $17^{2+2.1}$ 19
X_{29}^{295}	$\langle w_3w_7\rangle \cong S_2$	0 9 $11^{2.2+1}$ $13^{4.2+2.1}$ $15^{2.2+2.1}$ $17^{2.2+1}$ 19
X_{29}^{296}	$\langle w_4, w_1w_6, w_1w_6{w_\tau}^{w_3w_4w_5w_4w_3}\rangle \cong {S_2}^3$	0 11 13^{8+4} $15^{4+3.2}$ 17^4 19
X_{29}^{297}	$\langle w_1w_4w_6\rangle \cong S_2$	0 $11^{2.2+1}$ $13^{4.2}$ $15^{4.2+2.1}$ $17^{2.2}$ 19
X_{29}^{298}	$\langle w_2w_7, w_3w_5w_7\rangle \cong {S_2}^2$	0 $11^{2.2}$ $13^{2.4+2}$ $15^{4+3.2}$ 17^2 19^2

in the E_8 root system (continued)
W_X-orbits on X
$\{\substack{2465432\\3}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{124\cdot321\\2}\}, \{\substack{1233321\\\cdot}\}, \{\substack{\cdot34\cdot321\\2}\}, \{\substack{2354321\\2}, \substack{1354321\\3}\}, \{\substack{1233221\\2}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{1232221\\1}\}, \{\substack{12432\cdot1\\2}\}, \{\substack{0011111\\\cdot}\}, \{\substack{000\cdot111\\0}\}, \{\substack{\cdot3432\cdot1\\2}\}, \{\substack{1111111\\1}, \substack{0111111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{0000001\\0}\}, \{\substack{2343211\\2}, \substack{1232111\\\cdot}, \substack{1111111\\\cdot}, \substack{0000011\\0}\}, \{\substack{1221111\\1}, \substack{1122111\\1}\}, \{\substack{1\cdot43211\\2}, \substack{123\cdot211\\\cdot}, \substack{0\cdot11111\\\cdot}, \substack{000\cdot111\\0}\}, \{\substack{2343221\\2}\}, \{\substack{234\cdot321\\2}, \substack{1\cdot43221\\2}\}, \{\substack{1344321\\2}, \substack{1243321\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{12432\cdot1\\2}\}, \{\substack{1232111\\\cdot}\}, \{\substack{1121111\\1}, \substack{0122111\\1}\}, \{\substack{0011111\\\cdot}\}, \{\substack{\cdot354321\\\cdot}\}, \{\substack{124\cdot321\\2}\}, \{\substack{00000\cdot1\\0}\}, \{\substack{\cdot3432\cdot1\\2}\}, \{\substack{1233221\\2}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{1232221\\1}\}, \{\substack{000\cdot111\\0}\}, \{\substack{2344321\\2}, \substack{1343321\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{2454321\\2}\}, \{\substack{1233321\\1}\}, \{\substack{\cdot34\cdot321\\2}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{1233211\\1}, \substack{1232221\\1}\}, \{\substack{\cdot111111\\0}\}, \{\substack{\cdot3432\cdot1\\2}\}, \{\substack{124\cdot321\\2}\}, \{\substack{000\cdot111\\0}\}, \{\substack{1232111\\1}\}, \{\substack{0011111\\0}\}, \{\substack{12432\cdot1\\2}\}, \{\substack{1233221\\2}, \substack{1232211\\2}\}, \{\substack{\cdot111111\\1}\}$
$\{\substack{2465432\\3}\}, \{\substack{1233321\\1}\}, \{\substack{\cdot34\cdot321\\2}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{\cdot354321\\2}\}, \{\substack{1233221\\1}, \substack{1232211\\1}\}, \{\substack{\cdot111111\\0}\}, \{\substack{\cdot3432\cdot1\\2}\}, \{\substack{122\cdot111\\1}\}, \{\substack{000\cdot111\\0}\}, \{\substack{1233321\\2}\}, \{\substack{0011111\\0}\}, \{\substack{12432\cdot1\\2}\}, \{\substack{1233211\\2}, \substack{1232221\\2}\}, \{\substack{0011111\\1}\}$
$\{\substack{2465432\\3}\}, \{\substack{2\cdot54321\\2}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{234\cdot321\\2}\}, \{\substack{23432\cdot1\\2}\}, \{\substack{1233211\\1}, \substack{1232221\\1}\}, \{\substack{1354321\\2}\}, \{\substack{1232111\\1}\}, \{\substack{1344321\\2}, \substack{1243321\\2}\}, \{\substack{1343211\\2}, \substack{1243221\\2}\}, \{\substack{1222221\\1}, \substack{1122211\\1}\}, \{\substack{1221111\\1}, \substack{1122111\\1}\}, \{\substack{0\cdot11111\\0}\}, \{\substack{1233221\\2}, \substack{1232211\\2}\}, \{\substack{000\cdot111\\0}\}, \{\substack{1354321\\3}\}, \{\substack{1232111\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{\cdot3\cdot4321\\\cdot}, \substack{12\cdot3\cdot21\\\cdot}\}, \{\substack{12\cdot3211\\\cdot}, \substack{00\cdot1111\\\cdot}\}, \{\substack{\cdot343\cdot21\\2}\}, \{\substack{\cdot343211\\2}, \substack{0000\cdot11\\0}\}, \{\substack{1221111\\1}\}$
$\{\substack{2465432\\3}\}, \{\substack{24\cdot4321\\\cdot}, \substack{2343321\\2}, \substack{1343221\\2}, \substack{12\cdot2221\\\cdot}\}, \{\substack{2343221\\2}, \substack{1343321\\2}\}, \{\substack{2354321\\2}, \substack{1354321\\3}, \substack{\cdot344321\\2}, \substack{\cdot343211\\2}, \substack{1243\cdot21\\2}, \substack{1233221\\2}, \substack{1233321\\1}, \substack{1232211\\2}, \substack{1232111\\1}, \substack{1111111\\1}, \substack{0111111\\0}, \substack{0000\cdot11\\0}\}, \{\substack{1244321\\2}, \substack{0000001\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{2\cdot54321\\2}\}, \{\substack{1354321\\2}\}, \{\substack{1\cdot44321\\2}\}, \{\substack{1\cdot43321\\2}\}, \{\substack{12322\cdot1\\1}\}, \{\substack{2465321\\3}\}, \{\substack{2344321\\2}\}, \{\substack{23432\cdot1\\2}\}, \{\substack{1343221\\2}, \substack{1243211\\2}\}, \{\substack{1222211\\1}, \substack{1122221\\1}\}, \{\substack{0\cdot11111\\0}\}, \{\substack{1233321\\2}\}, \{\substack{1111111\\0}\}, \{\substack{12322\cdot1\\2}\}, \{\substack{0122111\\1}\}, \{\substack{0001111\\0}\}, \{\substack{1232111\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{2343321\\2}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{2\cdot54321\\2}\}, \{\substack{2344321\\2}\}, \{\substack{23432\cdot1\\2}\}, \{\substack{1\cdot43321\\2}\}, \{\substack{12322\cdot1\\1}\}, \{\substack{1222211\\1}, \substack{1122221\\1}\}, \{\substack{1233321\\2}\}, \{\substack{1111111\\0}\}, \{\substack{1\cdot44321\\2}\}, \{\substack{12322\cdot1\\2}\}, \{\substack{2465321\\3}\}, \{\substack{1111111\\1}\}, \{\substack{1343221\\2}, \substack{1243211\\2}\}, \{\substack{0\cdot11111\\0}\}, \{\substack{0001111\\0}\}, \{\substack{1232111\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{\cdot3\cdot4321\\2}, \substack{12\cdot3\cdot21\\2}\}, \{\substack{\cdot354321\\3}, \substack{1233\cdot21\\1}\}, \{\substack{12\cdot3211\\2}, \substack{00\cdot1111\\0}\}, \{\substack{2454321\\2}, \substack{1232221\\2}\}, \{\substack{2343321\\2}, \substack{1343221\\2}\}, \{\substack{1122111\\1}, \substack{0122211\\1}\}, \{\substack{\cdot343211\\2}, \substack{0000\cdot11\\0}\}, \{\substack{1221111\\1}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465\cdot21\\3}\}, \{\substack{\cdot343211\\2}\}, \{\substack{2454321\\2}\}, \{\substack{2354321\\2}, \substack{1344321\\2}\}, \{\substack{2344321\\2}, \substack{1354321\\2}\}, \{\substack{12\cdot3211\\2}\}, \{\substack{1232211\\1}, \substack{1222111\\1}\}, \{\substack{2343321\\2}, \substack{1343221\\2}\}, \{\substack{1232\cdot11\\2}\}, \{\substack{1122111\\1}, \substack{0122211\\1}\}, \{\substack{\cdot111111\\0}\}, \{\substack{2465431\\3}\}, \{\substack{1221111\\1}\}, \{\substack{1243221\\2}, \substack{1233321\\2}\}, \{\substack{00\cdot1111\\0}\}, \{\substack{1232221\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{23432\cdot1\\2}\}, \{\substack{1232111\\\cdot}\}, \{\substack{2\cdot54321\\\cdot}\}, \{\substack{1\cdot432\cdot1\\2}\}, \{\substack{1221111\\1}, \substack{1122111\\1}\}, \{\substack{1233221\\2}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{1232221\\1}\}, \{\substack{1354321\\\cdot}\}, \{\substack{234\cdot321\\2}\}, \{\substack{00000\cdot1\\0}\}, \{\substack{1344321\\2}, \substack{1243321\\2}\}, \{\substack{000\cdot111\\0}\}$

Table 5: Maximal abelian sets

X	W_X	Sig(X)
X_{29}^{299}	$\langle w_3w_5w_7\rangle \cong S_2$	$0\ 11^{2.1}\ 13^{5.2}\ 15^{4.2+2.1}\ 17^{3.2}$
X_{29}^{300}	$\langle w_1w_2w_6, w_3w_5w_7{w_\sigma}^{w_1w_3w_6w_7}\rangle \cong {S_2}^2$	$0\ 11^{2}\ 13^{4+3.2}\ 15^{2.4+2}\ 17^{4+2}$
X_{29}^{301}	$\langle w_3, w_2w_5, w_5w_7\rangle \cong {S_2}^3$	$0\ 11^{2}\ 13^{2.4+2}\ 15^{2.4+2}\ 17^{3.2}$
X_{29}^{302}	$\langle w_2, w_3w_5, w_3w_5{w_7}^{w_4}\rangle \cong S_2 \times Dih_8$	$0\ 13^{8+4+2}\ 15^{8+2}\ 17^{2}\ 19^{2}$
X_{29}^{303}	$\langle w_3w_5w_7, w_3{w_5}^{w_4}\rangle \cong {S_2}^2$	$0\ 11^{4+1}\ 13^{4+2.2}\ 15^{4+3.2}\ 17^{4}\ 19$
X_{29}^{304}	$\langle w_5w_7, w_1w_2w_5\rangle \cong {S_2}^2$	$0\ 13^{4.2}\ 15^{3.4+2.2}\ 17^{2.2}$
X_{29}^{305}	$\langle w_1w_5w_7\rangle \cong S_2$	$0\ 11\ 13^{6.2}\ 15^{4.2+2.1}\ 17^{2.2}\ 19$
X_{29}^{306}	$\langle w_2w_7, w_3w_5w_7\rangle \cong {S_2}^2$	$0\ 13^{4+2.2}\ 15^{2.4+4.2}\ 17^{2.2}$
X_{29}^{307}	$\langle w_1w_4w_6, w_2w_3w_5, w_2w_7w_\sigma\rangle = [2^5 3]$	$0\ 15^{24+4}$
X_{29}^{308}	$\langle w_1, w_\sigma, w_2w_5, w_4w_6, w_5w_7\rangle \cong S_3 \times Alt_5$	$0\ 15^{18+10}$
X_{29}^{309}	$\langle w_1, w_2, w_3, w_6\rangle \cong S_3 \times {S_2}^2$	$0\ 4\ 8^{2}\ 10^{6+3}\ 12^{6+4}\ 14^{2}\ 16^{3}\ 20$
X_{29}^{310}	$\langle w_1, w_3, w_2w_7\rangle \cong S_3 \times S_2$	$0\ 6^{2}\ 8\ 10^{6+3}\ 12^{3+2}\ 14^{6+2}\ 16^{2}\ 20$
X_{29}^{311}	$\langle w_3, w_6\rangle \cong {S_2}^2$	$0\ 6\ 8^{2+1}\ 10^{2.2+1}\ 12^{4+2.2}$ $14^{3.2+2.1}\ 16\ 18\ 20$
X_{29}^{312}	$\langle w_3, w_5\rangle \cong {S_2}^2$	$0\ 6\ 8^{2+1}\ 10^{2.2+1}\ 12^{4+2.2+1}$ $14^{2.2+1}\ 16^{2+2.1}\ 20$
X_{29}^{313}	$\langle w_1, w_2, w_3, w_4, w_7\rangle \cong S_5 \times S_2$	$0\ 6^{2}\ 10^{10}\ 12^{5}\ 14^{10}\ 22$
X_{29}^{314}	$\langle w_1, w_3, w_2w_6\rangle \cong S_3 \times S_2$	$0\ 6\ 8^{2}\ 10^{6}\ 12^{6+3}\ 14^{3+2.2}\ 16^{2}\ 22$

in the E_8 root system (continued)
W_X-orbits on X
$\{\substack{2465432\\3}\}, \{\substack{1354321\\2}\}, \{\substack{1233321\\1}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{2\cdot54321\\2}\}, \{\substack{1344321\\2}, \substack{1243321\\2}\}, \{\substack{1233211\\1}, \substack{1232221\\1}\}, \{\substack{0\cdot11111\\0}\}, \{\substack{1343211\\2}, \substack{1243221\\2}\}, \{\substack{1222221\\1}, \substack{1122211\\1}\}, \{\substack{012\cdot111\\1}\}, \{\substack{000\cdot111\\0}\}, \{\substack{1354321\\3}\}, \{\substack{1233321\\2}\}, \{\substack{23432\cdot1\\2}\}, \{\substack{1233221\\2}, \substack{1232211\\2}\}, \{\substack{1221111\\1}, \substack{1122111\\1}\}$
$\{\substack{2465432\\3}\}, \{\substack{2454321\\\cdot}\}, \{\substack{\cdot344321\\2}, \substack{\cdot343211\\2}\}, \{\substack{2465431\\3}, \substack{1221111\\1}\}, \{\substack{2343321\\2}, \substack{1343221\\2}\}, \{\substack{2343221\\2}, \substack{1343321\\2}\}, \{\substack{2354321\\2}, \substack{1354321\\3}, \substack{1111111\\1}, \substack{0111111\\0}\}, \{\substack{1233221\\2}, \substack{1233321\\1}, \substack{1232211\\2}, \substack{1232111\\1}\}, \{\substack{1233211\\\cdot}\}, \{\substack{1243\cdot21\\2}, \substack{0000\cdot11\\0}\}, \{\substack{1243211\\2}, \substack{0001111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{1354321\\\cdot}\}, \{\substack{1\cdot4\cdot321\\2}\}, \{\substack{1\cdot432\cdot1\\2}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{1233221\\2}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{1232221\\1}\}, \{\substack{0\cdot11111\\\cdot}\}, \{\substack{1232111\\\cdot}\}, \{\substack{234\cdot321\\2}\}, \{\substack{23432\cdot1\\2}\}, \{\substack{000\cdot111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2\cdot\cdot\cdot321\\\cdot}\}, \{\substack{1232111\\\cdot}, \substack{1111111\\\cdot}\}, \{\substack{1221111\\1}, \substack{1122111\\1}\}, \{\substack{1\cdot43221\\2}, \substack{123\cdot211\\\cdot}, \substack{1\cdot22221\\1}\}, \{\substack{00000\cdot1\\0}\}, \{\substack{1344321\\2}, \substack{1243321\\2}\}, \{\substack{01222\cdot1\\1}\}$
$\{\substack{2465432\\3}\}, \{\substack{2\cdot\cdot\cdot321\\2}\}, \{\substack{1111111\\0}\}, \{\substack{1233221\\1}, \substack{1232211\\1}, \substack{1222221\\1}, \substack{1122211\\1}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{23432\cdot1\\2}\}, \{\substack{0\cdot\cdot\cdot111\\0}\}, \{\substack{1344321\\2}, \substack{1243321\\2}\}, \{\substack{1343321\\2}, \substack{1244321\\2}\}, \{\substack{1232111\\1}, \substack{1111111\\1}\}, \{\substack{1343211\\2}, \substack{1243221\\2}, \substack{1233211\\2}, \substack{1232221\\2}\}, \{\substack{1232111\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{2354321\\2}, \substack{1354321\\3}\}, \{\substack{124\cdot321\\2}\}, \{\substack{12432\cdot1\\2}\}, \{\substack{\cdot34\cdot321\\2}\}, \{\substack{\cdot3432\cdot1\\2}\}, \{\substack{1233221\\2}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{1232221\\1}\}, \{\substack{1232111\\\cdot}\}, \{\substack{0011111\\\cdot}\}, \{\substack{1111111\\1}, \substack{0111111\\0}\}, \{\substack{000\cdot111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2454321\\2}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{\cdot354321\\2}\}, \{\substack{2344321\\2}, \substack{1343321\\2}\}, \{\substack{2343321\\2}, \substack{1344321\\2}\}, \{\substack{1233221\\1}, \substack{1232211\\1}\}, \{\substack{\cdot111111\\0}\}, \{\substack{2343221\\2}, \substack{1343211\\2}\}, \{\substack{122\cdot111\\1}\}, \{\substack{1122211\\1}, \substack{0122221\\1}\}, \{\substack{000\cdot111\\0}\}, \{\substack{1233321\\2}\}, \{\substack{0011111\\0}\}, \{\substack{12432\cdot1\\2}\}, \{\substack{1233211\\2}, \substack{1232221\\2}\}, \{\substack{1232111\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{2\cdot54321\\\cdot}\}, \{\substack{234\cdot321\\2}\}, \{\substack{23432\cdot1\\2}\}, \{\substack{1\cdot432\cdot1\\2}\}, \{\substack{1233211\\2}, \substack{1232221\\2}, \substack{1233221\\1}, \substack{1232211\\1}\}, \{\substack{1354321\\\cdot}\}, \{\substack{1343321\\2}, \substack{1244321\\2}\}, \{\substack{1233321\\\cdot}\}, \{\substack{1222111\\1}, \substack{1121111\\1}\}, \{\substack{000\cdot111\\0}\}, \{\substack{00000\cdot1\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2454321\\2}, \substack{2354321\\3}, \substack{1354321\\\cdot}, \substack{234\cdot321\\2}, \substack{\cdot\cdot\cdot3211\\2}, \substack{1\cdot43221\\2}, \substack{1233321\\2}, \substack{1233\cdot21\\1}, \substack{1232221\\2}, \substack{1232211\\1}, \substack{1222111\\1}, \substack{1121111\\1}, \substack{0111111\\1}, \substack{00\cdot\cdot\cdot11\\0}\}, \{\substack{2343221\\2}, \substack{1343321\\2}, \substack{1244321\\2}, \substack{0000001\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{\cdot3\cdot\cdot\cdot\cdot1\\\cdot}, \substack{00\cdot\cdot\cdot\cdot1\\\cdot}\}, \{\substack{12432\cdot1\\2}, \substack{1233321\\\cdot}, \substack{1233211\\2}, \substack{1232221\\2}, \substack{1233221\\1}, \substack{1232211\\1}, \substack{122\cdot111\\1}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{2465\cdot21\\3}\}, \{\substack{\cdot\cdot54321\\\cdot}\}, \{\substack{\cdot\cdot44321\\2}\}, \{\substack{\cdot\cdot43\cdot21\\2}\}, \{\substack{1233\cdot21\\\cdot}\}, \{\substack{1232221\\\cdot}\}, \{\substack{\cdot\cdot43211\\2}\}, \{\substack{0001111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{2465321\\3}\}, \{\substack{\cdot\cdot54321\\\cdot}\}, \{\substack{\cdot\cdot44321\\2}\}, \{\substack{\cdot\cdot43321\\2}\}, \{\substack{1233321\\\cdot}\}, \{\substack{\cdot\cdot432\cdot1\\2}\}, \{\substack{1233211\\2}, \substack{1233221\\1}\}, \{\substack{1232221\\2}, \substack{1232211\\1}\}, \{\substack{0001111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{2465\cdot21\\3}\}, \{\substack{2464321\\3}\}, \{\substack{2\cdot54321\\3}\}, \{\substack{2\cdot54321\\2}\}, \{\substack{1354321\\2}\}, \{\substack{1\cdot43\cdot21\\2}\}, \{\substack{2343\cdot21\\2}\}, \{\substack{1\cdot44321\\2}\}, \{\substack{1\cdot43211\\2}\}, \{\substack{1233\cdot21\\2}\}, \{\substack{1233\cdot21\\1}\}, \{\substack{2343211\\2}\}, \{\substack{1232221\\2}\}, \{\substack{1233211\\2}\}, \{\substack{0121111\\1}\}, \{\substack{1111111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{246\cdot321\\3}\}, \{\substack{2465421\\3}\}, \{\substack{2\cdot54321\\3}\}, \{\substack{2\cdot54321\\2}\}, \{\substack{1354321\\2}\}, \{\substack{1\cdot4\cdot321\\2}\}, \{\substack{234\cdot321\\2}\}, \{\substack{1\cdot43221\\2}\}, \{\substack{2343221\\2}\}, \{\substack{1\cdot43211\\2}\}, \{\substack{123\cdot221\\2}\}, \{\substack{1233321\\1}\}, \{\substack{123\cdot211\\2}\}, \{\substack{1232111\\1}\}, \{\substack{0122221\\1}\}, \{\substack{1111111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{\cdot\cdot\cdot4321\\\cdot}\}, \{\substack{\cdot\cdot\cdot3321\\\cdot}\}, \{\substack{\cdot\cdot\cdot32\cdot1\\\cdot}\}, \{\substack{0000111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{2465\cdot21\\3}\}, \{\substack{\cdot\cdot54321\\\cdot}\}, \{\substack{\cdot\cdot43\cdot21\\2}\}, \{\substack{\cdot\cdot44321\\2}\}, \{\substack{\cdot\cdot43211\\2}\}, \{\substack{1233221\\2}, \substack{1233321\\1}\}, \{\substack{1232221\\\cdot}\}, \{\substack{1232211\\2}, \substack{1232111\\1}\}, \{\substack{0001111\\0}\}$

Table 5: Maximal abelian sets

X	W_X	Sig(X)
X_{29}^{315}	$\langle w_6, w_5w_7\rangle \cong Dih_8$	0 8^{4+1} $10^{2.1}$ $12^{2.4+1}$ $14^{2.4}$ 16^2 18 20
X_{29}^{316}	$\langle w_3, w_5, w_7\rangle \cong {S_2}^3$	0 $8^{2.2}$ 10^{2+1} $12^{2.4+2}$ 14^{4+2+1} 16^{2+1} 22
X_{29}^{317}	$\langle w_1, w_2, w_3, w_6, w_7\rangle \cong {S_3}^2 \times S_2$	0 8^3 10^6 12^9 14^{6+3} 24
X_{29}^{318}	$\langle w_1, w_5, w_6, w_7, w^{\diamond}w_2w_3w_\sigma\rangle \cong S_4 \times Dih_8$	0 4 6 10^{16} 14^6 16^4
X_{29}^{319}	$\langle w_4, w_5, w_1w_7\rangle \cong S_3 \times S_2$	0 4 8^2 10^{6+3+1} 12^{6+2} 14^3 16^2 18^2
X_{29}^{320}	$\langle w_4, w_3w_5w_7\rangle \cong Dih_8$	0 6^2 8 $10^{2.4+2}$ 12^4 14^{4+2} 16^4 18
X_{29}^{321}	$\langle w_1, w_5, w_6\rangle \cong S_3 \times S_2$	0 6 10^{3+2} $12^{6+2.3+1}$ 14^3 16^{2+1} 18^2 20
X_{29}^{322}	$\langle w_1, w_2, w_4, w_1w_6{w_\tau}^{w_3w_4w_5w_4w_3}\rangle \cong S_3 \times Dih_8$	0 4 10^{6+4} 12^{12} 14 18^4
X_{29}^{323}	$\langle w_3w_5w_7\rangle \cong S_2$	0 6 8^{2+1} $10^{3.2}$ $12^{3.2+1}$ $14^{3.2}$ 16^{2+1} 18^2
X_{29}^{324}	$\langle w_1, w_2w_5\rangle \cong {S_2}^2$	0 6 8 $10^{4+2.2}$ 12^{4+2+1} $14^{2.2}$ $16^{2.2+1}$ 18^2
X_{29}^{325}	$\langle w_2, w_3, w_6\rangle \cong {S_2}^3$	0 6 8 $10^{4+2.2}$ 12^{4+2} 14^{4+2+1} 16^2 18^{2+1}
X_{29}^{326}	$\langle w_6\rangle \cong S_2$	0 6 8 $10^{2.2+3.1}$ $12^{2.2+4.1}$ $14^{2.2+2.1}$ $16^{3.1}$ 18 20
X_{29}^{327}	$\langle w_3, w_2w_5\rangle \cong {S_2}^2$	0 6 $10^{3.2}$ $12^{2.4+2+1}$ $14^{2.2}$ 16^2 $18^{2+2.1}$
X_{29}^{328}	$\langle w_4, w_6\rangle \cong {S_2}^2$	0 6 $10^{2.2+1}$ $12^{4+4.2+1}$ 14^{2+1} 16^{2+1} $18^{2.1}$ 20
X_{29}^{329}	$\langle w_1, w_6, w_5w_7\rangle \cong S_2 \times Dih_8$	0 6 8^{4+2} 12^{8+4+1} 14^4 16^2 18^2
X_{29}^{330}	$\langle w_1, w_4w_6\rangle \cong {S_2}^2$	0 6 8^{2+1} 10^{4+2} $12^{4+2.2}$ 14^{2+1} $16^{3.2}$ 18

in the E_8 root system (continued)
W_X-orbits on X
$\{\substack{2465432\\3}\}, \{\substack{246\cdot\cdot\cdot1\\3}\}, \{\substack{2454321\\3}\}, \{\substack{2454321\\2}\}, \{\substack{2354321\\3}\}, \{\substack{234\cdot\cdot\cdot1\\2}\}, \{\substack{134\cdot\cdot\cdot1\\2}\}, \{\substack{1354321\\2}\}, \{\substack{124\cdot\cdot\cdot1\\2}\}, \{\substack{1233\cdot21\\2}, \substack{1232\cdot11\\2}\}, \{\substack{1233211\\1}, \substack{1232221\\1}\}, \{\substack{0111111\\1}\}, \{\substack{1111111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{246\cdot321\\3}\}, \{\substack{2\cdot54321\\3}\}, \{\substack{1354321\\2}\}, \{\substack{1\cdot4\cdot321\\2}\}, \{\substack{1\cdot432\cdot1\\2}\}, \{\substack{2\cdot54321\\2}\}, \{\substack{123\cdot2\cdot1\\2}\}, \{\substack{23432\cdot1\\2}\}, \{\substack{1233321\\2}\}, \{\substack{012\cdot111\\1}\}, \{\substack{1233321\\1}\}, \{\substack{1111111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465\cdot\cdot1\\3}\}, \{\substack{\cdot\cdot54321\\\cdot}\}, \{\substack{\cdot\cdot43\cdot\cdot1\\2}\}, \{\substack{1232\cdot\cdot1\\\cdot}\}, \{\substack{\cdot\cdot44321\\2}\}, \{\substack{0001111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2454321\\3}\}, \{\substack{2454321\\2}\}, \{\substack{246\cdot\cdot\cdot1\\3}, \substack{\cdot34\cdot\cdot\cdot1\\2}, \substack{122\cdot\cdot\cdot1\\1}\}, \{\substack{123\cdot\cdot\cdot1\\2}\}, \{\substack{\cdot354321\\2}, \substack{\cdot111111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{1233321\\1}\}, \{\substack{\cdot111111\\0}\}, \{\substack{12\cdot\cdot2\cdot1\\1}\}, \{\substack{00\cdot\cdot111\\0}\}, \{\substack{2454321\\2}\}, \{\substack{\cdot3\cdot\cdot321\\2}\}, \{\substack{1122221\\1}, \substack{0122211\\1}\}, \{\substack{12\cdot\cdot321\\2}\}, \{\substack{00000\cdot1\\0}\}, \{\substack{2343211\\2}, \substack{1343221\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{1354321\\3}\}, \{\substack{2\cdot\cdot\cdot321\\3}\}, \{\substack{13\cdot4321\\2}, \substack{12\cdot3321\\2}\}, \{\substack{1343321\\2}, \substack{1244321\\2}\}, \{\substack{2\cdot\cdot\cdot321\\2}\}, \{\substack{1343211\\2}, \substack{12\cdot3221\\2}, \substack{1232211\\2}\}, \{\substack{01222\cdot1\\1}\}, \{\substack{1233211\\1}, \substack{12\cdot2221\\1}, \substack{1122211\\1}\}, \{\substack{1232111\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{246\cdot\cdot21\\3}\}, \{\substack{\cdot354321\\2}\}, \{\substack{\cdot34\cdot\cdot21\\2}\}, \{\substack{124\cdot\cdot21\\2}\}, \{\substack{123\cdot\cdot21\\1}\}, \{\substack{2454321\\3}\}, \{\substack{123\cdot\cdot21\\2}\}, \{\substack{\cdot343211\\2}\}, \{\substack{1243211\\2}\}, \{\substack{\cdot111111\\0}\}, \{\substack{0011111\\1}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{24\cdot4321\\\cdot}, \substack{12\cdot2221\\\cdot}\}, \{\substack{\cdot343\cdot21\\2}\}, \{\substack{\cdot3\cdot4321\\\cdot}, \substack{12\cdot3\cdot21\\\cdot}\}, \{\substack{1244321\\2}\}, \{\substack{\cdot343211\\2}, \substack{0000\cdot11\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{1233321\\1}\}, \{\substack{2\cdot54321\\2}\}, \{\substack{1111111\\0}\}, \{\substack{234\cdot321\\2}\}, \{\substack{1233221\\1}, \substack{1232211\\1}\}, \{\substack{1233211\\1}, \substack{1232221\\1}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{1222221\\1}, \substack{1122211\\1}\}, \{\substack{0\cdot11111\\0}\}, \{\substack{1354321\\2}\}, \{\substack{1344321\\2}, \substack{1243321\\2}\}, \{\substack{23432\cdot1\\2}\}, \{\substack{000\cdot111\\0}\}, \{\substack{1221111\\1}, \substack{1122111\\1}\}, \{\substack{1233321\\2}\}, \{\substack{1343211\\2}, \substack{1243221\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{2465421\\3}\}, \{\substack{\cdot34\cdot321\\2}\}, \{\substack{2454321\\\cdot}\}, \{\substack{1233321\\\cdot}\}, \{\substack{\cdot354321\\\cdot}\}, \{\substack{124\cdot321\\2}\}, \{\substack{1222221\\1}\}, \{\substack{\cdot343221\\2}\}, \{\substack{1233221\\2}, \substack{1232221\\1}\}, \{\substack{\cdot343211\\2}\}, \{\substack{1232211\\2}, \substack{1233211\\1}\}, \{\substack{1243221\\2}\}, \{\substack{000\cdot111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{2344321\\2}\}, \{\substack{1233\cdot21\\\cdot}\}, \{\substack{1\cdot44321\\2}\}, \{\substack{1\cdot22221\\1}\}, \{\substack{2\cdot54321\\\cdot}\}, \{\substack{2343\cdot21\\2}\}, \{\substack{1\cdot43\cdot21\\2}\}, \{\substack{1354321\\\cdot}\}, \{\substack{0001111\\0}\}, \{\substack{1233211\\\cdot}\}, \{\substack{0000\cdot11\\0}\}, \{\substack{2343211\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{2354321\\2}\}, \{\substack{2465\cdot21\\3}\}, \{\substack{2343\cdot21\\2}\}, \{\substack{2464321\\3}\}, \{\substack{2454321\\2}\}, \{\substack{2344321\\2}\}, \{\substack{1243\cdot21\\2}\}, \{\substack{1233\cdot21\\1}\}, \{\substack{2454321\\3}\}, \{\substack{1354321\\2}\}, \{\substack{1244321\\2}\}, \{\substack{1122221\\1}\}, \{\substack{1343\cdot21\\2}\}, \{\substack{1233\cdot21\\2}\}, \{\substack{2343211\\2}\}, \{\substack{1232221\\2}\}, \{\substack{1243211\\2}\}, \{\substack{1121111\\1}\}, \{\substack{1111111\\0}\}, \{\substack{1233211\\2}\}, \{\substack{0111111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{234\cdot321\\2}\}, \{\substack{1233321\\\cdot}\}, \{\substack{1\cdot22221\\1}\}, \{\substack{2\cdot54321\\\cdot}\}, \{\substack{1\cdot4\cdot321\\2}\}, \{\substack{1233221\\2}, \substack{1232221\\1}\}, \{\substack{2343221\\2}\}, \{\substack{1354321\\\cdot}\}, \{\substack{1\cdot43221\\2}\}, \{\substack{000\cdot111\\0}\}, \{\substack{1232211\\2}, \substack{1233211\\1}\}, \{\substack{2343211\\2}\}, \{\substack{0000011\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{2465\cdot21\\3}\}, \{\substack{23\cdot4321\\2}\}, \{\substack{1244321\\2}\}, \{\substack{12\cdot3\cdot21\\2}\}, \{\substack{24\cdot4321\\3}\}, \{\substack{13\cdot4321\\2}\}, \{\substack{2343\cdot21\\2}\}, \{\substack{1233\cdot21\\1}\}, \{\substack{1122221\\1}\}, \{\substack{1343\cdot21\\2}\}, \{\substack{1232221\\2}\}, \{\substack{12\cdot3211\\2}\}, \{\substack{2343211\\2}\}, \{\substack{1111111\\0}\}, \{\substack{0011111\\1}\}, \{\substack{0111111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2454321\\2}\}, \{\substack{246\cdot\cdot\cdot1\\3}\}, \{\substack{\cdot354321\\2}\}, \{\substack{\cdot34\cdot\cdot\cdot1\\2}\}, \{\substack{1233\cdot21\\1}, \substack{1232\cdot11\\1}\}, \{\substack{2454321\\3}\}, \{\substack{124\cdot\cdot\cdot1\\2}\}, \{\substack{\cdot111111\\0}\}, \{\substack{1233211\\2}, \substack{1232221\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{2454321\\2}\}, \{\substack{2465\cdot21\\3}\}, \{\substack{2465431\\3}\}, \{\substack{\cdot3\cdot4321\\2}\}, \{\substack{24\cdot4321\\3}\}, \{\substack{\cdot343\cdot21\\2}\}, \{\substack{\cdot343211\\2}\}, \{\substack{1233\cdot21\\1}\}, \{\substack{1232211\\1}, \substack{1222111\\1}\}, \{\substack{1244321\\2}\}, \{\substack{1243221\\2}, \substack{1233321\\2}\}, \{\substack{12\cdot3211\\2}\}, \{\substack{\cdot111111\\0}\}, \{\substack{1232221\\2}\}$

Table 5: Maximal abelian sets

X	W_X	$\mathrm{Sig}(X)$
X_{29}^{331}	$\langle w_5w_7\rangle \cong S_2$	$0\ 8^{2+1}\ 10^{2.2+2.1}\ 12^{2.2+1}\ 14^{3.2+1}\ 16^{2.2+1}\ 18^{2.1}$
X_{29}^{332}	$\langle w_1, w_5w_7\rangle \cong {S_2}^2$	$0\ 8^{2}\ 10^{2.2}\ 12^{4+2+2.1}\ 14^{4+2.2+1}\ 16^{2}\ 18^{2}\ 20$
X_{29}^{333}	$\langle w_4, w_1w_6\rangle \cong {S_2}^2$	$0\ 8^{2+1}\ 10^{4+2.1}\ 12^{2.2}\ 14^{4+3.2}\ 16^{2}\ 18^{2+1}$
X_{29}^{334}	$\langle w_1, w_2w_7\rangle \cong {S_2}^2$	$0\ 8^{2+1}\ 10^{4+2.1}\ 12^{3.2}\ 14^{2.2}\ 16^{4+2.2}\ 18$
X_{29}^{335}	$\langle w_4\rangle \cong S_2$	$0\ 8^{2.1}\ 10^{2+2.1}\ 12^{3.2+2.1}\ 14^{2.2+5.1}\ 16^{2.1}\ 18^{2.1}\ 20$
X_{29}^{336}	$\langle w_1, w_2w_5\rangle \cong {S_2}^2$	$0\ 8^{2}\ 10^{2.2}\ 12^{4+2.2+1}\ 14^{4+2}\ 16^{2.2+1}\ 18\ 20$
X_{29}^{337}	$\langle w_5w_7\rangle \cong S_2$	$0\ 8^{2+1}\ 10^{2.2+1}\ 12^{2.2+2.1}\ 14^{4.2+1}\ 16^{2+1}\ 18\ 20$
X_{29}^{338}	$\langle w_1w_5w_7\rangle \cong S_2$	$0\ 8^{2}\ 10^{2.2+1}\ 12^{3.2+1}\ 14^{3.2+1}\ 16^{2.2}\ 18^{2+1}$
X_{29}^{339}	$\langle w_3, w_2w_6\rangle \cong {S_2}^2$	$0\ 8\ 10^{4+2.2+1}\ 12\ 14^{4+3.2}\ 16^{3.2}\ 20$
X_{29}^{340}	$\langle w_5\rangle \cong S_2$	$0\ 8\ 10^{2+3.1}\ 12^{2.2+3.1}\ 14^{2.2+3.1}\ 16^{2+3.1}\ 18^{3.1}$
X_{29}^{341}	$\langle w_1w_2w_6\rangle \cong S_2$	$0\ 8\ 10^{2.2}\ 12^{4.2}\ 14^{4.2+1}\ 16^{2+1}\ 18^{2}\ 20$
X_{29}^{342}	$\langle w_5\rangle \cong S_2$	$0\ 8\ 10^{2+2.1}\ 12^{2.2+5.1}\ 14^{2+4.1}\ 16^{2.2+2.1}\ 18\ 20$

in the E_8 root system (continued)
W_X-orbits on X
$\{\substack{2465432\\3}\},\{\substack{24654\cdot1\\3}\},\{\substack{2454321\\2}\},\{\substack{246\cdot321\\3}\},\{\substack{234\cdot321\\2}\},\{\substack{2454321\\3}\},\{\substack{2354321\\2}\},$ $\{\substack{134\cdot321\\2}\},\{\substack{23432\cdot1\\2}\},\{\substack{1233321\\1}\},\{\substack{124\cdot321\\2}\},\{\substack{13432\cdot1\\2}\},\{\substack{1233221\\1},\substack{1232211\\1}\},\{\substack{1233321\\2}\},$ $\{\substack{12432\cdot1\\2}\},\{\substack{1233211\\2},\substack{1232221\\2}\},\{\substack{1111111\\0}\},\{\substack{1111111\\1}\},\{\substack{0111111\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{24654\cdot1\\3}\},\{\substack{246\cdot321\\3}\},\{\substack{\cdot354321\\2}\},\{\substack{\cdot34\cdot321\\2}\},\{\substack{124\cdot321\\2}\},\{\substack{2454321\\3}\},$ $\{\substack{1233321\\1}\},\{\substack{\cdot3432\cdot1\\2}\},\{\substack{12432\cdot1\\2}\},\{\substack{1233221\\1},\substack{1232211\\1}\},\{\substack{1233321\\2}\},\{\substack{1233211\\2},\substack{1232221\\2}\},$ $\{\substack{\cdot111111\\0}\},\{\substack{0011111\\1}\}$
$\{\substack{2465432\\3}\},\{\substack{2465\cdot21\\3}\},\{\substack{2465431\\3}\},\{\substack{\cdot3\cdot4321\\2}\},\{\substack{2454321\\2}\},\{\substack{1244321\\2}\},\{\substack{24\cdot4321\\3}\},$ $\{\substack{1233\cdot21\\1}\},\{\substack{12\cdot3\cdot21\\2}\},\{\substack{2343221\\2},\substack{1343321\\2}\},\{\substack{\cdot343211\\2}\},\{\substack{12\cdot3211\\2}\},\{\substack{1122211\\1},\substack{0122111\\1}\},$ $\{\substack{\cdot111111\\0}\},\{\substack{1232221\\2}\}$
$\{\substack{2465432\\3}\},\{\substack{24654\cdot1\\3}\},\{\substack{1244321\\2}\},\{\substack{\cdot354321\\\cdot}\},\{\substack{2465321\\3}\},\{\substack{1243321\\2}\},\{\substack{\cdot344321\\2}\},$ $\{\substack{12432\cdot1\\2}\},\{\substack{1233321\\\cdot}\},\{\substack{\cdot343321\\2}\},\{\substack{1233211\\2},\substack{1233221\\1}\},\{\substack{\cdot3432\cdot1\\2}\},\{\substack{1232221\\2},\substack{1232211\\1}\},$ $\{\substack{0011111\\\cdot}\},\{\substack{0001111\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{2465431\\3}\},\{\substack{2465421\\3}\},\{\substack{23\cdot4321\\2}\},\{\substack{2465321\\3}\},\{\substack{1244321\\2}\},\{\substack{24\cdot4321\\3}\},$ $\{\substack{13\cdot4321\\2}\},\{\substack{12\cdot3321\\2}\},\{\substack{2343321\\2}\},\{\substack{1233321\\1}\},\{\substack{12\cdot3221\\2}\},\{\substack{12\cdot3211\\2}\},\{\substack{2343221\\2}\},$ $\{\substack{1343321\\2}\},\{\substack{2343211\\2}\},\{\substack{1233221\\1}\},\{\substack{1122211\\1}\},\{\substack{1343211\\2}\},\{\substack{1232221\\2}\},\{\substack{1111111\\0}\},$ $\{\substack{0011111\\1}\},\{\substack{0111111\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{246\cdot321\\3}\},\{\substack{2454321\\\cdot}\},\{\substack{\cdot343221\\2}\},\{\substack{\cdot354321\\\cdot}\},\{\substack{\cdot343211\\2}\},\{\substack{1232111\\\cdot}\},$ $\{\substack{1243221\\2}\},\{\substack{\cdot34\cdot321\\2}\},\{\substack{1233221\\2},\substack{1232221\\1}\},\{\substack{124\cdot321\\2}\},\{\substack{1232211\\2},\substack{1233211\\1}\},\{\substack{1222221\\1}\},$ $\{\substack{0000011\\0}\},\{\substack{0000001\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{24654\cdot1\\3}\},\{\substack{2354321\\2}\},\{\substack{246\cdot321\\3}\},\{\substack{234\cdot321\\2}\},\{\substack{2454321\\2}\},\{\substack{23432\cdot1\\2}\},$ $\{\substack{124\cdot321\\2}\},\{\substack{2454321\\3}\},\{\substack{1354321\\2}\},\{\substack{134\cdot321\\2}\},\{\substack{12432\cdot1\\2}\},\{\substack{1233211\\1},\substack{1232221\\1}\},\{\substack{11222\cdot1\\1}\},$ $\{\substack{1233321\\2}\},\{\substack{1233221\\2},\substack{1232211\\2}\},\{\substack{1111111\\0}\},\{\substack{1232111\\2}\},\{\substack{0111111\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{24654\cdot1\\3}\},\{\substack{246\cdot321\\3}\},\{\substack{\cdot354321\\2}\},\{\substack{2454321\\2}\},\{\substack{2344321\\2},\substack{1343321\\2}\},\{\substack{2343321\\2},$ $\substack{1344321\\2}\},\{\substack{124\cdot321\\2}\},\{\substack{2454321\\3}\},\{\substack{2343211\\2},\substack{1343221\\2}\},\{\substack{12432\cdot1\\2}\},\{\substack{1233211\\1},\substack{1232221\\1}\},$ $\{\substack{1233321\\2}\},\{\substack{1233221\\2},\substack{1232211\\2}\},\{\substack{1122221\\1},\substack{0122211\\1}\},\{\substack{\cdot111111\\0}\},\{\substack{1232111\\2}\}$
$\{\substack{2465432\\3}\},\{\substack{2465431\\3}\},\{\substack{2\cdot54321\\\cdot}\},\{\substack{2465\cdot21\\3}\},\{\substack{2343\cdot21\\2}\},\{\substack{2344321\\2}\},\{\substack{2343211\\2}\},$ $\{\substack{1\cdot43\cdot21\\2}\},\{\substack{1\cdot44321\\2}\},\{\substack{1233221\\2},\substack{1233321\\1}\},\{\substack{1232221\\\cdot}\},\{\substack{1\cdot43211\\2}\},\{\substack{1232211\\2},\substack{1232111\\1}\},$ $\{\substack{1111111\\\cdot}\},\{\substack{0001111\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{2465431\\3}\},\{\substack{246\cdot321\\3}\},\{\substack{2465421\\3}\},\{\substack{2454321\\3}\},\{\substack{2454321\\2}\},\{\substack{234\cdot321\\2}\},$ $\{\substack{134\cdot321\\2}\},\{\substack{1354321\\2}\},\{\substack{2343221\\2}\},\{\substack{1343221\\2}\},\{\substack{124\cdot321\\2}\},\{\substack{123\cdot221\\2}\},\{\substack{2343211\\2}\},$ $\{\substack{1243221\\2}\},\{\substack{1233321\\1}\},\{\substack{123\cdot211\\2}\},\{\substack{1243211\\2}\},\{\substack{1232111\\1}\},\{\substack{1122221\\1}\},\{\substack{1111111\\0}\},$ $\{\substack{0111111\\1}\},\{\substack{0111111\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{2465431\\3}\},\{\substack{2465\cdot21\\3}\},\{\substack{2454321\\\cdot}\},\{\substack{2354321\\3},\substack{1354321\\2}\},\{\substack{\cdot344321\\2}\},\{\substack{2343321\\2},$ $\substack{1343221\\2}\},\{\substack{2343221\\2},\substack{1343321\\2}\},\{\substack{\cdot343211\\2}\},\{\substack{1243\cdot21\\2}\},\{\substack{1233221\\2},\substack{1233321\\1}\},\{\substack{1232221\\\cdot}\},$ $\{\substack{1244321\\2}\},\{\substack{1232211\\2},\substack{1232111\\1}\},\{\substack{1243211\\2}\},\{\substack{1111111\\0},\substack{0111111\\1}\},\{\substack{0001111\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{2465431\\3}\},\{\substack{246\cdot321\\3}\},\{\substack{2465421\\3}\},\{\substack{1354321\\2}\},\{\substack{134\cdot321\\2}\},\{\substack{124\cdot321\\2}\},$ $\{\substack{2454321\\3}\},\{\substack{2454321\\2}\},\{\substack{2354321\\2}\},\{\substack{1343221\\2}\},\{\substack{1243221\\2}\},\{\substack{123\cdot221\\2}\},\{\substack{2343221\\2}\},$ $\{\substack{1233321\\2}\},\{\substack{1243211\\2}\},\{\substack{1233321\\1}\},\{\substack{123\cdot211\\2}\},\{\substack{012\cdot111\\1}\},\{\substack{2343211\\2}\},\{\substack{1122221\\1}\},$ $\{\substack{0111111\\0}\},\{\substack{1111111\\0}\}$

Table 5: Maximal abelian sets

X	W_X	Sig(X)
X_{29}^{343}	$\langle w_1, w_6, w_5w_7\rangle \cong S_2 \times Dih_8$	0 8 10^{4+1} 12^8 14^4 $16^{4+2.2}$ 18^2
X_{29}^{344}	$\langle w_6, w_5w_7\rangle \cong Dih_8$	0 10^{4+1} 12^{4+1} $14^{3.4}$ 16^{2+1} $18^{2.1}$ 20
X_{29}^{345}	$\langle w_5, w_7\rangle \cong {S_2}^2$	0 $10^{2.2+1}$ $12^{2.2+1}$ $14^{4+3.2+2.1}$ 16^{2+1} $18^{2.1}$ 20
X_{29}^{346}	$\langle w_4, w_6\rangle \cong {S_2}^2$	0 8 $10^{2.2+1}$ $12^{4+2.2+1}$ $14^{3.2+2.1}$ 16^{2+1} 18 22
X_{29}^{347}	$\langle w_4, w_1w_6\rangle \cong {S_2}^2$	0 8 10^{2+1} $12^{2.4+2}$ $14^{3.2+2.1}$ $16^{2.2}$ 20^2
X_{29}^{348}	$\langle w_5, w_6, w_7, w^{\diamond}w_2w_3w_{\sigma}\rangle \cong S_4 \times S_2$	0 8 10^8 12^{8+1} 14^{6+1} 16^2 24
X_{29}^{349}	$\langle w_5, w_6, w_1w_2\rangle \cong S_3 \times S_2$	0 10^{3+2} 12^{6+2} 14^{6+3+2} 16 18^2 22
X_{29}^{350}	$\langle w_1, w_2, w_4, w_5\rangle \cong S_4 \times S_2$	0 10^{4+2} 12^{8+6} 14^4 16^2 18 24
X_{29}^{351}	$\langle w_6, w_1w_5w_7\rangle \cong Dih_8$	0 10^4 12^{4+2+1} $14^{2.4+2+1}$ 16^4 20^2
X_{29}^{352}	$\langle w_1, w_4, w_2w_5\rangle \cong S_2 \times Dih_8$	0 10^4 12^{8+2} $14^{2.4+2}$ 16^2 20 22
X_{29}^{353}	$\langle w_1, w_5w_7, w_4w_{\tau}\rangle \cong S_3 \times S_2$	0 6 8^3 10^6 12^6 14^{6+3} 18^3
X_{29}^{354}	$\langle w_3w_5w_7\rangle \cong S_2$	0 $8^{2.2+1}$ 10^{2+1} $12^{4.2}$ $14^{2.2+2.1}$ $16^{2.2}$ 18^2
X_{29}^{355}	$\langle w_2, w_4, w_1w_6{w_{\tau}}^{w_3w_4w_5w_4w_3w_1}\rangle \cong S_3 \times S_2$	0 6 8 10^{6+2} 12^6 14^{6+1} 16^2 18^{2+1}
X_{29}^{356}	$\langle w_3w_5w_7\rangle \cong S_2$	0 8^{2+1} $10^{2.2+1}$ $12^{3.2+1}$ $14^{3.2}$ $16^{3.2}$ 20
X_{29}^{357}	$\langle w_1w_2w_6, w_3{w_5}^{w_4w_3}\rangle \cong Dih_{16}$	0 8 10^{8+2} 14^8 16^8 18

in the E_8 root system (continued)
W_X-orbits on X
$\{\substack{2465432\\3}\}, \{\substack{2454321\\3}\}, \{\substack{246\cdot\cdot\cdot1\\3}\}, \{\substack{2454321\\2}\}, \{\substack{\cdot34\cdot\cdot\cdot1\\2}\}, \{\substack{1233\cdot21\\2}, \substack{1232\cdot11\\2}\}, \{\substack{124\cdot\cdot\cdot1\\2}\}, \{\substack{1233211\\1}, \substack{1232221\\1}\}, \{\substack{\cdot111111\\1}\}, \{\substack{\cdot111111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{246\cdot\cdot\cdot1\\3}\}, \{\substack{2454321\\3}\}, \{\substack{234\cdot\cdot\cdot1\\2}\}, \{\substack{2354321\\2}\}, \{\substack{134\cdot\cdot\cdot1\\2}\}, \{\substack{124\cdot\cdot\cdot1\\2}\}, \{\substack{1233\cdot21\\2}, \substack{1232\cdot11\\2}\}, \{\substack{1233211\\1}, \substack{1232221\\1}\}, \{\substack{1111111\\1}\}, \{\substack{1111111\\0}\}, \{\substack{0011111\\1}\}, \{\substack{0111111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{246\cdot321\\3}\}, \{\substack{2454321\\3}\}, \{\substack{234\cdot321\\2}\}, \{\substack{23432\cdot1\\2}\}, \{\substack{2454321\\2}\}, \{\substack{123\cdot2\cdot1\\2}\}, \{\substack{124\cdot321\\2}\}, \{\substack{13432\cdot1\\2}\}, \{\substack{12432\cdot1\\2}\}, \{\substack{1354321\\2}\}, \{\substack{1233321\\2}\}, \{\substack{112\cdot111\\1}\}, \{\substack{1233321\\1}\}, \{\substack{1111111\\0}\}, \{\substack{0111111\\1}\}, \{\substack{0111111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{2465\cdot21\\3}\}, \{\substack{23\cdot4321\\2}\}, \{\substack{1244321\\2}\}, \{\substack{12\cdot3\cdot21\\2}\}, \{\substack{24\cdot4321\\3}\}, \{\substack{2343\cdot21\\2}\}, \{\substack{1122221\\1}\}, \{\substack{13\cdot4321\\2}\}, \{\substack{12\cdot3211\\2}\}, \{\substack{1122\cdot11\\1}\}, \{\substack{2454321\\2}\}, \{\substack{1232221\\2}\}, \{\substack{1232\cdot11\\2}\}, \{\substack{0122221\\1}\}, \{\substack{1111111\\0}\}, \{\substack{0111111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{2465\cdot21\\3}\}, \{\substack{1244321\\2}\}, \{\substack{\cdot3\cdot4321\\2}\}, \{\substack{12\cdot3\cdot21\\2}\}, \{\substack{24\cdot4321\\3}\}, \{\substack{2343221\\2}, \substack{1343321\\2}\}, \{\substack{12\cdot3211\\2}\}, \{\substack{\cdot122221\\1}\}, \{\substack{2454321\\2}\}, \{\substack{1232221\\2}\}, \{\substack{1232\cdot11\\2}\}, \{\substack{1122211\\1}, \substack{0122111\\1}\}, \{\substack{\cdot111111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2354321\\2}\}, \{\substack{246\cdot\cdot\cdot1\\3}, \substack{234\cdot\cdot\cdot1\\2}\}, \{\substack{124\cdot\cdot\cdot1\\2}, \substack{112\cdot\cdot\cdot1\\1}\}, \{\substack{2454321\\3}\}, \{\substack{123\cdot\cdot\cdot1\\2}\}, \{\substack{2454321\\2}\}, \{\substack{1354321\\2}, \substack{1111111\\0}\}, \{\substack{0111111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{246\cdot\cdot21\\3}\}, \{\substack{2454321\\\cdot}\}, \{\substack{\cdot34\cdot\cdot21\\2}\}, \{\substack{2354321\\3}, \substack{1354321\\2}\}, \{\substack{123\cdot\cdot21\\\cdot}\}, \{\substack{124\cdot\cdot21\\2}\}, \{\substack{\cdot343211\\2}\}, \{\substack{1243211\\2}\}, \{\substack{1111111\\0}, \substack{0111111\\1}\}, \{\substack{0000001\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{24\cdot\cdot321\\\cdot}\}, \{\substack{\cdot343221\\2}\}, \{\substack{\cdot3\cdot\cdot321\\\cdot}\}, \{\substack{12\cdot\cdot221\\\cdot}\}, \{\substack{12\cdot\cdot321\\\cdot}\}, \{\substack{\cdot343211\\2}\}, \{\substack{0000011\\0}\}, \{\substack{0000001\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{246\cdot\cdot\cdot1\\3}\}, \{\substack{124\cdot\cdot\cdot1\\2}\}, \{\substack{\cdot354321\\2}\}, \{\substack{2454321\\3}\}, \{\substack{2344321\\2}, \substack{1343\cdot21\\2}, \substack{2343211\\2}\}, \{\substack{1233\cdot21\\2}, \substack{1232\cdot11\\2}\}, \{\substack{1233211\\2}, \substack{1232221\\2}\}, \{\substack{2454321\\2}\}, \{\substack{1122221\\1}, \substack{0122\cdot11\\1}, \substack{1121111\\1}\}, \{\substack{\cdot111111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{24\cdot\cdot321\\\cdot}\}, \{\substack{\cdot3\cdot\cdot321\\\cdot}\}, \{\substack{\cdot343221\\2}\}, \{\substack{12\cdot\cdot321\\\cdot}\}, \{\substack{12\cdot3221\\2}, \substack{12\cdot2221\\1}\}, \{\substack{\cdot343211\\2}\}, \{\substack{1232211\\2}, \substack{1233211\\1}\}, \{\substack{0000011\\0}\}, \{\substack{0000001\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2454321\\2}\}, \{\substack{24654\cdot1\\3}, \substack{1233321\\1}\}, \{\substack{\cdot3\cdot\cdot321\\2}\}, \{\substack{246\cdot321\\3}, \substack{1233221\\1}, \substack{12\cdot2211\\1}, \substack{1222221\\1}\}, \{\substack{\cdot3432\cdot1\\2}, \substack{\cdot111111\\0}\}, \{\substack{12\cdot\cdot321\\2}\}, \{\substack{1233211\\2}, \substack{1232221\\2}, \substack{0011111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{2\cdot54321\\2}\}, \{\substack{1233321\\1}\}, \{\substack{234\cdot321\\2}\}, \{\substack{1354321\\2}\}, \{\substack{246\cdot321\\3}\}, \{\substack{1344321\\2}, \substack{1243321\\2}\}, \{\substack{1343321\\2}, \substack{1244321\\2}\}, \{\substack{1233221\\1}, \substack{1232211\\1}\}, \{\substack{23432\cdot1\\2}\}, \{\substack{1222221\\1}, \substack{1122211\\1}\}, \{\substack{1233321\\2}\}, \{\substack{1111111\\0}\}, \{\substack{1343211\\2}, \substack{1243221\\2}\}, \{\substack{0\cdot11111\\0}\}, \{\substack{1233211\\2}, \substack{1232221\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{1244321\\2}\}, \{\substack{2465431\\3}\}, \{\substack{13\cdot4321\\\cdot}, \substack{12\cdot3321\\\cdot}\}, \{\substack{2465421\\3}, \substack{0122221\\1}\}, \{\substack{23\cdot4321\\\cdot}, \substack{12\cdot3221\\\cdot}\}, \{\substack{12\cdot3211\\\cdot}, \substack{00\cdot1111\\\cdot}\}, \{\substack{1343321\\2}\}, \{\substack{2343321\\2}, \substack{1343221\\2}\}, \{\substack{1343211\\2}, \substack{0000111\\0}\}, \{\substack{2343221\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{1354321\\2}\}, \{\substack{1344321\\2}, \substack{1243321\\2}\}, \{\substack{1343321\\2}, \substack{1244321\\2}\}, \{\substack{1354321\\3}\}, \{\substack{246\cdot321\\3}\}, \{\substack{2\cdot54321\\2}\}, \{\substack{01222\cdot1\\1}\}, \{\substack{1233321\\2}\}, \{\substack{234\cdot321\\2}\}, \{\substack{1343211\\2}, \substack{1243221\\2}\}, \{\substack{1233211\\1}, \substack{1232221\\1}\}, \{\substack{1233221\\2}, \substack{1232211\\2}\}, \{\substack{1222221\\1}, \substack{1122211\\1}\}, \{\substack{0\cdot11111\\0}\}, \{\substack{1232111\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{24\cdot4321\\\cdot}, \substack{2343321\\2}, \substack{1343221\\2}, \substack{12\cdot2221\\\cdot}\}, \{\substack{2343221\\2}, \substack{1343321\\2}\}, \{\substack{2354321\\2}, \substack{1354321\\3}, \substack{\cdot344321\\2}, \substack{1243\cdot21\\2}, \substack{1233221\\2}, \substack{1233321\\1}\}, \{\substack{\cdot343211\\2}, \substack{1232211\\2}, \substack{1232111\\1}, \substack{1111111\\1}, \substack{0111111\\0}, \substack{0000\cdot11\\0}\}, \{\substack{1244321\\2}\}$

Table 5: Maximal abelian sets

X	W_X	Sig(X)
X_{29}^{358}	$\langle w_3w_5, w_3w_5{w_7}^{w_4}\rangle \cong {S_2}^2$	0 8^2 10^{4+1} $12^{4+2.2}$ 14^4 16^{4+2+1} 18^2
X_{29}^{359}	$\langle w_1w_4w_7\rangle \cong S_2$	0 8 $10^{2.2+1}$ $12^{4.2}$ $14^{2.2}$ $16^{3.2+2.1}$ 18^2
X_{29}^{360}	$\langle w_3w_5, w_3w_5{w_7}^{w_4}\rangle \cong {S_2}^2$	0 8 10^{4+1} 12^{4+2} $14^{4+3.2}$ 16^2 18^4
X_{29}^{361}	$\langle w_1w_5w_7\rangle \cong S_2$	0 8 10 $12^{6.2}$ $14^{2.2+2.1}$ $16^{2.2}$ $18^{2.2}$
X_{29}^{362}	$\langle w_3, w_6, w_5w_7\rangle \cong S_2 \times Dih_8$	0 8^2 10^4 $12^{2.4}$ 14^{8+1} 16^2 18^2 20
X_{29}^{363}	$\langle w_3w_5w_7\rangle \cong S_2$	0 8^2 $10^{2.2+1}$ $12^{3.2+1}$ $14^{3.2+1}$ $16^{2.2}$ 18^{2+1}
X_{29}^{364}	$\langle w_3w_5, w_3w_5{w_7}^{w_4}\rangle \cong {S_2}^2$	0 10^{4+2} 12^{4+1} 14^{4+2+1} $16^{4+2.2}$ 18^2
X_{29}^{365}	$\langle w_3w_6\rangle \cong S_2$	0 8 $10^{2.2+1}$ $12^{3.2+1}$ $14^{2.2+3.1}$ $16^{2.2+1}$ 18^{2+1}
X_{29}^{366}	$\langle w_1, w_5, w^{\diamond}w_2w_3w_5w_7{w_\sigma}^{w_6}\rangle \cong {S_2}^2 \wr S_2$	0 6 10^8 12^{8+1} 14^{4+1} 16^4 22
X_{29}^{367}	$\langle w_5, w_3w_7\rangle \cong {S_2}^2$	0 8^2 $10^{4+2.1}$ $12^{3.2+1}$ $14^{4+2.2}$ $16^{2.2}$ 22
X_{29}^{368}	$\langle w_1, w_2, w_4, w_1w_6{w_\tau}^{w_3w_4w_5w_4w_3}\rangle \cong S_3 \times Dih_8$	0 10^{6+4} 12 14^{12} 16^4 22
X_{29}^{369}	$\langle w_3, w_2w_7\rangle \cong {S_2}^2$	0 10^{2+1} $12^{4+2.2}$ $14^{3.2}$ $16^{4+2.2+1}$ $18^{2.1}$
X_{29}^{370}	$\langle w_1w_2\rangle \cong S_2$	0 $10^{2.2+1}$ $12^{2.2+2.1}$ $14^{4.2+1}$ $16^{2.2+2.1}$ 18 20
X_{29}^{371}	$\langle w_5w_7\rangle \cong S_2$	0 8 $10^{2.2}$ $12^{4.2+1}$ $14^{2.2+2.1}$ $16^{2.2+2.1}$ 18 20

in the E_8 root system (continued)
W_X-orbits on X
$\{\substack{2465432\\3}\},\{\substack{23432\cdot1\\2}\},\{\substack{2\cdot\cdot\cdot321\\3}\},\{\substack{1232111\\2}\},\{\substack{2\cdot\cdot\cdot321\\2}\},\{\substack{1232111\\1},\substack{1111111\\1}\},\{\substack{1221111\\1},\substack{1122111\\1}\},\{\substack{1\cdot43211\\2},\substack{123\cdot221\\2}\},\{\substack{123\cdot221\\1},\substack{1\cdot22211\\1}\},\{\substack{00000\cdot1\\0}\},\{\substack{1354321\\3}\},\{\substack{1344321\\2},\substack{1243321\\2}\}$
$\{\substack{2465432\\3}\},\{\substack{1233321\\1}\},\{\substack{24654\cdot1\\3}\},\{\substack{\cdot343321\\2}\},\{\substack{2454321\\2}\},\{\substack{2354321\\2},\substack{1344321\\2}\},\{\substack{2344321\\2},\substack{1354321\\2}\},\{\substack{12\cdot3321\\2}\},\{\substack{1232211\\1},\substack{1222221\\1}\},\{\substack{1122211\\1},\substack{0122221\\1}\},\{\substack{\cdot111111\\0}\},\{\substack{2343221\\2},\substack{1343211\\2}\},\{\substack{12322\cdot1\\2}\},\{\substack{00\cdot1111\\0}\},\{\substack{2465321\\3}\},\{\substack{1221111\\1}\},\{\substack{1243221\\2},\substack{1233211\\2}\}$
$\{\substack{2465432\\3}\},\{\substack{1233321\\1}\},\{\substack{2\cdot\cdot\cdot321\\2}\},\{\substack{1111111\\0}\},\{\substack{123\cdot211\\1},\substack{1\cdot22221\\1}\},\{\substack{24654\cdot1\\3}\},\{\substack{0\cdot\cdot\cdot111\\0}\},\{\substack{1354321\\2},\substack{1233321\\2}\},\{\substack{1344321\\2},\substack{1243321\\2}\},\{\substack{23432\cdot1\\2}\},\{\substack{1221111\\1},\substack{1122111\\1}\},\{\substack{1\cdot43221\\2},\substack{123\cdot211\\2}\}$
$\{\substack{2465432\\3}\},\{\substack{1233321\\1}\},\{\substack{2454321\\2}\},\{\substack{24654\cdot1\\3}\},\{\substack{\cdot354321\\2}\},\{\substack{2344321\\2},\substack{1343321\\2}\},\{\substack{2343321\\2},\substack{1344321\\2}\},\{\substack{1233221\\1},\substack{1232211\\1}\},\{\substack{\cdot111111\\0}\},\{\substack{1122211\\1},\substack{0122221\\1}\},\{\substack{000\cdot111\\0}\},\{\substack{1233321\\2}\},\{\substack{0011111\\0}\},\{\substack{2343221\\2},\substack{1343211\\2}\},\{\substack{122\cdot111\\1}\},\{\substack{12432\cdot1\\2}\},\{\substack{1233211\\2},\substack{1232221\\2}\}$
$\{\substack{2465432\\3}\},\{\substack{2\cdot54321\\2}\},\{\substack{246\cdot\cdot\cdot1\\3}\},\{\substack{234\cdot\cdot\cdot1\\2}\},\{\substack{1233\cdot21\\1},\substack{1232\cdot11\\1}\},\{\substack{1\cdot4\cdot\cdot\cdot1\\2}\},\{\substack{1111111\\0}\},\{\substack{0\cdot11111\\0}\},\{\substack{1233211\\2},\substack{1232221\\2}\},\{\substack{1111111\\1}\}$
$\{\substack{2465432\\3}\},\{\substack{2\cdot54321\\2}\},\{\substack{24654\cdot1\\3}\},\{\substack{246\cdot321\\3}\},\{\substack{1354321\\2}\},\{\substack{234\cdot321\\2}\},\{\substack{23432\cdot1\\2}\},\{\substack{1233221\\1},\substack{1232211\\1}\},\{\substack{1233321\\1}\},\{\substack{1343321\\2},\substack{1244321\\2}\},\{\substack{1343221\\2},\substack{1243211\\2}\},\{\substack{1343211\\2},\substack{1243221\\2}\},\{\substack{1111111\\0}\},\{\substack{1222111\\1},\substack{1121111\\1}\},\{\substack{0\cdot11111\\0}\},\{\substack{1233211\\2},\substack{1232221\\2}\},\{\substack{1233321\\2}\}$
$\{\substack{2465432\\3}\},\{\substack{2\cdot\cdot\cdot321\\2}\},\{\substack{23432\cdot1\\2}\},\{\substack{2\cdot\cdot\cdot321\\3}\},\{\substack{1111111\\0}\},\{\substack{123\cdot221\\1},\substack{1\cdot22211\\1}\},\{\substack{1221111\\1},\substack{1122111\\1}\},\{\substack{1233321\\1}\},\{\substack{1\cdot43211\\2},\substack{123\cdot221\\2}\},\{\substack{1354321\\2},\substack{1233321\\2}\},\{\substack{1344321\\2},\substack{1243321\\2}\},\{\substack{00000\cdot1\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{2344321\\2}\},\{\substack{2465\cdot21\\3}\},\{\substack{2\cdot54321\\2}\},\{\substack{2465431\\3}\},\{\substack{2343\cdot21\\2}\},\{\substack{1\cdot44321\\2}\},\{\substack{1233\cdot21\\1}\},\{\substack{2343211\\2}\},\{\substack{1233\cdot21\\2}\},\{\substack{1222211\\1},\substack{1122111\\1}\},\{\substack{2464321\\3}\},\{\substack{1233211\\2}\},\{\substack{1111111\\0}\},\{\substack{1343221\\2},\substack{1243321\\2}\},\{\substack{1\cdot43211\\2}\},\{\substack{1111111\\1}\},\{\substack{0\cdot11111\\0}\},\{\substack{1232221\\2}\}$
$\{\substack{2465432\\3}\},\{\substack{2454321\\2}\},\{\substack{24654\cdot1\\3},\substack{\cdot34\cdot321\\2},\substack{12222\cdot1\\1}\},\{\substack{246\cdot321\\3},\substack{\cdot3432\cdot1\\2},\substack{122\cdot111\\1}\},\{\substack{1233321\\1}\},\{\substack{\cdot354321\\2},\substack{\cdot111111\\0}\},\{\substack{1233321\\2}\},\{\substack{123\cdot2\cdot1\\2}\},\{\substack{0011111\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{2\cdot54321\\2}\},\{\substack{123\cdot2\cdot1\\1}\},\{\substack{1354321\\2}\},\{\substack{1111111\\0}\},\{\substack{24654\cdot1\\3}\},\{\substack{234\cdot321\\2}\},\{\substack{0\cdot11111\\0}\},\{\substack{1232111\\1}\},\{\substack{1\cdot4\cdot321\\2}\},\{\substack{23432\cdot1\\2}\},\{\substack{1222211\\1},\substack{1122221\\1}\},\{\substack{1343221\\2},\substack{1243211\\2}\},\{\substack{000\cdot111\\0}\},\{\substack{1232111\\2}\}$
$\{\substack{2465432\\3}\},\{\substack{24\cdot4321\\\cdot},\substack{12\cdot2221\\\cdot}\},\{\substack{\cdot343\cdot21\\2}\},\{\substack{1221111\\1}\},\{\substack{\cdot3\cdot4321\\\cdot},\substack{12\cdot3\cdot21\\\cdot}\},\{\substack{\cdot343211\\2},\substack{0000\cdot11\\0}\},\{\substack{0000001\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{24654\cdot1\\3}\},\{\substack{2344321\\2}\},\{\substack{2\cdot54321\\\cdot}\},\{\substack{1\cdot44321\\2}\},\{\substack{1233321\\\cdot}\},\{\substack{23432\cdot1\\2}\},\{\substack{1\cdot43321\\2}\},\{\substack{1233221\\2},\substack{1233211\\1}\},\{\substack{1\cdot432\cdot1\\2}\},\{\substack{1232211\\2},\substack{1232221\\1}\},\{\substack{1111111\\\cdot}\},\{\substack{0001111\\0}\},\{\substack{0122111\\1}\},\{\substack{0000111\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{2454321\\\cdot}\},\{\substack{\cdot343221\\2}\},\{\substack{2465321\\3}\},\{\substack{\cdot344321\\2}\},\{\substack{1233221\\\cdot}\},\{\substack{2464321\\3}\},\{\substack{1222111\\1}\},\{\substack{2354321\\2},\substack{1354321\\3}\},\{\substack{\cdot343321\\2}\},\{\substack{\cdot343211\\2}\},\{\substack{1232221\\\cdot}\},\{\substack{1243221\\2}\},\{\substack{1233321\\\cdot}\},\{\substack{1111111\\1},\substack{0111111\\0}\},\{\substack{1244321\\2}\},\{\substack{0000011\\0}\},\{\substack{1243211\\2}\},\{\substack{0000001\\0}\}$
$\{\substack{2465432\\3}\},\{\substack{2454321\\2}\},\{\substack{24654\cdot1\\3}\},\{\substack{234\cdot321\\2}\},\{\substack{246\cdot321\\3}\},\{\substack{134\cdot321\\2}\},\{\substack{23432\cdot1\\2}\},\{\substack{12222\cdot1\\1}\},\{\substack{2354321\\2}\},\{\substack{13432\cdot1\\2}\},\{\substack{1233211\\1},\substack{1232221\\1}\},\{\substack{1233321\\2}\},\{\substack{1111111\\0}\},\{\substack{124\cdot321\\2}\},\{\substack{1233221\\2},\substack{1232211\\2}\},\{\substack{1111111\\1}\},\{\substack{0111111\\0}\},\{\substack{1232111\\2}\},\{\substack{0011111\\0}\}$

Table 5: Maximal abelian sets

X	W_X	Sig(X)
X_{29}^{372}	$\langle w_1, w_5w_7\rangle \cong {S_2}^2$	0 10^2 $12^{4+2.2}$ $14^{4+3.2+1}$ $16^{2.2}$ $18^{2.1}$ 20
X_{29}^{373}	$\langle w_3w_5w_7\rangle \cong S_2$	0 $10^{3.2}$ $12^{2.2}$ $14^{4.2+2.1}$ $16^{2.2+1}$ 18^{2+1}
X_{29}^{374}	$\langle w_4, w_3w_5w_7\rangle \cong Dih_8$	0 10 $12^{2.4+2}$ $14^{2.4+2}$ 16^{4+1} 20^2
X_{29}^{375}	$\langle w_3w_5w_7\rangle \cong S_2$	0 10^2 $12^{4.2+1}$ $14^{4.2}$ $16^{3.2+1}$ 18 20
X_{29}^{376}	$\langle w_2w_3w_6\rangle \cong S_2$	0 10 $12^{3.2+1}$ $14^{5.2+1}$ $16^{3.2}$ 18^{2+1}
X_{29}^{377}	$\langle w_7, w_1w_2, {w_3w_5}^{w_4w_3}\rangle \cong S_2 \times Dih_{10}$	0 10^5 12^{10} 14^5 16^{5+2} 22
X_{29}^{378}	$\langle w_1, w_5, w_6, w_7, w^\diamond w_2w_3w_\sigma\rangle \cong S_4 \times Dih_8$	0 12^{16} 14^6 16^4 20 22
X_{29}^{379}	$\langle w_6, w_7, w^\diamond w_2w_3w_\sigma\rangle \cong S_3 \times S_2$	0 10 $12^{2.6+2}$ 14^{6+1} 16^{2+1} 18^2 22
X_{29}^{380}	$\langle w_6, w_7, w_2w_3, {w^\diamond w_2w_3w_\sigma}^{w_4}\rangle \cong S_3 \times {S_2}^2$	0 12^{6+2} 14^{12+1} 16^4 18^2 20
X_{29}^{381}	$\langle w_2, w_1w_6, {w_1w_4w_6}^{w_3w_5w_4}\rangle \cong {S_3}^2 \times S_2$	0 6 12^{18+3} 18^6
X_{29}^{382}	$\langle w_1, w_4w_7\rangle \cong {S_2}^2$	0 $12^{4+2+2.1}$ $14^{4+4.2+1}$ $16^{2.2}$ 18^2 20
X_{29}^{383}	$\langle w_2w_3w_5\rangle \cong S_2$	0 10^{2+1} $12^{3.2+2.1}$ $14^{3.2}$ $16^{4.2+1}$ 18^2
X_{29}^{384}	$\langle w_2, w_1w_6, w_3w_7\rangle \cong S_3 \times S_2$	0 10^3 12^{6+1} 14^{6+3} 16^6 18^3
X_{29}^{385}	$\langle w_\tau, w_1w_6, w_3w_5\rangle \cong S_3 \times S_2$	0 10^3 12^6 $14^{2.6}$ 16^3 18^{3+1}
X_{29}^{386}	$\langle w_1w_4w_6\rangle \cong S_2$	0 10 $12^{3.2+1}$ $14^{5.2+1}$ $16^{3.2}$ 18^{2+1}

in the E_8 root system (continued)
W_X-orbits on X
$\{{2465432 \atop 3}\}, \{{24654\cdot1 \atop 3}\}, \{{\cdot34\cdot321 \atop 2}\}, \{{246\cdot321 \atop 3}\}, \{{\cdot354321 \atop 2}\}, \{{\cdot3432\cdot1 \atop 2}\}, \{{124\cdot321 \atop 2}\}, \{{1233211 \atop 1}, {1232221 \atop 1}\}, \{{12222\cdot1 \atop 1}\}, \{{1233321 \atop 2}\}, \{{1233221 \atop 2}, {1232211 \atop 2}\}, \{{\cdot111111 \atop 0}\}, \{{1232111 \atop 2}\}, \{{0011111 \atop 0}\}, \{{0011111 \atop 1}\}$
$\{{2465432 \atop 3}\}, \{{24654\cdot1 \atop 3}\}, \{{2\cdot54321 \atop 2}\}, \{{234\cdot321 \atop 2}\}, \{{246\cdot321 \atop 3}\}, \{{23432\cdot1 \atop 2}\}, \{{1344321 \atop 2}, {1243321 \atop 2}\}, \{{1343321 \atop 2}, {1244321 \atop 2}\}, \{{1233211 \atop 1}, {1232221 \atop 1}\}, \{{1222211 \atop 1}, {1122221 \atop 1}\}, \{{1233321 \atop 2}\}, \{{1111111 \atop 0}\}, \{{1343221 \atop 2}, {1243211 \atop 2}\}, \{{1233221 \atop 2}, {1232211 \atop 2}\}, \{{1111111 \atop 1}\}, \{{0\cdot11111 \atop 0}\}, \{{1232111 \atop 2}\}$
$\{{2465432 \atop 3}\}, \{{1233321 \atop 1}\}, \{{2\cdot\cdot\cdot321 \atop 2}\}, \{{13\cdot4321 \atop 2}, {12\cdot3321 \atop 2}\}, \{{1343321 \atop 2}, {1244321 \atop 2}\}, \{{2\cdot\cdot\cdot321 \atop 3}\}, \{{1233211 \atop 1}, {12\cdot2221 \atop 1}, {1122211 \atop 1}\}, \{{01222\cdot1 \atop 1}\}, \{{1343211 \atop 2}, {12\cdot3221 \atop 2}, {1232211 \atop 2}\}, \{{1111111 \atop 0}\}, \{{00000\cdot1 \atop 0}\}$
$\{{2465432 \atop 3}\}, \{{24654\cdot1 \atop 3}\}, \{{2\cdot54321 \atop 2}\}, \{{234\cdot321 \atop 2}\}, \{{1344321 \atop 2}, {1243321 \atop 2}\}, \{{1343321 \atop 2}, {1244321 \atop 2}\}, \{{1233321 \atop 2}\}, \{{246\cdot321 \atop 3}\}, \{{1233211 \atop 1}, {1232221 \atop 1}\}, \{{1222221 \atop 1}, {1122211 \atop 1}\}, \{{01222\cdot1 \atop 1}\}, \{{1343211 \atop 2}, {1243221 \atop 2}\}, \{{1233221 \atop 2}, {1232211 \atop 2}\}, \{{0\cdot11111 \atop 0}\}, \{{1111111 \atop 0}\}, \{{1111111 \atop 1}\}, \{{1232111 \atop 2}\}$
$\{{2465432 \atop 3}\}, \{{2465431 \atop 3}\}, \{{2465\cdot21 \atop 3}\}, \{{2454321 \atop 3}, {2354321 \atop 2}\}, \{{2343\cdot21 \atop 2}\}, \{{2344321 \atop 2}\}, \{{1\cdot44321 \atop 2}\}, \{{1343321 \atop 2}, {1243221 \atop 2}\}, \{{1343221 \atop 2}, {1243321 \atop 2}\}, \{{1233221 \atop 2}, {1233321 \atop 1}\}, \{{1232221 \atop \cdot}\}, \{{2343211 \atop 2}\}, \{{1\cdot43211 \atop 2}\}, \{{1232211 \atop 2}, {1232111 \atop 1}\}, \{{1111111 \atop \cdot}\}, \{{0111111 \atop 0}, {0011111 \atop 1}\}, \{{0001111 \atop 0}\}$
$\{{2465432 \atop 3}\}, \{{24\cdot4321 \atop \cdot}, {\cdot343321 \atop 2}\}, \{{\cdot3432\cdot1 \atop 2}, {12\cdot22\cdot1 \atop \cdot}\}, \{{1232111 \atop \cdot}, {1111111 \atop 1}, {0111111 \atop 0}, {0000111 \atop 0}\}, \{{2354321 \atop 2}, {1354321 \atop 3}, {\cdot344321 \atop 2}, {1243321 \atop 2}\}, \{{00000\cdot1 \atop 0}\}, \{{1244321 \atop 2}\}$
$\{{2465432 \atop 3}\}, \{{246\cdot\cdot\cdot1 \atop 3}, {\cdot34\cdot\cdot\cdot1 \atop 2}, {122\cdot\cdot\cdot1 \atop 1}\}, \{{123\cdot\cdot\cdot1 \atop 2}\}, \{{\cdot354321 \atop 2}, {\cdot111111 \atop 0}\}, \{{0011111 \atop 1}\}, \{{0011111 \atop 0}\}$
$\{{2465432 \atop 3}\}, \{{2454321 \atop 2}\}, \{{2465\cdot\cdot1 \atop 3}, {2343\cdot\cdot1 \atop 2}\}, \{{1343\cdot\cdot1 \atop 2}, {1222\cdot\cdot1 \atop 1}\}, \{{2464321 \atop 3}, {2344321 \atop 2}\}, \{{123\cdot\cdot\cdot1 \atop 2}\}, \{{0111111 \atop 1}\}, \{{1354321 \atop 2}, {1111111 \atop 0}\}, \{{0111111 \atop 0}\}, \{{1244321 \atop 2}, {1121111 \atop 1}\}, \{{0011111 \atop 0}\}$
$\{{2465432 \atop 3}\}, \{{2465\cdot\cdot1 \atop 3}, {2343\cdot\cdot1 \atop 2}\}, \{{2454321 \atop 3}, {2354321 \atop 2}\}, \{{1\cdot43\cdot\cdot1 \atop 2}, {1232\cdot\cdot1 \atop \cdot}\}, \{{2344321 \atop 2}\}, \{{1\cdot44321 \atop 2}, {1111111 \atop \cdot}\}, \{{0111111 \atop 0}, {0011111 \atop 1}\}, \{{0001111 \atop 0}\}$
$\{{2465432 \atop 3}\}, \{{0000001 \atop 0}\}, \{{\cdot\cdot43211 \atop 2}, {123\cdot\cdot11 \atop \cdot}, {\cdot\cdot11111 \atop \cdot}, {000\cdot\cdot11 \atop 0}\}, \{{1221111 \atop 1}, {1122111 \atop 1}, {0122211 \atop 1}\}, \{{\cdot344321 \atop 2}, {2343321 \atop 2}, {1343221 \atop 2}, {1243\cdot21 \atop 2}\}$
$\{{2465432 \atop 3}\}, \{{\cdot3\cdot4321 \atop 2}\}, \{{24654\cdot1 \atop 3}\}, \{{2465321 \atop 3}\}, \{{1233321 \atop 1}\}, \{{\cdot3432\cdot1 \atop 2}\}, \{{\cdot343321 \atop 2}\}, \{{1232211 \atop 1}, {1222221 \atop 1}\}, \{{12\cdot2111 \atop 1}\}, \{{\cdot111111 \atop 0}\}, \{{1244321 \atop 2}\}, \{{1243221 \atop 2}, {1233211 \atop 2}\}, \{{00\cdot1111 \atop 0}\}, \{{12322\cdot1 \atop 2}\}, \{{0011111 \atop 1}\}$
$\{{2465432 \atop 3}\}, \{{234\cdot321 \atop 2}\}, \{{2343221 \atop 2}\}, \{{2454321 \atop 3}, {2354321 \atop 2}\}, \{{2454321 \atop 2}, {2354321 \atop 3}\}, \{{1111111 \atop \cdot}\}, \{{2465431 \atop 3}\}, \{{2343211 \atop 2}\}, \{{1233321 \atop \cdot}\}, \{{1232221 \atop 2}, {1233221 \atop 1}\}, \{{1221111 \atop 1}, {1122111 \atop 1}\}, \{{1344321 \atop 2}, {1243321 \atop 2}\}, \{{1\cdot43221 \atop 2}\}, \{{1233211 \atop 2}, {1232211 \atop 1}\}, \{{000\cdot111 \atop 0}\}, \{{0000011 \atop 0}\}, \{{1\cdot43211 \atop 2}\}$
$\{{2465432 \atop 3}\}, \{{\cdot\cdot44321 \atop 2}\}, \{{1233\cdot\cdot1 \atop \cdot}\}, \{{0001111 \atop 0}\}, \{{\cdot\cdot54321 \atop \cdot}\}, \{{1222221 \atop 1}, {1122211 \atop 1}, {0122111 \atop 1}\}, \{{2343221 \atop 2}, {1\cdot43321 \atop 2}, {\cdot343211 \atop 2}, {1243221 \atop 2}\}, \{{0000\cdot\cdot1 \atop 0}\}$
$\{{2465432 \atop 3}\}, \{{\cdot\cdot54321 \atop 2}\}, \{{246\cdot\cdot21 \atop 3}, {123\cdot\cdot21 \atop 1}\}, \{{234\cdot321 \atop 2}, {1344321 \atop 2}, {1343221 \atop 2}, {1243\cdot21 \atop 2}\}, \{{\cdot\cdot43211 \atop 2}, {\cdot\cdot11111 \atop 0}\}, \{{1221111 \atop 1}, {1122111 \atop 1}, {0122211 \atop 1}\}, \{{123\cdot\cdot21 \atop 2}\}, \{{0000001 \atop 0}\}$
$\{{2465432 \atop 3}\}, \{{2454321 \atop 2}\}, \{{2465\cdot21 \atop 3}\}, \{{2354321 \atop 2}, {1344321 \atop 2}\}, \{{2344321 \atop 2}, {1354321 \atop 2}\}, \{{2465431 \atop 3}\}, \{{2343321 \atop 2}, {1343221 \atop 2}\}, \{{\cdot343211 \atop 2}\}, \{{12\cdot2221 \atop 1}\}, \{{1232111 \atop 1}, {1222211 \atop 1}\}, \{{\cdot111111 \atop 0}\}, \{{1244321 \atop 2}\}, \{{1243321 \atop 2}, {1233221 \atop 2}\}, \{{1122111 \atop 1}, {0122211 \atop 1}\}, \{{00\cdot1111 \atop 0}\}, \{{1232\cdot11 \atop 2}\}, \{{1232221 \atop 2}\}$

Table 5: Maximal abelian sets

X	W_X	Sig(X)
X_{29}^{387}	$\langle w_3w_5w_7\rangle \cong S_2$	0 10^{2+1} $12^{3.2+1}$ $14^{4.2+1}$ $16^{3.2}$ 18^{2+1}
X_{29}^{388}	$\langle w_4w_\tau, w_2w_3w_5\rangle \cong Dih_{16}$	0 10 12^{8} 14^{8} 16^{8+1} 18^{2}
X_{29}^{389}	$\langle w_3w_5, w_3w_5w_7{}^{w_4}\rangle \cong {S_2}^2$	0 12^{4+1} $14^{2.4+2.2+1}$ $16^{4+2.2}$ 18^{2}
X_{29}^{390}	$\langle w_1w_4, w_1w_4w_7{}^{w_3}\rangle \cong {S_2}^2$	0 12^{4+2} $14^{4+3.2}$ $16^{2.4+2+1}$ 18
X_{29}^{391}	$\langle w_1w_4w_6, w_2w_3w_5\rangle \cong Dih_{24}$	0 12 14^{12+3} 16^{12}
X_{29}^{392}	$\langle w_2w_3w_6, w_3w_5w_\tau{}^{w_2w_4w_3w_5w_7}\rangle \cong Dih_{18}$	0 $14^{2.9}$ 16^{9} 18
X_{29}^{393}	$\langle w_1, w_2, w_4, w_5\rangle \cong S_4 \times S_2$	0 3 9^{4+2} 11^{8+6} 13^{4} 17^{2} 19
X_{29}^{394}	$\langle w_1, w_3, w_4, w_5, w_7\rangle \cong S_5 \times S_2$	0 9^{5} 11^{10} 13^{10} 15^{2} 25
X_{29}^{395}	$\langle w_2, w_4, w_5, w_4w_\tau{}^{w_3w_5w_1w_6}\rangle \cong W(B_4)$	0 9 11^{16} 13^{8} 17^{2} 25
X_{29}^{396}	$\langle w_4, w_3w_5\rangle \cong Dih_8$	0 7 9^{4+1} 11^{4+1} $13^{2.4}$ 15^{4+1} 17^{2+1} 19
X_{29}^{397}	$\langle w_5, w_6, w_1w_2\rangle \cong S_3 \times S_2$	0 5 9^{3+2} 11^{6+2} 13^{6+3} 15^{2} 17 19^{2}
X_{29}^{398}	$\langle w_5w_7, w_1w_2w_5\rangle \cong {S_2}^2$	0 7^{2} $9^{2.2}$ 11^{4+2} $13^{4+2.2}$ 15^{4+2} 19^{2}
X_{29}^{399}	$\langle w_1, w_2, w_2w_5{}^{w_4}\rangle \cong S_2 \times Dih_8$	0 7 9^{4} 11^{8} $13^{4+2.2}$ 15^{4+2} 23
X_{29}^{400}	$\langle w_1, w_2, w_2w_5{}^{w_4}\rangle \cong S_2 \times Dih_8$	0 5 9^{4} 11^{8+2} $13^{2.4}$ 15^{2} 17^{2} 21
X_{29}^{401}	$\langle w_3, w_5\rangle \cong {S_2}^2$	0 5 7 $9^{2.2+1}$ $11^{4+2.2+1}$ $13^{2.2+1}$ $15^{2.2+1}$ 17 19
X_{29}^{402}	$\langle w_1, w_2w_6\rangle \cong {S_2}^2$	0 7^{2+1} $9^{2.2}$ 11^{4+1} $13^{4+3.2}$ 15^{2+1} 17^{2} 19

in the E_8 root system (continued)
W_X-orbits on X
$\{\substack{2465432\\3}\}, \{\substack{2\cdot54321\\2}\}, \{\substack{1111111\\0}\}, \{\substack{234\cdot321\\2}\}, \{\substack{23432\cdot1\\2}\}, \{\substack{1233211\\1}, \substack{1232221\\1}\}, \{\substack{1232111\\1}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{1222211\\1}, \substack{1122221\\1}\}, \{\substack{1221111\\1}, \substack{1122111\\1}\}, \{\substack{0\cdot11111\\0}\}, \{\substack{1354321\\2}\}, \{\substack{1344321\\2}, \substack{1243321\\2}\}, \{\substack{1343221\\2}, \substack{1243211\\2}\}, \{\substack{000\cdot111\\0}\}, \{\substack{1233221\\2}, \substack{1232211\\2}\}, \{\substack{1232111\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{2343221\\2}\}, \{\substack{2343211\\2}, \substack{12\cdot2111\\\cdot}, \substack{11\cdot1111\\\cdot}, \substack{0000011\\0}\}, \{\substack{2454321\\3}, \substack{23\cdot\cdot321\\2}, \substack{1\cdot\cdot3221\\2}, \substack{1232221\\1}\}, \{\substack{1\cdot43211\\2}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{0111111\\0}, \substack{0011111\\1}, \substack{000\cdot111\\0}\}, \{\substack{2465431\\3}\}, \{\substack{1343321\\2}, \substack{1244321\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{2\cdot\cdot\cdot321\\2}\}, \{\substack{1233321\\1}\}, \{\substack{2\cdot\cdot\cdot321\\3}\}, \{\substack{123\cdot221\\1}, \substack{1\cdot22211\\1}\}, \{\substack{1354321\\2}, \substack{1233321\\2}\}, \{\substack{1344321\\2}, \substack{1243321\\2}\}, \{\substack{1111111\\0}\}, \{\substack{1\cdot43211\\2}, \substack{123\cdot221\\2}\}, \{\substack{01222\cdot1\\1}\}, \{\substack{1221111\\1}, \substack{1122111\\1}\}, \{\substack{00000\cdot1\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{\cdot\cdot\cdot3321\\2}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{12\cdot2221\\1}, \substack{\cdot122211\\1}\}, \{\substack{2454321\\2}, \substack{1244321\\2}\}, \{\substack{2344321\\2}, \substack{1354321\\2}\}, \{\substack{12322\cdot1\\2}\}, \{\substack{\cdot343211\\2}, \substack{12\cdot3221\\2}\}, \{\substack{\cdot\cdot\cdot1111\\0}\}, \{\substack{1121111\\1}, \substack{0111111\\1}\}, \{\substack{2465321\\3}\}, \{\substack{1232111\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{2454321\\2}, \substack{2354321\\3}, \substack{1354321\\\cdot}, \substack{234\cdot321\\2}, \substack{1\cdot43221\\2}, \substack{1233321\\2}, \substack{1233\cdot21\\1}, \substack{1232221\\2}\}, \{\substack{2343221\\2}, \substack{1343321\\2}, \substack{1244321\\2}\}, \{\substack{\cdot\cdot\cdot3211\\2}, \substack{1232211\\1}, \substack{1222111\\1}, \substack{1121111\\1}, \substack{0111111\\1}, \substack{00\cdot\cdot\cdot11\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}, \substack{1\cdot44321\\2}, \substack{1343321\\2}, \substack{1243221\\2}, \substack{0111111\\1}, \substack{0011111\\0}, \substack{0000\cdot11\\0}\}, \{\substack{1343221\\2}, \substack{1243321\\2}, \substack{1233221\\2}, \substack{1233321\\1}, \substack{1232221\\\cdot}, \substack{0122\cdot11\\1}, \substack{0001111\\0}\}, \{\substack{2454321\\2}, \substack{2354321\\3}, \substack{2344321\\2}, \substack{1\cdot43211\\2}, \substack{1232211\\2}, \substack{1232111\\1}, \substack{1111111\\\cdot}\}, \{\substack{2343211\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{24\cdot\cdot321\\\cdot}\}, \{\substack{\cdot343221\\2}\}, \{\substack{\cdot3\cdot\cdot321\\\cdot}\}, \{\substack{12\cdot\cdot221\\\cdot}\}, \{\substack{12\cdot\cdot321\\\cdot}\}, \{\substack{\cdot343211\\2}\}, \{\substack{0000011\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{\cdot\cdot\cdot\cdot111\\0}\}, \{\substack{\cdot\cdot\cdot\cdot2\cdot1\\1}\}, \{\substack{\cdot\cdot\cdot\cdot321\\2}\}, \{\substack{00000\cdot1\\0}\}, \{\substack{1232111\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{2343221\\2}\}, \{\substack{2\cdot\cdot\cdot321\\\cdot}, \substack{1\cdot\cdot\cdot221\\\cdot}\}, \{\substack{1\cdot\cdot\cdot321\\\cdot}\}, \{\substack{2343211\\2}, \substack{0000011\\0}\}, \{\substack{0000001\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{2\cdot\cdot\cdot321\\3}\}, \{\substack{2465421\\3}\}, \{\substack{2\cdot\cdot\cdot321\\2}\}, \{\substack{2343221\\2}\}, \{\substack{13\cdot4321\\2}, \substack{12\cdot3321\\2}\}, \{\substack{1\cdot\cdot\cdot221\\2}\}, \{\substack{1\cdot\cdot\cdot211\\2}\}, \{\substack{1233321\\1}\}, \{\substack{1221111\\1}, \substack{1122111\\1}\}, \{\substack{0122221\\1}\}, \{\substack{1111111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{246\cdot\cdot21\\3}\}, \{\substack{2454321\\\cdot}\}, \{\substack{\cdot34\cdot\cdot21\\2}\}, \{\substack{2354321\\3}, \substack{1354321\\2}\}, \{\substack{123\cdot\cdot21\\\cdot}\}, \{\substack{124\cdot\cdot21\\2}\}, \{\substack{\cdot343211\\2}\}, \{\substack{1243211\\2}\}, \{\substack{1111111\\0}, \substack{0111111\\1}\}$
$\{\substack{2465432\\3}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{246\cdot321\\3}\}, \{\substack{2454321\\\cdot}\}, \{\substack{\cdot34\cdot321\\2}\}, \{\substack{2354321\\3}, \substack{1354321\\2}\}, \{\substack{\cdot3432\cdot1\\2}\}, \{\substack{124\cdot321\\2}\}, \{\substack{1233321\\\cdot}\}, \{\substack{1233221\\2}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{1232221\\1}\}, \{\substack{12432\cdot1\\2}\}, \{\substack{1111111\\0}, \substack{0111111\\1}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465421\\3}\}, \{\substack{24\cdot\cdot321\\\cdot}\}, \{\substack{\cdot3\cdot\cdot321\\\cdot}\}, \{\substack{12\cdot\cdot321\\\cdot}\}, \{\substack{\cdot343221\\2}\}, \{\substack{\cdot343211\\2}\}, \{\substack{123\cdot221\\\cdot}\}, \{\substack{1243211\\2}, \substack{1222211\\1}\}, \{\substack{0000001\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{24\cdot\cdot321\\\cdot}\}, \{\substack{\cdot3\cdot\cdot321\\\cdot}\}, \{\substack{\cdot343221\\2}\}, \{\substack{12\cdot\cdot321\\\cdot}\}, \{\substack{123\cdot221\\\cdot}\}, \{\substack{\cdot343211\\2}\}, \{\substack{1243211\\2}, \substack{1222211\\1}\}, \{\substack{0000011\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{2465421\\3}\}, \{\substack{246\cdot321\\3}\}, \{\substack{2\cdot54321\\2}\}, \{\substack{1354321\\2}\}, \{\substack{1\cdot4\cdot321\\2}\}, \{\substack{2\cdot54321\\3}\}, \{\substack{234\cdot321\\2}\}, \{\substack{1233321\\1}\}, \{\substack{1\cdot43221\\2}\}, \{\substack{123\cdot221\\1}\}, \{\substack{1233321\\2}\}, \{\substack{1\cdot43211\\2}\}, \{\substack{123\cdot221\\2}\}, \{\substack{2343211\\2}\}, \{\substack{0122211\\1}\}, \{\substack{1111111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465\cdot21\\3}\}, \{\substack{2465431\\3}\}, \{\substack{2454321\\\cdot}\}, \{\substack{\cdot344321\\2}\}, \{\substack{\cdot354321\\\cdot}\}, \{\substack{1244321\\2}\}, \{\substack{\cdot343\cdot21\\2}\}, \{\substack{\cdot343211\\2}\}, \{\substack{1233221\\2}, \substack{1233321\\1}\}, \{\substack{1233211\\\cdot}\}, \{\substack{1243\cdot21\\2}\}, \{\substack{1222221\\1}\}, \{\substack{1232211\\2}, \substack{1232111\\1}\}, \{\substack{0001111\\0}\}$

Table 5: Maximal abelian sets

X	W_X	Sig(X)
X_{29}^{403}	$\langle w_1, w_2w_7\rangle \cong {S_2}^2$	0 7^2 9^{2+1} $11^{4+2.2}$ 13^{4+2+1} $15^{3.2}$ 17 21
X_{29}^{404}	$\langle w_4, w_5, w_6, w_4{w_6}^{w_3w_5w_4w_2w_4w_3}\rangle \cong S_4 \times S_2$	0 11^{8+1} 13^{8+6} 15 17^2 19 23
X_{29}^{405}	$\langle w_2, w_4, w_2w_3{w_5}^{w_4w_3w_5}\rangle \cong S_3 \times S_2$	0 5 9 $11^{2.6+2}$ 13^{6+1} 17^{2+1} 19^2
X_{29}^{406}	$\langle w_2w_7, w_3w_5w_7\rangle \cong {S_2}^2$	0 7^2 $9^{2.2}$ $11^{2.4}$ $13^{2.2}$ 15^{4+2} $17^{2.2}$
X_{29}^{407}	$\langle w_1, w_6, w_5w_7\rangle \cong S_2 \times Dih_8$	0 9^{4+2} $13^{8+2.4+1}$ $17^{2.2}$ 21
X_{29}^{408}	$\langle w_1, w_4w_6, w_5w_7\rangle \cong S_2 \times Dih_{10}$	0 5 9^5 11^{10} 13^5 15^2 17^5
X_{29}^{409}	$\langle w_4w_6, w_5w_7\rangle \cong Dih_{10}$	0 9^5 11^{5+1} 13^5 $15^{2.5}$ 17 21
X_{29}^{410}	$\langle w_1w_2w_6\rangle \cong S_2$	0 7 9^2 $11^{4.2+1}$ $13^{3.2}$ $15^{2.2+1}$ $17^{2.2}$ 19
X_{29}^{411}	$\langle w_2w_7, w_3w_5w_7\rangle \cong {S_2}^2$	0 9^2 11^{4+2} $13^{4.2}$ $15^{2.4}$ 17^2 19^2
X_{29}^{412}	$\langle w_1w_2w_6, w_3w_5{w_7}^{w_4}\rangle \cong Dih_{14}$	0 9^7 $13^{2.7}$ 17^7
X_{29}^{413}	$\langle w_1, w_5w_7, w_4w_\tau\rangle \cong S_3 \times S_2$	0 9^3 11^6 13^{6+3} 15^6 17^3 21
X_{29}^{414}	$\langle w_5w_7\rangle \cong S_2$	0 7 $9^{2.2+1}$ $11^{2.2+1}$ $13^{3.2+2.1}$ $15^{2.2+1}$ 17^{2+1} 19
X_{29}^{415}	$\langle w_1, w_2w_5, w_5w_7\rangle \cong {S_2}^3$	0 $9^{2.2}$ 11^4 $13^{2.4+2}$ 15^{4+2} 17^2 19^2
X_{29}^{416}	$\langle w_\tau, w_1w_6, w_3w_5\rangle \cong S_3 \times S_2$	0 9^{3+1} 11^6 13^6 15^6 $17^{2.3}$
X_{29}^{417}	$\langle w_4, w_2w_5, w_3w_5w_7\rangle = [2^6]$	0 11^8 13^{8+2} 15^8 21^2
X_{29}^{418}	$\langle w_1, w_2, w_4, w_1{w_6}^{w_3w_4w_5w_4w_3}\rangle = [2^63]$	0 5 11^{16} 13^6 17^4 21
X_{29}^{419}	$\langle w_2w_5, w_3w_5w_7, w_3w_5{w_7}^{w_4}\rangle \cong S_2 \times Dih_8$	0 9^2 11^8 13^4 15^8 17^{4+2}

in the E_8 root system (continued)
W_X-orbits on X
$\{\substack{2465432\\3}\}, \{\substack{24654{\cdot}1\\3}\}, \{\substack{2454321\\{\cdot}}\}, \{\substack{2465321\\3}\}, \{\substack{{\cdot}354321\\{\cdot}}\}, \{\substack{{\cdot}344321\\2}\}, \{\substack{{\cdot}343321\\2}\}, \{\substack{{\cdot}3432{\cdot}1\\2}\}, \{\substack{1233321\\{\cdot}}\}, \{\substack{1243321\\2}\}, \{\substack{12432{\cdot}1\\2}\}, \{\substack{1233221\\2}, \substack{1233211\\1}\}, \{\substack{1232211\\2}, \substack{1232221\\1}\}, \{\substack{1221111\\1}\}, \{\substack{0001111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{24{\cdot}{\cdot}{\cdot}21\\3}, \substack{23{\cdot}{\cdot}{\cdot}21\\2}\}, \{\substack{2354321\\3}\}, \{\substack{13{\cdot}{\cdot}{\cdot}21\\2}, \substack{12{\cdot}{\cdot}{\cdot}21\\1}\}, \{\substack{12{\cdot}{\cdot}{\cdot}21\\2}\}, \{\substack{2343211\\2}\}, \{\substack{1343211\\2}, \substack{1111111\\0}\}, \{\substack{0011111\\1}\}, \{\substack{0000001\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{2465421\\3}\}, \{\substack{2{\cdot}{\cdot}4321\\{\cdot}}\}, \{\substack{13{\cdot}4321\\{\cdot}}, \substack{12{\cdot}3321\\{\cdot}}\}, \{\substack{1343321\\2}, \substack{1244321\\2}\}, \{\substack{12{\cdot}{\cdot}221\\{\cdot}}\}, \{\substack{2343221\\2}\}, \{\substack{1343211\\2}, \substack{1122211\\1}\}, \{\substack{2343211\\2}\}, \{\substack{0122111\\1}, \substack{0000111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{246{\cdot}321\\3}\}, \{\substack{23432{\cdot}1\\2}\}, \{\substack{1232111\\{\cdot}}\}, \{\substack{2{\cdot}54321\\{\cdot}}\}, \{\substack{1{\cdot}432{\cdot}1\\2}\}, \{\substack{1354321\\{\cdot}}\}, \{\substack{1221111\\1}, \substack{1122111\\1}\}, \{\substack{1233221\\2}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{1232221\\1}\}, \{\substack{234{\cdot}321\\2}\}, \{\substack{1344321\\2}, \substack{1243321\\2}\}, \{\substack{00000{\cdot}1\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{246{\cdot}{\cdot}{\cdot}1\\3}\}, \{\substack{{\cdot}354321\\2}\}, \{\substack{{\cdot}34{\cdot}{\cdot}{\cdot}1\\2}\}, \{\substack{124{\cdot}{\cdot}{\cdot}1\\2}\}, \{\substack{1233{\cdot}21\\1}, \substack{1232{\cdot}11\\1}\}, \{\substack{2454321\\3}\}, \{\substack{1233211\\2}, \substack{1232221\\2}\}, \{\substack{{\cdot}111111\\0}\}, \{\substack{0011111\\1}\}$
$\{\substack{2465432\\3}\}, \{\substack{2454321\\2}\}, \{\substack{24{\cdot}{\cdot}{\cdot}{\cdot}1\\3}\}, \{\substack{{\cdot}3{\cdot}{\cdot}{\cdot}{\cdot}1\\2}\}, \{\substack{1233{\cdot}21\\1}, \substack{1232211\\1}, \substack{122{\cdot}111\\1}\}, \{\substack{{\cdot}111111\\0}\}, \{\substack{1243221\\2}, \substack{1233321\\2}, \substack{12{\cdot}3211\\2}, \substack{1232221\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{24{\cdot}{\cdot}{\cdot}{\cdot}1\\3}\}, \{\substack{23{\cdot}{\cdot}{\cdot}{\cdot}1\\2}\}, \{\substack{2354321\\3}\}, \{\substack{13{\cdot}{\cdot}{\cdot}{\cdot}1\\2}\}, \{\substack{1243221\\2}, \substack{1233321\\2}, \substack{12{\cdot}3211\\2}, \substack{1232221\\2}\}, \{\substack{1233{\cdot}21\\1}, \substack{1232211\\1}, \substack{122{\cdot}111\\1}\}, \{\substack{1111111\\0}\}, \{\substack{0011111\\1}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{2454321\\{\cdot}}\}, \{\substack{2465{\cdot}21\\3}\}, \{\substack{{\cdot}344321\\2}\}, \{\substack{2343321\\2}, \substack{1343221\\2}\}, \{\substack{2343221\\2}, \substack{1343321\\2}\}, \{\substack{1222221\\1}\}, \{\substack{2354321\\2}, \substack{1354321\\3}\}, \{\substack{1233221\\2}, \substack{1233321\\1}\}, \{\substack{1232221\\{\cdot}}\}, \{\substack{{\cdot}343211\\2}\}, \{\substack{1243{\cdot}21\\2}\}, \{\substack{1244321\\2}\}, \{\substack{1232211\\2}, \substack{1232111\\1}\}, \{\substack{1111111\\1}, \substack{0111111\\0}\}, \{\substack{0001111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{24654{\cdot}1\\3}\}, \{\substack{2{\cdot}54321\\{\cdot}}\}, \{\substack{234{\cdot}321\\2}\}, \{\substack{1354321\\{\cdot}}\}, \{\substack{1344321\\2}, \substack{1243321\\2}\}, \{\substack{23432{\cdot}1\\2}\}, \{\substack{1233321\\{\cdot}}\}, \{\substack{1{\cdot}432{\cdot}1\\2}\}, \{\substack{1233221\\2}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{1232221\\1}\}, \{\substack{1221111\\1}, \substack{1122111\\1}\}, \{\substack{000{\cdot}111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465{\cdot}{\cdot}1\\3}, \substack{2454321\\{\cdot}}, \substack{{\cdot}344321\\2}\}, \{\substack{2354321\\2}, \substack{1354321\\3}, \substack{2343321\\2}, \substack{1343221\\2}, \substack{1233211\\{\cdot}}, \substack{1222221\\1}\}, \{\substack{2343221\\2}, \substack{1343321\\2}, \substack{1244321\\2}, \substack{{\cdot}343211\\2}, \substack{1233221\\2}, \substack{1233321\\1}\}, \{\substack{1243{\cdot}21\\2}, \substack{1232211\\2}, \substack{1232111\\1}, \substack{1111111\\1}, \substack{0111111\\0}, \substack{0001111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{24654{\cdot}1\\3}, \substack{1233321\\1}\}, \{\substack{{\cdot}3{\cdot}{\cdot}321\\2}\}, \{\substack{246{\cdot}321\\3}, \substack{1233221\\1}, \substack{1232211\\1}, \substack{12222{\cdot}1\\1}\}, \{\substack{12{\cdot}{\cdot}321\\2}\}, \{\substack{{\cdot}3432{\cdot}1\\2}, \substack{{\cdot}111111\\0}\}, \{\substack{1233211\\2}, \substack{1232221\\2}, \substack{0011111\\0}\}, \{\substack{0011111\\1}\}$
$\{\substack{2465432\\3}\}, \{\substack{2454321\\2}\}, \{\substack{24654{\cdot}1\\3}\}, \{\substack{234{\cdot}321\\2}\}, \{\substack{1233321\\1}\}, \{\substack{134{\cdot}321\\2}\}, \{\substack{12222{\cdot}1\\1}\}, \{\substack{2354321\\2}\}, \{\substack{246{\cdot}321\\3}\}, \{\substack{23432{\cdot}1\\2}\}, \{\substack{1233211\\1}, \substack{1232221\\1}\}, \{\substack{1233321\\2}\}, \{\substack{1111111\\0}\}, \{\substack{124{\cdot}321\\2}\}, \{\substack{13432{\cdot}1\\2}\}, \{\substack{0111111\\0}\}, \{\substack{1233221\\2}, \substack{1232211\\2}\}, \{\substack{1111111\\1}\}, \{\substack{0011111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{24654{\cdot}1\\3}\}, \{\substack{2454321\\{\cdot}}\}, \{\substack{{\cdot}34{\cdot}321\\2}\}, \{\substack{{\cdot}354321\\{\cdot}}\}, \{\substack{{\cdot}3432{\cdot}1\\2}\}, \{\substack{12222{\cdot}1\\1}\}, \{\substack{1233221\\2}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{1232221\\1}\}, \{\substack{124{\cdot}321\\2}\}, \{\substack{1232111\\{\cdot}}\}, \{\substack{000{\cdot}111\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{{\cdot}{\cdot}54321\\2}\}, \{\substack{2465431\\3}\}, \{\substack{246{\cdot}{\cdot}21\\3}, \substack{123{\cdot}{\cdot}21\\1}\}, \{\substack{234{\cdot}321\\2}, \substack{1344321\\2}, \substack{1343221\\2}, \substack{1243{\cdot}21\\2}\}, \{\substack{{\cdot}{\cdot}43211\\2}, \substack{{\cdot}{\cdot}11111\\0}\}, \{\substack{123{\cdot}{\cdot}21\\2}\}, \{\substack{1221111\\1}, \substack{1122111\\1}, \substack{0122211\\1}\}$
$\{\substack{2465432\\3}\}, \{\substack{2{\cdot}{\cdot}{\cdot}321\\{\cdot}}\}, \{\substack{1{\cdot}{\cdot}{\cdot}321\\{\cdot}}\}, \{\substack{23432{\cdot}1\\2}\}, \{\substack{1343211\\2}, \substack{12{\cdot}3221\\2}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{12{\cdot}2221\\1}, \substack{1122211\\1}\}, \{\substack{00000{\cdot}1\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{1244321\\2}\}, \{\substack{2465{\cdot}21\\3}, \substack{{\cdot}3{\cdot}4321\\{\cdot}}, \substack{12{\cdot}3{\cdot}21\\{\cdot}}, \substack{{\cdot}122221\\1}\}, \{\substack{12{\cdot}3211\\{\cdot}}, \substack{00{\cdot}1111\\{\cdot}}\}, \{\substack{{\cdot}343{\cdot}21\\2}\}, \{\substack{0000001\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{23432{\cdot}1\\2}\}, \{\substack{2{\cdot}{\cdot}{\cdot}321\\{\cdot}}\}, \{\substack{1232111\\{\cdot}}, \substack{1111111\\{\cdot}}\}, \{\substack{1343211\\2}, \substack{12{\cdot}3221\\2}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{12{\cdot}2221\\1}, \substack{1122211\\1}\}, \{\substack{1{\cdot}4{\cdot}321\\2}\}, \{\substack{00000{\cdot}1\\0}\}$

Table 5: Maximal abelian sets

X	W_X	Sig(X)
X_{29}^{420}	$\langle w_2, w_1w_6, w_2w_3w_5{}^{w_4w_3w_5}\rangle = [2^3 3^2]$	0 13^{18+3} 17^6 21
X_{29}^{421}	$\langle w_1w_2w_7, w_1w_4w_6{}^{w_3w_4w_5w_4w_3}\rangle \cong Dih_{18}$	0 9 $13^{2.9}$ 17^9
X_{29}^{422}	$\langle w_1w_2w_6, w_3w_5{}^{w_4w_3}$, $w_2w_5w_\sigma{}^{w_4w_3w_1w_3w_4w_2w_6w_5w_4w_2w_3w_4w_5w_6}\rangle = [2^6]$	0 13^8 15^{16} 17^4
X_{29}^{423}	$\langle w_2, w_3, w_4, w_5, w_6\rangle \cong W(D_5)$	0 10^{10} 12^{16} 16 26
X_{29}^{424}	$\langle w_2, w_3, w_4, w_4w_\tau{}^{w_3w_5w_1w_6}\rangle \cong W(B_4)$	0 2 8 10^{16} 12^8 18^2
X_{29}^{425}	$\langle w_4, w_5, w_6, w_4w_6{}^{w_3w_5w_4w_2w_4w_3}\rangle \cong S_4 \times S_2$	0 4 10^{8+1} 12^{8+6} 16 18^2 20
X_{29}^{426}	$\langle w_3, w_4, w_2w_3{}^{w_4w_5w_6}, w_2w_5{}^{w_4w_3w_1}\rangle \cong W(F_4)$	0 12^{24} 18^3 24
X_{29}^{427}	$\langle w_7, w_3w_5, w_1w_2{}^{w_4}\rangle \cong S_2 \times Dih_{10}$	0 8^5 10^2 12^{10} 14^5 16^5 20
X_{29}^{428}	$\langle w_1, w_2, w_3, w_4, w_5, w_6\rangle \cong W(E_6)$	0 11^{27} 27
X_{29}^{429}	$\langle w_3, w_4, w_2w_3{}^{w_4w_5w_6}, w_2w_5{}^{w_4w_3w_1}\rangle \cong W(F_4)$	0 3 11^{24} 19^3
X_{29}^{430}	$\langle w_1, w_2, w_3, w_4, w_5, w_6, w_8\rangle \cong S_2 \times W(E_6)$	0^2 10^{27}
X_{29}^{431}	$\langle w_1, w_3, w_2w_8, w_2w_5{}^{w_4}\rangle = [2^6 3]$	2^2 6 10^{16} 14^6 18^4
X_{29}^{432}	$\langle w_3, w_2w_5, w_1w_4w_7, w_2w_5{}^{w_4}\rangle = [2^7]$	2 4^2 10^{16} 16^8 18^2
X_{30}^{1}	$\langle w_5, w_4w_6, w_3w_7\rangle \cong W(C_3)$	1 7^6 10 11^6 14^{12} 17 18^3
X_{30}^{2}	$\langle w_3, w_4, w_6\rangle \cong S_3 \times S_2$	1 4 8^{3+2} 10^{3+2} 11 13^{6+3} 14 15^3 17^{2+1} 19
X_{30}^{3}	$\langle w_5, w_3w_7\rangle \cong S_2{}^2$	1 6^2 $8^{2.2}$ $10^{2.2}$ 11 12^2 13^{4+1} 15^{4+2+1} 17^{2+1} 19
X_{30}^{4}	$\langle w_4, w_3w_5w_7\rangle \cong Dih_8$	1 6^2 8^4 10^4 11 12^2 13^{4+2} 15^4 $17^{4+2.1}$
X_{30}^{5}	$\langle w_2, w_3, w_4, w_5, w_6, w_8\rangle \cong W(D_5) \times S_2$	1^2 9^{10} 12^{16} 17^2
X_{30}^{6}	$\langle w_3, w_4, w_2w_5\rangle \cong W(B_3)$	1 3 7 9^8 11 12^8 14^6 15 17 18^2
X_{30}^{7}	$\langle w_2, w_4, w_6\rangle \cong S_3 \times S_2$	1 5 7^2 $9^{2.3}$ 11^2 12^3 13 14^{6+2} 16^{3+1} 17 20
X_{30}^{8}	$\langle w_4, w_2w_5, w_3w_5w_7\rangle = [2^6]$	1 5^2 9^8 12^8 13^2 16^8 17

in the E_8 root system (continued)
W_X-orbits on X
$\{\substack{2465432\\3}\}, \{\substack{2\cdot\cdot4321\\\cdot}, \substack{13\cdot4321\\\cdot}, \substack{12\cdot3321\\\cdot}, \substack{12\cdot\cdot221\\\cdot}\}, \{\substack{2343221\\2}, \substack{1343321\\2}, \substack{1244321\\2}\}, \{\substack{\cdot343211\\2}, \substack{1122211\\1}, \substack{0122111\\1}, \substack{0000\cdot11\\0}\}, \{\substack{0000001\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{1221111\\1}\}, \{\substack{24\cdot4321\\\cdot}, \substack{1232221\\2}, \substack{1232211\\1}, \substack{1111111\\0}, \substack{0111111\\1}, \substack{00000\cdot1\\0}\}, \{\substack{\cdot343321\\2}, \substack{2343221\\2}, \substack{1343211\\2}, \substack{1232111\\\cdot}, \substack{1111111\\1}, \substack{0111111\\0}, \substack{0000111\\0}\}, \{\substack{\cdot344321\\2}, \substack{1243\cdot\cdot1\\2}, \substack{1233211\\2}, \substack{1233221\\1}, \substack{1122211\\1}, \substack{0122221\\1}\}$
$\{\substack{2465432\\3}\}, \{\substack{24\cdot4321\\\cdot}, \substack{2343321\\2}, \substack{1343221\\2}, \substack{12\cdot2221\\\cdot}\}, \{\substack{23\cdot4321\\2}, \substack{1354321\\3}, \substack{1344321\\2}, \substack{\cdot343211\\2}, \substack{1243321\\2}, \substack{12\cdot3221\\2}, \substack{1233321\\1}, \substack{1232211\\2}, \substack{1232111\\1}, \substack{1111111\\1}, \substack{0111111\\0}, \substack{0000\cdot11\\0}\}, \{\substack{1244321\\2}, \substack{1122111\\1}, \substack{0122211\\1}, \substack{0000001\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2\cdot\cdot\cdot\cdot21\\\cdot}\}, \{\substack{1\cdot\cdot\cdot\cdot21\\\cdot}\}, \{\substack{2343211\\2}\}, \{\substack{0000001\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{2343221\\2}\}, \{\substack{2\cdot\cdot\cdot321\\\cdot}, \substack{1\cdot\cdot\cdot221\\\cdot}\}, \{\substack{1\cdot\cdot\cdot321\\\cdot}\}, \{\substack{2343211\\2}, \substack{0000011\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{24\cdot\cdot\cdot21\\3}, \substack{23\cdot\cdot\cdot21\\2}\}, \{\substack{2354321\\3}\}, \{\substack{13\cdot\cdot\cdot21\\2}, \substack{12\cdot\cdot\cdot21\\1}\}, \{\substack{12\cdot\cdot\cdot21\\2}\}, \{\substack{2343211\\2}\}, \{\substack{1343211\\2}, \substack{1111111\\0}\}, \{\substack{0011111\\1}\}$
$\{\substack{2465432\\3}\}, \{\substack{\cdot\cdot\cdot\cdot321\\\cdot}, \substack{1\cdot\cdot\cdot221\\\cdot}\}, \{\substack{2343211\\2}, \substack{0122211\\1}, \substack{0000011\\0}\}, \{\substack{0000001\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{246\cdot321\\3}, \substack{\cdot\cdot54321\\2}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{\cdot\cdot432\cdot1\\2}, \substack{123\cdot2\cdot1\\1}\}, \{\substack{1354321\\3}, \substack{234\cdot321\\2}, \substack{1343321\\2}, \substack{1244321\\2}\}, \{\substack{1232111\\2}, \substack{1222111\\1}, \substack{1121111\\1}, \substack{0\cdot11111\\0}\}, \{\substack{1233321\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{\cdot\cdot\cdot\cdot\cdot21\\\cdot}\}, \{\substack{0000001\\0}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{\cdot\cdot\cdot\cdot321\\\cdot}, \substack{1\cdot\cdot\cdot221\\\cdot}\}, \{\substack{2343211\\2}, \substack{0122211\\1}, \substack{0000011\\0}\}$
$\{\substack{246543\cdot\\3}\}, \{\substack{\cdot\cdot\cdot\cdot\cdot21\\\cdot}\}$
$\{\substack{246543\cdot\\3}\}, \{\substack{2465421\\3}\}, \{\substack{\cdot\cdot\cdot\cdot321\\\cdot}\}, \{\substack{\cdot\cdot43221\\2}, \substack{\cdot\cdot22221\\1}\}, \{\substack{1233211\\2}, \substack{1232210\\2}, \substack{1233210\\1}, \substack{1232211\\1}\}$
$\{\substack{2465432\\3}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{\cdot\cdot\cdot\cdot321\\\cdot}\}, \{\substack{2343221\\2}, \substack{1\cdot43211\\2}, \substack{1233221\\2}, \substack{1232221\\1}, \substack{1\cdot22211\\1}, \substack{0122221\\1}\}, \{\substack{1232210\\2}, \substack{1233210\\1}\}$
$\{\substack{2465432\\3}\}, \{\substack{2\cdot\cdot\cdot\cdot\cdot1\\3}\}, \{\substack{1354321\\3}\}, \{\substack{2\cdot\cdot\cdot\cdot\cdot1\\2}\}, \{\substack{13\cdot\cdot\cdot21\\2}, \substack{124\cdot321\\2}, \substack{123\cdot221\\2}, \substack{12\cdot\cdot\cdot11\\2}\}, \{\substack{2343210\\2}\}, \{\substack{1232111\\1}, \substack{1222211\\1}, \substack{1122221\\1}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{2\cdot\cdot4321\\3}\}, \{\substack{2465\cdot21\\3}\}, \{\substack{2\cdot\cdot4321\\2}\}, \{\substack{2343\cdot21\\2}\}, \{\substack{1354321\\3}\}, \{\substack{1\cdot\cdot3\cdot21\\2}\}, \{\substack{1\cdot\cdot4321\\2}\}, \{\substack{2343211\\2}\}, \{\substack{1\cdot\cdot2221\\1}\}, \{\substack{1232\cdot11\\2}\}, \{\substack{2343210\\2}\}, \{\substack{1233211\\1}\}$
$\{\substack{2465432\\3}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{246\cdot321\\3}\}, \{\substack{2\cdot54321\\3}\}, \{\substack{2\cdot54321\\2}\}, \{\substack{234\cdot321\\2}\}, \{\substack{1354321\\3}\}, \{\substack{23432\cdot1\\2}\}, \{\substack{1\cdot4\cdot321\\2}\}, \{\substack{1354321\\2}\}, \{\substack{123\cdot2\cdot1\\2}\}, \{\substack{1343221\\2}, \substack{1243211\\2}\}, \{\substack{1233321\\1}\}, \{\substack{1222211\\1}, \substack{1122221\\1}\}, \{\substack{2343210\\2}\}, \{\substack{1232111\\1}\}$
$\{\substack{2465432\\3}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{2\cdot\cdot\cdot321\\3}\}, \{\substack{2\cdot\cdot\cdot321\\2}\}, \{\substack{1354321\\3}\}, \{\substack{23432\cdot1\\2}\}, \{\substack{13\cdot4321\\2}, \substack{12\cdot3321\\2}\}, \{\substack{1343321\\2}, \substack{1244321\\2}\}, \{\substack{1343221\\2}, \substack{12\cdot3211\\2}, \substack{1232221\\2}\}, \{\substack{1233221\\1}, \substack{12\cdot2211\\1}, \substack{1122221\\1}\}, \{\substack{2343210\\2}\}, \{\substack{1232111\\2}\}$
$\{\substack{246543\cdot\\3}\}, \{\substack{2\cdot\cdot\cdot\cdot21\\\cdot}\}, \{\substack{1\cdot\cdot\cdot\cdot21\\\cdot}\}, \{\substack{234321\cdot\\2}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{2465421\\3}\}, \{\substack{2\cdot\cdot\cdot321\\\cdot}\}, \{\substack{2343221\\2}\}, \{\substack{1\cdot\cdot\cdot321\\\cdot}\}, \{\substack{1\cdot\cdot3221\\2}, \substack{1\cdot\cdot2221\\1}\}, \{\substack{2343211\\2}\}, \{\substack{2343210\\2}\}, \{\substack{1232211\\2}, \substack{1233211\\1}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465431\\3}\}, \{\substack{2465\cdot21\\3}\}, \{\substack{24\cdot4321\\\cdot}\}, \{\substack{23\cdot4321\\\cdot}\}, \{\substack{2343\cdot21\\2}\}, \{\substack{13\cdot4321\\\cdot}\}, \{\substack{2343211\\2}\}, \{\substack{12\cdot3\cdot21\\\cdot}\}, \{\substack{1343\cdot21\\2}\}, \{\substack{12\cdot3211\\\cdot}\}, \{\substack{1122221\\1}\}, \{\substack{2343210\\2}\}, \{\substack{1221111\\1}\}$
$\{\substack{2465432\\3}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{2\cdot\cdot\cdot321\\\cdot}\}, \{\substack{1\cdot\cdot\cdot321\\\cdot}\}, \{\substack{23432\cdot1\\2}\}, \{\substack{1343211\\2}, \substack{12\cdot3221\\2}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{12\cdot2221\\1}, \substack{1122211\\1}\}, \{\substack{2343210\\2}\}$

Table 5: Maximal abelian sets

X	W_X	Sig(X)
X_{30}^{9}	$\langle w_3, w_2w_6\rangle \cong {S_2}^2$	1 5 7^2 $9^{4+2.1}$ 11^2 $12^{2.2}$ 13 14^{4+2} $16^{2.2}$ 17 18^2
X_{30}^{10}	$\langle w_2, w_3, w_6, w_7\rangle \cong S_3 \times {S_2}^2$	1 7^{3+1} 9^4 11^{3+1} 12^2 14^6 16^6 17 18^2
X_{30}^{11}	$\langle w_3, w_6, w_5w_7\rangle \cong S_2 \times Dih_8$	1 7^4 $9^{2.2}$ 11^4 12 14^8 16^{4+2} 17 20
X_{30}^{12}	$\langle w_2w_7, w_3w_5w_7\rangle \cong {S_2}^2$	1 $7^{2.2}$ 9^4 $11^{2.2}$ 12^2 14^{4+2} 16^{4+2} 17 18^2
X_{30}^{13}	$\langle w_3, w_4, w_5, w_7\rangle \cong S_4 \times S_2$	1 6^2 8^4 10^4 12^2 13^6 15^{8+1} 17 21
X_{30}^{14}	$\langle w_2, w_5\rangle \cong {S_2}^2$	1 6 $8^{2.2+1}$ $10^{2.2+1}$ 12 $13^{2.2}$ $15^{4+2+2.1}$ $17^{2+2.1}$ 19
X_{30}^{15}	$\langle w_2w_5, w_4w_6\rangle \cong Dih_{10}$	1 6 8^5 10^5 12 13^5 15^5 $17^{5+2.1}$
X_{30}^{16}	$\langle w_3, w_4, {w_3w_5w_7}^{w_4w_6w_5}\rangle \cong S_3 \wr S_2$	1 8^6 10^6 13 15^9 17^{6+1}
X_{30}^{17}	$\langle w_4w_6, w_3w_5w_7\rangle \cong Dih_{12}$	1 8^6 10^6 13 15^{6+3} 17^{6+1}
X_{30}^{18}	$\langle w_3, w_5, {w_2w_3w_6}^{w_4}\rangle = [2^43]$	1 5 9^{6+4} 13 14^{12} 17 18^4
X_{30}^{19}	$\langle w_3, w_4, w_5, w_6, w_7\rangle \cong S_6$	1 7^6 11^6 14^{15} 17 22
X_{30}^{20}	$\langle w_2, w_3w_5, {w_3w_5w_7}^{w_4}\rangle \cong S_2 \times Dih_8$	1 7^2 9^8 11^2 14^{4+2} 16^8 17 18^2
X_{30}^{21}	$\langle w_3, w_2w_7, w_5w_7\rangle \cong {S_2}^3$	1 7^2 $9^{4+2.2}$ 11^2 14^{4+2} $16^{2.4}$ 17 18^2
X_{30}^{22}	$\langle w_7, w_3w_5, {w_4w_7}^{w_6w_5w_6}\rangle = [2^43]$	1 8^6 10^6 15^{12} 17^{3+1} 19
X_{30}^{23}	$\langle w_2, w_3, w_4, {w_3w_5w_7}^{w_4w_6w_5}\rangle = [2^73^2]$	1 9^{12} 16^{16} 17
X_{30}^{24}	$\langle w_2w_7, w_4w_6, w_3w_5w_7\rangle = [2^63]$	1 9^{12} 16^{16} 17
X_{30}^{25}	$\langle w_2, w_3w_8, {w_2w_5}^{w_4}\rangle = [2^5]$	3^2 5 9^8 12^8 13 15^2 16^4 18^4
X_{31}^{1}	$\langle w_1, w_4, w_6\rangle \cong {S_2}^3$	2 5 $7^{2.2}$ 9 10^2 $12^{2.4}$ 14^2 15^{4+1} $17^{3.2}$ 19
X_{31}^{2}	$\langle w_1, w_2, w_4, w_5, w_6, w_8\rangle \cong S_5 \times {S_2}^2$	2^2 8^5 11^{10} 14^{10} 17^4
X_{31}^{3}	$\langle w_1, w_4, w_2w_5\rangle \cong S_2 \times Dih_8$	2 4 6 8^4 11^8 13^2 14^4 15^2 16^4 17^2 18^2

in the E_8 root system (continued)
W_X-orbits on X
$\{{2465432\atop 3}\},\{{2465431\atop 3}\},\{{2465\cdot21\atop 3}\},\{{2\cdot54321\atop \cdot}\},\{{2464321\atop 3}\},\{{2344321\atop 2}\},\{{2343\cdot21\atop 2}\},\{{1354321\atop \cdot}\},\{{1\cdot44321\atop 2}\},\{{2343211\atop 2}\},\{{1\cdot43\cdot21\atop 2}\},\{{1233221\atop 2},{1233321\atop 1}\},\{{1233211\atop \cdot}\},\{{1\cdot22221\atop 1}\},\{{2343210\atop 2}\},\{{1232211\atop 2},{1232111\atop 1}\}$
$\{{2465432\atop 3}\},\{{2465\cdot\cdot1\atop 3}\},\{{2464321\atop 3}\},\{{2\cdot54321\atop \cdot}\},\{{2343\cdot\cdot1\atop 2}\},\{{2344321\atop 2}\},\{{1354321\atop \cdot}\},\{{1\cdot43\cdot\cdot1\atop 2}\},\{{1233\cdot\cdot1\atop \cdot}\},\{{2343210\atop 2}\},\{{1\cdot21111\atop 1}\}$
$\{{2465432\atop 3}\},\{{246\cdot\cdot\cdot1\atop 3}\},\{{2\cdot54321\atop 3}\},\{{2\cdot54321\atop 2}\},\{{234\cdot\cdot\cdot1\atop 2}\},\{{1354321\atop 3}\},\{{1\cdot4\cdot\cdot\cdot1\atop 2}\},\{{1233\cdot21\atop 1},{1232\cdot11\atop 1}\},\{{1233211\atop 2},{1232221\atop 2}\},\{{2343210\atop 2}\},\{{1111111\atop 1}\}$
$\{{2465432\atop 3}\},\{{24654\cdot1\atop 3}\},\{{246\cdot321\atop 3}\},\{{2\cdot54321\atop \cdot}\},\{{234\cdot321\atop 2}\},\{{23432\cdot1\atop 2}\},\{{1354321\atop \cdot}\},\{{1\cdot432\cdot1\atop 2}\},\{{1344321\atop 2},{1243321\atop 2}\},\{{1233221\atop 2},{1232211\atop 2},{1233211\atop 1},{1232221\atop 1}\},\{{1233321\atop \cdot}\},\{{2343210\atop 2}\},\{{1221111\atop 1},{1122111\atop 1}\}$
$\{{2465432\atop 3}\},\{{24654\cdot1\atop 3}\},\{{2\cdot\cdot\cdot321\atop 3}\},\{{2\cdot\cdot\cdot321\atop 2}\},\{{23432\cdot1\atop 2}\},\{{1\cdot\cdot\cdot321\atop 2}\},\{{1\cdot\cdot\cdot2\cdot1\atop 2}\},\{{1233321\atop 1}\},\{{2343210\atop 2}\},\{{1111111\atop 0}\}$
$\{{2465432\atop 3}\},\{{2465421\atop 3}\},\{{246\cdot321\atop 3}\},\{{2454321\atop \cdot}\},\{{2465431\atop 3}\},\{{2354321\atop \cdot}\},\{{234\cdot321\atop 2}\},\{{2343211\atop 2}\},\{{2343221\atop 2}\},\{{1354321\atop \cdot}\},\{{134\cdot321\atop 2}\},\{{123\cdot211\atop \cdot}\},\{{124\cdot321\atop 2}\},\{{1343221\atop 2}\},\{{1222211\atop 1}\},\{{1232111\atop \cdot}\},\{{2343210\atop 2}\},\{{1222221\atop 1}\},\{{1122221\atop 1}\}$
$\{{2465432\atop 3}\},\{{2465431\atop 3}\},\{{24\cdot\cdot\cdot21\atop \cdot}\},\{{23\cdot\cdot\cdot21\atop \cdot}\},\{{2343211\atop 2}\},\{{13\cdot\cdot\cdot21\atop \cdot}\},\{{124\cdot321\atop 2},{1233221\atop 2},{12\cdot2221\atop 1}\},\{{1232\cdot11\atop 2},{1233211\atop 1},{1232111\atop 1},{1222211\atop 1}\},\{{2343210\atop 2}\},\{{1122221\atop 1}\}$
$\{{2465432\atop 3}\},\{{2\cdot\cdot\cdot\cdot\cdot1\atop 3}\},\{{2\cdot\cdot\cdot\cdot\cdot1\atop 2}\},\{{1354321\atop 3}\},\{{1\cdot\cdot3\cdot\cdot1\atop 2}\},\{{1233\cdot\cdot1\atop 1},{1\cdot\cdot1111\atop 1}\},\{{2343210\atop 2}\}$
$\{{2465432\atop 3}\},\{{2\cdot\cdot\cdot\cdot\cdot1\atop 3}\},\{{2\cdot\cdot\cdot\cdot\cdot1\atop 2}\},\{{1354321\atop 3}\},\{{1343\cdot21\atop 2},{1244321\atop 2},{12\cdot3211\atop 2},{1232221\atop 2}\},\{{1343211\atop 2},{1243221\atop 2},{1233321\atop 2}\},\{{1233\cdot21\atop 1},{1232211\atop 1},{1222111\atop 1},{11\cdot1111\atop 1}\},\{{2343210\atop 2}\}$
$\{{2465432\atop 3}\},\{{2465431\atop 3}\},\{{246\cdot\cdot21\atop 3},{234\cdot\cdot21\atop 2}\},\{{2\cdot54321\atop \cdot}\},\{{2343211\atop 2}\},\{{1354321\atop \cdot},{1\cdot4\cdot321\atop 2},{123\cdot221\atop \cdot},{1\cdot22221\atop 1}\},\{{2343210\atop 2}\},\{{1232111\atop \cdot},{1\cdot22211\atop 1}\}$
$\{{2465432\atop 3}\},\{{2\cdot\cdot\cdot\cdot\cdot1\atop 3}\},\{{2\cdot\cdot\cdot\cdot\cdot1\atop 2}\},\{{1\cdot\cdot\cdot\cdot\cdot1\atop 2}\},\{{2343210\atop 2}\},\{{1111111\atop 0}\}$
$\{{2465432\atop 3}\},\{{24654\cdot1\atop 3}\},\{{2\cdot\cdot\cdot321\atop \cdot}\},\{{23432\cdot1\atop 2}\},\{{1354321\atop \cdot},{1233321\atop \cdot}\},\{{1343321\atop 2},{1244321\atop 2}\},\{{1\cdot43221\atop 2},{123\cdot211\atop \cdot},{1\cdot22221\atop 1}\},\{{2343210\atop 2}\},\{{1222111\atop 1},{1121111\atop 1}\}$
$\{{2465432\atop 3}\},\{{24654\cdot1\atop 3}\},\{{2\cdot54321\atop \cdot}\},\{{246\cdot321\atop 3}\},\{{234\cdot321\atop 2}\},\{{23432\cdot1\atop 2}\},\{{1\cdot4\cdot321\atop 2}\},\{{1354321\atop \cdot}\},\{{1233211\atop 2},{1232221\atop 2},{1233221\atop 1},{1232211\atop 1}\},\{{1\cdot222\cdot1\atop 1}\},\{{2343210\atop 2}\},\{{1232111\atop \cdot}\}$
$\{{2465432\atop 3}\},\{{2\cdot\cdot\cdot\cdot\cdot1\atop 3}\},\{{2\cdot\cdot\cdot\cdot\cdot1\atop 2}\},\{{13\cdot4321\atop 2},{1\cdot\cdot\cdot2\cdot1\atop 2},{12\cdot3321\atop 2}\},\{{1233321\atop 1},{1221111\atop 1},{1122111\atop 1}\},\{{2343210\atop 2}\},\{{1111111\atop 0}\}$
$\{{2465432\atop 3}\},\{{2\cdot\cdot\cdot\cdot\cdot1\atop \cdot}\},\{{1\cdot\cdot4321\atop \cdot},{1\cdot\cdot2\cdot\cdot1\atop \cdot}\},\{{2343210\atop 2}\}$
$\{{2465432\atop 3}\},\{{2\cdot\cdot\cdot\cdot\cdot1\atop \cdot}\},\{{13\cdot4321\atop \cdot},{1243321\atop 2},{1233221\atop 2},{1232\cdot11\atop 2},{1233211\atop 1},{1\cdot\cdot2221\atop 1},{1232111\atop 1},{1222211\atop 1},{1122\cdot11\atop 1},{1221111\atop 1}\},\{{2343210\atop 2}\}$
$\{{246543\cdot\atop 3}\},\{{2465421\atop 3}\},\{{2\cdot\cdot\cdot321\atop \cdot}\},\{{1\cdot\cdot\cdot321\atop \cdot}\},\{{2343221\atop 2}\},\{{234321\cdot\atop 2}\},\{{123\cdot221\atop \cdot}\},\{{1343211\atop 2},{1243210\atop 2},{1222210\atop 1},{1122211\atop 1}\}$
$\{{2465432\atop 3}\},\{{2465431\atop 3}\},\{{2465\cdot21\atop 3}\},\{{24\cdot4321\atop 3}\},\{{2454321\atop 2}\},\{{\cdot354321\atop 3}\},\{{\cdot3\cdot4321\atop 2}\},\{{\cdot343\cdot21\atop 2}\},\{{\cdot343211\atop 2}\},\{{12\cdot3\cdot21\atop 2}\},\{{1244321\atop 2}\},\{{\cdot343210\atop 2}\},\{{12\cdot2221\atop 1}\},\{{1232\cdot11\atop 2}\},\{{1233211\atop 1}\}$
$\{{246543\cdot\atop 3}\},\{{24\cdot\cdot\cdot21\atop \cdot}\},\{{\cdot3\cdot\cdot\cdot21\atop \cdot}\},\{{12\cdot\cdot\cdot21\atop \cdot}\},\{{\cdot34321\cdot\atop 2}\}$
$\{{2465432\atop 3}\},\{{2465431\atop 3}\},\{{2465421\atop 3}\},\{{24\cdot\cdot321\atop \cdot}\},\{{\cdot3\cdot\cdot321\atop \cdot}\},\{{\cdot343221\atop 2}\},\{{12\cdot\cdot321\atop \cdot}\},\{{\cdot343211\atop 2}\},\{{12\cdot3221\atop 2},{12\cdot2221\atop 1}\},\{{\cdot343210\atop 2}\},\{{1232211\atop 2},{1233211\atop 1}\}$

Table 5: Maximal abelian sets		
X	W_X	Sig(X)
X_{31}^4	$\langle w_1, w_2, w_4, w_6, w_7\rangle \cong {S_3}^2 \times S_2$	$2\ 6^3\ 8^3\ 11^6\ 13^6\ 16^9\ 17^2\ 20$
X_{31}^5	$\langle w_1, w_2w_6, w_4w_7\rangle \cong S_3 \times S_2$	$2\ 6^3\ 8^3\ 11^6\ 13^6\ 14\ 16^6\ 17^2\ 18^3$
X_{31}^6	$\langle w_1, w_2w_5, w_4w_6, w_5w_7\rangle \cong S_2 \times Alt_5$	$2\ 7^6\ 12^{12}\ 17^{10+2}$
X_{31}^7	$\langle w_1, w_2w_5, w_4w_7, w_5w_8\rangle \cong S_4 \times S_2$	$4^3\ 8^4\ 11^8\ 14^4\ 15^6\ 18^6$
X_{32}^1	$\langle w_1, w_3, w_6, w_5w_7\rangle \cong S_3 \times Dih_8$	$3\ 6^4\ 9^3\ 11^3\ 14^{12}\ 17^{4+3}\ 19^2$
X_{32}^2	$\langle w_1, w_3, w_2w_5, w_2w_7\rangle \cong S_3 \times {S_2}^2$	$3\ 5^2\ 7^2\ 10^6\ 13^6\ 15^6\ 16^2\ 17^3\ 18^4$
X_{32}^3	$\langle w_1, w_2, w_3, w_5, w_6, w_8\rangle \cong {S_3}^2 \times {S_2}^2$	$3^2\ 7^3\ 10^6\ 13^9\ 16^6\ 17^6$
X_{33}	$\langle w_1, w_3, w_4, w_6, w_8\rangle \cong S_4 \times {S_2}^2$	$4^2\ 6^2\ 9^4\ 12^6\ 15^8\ 17^8\ 18^{2+1}$
X_{34}^1	$\langle w_1, w_3, w_4, w_5, w_7, w_8\rangle \cong S_5 \times S_3$	$5^3\ 8^5\ 14^{10}\ 17^{15}\ 20$
X_{34}^2	$\langle w_1, w_2, w_3, w_4, w_6, w_7, w_8\rangle \cong S_4 \times S_3$	$5^4\ 11^{10}\ 17^{20}$
X_{36}	$\langle w_1, w_3, w_4, w_5, w_6, w_7, w_8\rangle \cong S_8$	$7^8\ 17^{28}$

in the E_8 root system (continued)
W_X-orbits on X
$\{\substack{2465432\\3}\}, \{\substack{2465\cdot\cdot1\\3}\}, \{\substack{24\cdot4321\\\cdot}\}, \{\substack{\cdot3\cdot4321\\\cdot}\}, \{\substack{\cdot343\cdot\cdot1\\2}\}, \{\substack{12\cdot3\cdot\cdot1\\\cdot}\}, \{\substack{\cdot343210\\2}\}, \{\substack{1221111\\1}\}$
$\{\substack{2465432\\3}\}, \{\substack{2465\cdot\cdot1\\3}\}, \{\substack{24\cdot4321\\\cdot}\}, \{\substack{\cdot3\cdot4321\\\cdot}\}, \{\substack{\cdot343\cdot\cdot1\\2}\}, \{\substack{1244321\\2}\}, \{\substack{12\cdot3321\\2}, \substack{1243221\\2}, \substack{1233211\\\cdot}, \substack{1233221\\1}\}, \{\substack{\cdot343210\\2}\}, \{\substack{1232111\\2}, \substack{1232211\\1}, \substack{1222221\\1}\}$
$\{\substack{2465432\\3}\}, \{\substack{24\cdot\cdot\cdot\cdot1\\\cdot}\}, \{\substack{\cdot3\cdot\cdot\cdot\cdot1\\\cdot}\}, \{\substack{1243221\\2}, \substack{1233321\\2}, \substack{12\cdot3211\\2}, \substack{1233\cdot21\\1}, \substack{1232221\\2}, \substack{1232211\\1}, \substack{122\cdot111\\1}\}, \{\substack{\cdot343210\\2}\}$
$\{\substack{24654\cdot\cdot\\3}\}, \{\substack{24\cdot\cdot321\\\cdot}\}, \{\substack{\cdot3\cdot\cdot321\\\cdot}\}, \{\substack{12\cdot\cdot321\\\cdot}\}, \{\substack{\cdot3432\cdot\cdot\\2}\}, \{\substack{1243221\\2}, \substack{1233211\\2}, \substack{1232210\\2}, \substack{1233210\\1}, \substack{1232211\\1}, \substack{1222221\\1}\}$
$\{\substack{2465432\\3}\}, \{\substack{246\cdot\cdot\cdot1\\3}\}, \{\substack{\cdot\cdot54321\\3}\}, \{\substack{\cdot\cdot54321\\2}\}, \{\substack{\cdot\cdot4\cdot\cdot\cdot1\\2}\}, \{\substack{1233\cdot21\\2}, \substack{1232\cdot11\\2}\}, \{\substack{\cdot\cdot43210\\2}\}, \{\substack{1233211\\1}, \substack{1232221\\1}\}$
$\{\substack{2465432\\3}\}, \{\substack{24654\cdot1\\3}\}, \{\substack{246\cdot321\\3}\}, \{\substack{\cdot\cdot54321\\\cdot}\}, \{\substack{\cdot\cdot4\cdot321\\2}\}, \{\substack{\cdot\cdot432\cdot1\\2}\}, \{\substack{1233321\\\cdot}\}, \{\substack{\cdot\cdot43210\\2}\}, \{\substack{1233221\\2}, \substack{1232211\\2}, \substack{1233211\\1}, \substack{1232221\\1}\}$
$\{\substack{246543\cdot\\3}\}, \{\substack{246\cdot\cdot21\\3}\}, \{\substack{\cdot\cdot54321\\\cdot}\}, \{\substack{\cdot\cdot4\cdot\cdot21\\2}\}, \{\substack{123\cdot\cdot21\\\cdot}\}, \{\substack{\cdot\cdot4321\cdot\\2}\}$
$\{\substack{246543\cdot\\3}\}, \{\substack{2465\cdot21\\3}\}, \{\substack{\cdot\cdot\cdot4321\\3}\}, \{\substack{\cdot\cdot\cdot4321\\2}\}, \{\substack{\cdot\cdot\cdot3\cdot21\\2}\}, \{\substack{\cdot\cdot\cdot321\cdot\\2}\}, \{\substack{1233\cdot21\\1}\}, \{\substack{1232221\\2}\}$
$\{\substack{24654\cdot\cdot\\3}\}, \{\substack{\cdot\cdot\cdot\cdot321\\3}\}, \{\substack{\cdot\cdot\cdot\cdot321\\2}\}, \{\substack{\cdot\cdot\cdot\cdot2\cdot\cdot\\2}\}, \{\substack{1233321\\1}\}$
$\{\substack{2465\cdot\cdot\cdot\\3}\}, \{\substack{\cdot\cdot\cdot4321\\\cdot}\}, \{\substack{\cdot\cdot\cdot3\cdot\cdot\cdot\\\cdot}\}$
$\{\substack{\cdot\cdot\cdot\cdot\cdot\cdot\cdot\\3}\}, \{\substack{\cdot\cdot\cdot\cdot\cdot\cdot\cdot\\2}\}$

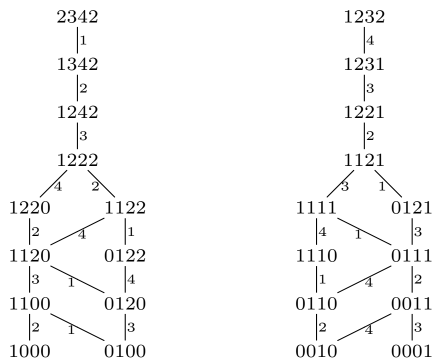
APPENDIX A

Root trees for root systems of exceptional type

In this appendix we provide what we call the ‘root tree’ for each of the root systems of exceptional type. This consists of the positive roots, arranged in levels according to height, with a line labelled with the index i drawn between each pair of roots mapped to each other by the simple reflection w_i; if there are two root lengths, we give separate arrangements for the long and short roots, and for the former we use the long height as explained in section 1.2. The information here is certainly not new, but is presented for the convenience of the reader; for example, in the sections proving the completeness of the sets $\mathcal{S}(\Phi)$, it may make it easier to follow some of the arguments which claim that one set of roots is obtained from another by applying a certain sequence of simple reflections.

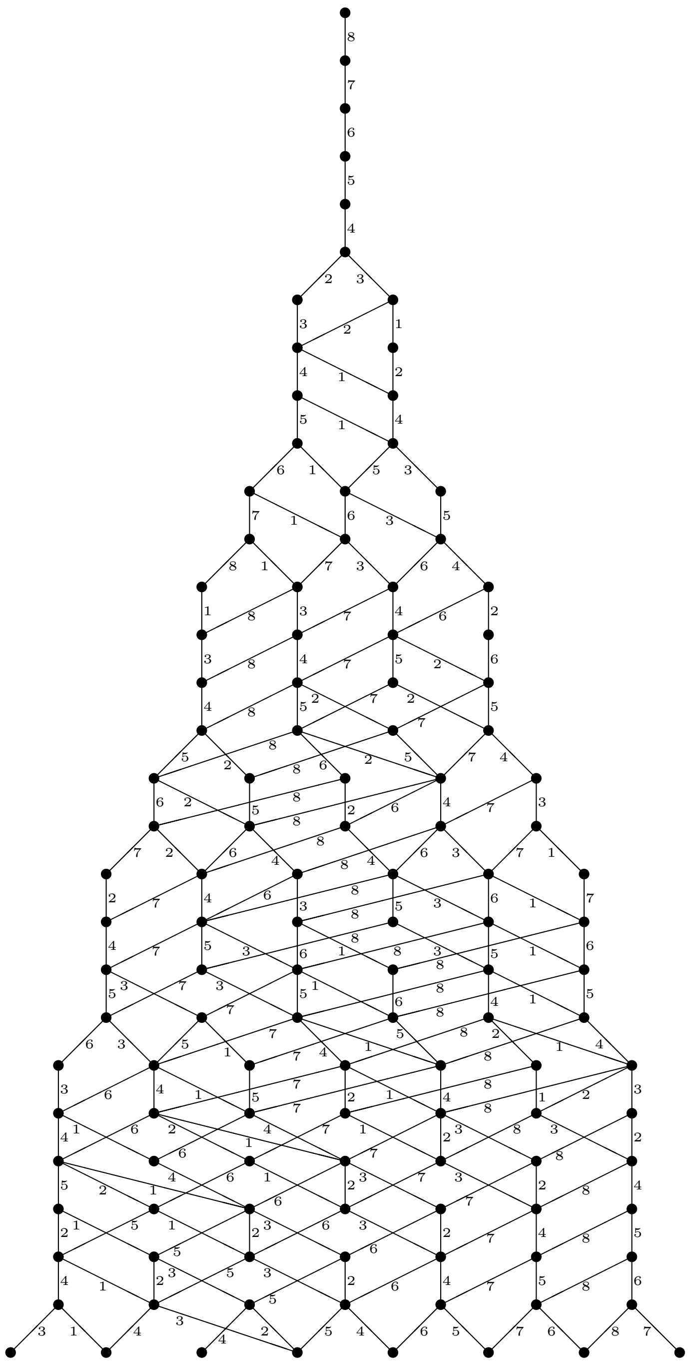
For the G_2 root system we have the following.

For the F_4 root system we have the following.



Since the E_6 and E_7 root systems are subsystems of the E_8 root system, we provide just one root tree to cover all three cases. Considerations of space mean that it is not really feasible both to position the roots on the page and to draw and label lines between them. We therefore present the information in two parts on the following pair of pages: the roots themselves are arranged on the left page (and in the interests of clarity we include lines demarcating the boundaries of the two smaller root systems here); on the right, we put nodes in place of the roots, and draw the labelled lines between these.

2465432 3							
2465431 3							
2465421 3							
2465321 3							
2464321 3							
2454321 3							
2454321 2	2354321 3						
2354321 2	1354321 3						
2344321 2	1354321 2						
2343321 2	1344321 2						
2343221 2	1343321 2	1244321 2					
2343211 2	1343221 2	1243321 2					
2343210 2	1343211 2	1243221 2	1233321 2				
1343210 2	1243211 2	1233221 2	1233321 1				
1243210 2	1233211 2	1232221 2	1233221 1				
1233210 2	1232211 2	1233211 1	1232221 1				
1232210 2	1233210 1	1232111 2	1232211 1	1222221 1			
1232110 2	1232210 1	1232111 1	1222211 1	1122221 1			
1232100 2	1232110 1	1222210 1	1222111 1	1122211 1	0122221 1		
1232100 1	1222110 1	1122210 1	1221111 1	1122111 1	0122211 1		
1222100 1	1221110 1	1122110 1	0122210 1	1121111 1	0122111 1		
1221100 1	1122100 1	1121110 1	0122110 1	1111111 1	0121111 1		
1221000 1	1121100 1	0122100 1	1111110 1	0121110 1	1111111 0	0111111 1	
1121000 1	1111100 1	0121100 1	1111110 0	0111110 1	0111111 0	0011111 1	
1111000 1	0121000 1	1111100 0	0111100 1	0111110 0	0011110 1	0011111 0	
1110000 1	1111000 0	0111000 1	0111100 0	0011100 1	0011110 0	0001111 0	
1110000 0	0110000 1	0111000 0	0011000 1	0011100 0	0001110 0	0000111 0	
1100000 0	0110000 0	0010000 1	0011000 0	0001100 0	0000110 0	0000011 0	
1000000 0	0100000 0	0000000 1	0010000 0	0001000 0	0000100 0	0000010 0	0000001 0



Bibliography

1. N. Bourbaki, *Groupes et algèbres de Lie, Chapitres 4, 5 et 6*, Hermann, Paris, 1975.
2. R.W. Carter, "Conjugacy classes in the Weyl group", *Compositio Math.* **25** (1972), 1–59.
3. J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker and R.A. Wilson, *Atlas of finite groups*, Clarendon Press, Oxford, 1985.
4. D. Gorenstein, R. Lyons and R.M. Solomon, *The classification of the finite simple groups, number 3*, Amer. Math. Soc., Providence, 1998.
5. R.M. Guralnick and G. Malle, "Classification of 2F-modules, I", *J. Alg.* **257** (2002), 348–372.
6. R.M. Guralnick and G. Malle, "Classification of 2F-modules, II", in *Finite Groups 2003, Proceedings of the Gainesville Conference on Finite Groups, March 6 - 12, 2003*, ed. C. Ho, P. Sin, P. Tiep and A. Turull, de Gruyter, Berlin, New York, 2004, 117–183.
7. R.M. Guralnick, R. Lawther and G. Malle, "The 2F-modules for nearly simple groups", *J. Alg.* **307** (2007), 643–676.
8. R. Lawther, "2F-modules, abelian sets of roots and 2-ranks", *J. Alg.* **307** (2007), 614–642; correction, *J. Alg.* **324** (2010), 3677.
9. R. Lawther, M.W. Liebeck and G.M. Seitz, "Fixed point ratios in actions of finite exceptional groups of Lie type", *Pac. J. Math.* **205** (2002), 393–464.
10. A. Mal'cev, "Commutative subalgebras of semisimple Lie algebras", *Izvestia Akad. Nauk SSSR Ser. Mat.* **9** (1945), 291–300; Amer. Math. Soc. Translations **40** (1951).
11. J. Pevtsova and J. Stark, "Varieties of elementary subalgebras of maxiaml dimension for modular Lie algebras", `arXiv:1503.01043v1 [math.RT]`.
12. G. Royle, `http://staffhome.ecm.uwa.edu.au/~00013890/remote/graphs/`.