Polynomials and Models of Type Theory

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This dissertation is the result of my own work and includes nothing that is the outcome of work done in collaboration except where specifically indicated in the text. No part of this dissertation has been submitted for any other qualification.

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Abstract

This thesis studies the structure of categories of polynomials, the diagrams that represent polynomial functors. Specifically, we construct new models of intensional dependent type theory based on these categories.

Firstly, we formalize the conceptual viewpoint that polynomials are built out of sums and products. Polynomial functors make sense in a category when there exist pseudomonads freely adding indexed sums and products to fibrations over the category, and a category of polynomials is obtained by adding sums to the opposite of the codomain fibration.

A fibration with sums and products is essentially the structure defining a categorical model of dependent type theory. For such a model the base category of the fibration should also be identified with the fibre over the terminal object. Since adding sums does not preserve this property, we are led to consider a general method for building new models of type theory from old ones, by first performing a fibrewise construction and then extending the base.

Applying this method to the polynomial construction, we show that given a fibration with sufficient structure modelling type theory, there is a new model in a category of polynomials. The key result is establishing that although the base category is not locally cartesian closed, this model has dependent product types.

Finally, we investigate the properties of identity types in this model, and consider the link with functional interpretations in logic.

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Contents

Introduction **1**

1 Polynomials, monads and fibrations **5**

1.1 Polynomials 5

1.2 Spans and internal categories 9

1.3 The arrow category 11

1.4 Fibrations and opfibrations 13

1.5 Distributivity 16

1.6 Opposites of fibrations 20

1.7 Fibrations with products 23

1.8 Polynomials 24

1.9 Polynomials in non-lcc categories 27

2 Categorical models of type theory **31**

2.1 Dependent type theory 31

2.2 Categories of types 32

2.3 Type constructors 36

2.3.1 The unit type 37

2.3.2 Dependent sum types 38

2.3.3 Dependent product types 42

2.3.4 The empty type 44

2.3.5 Binary sum types 45

2.3.6 Coherence 48

2.4 Identity types 49

2.5 Interaction of type constructors 57

2.5.1 Sums and products 57

2.5.2 Binary sums and products 58

2.5.3 Identities and sums 58

2.5.4 Identities and products 59

3 Constructing new models **63**
3.1 Extending the type theory 63
3.2 Adding sums 65
3.3 Dependent sum and product types 75
3.4 Identity types 80

4 A polynomial model **89**
4.1 Polynomials 89
4.2 A model of type theory 92
4.3 Dependent product types 94
4.4 Identity types 102
4.5 Function extensionality 104

5 Outlook **107**
5.1 Iterating polynomials 107
5.2 Dialectica-style interpretations 107
5.3 A model theory for type theory 109

A Some definitions **111**

References **119**

Introduction

The concept of a polynomial function on natural numbers, built out of sums and products, generalizes naturally to an abstract categorical setting. On sets, a polynomial functor is a functor

$$X \mapsto \sum_{a \in A} X^{B_a}$$

where $(B_a)_{a\in A}$ is a family of sets indexed by A and the sum is a disjoint union. From a computer science perspective, a functor of this form corresponds to a datatype: A defines a set of 'shapes' of data structures and for each shape a the exponent B_a is a set of 'positions' to be filled by elements of X.

Such a functor can be completely characterized by specifying just the indexing

$$B \xrightarrow{f} A \tag{0.1}$$

where $f^{-1}(a) = B_a$. The functor $\sum_{a\in A} X^{B_a}$ is then explicitly described in terms of the left and right adjoints Σ and Π of pullback functors as $\Sigma_A \Pi_f B^* : \mathbf{Set} \to \mathbf{Set}$.

In this form, polynomial functors make sense in any locally cartesian closed category $\mathcal{B}$. More generally, an indexed family of polynomials in multiple variables

$$(X_i)_{i\in I} \mapsto \left(\sum_{a\in A_j} \prod_{b \in B_a} X_{s(b)} \right)_{j\in J}$$

can be represented by a diagram

$$I \xleftarrow{s} B \xrightarrow{f} A \xrightarrow{t} J \tag{0.2}$$

in $\mathcal{B}$, which defines the polynomial functor $\Sigma_t \Pi_f s^* : \mathcal{B}/I \to \mathcal{B}/J$ on slice categories.

Notions of polynomial functors arise in a wide variety of fields (see [GK13] for examples). The categories formed by their polynomial diagrams provide a simplifying framework in which to work with such functors, and over the last decade the study of these categories has revealed a remarkably rich structure [AAG03, GK13, Hyv13].

In most cases this structure can be constructed by hand, but from a conceptual point of view it makes sense to see polynomials in terms of indexed sums and products. The category $\mathbb{P}\mathbf{oly}$ of diagrams of shape (0.2) in a category $\mathcal{B}$ is fibred over $\mathcal{B}$, and this fibration is constructed from the pseudomonads Σ and Π which freely add sums and products to fibrations. In fact the requirement that $\mathcal{B}$ be locally cartesian closed corresponds exactly to the existence of a pseudo-distributive law between Σ and Π giving $\Sigma\Pi$ the structure of a pseudomonad. The pseudomonad Π is itself a composite construction $(\Sigma(-)^{op})^{op}$, formed from Σ and the construction which takes the opposite of a fibration. From this we see that $\Sigma\Pi$ is two iterations of a more basic construction $Pol(-) = \Sigma(-)^{op}$. In particular the fibred version of the category of single-variable polynomials as in (0.1) is just given by applying Pol to the canonical indexing of $\mathcal{B}$ over itself. For a general fibration p, we think of $Pol(p)$ as the fibration of polynomials over p.

The monads Σ and Π also play a central role in the categorical perspective on dependent type theory. Type theories are formal systems used variously in studying foundations of mathematics, constructive mathematics and the formalization of programming languages. Categorical models provide a useful framework for describing semantics of these type theories. It is standard that a model of intuitionistic Martin-Löf type theory [ML84] can be essentially represented by a fibration of types over contexts, and the model has sum and product types when the fibration has the structure of a pseudoalgebra for Σ and Π (see e.g. [Jac99]).

Considering this link between the polynomials and type theories, it seems natural to ask if they can be combined in some way. Given a fibration modelling type theory with sums and products, can applying the polynomial construction to the fibration produce another such model?

There are certain points making this not quite straightforward. Firstly, in order that the base of the fibration represents the contexts of the corresponding type theory, it is necessary that the base be identified with the fibre over the terminal object. Constructions like adding sums or taking fibrewise opposites will not preserve this property. We therefore need a way of extending the fibration over a new base category to take into account the new contexts.

In addition, we would like the construction of the new model to interact with identity types of the type theory in a meaningful way. In a category, identity types correspond to certain factorizations of morphisms [AW09]. Since the base category of polynomials is cartesian closed [ALS10], it automatically has a trivial type theory structure using the fibration of product projections, in which there is no type dependency and the identity types internally identify all terms. On the other hand this category is not locally cartesian closed, so it cannot model extensional type theory, where the fibration consists of all morphisms in the category and identity is just categorical equality. Thus to get a reasonable notion of model we wish to find a class of maps intermediate between these, which is closed under dependent products and has suitable factorizations. This thesis shows that, when the original model of type theory has sufficient structure, we can in fact construct such a class of maps defining a polynomial model.

Outline of the thesis

To begin, Chapter 1 describes an abstract framework for defining polynomials. After recalling the usual construction of categories formed by polynomials and polynomial functors together with their morphisms and composition, we return to some foundations. To build up a conceptual picture of these categories, we review the basic notions of monads, fibrations and opposites of fibrations in the setting of a bicategory of spans. Taking as a template the interaction between the free fibration monad and free opfibration monad on a functor, we then see how a category of polynomials arises from the interaction between the free sum and product pseudomonads on a fibration. Finally we consider a way of making sense of this in a category which is not locally cartesian closed. Some 2-categorical concepts used are defined in Appendix A. While the constructions of sums, products and opposites for fibrations are well-known, they have not previously been studied in the context of bicategories, or for fibrations of internal categories. The connection here between distributive laws, local cartesian closure and polynomials is new.

Chapter 2 reviews some basic background on type theory and categorical models. This chapter does not contain new material, but motivates the form of categorical structures used in the rest of the thesis. There are various essentially equivalent ways of presenting a model of type theory in a category. More significantly, there are choices to be made about which type constructors to include and which rules type constructors should be required to satisfy, in particular when to admit an η-conversion rule corresponding to a strong universal property. We describe here one formulation, which is chosen to make the constructions in this thesis clearer rather than for philosophical reasons.

Chapter 3 investigates a general method for taking a model of type theory and constructing a new one from it. Thinking of models of type theory from the fibration point of view, we would like to perform some categorical constructions in the fibres. We then need to fix the base category to ensure it is identified with the fibre over the terminal object. Specifically, we freely add sums to a fibration with sufficient structure, and construct a new model by extending along the right adjoint of the fibration. Under suitable conditions the type constructors of the original model are also preserved.

Chapter 4 is the heart of the thesis. The construction of the previous chapter is applied to the opposite of a fibration, to give a model of type theory in a category of polynomials. There are many details which then need to be checked. The crucial step is showing that the display maps of this model are closed under dependent products; in doing so we also characterize the exponential morphisms in the category of polynomials over **Set**. We then construct identity types, and as an application of this model show that in constrast to many models of type theory currently studied, the principle of function extensionality does not hold in this case.

Finally, Chapter 5 explores some possible themes for future research. We look at the link between polynomials and Gödel's Dialectica interpretation, raising the question of potential extensions to other functional interpretations and how such models might fit into a general theory.

Chapter 1

Polynomials, monads and fibrations

1.1 Polynomials

We start by reviewing some of the theory of polynomials and polynomial functors [GK13, Abb03]. The setting for this section is a locally cartesian closed (lcc) category $\mathcal{B}$, so that for each object I of $\mathcal{B}$ the slice category $\mathcal{B}/I$ is cartesian closed. Equivalently, $\mathcal{B}$ has (chosen) pullbacks and for each morphism $f : I \to J$ in $\mathcal{B}$, the pullback functor $f^* : \mathcal{B}/J \to \mathcal{B}/I$ has left and right adjoints

$$\Sigma_f : \mathcal{B}/I \to \mathcal{B}/J$$
$$\Pi_f : \mathcal{B}/I \to \mathcal{B}/J$$

respectively. We also assume that $\mathcal{B}$ has a terminal object, so it has all finite limits and is cartesian closed.

Definition 1.1. A *polynomial* F in $\mathcal{B}$ is a diagram

$$\begin{tikzcd} & B \arrow[dl, "s"'] \arrow[r, "f"] & A \arrow[dr, "t"] & \\ I & & & J. \end{tikzcd}$$

The polynomial F induces a functor $P_F : \mathcal{B}/I \to \mathcal{B}/J$, called the *extension* of F, or the functor represented by F, which is the composite

$$P_F = \Sigma_t \, \Pi_f \, s^*.$$

A functor $\mathcal{B}/I \to \mathcal{B}/J$ is called a polynomial functor if it is isomorphic to one which has the above form.

In the internal language of a locally cartesian closed category (defined in Chapter 2), an object $X \to J$ of $\mathcal{B}/J$ can be thought of as a J-indexed family $(X_j)_{j\in J}$. The pullback $f^* : \mathcal{B}/J \to \mathcal{B}/I$ corresponds to reindexing, sending $(X_j)_{j\in J}$ to $(X_{f(i)})_{i\in I}$. The left adjoint Σ_f sums the components of each fibre I_j of f, sending $(X_i)_{i\in I}$ to $(\Sigma_{i\in I_j} X_i)_{j\in J}$, while the right adjoint Π_f sends $(X_i)_{i\in I}$ to the family of sections $(\Pi_{i\in I_j} X_i)_{j\in J}$. So the functor P_F takes the form of the polynomial

$$P_F : (X_i)_{i\in I} \mapsto \left(\sum_{a\in A_j} \prod_{b\in B_a} X_{s(b)} \right)_{j\in J} .$$

Example 1.2. (a) For an object A of $\mathcal{B}$, the identity functor $\mathcal{B}/A \to \mathcal{B}/A$ is represented by the polynomial

$$A \xleftarrow{=} A \xrightarrow{=} A \xrightarrow{=} A.$$

(b) The functor $A \times - : \mathcal{B} \to \mathcal{B}$ is represented by

$$1 \longleftarrow A \xrightarrow{=} A \longrightarrow 1.$$

(c) The free monoid monad $\Sigma_{n\in\mathbb{N}}(-)^n : \mathbf{Set} \to \mathbf{Set}$ is represented by

$$1 \longleftarrow \{(i,n) \mid i \leq n \in \mathbb{N}\} \xrightarrow{\pi_2} \mathbb{N} \longrightarrow 1,$$

since the fibre of π_2 over each $n \in \mathbb{N}$ is a set of size n.

The locally cartesian closed structure of $\mathcal{B}$ gives a canonical enrichment of each slice category $\mathcal{B}/I$ in $\mathcal{B}$ [Kel05]. In the internal language, the hom-object for a pair of objects $A \to I$ and $B \to I$ is

$$Hom_{\mathcal{B}/I}((A_i)_{i\in I}, (B_i)_{i\in I}) = \prod_{i\in I} B_i^{A_i}.$$

For a morphism $f : I \to J$ in $\mathcal{B}$, each of f^*, Σ_f and Π_f extends naturally to an enriched functor between the enriched slice categories. This means that all polynomial functors are enriched in $\mathcal{B}$, and the natural notion of morphism between them is an enriched natural transformation. The corresponding notion for polynomials is the following:

Definition 1.3. A *morphism of polynomials* $F \to F'$ is given by morphisms h, k, l, m in $\mathcal{B}$ making

$$\begin{tikzcd}
I \arrow[d, "h"'] & B \arrow[l, "s"'] \arrow[rr, "f"] & & A \arrow[r, "t"] \arrow[d, "k"] & J \arrow[d, "l"] \\
 & B' \times_{A'} A \arrow[u, "m"] \arrow[d] \arrow[urr] & & & \\
I' & B' \arrow[l, "s'"] \arrow[rr, "f'"'] & & A' \arrow[r, "t'"'] & J'
\end{tikzcd}$$

commute.

When h and l are identities, F and F' both represent polynomial functors $\mathcal{B}/I \to \mathcal{B}/J$. To make sense of this definition intuitively, we can think of $\mathcal{B}$ as the category of sets. For each I-indexed family $(X_i)_{i\in I}$, $P_F((X_i)_{i\in I})$ gives the J-indexed family $\left(\sum_{a\in A_j}\prod_{b\in B_a} X_{s(b)}\right)_{j\in J}$ of elements in A_j together with a function mapping each b in B_a to some $\phi(b)$ in $X_{s(b)}$. Then the morphism k defines an element $a' = k(a)$ in A'_j, and for each b' in $B_{k(a)}$ the element $\phi(m(b'))$ is in $X_{s'(b)}$. Thus we have a function

$$\left(\sum_{a\in A_j}\prod_{b\in B_a} X_{s(b)}\right)_{j\in J} \to \left(\sum_{a'\in A'_j}\prod_{b'\in B'_{a'}} X_{s'(b')}\right)_{j\in J}$$

or a component of a transformation $P_F \to P_{F'}$.

Proposition 1.4 ([GK13])**.** *Polynomials from I to J and morphisms of polynomials over I and J form a category $\mathbb{P}\mathbf{oly}_{\mathcal{B}}(I,J)$, which is equivalent to the category $\mathbb{P}\mathbf{olyFun}_{\mathcal{B}}(\mathcal{B}/I, \mathcal{B}/J)$ of polynomial functors $\mathcal{B}/I \to \mathcal{B}/J$ and enriched natural transformations.*

When h and l are not necessarily identities, a morphism as above corresponds to an enriched natural transformation between the composites with the left adjoints Σ_h and Σ_l:

$$\begin{tikzcd}
\mathcal{B}/I \arrow[r, "P_F"] \arrow[d, "\Sigma_h"'] \arrow[dr, phantom, "\swarrow"] & \mathcal{B}/J \arrow[d, "\Sigma_l"] \\
\mathcal{B}/I' \arrow[r, "P_{F'}"'] & \mathcal{B}/J'.
\end{tikzcd} \tag{1.1}$$

Given two polynomials $I \xleftarrow{s} B \xrightarrow{f} A \xrightarrow{t} J$ and $J \xleftarrow{u} D \xrightarrow{g} C \xrightarrow{v} K$ representing functors

$P_F : \mathcal{B}/I \to \mathcal{B}/J$ and $P_G : \mathcal{B}/J \to \mathcal{B}/K$, the composite functor $P_G P_F : \mathcal{B}/I \to \mathcal{B}/K$ is also polynomial. This follows using two principles which hold in locally cartesian closed categories which will reoccur throughout this thesis.

Proposition 1.5. Beck-Chevalley condition for sums (respectively products) (BCC): *For every pullback square*

$$\begin{tikzcd}
D \arrow[r, "h"] \arrow[d, "g"'] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[d, "f"] \\
C \arrow[r, "k"'] & A
\end{tikzcd}$$

in $\mathcal{B}$, the canonical map $\Sigma_g h^* \to k^* \Sigma_f$ (respectively $k^* \Pi_f \to \Pi_g h^*$) is an isomorphism.

Proposition 1.6. "Type-theoretic axiom of choice" (AC): *Given morphisms $X \xrightarrow{x} B$ and $B \xrightarrow{f} A$ in $\mathcal{B}$, there is a diagram*

$$\begin{tikzcd}
 & f^*\Pi_f x \arrow[dl, "\varepsilon"'] \arrow[r, "g"] \arrow[dd] \arrow[ddr, phantom, very near start, "\lrcorner"] & \Pi_f x \arrow[dd, "k"] \\
X \arrow[dr, "x"'] & & \\
 & B \arrow[r, "f"'] & A
\end{tikzcd}$$

where ε is the component at x of the counit of the adjunction $f^* \dashv \Pi_f$. Then the canonical morphism $\Sigma_k \Pi_g \varepsilon^* \to \Pi_f \Sigma_x$ is an isomorphism.

Using these, the composite $P_G P_F$ is represented by the polynomial

$$I \leftarrow M \to \Pi_g h \to K$$

as in the diagram

$$\begin{tikzcd}
 & & & M \arrow[r, "m"] \arrow[dddr, phantom, very near start, "\lrcorner"] \arrow[ddll, "p"'] & g^*\Pi_g h \arrow[r, "n"] \arrow[dl, "\varepsilon"'] \arrow[dd] \arrow[ddr, phantom, very near start, "\lrcorner"] & \Pi_g h \arrow[dd, "l"] & \\
 & & & E \arrow[dl, "k"'] \arrow[dr, "h"] \arrow[dd, phantom, very near start, "\vee"] & & & \\
 & B \arrow[r, "f"'] \arrow[dl, "s"'] & A \arrow[dr, "t"'] & & D \arrow[r, "g"'] \arrow[dl, "u"] & C \arrow[dr, "v"] & \\
I & & & J & & & K,
\end{tikzcd}$$

since

$$\begin{aligned}
P_G P_F &= \Sigma_v \, \Pi_g \, u^* \, \Sigma_t \, \Pi_f \, s^* \\
&\cong \Sigma_v \, \Pi_g \, \Sigma_h \, k^* \, \Pi_f \, s^* \quad (BCC) \\
&\cong \Sigma_v \, \Sigma_l \, \Pi_n \, \varepsilon^* k^* \, \Pi_f \, s^* \quad (AC) \\
&\cong \Sigma_v \, \Sigma_l \, \Pi_n \, \Pi_m \, p^* \, s^* \quad (BCC) \\
&\cong \Sigma_{vl} \, \Pi_{nm} \, (sp)^*.
\end{aligned}$$

Composition of polynomials is associative up to isomorphism and compatible with polynomial morphisms.

Proposition 1.7 ([GK13])**.** *Polynomials in $\mathcal{B}$ form the horizontal morphisms of a (pseudo) double category $\mathbb{P}\mathbf{oly}_{\mathcal{B}}$ which has $\mathcal{B}$ as its vertical category. It is equivalent as a double category to the double category $\mathbb{P}\mathbf{olyFun}_{\mathcal{B}}$ with slice categories as objects, polynomial functors as horizontal morphisms and enriched natural transformations as in (1.1) as 2-cells.*

Additionally, the double category $\mathbb{P}\mathbf{oly}_{\mathcal{B}}$ has the structure of a *framed bicategory* [Shu08] (equivalently a proarrow equipment [Woo82]). This says in particular that the functor

$$(\mathbb{P}\mathbf{oly}_{\mathcal{B}})_1 \to \mathcal{B} \times \mathcal{B}$$

projecting a polynomial onto its endpoints (I, J) is both a fibration and an opfibration.

In the rest of this chapter, we shall investigate how the structure of these categories of polynomials arises naturally in an abstract setting when considering monads and fibrations.

1.2 Spans and internal categories

While the polynomials described above correspond to functors on **Cat**, analogues of the pullbacks, sums and products used also make sense for internal categories in a setting other than **Set**. For example we might consider categories in other sheaf toposes. It is interesting to investigate what structure of **Set** is needed to develop the theory of polynomials. We will start by working merely with a category $\mathcal{E}$ with pullbacks, and add other conditions as they are required. The case $\mathcal{E} = \mathbf{Set}$ will be a running example throughout this section, and is the only case considered in later chapters.

As a first step towards the construction of polynomials, we recall the well-known construction of internal categories as monads. Let $\mathcal{E}$ be a category with (chosen) pullbacks. Then there is a bicategory $\mathbf{Span}(\mathcal{E})$ of spans in $\mathcal{E}$, where the objects are the objects of $\mathcal{E}$, 1-cells $X \to\!\!\!\!\!\to Y$ are spans of arrows

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ X & & Y, \end{tikzcd}$$

and 2-cells are maps of spans. Composition is given by pullback.

If $\mathcal{E}$ is a 2-category, then $\mathbf{Span}(\mathcal{E})$ is a bicategory enriched in **2-Cat** (as defined in Appendix A): each hom-category has the structure of a 2-category and this structure is compatible with horizontal composition. The 3-cells of $\mathbf{Span}(\mathcal{E})$ are the 2-cells in $\mathcal{E}$

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] \arrow[dd, bend right] \arrow[dd, bend left] & \\ X & \Rightarrow & Y \\ & B \arrow[ul] \arrow[ur] & \end{tikzcd}$$

which are vertical over X and Y.

To equip a 1-cell

$$\begin{tikzcd} & A_1 \arrow[dl, "d"'] \arrow[dr, "c"] & \\ A_0 & & A_0 \end{tikzcd}$$

in $\mathbf{Span}(\mathcal{E})$ with the structure of a monad $\mathcal{A}$ is exactly to equip $A_1 \overset{d}{\underset{c}{\rightrightarrows}} A_0$ with the structure of identities $A_0 \to A_1$ and composition $A_1 \times_{A_0} A_1 \to A_1$ of an internal category in $\mathcal{E}$.

Thus monads in $\mathbf{Span}(\mathcal{E})$ are the objects of the 2-category $\mathbf{Cat}(\mathcal{E})$ of internal categories. A 1-cell $f\colon \mathcal{A} \to \mathcal{B}$ is an internal functor, that is a diagram

$$\begin{tikzcd} A_0 \arrow[d, "f_0"'] & A_1 \arrow[l, "d"'] \arrow[r, "c"] \arrow[d, "f_1"] & A_0 \arrow[d, "f_0"] \\ B_0 & B_1 \arrow[l, "d"] \arrow[r, "c"'] & B_0 \end{tikzcd}$$

preserving the category structure; and a 2-cell $\alpha : f \Rightarrow g$ between 1-cells $\mathcal{A} \underset{g}{\overset{f}{\rightrightarrows}} \mathcal{B}$ is an internal natural transformation, that is a map $A_0 \to B_1$ satisfying the usual equations. Garner and Shulman show in [GS13] how internal categories and profunctors form a proarrow equipment arising from $\mathcal{E}$ together with $\mathbf{Span}(\mathcal{E})$, but we will not consider this here.

The category $\mathbf{Cat}(\mathcal{E})$ has pullbacks, so we can repeat the construction to form the **2-Cat**-enriched bicategory $\mathbf{Span}(\mathbf{Cat}(\mathcal{E}))$. Monads in $\mathbf{Span}(\mathbf{Cat}(\mathcal{E}))$ are internal categories in $\mathbf{Cat}(\mathcal{E})$, which are (strict) double categories.

1.3 The arrow category

Let $\mathcal{B} \in \mathbf{Cat}(\mathcal{E})$ be an internal category. We consider a particular monad on $\mathcal{B}$ in $\mathbf{Span}(\mathbf{Cat}(\mathcal{E}))$, i.e. a double category in $\mathcal{E}$.

Using pullbacks in $\mathcal{E}$, we can construct the internal category of arrows $\mathcal{B}^{\mathbb{2}}$. This is the cotensor of $\mathcal{B}$ with the category $\mathbb{2} = \bullet \to \bullet$, i.e. it is equipped with functors and a natural transformation

$$\begin{tikzcd} \mathcal{B}^{\mathbb{2}} \arrow[r, bend left, "d"] \arrow[r, bend right, "c"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{B} \end{tikzcd}$$

and is universal with this data.

This is equivalently described as the comma object

$$\begin{tikzcd} \mathcal{B}^{\mathbb{2}} \arrow[r, "c"] \arrow[d, "d"'] \arrow[dr, phantom, "\overset{\alpha}{\nearrow}"] & \mathcal{B} \arrow[d, "1_{\mathcal{B}}"] \\ \mathcal{B} \arrow[r, "1_{\mathcal{B}}"'] & \mathcal{B} \end{tikzcd}$$

over the identity cospan $\mathcal{B} \to \mathcal{B} \leftarrow \mathcal{B}$, or as the lax limit

$$\begin{tikzcd} & \mathcal{B}^{\mathbb{2}} \arrow[dl, "c"'] \arrow[dr, "d"] \arrow[d, phantom, "\underset{\alpha}{\Leftarrow}"] & \\ \mathcal{B} & & \mathcal{B} \arrow[ll, "1_{\mathcal{B}}"] \end{tikzcd}$$

of the identity arrow on $\mathcal{B}$.

This universal property applied to the natural transformations

$$
\begin{tikzcd}
& \mathcal{B} \arrow[dl, "1_{\mathcal{B}}"'] \arrow[dr, "1_{\mathcal{B}}"] \arrow[d, phantom, "="] & \\
\mathcal{B} & & \mathcal{B} \arrow[ll, "1_{\mathcal{B}}"]
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
& & \mathcal{B}^{\mathbb{2}} \times_{\mathcal{B}} \mathcal{B}^{\mathbb{2}} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[d, phantom, "\vee"] & & \\
& \mathcal{B}^{\mathbb{2}} \arrow[dl, "c"'] \arrow[dr, "d"] \arrow[rr, phantom, "="] \arrow[d, phantom, "\Leftarrow \alpha"] & & \mathcal{B}^{\mathbb{2}} \arrow[dl, "c"'] \arrow[dr, "d"] \arrow[d, phantom, "\Leftarrow \alpha"] & \\
\mathcal{B} & & \mathcal{B} \arrow[ll, "1_{\mathcal{B}}"] & & \mathcal{B} \arrow[ll, "1_{\mathcal{B}}"]
\end{tikzcd}
$$

determines maps $\eta : \mathcal{B} \to \mathcal{B}^{\mathbb{2}}$ and $\mu : \mathcal{B}^{\mathbb{2}} \times_{\mathcal{B}} \mathcal{B}^{\mathbb{2}} \to \mathcal{B}^{\mathbb{2}}$ giving the span

$$
\begin{tikzcd}
& \mathcal{B}^{\mathbb{2}} \arrow[dl, "c"'] \arrow[dr, "d"] & \\
\mathcal{B} & & \mathcal{B}
\end{tikzcd}
$$

the structure of a monad $\Phi_{\mathcal{B}}$ in $\mathbf{Span}(\mathbf{Cat}(\mathcal{E}))$.

Example 1.8 ($\mathcal{E} = \mathbf{Set}$). When $\mathcal{E}$ is $\mathbf{Set}$, $\mathbf{Cat}(\mathcal{E})$ is the category of small categories $\mathbf{Cat}$. The monad $\Phi_{\mathcal{B}}$ is given by the usual category of arrows and commutative squares $\mathcal{B}^{\mathbb{2}}$, with d and c the domain and codomain functors.

A monad in a **2-Cat**-enriched bicategory acts by composition as a 2-monad on each of the hom-2-categories. Thus $\Phi_{\mathcal{B}}$ defines by composition on one side a 2-monad on $\mathbf{Span}(\mathbf{Cat}(\mathcal{E}))(\mathcal{A}, \mathcal{B})$, and on the other a 2-monad on $\mathbf{Span}(\mathbf{Cat}(\mathcal{E}))(\mathcal{B}, \mathcal{C})$, for all $\mathcal{A}$, $\mathcal{C}$ in $\mathbf{Cat}(\mathcal{E})$. Moreover, the definition of $\Phi_{\mathcal{B}}$ as a limit in a 2-category gives these monads a form of uniqueness property which is characteristic of monads involving limits and colimits. Recall from [Koc95]:

Definition 1.9. A pseudomonad (T, η, μ) on a 2-category is *lax-idempotent* (also called Kock-Zöberlein) if the following equivalent conditions hold:

1. The multiplication μ is left adjoint to ηT with invertible counit,
2. The multiplication μ is right adjoint to $T\eta$ with invertible unit,
3. there is a modification $\delta : T\eta \to \eta T$ such that $\delta\eta = 1$ and $\mu\delta = 1$,
4. to give an object A a T-pseudoalgebra structure is exactly to give a left adjoint to $\eta_A : A \to TA$ with invertible counit.

Dually, a pseudomonad is *colax-idempotent* if the multiplication is right adjoint to ηT with invertible unit.

A pseudomonad in a **2-Cat**-enriched bicategory is called lax-idempotent if it acts as a lax-idempotent pseudomonad on the left, equivalently if it acts as a colax-idempotent pseudomonad on the right.

In particular, the 2-dimensional universal property of the arrow category $\mathcal{B}^{\mathbb{2}}$ determines a 3-cell

$$\begin{tikzcd} \mathcal{B}^{\mathbb{2}} \arrow[r, bend left, "\eta\Phi_{\mathcal{B}}"] \arrow[r, bend right, "\Phi_{\mathcal{B}}\eta"'] \arrow[r, phantom, "\Downarrow\delta"] & \mathcal{B}^{\mathbb{2}} \times_{\mathcal{B}} \mathcal{B}^{\mathbb{2}} \end{tikzcd}$$

satisfying $\delta\eta = 1$ and $\mu\delta = 1$, so $\Phi_{\mathcal{B}}$ is colax-idempotent.

1.4 Fibrations and opfibrations

We now take a closer look at the 2-monads that $\Phi_{\mathcal{B}}$ induces by composition.

Consider the slice $\mathbf{Cat}(\mathcal{E})/\mathcal{B}$ for an object $\mathcal{B}$. When $\mathcal{E}$ has a terminal object, then this can be identified with either of the hom-2-categories $\mathbf{Span}(\mathbf{Cat}(\mathcal{E}))(\mathcal{B}, 1)$ or $\mathbf{Span}(\mathbf{Cat}(\mathcal{E}))(1, \mathcal{B})$. So composing with monad $\Phi_{\mathcal{B}}$ gives two 2-monads on $\mathbf{Cat}(\mathcal{E})/\mathcal{B}$ which send an object $\mathcal{A} \to \mathcal{B}$ to the composites $d^*\mathcal{A} \to \mathcal{B}^{\mathbb{2}} \xrightarrow{c} \mathcal{B}$ and $c^*\mathcal{A} \to \mathcal{B}^{\mathbb{2}} \xrightarrow{d} \mathcal{B}$ respectively, as in the diagrams

$$\begin{tikzcd}[column sep=small] & & d^*\mathcal{A} \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45] & & \\ & \mathcal{B}^{\mathbb{2}} \arrow[dl, "c"'] \arrow[dr, "d"] & & \mathcal{A} \arrow[dl] & \\ \mathcal{B} & & \mathcal{B} & & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & c^*\mathcal{A} \arrow[dl] \arrow[dr] \arrow[d, phantom, "\lrcorner" rotate=-45] & & & \\ \mathcal{A} \arrow[dr] & & \mathcal{B}^{\mathbb{2}} \arrow[dl, "c"'] \arrow[dr, "d"] & & \\ & \mathcal{B} & & \mathcal{B}. & \end{tikzcd}$$

Definition 1.10. A module for $\Phi_{\mathcal{B}}$ acting on the left hom-2-category is called a *fibration* and on the right hom-2-category an *opfibration*; strict left and right modules are *strict fibrations* and *strict opfibrations* respectively.

Note that since $\Phi_{\mathcal{B}}$ is colax-idempotent these are 'property-like' structures – a morphism can have at most one module structure up to isomorphism.

Example 1.11 ($\mathcal{E} = \mathbf{Set}$). To give a functor $\mathcal{A} \xrightarrow{p} \mathcal{B}$ in $\mathbf{Cat}$ the structure of a left $\Phi_{\mathcal{B}}$-module is exactly to give p the structure of a cloven Grothendieck fibration, i.e. to give a chosen cartesian lifting $f^*J \to J$ for each morphism $f : I \to pJ$ in $\mathcal{B}$. Likewise to give $\mathcal{A} \xrightarrow{p} \mathcal{B}$ the structure of a right $\Phi_{\mathcal{B}}$-module is to give p the structure of a cloven Grothendieck opfibration, i.e. a chosen opcartesian lifting $I \to f_!I$ for each morphism $f : pI \to J$ in $\mathcal{B}$. Strict fibrations and opfibrations correspond to split Grothendieck fibrations and opfibrations.

The morphism $\mathcal{B}^{\mathbb{2}} \xrightarrow{d} \mathcal{B}$ is naturally a fibration, and $\mathcal{B}^{\mathbb{2}} \xrightarrow{c} \mathcal{B}$ is an opfibration.

Definition 1.12. The internal category $\mathcal{B}$ *has pullbacks* if c is also a fibration.

This definition is a generalization of the case in **Cat**:

Example 1.13 ($\mathcal{E} = \mathbf{Set}$). A functor

$$\begin{tikzcd}
\mathcal{B}^{\mathbb{2}} \times_{\mathcal{B}} \mathcal{B}^{\mathbb{2}} \arrow[dd, "e"'] \arrow[dr, "\pi_2"] & & \\
& \mathcal{B}^{\mathbb{2}} \arrow[dr, "d"] & \\
\mathcal{B}^{\mathbb{2}} \arrow[rr, "c"'] & & \mathcal{B}
\end{tikzcd}$$

in **Cat** gives c the structure of a left $\Phi_{\mathcal{B}}$-module exactly when e sends a cospan $I \xrightarrow{f} K \xleftarrow{g} J$ in $\mathcal{B}$ to a pullback of f along g.

Definition 1.14. A span $\mathcal{A} \xleftarrow{q} \mathcal{M} \xrightarrow{p} \mathcal{B}$ is a *two-sided fibration* if it is a Φ-bimodule, i.e. has the structure of a right $\Phi_{\mathcal{A}}$-module and left $\Phi_{\mathcal{B}}$-module in a compatible way (See Definition A.6).

Example 1.15 ($\mathcal{E} = \mathbf{Set}$). In **Cat**, a span $\mathcal{A} \xleftarrow{q} \mathcal{M} \xrightarrow{p} \mathcal{B}$ is a two-sided fibration iff:

- p is a cloven fibration with q-vertical cartesian liftings $f^*J \to J$ for each morphism $I \xrightarrow{f} pJ$ in $\mathcal{B}$,
- q is a cloven opfibration with p-vertical opcartesian liftings $J \to g_!J$ for each $qJ \xrightarrow{g} K$ in $\mathcal{A}$,
- each canonical morphism $g_!f^*J \to f^*g_!J$ is an isomorphism.

In particular, every morphism $I \to J$ in the category $\mathcal{M}$ factors into three

$$I \xrightarrow{\alpha} \bullet \xrightarrow{\beta} \bullet \xrightarrow{\gamma} J$$

where α is q-opcartesian p-vertical, β is p, q-vertical, and γ is p-cartesian q-vertical, and this factorization is unique up to unique vertical isomorphisms.

For a general 2-category $\mathcal{E}$ with pullbacks, two-sided fibrations in $\mathbf{Cat}(\mathcal{E})$ can be defined representably: A span $\mathcal{B} \xleftarrow{q} \mathcal{M} \xrightarrow{p} \mathcal{A}$ is a two-sided fibration iff

$$\mathbf{Cat}(\mathcal{E})(\mathcal{C}, \mathcal{B}) \xleftarrow{q_*} \mathbf{Cat}(\mathcal{E})(\mathcal{C}, \mathcal{M}) \xrightarrow{p_*} \mathbf{Cat}(\mathcal{E})(\mathcal{C}, \mathcal{A})$$

is a two-sided fibration in **Cat** for each $\mathcal{C}$ in $\mathbf{Cat}(\mathcal{E})$, and for each $f : \mathcal{C} \to \mathcal{D}$ in $\mathbf{Cat}(\mathcal{E})$ the functor $\mathbf{Cat}(\mathcal{E})(\mathcal{D}, \mathcal{M}) \xrightarrow{-\circ f} \mathbf{Cat}(\mathcal{E})(\mathcal{C}, \mathcal{M})$ preserves p-cartesian and q-opcartesian morphisms.

Two-sided fibrations were defined by Street in [Str74], under the name *bifibrations*. For each pair of objects $\mathcal{A}$ and $\mathcal{B}$, the two-sided fibrations from $\mathcal{A}$ to $\mathcal{B}$ assemble into a 2-category $\mathbf{Fib}(\mathcal{E})(\mathcal{A}, \mathcal{B})$. It has as objects bimodules, as 1-cells the maps of spans which preserve cartesian and opcartesian morphisms, and as 2-cells the 2-cells of $\mathbf{Span}(\mathbf{Cat}(\mathcal{E}))(\mathcal{A}, \mathcal{B})$.

Moreover, when $\mathcal{E}$ has sufficient structure, these 2-categories form the hom-2-categories of a **2-Cat**-enriched bicategory $\mathbf{Fib}(\mathcal{E})$. The composite

$$\begin{tikzcd} & \mathcal{N} \otimes \mathcal{M} \arrow[dl] \arrow[dr] & \\ \mathcal{A} & & \mathcal{C} \end{tikzcd}$$

(also written as $\mathcal{A} \leftarrow \mathcal{N}\mathcal{M} \to \mathcal{C}$) of bimodules $\mathcal{A} \leftarrow \mathcal{M} \to \mathcal{B}$ and $\mathcal{B} \leftarrow \mathcal{N} \to \mathcal{C}$ is given by composing as spans and then quotienting out by the action of $\Phi_{\mathcal{B}}$, so in other words it is the following coequalizer:

$$\begin{tikzcd} \mathcal{M} \times_{\mathcal{B}} \mathcal{B}^{\mathbb{2}} \times_{\mathcal{B}} \mathcal{N} \arrow[r, shift left] \arrow[r, shift right] & \mathcal{M} \times_{\mathcal{B}} \mathcal{N} \arrow[r] & \mathcal{N} \otimes \mathcal{M}. \end{tikzcd}$$

The identity for composition is the span $\Phi_{\mathcal{B}} = \mathcal{B} \xleftarrow{c} \mathcal{B}^{\mathbb{2}} \xrightarrow{d} \mathcal{B}$. Composition is associative (up to isomorphism) because c and d are an opfibration and fibration respectively, so are both exponentiable in $\mathbf{Cat}(\mathcal{E})$ when $\mathcal{E}$ is locally cartesian closed [Gir64, Joh77], and pulling back along either morphism commutes with coequalizers. The required reflexive coequalizers exist in $\mathbf{Cat}(\mathcal{E})$ when $\mathcal{E}$ has pullback-stable finite colimits and free cartesian monoids. Thus for example the **2-Cat**-enriched bicategory $\mathbf{Fib}(\mathcal{E})$ is defined whenever $\mathcal{E}$ is locally cartesian closed and has countable colimits, or when $\mathcal{E}$ is a topos with a natural numbers object [Joh77].

By the symmetry of $\mathbf{Span}(\mathbf{Cat}(\mathcal{E}))$, reversing $\Phi_{\mathcal{B}}$ gives a span

$$\Psi_{\mathcal{B}} = \begin{tikzcd} & \mathcal{B}^{\mathbb{2}} \arrow[dl, "d"'] \arrow[dr, "c"] & \\ \mathcal{B} & & \mathcal{B}, \end{tikzcd}$$

which is also a monad in $\mathbf{Span}(\mathbf{Cat}(\mathcal{E}))$, and is lax-idempotent. A left module for $\Psi_{\mathcal{B}}$ is the reverse of a right $\Phi_{\mathcal{B}}$-module, or in other words a span $\mathcal{A} \xleftarrow{q} \mathcal{M} \xrightarrow{p} \mathcal{B}$ where

p has the structure of a cloven opfibration and the structure map commutes with q. Similarly a right $\Psi_{\mathcal{B}}$-module is a span $\mathcal{B} \xleftarrow{q} \mathcal{M} \xrightarrow{p} \mathcal{C}$ where q has the structure of a cloven fibration and the structure map commutes with p.

The span $\Psi_{\mathcal{B}}$ is not a two-sided fibration, even when c is a fibration, as the compatibility condition between c and d does not hold.

1.5 Distributivity

Although $\Psi_{\mathcal{B}}$ is not a $\Phi_{\mathcal{B}}$-module, we can still study the combination of module structures for $\Phi_{\mathcal{B}}$ and $\Psi_{\mathcal{B}}$ by considering pseudo-distributive laws between the two monads.

Definition 1.16. A *pseudo-distributive law* of a monad S over a monad T in a **2-Cat**-enriched bicategory consists of a 2-cell $\lambda\colon ST \to TS$ and invertible 3-cells

$$\begin{tikzcd}
S^2T \arrow[r, "S\lambda"] \arrow[d, "\mu_S T"'] & STS \arrow[r, "\lambda S"] \arrow[d, phantom, "\Swarrow\alpha"] & TS^2 \arrow[d, "T\mu_S"] \\
ST \arrow[rr, "\lambda"'] & & TS
\end{tikzcd}
\qquad
\begin{tikzcd}
ST^2 \arrow[r, "\lambda T"] \arrow[d, "S\mu_T"'] & TST \arrow[r, "T\lambda"] \arrow[d, phantom, "\beta\Nearrow"] & T^2S \arrow[d, "\mu_T S"] \\
ST \arrow[rr, "\lambda"'] & & TS
\end{tikzcd}$$

$$\begin{tikzcd}
T \arrow[d, "\eta_S T"'] \arrow[dr, "T\eta_S"] \arrow[dr, phantom, shift right=3ex, "\Swarrow\gamma"] & \\
ST \arrow[r, "\lambda"'] & TS
\end{tikzcd}
\qquad
\begin{tikzcd}
S \arrow[d, "S\eta_T"'] \arrow[dr, "\eta_T S"] \arrow[dr, phantom, shift right=3ex, "\Swarrow\delta"] & \\
ST \arrow[r, "\lambda"'] & TS
\end{tikzcd}$$

satisfying 8 coherence conditions given by Marmolejo in [Mar99] (Definition A.7). Here we have suppressed the associativity and unit constraints for S and T.

In the case when S is colax-idempotent and T is lax-idempotent, such as for $S = \Phi_{\mathcal{B}}$ and $T = \Psi_{\mathcal{B}}$ here, less data is required [Mar99]: a pseudo-distributive law is unique up to isomorphism if it exists, and to define one it suffices to give λ and γ subject to 5 conditions (Proposition A.8).

To give such a pseudo-distributive law $\lambda : \Phi_{\mathcal{B}}\Psi_{\mathcal{B}} \to \Psi_{\mathcal{B}}\Phi_{\mathcal{B}}$ of $\Phi_{\mathcal{B}}$ over $\Psi_{\mathcal{B}}$ is equivalent to giving a lifting of $\Psi_{\mathcal{B}}$ to a pseudomonad on each 2-category $\Phi_{\mathcal{B}}$-$Mod(\mathcal{A}, \mathcal{B})$ of left $\Phi_{\mathcal{B}}$-modules, pseudonaturally in $\mathcal{A}$, as shown in [CHP04] and [Mar04]. $\Psi_{\mathcal{B}}\Phi_{\mathcal{B}}$ then has the structure of a pseudomonad on $\mathbf{Span}(\mathcal{E})(\mathcal{A}, \mathcal{B})$, with $\Psi_{\mathcal{B}}\Phi_{\mathcal{B}}$-$Mod(\mathcal{A}, \mathcal{B})$ biequivalent to the 2-category of left modules for this lifted pseudomonad. $\Psi_{\mathcal{B}}$ in fact lifts to a pseudomonad on categories of two-sided fibrations $\mathbf{Fib}(\mathcal{E})(\mathcal{A},\mathcal{B})$, since composition of spans with the pseudo-distributive law will not affect the right $\Phi_{\mathcal{A}}$-module structure.

If such a pseudo-distributive law from $\Phi_{\mathcal{B}}$ to $\Psi_{\mathcal{B}}$ exists, then since the identity $\mathcal{B} \xrightarrow{=} \mathcal{B}$ is canonically a fibration,

$$\Psi_{\mathcal{B}}(\mathcal{B} \xrightarrow{\;=\;} \mathcal{B}) \quad = \quad \mathcal{B}^{\mathbb{2}} \xrightarrow{\;c\;} \mathcal{B}$$

will also be a fibration. In other words, $\mathcal{B}$ has pullbacks.

Conversely, having pullbacks suffices for such a pseudo-distributive law to exist. We first consider the case in **Set**:

Proposition 1.17 ($\mathcal{E} = \mathbf{Set}$)**.** *There is a pseudo-distributive law of $\Phi_{\mathcal{B}}$ over $\Psi_{\mathcal{B}}$ in* $\mathbf{Span}(\mathbf{Cat})$ *iff the category $\mathcal{B}$ has pullbacks.*

Proof. Assume $\mathcal{B}$ has pullbacks. The map sending a cospan in $\mathcal{B}$ to its (chosen) pullback extends to a functor $\lambda : \Phi_{\mathcal{B}}\Psi_{\mathcal{B}} \to \Psi_{\mathcal{B}}\Phi_{\mathcal{B}}$:

$$
\begin{tikzcd}[column sep=small]
\Phi_{\mathcal{B}}\Psi_{\mathcal{B}} \arrow[dddd] & & \mathcal{B}^{\mathbb{2}} \arrow[dl, "d"'] \arrow[dr, "c"] & & \mathcal{B}^{\mathbb{2}} \arrow[dl, "c"'] \arrow[dr, "d"] & & \cdot \arrow[dr] & & \cdot \arrow[dl] \\
 & \mathcal{B} & & \mathcal{B} & & \mathcal{B} & & \cdot \arrow[dd, mapsto] & \\
 & & & & & & & & \\
 & & \mathcal{B}^{\mathbb{2}} \arrow[dl, "c"'] \arrow[dr, "d"] & & \mathcal{B}^{\mathbb{2}} \arrow[dl, "d"'] \arrow[dr, "c"] & & & \cdot \arrow[dl] \arrow[dr] & \\
\Psi_{\mathcal{B}}\Phi_{\mathcal{B}} & \mathcal{B} & & \mathcal{B} & & \mathcal{B} & \cdot \arrow[dr, dotted] & & \cdot \arrow[dl, dotted] \\
 & & & & & & & \cdot &
\end{tikzcd}
$$

λ is clearly a 2-cell in $\mathbf{Span}(\mathbf{Cat})(\mathcal{B},\mathcal{B})$. The required invertible 3-cell γ in

$$
\begin{tikzcd}
\Psi_{\mathcal{B}} \arrow[d, "\eta\Psi_{\mathcal{B}}"'] \arrow[dr, "\Psi_{\mathcal{B}}\eta"] & \\
\Phi_{\mathcal{B}}\Psi_{\mathcal{B}} \arrow[r, "\lambda"'] \arrow[ur, phantom, "\Downarrow\gamma"] & \Psi_{\mathcal{B}}\Phi_{\mathcal{B}}
\end{tikzcd}
$$

is defined for each object $A \xrightarrow{f} B$ of $\mathcal{B}^{\mathbb{2}}$ to be the unique isomorphism of spans

$$\begin{tikzcd}
& A \arrow[d, "\cong"] \arrow[ddl, "1_A"'] \arrow[ddr, "f"] & \\
& A \times_B B \arrow[dl, "f^*(1_B)"] \arrow[dr, "(1_B)^* f"'] & \\
A \arrow[dr, dotted, "f"'] & & B \arrow[dl, dotted, "1_B"] \\
& B &
\end{tikzcd}$$

Similarly the invertible 3-cells in the coherence conditions (1)-(3) of Proposition A.8 are given by the natural isomorphisms relating $(1_B)^*g$ to g, h^*g^*f to $(gh)^*f$, and k^*f^*g to $(fk)^*g$ for any morphisms $E \xrightarrow{k} A \xrightarrow{f} B \xleftarrow{g} C \xleftarrow{h} D$ in $\mathcal{B}$.

The remaining two coherence conditions required for a pseudo-distributive law hold since there is a unique 3-cell fitting into each diagram. $\square$

More generally:

Proposition 1.18. *There is a pseudo-distributive law of $\Phi_{\mathcal{B}}$ over $\Psi_{\mathcal{B}}$ in* $\mathbf{Span}(\mathbf{Cat}(\mathcal{E}))$ *iff $\mathcal{B}$ has pullbacks.*

Proof. We reconstruct the above definition of λ and γ internally in $\mathbf{Cat}(\mathcal{E})$. Assuming $\mathcal{B}$ has pullbacks, there is a $\Phi_{\mathcal{B}}$-module structure map

$$\mathcal{B}^{\mathbb{2}} \times_{\mathcal{B}} \mathcal{B}^{\mathbb{2}} \xrightarrow{e} \mathcal{B}^{\mathbb{2}}$$

in $\mathbf{Cat}(\mathcal{E})$ as in Example 1.13. Since $\Phi_{\mathcal{B}}$ is colax-idempotent, e is right adjoint to $(1, \eta c) : \mathcal{B}^{\mathbb{2}} \to \mathcal{B}^{\mathbb{2}} \times_{\mathcal{B}} \mathcal{B}^{\mathbb{2}}$ with invertible unit. Composing the counit ε of this adjunction with the map $d\pi_1 : \mathcal{B}^{\mathbb{2}} \times_{\mathcal{B}} \mathcal{B}^{\mathbb{2}} \to \mathcal{B}$ gives a 2-cell

$$\begin{tikzcd}
\mathcal{B}^{\mathbb{2}} \times_{\mathcal{B}} \mathcal{B}^{\mathbb{2}} \arrow[r, bend left, "de"] \arrow[r, bend right, "d\pi_1"'] \arrow[r, phantom, "\Downarrow"] & \mathcal{B},
\end{tikzcd}$$

which by the universal property of the arrow category $\mathcal{B}^{\mathbb{2}}$ (Section 1.3) corresponds to a map $\tau : \mathcal{B}^{\mathbb{2}} \times_{\mathcal{B}} \mathcal{B}^{\mathbb{2}} \to \mathcal{B}^{\mathbb{2}}$ satisfying $d\tau = de$, $c\tau = d\pi_1$, and $\alpha\tau = d\pi_1\varepsilon$.

The morphism

$$\mathcal{B}^{\mathbb{2}} \times_{\mathcal{B}} \mathcal{B}^{\mathbb{2}} \xrightarrow{(\tau, e)} \mathcal{B}^{\mathbb{2}} \times_{\mathcal{B}} \mathcal{B}^{\mathbb{2}}$$

is then a map of spans $\Phi_{\mathcal{B}}\Psi_{\mathcal{B}} \to \Psi_{\mathcal{B}}\Phi_{\mathcal{B}}$, which we define to be λ.

To construct the 3-cell γ in

$$\begin{tikzcd} \mathcal{B}^{\mathbb{2}} \arrow[d, "{(1,\eta c)}"'] \arrow[dr, "{(\eta d,1)}"] & \\ \mathcal{B}^{\mathbb{2}} \times_{\mathcal{B}} \mathcal{B}^{\mathbb{2}} \arrow[r, "{(\tau,e)}"'] \arrow[r, phantom, bend left, "\Downarrow\gamma"] & \mathcal{B}^{\mathbb{2}} \times_{\mathcal{B}} \mathcal{B}^{\mathbb{2}}, \end{tikzcd}$$

we require invertible 2-cells in $\mathbf{Cat}(\mathcal{E})$ of the form $\tau(1,\eta c) \Rightarrow \eta d$ and $e(1,\eta c) \Rightarrow 1_{\mathcal{B}^{\mathbb{2}}}$. The second of these is the invertible unit of the adjunction $(1,\eta c) \dashv e$, and the first is again given by the universal property of $\mathcal{B}^{\mathbb{2}}$ since $\alpha\tau(1,\eta c) = d\pi_1\varepsilon(1,\eta c) \cong 1_d = \alpha\eta d$ by the triangular identity of the adjunction $(1,\eta c) \dashv e$.

The fact that λ and γ satisfy the coherence conditions required for a pseudo-distributive law now follows from the case $\mathcal{E} = \mathbf{Set}$ by the $\mathbf{Cat}$-enriched Yoneda embedding. All the constructions used to form $\mathbf{Span}(\mathbf{Cat}(\mathcal{E}))$, $\mathcal{B}^{\mathbb{2}}$, $\Phi_{\mathcal{B}}$ and $\Psi_{\mathcal{B}}$ are defined in terms of limits, and are preserved by each hom 2-functor $\mathbf{Cat}(\mathcal{E})(\mathcal{A},-) : \mathbf{Cat}(\mathcal{E}) \to \mathbf{Cat}$. $\square$

Thus $\Psi_{\mathcal{B}}$ lifts to a pseudomonad $\Psi'_{\mathcal{B}}$ on each $\mathbf{Fib}(\mathcal{E})(\mathcal{A},\mathcal{B})$ exactly when $\mathcal{B}$ has pullbacks. Since $\Phi_{\mathcal{B}}$ and $\Psi_{\mathcal{B}}$ are colax-idempotent and lax-idempotent respectively, such a lifting is unique up to isomorphism if it exists. Suppose now that this is the case.

Definition 1.19. A fibration *has sums* if it has the structure of a left $\Psi'_{\mathcal{B}}$-module.

Recall that composition in $\mathbf{Fib}(\mathcal{E})$ is given by bimodule tensor $\otimes$, in other words by a coequalizer of composites of spans. Since $\Psi'_{\mathcal{B}}$ is given by composition with a span and pullback along d preserves coequalizers, $\Psi'_{\mathcal{B}}$ has a tensorial strength: that is a family of maps

$$\Psi'_{\mathcal{B}}(\mathcal{N}) \otimes \mathcal{M} \xrightarrow{\cong} \Psi'_{\mathcal{B}}(\mathcal{N} \otimes \mathcal{M})$$

natural in spans $\mathcal{M}\colon \mathcal{C} \twoheadrightarrow \mathcal{A}$ and $\mathcal{N}\colon \mathcal{A} \twoheadrightarrow \mathcal{B}$, which satisfy unit and associativity conditions. Setting $\mathcal{N}$ to be the identity two-sided fibration $\Phi_{\mathcal{B}}$ shows that the monad $\Psi'_{\mathcal{B}}$ is given by composition in $\mathbf{Fib}(\mathcal{E})$ with the span $\Sigma_{\mathcal{B}} :\equiv \Psi'_{\mathcal{B}}(\Phi_{\mathcal{B}})$. In other words, $\Sigma_{\mathcal{B}}$ is a lax-idempotent pseudomonad in the $\mathbf{2}$-$\mathbf{Cat}$-enriched bicategory $\mathbf{Fib}(\mathcal{E})$, and composing with $\Sigma_{\mathcal{B}}$ on the right freely adds sums to fibrations.

In $\mathbf{Cat}$, this definition of fibrations with sums reduces to the well-known one [Jac99]:

Example 1.20 ($\mathcal{E} = \mathbf{Set}$). To give a cloven fibration $1 \leftarrow \mathcal{M} \xrightarrow{p} \mathcal{B}$ in $\mathbf{Cat}$ sums is to give a left adjoint $\coprod_f$ for each reindexing functor $f^* : \mathcal{M}^J \to \mathcal{M}^I$, which satisfy the

Beck-Chevalley condition. The pseudomonad $\Sigma_{\mathcal{B}}$ is the span

$$\begin{tikzcd}
& \mathcal{B}^{\leftarrow\cdot\rightarrow} \arrow[dl, "l"'] \arrow[dr, "r"] & \\
\mathcal{B} & & \mathcal{B},
\end{tikzcd}$$

where the category $\mathcal{B}^{\leftarrow\cdot\rightarrow}$ has as objects the spans $I \leftarrow A \rightarrow J$ in $\mathcal{B}$ and as morphisms commuting diagrams

$$\begin{tikzcd}
I \arrow[d] & A \arrow[l] \arrow[r] \arrow[d] & J \arrow[d] \\
I' & A' \arrow[l] \arrow[r] & J'.
\end{tikzcd}$$

The functors l and r send such a morphism to $I \to I'$ and $J \to J'$ respectively.

1.6 Opposites of fibrations

In the previous section, when the monad $\Phi_{\mathcal{B}}$ is considered as an internal category in $\mathbf{Span}(\mathcal{E})$, constructing the reversed span $\Psi_{\mathcal{B}}$ from $\Phi_{\mathcal{B}}$ corresponds to taking the opposite internal category. There is an analogous construction for fibrations. We show how strict fibrations can also be seen as internal categories in a particular category, and so we find a natural definition of the opposite of a fibration.

Let $\mathcal{B}$ be an internal category in $\mathcal{E}$, so we have objects B_0, B_1, $B_2 = B_1 \times_{B_0} B_1$ and morphisms

$$\begin{tikzcd}
B_2 \arrow[r, shift left=2, "s"] \arrow[r, "m" description] \arrow[r, shift right=2, "t"'] & B_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & B_0 \arrow[l, "i" description]
\end{tikzcd}$$

satisfying the required equations. Then this diagram also represents a category object internal to $\mathbf{Cat}(\mathcal{E})$, when B_0, B_1 and B_2 are considered to be discrete categories.

$\mathcal{B}$ is the *lax codescent object* of this diagram in $\mathbf{Cat}(\mathcal{E})$, i.e. equipped with a functor $u : B_0 \to \mathcal{B}$ (the inclusion of the discrete category) and natural transformation $\alpha : ud \Rightarrow uc$ such that $\alpha i = 1_u$ and

$$\begin{tikzcd}
& B_1 \arrow[rr, "d"] & & B_0 \arrow[dr, "u"] & \\
B_2 \arrow[ur, "s"] \arrow[rr, "m"] \arrow[dr, "t"'] & & B_1 \arrow[ur, "d"] \arrow[dr, "c"'] \arrow[rr, phantom, "\Downarrow\alpha"] & & \mathcal{B} \\
& B_1 \arrow[rr, "c"'] & & B_0 \arrow[ur, "u"'] &
\end{tikzcd}
=
\begin{tikzcd}
& B_1 \arrow[rr, "d"] \arrow[dr, "c"'] & & B_0 \arrow[dr, "u"] \arrow[d, phantom, "\Swarrow\alpha"] & \\
B_2 \arrow[ur, "s"] \arrow[dr, "t"'] & & B_0 \arrow[rr, "u" description] & {} & \mathcal{B} \\
& B_1 \arrow[ur, "d"] \arrow[rr, "c"'] & & B_0 \arrow[ur, "u"'] \arrow[u, phantom, "\Searrow\alpha"] &
\end{tikzcd}$$

and is universal with this property [Lac02].

For any functor $\mathcal{M} \xrightarrow{p} \mathcal{B}$ in $\mathbf{Cat}(\mathcal{E})$, we can construct (not necessarily discrete) categories $\mathcal{M}_0$, $\mathcal{M}_1$ and $\mathcal{M}_2$ and functors

$$\begin{tikzcd}
\mathcal{M}_2 \arrow[r, shift left, "m'"] \arrow[r, shift right, "t'"'] \arrow[d] & \mathcal{M}_1 \arrow[r, shift right, "c'"'] \arrow[d] & \mathcal{M}_0 \arrow[l, shift right, "i'"'] \arrow[r, hook, "u'"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{M} \arrow[d, "p"] \\
B_2 \arrow[r, shift left=2, "s"] \arrow[r, "m" description] \arrow[r, shift right=2, "t"'] & B_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & B_0 \arrow[l, "i" description] \arrow[r, hook, "u"] & \mathcal{B}
\end{tikzcd} \tag{1.2}$$

such that all corresponding squares are pullbacks. p is a strict fibration exactly if there are functors

$$\mathcal{M}_2 \xrightarrow{s'} \mathcal{M}_1 \xrightarrow{d'} \mathcal{M}_0$$

such that the corresponding squares involving s and d commute and make the top row of (1.2) into a category object in $\mathbf{Cat}(\mathcal{E})$. The natural transformation α then induces $\alpha' : u'd' \Rightarrow u'c'$ making $\mathcal{M}$ into the lax codescent object of the top row.

Example 1.21 ($\mathcal{E} = \mathbf{Set}$). In $\mathbf{Cat}$, the category $\mathcal{M}_0$ consists of the objects of $\mathcal{M}$ with the morphisms of $\mathcal{M}$ that are p-vertical. The objects of $\mathcal{M}_1$ are pairs ($J \in \mathcal{B}$, $f : I \to pJ \in \mathcal{M}$), which d' sends to the domain of the chosen cartesian lifting $f^*J \to J$. This lifting is the corresponding component of the natural transformation α'.

Projecting onto the object, morphism, and composable morphism parts of the categories in (1.2) gives internal diagrams in $\mathcal{E}$ over $\mathcal{B}$, as defined in [Joh77]. Thus a strict fibration over $\mathcal{B}$ corresponds exactly to an internal category in the category $\mathcal{E}^{\mathcal{B}}$ of such diagrams.

Taking the opposite of this category corresponds to taking the opposites of all the categories and functors in Diagram (1.2). This will not affect the bottom row, but the top row will have a new lax codescent object

$$\begin{tikzcd}
\mathcal{M}_2^{op} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathcal{M}_1^{op} \arrow[r, shift left=2] \arrow[r, shift right=2] & \mathcal{M}_0^{op} \arrow[l] \arrow[r] & \mathcal{M}^{\circ}.
\end{tikzcd}$$

Such a lax codescent object always exists in $\mathbf{Cat}(\mathcal{E})$, as shown by Weber in [Web15] for any internal category diagram where the internal codomain functor $\mathcal{M}_1^{op} \to \mathcal{M}_0^{op}$ is the pullback of a functor of discrete categories. The universal property of the colimit then induces a functor $\mathcal{M}^{\circ} \xrightarrow{p^{op}} \mathcal{B}$. This gives p^{op} the structure of a fibration, and it is called the *opposite* of the fibration p.

In **Cat**, this gives the usual construction of the opposite of a split Grothendieck fibration:

Example 1.22 ($\mathcal{E} = \mathbf{Set}$). The opposite of $\mathcal{M} \xrightarrow{p} \mathcal{B}$ in **Cat** is given by reversing the arrows of $\mathcal{M}$ which are vertical over $\mathcal{B}$. The category $\mathcal{M}^\circ$ has the same objects as $\mathcal{M}$, and as morphisms $A \to B$ over $pA \xrightarrow{u} pB$ the spans $A \xleftarrow{\alpha} M \xrightarrow{\beta} B$ in $\mathcal{M}$ where α is p-vertical and β is a chosen p-cartesian lifting of u.

Opposites for non-strict cloven fibrations are defined in the same way, except that the $\mathcal{M}_i$ no longer form a strict category object, with isomorphisms $d'i' \cong 1$ and $d'm' \cong d's'$ rather than equalities.

Example 1.23 ($\mathcal{E} = \mathbf{Set}$). For a cloven fibration $\mathcal{M} \xrightarrow{p} \mathcal{B}$ in **Cat**, the category $\mathcal{M}^\circ$ has the same objects as $\mathcal{M}$. Morphisms are spans $A \xleftarrow{\alpha} M \xrightarrow{\beta} B$ in $\mathcal{M}$ where α is p-vertical and β is p-cartesian, considered up to the equivalence relation relating two such spans (α, β) and (α', β') if there is a vertical isomorphism $M \to M'$ forming a morphism of spans.

For two-sided fibrations $\mathcal{A} \xleftarrow{q} \mathcal{M} \xrightarrow{p} \mathcal{B}$, consider the internal category $\mathcal{A} \times \mathcal{B}$ and the diagram

$$\begin{tikzcd}
\mathcal{M}_2 \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathcal{M}_1 \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathcal{M}_0 \arrow[r, hook, "u'"] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathcal{M} \arrow[d, "{(q,\, p)}"] \\
A_2 \times B_2 \arrow[r, "m \times m"] & A_1 \times B_1 \arrow[r, "d \times c"] & A_0 \times B_0 \arrow[r, hook, "u"] & \mathcal{A} \times \mathcal{B}.
\end{tikzcd}$$

The fibration structure of p and the opfibration structure of q induce morphisms d', $c' : \mathcal{M}_1 \to \mathcal{M}_0$ respectively, and the compatibility between the structures ensures that these can be extended to give a (weak) category object in $\mathbf{Cat}(\mathcal{E})$ with lax codescent object $\mathcal{M}$. Taking opposites of categories as above gives a functor $\mathcal{M}^\circ \xrightarrow{(q^{op},\, p^{op})} \mathcal{A} \times \mathcal{B}$, which defines a new two-sided fibration between $\mathcal{A}$ and $\mathcal{B}$.

Definition 1.24. The span

$$\begin{tikzcd}
& \mathcal{M}^\circ \arrow[dl, "q^{op}"'] \arrow[dr, "p^{op}"] & \\
\mathcal{A} & & \mathcal{B}
\end{tikzcd}$$

is called the *opposite two-sided fibration* of $\mathcal{A} \xleftarrow{q} \mathcal{M} \xrightarrow{p} \mathcal{B}$.

Example 1.25 ($\mathcal{E} = \mathbf{Set}$). In **Cat** this corresponds to reversing the arrows of $\mathcal{M}$ which are vertical over both $\mathcal{A}$ and $\mathcal{B}$.

Using the universal property of the codescent objects, taking opposites extends to a pseudofunctor

$$(-)^{op}\colon \mathbf{Fib}(\mathcal{E})(\mathcal{A},\mathcal{B}) \to \mathbf{Fib}(\mathcal{E})(\mathcal{A},\mathcal{B})$$

for each $\mathcal{A}$ and $\mathcal{B}$, and $((-)^{op})^{op} \cong 1$. When $\mathcal{E}$ is either locally cartesian closed with countable colimits or a topos with a natural numbers object, then opposites commute with the composition of 2-sided fibrations defined in Section 1.4, i.e. $\mathcal{N}^{op} \otimes \mathcal{M}^{op} \cong (\mathcal{N} \otimes \mathcal{M})^{op}$ naturally in $\mathcal{M}$ and $\mathcal{N}$.

1.7 Fibrations with products

Having defined sums and opposites for fibrations, we can now consider their combination.

Definition 1.26. A (two-sided) fibration *has products* if its opposite has sums.

Example 1.27 ($\mathcal{E} = \mathbf{Set}$). In **Cat**, a cloven fibration $1 \leftarrow \mathcal{M} \xrightarrow{p} \mathcal{B}$ has products if each reindexing functor $f^* : \mathcal{M}^J \to \mathcal{M}^I$ has a right adjoint Π_f satisfying the Beck-Chevalley condition. In particular, this holds for the codomain functor $\mathcal{B}^{\mathbb{2}} \xrightarrow{c} \mathcal{B}$ if and only if $\mathcal{B}$ is locally cartesian closed.

Thus in general we define:

Definition 1.28. A category $\mathcal{B}$ with pullbacks is *locally cartesian closed* if the codomain fibration c has products.

Given a two-sided fibration $\mathcal{M}$, we can freely add products to $\mathcal{M}$ by taking the opposite fibration, adding sums, and then taking the opposite again. Since

$$(\Sigma_{\mathcal{B}} \otimes \mathcal{M}^{op})^{op} \cong (\Sigma_{\mathcal{B}})^{op} \otimes \mathcal{M},$$

the span $\Pi_{\mathcal{B}} :\equiv (\Sigma_{\mathcal{B}})^{op}$ is a colax-idempotent pseudomonad in $\mathbf{Fib}(\mathcal{E})$ which freely adds products by composition on the right. Thus a fibration has products if it has the structure of a left $\Pi_{\mathcal{B}}$-module.

Example 1.29 ($\mathcal{E} = \mathbf{Set}$). In **Cat**, the pseudomonad $\Pi_{\mathcal{B}}$ is a span

$$\begin{tikzcd} & (\mathcal{B}^{\leftarrow\cdot\rightarrow})^{\circ} \arrow[dl, "l^{op}"'] \arrow[dr, "r^{op}"] & \\ \mathcal{B} & & \mathcal{B}, \end{tikzcd}$$

where the category $(\mathcal{B}^{\leftarrow\cdot\rightarrow})^{\circ}$ is given by reversing the arrows of $\mathcal{B}^{\leftarrow\cdot\rightarrow}$ that are vertical for both projections onto $\mathcal{B}$. So $(\mathcal{B}^{\leftarrow\cdot\rightarrow})^\circ$ has as objects the spans $I \leftarrow A \rightarrow J$ in $\mathcal{B}$ and as morphisms commuting diagrams

$$\begin{tikzcd} I \arrow[d] & A \arrow[l] \arrow[r] & J \arrow[d] \\ & E \arrow[u] \arrow[ur] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \\ I' & A' \arrow[l] \arrow[r] & J'. \end{tikzcd}$$

The functors l^{op} and r^{op} send such a morphism to $I \rightarrow I'$ and $J \rightarrow J'$ respectively.

1.8 Polynomials

We now have two monads in $\mathbf{Fib}(\mathcal{E})$: $\Sigma_\mathcal{B}$ adding sums and its opposite $\Pi_\mathcal{B}$ adding products. Mirroring the situation of $\Phi_\mathcal{B}$ and $\Psi_\mathcal{B}$, we consider the interaction between $\Sigma_\mathcal{B}$ and $\Pi_\mathcal{B}$.

$\Sigma_\mathcal{B}\Pi_\mathcal{B}$ will be a pseudomonad in $\mathbf{Fib}(\mathcal{E})$ if there is a pseudo-distributive law

$$\lambda : \Pi_\mathcal{B}\Sigma_\mathcal{B} \rightarrow \Sigma_\mathcal{B}\Pi_\mathcal{B}.$$

If such a law exists of $\Pi_\mathcal{B}$ over $\Sigma_\mathcal{B}$, then as before $\Sigma_\mathcal{B}$ lifts to a pseudomonad on left $\Pi_\mathcal{B}$-modules. Since the identity $\mathcal{B} \xrightarrow{=} \mathcal{B}$ canonically has products,

$$\Sigma_\mathcal{B}(\mathcal{B} \xrightarrow{\;=\;} \mathcal{B}) \quad = \quad \mathcal{B}^\mathbb{2} \xrightarrow{\;c\;} \mathcal{B}$$

will also have products. In other words, $\mathcal{B}$ is locally cartesian closed.

In **Cat**, the converse holds:

Proposition 1.30 ($\mathcal{E} = \mathbf{Set}$)**.** *There exists a pseudo-distributive law of* $\Pi_\mathcal{B}$ *over* $\Sigma_\mathcal{B}$ *in* **Fib** *exactly when* $\mathcal{B}$ *is locally cartesian closed.*

Proof. In **Cat**, the composite fibration $\Sigma_\mathcal{B}\Pi_\mathcal{B}$ is a span $\mathcal{B} \leftarrow \mathcal{M} \rightarrow \mathcal{B}$ where the category $\mathcal{M}$ has as objects diagrams $I \leftarrow B \rightarrow A \rightarrow J$ in $\mathcal{B}$, i.e. polynomials, and as morphisms the morphisms of polynomials

$$\begin{tikzcd}
I \arrow[dd] & B \arrow[l] \arrow[r] & A \arrow[r] \arrow[dd] & J \arrow[dd] \\
 & E \arrow[u] \arrow[ur] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & \\
I' & B' \arrow[l] \arrow[r] & A' \arrow[r] & J'.
\end{tikzcd}$$

The composite $\Pi_{\mathcal{B}}\Sigma_{\mathcal{B}}$ is a span $\mathcal{B} \leftarrow \mathcal{N} \to \mathcal{B}$ where $\mathcal{N}$ has the same objects as $\mathcal{M}$ and as morphisms the commuting diagrams

$$\begin{tikzcd}
I \arrow[dd] & B \arrow[l] \arrow[r] & A \arrow[r] & J \arrow[dd] \\
 & E \arrow[u] \arrow[ur, phantom, "\urcorner", very near start] \arrow[r] \arrow[d] & F \arrow[u] \arrow[ur] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \\
I' & B' \arrow[l] \arrow[r] & A' \arrow[r] & J'.
\end{tikzcd}$$

If $\mathcal{B}$ is locally cartesian closed, there is a functor $\lambda : \mathcal{N} \to \mathcal{M}$ sending a diagram $I \xleftarrow{s} B \xrightarrow{f} A \xrightarrow{t} J$ to the polynomial

$$I \leftarrow t^*\Pi_t f \to \Pi_t f \to J$$

as in the diagram

$$\begin{tikzcd}
 & & t^*\Pi_t f \arrow[dl, "\varepsilon"'] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Pi_t f \arrow[d] \\
 & B \arrow[dl, "s"'] \arrow[r, "f"] & A \arrow[r, "t"'] & J \\
I & & &
\end{tikzcd}$$

where ε is the component at f of the counit of the adjunction $t^* \dashv \Pi_t$. The Beck-Chevalley condition for Π ensures that λ preserves the cartesian and opcartesian morphisms in $\mathcal{N}$, so it defines a morphism $\Pi_{\mathcal{B}}\Sigma_{\mathcal{B}} \to \Sigma_{\mathcal{B}}\Pi_{\mathcal{B}}$ in $\mathbf{Span}(\mathbf{Cat})(\mathcal{B}, \mathcal{B})$.

The components of the 3-cell γ in the diagram

$$\begin{tikzcd}
\mathcal{B}^{\leftarrow\cdot\rightarrow} \arrow[d, "\Pi_{\mathcal{B}}\eta"'] \arrow[dr, "\eta\Pi_{\mathcal{B}}"] & \\
\mathcal{N} \arrow[r, "\lambda"'] \arrow[ur, phantom, "\Swarrow\gamma"] & \mathcal{M}
\end{tikzcd}$$

are defined for each span $I \xleftarrow{s} B \xrightarrow{f} A$ as the unique isomorphism of polynomials

$$\begin{tikzcd}
I \arrow[d, "="'] & (1_A)^*\Pi_{1_A} f \arrow[l, "s"'] \arrow[r] & \Pi_{1_A} f \arrow[r] \arrow[d, "\cong"] & A \arrow[d, "="] \\
& E \arrow[u, "\cong"] \arrow[ur] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & \\
I & B \arrow[l] \arrow[r, "="'] & B \arrow[r, "f"'] & A.
\end{tikzcd}$$

Similarly the first coherence condition of Proposition A.8 for a pseudo-distributive law corresponds to giving the isomorphisms $\Pi_t(1_A) \cong J$ for any $A \xrightarrow{t} J$. The second coherence condition follows from the canonical isomorphisms $\Pi_{mt}f \cong \Pi_m\Pi_t f$ for any morphisms $B \xleftarrow{f} A \xrightarrow{t} J \xrightarrow{m} K$. The third coherence condition reduces to the 'type-theoretic axiom of choice' of Proposition 1.6.

As in the proof of Proposition 1.17, the remaining two coherence conditions follow by uniqueness, since by the universal properties of Π and pullback there is a unique 3-cell fitting into each of the diagrams. □

Remark 1.31. Unlike the case of the distributive law for $\Phi_{\mathcal{B}}$ and $\Psi_{\mathcal{B}}$, the previous proposition does not extend by representability to arbitrary $\mathbf{Fib}(\mathcal{E})$. As hom-functors do not preserve coequalizers in general, composition in the **2-Cat**-enriched bicategory $\mathbf{Fib}(\mathcal{E})$ is not representably defined. However, it might still be possible though computationally challenging to internalize the proof of the proposition for **Set** and check the coherence conditions by hand. It would also be interesting to investigate a more conceptual proof by relating these pseudomonads to *clubs* defined by Kelly in [Kel92], which are monads interacting well with pullbacks. The two propositions 1.30 and 1.18 have a similar form, stating that to give a distributive law $ST \to TS$ between two monads it suffices to give a S-module structure to T acting on a terminal object (in this case the terminal object $1 \leftarrow \mathcal{B} \xrightarrow{=} \mathcal{B}$ of $\mathbf{Fib}(\mathcal{E})(1, \mathcal{B})$), and a theorem of this form was proved by Garner in [Gar08] using an generalization of clubs.

From now on we focus only on the case $\mathcal{E} = \mathbf{Set}$. Since $\Pi_{\mathcal{B}}$ is colax-idempotent and $\Sigma_{\mathcal{B}}$ is lax-idempotent, if such a pseudo-distributive law exists then it is unique up to isomorphism. Suppose that this is the case, then $\Sigma_{\mathcal{B}}\Pi_{\mathcal{B}}$ has the structure of a pseudomonad with composition

$$(\Sigma\Pi)(\Sigma\Pi) \cong \Sigma(\Pi\Sigma)\Pi \xrightarrow{\Sigma\lambda\Pi} \Sigma\Sigma\Pi\Pi \xrightarrow{\mu\mu} \Sigma\Pi$$

sending two polynomials $I \xleftarrow{s} B \xrightarrow{f} A \xrightarrow{t} J$ and $J \xleftarrow{u} D \xrightarrow{g} C \xrightarrow{v} K$ to

$$I \leftarrow P \rightarrow \Pi_g h \rightarrow K$$

as in the diagram

$$\begin{tikzcd}
 & & & P \arrow[r] \arrow[ddll] \arrow[dr, phantom, "\lrcorner", very near start] & g^*\Pi_g h \arrow[r] \arrow[dl, "\varepsilon"'] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & \Pi_g h \arrow[dd] & \\
 & & & E \arrow[dl] \arrow[dr, "h"] \arrow[d, phantom, "\vee", very near start] & & & \\
 & B \arrow[r, "f"'] \arrow[dl, "s"'] & A \arrow[dr, "t"'] & & D \arrow[r, "g"'] \arrow[dl, "u"'] & C \arrow[dr, "v"] & \\
I & & & J & & & K,
\end{tikzcd}$$

i.e. exactly their composite as polynomials.

Thus we have:

Proposition 1.32 ($\mathcal{E}$ = **Set**)**.** *The pseudo double category $\mathbb{P}\mathbf{oly}_{\mathcal{B}}$ of polynomials in a locally cartesian closed category $\mathcal{B}$, as defined in Proposition 1.7, is exactly the pseudomonad $\Sigma_{\mathcal{B}}\Pi_{\mathcal{B}}$.* □

1.9 Polynomials in non-lcc categories

When a category $\mathcal{B}$ (in **Set**) is not locally cartesian closed, it can still make sense to consider polynomials in $\mathcal{B}$, as long as we restrict to those diagrams

$$I \xleftarrow{s} B \xrightarrow{f} A \xrightarrow{t} J$$

for which s^*, Π_f and Σ_t are defined. For example, Weber [Web14] examines the case of a category with pullbacks, in which the polynomials are all the diagrams of this shape such that the middle morphism f is exponentiable. Here we generalize in a slightly different direction, motivated by the above analysis of sums and products as monads on slice categories. In a non-locally cartesian closed category, we will not require the associated functor $\Sigma_t \Pi_f s^*$ of a polynomial to be defined on the full slice category $\mathcal{B}/I$, but only on a subcategory of it. Polynomial diagrams should then consist of morphisms for which pullback and its adjoints Σ and Π are defined on this subcategory.

In detail, we start with a class of morphisms in $\mathcal{B}$ which contains identities and is closed under composition. This means that these morphisms are the objects of a full subcategory $\mathcal{F}$ of $\mathcal{B}^2$, such that the spans

$$\Phi_{\mathcal{F}} = \begin{tikzcd} & \mathcal{F} \arrow[dl, "c"'] \arrow[dr, "d"] & \\ \mathcal{B} & & \mathcal{B} \end{tikzcd} \quad \text{and} \quad \Psi_{\mathcal{F}} = \begin{tikzcd} & \mathcal{F} \arrow[dl, "d"'] \arrow[dr, "c"] & \\ \mathcal{B} & & \mathcal{B} \end{tikzcd}$$

are submonads in **Span** of $\Phi_{\mathcal{B}}$ and $\Psi_{\mathcal{B}}$ respectively. In diagrams, objects in $\mathcal{F}$ will be denoted by double-headed arrows $\twoheadrightarrow$.

Just as before (Proposition 1.18) the existence of pullbacks corresponds to a pseudo-distributive law.

Proposition 1.33. *The following are equivalent:*

1. *$\Psi_{\mathcal{F}}$ lifts to a lax-idempotent pseudomonad $\Sigma_{\mathcal{F}}$ in* **Fib***,*
2. *there is a pseudo-distributive law $\lambda : \Phi_{\mathcal{B}}\Psi_{\mathcal{F}} \to \Psi_{\mathcal{F}}\Phi_{\mathcal{B}}$,*
3. *the codomain functor $c : \mathcal{F} \to \mathcal{B}$ is a fibration,*
4. *for every morphism f in $\mathcal{F}$ and morphism g in $\mathcal{B}$, there exists a pullback*

$$\begin{tikzcd} C \arrow[r] \arrow[d, "h"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "f"] \\ D \arrow[r, "g"'] & A \end{tikzcd}$$

such that h is in $\mathcal{F}$.

□

Definition 1.34. A fibration $p : \mathcal{M} \to \mathcal{B}$ *has $\mathcal{F}$-sums* if it has the structure of a left module for $\Sigma_{\mathcal{F}}$ considered as a pseudomonad on $\mathbf{Fib}/\mathcal{B} \cong \mathbf{Fib}(1, \mathcal{B})$. That is, for every $\mathcal{F}$-map $f : B \twoheadrightarrow A$ in $\mathcal{B}$, the reindexing functor $f^* : \mathcal{M}_A \to \mathcal{M}_B$ has a left adjoint Σ_f and the Beck-Chevalley condition holds in the form: For every pullback square

$$\begin{tikzcd} D \arrow[r, "h"] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "f"] \\ C \arrow[r, "k"'] & A \end{tikzcd}$$

in $\mathcal{B}$ with f (and hence g) in $\mathcal{F}$, the canonical map $\Sigma_g h^* \to k^* \Sigma_f$ is an isomorphism.

In particular, the fibration $\mathcal{F} \to \mathcal{B}$ itself has $\mathcal{F}$-sums, with the left adjoint Σ_f for a morphism $f \in \mathcal{F}$ given by composition with f.

Dually, a fibration *has $\mathcal{F}$-products* if it has the structure of a left module for the opposite pseudomonad $\Pi_{\mathcal{F}}$, so each $f^* : \mathcal{M}_A \to \mathcal{M}_B$ has a right adjoint Π_f such that for any pullback square as above the canonical map $k^*\Pi_f \to \Pi_g h^*$ is an isomorphism.

The results about $\Sigma_{\mathcal{B}}$ and $\Pi_{\mathcal{B}}$ (Proposition 1.30) now generalize to the monads $\Sigma_{\mathcal{F}}$ and $\Pi_{\mathcal{F}}$.

Proposition 1.35. *To give a pseudo-distributive law of $\Pi_{\mathcal{F}}$ over $\Sigma_{\mathcal{F}}$ is exactly to give the fibration $\mathcal{F} \xrightarrow{c} \mathcal{B}$ the structure of $\mathcal{F}$-products.* □

Remark 1.36. 1. Such a pseudo-distributive law is constructed by Hofstra [Hof11] for the case when $\mathcal{F}$ is the class of product projections in a cartesian closed category.

2. If $\mathcal{F} \xrightarrow{c} \mathcal{B}$ has $\mathcal{F}$-products, then the functor Π_f is additionally a partial right adjoint to the pullback functor on slice categories $f^* : \mathcal{B}/A \to \mathcal{B}/B$. In other words, there is a bijection of morphisms

$$\begin{tikzcd} f^*C \arrow[rr] \arrow[dr] & & E \arrow[dl, two heads] \\ & B & \end{tikzcd} \quad \Leftrightarrow \quad \begin{tikzcd} C \arrow[rr] \arrow[dr, "k"'] & & \Pi_f E \arrow[dl, two heads] \\ & A & \end{tikzcd}$$

even when $k : C \to A$ is not in $\mathcal{F}$, since by the Beck-Chevalley condition they both correspond to morphisms $C \to k^*\Pi_f E$ in the fibre $\mathcal{F}/C$.

To summarize, when $\mathcal{F}$ is a class of morphisms which is closed under composition and identities and $\mathcal{F} \xrightarrow{c} \mathcal{B}$ is a fibration with $\mathcal{F}$-products, there is a pseudomonad $\Sigma_{\mathcal{F}}\Pi_{\mathcal{F}}$ in **Fib**. In other words, we get a double category $\mathbb{P}\mathbf{oly}_{\mathcal{F}}$ of polynomials. The objects are all objects of $\mathcal{B}$, and the horizontal morphisms are polynomials

$$\begin{tikzcd} & B \arrow[r, two heads, "f"] \arrow[dl, "s"'] & A \arrow[dr, two heads, "t"] & \\ I & & & J \end{tikzcd}$$

where t and f are in $\mathcal{F}$. The 2-cells and horizontal composition correspond to morphisms and composition of polynomials as before.

Now consider this pseudomonad acting on the slice category $\mathbf{Fib}/\mathcal{B} \cong \mathbf{Fib}(1, \mathcal{B})$ of fibrations over $\mathcal{B}$. Applying it to the domain fibration $d : \mathcal{F} \to \mathcal{B}$ gives the category

$(\mathbb{P}\mathbf{oly}_{\mathcal{F}})_1$ of polynomials and polynomial morphisms described above. Considering just the part fibred over J and not I, we get the fibration

$$\Sigma_{\mathcal{F}}\Pi_{\mathcal{F}}(\mathcal{B} \xrightarrow{=} \mathcal{B}) = \Sigma_{\mathcal{F}}(\mathcal{F} \xrightarrow{c} \mathcal{B})^{op} = \Sigma_{\mathcal{F}}\mathcal{F}^{op}.$$

For a general fibration $\mathcal{M} \xrightarrow{p} \mathcal{B}$, we have

$$\Sigma_{\mathcal{F}}\Pi_{\mathcal{F}}p \cong \Sigma_{\mathcal{F}}(\Sigma_{\mathcal{F}}(p)^{op})^{op},$$

so the pseudomonad $\Sigma_{\mathcal{F}}\Pi_{\mathcal{F}}$ is given by two iterations of the construction $\Sigma_{\mathcal{F}}(-)^{op}$, as observed by Hyland in [Hyl07]. Thus we think of $Pol(-) :\equiv \Sigma_{\mathcal{F}}(-)^{op} = (\Pi_{\mathcal{F}}(-))^{op}$ as being the basic construction of polynomials over a fibration, and study the structure of the fibration $Pol(\mathcal{F})$ further in Chapter 4.

Chapter 2

Categorical models of type theory

2.1 Dependent type theory

We give here an informal account of the language of dependent type theory [ML84], and how it is interpreted in category theory.

The basic objects of type theory are *types*, and *terms* of each type. The notation $a : A$ denotes that a is a term of type A. In dependent type theory, types and terms can depend on terms of other types, so types and terms are always defined in *context*, written

$$\Gamma \vdash A : \mathrm{Type} \qquad \text{and} \qquad \Gamma \vdash a : A, \tag{2.1}$$

where a context Γ is a finite (possibly empty) list of distinct typed variables

$$x_1 : A_1,\ x_2 : A_2,\ \ldots,\ x_n : A_n$$

that A and a can depend on, and each A_i depends only on the previous x_j, $j < i$.

Two types or two terms can be *definitionally equal*, written

$$\Gamma \vdash A = B : \mathrm{Type} \qquad \text{and} \qquad \Gamma \vdash a = b : A. \tag{2.2}$$

The statements to the right of the turnstiles in (2.1) and (2.2) are the basic forms of *judgements* that can be made in type theory.

Defined terms can be substituted for the variables in a judgement, so for example if b is a term of type B in context $x : A$ and a is a term of type A, then $b[a/x]$ is a term of type $B[a/x]$. This is written as a *rule of inference*

$$\frac{\Gamma, x : A, \Delta \vdash \mathcal{J} \qquad \Gamma \vdash a : A}{\Gamma, \Delta[a/x] \vdash \mathcal{J}[a/x]},$$

where $\mathcal{J}$ stands for any judgement, asserting that the conclusion below follows from the hypotheses above the line.

Further rules state that the hypotheses of a judgement can be weakened:

$$\frac{\Gamma, \Delta \vdash \mathcal{J} \qquad \Gamma \vdash A : \mathrm{Type}}{\Gamma, x : A, \Delta \vdash \mathcal{J}}$$

(when x is a new variable); that context variables are valid terms in context:

$$\frac{\Gamma \vdash A : \mathrm{Type}}{\Gamma, x : A \vdash x : A},$$

that definitional equality is compatible with typing judgements:

$$\frac{\Gamma \vdash a : A \qquad \Gamma \vdash A = B : \mathrm{Type}}{\Gamma \vdash a : B} \qquad \frac{\Gamma \vdash a = b : A \qquad \Gamma \vdash A = B : \mathrm{Type}}{\Gamma \vdash a = b : B},$$

and that definitional equality of types and terms are equivalence relations.

A particular instance of type theory consists of judgements in context given as axioms, plus all the judgements in context that can be derived using the above rules of inference.

2.2 Categories of types

Given a type theory, we can study it categorically by thinking of the types as objects of a category and the terms as morphisms. More formally, we can construct the *term model* of a type theory, which is a category of types $\mathcal{T}$ fibred over a category of contexts $\mathcal{C}$ (See e.g. [Jac99]).

$\mathcal{C}$ has as objects the contexts Γ of the type theory. A morphism $\Gamma \to \Delta$ in $\mathcal{C}$, where Δ is a context $y_1 : B_1,\ y_2 : B_2,\ \ldots,\ y_m : B_m$, is a tuple of terms $\vec{t} = (t_1,\ \ldots,\ t_m)$ satisfying

$$\Gamma \vdash t_i : B_i[t_1/y_1, \ldots, t_{i-1}/y_{i-1}] \qquad \text{for each } 1 \leq i \leq m.$$

We implicitly identify contexts, types and terms if they are the same up to substitution of definitionally equal types and terms, as in [Jac99]. The identity morphism on the context Δ is the tuple of variables $(y_1, \ldots, y_m)$. The composite of two morphisms $\vec{t} : \Gamma \to \Delta$ and $\vec{s} : \Delta \to \Theta$ is given by substituting each term t_i for the variable y_i in each component of $\vec{s}$. The empty context is a terminal object in $\mathcal{C}$.

The category of types $\mathcal{T}$ has as objects types in context $(\Gamma \vdash A : \text{Type})$. A morphism $(\Gamma \vdash A : \text{Type}) \to (\Delta \vdash B : \text{Type})$ consists of a morphism $t_1, \ldots, t_m : \Gamma \to \Delta$ in $\mathcal{C}$ and a term b where

$$\Gamma,\ x : A \vdash b : B[t_1/y_1, \ldots, t_m/y_m].$$

Proposition 2.1. *The forgetful functor $p : \mathcal{T} \to \mathcal{C}$ sending a type in context $(\Gamma \vdash A : \text{Type})$ to Γ is a split fibration.*

Proof. Reindexing is given by substitution: If $\vec{t}$ is a morphism of contexts $\Gamma \to \Delta$, then the reindexing functor $\vec{t}^*$ sends an object $(\Delta \vdash B : \text{Type})$ over Δ to $(\Gamma \vdash B[t_1/y_1, \ldots, t_m/y_m] : \text{Type})$.

$$\begin{tikzcd}
\Gamma \vdash B[t_1/y_1, \ldots, t_m/y_m] : \text{Type} \arrow[r] \arrow[d, dashed] & \Delta \vdash B : \text{Type} \arrow[d, dashed] \\
\Gamma \arrow[r, "\vec{t}"'] & \Delta
\end{tikzcd} \tag{2.3}$$

The cartesian morphism above $\vec{t}$ in $\mathcal{T}$ is $(\vec{t}, z)$, where z is a new variable of type $B[t_1/y_1, \ldots, t_m/y_m]$, representing the projection

$$\Gamma,\ z : B[t_1/y_1, \ldots, t_m/y_m] \vdash z : B[t_1/y_1, \ldots, t_m/y_m].$$

Splitness of the fibration follows since repeated substitution of terms is associative. $\square$

Each type A in context Γ determines a morphism $\mathfrak{P}_A : (\Gamma, x : A) \to \Gamma$ in $\mathcal{C}$ by projection. Such morphisms are called *display maps* (and denoted by arrows $\twoheadrightarrow$ in diagrams). Terms of type A in context Γ correspond to the morphisms in $\mathcal{C}$ which are sections of the display map $\Gamma, x : A \twoheadrightarrow \Gamma$.

$$\begin{tikzcd}
\Gamma \arrow[rr, dotted, "a:A"] \arrow[dr, "="'] & & \Gamma, x : A \arrow[dl, "\mathfrak{P}_A"] \\
& \Gamma &
\end{tikzcd}$$

A cartesian morphism as in Diagram 2.3 gives a commuting diagram between the display maps

$$\begin{tikzcd} \Gamma,\ z : B[t_1/y_1, \dots, t_m/y_m] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Delta,\ z : B \arrow[d] \\ \Gamma \arrow[r, "\vec{t}"'] & \Delta, \end{tikzcd}$$

which has the universal property of a pullback square in $\mathcal{C}$.

There are various essentially equivalent ways to formalize this categorical structure, such as comprehension categories [Jac93], categories with families [Dyb96], categories with attributes, contextual categories [Car86], D-categories [Ehr88] and type categories [Pit00].

Definition 2.2. A *full split comprehension category* consists of a split fibration $p : \mathcal{E} \to \mathcal{B}$, where the category $\mathcal{B}$ has a terminal object, together with a full and faithful functor $\mathfrak{P} \colon \mathcal{E} \to \mathcal{B}^{\mathbb{2}}$ such that

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "\mathfrak{P}"] \arrow[dr, "p"'] & & \mathcal{B}^{\mathbb{2}} \arrow[dl, "c"] \\ & \mathcal{B} & \end{tikzcd}$$

commutes and $\mathfrak{P}$ preserves cartesian morphisms.

The term model of a type theory is a full split comprehension category; conversely from a split comprehension category $p : \mathcal{E} \to \mathcal{B}$ we can construct a type theory $\mathbb{T}_p$. Contexts are represented by certain objects of $\mathcal{B}$, and the objects in the fibre $\mathcal{E}_\Gamma$ over a context Γ are regarded as the types in context Γ. The terminal object of $\mathcal{B}$ represents the empty context, and further contexts are generated successively from it: if A is a type in context Γ then the domain of the display map $\mathfrak{P}_A$ represents the extended context $\Gamma, x : A$. Sections of the display map $\mathfrak{P}_A$ represent the terms of type A.

We usually restrict attention to those comprehension categories such that every object of $\mathcal{B}$ appears as a context in the above construction of the type theory, called *reachable* comprehension categories. In this case, these constructions are inverses up to isomorphism: the correspondence sending a type theory to its term model and a comprehension category p to the theory $\mathbb{T}_p$ extends to an equivalence between the category of full split reachable comprehension categories and structure-preserving functors, and a suitable category of type theories [Car86, Pit00].

A crucial role in the term model is played by the display maps in the context category $\mathcal{B}$, which essentially determine the type theory. Each display map defines a type in context, with terms given by sections. Substitution of terms is the operation of taking a pullback of a display map to get another display map. In other words we have the following structure:

Definition 2.3 ([Tay99, HP89])**.** A *class of display maps* in a category $\mathcal{B}$ with terminal object is a class of morphisms $\mathcal{F} \subseteq \mathcal{B}^{\mathbb{2}}$ such that $\mathcal{F}$ is *stable*: pullbacks of display maps along any morphism in $\mathcal{B}$ exist and are in $\mathcal{F}$.

Remark 2.4. Such a class of morphisms is frequently called a *class of fibrations*, but we will not use this terminology here to avoid confusion with Grothendieck fibrations.

Given a comprehension category $\mathfrak{P}\colon \mathcal{E} \to \mathcal{B}^{\mathbb{2}}$, the closure of the image of $\mathfrak{P}$ in $\mathcal{B}^{\mathbb{2}}$ under isomorphism is a class of display maps. Conversely, given a class of display maps $\mathcal{F}$, the full subcategory of $\mathcal{B}^{\mathbb{2}}$ spanned by $\mathcal{F}$ defines a fibration

$$\begin{tikzcd} \mathcal{F} \arrow[rr, hook] \arrow[dr] & & \mathcal{B}^{\mathbb{2}} \arrow[dl, "c"] \\ & \mathcal{B}. & \end{tikzcd}$$

This is not a split comprehension category unless pullbacks are strictly associative. In general pullbacks are only associative up to isomorphism, so this does not give a sound interpretation of the strictly associative substitution of terms into types. However, such a fibration is equivalent to a split comprehension category:

Proposition 2.5 ([Gir71])**.** *The forgetful 2-functor from split fibrations to cloven fibrations over a category $\mathcal{B}$ has a left 2-adjoint F*

$$\begin{tikzcd} \mathbf{SplitFib}/\mathcal{B} \arrow[r, shift left, "U"] \arrow[r, phantom, "\top"] & \mathbf{Fib}/\mathcal{B} \arrow[l, shift left, "F"] \end{tikzcd}$$

such that each component of the unit is an equivalence.

Here the morphisms of the left-hand category are functors over $\mathcal{B}$ which preserve the splitting exactly, while the morphisms of the right-hand category are functors over $\mathcal{B}$ which preserve cartesian morphisms but not necessarily the cleavage. The 2-cells are natural transformations over $\mathcal{B}$. Thus every class of display maps is equivalent as a fibration to a split fibration strictly modelling the rules of type theory.

Example 2.6. (a) If $\mathcal{B}$ has all pullbacks, so the codomain functor $\mathcal{B}^{\mathbb{2}} \to \mathcal{B}$ is a fibration, then the class of all maps in $\mathcal{B}$ is a class of display maps.

(b) The class of all isomorphisms in any category is a class of display maps.

(c) If $\mathcal{B}$ has pullbacks, or at least pullbacks along monomorphisms, then the class of monomorphisms in $\mathcal{B}$ is a class of display maps. In this case a dependent type $a : A \vdash B : \text{Type}$ can be thought of as a predicate on the type A, where each fibre $B(a)$ is either empty or uniquely inhabited. However, this and the previous example are in general not reachable classes.

(d) If $\mathcal{B}$ has finite products, then the class of all product projections $A \times B \to B$ is a class of display maps, and is the smallest class which is reachable. This is in some sense a trivial example of dependent type theory: the fibres of a dependent type are constant, so the type dependency plays no role. The fibration $s(\mathcal{B}) \to \mathcal{B}$ of such product projections is called the simple fibration, as the corresponding syntax is simple type theory.

2.3 Type constructors

In addition to the basic rules of inference of Section 2.1, a type theory can be extended by rules to construct and manipulate new kinds of types.

Each kind of type has associated rules following a similar pattern: there are rules detailing how to form these types from other given types (*formation*), how to construct terms of these types (*introduction*), how to derive new judgements from judgements involving the introduced terms (*elimination*), and how to combine introduction and elimination (*conversion*). In particular, we use both β-conversion rules, for simplifying an introduction followed by an elimination, and the dual η-conversion rules for an elimination followed by introduction. Most of the type constructors given here are presented as *positive types*, where the introduction rules 'generate' all terms, in the sense that judgements depending on a general term are specified by the case when the term is one of the basic terms introduced.

We assume additional rules requiring the new types and terms to interact as expected with substitution and definitional equality, but these will be left implicit. For clarity context variables shared by the hypotheses and conclusion of an inference rule will be omitted when stating the rules.

The new types give additional categorical structure to the term model of the theory. The presence of η-rules in the type theory means that this usually takes the form of some kind of universal property. We shall recall some standard type constructors and motivate what additional structure should be required of the corresponding class of fibrations interpreting the theory.

2.3.1 The unit type

A *unit type* is a specified type which has a unique term. These are constructed by the rules:

$$\frac{}{\vdash 1 : \mathrm{Type}} \quad \text{1-F}\textsc{ormation}$$

$$\frac{}{\vdash * : 1} \quad \text{1-I}\textsc{ntroduction}$$

The elimination rule states that types depending on a variable of unit type are determined by the case for $*$:

$$\frac{x : 1 \vdash C : \mathrm{Type} \qquad \vdash c : C[*/x]}{x : 1 \vdash \mathit{case}(c) : C} \quad \text{1-E}\textsc{limination}$$

Introduction and elimination are inverse processes:

$$\frac{x : 1 \vdash C : \mathrm{Type} \qquad \vdash c : C[*/x]}{\vdash \mathit{case}(c)[*/x] = c : C[*/x]} \quad \text{1-}\beta\text{-C}\textsc{onversion}$$

$$\frac{x : 1 \vdash C : \mathrm{Type} \qquad x : 1 \vdash c : C}{x : 1 \vdash \mathit{case}(c[*/x]) = c : C} \quad \text{1-}\eta\text{-C}\textsc{onversion}$$

While the last three rules follow the general pattern for positive type constructors, they can be replaced by a simpler one. Applying the β-conversion rule for the type $C = 1$ when c is either u or $*$ gives the rule:

$$\frac{\vdash u : 1}{\vdash u = * : 1} \quad \text{1-U}\textsc{niqueness}$$

In a categorical model, this says that there is an object 1 in the fibre above the (terminal) empty context $\{\}$, such that for any context Γ the display map representing

$(\Gamma \vdash 1 : \text{Type})$ has a unique section.

$$\begin{tikzcd}
\Gamma, x : 1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & x : 1 \arrow[d, "\mathfrak{P}_1"] \\
\Gamma \arrow[r] \arrow[u, dotted, bend left, "*{:}1"] & \{\} \arrow[u, dotted, bend left, "*{:}1"]
\end{tikzcd}$$

In other words, every Γ has a unique morphism to the context $x : 1$, so $x : 1$ is terminal in $\mathcal{B}$ and the display map $\mathfrak{P}_1$ is an isomorphism.

Since a class of display maps is closed under pullbacks and every isomorphism appears as a pullback of $\mathfrak{P}_1$ we define:

Definition 2.7. A class of display maps $\mathcal{F} \subseteq \mathcal{B}^2$ *has a unit type* if $\mathcal{F}$ contains all isomorphisms in $\mathcal{B}$.

Example 2.8. If $\mathcal{B}$ has finite limits, the classes of display maps consisting of all morphisms, monomorphisms, and product projections respectively all have a unit type.

2.3.2 Dependent sum types

Given a type B depending on a variable in A, the *dependent sum type* $\Sigma x{:}A.B$ represents the disjoint union of the types $B(x)$ as x ranges over A. Objects of $\Sigma x{:}A.B$ are pairs of terms a in A and b in the corresponding type $B[a/x]$:

$$\frac{x : A \vdash B : \text{Type}}{\vdash \Sigma x{:}A.B : \text{Type}} \quad \Sigma\text{-}\textsc{Formation}$$

$$\frac{x : A \vdash B : \text{Type} \qquad \vdash a : A \qquad \vdash b : B[a/x]}{\vdash (a, b) : \Sigma x{:}A.B} \quad \Sigma\text{-}\textsc{Introduction}$$

The elimination rule states that for types and terms depending on a variable of the sum type, it is sufficient to know what happens when the variable is a pair:

$$\frac{\begin{array}{c}\vdash p : \Sigma x{:}A.B \qquad z : \Sigma x{:}A.B \vdash C : \text{Type} \\ x : A, y : B \vdash d : C[(x, y)/z]\end{array}}{\vdash \mathit{case}(p, (x, y).d) : C[p/z]} \quad \Sigma\text{-}\textsc{Elimination}$$

where the notation $(x, y).d$ means that the variables x and y are bound. The conversion rules ensure that the pairing and case functions are compatible:

$$\frac{\vdash a : A \qquad \vdash b : B[a/x] \qquad z : \Sigma x{:}A.B \vdash C : \text{Type} \qquad x : A, y : B \vdash d : C[(x,y)/z]}{\vdash \mathit{case}((a,b),(x,y).d) = d[a/x, b/y] : C[(a,b)/z]}\ \Sigma\text{-}\beta\text{-}\textsc{Conversion}$$

$$\frac{z : \Sigma x{:}A.B \vdash C : \text{Type} \qquad \vdash p : \Sigma x{:}A.B \qquad z : \Sigma x{:}A.B \vdash d : C}{\vdash \mathit{case}(p,(x,y).d[(x,y)/z]) = d[p/z] : C[p/z]}\ \Sigma\text{-}\eta\text{-}\textsc{Conversion}$$

Remark 2.9. 1. If the type B does not depend on the variable x in A, then terms of type $\Sigma x{:}A.B$ are pairs of terms $(a : A,\ b : B)$. The type $\Sigma x{:}A.B$ is then written as $A \times B$ and called a binary product type.

2. These rules define *strong* sum types. If the type C in the elimination and conversion rules is not allowed to depend on a variable in $\Sigma x{:}A.B$, the corresponding types are called *weak* sum types.

Categorically, the rules for weak sums say that for any types A and C in context Γ and type B in the extended context $\Gamma, x : A$, there is a bijection between terms of C depending on a variable $z : (\Sigma x{:}A.B)$ and terms of C depending on variables $x : A, y : B$. In other words there is a bijection between maps in the fibre categories:

$$\begin{tikzcd} \Sigma x{:}A.B \arrow[rr] \arrow[dr] & & C \arrow[dl, two heads] \\ & \Gamma & \end{tikzcd} \quad \Leftrightarrow \quad \begin{tikzcd} B \arrow[rr] \arrow[dr] & & C \arrow[dl, two heads] \\ & \Gamma, x : A & \end{tikzcd}$$

Thus the functor $\Sigma_A : \mathcal{F}/(\Gamma, x : A) \to \mathcal{F}/\Gamma$ sending a type B to $\Sigma x{:}A.B$ is left adjoint to the pullback functor $\mathfrak{P}_A^*$ sending C to C in the weakened context.

The fact that sum types are compatible with substitution corresponds to the Beck-Chevalley condition: Consider a pullback square of the form

$$\begin{tikzcd} \Gamma, x : A \arrow[r, "t"] \arrow[d, two heads, "\mathfrak{P}_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Delta, y : B \arrow[d, two heads, "\mathfrak{P}_B"] \\ \Gamma \arrow[r, "s"'] & \Delta \end{tikzcd}$$

in the context category $\mathcal{B}$. The type A is given by substituting the terms making up the morphism s into the type B. For a type C depending on B, there is an isomorphism of types in context

$$(\Sigma y{:}B.C)[s_1/x_1,\ldots,s_n/x_n] \cong \Sigma y{:}B[s_1/x_1,\ldots,s_n/x_n].C[s_1/x_1,\ldots,s_n/x_n],$$

i.e. the canonical map $\Sigma_A t^* \to s^*\Sigma_B$ is an isomorphism.

Definition 2.10. A class of display maps $\mathcal{F}$ *has weak dependent sum types* if the fibration $\mathcal{F} \to \mathcal{B}$ has sums along display maps. In other words, for each $f \in \mathcal{F}$ the reindexing functor f^* has a left adjoint Σ_f, and the Beck-Chevalley condition holds.

Strong sums correspond to composition of display maps: a pair of composable display maps represents types $\Gamma \vdash A : \text{Type}$ and $\Gamma, x : A \vdash B : \text{Type}$. If the type theory has weak sum types, then there is a canonical context map t in

$$\begin{tikzcd}
\Gamma, x : A, y : B \arrow[r, "t"] \arrow[d, two heads] & \Gamma, z : (\Sigma x{:}A.B) \arrow[d, two heads] \\
\Gamma, x : A \arrow[r, two heads] & \Gamma
\end{tikzcd}$$

given by pairing. If it has strong sum types, then there is a map in the reverse direction given by the terms

$$\begin{aligned}
\Gamma, z : (\Sigma x{:}A.B) &\vdash \mathit{case}(z,(x,y).x) : A && \text{(called } \pi_1(z)\text{)} \\
\Gamma, z : (\Sigma x : {:}A.B) &\vdash \mathit{case}(z,(x,y).y) : B[\mathit{case}(z,(x,y).x)/z] && \text{(called } \pi_2(z)\text{)},
\end{aligned} \tag{2.4}$$

and the conversion rules ensure that it is an inverse to pairing. Since display maps are closed under composing with isomorphisms, the composite is a display map.

Definition 2.11. A class of display maps $\mathcal{F}$ has *strong dependent sum types* if it is closed under composition.

Remark 2.12. If $\mathcal{F}$ has strong dependent sum types, then the left adjoint Σ_f to reindexing along $f \in \mathcal{F}$ is just given by composition with f. In what follows, 'dependent sum types' will refer to the strong version.

In other words, a class of display maps has unit and sum types iff the span

$$\Psi_{\mathcal{F}} = \begin{tikzcd}
& \mathcal{F} \arrow[dl, "d"'] \arrow[dr, "c"] & \\
\mathcal{B} & & \mathcal{B}
\end{tikzcd}$$

is a submonad of

$$\Psi_B = \begin{tikzcd} & \mathcal{B}^{\mathbf{2}} \arrow[dl, "d"'] \arrow[dr, "c"] & \\ \mathcal{B} & & \mathcal{B} \end{tikzcd}$$

in the bicategory **Span**(**Cat**). As in Proposition 1.33, the fact that $\mathcal{F}$ is closed under pullbacks corresponds to the existence of a pseudo-distributive law $\Phi_B \Psi_{\mathcal{F}} \to \Psi_{\mathcal{F}} \Phi_B$ between $\mathcal{F}$ and the monad Φ_B given by reversing the span Ψ_B.

Having sum types allows us to blur the distinction between contexts and types. Up to isomorphism, any non-empty context $x_1 : A_1,\ x_2 : A_2,\ \ldots,\ x_n : A_n$ can be identified with the single-type context

$$x : (\Sigma x_1 {:} A_1.(\Sigma x_2 {:} A_2.(\cdots (\Sigma x_{n-1} {:} A_{n-1}.A_n) \cdots))).$$

If there is additionally a unit type, identified with the empty context, then every context corresponds to a closed type, and every context morphism corresponds to a single term. Thus the condition that a categorical model with unit and sum types be reachable reduces to requiring that there is a display map from each object to the terminal object.

Definition 2.13. A class of display maps in a category $\mathcal{B}$ is called *well-rooted* if the unique morphism from each object in $\mathcal{B}$ to the terminal object is a display map.

Since every isomorphism in $\mathcal{B}$ is a pullback of the identity morphism $1 \to 1$, a well-rooted class of display maps automatically has a unit type.

Example 2.14. The classes of display maps consisting of all morphisms, monomorphisms, and product projections respectively always have (strong) sum types.

Remark 2.15. Recall the binary product types described in Remark 2.9. The type $\Gamma \vdash A \times B$ is the product of A and B in the fibre over Γ. The projections are given by terms $\pi_1(z) : A$ and $\pi_2(z) : B$ defined as in (2.4), where $\pi_1((a, b)) = a$ and $\pi_2((a, b)) = b$ for terms $a : A,\ b : B$ by the conversion rules. A map $C \to A \times B$ in the fibre is a term $\Gamma, z : C \vdash p : A \times B$, and for such a p two applications of the η-rule show that $(\pi_1(p), \pi_2(p)) = p$. Thus a class of display maps with dependent sum types in particular has fibred binary products.

2.3.3 Dependent product types

Given a type B depending on a variable in A, the *dependent product type* $\Pi x{:}A.B$ represents the collection of functions mapping each term a in A to a term in the corresponding type $B[a/x]$.

$$\frac{x : A \vdash B : \mathrm{Type}}{\vdash \Pi x{:}A.B : \mathrm{Type}} \quad \Pi\text{-}\textsc{Formation}$$

Terms of the dependent product are formed by abstracting the variable of dependent terms in B:

$$\frac{x : A \vdash B : \mathrm{Type} \qquad x : A \vdash b : B}{\vdash \lambda x{:}A.b : \Pi x{:}A.B} \quad \Pi\text{-}\textsc{Introduction}$$

The dependent product is not a positive type, but terms of the dependent product can be applied as functions to terms in A:

$$\frac{\vdash f : \Pi x{:}A.B \qquad \vdash a : A}{\vdash fa : B[a/x]} \quad \Pi\text{-}\textsc{Elimination}$$

The conversion rules assert that abstraction and application are inverse processes:

$$\frac{\vdash a : A \qquad \vdash b : B[a/x]}{\vdash (\lambda x{:}A.b)a = b[a/x] : B[a/x]} \quad \Pi\text{-}\beta\text{-}\textsc{Conversion}$$

$$\frac{\vdash p : \Pi x{:}A.B}{\vdash p = \lambda x{:}A.px : \Pi x{:}A.B} \quad \Pi\text{-}\eta\text{-}\textsc{Conversion}$$

Remark 2.16. If the type B does not depend on the variable x in A, then terms of type $\Pi x{:}A.B$ correspond to functions from the terms of type A to the terms of type B. The type $\Pi x{:}A.B$ is then written as $A \Rightarrow B$ and called a function type.

Similarly to the case for sum types, the rules for dependent products give a natural bijection between terms of the product type $\Pi x{:}A.B$ in context $\Gamma, y : C$ and terms of

B in context $\Gamma, x : A, y : C$. This gives a bijection of maps

$$\begin{tikzcd} C \arrow[rr] \arrow[dr, two heads] & & \Pi x{:}A.B \arrow[dl, two heads] \\ & \Gamma & \end{tikzcd} \quad \Leftrightarrow \quad \begin{tikzcd} C \arrow[rr] \arrow[dr, two heads] & & B \arrow[dl, two heads] \\ & \Gamma, x : A & \end{tikzcd}$$

for any type C in the fibre category $\mathcal{F}/\Gamma$, and so the functor $\mathcal{F}/(\Gamma, x : A) \to \mathcal{F}/\Gamma$ sending a type B to $\Pi x{:}A.B$ is right adjoint to the pullback functor $\mathfrak{P}_A^*$.

As for sum types, the Beck-Chevalley condition corresponds to compatibility of product types with substitution. For every pullback square of the form

$$\begin{tikzcd} \Gamma, x : A \arrow[r, "t"] \arrow[d, two heads, "\mathfrak{P}_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Delta, y : B \arrow[d, two heads, "\mathfrak{P}_B"] \\ \Gamma \arrow[r, "s"'] & \Delta \end{tikzcd}$$

and type C, there is always an isomorphism of types

$$(\Pi y{:}B.C)[s_1/x_1, \ldots, s_n/x_n] \cong \Pi y{:}B[s_1/x_1, \ldots, s_n/x_n].C[s_1/x_1, \ldots, s_n/x_n].$$

Thus the canonical map $s^*\Pi_B \to \Pi_A t^*$ is an isomorphism.

Definition 2.17. A class of display maps has *dependent product types* if for each morphism f in $\mathcal{F}$ the reindexing functor f^* along the display map has a right adjoint Π_f satisfying the Beck-Chevalley condition; in other words, if the fibration $\mathcal{F} \to \mathcal{B}$ has $\mathcal{F}$-products.

Example 2.18. (a) The codomain fibration has product types if and only if the pullback functor f^* for each $f : A \to B$ in $\mathcal{B}$ has a right adjoint $\Pi_f : \mathcal{B}/A \to \mathcal{B}/B$, i.e. iff $\mathcal{B}$ is locally cartesian closed. The Beck-Chevalley condition holds by taking right adjoints in the corresponding condition for sums, which is automatically satisfied.

(b) When $\mathcal{B}$ has finite products, the class of display maps given by product projections has product types if and only if the functor $A \times (-) : \mathcal{B} \to \mathcal{B}$ has a right adjoint for each A in $\mathcal{B}$, i.e. iff $\mathcal{B}$ is cartesian closed. The Beck-Chevalley condition is again automatically satisfied. The corresponding product types are known as simple products.

Remark 2.19. 1. If a class of display maps has dependent product and sum types, then the function type $\Gamma \vdash (A \Rightarrow B)$ in Remark 2.16 is an exponential in the fibre category $\mathcal{F}/\Gamma$ over Γ. Morphisms $C \to (A \Rightarrow B)$ in $\mathcal{F}/\Gamma$ correspond to morphisms $C \to B$ in the fibre over $\Gamma,\ x : A$, which by the adjunction for sums correspond to morphisms $A \times C \to B$ in $\mathcal{F}/\Gamma$. Thus a class of display maps with unit, sum and product types is a fibred cartesian closed category.

2. As in Remark 1.36, the Beck-Chevalley condition says equivalently that the functor Π_f is additionally a partial right adjoint to the pullback functor on slice categories $f^* : \mathcal{B}/A \to \mathcal{B}/B$, or in other words that each inclusion $\mathcal{F}/A \hookrightarrow \mathcal{B}/A$ preserves exponentials.

By analogy with Example 2.18(a), a class of display maps with unit, dependent sum and dependent product types is called a *relatively cartesian closed category* [Tay99]. We see that a relatively cartesian closed category is exactly the structure required to model polynomials, in the sense described in Section 1.9. As in that section, in what follows we frequently write the sum $\Sigma x{:}A.B$ using subscripts as $\sum_{x:A} B$ or $\sum_{x\in A} B$ to mimic polynomials in **Set**, and similarly for products.

2.3.4 The empty type

An *empty type* is a specified type with no terms. It has no introduction rule, but has rules stating that the existence of a term of this type would imply that all types are uniquely inhabited.

$$\frac{}{\vdash 0 : \mathrm{Type}} \quad 0\text{-}\textsc{Formation}$$

$$\frac{\vdash C : \mathrm{Type} \qquad \vdash a : 0}{\vdash \mathit{empty}(a) : C} \quad 0\text{-}\textsc{Elimination}$$

$$\frac{\vdash C : \mathrm{Type} \qquad \vdash a : 0 \qquad x : 0 \vdash c : C}{\vdash \mathit{empty}(a) = c[a/x] : C} \quad 0\text{-}\eta\text{-}\textsc{Conversion}$$

For any context Γ, the empty type 0 is an object in the fibre $\mathcal{F}/\Gamma$ with a unique map to any other object C given by *empty*. 0 is preserved by substitution, i.e. reindexing.

Definition 2.20. A class of display maps *has an empty type* if the fibration $\mathcal{F} \to \mathcal{B}$ has a fibred initial object; equivalently if the category $\mathcal{B}$ has an initial object 0 which is preserved by pullback.

2.3.5 Binary sum types

If A and B are types, the *binary sum type* $A+B$ represents the disjoint union of A and B. Each term in A and B gives a term in the sum type:

$$\frac{\vdash A : \text{Type} \qquad \vdash B : \text{Type}}{\vdash A+B : \text{Type}} \quad +\text{-Formation}$$

$$\frac{\vdash a : A}{\vdash \mathit{inl}(a) : A+B} \qquad \frac{\vdash b : B}{\vdash \mathit{inr}(b) : A+B} \quad +\text{-Introduction}$$

Analogously to the rules for dependent sum types, the elimination and conversion rules assert that types and terms depending on a variable of the binary sum type are completely determined by the case when the variable comes from either A or B:

$$\frac{\begin{array}{c}\vdash p : A+B \qquad z : A+B, \Delta \vdash C : \text{Type} \\ x : A, \Delta[\mathit{inl}(x)/z] \vdash c : C[\mathit{inl}(x)/z] \\ y : B, \Delta[\mathit{inr}(y)/z] \vdash d : C[\mathit{inr}(y)/z]\end{array}}{\Delta[p/z] \vdash \mathit{case}(p, x.c, y.d) : C[p/z]} \quad +\text{-Elimination}$$

$$\frac{\begin{array}{c}\vdash a : A \qquad z : A+B, \Delta \vdash C : \text{Type} \\ x : A, \Delta[\mathit{inl}(x)/z] \vdash c : C[\mathit{inl}(x)/z] \\ y : B, \Delta[\mathit{inr}(y)/z] \vdash d : C[\mathit{inr}(y)/z]\end{array}}{\Delta[\mathit{inl}(a)/z] \vdash \mathit{case}(\mathit{inl}(a), x.c, y.d) = c[a/x] : C[\mathit{inl}(a)/z]}$$

$$\frac{\begin{array}{c}\vdash b : B \qquad z : A+B, \Delta \vdash C : \text{Type} \\ x : A, \Delta[\mathit{inl}(x)/z] \vdash c : C[\mathit{inl}(x)/z] \\ y : B, \Delta[\mathit{inr}(y)/z] \vdash d : C[\mathit{inr}(y)/z]\end{array}}{\Delta[\mathit{inr}(b)/z] \vdash \mathit{case}(\mathit{inr}(b), x.c, y.d) = d[b/y] : C[\mathit{inr}(b)/z]} \quad +\text{-}\beta\text{-Conversion}$$

$$\frac{\begin{array}{c}z : A+B, \Delta \vdash C : \text{Type} \\ \vdash p : A+B \qquad z : A+B \vdash c : C\end{array}}{\Delta[p/z] \vdash \mathit{case}(p, x.c[\mathit{inl}(x)/z], y.c[\mathit{inr}(y)/z]) = c[p/z] : C[p/z]} \quad +\text{-}\eta\text{-Conversion}$$

Remark 2.21. These rules define *strong* binary sum types. If the type C in the elimination and conversion rules does not depend on a variable in $A+B$, the corresponding types are called *weak* binary sum types.

The formation and introduction rules for weak sums define inclusion maps $A \to$

$A + B \leftarrow B$ in the fibre category $\mathcal{F}/\Gamma$ for each context Γ. The elimination and conversion rules state that there is a bijection between maps $A + B \to C$ and pairs of maps $A \to C$, $B \to C$ for any C in the fibre category, and compatibility with substitution corresponds to stability of coproducts under reindexing.

Definition 2.22. A class of display maps *has weak binary sum types* if the fibration $\mathcal{F} \to \mathcal{B}$ has fibred binary coproducts, i.e. the category $\mathcal{B}$ has binary coproducts which are stable under pullback.

The additional elimination and conversion rules for strong binary sum types assert that for any types C and D in the fibre category $\mathcal{F}/(\Gamma, z : A + B)$, maps $D \to C$ above $A + B$ correspond bijectively to pairs of maps between the types reindexed along the coproduct inclusions.

Definition 2.23. A class of display maps *has strong binary sum types* if it has weak binary sums and for any diagram

$$\begin{tikzcd} D \arrow[r] \arrow[d, two heads] & C \arrow[d, two heads] & E \arrow[l] \arrow[d, two heads] \\ A \arrow[r] & A + B & B \arrow[l] \end{tikzcd} \tag{2.5}$$

in $\mathcal{B}$ where the bottom row is a coproduct diagram and all the vertical maps are display maps, the two squares are pullbacks iff the top row is also a coproduct diagram. This says equivalently that the canonical functor $\mathcal{F}/(A+B) \to \mathcal{F}/A \times \mathcal{F}/B$ given by reindexing along the coproduct inclusions is full and faithful.

Remark 2.24. A category with finite coproducts satisfying the above condition when the vertical morphisms are not required to be display maps is said to be *extensive*. This is equivalent to the conditions that pullbacks of coproduct inclusions along any morphisms exist and finite coproducts are disjoint and stable under pullback [CLW93].

The above rules for binary sums relate terms depending on variables of A and B with terms depending on a variable of $A + B$. Further rules make it possible to do the same for types:

$$\frac{\begin{array}{c}\vdash p : A + B \\ x : A \vdash C : \mathrm{Type} \qquad y : B \vdash D : \mathrm{Type}\end{array}}{\vdash \mathit{Case}(p, x.C, y.D) : \mathrm{Type}} \quad +\text{-}\textsc{Type-Elimination}$$

$$\frac{\vdash a : A \quad x : A \vdash C : \text{Type} \quad y : B \vdash D : \text{Type}}{\vdash \mathit{Case}(\mathit{inl}(a), x.C, y.D) = C[a/x] : \text{Type}}$$

$$\frac{\vdash b : B \quad x : A \vdash C : \text{Type} \quad y : B \vdash D : \text{Type}}{\vdash \mathit{Case}(\mathit{inr}(b), x.C, y.D) = D[b/y] : \text{Type}} \quad \text{+-Type-}\beta\text{-Conversion}$$

Remark 2.25.

1. The η-conversion rule, which would assert that the type $\mathit{Case}(p, x.C[\mathit{inl}(x)/z], y.C[\mathit{inl}(y)/z])$ is the same as $C[p/z]$ for any $z : A + B$ and type C depending on $A + B$, holds automatically up to isomorphism for strong binary sums since by previous rules there is a bijection between their terms.
2. Similar rules could be defined for types over dependent sums. However these are automatically satisfied by a type theory with strong dependent sums, using the projection maps of Equation 2.4.

In particular, assuming there are unit and empty types, the additional rules mean that for any types A and B there is a type

$$p : A + B \vdash \tilde{A}(p) \equiv \mathit{Case}(p, x.1, y.0)$$

such that $\tilde{A}(\mathit{inl}(a)) = 1$ and $\tilde{A}(\mathit{inr}(b)) = 0$ for $a : A$, $b : B$. In a categorical model with strong sums for types, this corresponds to the coproduct inclusion $A \to A + B$ being a display map. Conversely, if types of the form $\tilde{A}$ and $\tilde{B}$ exist, then the general form of the +-Type-elimination rule can be obtained by setting $\mathit{Case}(p, x.C, y.D)$ to be the type

$$\Sigma a{:}\tilde{A}(p).C(\mathit{case}(p, x.x, y.\mathit{empty}(a))) + \Sigma b{:}\tilde{B}(p).D(\mathit{case}(p, x.\mathit{empty}(b), y.y)).$$

If all coproduct inclusions are display maps, then considering a diagram of the form in (2.5) when the top row is the coproduct diagram

$$0 \longrightarrow B \longleftarrow B$$

shows that coproducts are always disjoint, so the category is extensive.

Definition 2.26. A class of display maps $\mathcal{F}$ in $\mathcal{B}$ *has strong sums for types* if it has binary sum types and $\mathcal{F}$ contains all coproduct inclusions; equivalently if $\mathcal{B}$ is extensive and $\mathcal{F}$ contains all coproduct inclusions.

Remark 2.27. This definition says equivalently that for every diagram of the form in (2.5), if the horizontal rows are coproduct diagrams and the outer vertical maps are display maps, then the central vertical map $D + E \to A + B$ is also a display map. In other words, the functor $\mathcal{F}/(A + B) \to \mathcal{F}/A \times \mathcal{F}/B$ is an equivalence.

Example 2.28. (a) A locally cartesian closed category is extensive iff it has finite disjoint coproducts, since pullback functors are left adjoints and preserve all colimits that exist.

(b) The class of all product projections in a category $\mathcal{B}$ cannot have strong sums for types unless $\mathcal{B}$ is trivial, since if the coproduct inclusions $1 \to 1 + 1$ are also product projections then $1 \cong 0$.

2.3.6 Coherence

For each type constructor with the rules considered here, the condition a fibration must satisfy to model these types takes a similar form. We require the existence in each fibre of objects and morphisms with some universal property, with a stability condition for reindexing. A variation on this formulation of constructors, which is sometimes used in homotopy type theory and proof assistants for example, is to remove the η-conversion rule. In category theory terms, the corresponding objects and morphisms would then require a weak universal property, which asserts existence but not uniqueness. The use of η-conversion is chosen here because it simplifies calculations and fits naturally with the models used; modifications of the results for other formulations may also be possible.

However, even a universal property will only define an object up to isomorphism. In a strict categorical model of type theory, to satisfy the rules in the associated type theory of Section 2.2 it is necessary to make a choice of each such object in such a way that they are strictly stable under substitution. The following coherence results by Lumsdaine and Warren [LW14] mean that this can always be done.

Proposition 2.29. *If $\mathcal{F}$ is class of display maps with at least unit, dependent sum and dependent product types, then the equivalent strict model of type theory given by Proposition 2.5 can be given the structure of unit, dependent sum and dependent product types, including strict associativity.*

If $\mathcal{F}$ also has an empty type or binary sum types, then the strict model of type theory can be given the structure of an empty type or binary sum types respectively. □

Remark 2.30. An equivalent coherence result was originally shown by Hofmann [Hof94] using a right adjoint to the forgetful functor $\mathbf{SplitFib}/\mathcal{C} \to \mathbf{Fib}/\mathcal{C}$ in Proposition 2.5 instead of a left adjoint. However, this approach does not seem to extend to the intensional identity types of the next section.

2.4 Identity types

If types are considered to be propositions, then under the Curry-Howard correspondence terms correspond to proofs of the type they inhabit. The unit type represents truth, and the empty type, which has no proofs, represents falsehood. Dependent sum types $\Sigma x{:}A.B$ correspond to existential quantifiers: to give a proof of $\exists x \in A.B$ is to give a witness a in A and a proof of $B[a/x]$. Dependent product types correspond to universal quantifiers: to give a proof of $\forall x \in A.B$ is to give a function assigning a proof of $B[a/x]$ to each a in A.

While types and terms can be judgementally equal, it is not possible to reason about proofs of equality in the same way. Thus the *identity type* constructor provides a type $Id_A(a, b)$ for each pair of terms a, b of type A, which represents the type of proofs that a equals b [ML84]. Terms a and b are called *propositionally equal* if $Id_A(a, b)$ is inhabited.

Alternatively, in the homotopy type theory interpretation [The13] where types are considered to be spaces, terms represent points of the spaces and the identity type $Id_A(a, b)$ corresponds to the space of paths from point a to point b in A.

The formation rule for identity types defines an identity type for any two elements of the same type:

$$\frac{\vdash A : \mathrm{Type} \qquad \vdash a : A \qquad \vdash b : A}{\vdash Id_A(a, b) : \mathrm{Type}} \ \textsc{Id-Formation}$$

For each term a of A there is a specified term of the identity type $Id_A(a,a)$, corresponding to a proof of the reflexivity of equality, or to a constant path:

$$\frac{\vdash a : A}{\vdash r(a) : Id_A(a,a)} \quad \text{Id-Introduction}$$

The elimination and conversion rules state that given a proof p that terms a and b are equal, terms of types depending on p are inductively generated by terms of the type for the reflexivity case:

$$\frac{\begin{array}{c}\vdash a : A \qquad \vdash b : A \qquad \vdash p : Id_A(a,b) \\ x, y : A, u : Id_A(x,y) \vdash C : \text{Type} \\ x : A \vdash d : C[x/y, r(x)/u]\end{array}}{\vdash J(a,b,p,x.d) : C[a/x, b/y, p/u]} \quad \text{Id-Elimination}$$

$$\frac{\begin{array}{c}\vdash a : A \\ x, y : A, u : Id_A(x,y) \vdash C : \text{Type} \\ x : A \vdash d : C[x/y, r(x)/u]\end{array}}{\vdash J(a,a,r(a),x.d) = d[a/x] : C[a/x, a/y, r(a)/u]} \quad \text{Id-}\beta\text{-Conversion}$$

In contrast to the type constructors of the previous section, we do not assume an η-conversion rule for identity types, which would state that the J constructor uniquely determines all terms of types depending on identity types:

$$\frac{\begin{array}{c}\vdash a : A \qquad \vdash b : A \qquad \vdash p : Id_A(a,b) \\ x, y : A, u : Id_A(x,y) \vdash C : \text{Type} \\ x, y : A, u : Id_A(x,y) \vdash d : C\end{array}}{\vdash J(a,b,p,x.d[x/y, r(x)/u]) = d[a/x, b/y, p/u] : C[a/x, b/y, p/u]} \quad \text{Id-}\eta\text{-Conversion}$$

This is because such a rule would force the structure of the identity types to be trivial – each would be uniquely inhabited or empty and so we would lose the intended computational interpretation identifying terms and proofs:

Proposition 2.31. *[Str93] A type theory with identity types satisfies the η-conversion* *rule if and only if it satisfies*

$$\frac{\vdash a : A \qquad \vdash b : A \qquad \vdash p : Id_A(a,b)}{\vdash a = b : A} \quad \textsc{Id-Reflection}$$

It then also satisfies

$$\frac{\vdash p : Id_A(a,b) \qquad \vdash q : Id_A(a,b)}{\vdash p = q : Id_A(a,b)} \quad \textsc{Uniqueness of Identity Proofs}$$

Proof. Applying the η-rule when the type C is A and d is x gives the judgement $J(a,b,p,x.x) = a : A$. However when C is A and d is y it gives $J(a,b,p,x.x) = b : A$.

Similarly, applying the rule when C is $Id_A(x,y)$ and d is u gives $J(a,b,p,x.r(x)) = p : Id_A(a,a)$, which is well-typed since $a = b$. When d is $r(x)$ it gives $J(a,b,p,x.r(x)) = r(a) : Id_A(a,a)$, so repeating this for q shows $p = r(a) = q : Id_A(a,b)$.

Conversely, given the reflection rule, it suffices to prove the η-rule up to propositional equality, i.e. to give a term of the identity type

$$x, y : A, u : Id_A(x,y) \vdash Id_C(J(x,y,u,x.d[x/y, r(x)/u]), d) : \text{Type}.$$

When $x = y$ and $u = r(x)$, the two terms of C are equal by the β-rule so reflexivity gives a term of the identity type, and the general case follows by the elimination rule. □

The uniqueness of identity proofs and reflection rules are independent of the other rules for identity types, first demonstrated by Hofmann and Streicher using a model in the category of groupoids [HS98]. Dependent type theory with identity types satisfying the reflection rule is called *extensional*, and *intensional* or *Martin-Löf* type theory otherwise.

In a categorical model with the type constructors of the previous section, the formation rule gives an object Id_A for each type A in context Γ, with a display map to the object $A \times_\Gamma A$ representing the context $\Gamma, x : A, y : A$.

The reflexivity term corresponds to a morphism r_A making the diagram

$$\begin{tikzcd} A \arrow[r, "r_A"] \arrow[dr, "\delta"'] & Id_A \arrow[d, "\mathfrak{P}_{Id_A}"] \\ & A \times_\Gamma A \end{tikzcd}$$

commute, where δ is the diagonal map.

Given any display map over Id_A with a commuting square as in the outside of the diagram

$$\begin{tikzcd} A \arrow[r, "d"] \arrow[d, "r_A"'] & C \arrow[d, "\mathfrak{P}_C"] \\ Id_A \arrow[r, equal] \arrow[ur, dashed, "J"] & Id_A, \end{tikzcd}$$

the elimination rule defines a diagonal morphism $J : Id_A \to C$. The bottom triangle commutes since J represents a term, and the top triangle commutes by the β-conversion rule.

More generally, any commuting square between r_A and a display map can be given a diagonal filler, by pulling back the display map along the bottom morphism and then applying the elimination rule.

$$\begin{tikzcd} A \arrow[r, "d"] \arrow[d, "r_A"'] & D \arrow[d, "\mathfrak{P}_D"] \\ Id_A \arrow[r, "e"'] & E \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[r] \arrow[d, "r_A"'] & C \arrow[r, "k"] \arrow[d, "\mathfrak{P}_C"] \arrow[dr, phantom, very near start, "\lrcorner"] & D \arrow[d, "\mathfrak{P}_D"] \\ Id_A \arrow[r, equal] \arrow[ur, dashed, "J"] & Id_A \arrow[r, "e"'] & E \end{tikzcd} \tag{2.6}$$

The morphism kJ is then a filler for the original square.

Definition 2.32. A morphism f has the *left lifting property* with respect to g, or equivalently g has the *right lifting property* with respect to f, written $f \boxslash g$, if for every commutative square of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & B \arrow[d, "g"] \\ C \arrow[r] & D, \end{tikzcd}$$

there exists a diagonal filler $C \to B$ making both triangles commute.

The elimination and β-conversion rules thus imply that the reflexivity map r_A has the left lifting property with respect to all display maps. The compatibility of identity types with substitution should mean that this property is stable under pullback, i.e. that for any morphism $s : \Delta \to \Gamma$,

$$\begin{tikzcd}
s^*A \arrow[rr] \arrow[dr, "s^*(r_A)"] \arrow[dd, two heads] \arrow[dr, phantom, very near start, "\lrcorner"] & & A \arrow[dr, "r_A"] \arrow[dd, two heads] & \\
 & s^*Id_A \arrow[rr] \arrow[dl, two heads] & & Id_A \arrow[dl, two heads] \\
\Delta \arrow[rr, "s"'] & & \Gamma &
\end{tikzcd} \tag{2.7}$$

the pullback $s^*(r_A)$ also has the left lifting property with respect to all display maps. We consider the class of all such morphisms in $\mathcal{B}$.

Definition 2.33. A class of morphisms $\mathcal{R}$ in a category $\mathcal{B}$ is called *factorizing* if every morphism f in $\mathcal{B}$ can be factored as $f = \rho \circ \lambda$

$$\begin{tikzcd}
 & Kf \arrow[dr, "\rho"] & \\
B \arrow[ur, "\lambda"] \arrow[rr, "f"'] & & A
\end{tikzcd} \tag{2.8}$$

such that

- $\rho \in \mathcal{R}$,
- $\lambda \in {}^{\boxslash}\mathcal{R}$, the class of morphisms with the left lifting property with respect to every map in $\mathcal{R}$.

If additionally $\mathcal{R} = ({}^{\boxslash}\mathcal{R})^{\boxslash}$, the class of morphisms with the right lifting property with respect to every map in ${}^{\boxslash}\mathcal{R}$, then $({}^{\boxslash}\mathcal{R}, \mathcal{R})$ form a *weak factorization system* on $\mathcal{B}$.

A factorizing class of morphisms is *functorial* if a chosen factorization as in (2.8) is given for each morphism in $\mathcal{B}$, which extends to a functor $(L, R) : \mathcal{B}^{\mathbb{2}} \to \mathcal{B}^{\mathbb{2}} \times_{\mathcal{B}} \mathcal{B}^{\mathbb{2}}$. It is an *orthogonal factorization system* if the filler for each square

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "({}^{\boxslash}\mathcal{R})\ni"'] & B \arrow[d, "\in\mathcal{R}"] \\
C \arrow[r] \arrow[ur, dashed, "!"] & D
\end{tikzcd}$$

is unique.

Example 2.34. (a) In any category the classes of isomorphisms and all morphisms form a functorial orthogonal factorization system.

(b) If $\mathcal{E} \to \mathcal{B}$ is a fibration, then the classes of vertical and cartesian morphisms form an orthogonal factorization system in $\mathcal{E}$.

(c) In an extensive category, the classes of coproduct inclusions and split epimorphisms form a functorial weak factorization system. The factorization of a morphism $f : B \to A$ is given by $B \hookrightarrow A + B \xrightarrow{\langle 1,f \rangle} A$.

The following consequence of identity types was first shown for the term model by Gambino and Garner [GG08], and in general by Shulman. Emmenegger [Emm14] extended it to avoid the hypothesis of dependent products, as long as pullbacks of r_A along all display maps have the left lifting property.

Proposition 2.35 ([Shu13])**.** *If a model of type theory with dependent sum, dependent product and unit types in a category $\mathcal{B}$ has factorizations of each diagonal satisfying the stability condition (2.7), then the class of display maps $\mathcal{D} \subseteq \mathcal{B}^2$ is factorizing. It follows that $({}^{\boxslash}\mathcal{D}, ({}^{\boxslash}\mathcal{D})^{\boxslash})$ is a weak factorization system.*

Sketch of Proof. Given a map $f : B \to A$, define the factorization of f to be $B \xrightarrow{l} Id(f) \xrightarrow{t} A$ given by the pullback

$$\begin{tikzcd}
B \arrow[r, "f"] \arrow[d, "l"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "r_A"] \\
Id(f) \arrow[r] \arrow[d, "{(s,t)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & Id_A \arrow[d, "\mathfrak{P}_{Id_A}=(s_A,t_A)"] \\
B \times A \arrow[r, "f \times 1"'] & A \times A.
\end{tikzcd}$$

In other words, $Id(f)$ represents the type $\sum_{b:B,a:A} Id_A(fb, a)$.

The span $B \xleftarrow{s} Id(f) \xrightarrow{t} A$ is equivalently described as a composite in $\mathbf{Span}(\mathcal{B})$ with the span $A \xleftarrow{s_A} Id_A \xrightarrow{t_A} A$:

$$\begin{tikzcd}
 & & Id(f) \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \\
 & B \arrow[dl, "="'] \arrow[dr, "f"] & & Id_A \arrow[dl, "s_A"'] \arrow[dr, "t_A"] & \\
B & & A & & A.
\end{tikzcd}$$

Since $\mathcal{D}$ contains product projections and is closed under composition and pullback, t is a display map. The morphism l can be shown to be in ${}^{\boxslash}\mathcal{D}$ because r_A is and $\mathcal{D}$ has product types.

As $\mathcal{D} \subseteq ({}^{\boxslash}\mathcal{D})^{\boxslash}$, the class $({}^{\boxslash}\mathcal{D})^{\boxslash}$ is also factorizing, and ${}^{\boxslash}(({}^{\boxslash}\mathcal{D})^{\boxslash}) = {}^{\boxslash}\mathcal{D}$, so $({}^{\boxslash}\mathcal{D})^{\boxslash}$ is the right class of a weak factorization system. $\square$

In particular, the identity Id_B appears as the factorization of the identity morphism on B

$$\begin{tikzcd}
 & Id_B \arrow[dr, "t_B", two heads] & \\
B \arrow[ur, "r_B"] \arrow[rr, "1_B"'] & & B.
\end{tikzcd}$$

Conversely, given a class of display maps $\mathcal{F}$ which is factorizing, the factorizations of diagonal maps required for identity types exist. The stability condition (2.7) holds if $\mathcal{F}$ has dependent product types [Shu13]. Thus we define:

Definition 2.36. A class of display maps $\mathcal{F} \subseteq \mathcal{B}$ with sum, unit and product types *has identity types* if $\mathcal{F}$ is factorizing, with a chosen factorization for each morphism in $\mathcal{B}$.

Remark 2.37. Unlike the other type constructors considered in this chapter, identity types in a categorical model are not unique up to isomorphism. However, a choice of identity types induces a weaker notion of equivalence in the category. Two morphisms $f, g : B \to A$ are said to be *homotopic*, written $f \sim g$, if $(f, g) : B \to A \times A$ factors through the identity type $(s_A, t_A) : Id_A \twoheadrightarrow A \times A$. In the type theory, this says that the type

$$\prod_{b:B} Id_A(fb, gb)$$

is inhabited. Two objects B and A are *(homotopy) equivalent*, written $B \simeq A$, if there are morphisms $f : B \to A$ and $h : A \to B$ such that the composites fh and hf

are homotopic to identity morphisms.

The object $Id(f)$ then plays the role of a limit of f 'up to homotopy': there is a homotopy $fs \sim t$

$$\begin{tikzcd} & Id(f) \arrow[dl, two heads, "s"'] \arrow[dr, two heads, "t"] \arrow[d, phantom, "\sim"] & \\ B \arrow[rr, "f"'] & & A \end{tikzcd}$$

and $Id(f), t, s$ are universal with this property.

Example 2.38. In a model of extensional identity types, the η-conversion rule requires the filler J to be unique for any commutative square of the form in Diagram 2.6. In this case in the square

$$\begin{tikzcd} B \arrow[r, "r_B"] \arrow[d, "r_B"'] & Id_B \arrow[d, two heads, "s_B"] \\ Id_B \arrow[r, two heads, "s_B"'] & B \end{tikzcd}$$

where s_B is the first projection of $\mathfrak{P}_{Id_B}$, both 1_{Id_B} and $r_B s_B$ are suitable fillers so they must be equal and r_B must be an isomorphism. Thus in a class of display maps with extensional identity types all diagonal morphisms are display maps. If it is also well-rooted then since any morphism $f : B \to A$ factors as

$$B \xrightarrow{(1,f)} B \times A \xrightarrow{\pi_2} A$$

which are pullbacks of the display maps $A \twoheadrightarrow A \times A$ and $B \twoheadrightarrow 1$ respectively, every morphism is a display map.

Conversely identity types in the codomain comprehension category are automatically extensional, since the reflexivity map r_A has the left lifting property with respect to all maps and so must be an isomorphism [See84].

In the intensional case, the lack of uniqueness of fillers means choices have to be made for each term J. This gives another coherence problem in addition to the stability under pullback considered in Section 2.3.6. In a strict categorical model of type theory, to give identity types requires not only specifying types Id and terms r_A which are stable, but also specifying extra data to give a filler J for each square in such a way that they are compatible with substitution. This problem is considered in detail in [War08] and [vdBG12]. However, for the associated strict model of type theory of Proposition 2.5, it is always possible to choose these fillers in a coherent way:

Proposition 2.39 ([LW14]). *If $\mathcal{F}$ is well-rooted class of display maps with dependent sum and product types and identity types, then the associated comprehension category models identity types.* □

2.5 Interaction of type constructors

We now consider types built out of more than one type constructor, and the properties that can be deduced.

2.5.1 Sums and products

Proposition 2.40. *For types $C \twoheadrightarrow B \twoheadrightarrow A$ in a context Γ, there is an isomorphism*

$$\prod_{a:A} \sum_{b:B(a)} C(a,b) \quad \cong \quad \sum_{f:\prod_{a:A} B(a)} \prod_{a:A} C(a, fa).$$

Proof. Let φ be a term of type $\prod_{a:A} \sum_{b:B(a)} C(a,b)$. Then for any $a : A$, we have terms

$$\begin{aligned} &\pi_1(\varphi a) : B(a) \quad \text{and} \\ &\pi_2(\varphi a) : C(a, \pi_1(\varphi a)). \end{aligned}$$

So the term $(\lambda a.\pi_1(\varphi a), \lambda a.\pi_2(\varphi a))$ has the type of the right-hand side.

Conversely, given a term of type $\sum_{f:\prod_{a:A} B(a)} \prod_{a:A} C(a, fa)$, it suffices by the rules for sums to assume that it has the form (f, ψ) where $f : \prod_{a:A} B(a)$ and $\psi : \prod_{a:A} C(a, fa)$. Then the term $\lambda a.(fa, \psi a)$ has the type of the left-hand side. The conversion rules for products and sums ensure that the two constructions are inverse to each other. □

Translating this into a categorical model gives exactly the distributive law between the monads $\Sigma_{\mathcal{F}}$ and $\Pi_{\mathcal{F}}$ in Proposition 1.35.

Remark 2.41. Under the propositions-as-types interpretation of type theory, reading Σ as $\exists$ and Π as $\forall$, the statement above corresponds to a form of the axiom of choice. However, to give a term of a sum type $\Sigma_{a:A} P(a)$ involves specifying a witness a, so there is no actual choice involved.

2.5.2 Binary sums and products

Proposition 2.42. *For types C and B in the same context Γ, there is an isomorphism*

$$C + B \;\cong\; \sum_{s:1+B} (\tilde{1}(s) \Rightarrow C)$$

over Γ, where $\tilde{1} \twoheadrightarrow 1 + B$ is the type corresponding to the coproduct inclusion.

Proof. To construct a term of type $\sum_{s:1+B}(\tilde{1}(s) \Rightarrow C)$ from a term p of type $C+B$, it suffices by the rules for binary sums to consider the cases $p = inl(c)$ and $p = inr(b)$ for terms $c : C$, $b : B$. If $p = inl(c)$, then since $\tilde{1}(inl(*)) = 1$, we get that $(inl(*), \lambda x.c)$ is a term of type $\sum_{s:1+B}(\tilde{1}(s) \Rightarrow C)$. If $p = inr(b)$, then since $\tilde{1}(inr(b)) = 0$, $(inr(b), \lambda x.empty(x))$ is a term of type $\sum_{s:1+B}(\tilde{1}(s) \Rightarrow C)$.

Conversely, given a term p of type $\sum_{s:1+B}(\tilde{1}(s) \Rightarrow C)$, it suffices to consider $p = (inl(*), \phi : 1 \Rightarrow C)$ and $p = (inr(b), \phi : 0 \Rightarrow C)$. In the first case, $inl(\phi(*))$ is a term of type $C + B$, and in the second case $inr(b)$ is. The two constructions are inverse to each other. □

2.5.3 Identities and sums

As the lack of η-conversion rules means that identity types are not defined uniquely, we cannot expect that identity types will commute up to isomorphism with other type constructors. However, at least for sum types they do commute up to homotopy equivalence, i.e. up to a term of a corresponding identity type.

Proposition 2.43 ([The13])**.** *For types $B \twoheadrightarrow A$ and terms*

$$x, y : \sum_{a:A} B(a),$$

there is an equivalence of types

$$Id_{\sum_{a:A} B(a)}(x, y) \;\simeq\; \sum_{p:Id_A(\pi_1 x, \pi_1 y)} Id_{B(\pi_1 y)}(p_*(\pi_2 x), \pi_2 y),$$

where for any $a_1, a_2 : A$, the path transport map $p_* : B(a_2) \Rightarrow B(a_1)$ is defined using the identity type rules to be the term $J(a_2, a_1, x.1_{B(x)})$.

Proof. To construct a term $f(q)$ of the type on the right-hand side from a term q of type $Id_{\sum_{a:A} B(a)}(x, y)$, it suffices to assume that $x = y$ and $q = r(x)$. But then $r(\pi_1 x)$ is a term of type $Id_A(\pi_1 x, \pi_1 y)$ and $r(\pi_1 x)_*$ is the identity function by β-conversion, and so $(r(\pi_1 x), r(\pi_2 x))$ is a term of the right-hand side as required.

Conversely, given $p : Id_A(\pi_1 x, \pi_1 y)$ and $s : Id_{B(\pi_1 y)}(p_*(\pi_2 x), \pi_2 y)$, using the identity rules twice we can assume that $\pi_1 x = \pi_1 y$ and $p = r(\pi_1 x)$, and also that $\pi_2 x = \pi_2 y$ and $s = r(\pi_2 x)$. Since $x = (\pi_1 x, \pi_2 x) = (\pi_1 y, \pi_2 y) = y$, $r(x)$ is then a term $g(p, s)$ of the left-hand side.

To show that f and g form an equivalence, we need to find terms of the identity types $Id(q, gf(q))$ and $Id((p, s), fg(p, s))$ for q, p, s as above. Again, it suffices to assume $q = r(x)$, so $gf(q) = g(r(\pi_1 x), r(\pi_2 x)) = r(x) = q$. Similarly, assuming $p = r(\pi_1 x)$ and $s = r(\pi_2 x)$, we get $fg(p, s) = f(r(x)) = (r(\pi_1 x), r(\pi_2 x)) = (p, s)$ as required. $\square$

In a similar way, identities for binary sum types and the unit type are characterized up to homotopy by their constituent types.

Proposition 2.44 ([The13])**.** *For types A, B and terms $a_1, a_2, a : A$, $b : B$ there are equivalences of types*

$$\begin{aligned} Id_{A+B}(inl(a_1), inl(a_2)) &\simeq Id_A(a_1, a_2) \\ Id_{A+B}(inl(a), inr(b)) &\simeq 0 \\ Id_1(*, *) &\simeq 1. \end{aligned}$$

2.5.4 Identities and products

In the case of product types, identities are not constrained in the same way. For any types $B \twoheadrightarrow A$ and any dependent functions

$$f, g : \prod_{a:A} B(a),$$

there is a map

$$happly : Id_{\prod_{a:A} B(a)}(f, g) \rightarrow \prod_{a:A} Id_{B(a)}(f(a), g(a)).$$

To construct it, it suffices by the rules for identity types to assume that $f = g$ and $r(f)$ is a term of the left-hand side, in which case $\lambda a.r(f(a))$ is a term of the right-hand side.

However, unlike the case for sum types, the rules of type theory do not necessarily imply that this is an equivalence.

Definition 2.45. A type theory satisfies the principle of *function extensionality* if for all $f, g : \prod_{a:A} B(a)$, *happly* is an equivalence

$$Id_{\prod_{a:A} B(a)}(f, g) \;\simeq\; \prod_{a:A} Id_{B(a)}(f(a), g(a)).$$

Intuitively, function extensionality says that functions which have equal values everywhere are equal. In a type theory without this principle, functions can be *intensionally* different despite being *extensionally* the same – the identity types of function spaces are not fixed by the identity types of their images. Assuming function extensionality as an axiom determines these function space identity types up to equivalence.

By a result of Streicher [Str93], function extensionality cannot be derived from the rules for Π-types, Σ-types and Id-types given above. Indeed, Streicher constructed a model where function extensionality fails, obtained by glueing the global sections functor of the category of assemblies. This model has intensional identity types satisfying the propositional version of the uniqueness of identity proofs condition, i.e. such that any two terms of an identity type are propositionally equal. It was previously known that function extensionality is independent of the rules of type theory without η-rules for product types, see [TvD88]. Hofmann [Hof95] gives an informal explanation for why it would not be expected to hold in general, using the normalization property of type theory.

It has been shown by Voevodsky (see [Lum11]) that to ensure *happly* is an equivalence, it is sufficient to construct a function in the opposite direction:

Proposition 2.46. *Function extensionality holds iff for all $f, g : \prod_{a:A} B(a)$ there exists a map*

$$\prod_{a:A} Id_{B(a)}(f(a), g(a)) \;\to\; Id_{\prod_{a:A} B(a)}(f, g). \tag{2.9}$$

□

Example 2.47. (a) For an extensive type theory modelled by a locally cartesian closed category, the identity type of a dependent type $B \twoheadrightarrow A$ is represented by the diagonal map

$$B \xrightarrow{(1,1)} B \times_A B\,.$$

There is a (unique) term of type $Id(f, g)$ for functions f and g if and only f and g

are equal as terms of the function type, so function extensionality automatically holds.

(b) For the class of display maps consisting of product projections, identity types are trivial: given a dependent type $B \twoheadrightarrow A$, the diagonal $B \to B \times_A B$ is a split monomorphism so has the left lifting property with respect to all product projections. Thus the identity type is just $B \times_A B$ and there is a (unique) identity path between any two terms in a fibre. Both sides of (2.9) represent the product $\prod_{a:A} B(a) \times \prod_{a:A} B(a)$, so function extensionality always holds.

Remark 2.48. 1. A well-rooted class of fibrations with dependent sum and product types and identity types is called a *tribe* by Joyal [Joy14], a *typical category* by Awodey [Awo14], and a *type-theoretic fibration category* by Shulman [Shu13]. A tribe satisfying function extensionality is called a *Martin-Löf tribe* [Joy14].

2. In the models of type theory usually studied from a homotopy type theory perspective, function extensionality holds. A term of the identity type between two functions can then be thought of as a path or a continuous homotopy. In particular, function extensionality is implied by Voevodsky's *univalence axiom* [The13]. This asserts the existence of a universe type, whose terms are types, such that the identity type between two types corresponds to the type of homotopy equivalences between them.

Chapter 3

Constructing new models

3.1 Extending the type theory

Suppose we have a well-rooted class of display maps

$$\begin{tikzcd}
\mathcal{F} \arrow[rr, hook] \arrow[dr] & & \mathcal{B}^{\mathbf{2}} \arrow[dl, "c"] \\
& \mathcal{B} &
\end{tikzcd}$$

representing a type theory. The aim of this chapter is to use such a model of type theory to build other models with possibly different properties, or in other words to construct a different well-rooted class of display maps $\mathcal{G}$ in some related category $\mathcal{C}$.

We start with some informal motivation for the form this construction will take.

The model $\mathcal{F}$ represents some particular instance of type theory $\mathbb{T}$. If we were to modify the type theory in some way, this should give a corresponding change in the categorical model, and conversely a categorical construction on $\mathcal{F}$ which produces a new class of display maps should correspond to a type-theoretic construction on $\mathbb{T}$.

In particular, consider the process of adding new types to the theory. For each context Γ in $\mathbb{T}$, suppose there is an embedding

$$\mathcal{F}_\Gamma \hookrightarrow \mathcal{E}_\Gamma$$

of the fibre $\mathcal{F}_\Gamma$ into some category which we want to think of as additional types in context Γ.

In order for the new types to respect substitution along morphisms in $\mathcal{B}$, the categories $\mathcal{E}_\Gamma$ should also assemble into a fibration $\mathcal{E} \to \mathcal{B}$, with the inclusions of original types forming a full and faithful fibred functor from $\mathcal{F}$

$$\begin{tikzcd} \mathcal{F} \arrow[rr, hook, "\Phi"] \arrow[dr, "c"'] & & \mathcal{E} \arrow[dl, "q"] \\ & \mathcal{B}. & \end{tikzcd}$$

However, if types have been added to the fibre $\mathcal{F}_1$ over the terminal object, then we have added 'closed types' in the new theory which no longer correspond to the category of contexts $\mathcal{B}$. To get a model of type theory it is necessary to add contexts as well, and 'extend' the fibration q along the functor

$$\mathcal{B} \cong \mathcal{F}_1 \overset{\Phi_1}{\hookrightarrow} \mathcal{E}_1 = \mathcal{C}$$

in some way to construct a well-rooted class of display maps $\mathcal{G}$ in the category $\mathcal{C}$. The functor Φ_1 should preserve the terminal object and pullbacks of display maps to preserve the existing types. Requiring that the pullback of $\mathcal{G}$ along Φ_1 is just $\mathcal{E}$ then ensures that for an original context Γ coming from $\mathcal{B}$, $\mathcal{E}_\Gamma$ represents exactly the category of types over Γ in the new theory.

In summary, the proposed construction of a new model proceeds according to the following scheme. We start with a model of type theory $\mathcal{F}$ in $\mathcal{B}$. Then, given a fibred inclusion $\Phi : \mathcal{F} \to \mathcal{E}$ over $\mathcal{B}$ such that $\Phi_1 : \mathcal{B} \to \mathcal{C}$ preserves finite limits where $\mathcal{C} = \mathcal{E}_1$, we obtain (under appropriate assumptions) a new model of the form

$$\begin{tikzcd} \mathcal{G} \arrow[rr, hook] \arrow[dr] & & \mathcal{C}^{\mathbb{2}} \arrow[dl] \\ & \mathcal{C} & \end{tikzcd}$$

such that the square

$$\begin{tikzcd} \mathcal{E} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{G} \arrow[d] \\ \mathcal{B} \arrow[r, hook, "\Phi_1"'] & \mathcal{C} \end{tikzcd} \tag{3.1}$$

is a pullback.

3.2 Adding sums

This section describes a family of new models of type theory which fit into the above framework. In particular, we consider models based on the monad Σ which freely adds sums to fibrations. We give two variants of the construction, one of which is a generalization of the other.

Given the class of display maps $\mathcal{F}$ in $\mathcal{B}$, let $\psi : \mathcal{C} \to \mathcal{B}$ be any cloven fibration over $\mathcal{B}$. To relate this to the above setting, we assume ψ has the following property.

Proposition 3.1. *The following are equivalent:*

1. *ψ has a full and faithful right adjoint*

$$\begin{tikzcd} \mathcal{B} \arrow[r, hook, shift left, "\phi"] \arrow[r, phantom, "\top"] & \mathcal{C} \arrow[l, shift left, "\psi"] \end{tikzcd} \tag{3.2}$$

   making $\mathcal{B}$ into a reflective subcategory of $\mathcal{C}$,

2. *$\mathcal{C}$ has a terminal object preserved by ψ,*

3. *ψ has fibrewise terminal objects.*

Proof. (1) $\Rightarrow$ (2) : $\mathcal{B}$ has a terminal object 1 preserved by the right adjoint ϕ. Then $\psi\phi(1) \cong 1$ by the counit of the reflection.
(2) $\Rightarrow$ (3) : For any object A of $\mathcal{B}$, the reindexing of the terminal object 1 of $\mathcal{C}$ along the unique morphism $A \to \psi(1)$ is terminal in the fibre over A, and is stable under reindexing.
(3) $\Rightarrow$ (1) : For an object A of $\mathcal{B}$, define $\phi(A)$ to be the terminal object of the fibre of ψ over A. This extends to an adjunction $\psi \dashv \phi$ where for D in $\mathcal{C}$ the component of the unit $\eta_D : D \to \phi\psi(D)$ is the unique vertical arrow over $\psi(D)$, and the counit is $\psi\phi = 1$. $\square$

Assume now that the class of display maps $\mathcal{F}$ has (strong) dependent sum types, so the monad $\Sigma_{\mathcal{F}}$ is defined which adds sums along morphisms in $\mathcal{F}$ to fibrations over $\mathcal{B}$ as in Definition 1.34.

Proposition 3.2. *Given ϕ and ψ as above, there is a fibred adjunction*

$$\begin{tikzcd}
\mathcal{F} \arrow[r, hook, shift left, "\Phi"] \arrow[r, phantom, "\top"] \arrow[drr] & \sum_{\mathcal{F}} \mathcal{C} \arrow[l, shift left, "\Psi"] \arrow[dr] & \\
& & \mathcal{B} \arrow[r, hook, shift left, "\phi"] \arrow[r, phantom, "\top"] & \mathcal{C} \arrow[l, shift left, "\psi"]
\end{tikzcd}$$

such that the restrictions of Φ and Ψ to the fibres over the terminal object of $\mathcal{B}$ are ϕ and ψ respectively.

Proof. The fibration $\Sigma_{\mathcal{F}}\mathcal{C} \to \mathcal{B}$ is given by pullback:

$$\begin{tikzcd}
& & \Sigma_{\mathcal{F}}\mathcal{C} \arrow[dl, "\Psi"'] \arrow[dr] \arrow[dd, phantom, "\vee" near start] & \\
& \mathcal{F} \arrow[dl, "c"'] \arrow[dr, "d"] & & \mathcal{C} \arrow[dl, "\psi"] \\
\mathcal{B} & & \mathcal{B} &
\end{tikzcd}$$

The fibre Ψ_f of Ψ over an object f of $\mathcal{F}$ is the fibre ψ_{df} of ψ, with reindexing along a morphism $h : f \to g$ given by $(dh)^* : \psi_{dg} \to \psi_{df}$. The cartesian morphisms for the fibration $c\Psi : \Sigma_{\mathcal{F}}\mathcal{C} \to \mathcal{B}$ are the pairs $(h \in \mathcal{F}^{\mathbf{2}}, g \in \mathcal{C}^{\mathbf{2}})$ such that g is ψ-cartesian over dh and h is c-cartesian, so Ψ preserves cartesian morphisms and hence defines a fibred functor over $\mathcal{B}$. Because $\mathcal{F}$ is well-rooted, the restriction of d to the fibre of c over 1 in $\mathcal{B}$ is an isomorphism, so Ψ restricts to ψ.

Each fibre category of ψ has a terminal object preserved by reindexing, so the same holds for Ψ. This defines a full and faithful functor Φ which is right adjoint to Ψ and restricts to the right adjoint ϕ over 1 in $\mathcal{B}$. Φ preserves cartesian morphisms and the unit and counit of the adjunction are vertical over $\mathcal{B}$. $\square$

Thus we have a fibred inclusion of $\mathcal{F}$ into $\Sigma_{\mathcal{F}}\mathcal{C}$ as described in the previous section. We want to construct a model of type theory $\mathcal{F}_{\mathcal{C}}$ in $\mathcal{C}$, which extends $\Sigma_{\mathcal{F}}\mathcal{C}$ in the sense outlined there.

Firstly, the display maps over an object ϕD in $\mathcal{C}$ should correspond to the fibre of $\Sigma_{\mathcal{F}}\mathcal{C}$ over D, which is just the set of pairs

$$\{(C, f : \psi C \to D) \mid C \in \mathcal{C},\ f \in \mathcal{F}\}.$$

To give such a pair it suffices to give the transpose $\bar{f} : C \to \phi D$ of f under the adjunction $\psi \dashv \phi$. This means that the display maps of $\mathcal{F}_\mathcal{C}$ over objects in the image of ϕ should be exactly

$$\overline{\mathcal{F}} = \{f : C \to \phi D \mid (\bar{f} : \psi C \to D) \in \mathcal{F}\}, \tag{3.3}$$

that is, the class of morphisms in $\mathcal{C}$ whose transpose under the adjunction is in $\mathcal{F}$.

If $\mathcal{F}_\mathcal{C}$ is to form a class of display maps, then pullbacks along these morphisms must exist in $\mathcal{C}$. We therefore assume that the following equivalent conditions hold.

Proposition 3.3. *Given $\mathcal{F}$ and an adjunction as above, the following are equivalent:*

1. *$\mathcal{C}$ has and ψ preserves finite products,*
2. *the pullback of a morphism $f \in \overline{\mathcal{F}}$ along any morphism in $\mathcal{C}$ exists, and ψ preserves this pullback.*

Proof. (2) $\Rightarrow$ (1) : Since ϕ preserve terminal objects, the product of objects A and B in $\mathcal{C}$ is the pullback

$$\begin{tikzcd} A \times B \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "!_A"] \\ B \arrow[r] & \phi 1, \end{tikzcd}$$

where the transpose $\psi A \to 1$ of the morphism $!_A$ is in the well-rooted class of maps $\mathcal{F}$. ψ preserves this product because it also preserves the terminal object.
(1) $\Rightarrow$ (2): Let $f : C \to \phi\psi A$ in $\overline{\mathcal{F}}$ and $g : B \to \phi\psi A$ be morphisms for which we want to construct a pullback in $\mathcal{C}$. The pullback

$$\begin{tikzcd} P \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \psi C \arrow[d, "\bar{f}"] \\ \psi B \arrow[r, "\bar{g}"'] & \psi A \end{tikzcd} \tag{3.4}$$

exists in $\mathcal{B}$ since $\bar{f}$ is in the class of display maps $\mathcal{F}$. Now since ψ preserves binary products, $\psi B \times \psi C \cong \psi(B \times C)$. Consider the morphism

$$P \xrightarrow{(p,q)} \psi B \times \psi C \xrightarrow{\cong} \psi(B \times C)$$

in $\mathcal{B}$. ψ is a fibration, so this morphism has a cartesian lifting $(m, n) : Q \to B \times C$ in $\mathcal{C}$.

This satisfies $\psi Q = P$, $\psi m = p$ and $\psi n = q$, and so also $fn = gm$ by transposing. The universal property of the lifting is the condition that a morphism $(h, k) : K \to B \times C$ factors uniquely through (m, n) if and only if $\psi(h, k)$ factors through (p, q),

$$\begin{tikzcd}
K \arrow[dd, dashed] \arrow[dr, "s"] \arrow[drrr, "{(h,k)}"] & & & \\
& Q \arrow[rr, "{(m,n)}"'] \arrow[dd, dashed] & & B \times C \arrow[dd, dashed] \\
\psi K \arrow[dr, "t"] \arrow[drrr] & & & \\
& P \arrow[r, "{(p,q)}"'] & \psi B \times \psi C \arrow[r, "\cong"'] & \psi(B \times C),
\end{tikzcd}$$

if and only if $\bar{f}\psi k = \bar{g}\psi h$, if and only if $fk = gh$ (and then automatically ψ applied to the morphism $s : K \to Q$ must give t by uniqueness of t). In other words, it is exactly the condition that

$$\begin{tikzcd}
Q \arrow[r, "n"] \arrow[d, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "f"] \\
B \arrow[r, "g"'] & \phi\psi A
\end{tikzcd}$$

is a pullback square in $\mathcal{C}$. The image of this square under ψ is the pullback of $\bar{f}$ and $\bar{g}$ in (3.4), so ψ preserves this pullback. $\square$

Remark 3.4. When $f \in \overline{\mathcal{F}}$ is in the image of ϕ, so it is of the form $\phi\psi f : \phi\psi C \to \phi\psi A$, then condition (2) always holds: the pullback of $\phi\psi f$ along g is given by

$$\begin{tikzcd}
Q \arrow[r, "\eta_Q"] \arrow[d, "m"'] & \phi P \arrow[r, "\phi q"] \arrow[d, "\phi p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \phi\psi C \arrow[d, "\phi\psi f"] \\
B \arrow[r, "\eta_B"'] & \phi\psi B \arrow[r, "\phi\psi g"'] & \phi\psi A
\end{tikzcd}$$

where $Q \xrightarrow{m} B$ is the cartesian lifting of $P \xrightarrow{p} \psi B$. This property makes the adjunction $\psi \dashv \phi$ together with the classes of morphisms $\mathcal{F}$ and $\mathcal{F}_{\mathcal{C}}$ into an *admissible Galois structure* defined by Janelidze in [Jan89]. An adjunction which satisfies condition (2) in the case when $\mathcal{F}$ consists of all morphisms of $\mathcal{C}$ is said to have *stable units* in [CHK85].

We can now define the new model of type theory.

Proposition 3.5. *Suppose we are given* *(i) a well-rooted class of display maps $\mathcal{F} \subseteq \mathcal{B}^{\mathbb{2}}$ with dependent sums,*

(ii) an adjunction

$$\begin{tikzcd} \mathcal{B} \arrow[r, hook, shift left, "\phi"] \arrow[r, phantom, "\top"] & \mathcal{C} \arrow[l, shift left, "\psi"] \end{tikzcd}$$

where ψ is a finite-product-preserving fibration.

Let $\overline{\mathcal{F}} \subseteq \mathcal{C}^{\mathbb{2}}$ be the class of transposed maps defined in (3.3), and let $\mathcal{F}_{\mathcal{C}}$ be the closure of $\overline{\mathcal{F}}$ under pullback. Then $\mathcal{F}_{\mathcal{C}}$ is a well-rooted class of display maps in $\mathcal{C}$, and the restriction of $\mathcal{F}_{\mathcal{C}}$ along ϕ is $\Sigma_{\mathcal{F}}\mathcal{C}$.

Proof. Since all pullbacks of morphisms in $\mathcal{F}_{\mathcal{C}}$ exist and are in $\mathcal{F}_{\mathcal{C}}$ by definition, it is a stable class of maps. The right adjoint ϕ preserves the terminal object of $\mathcal{B}$, so any morphism $A \to 1$ in $\mathcal{C}$ corresponds to $\psi A \to 1$ in $\mathcal{B}$, which is in $\mathcal{F}$ because $\mathcal{F}$ is well-rooted. The class $\mathcal{F}_{\mathcal{C}}$ contains all isomorphisms since $1 \to 1$ is in $\overline{\mathcal{F}}$. □

Proposition 3.6. *$\mathcal{F}_{\mathcal{C}}$ is exactly the class of morphisms in $\mathcal{C}$ which appear as the left vertical morphism in a pullback square of the form*

$$\begin{tikzcd} D \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "\in\overline{\mathcal{F}}"] \\ A \arrow[r, "\eta_A"'] & \phi\psi A. \end{tikzcd}$$

Proof. This class of morphisms is clearly contained in $\mathcal{F}_{\mathcal{C}}$. Conversely, given a morphism $h : D \to B$ in $\mathcal{F}_{\mathcal{C}}$ which arises as a pullback

$$\begin{tikzcd} D \arrow[r] \arrow[d, "h"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "f"] \\ B \arrow[r, "g"'] & \phi\psi A, \end{tikzcd}$$

this factors by naturality as

$$\begin{tikzcd} D \arrow[r] \arrow[d, "h"'] \arrow[dr, phantom, "\lrcorner", very near start] & C' \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "f"] \\ B \arrow[r, "\eta_B"'] & \phi\psi B \arrow[r, "\phi\psi g"'] & \phi\psi A. \end{tikzcd}$$

Since ψ preserves the right pullback, the transpose $\bar{f}' : \psi C' \to \psi B$ of f' is a pullback of $\bar{f} \in \mathcal{F}$ in $\mathcal{B}$. Hence $\bar{f}'$ is in $\mathcal{F}$, i.e. $f' \in \overline{\mathcal{F}}$. □

Example 3.7. (a) Let $\mathcal{C}$ be any category with finite products. Then $\mathcal{C} \to 1$ is trivially a fibration preserving finite products and the terminal object defines a unique adjunction

$$\begin{tikzcd} 1 \arrow[r, shift left, "1"] \arrow[r, phantom, "\top"] & \mathcal{C}. \arrow[l, shift left] \end{tikzcd}$$

There is a unique well-rooted class of display maps $\mathcal{F}$ on the category 1. The class of transposed morphisms $\overline{\mathcal{F}}$ in $\mathcal{C}$ consists of all maps into the terminal object, so applying the construction of Proposition 3.5 to this adjunction gives the simple model of type theory of Example 2.6(d) where the display maps in $\mathcal{C}$ are the product projections.

(b) For a class of display maps $\mathcal{F}$ with dependent sums, the functor $c : \mathcal{F} \to \mathcal{B}$ is itself a fibration such that $\mathcal{F}$ has and c preserves finite products. This means we can construct a model of type theory $\mathcal{F}_{\mathcal{F}}$ in the category $\mathcal{F}$. The morphisms in $\overline{\mathcal{F}}$ in this model are commutative squares

$$\begin{tikzcd} B \arrow[r, two heads] \arrow[d] & A \arrow[d, "\in\mathcal{F}"] \\ C \arrow[r, two heads, "="'] & C. \end{tikzcd}$$

Thus the display maps between objects $(B \twoheadrightarrow A)$ and $(D \twoheadrightarrow C)$ in $\mathcal{F}$ are the commutative squares

$$\begin{tikzcd} B \arrow[r, two heads] \arrow[d, "f"'] & A \arrow[d, "g"] \\ D \arrow[r, two heads] & C \end{tikzcd}$$

with g in $\mathcal{F}$ which arise as a pullback

$$\begin{tikzcd} (B \twoheadrightarrow A) \arrow[r] \arrow[d, "{(f,g)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & (E \twoheadrightarrow A) \arrow[d] \\ (D \twoheadrightarrow C) \arrow[r] & (C \twoheadrightarrow C) \end{tikzcd}$$

for some $(E \twoheadrightarrow A)$ in $\mathcal{F}$, in other words such that B is the pullback $D \times_C E$ and

f is the right morphism in

$$\begin{tikzcd}
B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d] \\
D \times_C A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "g"] \\
D \arrow[r, two heads] & C.
\end{tikzcd}$$

(c) The category of small categories **Cat** is fibred over **Set** via the objects functor, where the cartesian morphisms are the full and faithful functors. This fibration preserves finite limits and has a full and faithful right adjoint which sends a set to the indiscrete category on that set. Using the locally cartesian closed structure of **Set**, we get a well-rooted class of display maps in **Cat**, albeit a not very interesting one: the display maps are all pullbacks of functors with codomain an indiscrete category.

We now look again at the construction of display maps from a different viewpoint. In the hypotheses of Proposition 3.5, ψ is a fibration, so every morphism f in $\mathcal{C}$ factors uniquely (up to ψ-vertical isomorphism) as a vertical morphism f^v followed by a morphism f^c which is cartesian over ψf. In other words, f factors through a pullback

$$\begin{tikzcd}
B \arrow[dr, "f^v"] \arrow[drr, "\eta_B"] \arrow[ddr, "f"'] & & \\
& P \arrow[r, "\eta_P"'] \arrow[d, "f^c"] \arrow[dr, phantom, "\lrcorner", very near start] & \phi\psi B \arrow[d, "\phi\psi f"] \\
& A \arrow[r, "\eta_A"'] & \phi\psi A.
\end{tikzcd}$$

This gives another characterization of the display maps of the new model:

Proposition 3.8. *$\mathcal{F}_\mathcal{C}$ is exactly the class of maps $f : B \to A$ in $\mathcal{C}$ such that $\psi f \in \mathcal{F}$ and the vertical comparison map f^v is a product projection in the fibre of ψ over ψB.*

Proof. From the description of display maps in Proposition 3.6, f is a display map iff $\psi f \in \mathcal{F}$ and

$$\begin{tikzcd} B \arrow[r] \arrow[d, "f^v"'] & C \arrow[d, "\eta_C"] \\ P \arrow[r, "\eta_P"'] & \phi\psi B \end{tikzcd}$$

is a pullback for some C in the fibre of ψ over ψB. But this universal property makes B into the product $P \times C$ in the fibre category. $\square$

Generalizing from the class of product projections to other classes of display maps in the fibre categories, we get a more general construction of new models of type theory extending $\Sigma_{\mathcal{F}}\mathcal{C}$.

Proposition 3.9. *Suppose we have*

(i) a well-rooted class of display maps $\mathcal{F} \subseteq \mathcal{B}^{\mathbf{2}}$ with dependent sums,

(ii) an adjunction

$$\begin{tikzcd} \mathcal{B} \arrow[r, hook, shift left, "\phi"] \arrow[r, phantom, "\top"] & \mathcal{C} \arrow[l, shift left, "\psi"] \end{tikzcd}$$

where ψ is a finite-product-preserving fibration.

Assume additionally that for each $D \in \mathcal{C}$, the fibre category ψ_D of ψ has a class of morphisms $\mathcal{R}_D$ such that

(iii) $\mathcal{R}_D$ is a well-rooted class of display maps with dependent sums,

(iv) reindexing preserves these classes, i.e. for any $f : B \to A$ in $\mathcal{R}_{\psi B}$ and cartesian $g : D \to B$ in $\mathcal{B}$, the induced vertical morphism $h : D \to E$ in

$$\begin{tikzcd} D \arrow[r, "g", "cart"'] \arrow[d, "h"'] & B \arrow[d, "f"] \\ E \arrow[r, "cart"'] & A \end{tikzcd}$$

is in $\mathcal{R}_{\psi E}$,

(v) reindexing preserves the pullbacks of maps in $\mathcal{R}_D$.

Then

$$\mathcal{G} = \{f : B \to A \in \mathcal{C} \mid \psi f \in \mathcal{F} \text{ and the comparison map } f^v \in \mathcal{R}_{\psi B}\}$$

is a well-rooted class of display maps in $\mathcal{C}$. The restriction of $\mathcal{G}$ along ϕ is again $\Sigma_{\mathcal{F}}\mathcal{C}$.

Proof. Firstly, since $\mathcal{R}_{\psi B}$ contains all vertical isomorphisms and is closed under composition, $\mathcal{G}$ is well-defined whatever the choice of f^v. $\mathcal{G}$ is closed under composition with isomorphisms: given

$$D \xrightarrow[\cong]{\beta} B \xrightarrow[\in\mathcal{G}]{f} A \xrightarrow[\cong]{\alpha} E$$

in $\mathcal{C}$, $\psi(\alpha f \beta) \in \mathcal{F}$ since $\mathcal{F}$ is a class of display maps. All isomorphisms in $\mathcal{C}$ are cartesian, and the stability property of $\mathcal{R}$ ensures that $f\beta$ factors as a cartesian morphism composed with a vertical map in $\mathcal{R}_{\psi D}$.

To show that pullbacks along morphisms in $\mathcal{G}$ exist, let $f : B \to A$ be in $\mathcal{G}$. The cartesian part of f is stable under pullback as in Proposition 3.5, so assume f is vertical and in $\mathcal{R}_{\psi B}$. For any cartesian morphism $g : C \to A$, the square

$$\begin{tikzcd} P \arrow[r, "p"] \arrow[d, "q"'] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & A \end{tikzcd}$$

is a pullback where $P \xrightarrow{p} B$ is a cartesian lifting of ψg, and $q \in \mathcal{R}_{\psi C}$ by the stability property. For any vertical morphism $g : C \to A$, the pullback of f along g exists in the fibre category $\psi_{\psi B}$ and is in $\mathcal{R}_{\psi B}$. Since reindexing preserves this pullback, it also has the universal property of a pullback in $\mathcal{C}$.

For any A in $\mathcal{C}$, $\phi\psi A$ is terminal in the fibre of ψ over ψA, so each component $\eta_A : A \to \phi\psi A$ of the unit of the adjunction is in the well-rooted class $\mathcal{R}_{\psi A}$ and so is in $\mathcal{G}$. Thus $\bar{\mathcal{F}} \subseteq \mathcal{G}$ and $\mathcal{G}$ is well-rooted. Similarly, a morphism $f : B \to \phi\psi A$ in $\mathcal{C}$ is in $\mathcal{G}$ iff $\psi f \in \mathcal{F}$, so the class of display maps extends $\Sigma_{\mathcal{F}}\mathcal{C}$. $\square$

Remark 3.10. Intuitively, this construction based on the sums monad $\Sigma_{\mathcal{F}}$ gives a type theory whose types look like elements of a dependent sum – a type A in the new theory is a pair $(\underline{A}, \overline{A})$, where $\underline{A} = \psi A$ is a type in the theory corresponding to $\mathcal{F}$, and $\overline{A}$ is a type in the theory corresponding to $\mathcal{R}_{\underline{A}}$. A dependent type $f : B \twoheadrightarrow A$ consists of a dependent type $\underline{f} : \underline{B} \twoheadrightarrow \underline{A}$ in $\mathcal{F}$ (which is the morphism ψf), together with a dependent type $\overline{B} \twoheadrightarrow (\underline{f})^*\overline{A}$ in $\mathcal{R}_{\underline{B}}$ (which is f^v). We can think of such a $\overline{B}$ as having two kinds of type dependency, on $\underline{B}$ and $\overline{A}$ respectively, and write $B \twoheadrightarrow A$ in type theory notation as

$$(\underline{B}(\underline{a}),\ \overline{B}(\underline{a},\underline{b};\overline{a})) \twoheadrightarrow (\underline{A},\ \overline{A}(\underline{a})).$$

A term of this type is a pair $(\underline{b},\overline{b})$, where $\underline{b} : \underline{A} \to \underline{B}$ is a term in $\mathcal{F}$ and $\overline{b} : \overline{A} \to (\underline{b})^*\overline{B}$ is a term of the reindexed type in $\mathcal{R}_{\underline{A}}$.

Example 3.11. (a) When each class of display maps $\mathcal{R}_D$ consists of the product projections in the fibre ψ_D, they each have dependent sums and are stable under reindexing exactly when ψ preserves finite products. So the model $\mathcal{F}_{\mathcal{C}}$ is indeed a special case of this construction.

(b) Consider again the fibration $c : \mathcal{F} \to \mathcal{B}$ in Example 3.7(b). Each fibre of c is a slice category $\mathcal{F}/B$, which has a class of display maps $(\mathcal{F}/B)_{\mathcal{F}}$ consisting of the commuting triangles with all morphisms in $\mathcal{F}$:

$$\begin{tikzcd}
A \arrow[rr, two heads] \arrow[dr, two heads] & & C \arrow[dl, two heads] \\
& B. &
\end{tikzcd}$$

This has dependent sums, and is stable under reindexing because $\mathcal{F}$ is stable under pullback, so we can construct a model of type theory $\mathcal{G}$ in the category $\mathcal{F}$ which has more display maps than the model $\mathcal{F}_{\mathcal{F}}$. This class of display maps is described by Shulman in [Shu13]. Cartesian morphisms for c in $\mathcal{C}$ are pullback squares, so display maps between objects $(B \twoheadrightarrow A)$ and $(D \twoheadrightarrow C)$ in $\mathcal{F}$ are the commutative squares

$$\begin{tikzcd}
B \arrow[r, two heads] \arrow[d, "f"'] & A \arrow[d, "g"] \\
D \arrow[r, two heads] & C
\end{tikzcd}$$

with g in $\mathcal{F}$ such that the comparison map h in

$$\begin{tikzcd}
B \arrow[dr, "h"] \arrow[drr, bend left] \arrow[ddr, "f"'] & & \\
& D \times_C A \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, two heads, "g"] \\
& D \arrow[r, two heads] & C
\end{tikzcd}$$

is also in $\mathcal{F}$.

3.3 Dependent sum and product types

In order to construct the new models, the original class of display maps $\mathcal{F}$ was assumed to have at least dependent sum types. We now consider sum, product and identity types in $\mathcal{F}$, and investigate conditions for which the extended models inherit this structure.

Proposition 3.12. *The class of display maps $\mathcal{G}$ has dependent sum types, that is $\mathcal{G}$ is closed under composition.*

Proof. Let $g : C \twoheadrightarrow B$ and $f : B \twoheadrightarrow A$ be display maps in $\mathcal{G}$. Then $\psi(fg) = \psi(f)\psi(g)$ is in $\mathcal{F}$ because $\mathcal{F}$ has dependent sum types. The morphisms g and f factor as $C \xrightarrow{g^v} P \xrightarrow{g^c} B$ and $B \xrightarrow{f^v} Q \xrightarrow{f^c} A$ for some cartesian g^c, f^c and $g^v \in \mathcal{R}_{\psi C}$, $f^v \in \mathcal{R}_{\psi B}$. If $M \xrightarrow{m} Q$ is a cartesian lifting of $\psi(g) = \psi(g^c)$ and n the induced vertical map in

$$\begin{tikzcd} P \arrow[r, "g^c"] \arrow[d, "n"'] & B \arrow[d, "f^v"] \\ M \arrow[r, "m"'] & Q, \end{tikzcd}$$

then $n \in \mathcal{R}_{\psi C}$ by stability. Thus

$$C \xrightarrow{ng^v} M \xrightarrow{f^c m} A$$

is a cartesian-vertical factorization of gf with $ng^v \in \mathcal{R}_{\psi C}$, so gf is in $\mathcal{G}$. □

In the notation of Remark 3.10, given display maps $C \xrightarrow{g} B \xrightarrow{f} A$ corresponding to types

$$(\underline{C}(\underline{a}, \underline{b}),\ \overline{C}(\underline{a}, \underline{b}, \underline{c}; \overline{a}, \overline{b})) \twoheadrightarrow (\underline{B}(\underline{a}),\ \overline{B}(\underline{a}, \underline{b}; \overline{a})) \twoheadrightarrow (\underline{A},\ \overline{A}(\underline{a})),$$

the sum $\Sigma_{b:B}C(a, b) \twoheadrightarrow A$ is just calculated componentwise as

$$\left(\sum_{\underline{b}:\underline{B}(\underline{a})} \underline{C}(\underline{a}, \underline{b}),\ \sum_{\overline{b}:\overline{B}(\underline{a}, \underline{b}; \overline{a})} \overline{C}(\underline{a}, \underline{b}, \underline{c}; \overline{a}, \overline{b}) \right) \twoheadrightarrow (\underline{A},\ \overline{A}(\underline{a})).$$

Here the first Σ refers to the sum types of the model $\mathcal{F}$, and the second to the sum types of the model $\mathcal{R}_{\underline{C}}$.

Lemma 3.13. *If the class of display maps $\mathcal{F}$ has dependent product types and the fibration ψ has $\mathcal{F}$-products which preserve the display maps $\mathcal{R}_{\mathcal{D}}$, then $\mathcal{G}$ has products along all cartesian morphisms f in $\mathcal{C}$ with $\psi f \in \mathcal{F}$.*

Proof. If $f : B \to A$ is cartesian, it appears as a pullback

$$\begin{tikzcd} B \arrow[r, "\eta_B"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & \phi\psi B \arrow[d, "\phi\psi f"] \\ A \arrow[r, "\eta_A"'] & \phi\psi A. \end{tikzcd}$$

Firstly, consider products along $\phi\psi f$. Since $\mathcal{G}$ restricts to $\Sigma_{\mathcal{F}}\mathcal{C}$ along ϕ, to give a right adjoint $\Pi_{\phi\psi f}$ for the reindexing functor

$$(\phi\psi f)^* : (\mathcal{G})_{\phi\psi A} \to (\mathcal{G})_{\phi\psi B}$$

is equivalent to giving a right adjoint $\Pi_{\psi f}$ for

$$(\psi f)^* : (\Sigma_{\mathcal{F}}\mathcal{C})_{\psi A} \to (\Sigma_{\mathcal{F}}\mathcal{C})_{\psi B}.$$

But ψf is in $\mathcal{F}$, and if the fibration $\psi : \mathcal{C} \to \mathcal{B}$ has products along morphisms in $\mathcal{F}$ then so does $\Sigma_{\mathcal{F}}\mathcal{C} \to \mathcal{B}$, by the distributivity law for $\mathcal{F}$. The Beck-Chevalley condition holds for morphisms of this form in $\mathcal{G}$ since it holds for ψ.

Let $g : D \to B$ be another morphism in $\mathcal{G}$ for which we want to construct the product $\Pi_f g$. Since $\mathcal{G}$ contains η_A and is closed under composition, g is also a morphism $\eta_A f g \to \eta_A f$ in the slice category $\mathcal{G}/(\phi\psi A)$. The fact that products exist along $\phi\psi f$ means that $\phi\psi f$ is an exponentiable object in this slice category. As products for ψ preserve display maps, $g^{\phi\psi f}$ is in $\mathcal{G}$ and we can form the pullback

$$\begin{tikzcd} P \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & D^{\phi\psi f} \arrow[d, "g^{\phi\psi f}"] \\ A \arrow[r, "a"'] & B^{\phi\psi f} \end{tikzcd}$$

in $\mathcal{G}/(\phi\psi A)$, where $A \xrightarrow{a} B^{\phi\psi f}$ is the transpose of $A \times \phi\psi f \xrightarrow{\cong} B$. Given any $K \xrightarrow{k} A$ in $\mathcal{G}$, there are natural correspondences between morphisms

$$
\begin{array}{rcll}
k & \to & p & \text{in } \mathcal{C}/A \\
\hline
ak & \to & g^{\phi\psi f} & \text{in } \mathcal{C}/(B^{\phi\psi f}) \\
\hline
k \times \phi\psi f & \to & g & \text{in } \mathcal{C}/(\phi\psi A) \\
\hline
f^* k & \to & g & \text{in } \mathcal{C}/B.
\end{array}
$$

In other words, the morphism $P \xrightarrow{p} A$ has the universal property of the product $\Pi_f g$. The Beck-Chevalley condition holds since it holds for morphisms of the form $\phi\psi f$. $\square$

Proposition 3.14. *If $\mathcal{F}$ has dependent product types, each class of display maps $\mathcal{R}_D$ has dependent product types which are preserved by reindexing, and the fibration ψ has $\mathcal{F}$-products which preserve $\mathcal{R}_D$-maps, then $\mathcal{G}$ has dependent product types.*

Proof. Using the above lemma, it remains to construct products along vertical maps in $\mathcal{G}$. Consider morphisms $g : D \to B$ and $f : B \to A$ in $\mathcal{G}$ where f is ψ-vertical, $f \in \mathcal{R}_{\psi B}$, for which we we want to construct $\Pi_f g$. Let $q : Q \to A$ be a cartesian lifting of ψg, so there is an induced factorization of g

$$
\begin{tikzcd}
D \arrow[d, "g^v"'] \arrow[dr] & \\
P \arrow[r, "h"] \arrow[d, "g^c"'] \arrow[dr, phantom, very near start, "\lrcorner"] & Q \arrow[d, "q"] \\
B \arrow[r, "f"'] & A
\end{tikzcd}
$$

where h and g^v are vertical. Then $q : Q \to A$ has the universal property of the product $\Pi_f g^c$ in $\mathcal{C}$. To show this, take any other morphism $k : K \to A \in \mathcal{G}$. To give a morphism $K \to Q$ over A corresponds to giving a morphism $n : \psi K \to \psi Q$ in $\mathcal{B}$ such that $\psi k = (\psi q)n$. Since g^c is also cartesian over ψg, this corresponds to giving a morphism $f^*K \to P$ over B.

By the stability under reindexing of the class $\mathcal{R}_{\psi B}$, $h \in \mathcal{R}_{\psi P}$, so we can form the product $\Pi_h g^v \to Q$ in the fibre category over ψP. This is in fact a product in the category $\mathcal{C}$ too: given any morphism $k : K \to Q$ in $\mathcal{C}$, form the diagram

$$\begin{tikzcd}
D' \arrow[rr] \arrow[dd, "(g^v)'"'] \arrow[dr, phantom, "\lrcorner", very near start] & & D \arrow[dd, "g^v"] & \\
& (\Pi_h g^v)' \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & \Pi_h g^v \arrow[dd] \\
P' \arrow[rr] \arrow[dr, "h'"'] & & P \arrow[dr, "h"] & \\
& Q' \arrow[rr] & & Q
\end{tikzcd}$$

where all the horizontal morphisms are cartesian over ψk. Then to give a map $K \to \Pi_h g^v$ over Q corresponds to giving a map $K \to (\Pi_h g^v)'$ over Q', which since fibrewise products are stable under reindexing corresponds to giving $(h')^* K \to D'$ over P', i.e. a morphism $h^* K \to D$ over P.

Putting this together, the morphism $\Sigma_q \Pi_h g^v \to A$ has the universal property of the product $\Pi_f g \to A$. For any other morphism $K \to A \in \mathcal{G}$, a morphism $K \to \Sigma_q \Pi_h g^v$ over A corresponds to a morphism $K \to Q$ over A together with a morphism $K \to \Pi_h g^v$ over Q, i.e. a morphism $f^* K \to P$ over B and $h^* K \to D$ over P, which corresponds to just a morphism $f^* K \to D$ over B as required. $\square$

Intuitively, a cartesian display map $f : B \twoheadrightarrow A$ in $\mathcal{G}$ takes the form $(\underline{B}(\underline{a}),\ \overline{A}(\underline{a})) \twoheadrightarrow (\underline{A},\ \overline{A}(\underline{a}))$. For another morphism $g : C \twoheadrightarrow B$, the construction of $\prod_f g = \prod_{b:B(a)} C(a,b)$ in Lemma 3.13 gives

$$\left(\prod_{\underline{b}:\underline{B}(\underline{a})} \underline{C}(\underline{a},\underline{b}),\ \prod_{\underline{b}:\underline{B}(\underline{a})} \overline{C}(\underline{a},\underline{b},\varphi(\underline{b});\overline{a}) \right).$$

The first Π refers to the product types of the model of type theory $\mathcal{F}$, and the second to the structure of $\mathcal{F}$-products of the fibration ψ. When f is vertical, so it has the form $(\underline{A},\ \overline{B}(\underline{a};\overline{a})) \twoheadrightarrow (\underline{A},\ \overline{A}(\underline{a}))$, the product type constructed in Proposition 3.14 is

$$\left(\underline{C}(\underline{a}),\ \prod_{\overline{b}:\overline{B}(\underline{a};\overline{a})} \overline{C}(\underline{a},\underline{c};\overline{a},\overline{b}) \right).$$

The Π here refers to the product types of the model $\mathcal{R}_{\underline{C}}$.

Combining the two cases, the general form of a product type $\prod_{b:B(a)} C(a,b)$ in the extended model is

$$\left(\prod_{\underline{b}:\underline{B}(\underline{a})} \underline{C}(\underline{a}, \underline{b}), \prod_{\underline{b}:\underline{B}(\underline{a})} \prod_{\overline{b}:\overline{B}(\underline{a},\underline{b};\overline{a})} \overline{C}(\underline{a}, \underline{b}, \varphi(\underline{b}); \overline{a}, \overline{b}) \right).$$

In the case of the model $\mathcal{F}_{\mathcal{C}}$ where the classes of display maps $\mathcal{R}_D$ consist of the product projections in the fibres, each $\mathcal{R}_D$ has dependent product types iff the fibre ψ_D is cartesian closed. $\mathcal{F}$-products for ψ are right adjoints and so always preserve $\mathcal{R}_D$-maps. So we have as a special case of Proposition 3.14:

Corollary 3.15. *If $\mathcal{F}$ has dependent product types and the fibration $\psi : \mathcal{C} \to \mathcal{B}$ has $\mathcal{F}$-products and fibred exponentials (that is, each fibre has exponentials which are preserved by reindexing), then the class of display maps $\mathcal{F}_{\mathcal{C}}$ has dependent product types.*

Remark 3.16. (a) When $\mathcal{F}_{\mathcal{C}}$ has dependent product types, then in particular it has products along maps to the terminal object, so the base category $\mathcal{C}$ is cartesian closed. In this case the cartesian closed structure of the original category $\mathcal{B}$ is inherited from that of $\mathcal{C}$ and preserved by the adjunction: Day's reflection theorem states that a reflective subcategory of a cartesian closed category is an exponential ideal if and only if the reflector preserves finite products [Day72].

(b) In the case of Corollary 3.15, the previous remark can be demonstrated directly. As shown by Hermida in [Her99], if a fibration $\mathcal{C} \to \mathcal{B}$ over a cartesian closed category $\mathcal{B}$ has simple products and fibred exponentials then the total category $\mathcal{C}$ is cartesian closed. For objects A, B in $\mathcal{C}$ the exponential $B \Rightarrow A$ is given by

$$\prod_{\pi_1} ((\pi_2)^* B \Rightarrow (ev)^* A)$$

where $ev : (\psi B \Rightarrow \psi A) \times \psi B \to \psi A$ is the evaluation map in $\mathcal{B}$. Thus $\psi(B \Rightarrow A) = (\psi B \Rightarrow \psi A)$, and ψ preserves the cartesian closed structure.

Example 3.17. (a) If $\mathcal{B}$ is cartesian closed, then the functor $\mathcal{B} \to 1$ clearly has fibred exponentials so the the model of type theory consisting of product projections has dependent product types.

(b) If $\mathcal{F}$ has dependent product types, then each slice category $\mathcal{F}/B$ has exponentials which are preserved by pullback. Thus the class of display maps $\mathcal{F}_{\mathcal{F}}$ adding sums along the fibration $\mathcal{F} \to \mathcal{B}$ has dependent product types.

3.4 Identity types

Proposition 3.18. *If each class of display maps $\mathcal{F}$ and $\mathcal{R}_A$ has identity types, and ψ is an opfibration as well as a fibration, then $\mathcal{G}$ has identity types.*

Proof. This was proved by Stanculescu in [Sta12], where $\mathcal{F}$ and $\mathcal{R}_A$ are the right classes of weak factorization systems (i.e. closed under retracts).

Given a morphism $f : B \to A$ in $\mathcal{C}$, we require a factorization $f = \rho_f \circ \lambda_f$ where $\rho_f \in \mathcal{G}$, $\lambda_f \in {}^{\boxslash}\mathcal{G}$.

The image ψf in $\mathcal{B}$ has a factorization $\rho_{\psi f} \circ \lambda_{\psi f}$ with $\rho_{\psi f} \in \mathcal{F}$, $\lambda_{\psi f} \in {}^{\boxslash}\mathcal{F}$. If $p : P \to A$ is a cartesian lifting of $\rho_{\psi f}$, then f factors through p:

$$
\begin{tikzcd}
B \arrow[rr, "\eta_B"] \arrow[dr, "l"] \arrow[ddr, "f"'] & & \phi\psi B \arrow[d, "\phi\lambda_{\psi f}"] \\
& P \arrow[r, "\eta_P"] \arrow[d, "p"] \arrow[dr, phantom, "\lrcorner", very near start] & \phi K\psi f \arrow[d, "\phi\rho_{\psi f}"] \\
& A \arrow[r, "\eta_A"'] & \phi\psi A
\end{tikzcd}
$$

p is in the class of display maps $\mathcal{G}$. The cartesian property of p means it has the right lifting property with respect to all morphisms m such that $\psi m \in {}^{\boxslash}\mathcal{F}$: There exists a filler for a square

$$
\begin{tikzcd}
E \arrow[r] \arrow[d, "m"'] & P \arrow[d, two heads, "p"] \\
M \arrow[r] & A
\end{tikzcd}
$$

if there exists a filler for

$$
\begin{tikzcd}
E \arrow[r] \arrow[d, "m"'] & \phi\psi P \arrow[d, two heads, "\phi\psi p"] \\
M \arrow[r] & \phi\psi A,
\end{tikzcd}
$$

so by transposing, if there exists a filler for

$$\begin{tikzcd}
\psi E \arrow[r] \arrow[d, "\psi m"'] & \psi P \arrow[d, "\psi p"] \\
\psi M \arrow[r] & \psi A.
\end{tikzcd}$$

Since ψ is also an opfibration, l factors as

$$\begin{tikzcd}
B \arrow[rr, "l"] \arrow[dr, "m"'] & & P \\
& Q \arrow[ur, "v"'] &
\end{tikzcd}$$

where m is cocartesian over $\phi\lambda_{\psi f}$ and v is ψ-vertical. Dually to the above, the cocartesian property of m means it has the left lifting property with respect to all morphisms g such that $\psi g \in \mathcal{F}$, so in particular $m \in {}^{\boxslash}\mathcal{G}$.

Since $\mathcal{R}_{\psi P}$ has identity types, v factors as $Q \xrightarrow{x} Kv \xrightarrow{y} P$ for some $y \in \mathcal{R}_{\psi P}$ and $x \in {}^{\boxslash}\mathcal{R}_{\psi P}$. Since the classes of morphisms $\mathcal{R}$ are stable under reindexing, x will in fact have the left lifting property with respect to $\mathcal{R}_A$-maps for any A. Then

$$\begin{tikzcd}
B \arrow[rr, "f"] \arrow[dr, "xm"'] & & A \\
& Kv \arrow[ur, "py"'] &
\end{tikzcd}$$

is a factorization of f as required. $\square$

In most of the examples we have considered, ψ does not have sums along morphisms in ${}^{\boxslash}\mathcal{F}$, so it is not an opfibration. However, in some cases it is still possible to get a factorization of the morphism l as a map in ${}^{\boxslash}\mathcal{G}$ followed by a vertical morphism, by requiring that the adjunction $\psi \dashv \phi$ commutes suitably with the identity types in $\mathcal{F}$.

Proposition 3.19. *The following are equivalent:*

1. *For any l in ${}^{\boxslash}\mathcal{F}$, $\phi l \in {}^{\boxslash}\mathcal{G}$,*
2. *For any $l : B \to \psi C$ in ${}^{\boxslash}\mathcal{F}$, $\phi l \boxslash \eta_C$,*
3. *For any $h : \phi B \to C$ in $\mathcal{C}$ such that $\psi h \in {}^{\boxslash}\mathcal{F}$, h factors as $t \circ \phi\psi h$ for some vertical $t : \phi\psi C \to C$.*

Proof. (1) clearly implies (2). Given (2) and a morphism $l : B \to A$ in ${}^{\boxslash}\mathcal{F}$, as above ϕl lifts against all cartesian morphisms in $\mathcal{G}$. It also lifts against all $\mathcal{R}$-maps, since for any commutative diagram

$$\begin{tikzcd} \phi B \arrow[r, "k"] \arrow[d, "\phi l"'] & E \arrow[d, two heads, "r \in \mathcal{R}_{\psi D}"] \\ \phi A \arrow[r, "g"'] & D, \end{tikzcd}$$

there exists a filler by reindexing along g

$$\begin{tikzcd} \phi B \arrow[r, "h"] \arrow[d, "\phi l"'] & C \arrow[r] \arrow[d, two heads, "\eta_C"] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, two heads, "r"] \\ \phi A \arrow[r, equal] \arrow[ur, dashed] & \phi\psi C \arrow[r, "g"'] & D. \end{tikzcd}$$

Condition (3) is a restatement of (2), saying that if a square such as the left one in the above diagram commutes, then it has a filler $t : \phi\psi C \to C$. $\square$

Definition 3.20. The functor ϕ *preserves left morphisms* if the above equivalent conditions hold.

Proposition 3.21. *If $\mathcal{G}$ has dependent product types, each class of display maps $\mathcal{F}$ and $\mathcal{R}_A$ has identity types and ϕ preserves left morphisms, then $\mathcal{G}$ has identity types.*

To show this we need the following result about product types and identities:

Lemma 3.22. *When a class of display maps $\mathcal{F}$ has product types, the class of morphisms ${}^{\boxslash}\mathcal{F}$ is stable under pullback along $\mathcal{F}$-maps.*

Proof. Given a pullback

$$\begin{tikzcd} f^*U \arrow[r] \arrow[d, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & U \arrow[d, "l \in {}^{\boxslash}\mathcal{F}"] \\ W \arrow[r, two heads, "f"'] & V, \end{tikzcd}$$

to show that m is in ${}^{\boxslash}\mathcal{F}$ it suffices to show there exists a filler for every commutative square

$$\begin{tikzcd} f^*U \arrow[r] \arrow[d, "m"'] & Z \arrow[d, "r\in\mathcal{F}"] \\ W \arrow[r, equal] & W. \end{tikzcd}$$

But such a filler corresponds under the adjunction $f^* \dashv \Pi_f$ to a filler for

$$\begin{tikzcd} U \arrow[r] \arrow[d, "l"'] & \Pi_f Z \arrow[d, "\Pi_f r\in\mathcal{F}"] \\ V \arrow[r, equal] & V, \end{tikzcd}$$

which exists since $l \in {}^{\boxslash}\mathcal{F}$. □

Proof of Proposition 3.21. Given a morphism $f : B \to A$, we can factorize ψf as $\rho_{\psi f}\lambda_{\psi f}$ and construct $p \in \mathcal{G}$ and l as in the previous proposition. Recall from the construction of factorizations from identity types (Proposition 2.35) that in general we work with a factorization of the morphism $(1, g) : B \to A \times A$ rather than g itself. In other words, the factorization $\rho\lambda$ of g is chosen in such a way that λ has a retraction $s : A \to B$. In particular, we can do this for the factorization of ψf in $\mathcal{B}$. We therefore have pullbacks

$$\begin{tikzcd} B \arrow[r, "\eta_B"] \arrow[d, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & \phi\psi B \arrow[d, "\phi\lambda_{\psi f}"] \\ Q \arrow[r, "\eta_Q"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & \phi K\psi f \arrow[d, "\phi s_{\psi f}"] \\ B \arrow[r, "\eta_B"'] & \phi\psi B \end{tikzcd}$$

in $\mathcal{C}$, where m and q are cartesian over $\lambda_{\psi f}$ and $s_{\psi f}$ respectively.

Since ϕ preserves left morphisms, $\phi\lambda_{\psi f}$ is in ${}^{\boxslash}\mathcal{G}$. The morphism m is then also in ${}^{\boxslash}\mathcal{G}$ by the above lemma, so in other words m lifts against all display maps. In particular there is a filler v for the square

$$\begin{tikzcd} B \arrow[r, "l"] \arrow[d, "m"'] & P \arrow[d, "\eta_P"] \\ Q \arrow[ur, dashed, "v"] \arrow[r, "\eta_Q"'] & \phi K \psi f. \end{tikzcd} \tag{3.5}$$

Now v is ψ-vertical, so just as in the proof of Proposition 3.18 it can be factored using the identity types of $\mathcal{R}_{\psi P}$ as $Q \xrightarrow{x} Kv \xrightarrow{y} P$ for some $y \in \mathcal{R}_{\psi P}$ and $x \in {}^{\boxslash}\mathcal{R}_{\psi P}$. Then $B \xrightarrow{xm} Kv \xrightarrow{py} A$ is the required factorization of f. $\square$

In other words, the factorization of a morphism $f : B \twoheadrightarrow A$ is constructed by factorizing the projection ψf in $\mathcal{B}$, and then factorizing the induced morphism v in the fibre of ψ:

$$\begin{tikzcd}
& & Kv \arrow[dl] \arrow[dr] \arrow[ddd, dashed] & & & \\
& Q \arrow[ur, bend left] \arrow[dl] \arrow[rr, "v"] \arrow[ddr, dashed] & & P \arrow[dr] \arrow[ddl, dashed] & & \mathcal{C} \arrow[ddd, "\psi"] \\
B \arrow[ur, bend left] \arrow[rrrr, bend right, "f"] \arrow[dd, dashed] & & & & A \arrow[dd, dashed] & \\
& & K\psi f \arrow[dll] \arrow[drr] & & & \\
\psi B \arrow[urr, bend left] \arrow[rrrr, bend right, "\psi f"] & & & & \psi A & \mathcal{B}
\end{tikzcd}$$

In the notation of Remark 3.10, the type $Id_A(fb, a) = Kf \twoheadrightarrow B \times A$ looks like

$$\left(Id_{\underline{A}}(\underline{f}\underline{b}, \underline{a}),\ Id_{\overline{A}(\underline{a})}(v\overline{b}, \overline{a})\right) \twoheadrightarrow \left(\underline{B} \times \underline{A},\ \overline{B}(\underline{b}) \times \overline{A}(\underline{a})\right)$$

where v is the map $\overline{B}(\underline{b}) \to \overline{A}(\underline{a})$ induced by f and a term of the identity $Id_{\underline{A}}(\underline{f}\underline{b}, \underline{a})$. This description matches what we might expect for the identity type of an element of a dependent sum as in Section 2.5.3.

Remark 3.23. If $\mathcal{G}$ has identity types and also dependent product types as constructed in Proposition 3.21, then for dependent functions $f, g : \prod_{a:A} B(a)$ the type $\prod_{a:A} Id_{B(a)}(f(a), g(a))$ could be thought of as

$$\left(\prod_{\underline{a}:\underline{A}} Id_{\underline{B}(\underline{a})}(\underline{f}(\underline{a}), \underline{g}(\underline{a})),\ \prod_{\underline{a}:\underline{A}} \prod_{\overline{a}:\overline{A}(\underline{a})} Id_{\overline{B}(\underline{a}, \underline{g}(\underline{a}, \overline{a}), \overline{a})}(v\overline{f}(\underline{a}, \overline{a}), \overline{g}(\underline{a}, \overline{a}))\right).$$

On the other hand the type $Id_{\prod_{a:A} B(a)}(f, g)$ would look like

$$\left(Id_{\prod_{\underline{a}:\underline{A}} \underline{B}(\underline{a})}(\underline{f}, \underline{g}),\ Id_{\prod_{\underline{a}:\underline{A}} \prod_{\overline{a}:\overline{A}(\underline{a})} \overline{B}(\underline{a}, \underline{g}(\underline{a}), \overline{a})}(v\overline{f}, \overline{g})\right).$$

Thus if function extensionality holds for each class $\mathcal{F}$ and $\mathcal{R}_A$, and the $\mathcal{F}$-products of ψ preserve the identity types of $\mathcal{R}_A$, then the products and identity types in each component would commute, so that the types $\prod_{a:A} Id_{B(a)}(f(a), g(a))$ and $Id_{\prod_{a:A} B(a)}(f, g)$ would have equivalent descriptions. In other words in this case we would expect that function extensionality should also hold for $\mathcal{G}$. In contrast, in Section 4.5 we shall see in a model in a category of polynomials that without these assumptions, this no longer holds.

Example 3.24. (a) For a class of display maps $\mathcal{F}$ with dependent products and identities, consider the model of type theory over the category $\mathcal{F}$ constructed in Example 3.11(b). The inclusion $1 : \mathcal{B} \to \mathcal{F}$ preserves left morphisms. To show this, let $h : B \to C$ be a morphism in ${}^{\boxslash}\mathcal{F}$ and $D \twoheadrightarrow C$ an object of $\mathcal{F}$. We require a filler for all squares of the form

$$\begin{tikzcd}
(B \twoheadrightarrow B) \arrow[r] \arrow[d, "1(h)"'] & (D \twoheadrightarrow C) \arrow[d, "\eta_{D \twoheadrightarrow C}"] \\
(C \twoheadrightarrow C) \arrow[r, equal] & (C \twoheadrightarrow C).
\end{tikzcd}$$

To give such a square is to give a morphism k in $\mathcal{B}$ such that

$$\begin{tikzcd}
B \arrow[r, equal] \arrow[d, "k"'] & B \arrow[d, "h"] \\
D \arrow[r, two heads] & C
\end{tikzcd}$$

commutes. Using the left lifting property of h, the square

$$\begin{tikzcd}
B \arrow[r, "k"] \arrow[d, "h"'] & D \arrow[d, two heads] \\
C \arrow[r, equal] \arrow[ur, dashed, "g"] & C
\end{tikzcd}$$

then has a diagonal filler g, and the morphism $(g, 1_C)$ is a filler for the original square. Thus this model has identity types. It also has dependent product types by Proposition 3.14, and the model satisfies function extensionality, as shown by

Shulman in [Shu13].

(b) Let $\mathcal{G}$ be a class of display maps in a category $\mathcal{C}$. If $\mathcal{G}$ has a functorial choice of identity types such that the functor $Id : \mathcal{C} \to \mathcal{C}$ preserves coequalizers, then this model arises naturally as an example of the construction of this chapter. In other words there is a particular adjunction $\phi \dashv \psi : \mathcal{B} \to \mathcal{C}$ and class of display maps $\mathcal{F}$ for which $\mathcal{G}$ is an extended type theory. Specifically, let $\mathcal{B}$ be the full subcategory $\mathcal{C}_{disc}$ of objects A which are internally discrete, i.e. for which the two morphisms

$$\begin{tikzcd} Id(A) \arrow[r, shift left, "t_A"] \arrow[r, shift right, "s_A"'] & A \end{tikzcd}$$

are equal. The restriction $\mathcal{F}$ of $\mathcal{G}$ to $\mathcal{B}$ gives a model of type theory in which identity types are trivial, as internally all paths are constant. The inclusion $\mathcal{C}_{disc} \hookrightarrow \mathcal{C}$ has a left adjoint L which sends an object A to the coequalizer of s_A and t_A, the discrete reflection of A. LA can be thought of as the set of 'path components' of A. Non-constant paths in $\mathcal{G}$ are determined by the identity types of the display maps $\mathcal{R}$ in each fibre of L, in other words by the paths in each connected component. A similar construction is used by van Oosten [vO10] to describe a model of type theory in the effective topos **Eff** which arises from the category of discrete objects $\mathbf{Eff}_{disc}$ studied by Hyland, Robinson and Rosolini in [HRR90].

In the case of Proposition 3.21 when each class of display maps $\mathcal{R}_A$ consists of product projections, $\mathcal{R}_A$ automatically has identity types. The factorization in the extended class of display maps $\mathcal{F}_\mathcal{C}$ can then be equivalently described by a pullback in $\mathcal{C}$:

Proposition 3.25. *If $\mathcal{F}$ has identity types and ϕ preserves left morphisms, then $\mathcal{F}_\mathcal{C}$ has identity types, where the factorization $B \xrightarrow{\lambda_f} Kf \xrightarrow{s_f} A$ of a morphism f is given by*

$$\begin{tikzcd} B \arrow[r, "\eta_B"] \arrow[d, "\lambda_f"] \arrow[dd, bend right, "(1,f)"'] & \phi\psi B \arrow[d, "\phi\lambda_{\psi f}"] \\ Kf \arrow[r, "\eta_{Kf}"] \arrow[d, "(s_f,\rho_f)"] \arrow[dr, phantom, "\lrcorner", very near start] & \phi K\psi f \arrow[d, "(\phi s_{\psi f},\phi\rho_{\psi f})"] \\ B \times A \arrow[r, "\eta_{B\times A}"'] & \phi\psi B \times \phi\psi A \end{tikzcd}$$

Proof. In the proof of the previous proposition, the factorization of the morphism f is constructed by forming the pullbacks P and Q of $\phi s_{\psi f}$ and $\phi\rho_{\psi f}$ along components of the unit η and then factoring the morphism $v : Q \to P$ (Diagram 3.5) in the fibre over $K\psi f$. In this model $\mathcal{F}_{\mathcal{C}}$, the factorization of v is given by

$$Q \xrightarrow{(1,v)} Q \times P \xrightarrow{\pi_2} P,$$

using the product in the fibre of ψ. This fibrewise product is constructed in the total category $\mathcal{C}$ by the pullback of η_Q and η_P, as in the diagram

$$\begin{tikzcd}
 & & Q \times_{\phi K\psi f} P \arrow[dl] \arrow[dr] & & \\
 & Q \arrow[dl, "q"'] \arrow[dr, "\eta_Q"] & & P \arrow[dl, "\eta_P"'] \arrow[dr, "p"] & \\
B \arrow[dr, "\eta_B"'] & & \phi K\psi f \arrow[dl, "\phi s_{\psi f}"'] \arrow[dr, "\phi\rho_{\psi f}"] & & A, \arrow[dl, "\eta_A"] \\
 & \phi\psi B & & \phi\psi A &
\end{tikzcd}$$

which is equivalently described as the pullback of $(\phi s_{\psi f}, \phi\rho_{\psi f})$ along $\eta_{B\times A}$. $\square$

Remark 3.26. In this case, because the identity types of each class of display maps $\mathcal{R}_{\psi A}$ are trivial, to give a term of an identity type Id_A is just to give a term of the projected identity type $Id_{\psi A}$ in $\mathcal{B}$. Two distinct terms of A can have equal values under ψ, so the identity types of $\mathcal{F}_{\mathcal{C}}$ will not in general be extensional, even if the identity types in $\mathcal{F}$ are. The principle of uniqueness of identity proofs holds if and only it holds for $\mathcal{F}$.

Chapter 4

A polynomial model

4.1 Polynomials

In the previous chapter, new models of type theory were constructed by applying the sums monad $\Sigma_{\mathcal{F}}$ to a fibration. We now combine this with the other key component of the polynomial construction – the opposite of a fibration.

In particular, assume we have a fixed model

$$\begin{tikzcd} \mathcal{F} \arrow[rr, hook] \arrow[dr, "p"'] & & \mathcal{B}^{\mathbb{2}} \arrow[dl] \\ & \mathcal{B}. & \end{tikzcd}$$

As p is a fibration we can also form the fibration of polynomials

$$Pol(\mathcal{F} \to \mathcal{B}) = \Sigma_{\mathcal{F}}(p^{op})$$

over $\mathcal{B}$, where the fibre over the terminal object of this fibration is $\mathbf{Poly}_{\mathcal{F}} :\equiv \mathcal{F}^{\circ}$. The objective of this chapter is to use the techniques of Chapter 3 to extend this along the opposite fibration $p^{op} : \mathbf{Poly}_{\mathcal{F}} \to \mathcal{B}$ and construct a new model

$$\begin{tikzcd} \mathcal{F}_{\mathbf{Poly}} \arrow[rr, hook] \arrow[dr] & & (\mathbf{Poly}_{\mathcal{F}})^{\mathbb{2}} \arrow[dl] \\ & \mathbf{Poly}_{\mathcal{F}} & \end{tikzcd}$$

where $\mathcal{F}_{\mathbf{Poly}}$ is a suitable class of display maps in $\mathbf{Poly}_{\mathcal{F}}$.

The base category of the new type theory $\mathbf{Poly}_{\mathcal{F}}$ is also known as the category of *containers* and studied in the case when $\mathcal{B}$ is locally cartesian closed in [Abb03, AAG03]. It is the fibre over the terminal objects of the 2-sided fibration

$$\begin{tikzcd} & \Sigma_{\mathcal{F}}\Pi_{\mathcal{F}}\mathcal{F} \arrow[dl] \arrow[dr] & \\ \mathcal{B} & & \mathcal{B} \end{tikzcd}$$

constructed in Section 1.9, and corresponds to the category of polynomial functors $\mathcal{B} \to \mathcal{B}$.

An object of $\mathbf{Poly}_{\mathcal{F}}$ is a display map $(B \twoheadrightarrow A)$ in $\mathcal{F}$. Using the type theory structure of $\mathcal{F}$, it can be thought of as an indexed family over A and written as

$$\sum_{a:A} B(a) \twoheadrightarrow A.$$

It represents the polynomial functor $\mathcal{B} \to \mathcal{B}$ given by

$$X \mapsto \sum_{a:A} X^{B(a)}.$$

A morphism from a display map $(B \twoheadrightarrow A)$ to $(D \twoheadrightarrow C)$ in $\mathbf{Poly}_{\mathcal{F}}$ is a pair of morphisms (f, φ) making the diagram

$$\begin{tikzcd} D_f \arrow[r, "\varphi"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[r, two heads] & A \arrow[d, "f"] \\ D \arrow[rr, two heads] & & C \end{tikzcd}$$

commute (where the subscript of D_f refers to the pullback along f). In other words, to give such a morphism is to give a pair of terms

$$\begin{aligned} &f : A \to C \\ &\varphi : \prod_{a:A} (D(fa) \to B(a)), \end{aligned}$$

which by the type-theoretic axiom of choice is equivalent to giving a term of type

$$\prod_{a:A} \sum_{c:C} (D(c) \to B(a)).$$

As in Chapter 1, $\mathbf{Poly}_{\mathcal{F}}$ is equivalent to a full subcategory of the category $[\mathcal{B}, \mathcal{B}]$ of enriched endofunctors and enriched natural transformations. In the internal language of $\mathcal{B}$, a morphism as above corresponds to a natural transformation of polynomial functors

$$\sum_{a:A} X^{B(a)} \to \sum_{c:C} X^{D(c)}$$

defined on terms by $X^{\varphi(a)} : X^{B(a)} \to X^{D(fa)}$.

To construct a new model of type theory in $\mathbf{Poly}_{\mathcal{F}}$ using the methods of the previous chapter, Proposition 3.5 requires an adjunction

$$\begin{tikzcd} \mathcal{B} \arrow[r, hook, shift left, "\phi"] \arrow[r, phantom, "\top"] & \mathbf{Poly}_{\mathcal{F}} \arrow[l, shift left, "\psi"] \end{tikzcd}$$

where ψ is a finite-product-preserving fibration.

Proposition 4.1. *Such an adjunction exists when the class of display maps $\mathcal{F}$ has an empty type and weak binary sum types.*

Proof. The fibration $\psi = p^{op} : \mathbf{Poly}_{\mathcal{F}} \to \mathcal{B}$ sends $(B \twoheadrightarrow A)$ to A and (f, φ) to f. This fibration has fibred finite products exactly when its opposite $\mathcal{F} \to \mathcal{B}$ has fibred finite coproducts, so when $\mathcal{B}$ has an initial object 0 and binary coproducts which are stable under pullback. As shown in Section 2.3, these correspond to an empty type and weak binary sum types in the type theory interpreted by $\mathcal{F}$. Then $(0 \twoheadrightarrow 1)$ is clearly terminal in the category $\mathcal{F}^{op}$, and the functor

$$\phi : A \mapsto (0 \twoheadrightarrow A)$$

is a full and faithful right adjoint to ψ. The product of display maps $(B \twoheadrightarrow A)$ and $(D \twoheadrightarrow C)$ is

$$B \times C + A \times D \twoheadrightarrow A \times C,$$

or in type theory notation,

$$\sum_{(a,c):A\times C} B(a) + D(c) \twoheadrightarrow A \times C.$$

□

As well as products, the category $\mathbf{Poly}_{\mathcal{F}}$ has some pullbacks. Given morphisms $(f, \varphi) : (B \twoheadrightarrow A) \to (D \twoheadrightarrow C)$ and $(g, \gamma) : (F \twoheadrightarrow E) \to (D \twoheadrightarrow C)$, assume the pullback P of f and g exists in $\mathcal{B}$. Then whenever the following pushout exists in $\mathcal{F}/P$ and is stable under pullback,

$$\begin{tikzcd} P \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "f"] \\ E \arrow[r, "g"'] & C \end{tikzcd} \qquad \begin{tikzcd} D_{f\pi_1} = D_{g\pi_2} \arrow[r, "\varphi\pi_1"] \arrow[d, "\gamma\pi_2"'] & B_{\pi_1} \arrow[d, "\iota_1"] \\ F_{\pi_2} \arrow[r, "\iota_2"'] & Q \arrow[ul, phantom, very near start, "\ulcorner"] \end{tikzcd}$$

the induced display map $(Q \twoheadrightarrow P)$ has the universal property of a pullback of (f, φ) and (g, γ). The projections onto $(B \twoheadrightarrow A)$ and $(F \twoheadrightarrow E)$ are the morphisms (π_1, ι_1) and (π_2, ι_2) respectively. In particular, if $\mathcal{B}$ has stable coproducts then we always have pullbacks along a morphism (f, ι) when f is in $\mathcal{F}$ and ι is a coproduct inclusion $D_f \hookrightarrow D_f + B$ for some display map $(B \twoheadrightarrow A)$. The pullback of (g, γ) in this case is the polynomial $(F_{\pi_2} + B_{\pi_1} \twoheadrightarrow P)$.

4.2 A model of type theory

Applying Proposition 3.5 when $\mathbf{Poly}_{\mathcal{F}} \to \mathcal{B}$ has fibred finite products as above gives:

Proposition 4.2. *If $\mathcal{F}$ is a class of display maps representing a type theory with unit, dependent sum, dependent product, empty and weak binary sum types, then there is a model of type theory $\mathcal{F}_{\mathbf{Poly}} \subseteq (\mathbf{Poly}_{\mathcal{F}})^{\mathbf{2}}$ in the category of polynomials.* □

The display maps in the new model are those appearing as the left vertical morphism in a pullback of the form

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & (B \twoheadrightarrow A) \arrow[d, "(f,\iota)"] \\ (D \twoheadrightarrow C) \arrow[r, "\eta_{D\to C}"'] & (0 \twoheadrightarrow C) \end{tikzcd}$$

where f is in $\mathcal{F}$. This is a morphism

$$\begin{tikzcd} D_f \arrow[r, hook, "\iota"] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & D_f + B \arrow[r, two heads] & A \arrow[d, two heads, "f"] \\ D \arrow[rr, two heads] & & C. \end{tikzcd}$$

In other words, the display maps are the morphisms (f, φ) such that $f \in \mathcal{F}$ and φ is a coproduct inclusion.

Under the correspondence with polynomial functors, a display map represents a natural transformation of the form

$$\sum_{(c,a):\sum_{c:C} A(c)} X^{D(c)+B(c,a)} \to \sum_{c:C} X^{D(c)}$$

which is termwise just a projection $X^{D(c)+B(c,a)} \to X^{D(c)}$.

In order for this construction to be stable, so that the new model of type theory has the type constructors of the original, we can strengthen the binary sum type requirement:

Proposition 4.3. *The model of type theory in Proposition 4.2 has unit, dependent sum, and empty types. If $\mathcal{F}$ has strong binary sum types and binary sum types for types, then so does the new model.*

Proof. As a class of display maps the new model automatically has unit and dependent sum types. The polynomial $(0 \twoheadrightarrow 0)$ is initial in $\mathbf{Poly}_{\mathcal{F}}$ since 0 is stable under pullback and is therefore a strict initial object in $\mathcal{B}$, and this polynomial is stable under pullback. The binary sum of polynomials $(B \twoheadrightarrow A)$ and $(D \twoheadrightarrow C)$ is

$$B + D \twoheadrightarrow A + C,$$

which is a display map by the extensivity property of $\mathcal{F}$. The sum is then represented as

$$\sum_{s:A+C} \tilde{B}(s) + \tilde{D}(s) \twoheadrightarrow A + C$$

where $\tilde{B}$ is the type B considered as a dependent type over $A + C$. The coproduct inclusions from $(B \twoheadrightarrow A)$ and $(D \twoheadrightarrow C)$ are the morphisms $(\iota_A, 1_B)$ and $(\iota_C, 1_D)$ respectively. Given a display map into the sum

$$(H \twoheadrightarrow G) \to (B + D \twoheadrightarrow A + C)$$

for another object of $\mathbf{Poly}_{\mathcal{F}}$, we have $G \cong G_1 + G_2$ for some $(G_1 \twoheadrightarrow A)$ and $(G_2 \twoheadrightarrow C)$

and so $H \cong H_1 + H_2$ for some $(H_1 \twoheadrightarrow G_1)$ and $(H_2 \twoheadrightarrow G_2)$, giving pullbacks

$$\begin{tikzcd}
(H_1 \twoheadrightarrow G_1) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, very near start, "\lrcorner"] & (H \twoheadrightarrow G) \arrow[d, two heads] & (H_2 \twoheadrightarrow G_2) \arrow[l] \arrow[d, two heads] \arrow[dl, phantom, very near start, "\llcorner"] \\
(B \twoheadrightarrow A) \arrow[r, "\iota"'] & (D + B \twoheadrightarrow A + C) & (D \twoheadrightarrow C). \arrow[l, "\iota"]
\end{tikzcd}$$

Conversely if the two outer maps are given and $(H \twoheadrightarrow G)$ is defined as the coproduct then the centre map is clearly a display map in $\mathbf{Poly}_{\mathcal{F}}$, so the new model has strong binary sum types and sum types for types. $\square$

4.3 Dependent product types

The binary sums and products in Section 4.1 can alternatively be constructed by considering $\mathbf{Poly}_{\mathcal{F}}$ as a full subcategory of the functor category $[\mathcal{B}, \mathcal{B}]$. Limits and colimits are calculated pointwise in this category, so given polynomials $(D \twoheadrightarrow C)$ and $(B \twoheadrightarrow A)$ representing functors P and Q respectively,

$$\begin{aligned}
QP(X) &= \sum_{a:A} X^{B(a)} \times \sum_{c:C} X^{D(c)} \cong \sum_{(a,c):A\times C} X^{B(a)+D(c)} \\
(Q+P)(X) &= \sum_{a:A} X^{B(a)} + \sum_{c:C} X^{D(c)} \cong \sum_{s:A+C} X^{\tilde{B}(s)+\tilde{D}(s)},
\end{aligned}$$

which are again polynomial functors. Thus $\mathbf{Poly}_{\mathcal{F}}$ is closed under finite sums and products.

Exponential objects in $\mathbf{Poly}_{\mathcal{F}}$ can be calculated similarly. If P^Q exists in $[\mathcal{B}, \mathcal{B}]$ and is also represented by a polynomial, then this should be $(D \twoheadrightarrow C)^{(B \twoheadrightarrow A)}$ in $\mathbf{Poly}_{\mathcal{F}}$.

Theorem 4.4 ([ALS10]). *The category* $\mathbf{Poly}_{\mathcal{F}}$ *is cartesian closed.*

Proof. We use the Yoneda lemma to motivate the form that an exponential in $\mathbf{Poly}_{\mathcal{F}}$ should take. When $\mathcal{B}$ is locally small, the category $[\mathcal{B}, \mathcal{B}]$ is enriched in $\mathcal{B}$, with internal hom given by the end

$$Hom_{[\mathcal{B},\mathcal{B}]}(P, Q) = \int_{X \in \mathcal{B}} Hom_{\mathcal{B}}(P(X), Q(X)).$$

First consider the case when Q is represented by $(B \twoheadrightarrow 1)$, so Q is internally a representable functor $Hom_{\mathcal{B}}(B, -)$. When the exponential P^Q exists, it follows from the enriched Yoneda lemma that it must have the form

$$\begin{aligned}
P^Q(X) &\cong Hom_{[\mathcal{B},\mathcal{B}]}(Hom_{\mathcal{B}}(X,-), P^Q) \\
&\cong Hom_{[\mathcal{B},\mathcal{B}]}(Hom_{\mathcal{B}}(X,-) \times Q, P) \\
&\cong Hom_{[\mathcal{B},\mathcal{B}]}(Hom_{\mathcal{B}}(X,-) \times Hom_{\mathcal{B}}(B,-), P) \\
&\cong Hom_{[\mathcal{B},\mathcal{B}]}(Hom_{\mathcal{B}}(X+B,-), P) \\
&\cong P(X+B).
\end{aligned}$$

The extensivity property of $\mathcal{F}$ ensures that the functor $X \mapsto X+B$ can be formalized as expected as a polynomial functor. We have that

$$X + B \cong \sum_{s:1+B} X^{\tilde{1}(s)}$$

corresponding to the display map $1 \twoheadrightarrow 1 + B$ in $\mathcal{B}$.

A general Q is the sum of representable functors, and so

$$\begin{aligned}
P^Q(X) &\cong P^{\sum_{a:A} Hom_{\mathcal{B}}(B(a),-)}(X) \\
&\cong \prod_{a:A} P^{Hom_{\mathcal{B}}(B(a),-)}(X) \\
&\cong \prod_{a:A} P(X + B(a)) \\
&\cong \prod_{a:A} \sum_{c:C} \prod_{d:D(c)} \sum_{s:1+B(a)} X^{\tilde{1}(s)}
\end{aligned}$$

which can be rearranged by the axiom of choice to give

$$\begin{aligned}
&\cong \prod_{a:A} \sum_{c:C} \sum_{\phi:D(c)\to 1+B(a)} \prod_{d:D(c)} X^{\tilde{1}(\phi(d))} \\
&\cong \sum_{\sigma:S} \prod_{a:A} \prod_{d:D(\sigma(a)_1)} X^{\tilde{1}(\sigma(a)_2(d))}
\end{aligned}$$

where S is the type

$$\prod_{a:A} \sum_{c:C} (D(c) \to 1 + B(a)).$$

Thus when it exists, P^Q is a polynomial endofunctor represented by the display map

$$\sum_{\sigma:S}\left(\sum_{a:A}\sum_{d:D(\sigma(a)_1)}\tilde{1}(\sigma(a)_2(d))\right) \twoheadrightarrow S.$$

A direct calculation shows that this does indeed have the universal property of an exponential: a morphism in $\mathbf{Poly}_{\mathcal{F}}$ from another object $(H \twoheadrightarrow G)$ into this polynomial corresponds to a term of type

$$\begin{aligned}
&\prod_{g:G}\sum_{\sigma:S}\left(\sum_{a:A}\sum_{d:D(\sigma(a)_1)}\tilde{1}(\sigma(a)_2(d))\right) \to H(g) \\
\cong&\prod_{g:G}\sum_{\sigma:S}\prod_{a:A}\prod_{d:D(\sigma(a)_1)}\big(\tilde{1}(\sigma(a)_2(d)) \to H(g)\big) \\
\cong&\prod_{g:G}\prod_{a:A}\sum_{c:C}\sum_{\tau:D(c)\to 1+B(a)}\prod_{d:D(c)}\big(\tilde{1}(\tau(d)) \to H(g)\big) && \text{(AC)} \\
\cong&\prod_{g:G}\prod_{a:A}\sum_{c:C}\prod_{d:D(c)}\sum_{s:1+B(a)}\big(\tilde{1}(s) \to H(g)\big) && \text{(AC)} \\
\cong&\prod_{(g,a):G\times A}\sum_{c:C}(D(c) \to H(g)+B(a))
\end{aligned}$$

which corresponds to a morphism $(B \twoheadrightarrow A) \times (H \twoheadrightarrow G) \to (D \twoheadrightarrow C)$. □

Recall from Corollary 3.15 that when a fibration $\psi : \mathcal{C} \to \mathcal{B}$ has fibrewise exponentials, the resulting extended model of type theory in $\mathcal{C}$ has dependent product types. Unfortunately, we cannot apply this result to get dependent product types in this case: if $\psi : \mathbf{Poly}_{\mathcal{F}} \to \mathcal{B}$ had fibrewise exponentials then the functor ψ would preserve the cartesian closed structure, but the type

$$\psi((D \twoheadrightarrow C)^{(B\twoheadrightarrow A)}) \quad = \quad \prod_{a:A}\sum_{c:C}(D(c) \to 1+B(a))$$

is clearly not in general isomorphic to $\prod_{a:A} C$. And while Theorem 4.4 shows that dependent products exist along morphisms to 1 in $\mathbf{Poly}_{\mathcal{F}}$, the theorem cannot be extended to products along all morphisms in $\mathbf{Poly}_{\mathcal{F}}$, as Altenkirch, Levy and Staton [ALS10] have shown:

Theorem 4.5. *The category $\mathbf{Poly}_{\mathcal{F}}$ is not locally cartesian closed, even when the original underlying category $\mathcal{B}$ is.*

This is proved in [ALS10] by assuming that the unique morphism from $(1 \twoheadrightarrow 1)$ to $(2 \twoheadrightarrow 1)$ is exponentiable and deriving a contradiction. In fact, a similar contradiction can be obtained for a larger class of morphisms in $\mathbf{Poly}_{\mathcal{F}}$:

Proposition 4.6. *If φ is a display map which is not monic, then the morphism (f, φ) is not exponentiable in $\mathbf{Poly}_{\mathcal{F}}$.*

Proof. If it is exponentiable, then the pullback functor

$$(f, \varphi)^* : \mathbf{Poly}_{\mathcal{F}}/(D \twoheadrightarrow C) \to \mathbf{Poly}_{\mathcal{F}}/(B \twoheadrightarrow A)$$

exists and has a right adjoint $\prod_{(f,\varphi)}$. Then $(f, \varphi)^*$ preserves all colimits which exist. We shall construct a coequalizer in $\mathbf{Poly}_{\mathcal{F}}/(D \twoheadrightarrow C)$ for which this does not hold.

Since $\varphi : D_f \twoheadrightarrow B$ is a display map, its kernel pair

$$\begin{tikzcd} K \arrow[r, two heads, shift left, "s"] \arrow[r, two heads, shift right, "t"'] & D_f \end{tikzcd}$$

exists in $\mathcal{B}$, with equalizer the unique morphism $e : D_f \to K$ such that $se = te = 1_{D_f}$. As φ is not monic e is not an isomorphism.

The diagram

$$\begin{tikzcd} D_f + D_f \arrow[r, hook, "1+e"] \arrow[dr] & D_f + K \arrow[r, shift left, "\langle 1,s \rangle"] \arrow[r, shift right, "\langle 1,t \rangle"'] \arrow[d] & D_f \arrow[dl] \\ & A & \end{tikzcd}$$

is then an equalizer diagram in $\mathcal{F}/A$. The fibre of $\mathbf{Poly}_{\mathcal{F}} \to \mathcal{B}$ over A is just the opposite category $(\mathcal{F}/A)^{op}$, so the top row of the diagram

$$\begin{tikzcd} (D_f \twoheadrightarrow A) \arrow[r, shift left, "(1,\langle 1,s \rangle)"] \arrow[r, shift right, "(1,\langle 1,t \rangle)"'] \arrow[dr] & (D_f + K \twoheadrightarrow A) \arrow[r, "(1,1+e)"] \arrow[d] & (D_f + D_f \twoheadrightarrow A) \arrow[dl] \\ & (D \twoheadrightarrow C) & \end{tikzcd}$$

is a coequalizer in $\mathbf{Poly}_{\mathcal{F}}$. But the forgetful functor

$$\mathbf{Poly}_{\mathcal{F}}/(D \twoheadrightarrow C) \to \mathbf{Poly}_{\mathcal{F}}$$

is comonadic and hence creates coequalizers, where the comonad on $\mathbf{Poly}_{\mathcal{F}}$ is given by $(D \twoheadrightarrow C) \times (-)$, so the whole diagram is a coequalizer in $\mathbf{Poly}_{\mathcal{F}}/(D \twoheadrightarrow C)$.

The pullback of this coequalizer along (f, φ) is constructed by forming the pullback

$$\begin{tikzcd}
A_f \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "f"] \\
A \arrow[r, "f"'] & C
\end{tikzcd}$$

and the pushout along φ_{π_2}

$$\begin{tikzcd}
D_{f\pi_2} \arrow[r, hook] \arrow[d, "\varphi_{\pi_2}"'] & D_{f\pi_2} + D_{f\pi_2} \arrow[r, hook, "1+e_{\pi_2}"] \arrow[d] & D_{f\pi_2} + K_{\pi_2} \arrow[r, shift left, "\langle 1, s_{\pi_2} \rangle"] \arrow[r, shift right, "\langle 1, t_{\pi_2} \rangle"'] \arrow[d] & D_{f\pi_2} \arrow[d] \\
B_{\pi_2} \arrow[r, hook] & B_{\pi_2} \arrow[ul, phantom, very near start, "\ulcorner"] + D_{f\pi_2} \arrow[r, hook, "1+e_{\pi_2}"] & B_{\pi_2} \arrow[ul, phantom, very near start, "\ulcorner"] + K_{\pi_2} \arrow[r, shift left, "\langle 1, (\varphi s)_{\pi_2} \rangle"] \arrow[r, shift right, "\langle 1, (\varphi t)_{\pi_2} \rangle"'] & B_{\pi_2} \arrow[ul, phantom, very near start, "\ulcorner"]
\end{tikzcd}$$

to give

$$\begin{tikzcd}
(B_{\pi_2} \twoheadrightarrow A_f) \arrow[r, shift left, "(1, \langle 1, \varphi_{\pi_2} s_{\pi_2} \rangle)"] \arrow[r, shift right, "(1, \langle 1, \varphi_{\pi_2} t_{\pi_2} \rangle)"'] \arrow[dr] & (B_{\pi_2} + K_{\pi_2} \twoheadrightarrow A_f) \arrow[r, "(1, 1+e_{\pi_2})"] \arrow[d] & (B_{\pi_2} + D_{f\pi_2} \twoheadrightarrow A_f) \arrow[dl] \\
& (B \twoheadrightarrow A) &
\end{tikzcd}$$

in $\mathbf{Poly}_{\mathcal{F}}/(B \twoheadrightarrow A)$.

This is a coequalizer if and only if the bottom row of the pushout diagram is an equalizer. The two morphisms $B_{\pi_2} + K_{\pi_2} \to B_{\pi_2}$ in the bottom row are equal, so $e_{\pi_2} : D_{f\pi_2} \to K_{\pi_2}$ must be an isomorphism. However, e is a retract of e_{π_2} and is not an isomorphism, so this is a contradiction. $\square$

For the display maps in the model of Proposition 4.2 however, φ is a coproduct inclusion, and so is a display map in $\mathcal{F}$ which is monic. Surprisingly, in this case the morphism (f, φ) is exponentiable, so dependent product types can still be defined. Note that unlike the other type constructors considered so far this does not follow from the general methods of Chapter 3, but is shown by a direct calculation.

Proposition 4.7. *The polynomial model of type theory $\mathcal{F}_{\mathbf{Poly}}$ has dependent product types.*

Proof. Let (f, ι) and (p, ι) be display maps in $\mathbf{Poly}_{\mathcal{F}}$

$$\begin{tikzcd}
(D_{fp} + B_p + F \twoheadrightarrow E) \arrow[d, "(p,\iota)"] & \\
(D_f + B \twoheadrightarrow A) \arrow[r, two heads, "(f,\iota)"'] & (D \twoheadrightarrow C),
\end{tikzcd} \tag{4.1}$$

for which we want to form the dependent product $\prod_{(f,\iota)}(p, \iota)$.

These can be thought of as a collection of types

$$\begin{tikzcd}
(\sum_{(c,a,e)} D(c) + B(c,a) + F(c,a,e) \twoheadrightarrow \sum_{(c,a)} E(c,a)) \arrow[d] & \\
(\sum_{(c,a)} D(c) + B(c,a) \twoheadrightarrow \sum_c A(c)) \arrow[r, two heads] & (\sum_c D(c) \twoheadrightarrow C).
\end{tikzcd}$$

As in the previous chapter, display maps in this model factor naturally into a cartesian and a vertical part, and we can consider each case separately. (f, ι) factors as

$$\begin{tikzcd}
 & D_f \arrow[r, hook] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner", very near start] & D_f + B \arrow[r, two heads] & A \arrow[d, equal] & & (D_f + B \twoheadrightarrow A) \arrow[d, "(1_A,\iota)"] & \text{Case 2} \\
D_f \arrow[r, equal] \arrow[d] & D_f \arrow[rr, two heads] & & A \arrow[d, "f"] & & (D_f \twoheadrightarrow A) \arrow[d, "(f,1_{D_f})"] & \text{Case 1} \\
D \arrow[rrr, two heads] & & & C & & (D \twoheadrightarrow C). &
\end{tikzcd}$$

Case 1: Since the fibration $\mathcal{F} \to \mathcal{B}$ has $\mathcal{F}$-sums, its opposite $\mathbf{Poly}_{\mathcal{F}} \to \mathcal{B}$ has $\mathcal{F}$-products. For such a fibration, Lemma 3.13 states that morphisms in $\mathbf{Poly}_{\mathcal{F}}$ which are cartesian over $\mathcal{F}$-maps are exponentiable. Thus the lower morphism

$$(D_f \twoheadrightarrow A) \xrightarrow{(f,1)} (D \twoheadrightarrow C)$$

is exponentiable, and the product along it of a morphism

$$(D_f + F \twoheadrightarrow E) \xrightarrow{(p,\iota)} (D_f \twoheadrightarrow A)$$

is

$$\sum_{(c,\phi)} \sum_{a:A(c)} D(c) + F(c, a, \phi(a)) \twoheadrightarrow \sum_{c:C} \prod_{a:A(c)} E(c, a).$$

Case 2: Since every display map in this model arises as a pullback along a component of the unit η, it is enough to construct the product in the case when $D = 0$. The full product $\prod_{(f,\iota_{D_f})}(p, \iota_{D_{fp}+B_p})$ is then given by the pullback $(\eta_{D\to C})^* \prod_{(f,\iota_0)}(p, \iota_{B_p})$.

For the upper morphism

$$(B \twoheadrightarrow A) \xrightarrow{\eta_{B\to A}} (0 \twoheadrightarrow A),$$

we can factor the display map (p, ι) again into a vertical and cartesian component. In the proof of Corollary 3.15, which constructed dependent products along vertical morphisms, the assumption of products in the fibre categories was only used when both morphisms were vertical. So it suffices to construct a product of the form $\prod_{\eta_{B\to E}}(1, \iota)$, where the map $(1, \iota)$ is a pullback of a component of the unit η:

$$\begin{tikzcd}
(B + F \twoheadrightarrow E) \arrow[r] \arrow[d, "{(1,\iota)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & (F \twoheadrightarrow E) \arrow[d, "\eta_{F\to E}"] \\
(B \twoheadrightarrow E) \arrow[r, two heads, "\eta_{B\to E}"'] & (0 \twoheadrightarrow E)
\end{tikzcd}$$

Then the product $\prod_{\eta_{B\to A}}(p, \iota)$ is given by $\sum_{(p,1_0)} \prod_{\eta_{B_p\to E}}(1, \iota)$.

So, let $(H \twoheadrightarrow G)$ be another object in $\mathbf{Poly}_{\mathcal{F}}$ with a morphism (k, ι) into $(0 \twoheadrightarrow E)$ (not necessarily a display map). The pullback of (k, ι) along $\eta_{B\to E}$ exists and is given by

$$\sum_{g:G} B(k(g)) + H(g) \twoheadrightarrow G.$$

To give a morphism from this pullback into $(B + F \twoheadrightarrow E)$ over $(B \twoheadrightarrow E)$ is to give a term of type

$$\begin{aligned}
&\prod_{g:G}(F(g) \to B(k(g)) + H(g)) \\
\cong\ &\prod_{g:G} \prod_{f:F(k(g))} \sum_{s:1+B(k(g))} (\tilde{1}(s) \to H(g)) \\
\cong\ &\prod_{g:G} \sum_{w:F(k(g))\to 1+B(k(g))} \prod_{f:F(k(g))} \prod_{t:\tilde{1}(w(f))} H(g) \\
\cong\ &\prod_{g:G} \sum_{w:W(k(g))} (Z(k(g), w) \to H(g))
\end{aligned}$$

where we define the dependent types $(W \twoheadrightarrow E)$ and $(Z \twoheadrightarrow W)$ by

$$\begin{aligned} W(e) &= F(e) \to 1 + B(e) \\ Z(e, w) &= \sum_{f:F(e)} \tilde{1}(w(f)). \end{aligned}$$

This corresponds to the type of morphisms (m, φ) in $\mathbf{Poly}_{\mathcal{F}}$ making the diagram

$$\begin{tikzcd} (H \twoheadrightarrow G) \arrow[rr, "(m,\varphi)"] \arrow[dr, "(k,\iota)"'] & & (Z \twoheadrightarrow W) \arrow[dl] \\ & (0 \twoheadrightarrow E) & \end{tikzcd}$$

commute, so in other words the polynomial $(Z \twoheadrightarrow W) =$

$$\sum_{e:E} \sum_{w:F(e)\to 1+B(e)} \sum_{f:F(e)} \tilde{1}(w(f)) \twoheadrightarrow \sum_{e:E} (F(e) \to 1 + B(e))$$

has the universal property of the product $\prod_{\eta_{B\to E}}(1, \iota)$.

Putting together all the above cases, the general form of the dependent product $\prod_{(f,\iota)}(p, \iota)$ for morphisms as in Diagram 4.1 is the polynomial

$$\sum_{(c,\phi):S} \left(D(c) + \sum_{a:A(c)} \sum_{f:F(c,a,\phi(a)_1)} \tilde{1}(\phi(a)_2(f)) \right) \twoheadrightarrow S \tag{4.2}$$

where S is

$$\sum_{c:C} \prod_{a:A(c)} \sum_{e:E(c,a)} (F(c, a, e) \to 1 + B(c, a)).$$

Finally, we look at the Beck-Chevalley condition for the dependent products. Consider a pullback square in $\mathbf{Poly}_{\mathcal{F}}$, which will be a diagram of polynomials of the form

$$\begin{tikzcd} (N_g + B_h + F \twoheadrightarrow P) \arrow[r, "(h,H)"] \arrow[d, "(g,\iota)"'] \arrow[dr, phantom, "\lrcorner", very near start] & (D_f + B \twoheadrightarrow A) \arrow[d, "(f,\iota)"] \\ (N \twoheadrightarrow M) \arrow[r, "(k,K)"'] & (D \twoheadrightarrow C). \end{tikzcd}$$

We require the canonical morphism $(k, K)^*\Pi_{(f,\iota)} \to \Pi_{(g,\iota)}(h, H)^*$ to be an isomorphism.

Because the original class of display maps $\mathcal{F}$ has dependent types satisfying Beck-Chevalley, we can use the type theory notation to manipulate expressions such as (4.2) directly, where substitution of terms commutes naturally with sums and products. Then if (p, ι) is a dependent type over $(D_f + B \twoheadrightarrow A)$ as before, $(k, K)^*\Pi_{(f,\iota)}(p, \iota)$ and $\Pi_{(g,\iota)}(h, H)^*(p, \iota)$ both represent the polynomial

$$\sum_{(m,\phi):S'} \left(N(m) + \sum_{a:A(km)} \sum_{f:F(km,a,\phi(a)_1)} \tilde{1}(\phi(a)_2(f)) \right) \twoheadrightarrow S'$$

for

$$S' = \sum_{m:M} \prod_{a:A(km)} \sum_{e:E(km,a)} (F(km, a, e) \to 1 + B(km, a))$$

in a canonical way. Thus the Beck-Chevalley condition holds. □

Combining this result with Proposition 4.6 shows that when the underlying category $\mathcal{B}$ is **Set**, the class of fibrations given by this construction is the largest class of morphisms which can be given the structure of a model with dependent product types:

Corollary 4.8. *If $\mathcal{B}$ is a locally cartesian closed Boolean category, then a morphism (f, φ) is exponentiable in* **Poly** *if and only if φ is a monomorphism.* □

4.4 Identity types

Recall from Proposition 3.21 that we can construct identity types in the extended model as long as it has dependent product types, the original class of display maps $\mathcal{F}$ has identity types, and the left class ${}^{\boxslash}\mathcal{F}$ is suitably preserved by the functor along which we are extending.

Proposition 4.9. *The inclusion $\phi : \mathcal{B} \hookrightarrow \mathbf{Poly}_{\mathcal{F}}$ preserves left morphisms.*

Proof. Given a morphism $h : B \to C$ in ${}^{\boxslash}\mathcal{F}$ and an object $(D \twoheadrightarrow C)$ in $\mathbf{Poly}_{\mathcal{F}}$, we require a filler for all squares

$$\begin{tikzcd}
(0 \twoheadrightarrow B) \arrow[r] \arrow[d, "\phi h"'] \arrow[dr, phantom, "\lrcorner", very near start] & (D \twoheadrightarrow C) \arrow[d, "\eta_{D \twoheadrightarrow C}"] \\
(0 \twoheadrightarrow C) \arrow[r, equal] & (0 \twoheadrightarrow C)
\end{tikzcd}$$

in $\mathbf{Poly}_{\mathcal{F}}$. The top horizontal morphism of such a square must take the form

$$\begin{tikzcd} D_h \arrow[r, "H"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[r, two heads] & B \arrow[d, "h"] \\ D \arrow[rr, two heads] & & C \end{tikzcd}$$

in $\mathcal{B}$ for some morphism H. But 0 is a strict initial object, so $D_h \cong 0$. Recall from Lemma 3.22 that ${}^{\boxslash}\mathcal{F}$ is stable under pullback along display maps, so the morphism $0 \to D$ must be in ${}^{\boxslash}\mathcal{F}$. By constructing a filler for the square

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ D \arrow[r] & 1, \end{tikzcd}$$

it then follows that $D \cong 0$. This gives an isomorphism $(0 \twoheadrightarrow C) \to (D \twoheadrightarrow C)$ in $\mathbf{Poly}_{\mathcal{F}}$, which is the required filler. $\square$

From Proposition 3.21 we therefore get:

Corollary 4.10. *If $\mathcal{F}$ has identity types, then so does the model of type theory $\mathcal{F}_{\mathbf{Poly}}$.* $\square$

The identity type for an object $(B \twoheadrightarrow A)$ in $\mathbf{Poly}_{\mathcal{F}}$ is given as in Proposition 3.25 by the pullback

$$\begin{tikzcd} Id_{B \twoheadrightarrow A} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (0 \twoheadrightarrow Id_A) \arrow[d, "((s,1),(t,1))"] \\ (B \twoheadrightarrow A) \times (B \twoheadrightarrow A) \arrow[r, "\eta"'] & (0 \twoheadrightarrow A) \times (0 \twoheadrightarrow A). \end{tikzcd}$$

This is the fibration $(B_s + B_t \twoheadrightarrow Id_A)$, i.e.

$$\sum_{p \in Id_A} B(s(p)) + B(t(p)) \twoheadrightarrow Id_A,$$

with projection

$$\begin{tikzcd}
B_s + B_t \arrow[r, equal] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & B_s + B_t \arrow[r, two heads] & Id_A \arrow[d, "(s,t)"] & (B_s + B_t \twoheadrightarrow Id_A) \arrow[d] \\
B \times A + A \times B \arrow[rr, two heads] & & A \times A & (B \twoheadrightarrow A) \times (B \twoheadrightarrow A).
\end{tikzcd}$$

4.5 Function extensionality

Because dependent products in this polynomial model are not preserved by the fibration $\mathbf{Poly}_{\mathcal{F}} \to \mathcal{B}$, we cannot use the argument of Remark 3.23 to conclude that function extensionality in $\mathcal{F}$ implies function extensionality in $\mathcal{F}_{\mathbf{Poly}}$. And in fact, it does not hold in general. The following proposition gives a new proof of the independence of this principle from the rules for product, sum and identity types, originally shown by Streicher (see Section 2.5.4).

Proposition 4.11. *Function extensionality fails in this model of type theory in* $\mathbf{Poly}_{\mathcal{F}}$.

Proof. Recall from Proposition 2.46 that function extensionality holds if and only if for all dependent types $B \twoheadrightarrow A$ and terms $f, g : \prod_{a:A} B(a)$ there exists a morphism

$$\prod_{a:A} Id_{B(a)}(f(a), g(a)) \quad \to \quad Id_{\prod_{a:A} B(a)}(f, g).$$

In this model, let the types A and B be polynomials $1 + 1 \twoheadrightarrow 1$ and $1 + (1 + 1) \twoheadrightarrow 1$ respectively. There is a display map $B \twoheadrightarrow A$ in $\mathcal{F}_{\mathbf{Poly}}$ given by the coproduct inclusion $1 + 1 \hookrightarrow 1 + (1 + 1)$.

The product type $\prod_{a:A} B(a)$ is the polynomial $\sum_{s:1+(1+1)} \tilde{1}(s) \twoheadrightarrow 1 + (1 + 1)$ where $\tilde{1}$ refers to the first coproduct inclusion. The type $1 + 1$ has two distinct terms $inl(*)$ and $inr(*)$ with $Id_{1+1}(inl(*), inr(*)) \simeq 0$, and since 0 is a strict initial object in a categorical model this must actually be an isomorphism. These define two terms f, g of the product type

$$\begin{tikzcd}
0 \arrow[r, "0"] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & 0 \arrow[r, two heads] & 1 \arrow[d] \\
\sum_{s:1+1+1} \tilde{1}(s) \arrow[rr, two heads] & & 1 + (1 + 1)
\end{tikzcd}$$

where the right vertical morphism is $inr(inl(*))$ or $inr(inr(*))$ respectively. The type $Id_{\prod_{a:A} B(a)}(f, g)$ is then represented by the polynomial

$$\begin{aligned} & 0 \twoheadrightarrow Id_{1+(1+1)}\left(inr(inl(*)), inr(inr(*))\right) \\ \simeq\ & 0 \twoheadrightarrow 0. \end{aligned}$$

Intuitively, this says there are no 'global paths' between the terms f and g.

We now consider the 'pointwise paths'. The dependent type

$$Id_{B(a)}(b_1, b_2) \twoheadrightarrow B \times_A B$$

corresponds to the morphism of polynomials

$$\begin{tikzcd} \sum_{p:Id_1(*,*)} 1 + 1 + (1+1) \arrow[r, "="] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \sum_{p:Id_1(*,*)} 1 + 1 + (1+1) \arrow[r, two heads] & Id_1(*,*) \arrow[d] \\ 1 + 1 + (1+1) \arrow[rr, two heads] & & 1, \end{tikzcd}$$

so $Id_{B(a)}(f(a), g(a)) \twoheadrightarrow A$ is

$$\left(\sum_{p:Id_1(*,*)} (1+1) \twoheadrightarrow Id_1(*,*) \right) \twoheadrightarrow (1 + 1 \twoheadrightarrow 1),$$

and then using the construction of product types in Proposition 4.7, the type $\prod_{a:A} Id_{B(a)}(f(a), g(a))$ is the polynomial

$$\begin{aligned} & 0 \twoheadrightarrow Id_1(*,*) \\ \simeq\ & 0 \twoheadrightarrow 1. \end{aligned}$$

There is clearly no morphism from this to $Id_{\prod_{a:A} B(a)}(f, g)$, and function extensionality fails. $\square$

Remark 4.12. More generally, given a polynomial $A = (\sum_{c:C} D(c) \twoheadrightarrow C)$ with a dependent type $B = (\sum_{c,e} D(c) + F(c,e) \twoheadrightarrow \sum_{c:C} E(c))$ over it, terms f, g of the product type $\prod_{a:A} B(a)$ corresponds to certain terms s, t of type

$$T = \prod_{c:C} \sum_{e:E(c)} (F(c,e) \Rightarrow 1 + D(c))$$

in $\mathcal{B}$. The type $\prod_{a:A} Id_{B(a)}(f(a), g(a))$ is then represented by

$$0 \twoheadrightarrow \prod_{c:C} Id_{E(c)}(s(c)_1, t(c)_1),$$

while $Id_{\prod_{a:A} B(a)}(f, g)$ is

$$0 \twoheadrightarrow Id_T(s, t).$$

If function extensionality holds in $\mathcal{B}$, to give a term of this latter type corresponds to giving for each $c : C$ a term p of type $Id_{E(c)}(s(c)_1, t(c)_1)$, together with a term of type

$$Id_{F(c,t(c)_1) \Rightarrow 1+D(c)}(p_* s(c)_2, t(c)_2).$$

This contains more information than just a term of type $\prod_{a:A} Id_{B(a)}(f(a), g(a))$, and in general there is no reason to expect the types to be equivalent.

To summarize, given a categorical model of type theory with unit, dependent sum, dependent product, binary sum, and identity types, (for example the category **Set**, or the groupoid model), the polynomial construction gives a model of type theory with the same type constructors. This model has identity types which need not be extensional or satisfy the uniqueness of identity proofs, and function extensionality can fail to hold. In particular, this gives a new semantic proof that the function extensionality axiom is independent of the other rules of intensional type theory.

Chapter 5

Outlook

5.1 Iterating polynomials

The existence of a polynomial model suggests various avenues which could be investigated in future work.

For instance, consider again the pseudomonad $\Sigma\Pi$ acting on fibrations, which is constructed in Chapter 1. Applying this to a fibration representing a model of type theory, we could try to build a new model using the general method of Chapter 3 to extend the base. Alternatively, since $\Sigma\Pi \cong \Sigma(\Sigma(-)^{op})^{op} = Pol^2$, we could use the construction of a polynomial model twice. The two resulting fibrations would be different in general, with different base categories, and it might be worthwhile to compare them.

5.2 Dialectica-style interpretations

The original motivation for studying these iterated constructions comes from the "Dialectica interpretation", which Gödel introduced to provide a relative consistency proof for Heyting Arithmetic [Göd58]. Each formula α of Heyting Arithmetic is assigned a formula

$$\alpha^D \;=\; \exists u\, \forall x\; \alpha_D(u, x)$$

in a simply-typed system of computable functionals, where α_D is quantifier-free and decidable and defined by induction on the structure of α. A crucial step is the interpretation of implication, where for $\beta^D = \exists v\; \forall y\; \beta_D(v, y)$ the formula $(\alpha \to \beta)^D$

is defined to be

$$\exists f, F\ \forall u, y\ \ (\alpha_D(u, F(u, y)) \rightarrow \beta_D(f(u), y))\,.$$

An abstract version of this was described by de Paiva [dP89], forming a category where objects correspond to formulae and morphisms correspond to proofs under this style of implication. Given a fibration $p : \mathcal{P} \rightarrow \mathcal{T}$ (originally taken to be the subobject fibration), the category $Dial(p)$ has as objects triples $(U \in \mathcal{T}, X \in \mathcal{T}, \alpha \in \mathcal{P}(U \times X))$, which are thought of as formulae $\exists u \in U.\forall x \in X.\alpha(u, x)$. We can represent this as a diagram

$$\begin{tikzcd} U & U \times X \arrow[l, "\pi_2"'] & A. \arrow[l, dashed, "\alpha"'] \end{tikzcd}$$

A morphism $(U, X, \alpha) \rightarrow (V, Y, \beta)$ in $Dial(p)$ consists of $f : U \rightarrow V$ and $F : U \times Y \rightarrow V$ in $\mathcal{T}$ together with $\varphi : A(u, F(u, y)) \rightarrow B(f(u), y)$ in the fibre of $\mathcal{P}$ over $U \times Y$, as in the diagram

$$\begin{tikzcd}
U \arrow[dddd, "f"'] & U \times X \arrow[l] & A \arrow[l, dashed, "\alpha"'] \\
 & & (\pi_1, F)^*A \arrow[u] \arrow[dl, dashed] \arrow[dd, "\varphi"] \\
 & U \times Y \arrow[uul] \arrow[uu, "(\pi_1,F)"'] \arrow[dd, "f\times 1"] \arrow[ddl, phantom, "\llcorner", very near start] & \\
 & & (f \times 1)^*B \arrow[ul, dashed] \arrow[d] \\
V & V \times Y \arrow[l] & B. \arrow[l, dashed, "\beta"']
\end{tikzcd}$$

In other words, $Dial(p)$ is the fibre over 1 of the fibration $\Sigma_S\Pi_S p$ which adds sums and products along product projections to p. This correspondence is explained by Hofstra in [Hof11]. Thus iterating the type theory construction as described above should give some kind of model of type theory in the indexed Dialectica category. It would be interesting to study the properties of this model and see in what sense it corresponds to the original interpretation.

Several variants on the Dialectica interpretation have been proposed for proof-theoretic reasons, and some of these have also been shown to naturally give rise to categories. For example the Diller-Nahm interpretation, which does not require that atomic formulae be decidable, corresponds to the Kleisli category $Dial_{DN}(p)$ for a comonad on $Dial(p)$ induced by the free commutative monoid monad on p [Hyl02]. Taking the Kleisli category $Dial^+(p)$ for the comonad induced by the monad $(- + 1)$ corresponds to Dialectica with exception passing [Hyl07, Bie08]. These fibrations have good categorical properties if the original fibration p has sufficient structure: unlike $Dial(p)$ the fibres of $Dial_{DN}(p)$ are cartesian closed and those of $Dial^+(p)$ weakly cartesian closed. It seems reasonable to try to extend them to models of type theory.

5.3 A model theory for type theory

In addition to constructing new models of type theory, it is useful to study the relationships between them. With a general theory of models we could compare and contrast different type theories and understand interpretations of one theory in another.

One of many viewpoints on the theory of toposes is that it provides a model theory for higher-order intuitionistic type theory [Joh02]. It seems natural to look for an analogous categorical model theory for dependent type theory. However, there are some apparent differences.

Primary examples of toposes are given by categories of sheaves. Dependent type theory has presheaf models, such as those in simplicial sets [KLV12] and cubical sets [BCH14], but it is not evident how to extend constructions of this kind to sheaves. There are also toposes constructed from notions of realizability. Although realizability is a form of functional interpretation just as the Dialectica interpretation is, and Hofstra and Warren [HW13] have constructed models of type theory from realizers in a slightly different sense, there is no clear type-theoretic analogue of realizability toposes. Even describing a model theory for extensional type theories is not straightforward, as for example considered by van den Berg in [vdB06]. In any case constructions such as forming polynomials as described in this thesis do not necessarily preserve extensionality of the type theory. It seems there is still much to explore in this area.

Appendix A

Some definitions

In this appendix, we spell out for completeness some of the categorical definitions used in the previous chapters.

A bicategory $\mathfrak{B}$ is enriched in **2-Cat** (Section 1.2) when each hom-category $\mathfrak{B}(X,Y)$ has the structure of a 2-category, and this structure is preserved by horizontal composition. In detail:

Definition A.1 ([Car95]). A ***2-Cat**-enriched bicategory* $\mathfrak{B}$ consists of

- a collection of objects $ob\,\mathfrak{B}$,
- a 2-category $\mathfrak{B}(X,Y)$ for each pair of objects X,Y in $\mathfrak{B}$, whose objects are called *1-cells* and written $f : X \to\!\!\!\!\!\!+ \; Y$, whose morphisms are *2-cells*, and whose 2-cells are *3-cells* of $\mathfrak{B}$,
- a composition 2-functor

$$\mathfrak{B}(Y,Z) \times \mathfrak{B}(X,Y) \xrightarrow{\circ_{X,Y,Z}} \mathfrak{B}(X,Z)$$

  for each triple of objects X,Y,Z,
- an identity 2-functor $1 \xrightarrow{1_X} \mathfrak{B}(X,X)$ for each object X,
- a 2-natural isomorphism

$$\begin{tikzcd}
\mathfrak{B}(Z,W) \times \mathfrak{B}(Y,Z) \times \mathfrak{B}(X,Y) \arrow[r, "\circ\times 1"] \arrow[d, "1\times\circ"'] \arrow[dr, phantom, "\cong \Downarrow \alpha_{X,Y,Z,W}"] & \mathfrak{B}(Y,W) \times \mathfrak{B}(X,Y) \arrow[d, "\circ"] \\
\mathfrak{B}(Z,W) \times \mathfrak{B}(X,Z) \arrow[r, "\circ"'] & \mathfrak{B}(X,W)
\end{tikzcd}$$

for each quadruple of objects X, Y, Z, W,

- two 2-natural isomorphisms

$$
\begin{tikzcd}
& 1 \times \mathfrak{B}(X,Y) \arrow[dl, "\cong"'] \arrow[d, "1_Y \times 1"] \arrow[dl, phantom, shift left=4ex, "\cong \Nwarrow r_{X,Y}"] \\
\mathfrak{B}(X,Y) & \mathfrak{B}(Y,Y) \times \mathfrak{B}(X,Y) \arrow[l, "\circ"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{B}(X,Y) \times 1 \arrow[d, "1 \times 1_X"'] \arrow[dr, "\cong"] \arrow[dr, phantom, shift right=4ex, "l_{X,Y} \Nearrow \cong"] & \\
\mathfrak{B}(X,Y) \times \mathfrak{B}(X,X) \arrow[r, "\circ"'] & \mathfrak{B}(X,Y)
\end{tikzcd}
$$

for each pair of objects X, Y,

such that the diagrams

$$
\begin{tikzcd}
& ((k \circ h) \circ g) \circ f \arrow[r, "\alpha \circ 1"] \arrow[dl, "\alpha"'] & (k \circ (h \circ g)) \circ f \arrow[dr, "\alpha"] & \\
(k \circ h) \circ (g \circ f) \arrow[drr, "\alpha"'] & & & k \circ ((h \circ g) \circ f) \arrow[dl, "1 \circ \alpha"] \\
& & k \circ (h \circ (g \circ f)) &
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
(g \circ 1) \circ f \arrow[rr, "\alpha"] \arrow[dr, "l \circ 1"'] & & g \circ (1 \circ f) \arrow[dl, "1 \circ r"] \\
& g \circ f &
\end{tikzcd}
$$

commute for all 1-cells f, g, h, k such that the necessary composites are defined.

Example A.2. (a) Any bicategory can be considered as a **2-Cat**-enriched bicategory by regarding the hom-categories as locally discrete 2-categories. Conversely, any **2-Cat**-enriched bicategory $\mathfrak{B}$ has an *underlying bicategory* $\mathfrak{B}_u$, obtained by forgetting the 3-cells.

(b) Any strict 3-category can be considered as a **2-Cat**-enriched bicategory with identities as the 2-natural isomorphisms in Definition A.1.

(c) Reversing the 1-cells of a **2-Cat**-enriched bicategory $\mathfrak{B}$ gives another **2-Cat**-enriched bicategory $\mathfrak{B}^{op}$. In other words, the hom-2-category $\mathfrak{B}^{op}(X,Y)$ is $\mathfrak{B}(Y,X)$.

Remark A.3. Composition $\circ$ is usually denoted by juxtaposition and the associativity and unit isomorphisms for $\mathfrak{B}$ are suppressed.

Definition A.4 ([CHP04])**.** In a **2-Cat**-enriched bicategory $\mathfrak{B}$, a *pseudomonad* consists of

- a 1-cell $T : X \twoheadrightarrow X$,
- 2-cells $\mu : T^2 \to T$ and $\eta : 1_X \to T$,
- invertible 3-cells

$$\begin{tikzcd}
T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] \arrow[dr, phantom, "\Swarrow\tau"] & T^2 \arrow[d, "\mu"] \\
T^2 \arrow[r, "\mu"'] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[r, "T\eta"] \arrow[dr, "="'] & T^2 \arrow[d, "\mu"] \arrow[dl, phantom, "\Swarrow\lambda", near start] \arrow[dr, phantom, "\Searrow\rho", near start] & T \arrow[l, "\eta T"'] \arrow[dl, "="] \\
& T &
\end{tikzcd}$$

such that the following pasting diagrams of 3-cells are equal:

$$\begin{tikzcd}
T^4 \arrow[r, "T^2\mu"] \arrow[d, "\mu T^2"'] \arrow[dr, "T\mu T"] & T^3 \arrow[dr, "T\mu"] \arrow[d, phantom, "\Swarrow T\tau"] & \\
T^3 \arrow[dr, "\mu T"'] \arrow[r, phantom, "\Swarrow\tau T"] & T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"] \arrow[dr, phantom, "\Swarrow\tau"] & T^2 \arrow[d, "\mu"] \\
& T^2 \arrow[r, "\mu"'] & T
\end{tikzcd}
=
\begin{tikzcd}
T^4 \arrow[r, "T^2\mu"] \arrow[d, "\mu T^2"'] \arrow[dr, phantom, "\cong"] & T^3 \arrow[d, "\mu T"] \arrow[dr, "T\mu"] & \\
T^3 \arrow[r, "T\mu"] \arrow[dr, "\mu T"'] & T^2 \arrow[dr, "\mu"] \arrow[d, phantom, "\Swarrow\tau"] \arrow[r, phantom, "\Swarrow\tau"] & T^2 \arrow[d, "\mu"] \\
& T^2 \arrow[r, "\mu"'] & T
\end{tikzcd}$$

$$\begin{tikzcd}
T^2 \arrow[r, "T\eta T"] \arrow[dr, "="'] & T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"] \arrow[dl, phantom, "\Swarrow\lambda T", near start] \arrow[dr, phantom, "\Swarrow\tau"] & T^2 \arrow[d, "\mu"] \\
& T^2 \arrow[r, "\mu"'] & T
\end{tikzcd}
=
\begin{tikzcd}
T^2 \arrow[r, "T\eta T"] \arrow[dr, "="'] & T^3 \arrow[r, "T\mu"] \arrow[d, "T\mu"] \arrow[dl, phantom, "\Swarrow T\rho", near start] \arrow[dr, phantom, "="] & T^2 \arrow[d, "\mu"] \\
& T^2 \arrow[r, "\mu"'] & T.
\end{tikzcd}$$

It is a *monad* if the 3-cells τ, λ and ρ are identities, in which case the coherence axioms are automatically satisfied.

Composition with a 1-cell $T : X \twoheadrightarrow X$ defines a strict 2-functor on each hom-2-category $\mathfrak{B}(Z, X)$ and $\mathfrak{B}(X, Y)$. If (T, μ, η) is a pseudomonad, then composition with μ and η define 2-natural transformations giving these 2-functors the structure of pseudomonads. They are 2-monads if T is a monad.

Definition A.5. A *left module* for a pseudomonad $T : X \twoheadrightarrow X$ is a pseudoalgebra for T acting on the left hom-2-category, so in other words consists of a 1-cell $E : A \twoheadrightarrow X$

with a 2-cell $e : TE \to E$ and invertible 3-cells

$$\begin{tikzcd}
T^2E \arrow[r, "Te"] \arrow[d, "\mu E"'] \arrow[dr, phantom, "\swarrow\varepsilon"] & TE \arrow[d, "e"] \\
TE \arrow[r, "e"'] & E
\end{tikzcd}
\qquad
\begin{tikzcd}
E \arrow[r, "\eta E"] \arrow[dr, "="'] \arrow[dr, phantom, shift left=3, "\swarrow\bar\varepsilon"] & TE \arrow[d, "e"] \\
& E
\end{tikzcd}$$

satisfying the coherence axioms:

$$\begin{tikzcd}
T^3E \arrow[r, "T^2e"] \arrow[d, "\mu TE"'] \arrow[dr, "T\mu E"] & T^2E \arrow[dr, "Te"] \arrow[d, phantom, "\Downarrow T\varepsilon"] & \\
T^2E \arrow[r, phantom, "\swarrow\tau E"] \arrow[dr, "\mu E"'] & T^2E \arrow[r, "Te"] \arrow[d, "\mu E"'] \arrow[dr, phantom, "\swarrow\varepsilon"] & TE \arrow[d, "e"] \\
& TE \arrow[r, "e"'] & E
\end{tikzcd}
=
\begin{tikzcd}
T^3E \arrow[r, "T^2e"] \arrow[d, "\mu TE"'] \arrow[dr, phantom, "\cong"] & T^2E \arrow[d, "\mu E"] \arrow[dr, "Te"] & \\
T^2E \arrow[r, "Te"] \arrow[dr, "\mu E"'] & TE \arrow[r, phantom, "\swarrow\varepsilon"] \arrow[dr, "e"] \arrow[d, phantom, "\Downarrow\varepsilon"] & TE \arrow[d, "e"] \\
& TE \arrow[r, "e"'] & E
\end{tikzcd}$$

$$\begin{tikzcd}
TE \arrow[r, "T\eta E"] \arrow[dr, "="'] \arrow[dr, phantom, shift left=3, "\swarrow\lambda E"] & T^2E \arrow[r, "Te"] \arrow[d, "\mu E"] \arrow[dr, phantom, "\swarrow\varepsilon"] & TE \arrow[d, "e"] \\
& TE \arrow[r, "e"'] & E
\end{tikzcd}
=
\begin{tikzcd}
TE \arrow[r, "T\eta E"] \arrow[dr, "="'] \arrow[dr, phantom, shift left=3, "\swarrow T\bar\varepsilon"] & T^2E \arrow[r, "Te"] \arrow[d, "Te"] \arrow[dr, phantom, "="] & TE \arrow[d, "e"] \\
& TE \arrow[r, "e"'] & E.
\end{tikzcd}$$

It is a *strict left T-module* if ε and $\bar\varepsilon$ are identities, in which case the coherence axioms are automatically satisfied.

A *right T-module* is a pseudoalgebra for T acting on a right hom-2-category, or equivalently a left module for T in $\mathfrak{B}^{op}$.

Definition A.6. A *bimodule* for pseudomonads $S : Y \to\!\!\!\!\!\to Y$ and $T : X \to\!\!\!\!\!\to X$ is a 1-cell $M : Y \to\!\!\!\!\!\to X$ with the structure $(d, \delta, \bar\delta)$ of a right S-module and the structure $(e, \varepsilon, \bar\varepsilon)$ of a left T-module, together with an invertible 3-cell

$$\begin{tikzcd}
TMS \arrow[r, "Td"] \arrow[d, "eS"'] \arrow[dr, phantom, "\swarrow\gamma"] & TM \arrow[d, "e"] \\
MS \arrow[r, "d"'] & M
\end{tikzcd}$$

which is compatible with $\delta, \bar{\delta}, \varepsilon, \bar{\varepsilon}$, i.e. satisfies the coherence axioms:

$$
\begin{tikzcd}
TMS^2 \arrow[r, "TdS"] \arrow[d, "eS^2"'] \arrow[dr, "TM\mu"] & TMS \arrow[dr, "Td"] \arrow[d, phantom, "\Downarrow T\delta"] & \\
TMS \arrow[r, phantom, "\cong"] \arrow[dr, "M\mu"'] & TMS \arrow[r, "Td"] \arrow[d, "eS"] \arrow[dr, phantom, "\Downarrow\gamma"] & TM \arrow[d, "e"] \\
 & MS \arrow[r, "d"'] & M
\end{tikzcd}
=
\begin{tikzcd}
TMS^2 \arrow[r, "TdS"] \arrow[d, "eS^2"'] \arrow[dr, phantom, "\Downarrow\gamma S"] & TMS \arrow[d, "eS"] \arrow[dr, "Td"] & \\
TMS \arrow[r, "dS"] \arrow[dr, "M\mu"'] \arrow[r, phantom, shift right=4ex, "\Downarrow\delta"] & MS \arrow[r, phantom, "\Downarrow\gamma"] \arrow[dr, "d"] & TM \arrow[d, "e"] \\
 & MS \arrow[r, "d"'] & M
\end{tikzcd}
$$

$$
\begin{tikzcd}
T^2MS \arrow[r, "T^2d"] \arrow[d, "TeS"'] \arrow[dr, "\mu MS"] & T^2M \arrow[dr, "\mu M"] \arrow[d, phantom, "\cong"] & \\
TMS \arrow[r, phantom, "\Downarrow(\varepsilon S)^{-1}"] \arrow[dr, "eS"'] & TMS \arrow[r, "Td"] \arrow[d, "eS"] \arrow[dr, phantom, "\Downarrow\gamma"] & TM \arrow[d, "e"] \\
 & MS \arrow[r, "d"'] & M
\end{tikzcd}
=
\begin{tikzcd}
T^2MS \arrow[r, "T^2d"] \arrow[d, "TeS"'] \arrow[dr, phantom, "\Downarrow T\gamma"] & T^2M \arrow[d, "Te"] \arrow[dr, "\mu M"] & \\
TMS \arrow[r, "Td"] \arrow[dr, "eS"'] \arrow[r, phantom, shift right=4ex, "\Downarrow\gamma"] & TM \arrow[r, phantom, "\Downarrow\varepsilon^{-1}"] \arrow[dr, "e"] & TM \arrow[d, "e"] \\
 & MS \arrow[r, "d"'] & M
\end{tikzcd}
$$

$$
\begin{tikzcd}
TM \arrow[d, "e"'] \arrow[dr, "TM\eta"] \arrow[drr, "="] & & \\
M \arrow[r, phantom, "\cong"] \arrow[dr, "M\eta"'] & TMS \arrow[r, "Td"'] \arrow[d, "eS"] \arrow[u, phantom, "\Downarrow(T\bar{\delta})^{-1}"] \arrow[dr, phantom, "\Downarrow\gamma"] & TM \arrow[d, "e"] \\
 & MS \arrow[r, "d"'] & M
\end{tikzcd}
=
\begin{tikzcd}
TM \arrow[d, "e"'] \arrow[drr, "="] & & \\
M \arrow[rr, phantom, "="] \arrow[dr, "M\eta"'] \arrow[drr, "="] & & TM \arrow[d, "e"] \\
 & MS \arrow[r, "d"'] \arrow[u, phantom, "\Downarrow\bar{\delta}^{-1}"] & M
\end{tikzcd}
$$

$$
\begin{tikzcd}
MS \arrow[r, "d"] \arrow[dr, "\eta MS"] \arrow[ddr, "="'] & M \arrow[dr, "\eta M"] \arrow[d, phantom, "\cong"] & \\
 & \Downarrow\bar{\varepsilon}S\; TMS \arrow[r, "Td"'] \arrow[d, "eS"] \arrow[dr, phantom, "\Downarrow\gamma"] & TM \arrow[d, "e"] \\
 & MS \arrow[r, "d"'] & M
\end{tikzcd}
=
\begin{tikzcd}
MS \arrow[r, "d"] \arrow[ddr, "="'] & M \arrow[dr, "\eta M"] \arrow[ddr, "="'] & \\
 & = & \Downarrow\bar{\varepsilon}\; TM \arrow[d, "e"] \\
 & MS \arrow[r, "d"'] & M.
\end{tikzcd}
$$

Definition A.7. A *pseudo-distributive law* of a pseudomonad $S : X \nrightarrow X$ over a pseudomonad $T : X \nrightarrow X$ in a **2-Cat**-enriched bicategory consists of a 2-cell

$\lambda\colon ST \to TS$ and invertible 3-cells

$$
\begin{tikzcd}
S^2T \arrow[r, "S\lambda"] \arrow[d, "\mu_S T"'] & STS \arrow[r, "\lambda S"] \arrow[d, phantom, "\Swarrow\alpha"] & TS^2 \arrow[d, "T\mu_S"] \\
ST \arrow[rr, "\lambda"'] & & TS
\end{tikzcd}
\qquad
\begin{tikzcd}
ST^2 \arrow[r, "\lambda T"] \arrow[d, "S\mu_T"'] & TST \arrow[r, "T\lambda"] \arrow[d, phantom, "\beta\Nearrow"] & T^2S \arrow[d, "\mu_T S"] \\
ST \arrow[rr, "\lambda"'] & & TS
\end{tikzcd}
$$

$$
\begin{tikzcd}
T \arrow[d, "\eta_S T"'] \arrow[dr, "T\eta_S"] \arrow[dr, phantom, shift right=3ex, "\Swarrow\gamma"] & \\
ST \arrow[r, "\lambda"'] & TS
\end{tikzcd}
\qquad
\begin{tikzcd}
S \arrow[d, "S\eta_T"'] \arrow[dr, "\eta_T S"] \arrow[dr, phantom, shift right=3ex, "\Swarrow\delta"] & \\
ST \arrow[r, "\lambda"'] & TS
\end{tikzcd}
$$

satisfying the coherence axioms:

1.

$$
\begin{tikzcd}
S^3T \arrow[r, "S^2\lambda"] \arrow[dd, "\mu ST"'] \arrow[ddr, "S\mu T"] & S^2TS \arrow[dr, "S\lambda S"] \arrow[d, phantom, "\Swarrow S\alpha"] & \\
& & STS^2 \arrow[d, "ST\mu"] \\
S^2T \arrow[ddr, "\mu T"'] \arrow[r, phantom, "\Swarrow\tau T"] & S^2T \arrow[r, "S\lambda"] \arrow[dd, "\mu T"'] \arrow[dr, phantom, "\Swarrow\alpha"] & STS \arrow[d, "\lambda S"] \\
& & TS^2 \arrow[d, "T\mu"] \\
& ST \arrow[r, "\lambda"'] & TS
\end{tikzcd}
=
\begin{tikzcd}
S^3T \arrow[r, "S^2\lambda"] \arrow[dd, "\mu ST"'] \arrow[dr, phantom, "\cong"] & S^2TS \arrow[dr, "S\lambda S"] \arrow[dd, "\mu TS"'] \arrow[d, phantom, "\Swarrow S\alpha"] & & \\
& & STS^2 \arrow[d, "\lambda S^2"'] \arrow[dr, "ST\mu"] & \\
S^2T \arrow[r, "S\lambda"] \arrow[ddr, "\mu T"'] & STS \arrow[dr, "\lambda S"'] \arrow[d, phantom, "\Swarrow\alpha"] & TS^3 \arrow[r, phantom, "\cong"] \arrow[d, "T\mu S"'] \arrow[dr, "TS\mu"] & STS \arrow[d, "\lambda S"] \\
& & TS^2 \arrow[dr, "T\mu"'] & TS^2 \arrow[d, "T\mu"] \arrow[l, phantom, "\Swarrow\tau T"] \\
& ST \arrow[rr, "\lambda"'] & & TS
\end{tikzcd}
$$

2.

$$
\begin{tikzcd}
ST \arrow[dr, "S\eta T"'] \arrow[drr, "ST\eta"] \arrow[dddr, "1"'] \arrow[r, phantom, "\Downarrow S\gamma"] & & \\
& S^2T \arrow[r, "S\lambda"'] \arrow[dd, "\mu T"] \arrow[dl, phantom, "\Swarrow\lambda T"] & STS \arrow[d, "\lambda S"] \\
& & TS^2 \arrow[d, "T\mu"] \arrow[l, phantom, "\Swarrow\alpha"] \\
& ST \arrow[r, "\lambda"'] & TS
\end{tikzcd}
=
\begin{tikzcd}
ST \arrow[d, "\lambda"'] \arrow[dr, "ST\eta"] & \\
TS \arrow[r, phantom, "\cong"] \arrow[dr, "TS\eta"] \arrow[ddr, "1"'] & STS \arrow[d, "\lambda S"] \\
& TS^2 \arrow[d, "T\mu"] \arrow[dl, phantom, "\Swarrow T\lambda"] \\
& TS
\end{tikzcd}
$$

3.

$$
\begin{tikzcd}
S^2 \arrow[r, "S^2\eta"] \arrow[drr, "S\eta S"'] \arrow[ddrr, "\eta S^2"'] \arrow[dd, "\mu"'] & S^2T \arrow[dr, "S\lambda"] \arrow[d, phantom, "S\delta \Uparrow"] & \\
& \arrow[dr, phantom, "\delta S \Uparrow"] & STS \arrow[d, "\lambda S"] \\
S \arrow[drr, "\eta S"'] \arrow[rr, phantom, "\cong"] & & TS^2 \arrow[d, "T\mu"] \\
& & TS
\end{tikzcd}
=
\begin{tikzcd}
S^2 \arrow[r, "S^2\eta"] \arrow[dd, "\mu"'] \arrow[ddr, phantom, "\cong"] & S^2T \arrow[dr, "S\lambda"] \arrow[dd, "\mu T"'] & \\
& & STS \arrow[d, "\lambda S"] \arrow[dl, phantom, "\alpha^{-1} \Uparrow"] \\
S \arrow[r, "S\eta"] \arrow[drr, "\eta S"'] & ST \arrow[dr, "\lambda"] \arrow[d, phantom, "\delta \Uparrow"] & TS^2 \arrow[d, "T\mu"] \\
& \; & TS
\end{tikzcd}
$$

4.

$$
\begin{tikzcd}
1_X \arrow[r, "\eta"] \arrow[d, "\eta"'] \arrow[dr, phantom, "\cong"] & T \arrow[d, "\eta T"'] \arrow[ddr, "T\eta"] & \\
S \arrow[r, "S\eta"] \arrow[drr, "\eta S"'] & ST \arrow[dr, "\lambda"] \arrow[r, phantom, "\gamma^{-1} \Uparrow"] \arrow[d, phantom, "\delta \Uparrow"] & \; \\
& \; & TS
\end{tikzcd}
=
\begin{tikzcd}
1_X \arrow[r, "\eta"] \arrow[d, "\eta"'] & T \arrow[ddr, "T\eta"] \arrow[d, phantom, "\cong"] & \\
S \arrow[drr, "\eta S"'] & \; & \\
& & TS
\end{tikzcd}
$$

5.

$$
\begin{tikzcd}
S^2T^2 \arrow[rr, "S^2\mu"] \arrow[dr, "S\lambda T"] \arrow[ddddd, "\mu T^2"'] & & S^2T \arrow[ddr, "S\lambda"] \arrow[d, phantom, "\Downarrow S\beta"] & \\
& STST \arrow[dr, "ST\lambda"] \arrow[dd, "\lambda ST"'] & \; & \\
& \; \arrow[r, phantom, "\cong"] & ST^2 \arrow[r, "S\mu S"] \arrow[dd, "\lambda TS"] & STS \arrow[ddd, "\lambda S"] \\
\arrow[r, phantom, "\Downarrow \alpha T"] & TS^2T \arrow[dr, "TS\lambda"] \arrow[ddd, "T\mu T"'] & \; \arrow[r, phantom, "\Downarrow \beta S"] & \; \\
& \arrow[d, phantom, "\Downarrow T\alpha"] & TSTS \arrow[d, "T\lambda S"] & \\
ST^2 \arrow[dr, "\lambda T"'] & \; & T^2S^2 \arrow[r, "\mu S^2"] \arrow[dd, "T^2\mu"] & TS^2 \arrow[dd, "T\mu"] \\
& TST \arrow[dr, "T\lambda"'] & \; \arrow[r, phantom, "\cong"] & \; \\
& & T^2S \arrow[r, "\mu S"'] & TS
\end{tikzcd}
=
\begin{tikzcd}
S^2T^2 \arrow[rr, "S^2\mu"] \arrow[ddd, "\mu T^2"'] \arrow[dddrr, phantom, "\cong"] & & S^2T \arrow[dr, "S\lambda"] \arrow[ddd, "\mu T"] & \\
& & & STS \arrow[dd, "\lambda S"] \\
& & \; \arrow[r, phantom, "\Downarrow \alpha"] & \; \\
ST^2 \arrow[rr, "S\mu"] \arrow[dr, "\lambda T"'] & & ST \arrow[ddr, "\lambda"] \arrow[dd, phantom, "\Downarrow \beta"] & TS^2 \arrow[dd, "T\mu"] \\
& TST \arrow[dr, "T\lambda"'] & & \\
& & T^2S \arrow[r, "\mu S"'] & TS
\end{tikzcd}
$$

6.

$$
\begin{tikzcd}
T^2 \arrow[rr, "\mu"] \arrow[d, "\eta T^2"'] \arrow[drr, phantom, "\cong"] & & T \arrow[d, "\eta T"'] \arrow[dddr, bend left, "T\eta"] & \\
ST^2 \arrow[rr, "S\mu"] \arrow[dr, "\lambda T"'] & & ST \arrow[ddr, "\lambda"] \arrow[dr, phantom, "\Swarrow\gamma"] & \\
& TST \arrow[dr, "T\lambda"'] \arrow[r, phantom, "\Swarrow\beta"] & & \\
& & T^2S \arrow[r, "\mu S"'] & TS
\end{tikzcd}
=
\begin{tikzcd}
T^2 \arrow[rr, "\mu"] \arrow[d, "\eta T^2"'] \arrow[dr, bend left, "T\eta T"] \arrow[ddr, bend left, "T^2\eta"] \arrow[dl, phantom, "\Swarrow\gamma T"] & & T \arrow[ddr, bend left, "T\eta"] \arrow[d, phantom, "\cong"] & \\
ST^2 \arrow[r, "\lambda T"'] & TST \arrow[dr, "T\lambda"'] \arrow[r, phantom, "\Swarrow T\gamma"] & & \\
& & T^2S \arrow[r, "\mu S"'] & TS
\end{tikzcd}
$$

7.

$$
\begin{tikzcd}
ST^3 \arrow[r, "ST\mu"] \arrow[d, "\lambda T^2"'] \arrow[dr, "S\mu T"'] & ST^2 \arrow[r, "S\mu"] \arrow[d, phantom, "\Downarrow S\tau"] & ST \arrow[dddd, phantom, ""] \arrow[ddd, "\lambda"] \\
TST^2 \arrow[d, "T\lambda T"'] \arrow[r, phantom, "\Swarrow\beta T"] & ST^2 \arrow[ur, "S\mu"'] \arrow[dd, "\lambda T"] \arrow[dr, phantom, "\Swarrow\beta"] & \\
T^2ST \arrow[d, "T^2\lambda"'] \arrow[dr, "\mu ST"] & & \\
T^3S \arrow[dr, "\mu TS"'] \arrow[r, phantom, "\cong"] & TST \arrow[d, "T\lambda"] & TS \\
& T^2S \arrow[ur, "\mu S"'] &
\end{tikzcd}
=
\begin{tikzcd}
ST^3 \arrow[r, "ST\mu"] \arrow[d, "\lambda T^2"'] \arrow[dr, phantom, "\cong"] & ST^2 \arrow[r, "S\mu"] \arrow[d, "\lambda T"] \arrow[dr, phantom, "\Swarrow\beta"] & ST \arrow[ddd, "\lambda"] \\
TST^2 \arrow[r, "TS\mu"'] \arrow[d, "T\lambda T"'] \arrow[dr, phantom, "\Swarrow T\beta"] & TST \arrow[dd, "T\lambda"] & \\
T^2ST \arrow[d, "T^2\lambda"'] & & \\
T^3S \arrow[r, "T\mu S"] \arrow[dr, "\mu TS"'] & T^2S \arrow[r, "\mu S"] \arrow[d, phantom, "\Downarrow\tau S"] & TS \\
& T^2S \arrow[ur, "\mu S"'] &
\end{tikzcd}
$$

8.

$$
\begin{tikzcd}
ST \arrow[rr, "1"] \arrow[dr, "S\eta T"] \arrow[dd, "\lambda"'] \arrow[ddr, "\eta ST"'] \arrow[r, phantom, "\Uparrow S\rho"] & & ST \arrow[dd, "\lambda"] \\
& ST^2 \arrow[ur, "S\mu"'] \arrow[d, "\lambda T"] \arrow[l, phantom, "\delta T\,\Nearrow"] \arrow[r, phantom, "\beta^{-1}\Nearrow"] & \\
TS \arrow[dr, "\eta TS"'] \arrow[r, phantom, "\cong"] & TST \arrow[d, "T\lambda"] & TS \\
& T^2S \arrow[ur, "\mu S"'] &
\end{tikzcd}
=
\begin{tikzcd}
ST \arrow[rr, "1"] \arrow[d, "\lambda"'] \arrow[drr, phantom, "="] & & ST \arrow[d, "\lambda"] \\
TS \arrow[rr, "1"] \arrow[dr, "\eta TS"'] & \arrow[d, phantom, "\rho S\,\Uparrow"] & TS \\
& T^2S \arrow[ur, "\mu S"'] &
\end{tikzcd}
$$

Pseudo-distributive laws between pseudomonads are defined with nine coherence conditions in [Mar99], and the ninth is shown to follow from the others in [MW08]. The conditions are summarized in [Gam09].

Proposition A.8 ([Mar99])**.** *If S is a colax-idempotent pseudomonad and T is a lax-idempotent pseudomonad, then to give a pseudo-distributive law of S over T it suffices to give the 2-cell $\lambda\colon ST \to TS$ and the invertible 3-cell γ, satisfying the conditions*

1. *the 3-cell $\delta =$*

$$
\begin{tikzcd}
 & TS \arrow[r, "1"] \arrow[dr, "\eta TS"] \arrow[d, phantom, "\cong"] & TS \arrow[dr, "T\eta S"'] \arrow[drr, "1"] \arrow[d, phantom, "\cong \Downarrow \gamma S"] \arrow[drr, phantom, "\cong"] & & \\
S \arrow[ur, "\eta S"] \arrow[dr, "S\eta"'] \arrow[r, bend left, "\eta S"] \arrow[r, bend right, "S\eta"'] \arrow[r, phantom, "\Downarrow\nu"] & S^2 \arrow[r, "S\eta S"'] \arrow[d, phantom, "\cong"] & STS \arrow[r, "\lambda S"'] \arrow[d, phantom, "\cong"] & TS^2 \arrow[r, "T\mu"'] \arrow[d, phantom, "\cong"] & TS \\
 & ST \arrow[r, "\lambda"'] \arrow[ur, "ST\eta"'] & TS \arrow[ur, "TS\eta"] \arrow[urr, "1"'] & &
\end{tikzcd}
$$

   is invertible (equivalently, the composite $(T\mu)(\lambda S)(S\eta S)\nu$ is invertible),

2. *the 3-cell $(T\mu)(\lambda S)(S\lambda)(\sigma T)$ is invertible, where $\sigma : 1_{S^2} \Rightarrow (S\eta)\mu$ is the unit of the adjunction $\mu \dashv S\eta$,*

3. *the 3-cell $(\mu S)(T\lambda)(\lambda T)(S\rho)$ is invertible, where $\rho : (T\eta)\mu \Rightarrow 1_{T^2}$ is the counit of the adjunction $T\eta \dashv \mu$,*

4.

$$
\begin{tikzcd}
ST \arrow[rr, "\lambda"] \arrow[d, bend right, "S\eta T"'] \arrow[d, bend left, "\eta ST"] \arrow[d, phantom, "\Leftarrow \nu T"] \arrow[dr, phantom, "\cong"] & & TS \arrow[dl, "\eta TS"'] \arrow[d, "T\eta S"] \arrow[dl, phantom, "\Swarrow \gamma S"] & \\
S^2T \arrow[r, "S\lambda"'] & STS \arrow[r, "\lambda S"'] & TS^2 \arrow[r, "T\mu"'] & TS
\end{tikzcd}
=
\begin{tikzcd}
ST \arrow[rr, "\lambda"] \arrow[d, "S\eta T"'] \arrow[dr, "ST\eta"] \arrow[d, phantom, "\Swarrow S\gamma"] \arrow[drr, phantom, "\cong"] & & TS \arrow[d, bend right, "TS\eta"'] \arrow[d, bend left, "T\eta S"] \arrow[d, phantom, "\Leftarrow T\nu"] & \\
S^2T \arrow[r, "S\lambda"'] & STS \arrow[r, "\lambda S"'] & TS^2 \arrow[r, "T\mu"'] & TS
\end{tikzcd}
$$

5.

$$
\begin{tikzcd}
ST \arrow[rr, "\lambda"] \arrow[d, bend right, "S\eta T"'] \arrow[d, bend left, "ST\eta"] \arrow[d, phantom, "\Leftarrow S\nu"] \arrow[dr, phantom, "\cong"] & & TS \arrow[dl, "TS\eta"'] \arrow[d, "T\eta S"] \arrow[dl, phantom, "\Swarrow T\delta"] & \\
ST^2 \arrow[r, "\lambda T"'] & TST \arrow[r, "T\lambda"'] & T^2S \arrow[r, "\mu S"'] & TS
\end{tikzcd}
=
\begin{tikzcd}
ST \arrow[rr, "\lambda"] \arrow[d, "S\eta T"'] \arrow[dr, "\eta ST"] \arrow[d, phantom, "\Swarrow \delta T"] \arrow[drr, phantom, "\cong"] & & TS \arrow[d, bend right, "\eta TS"'] \arrow[d, bend left, "T\eta S"] \arrow[d, phantom, "\Leftarrow \nu S"] & \\
ST^2 \arrow[r, "\lambda T"'] & TST \arrow[r, "T\lambda"'] & T^2S \arrow[r, "\mu S"'] & TS.
\end{tikzcd}
$$

In this case a pseudo-distributive law is unique up to isomorphism if it exists.

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