# Particle relabelling transformations in elastodynamics

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## SUMMARY

The motion of a self-gravitating hyperelastic body is described through a time-dependent mapping from a reference body into physical space, and its material properties are determined by a referential density and strain-energy function defined relative to the reference body. Points within the reference body do not have a direct physical meaning, but instead act as particle labels that could be assigned in different ways. We use Hamilton's principle to determine how the referential density and strain-energy functions transform when the particle labels are changed, and describe an associated 'particle relabelling symmetry'. We apply these results to linearized elastic wave propagation and discuss their implications for seismological inverse problems. In particular, we show that the effects of boundary topography on elastic wave propagation can be mapped exactly into volumetric heterogeneity while preserving the form of the equations of motion. Several numerical calculations are presented to illustrate our results.

**Key words:** Numerical Solutions; Seismic tomography; Theoretical seismology.

## 1 INTRODUCTION

In this paper, we consider the motion of a self-gravitating elastic body and obtain *particle relabelling transformations* describing how the elastodynamic equations transform when the reference configuration is changed. The term 'particle relabelling transformation' has been borrowed from a somewhat analogous property in Hamiltonian formulations of fluid mechanics (e.g. Salmon 1988; Bokhove & Oliver 2006). This work has largely been motivated by a computational problem in normal-mode seismology, but it also has wider applications in seismology and related fields. Our main result is that the *form* of the elastodynamic equations is invariant under a change of reference configuration, and we obtain formulae showing how the material parameters and initial conditions transform. Within this introduction, we outline our reasons for introducing such transformations and describe their relation to some existing results.

### 1.1 Incorporation of boundary topography into normal-mode coupling calculations

Normal-mode coupling calculations are the standard method for computing long-period seismograms and spectra in laterally heterogeneous earth models; see Dahlen & Tromp (1998) for a comprehensive discussion. In broad terms, these calculations proceed by expanding the wavefield in a laterally heterogeneous earth model in terms of the eigenfunctions of a suitable spherically symmetric earth model, and so reduce the elastodynamic equations to a system of coupled ordinary differential equations for the expansion coefficients. We will say that a laterally heterogeneous earth model is

'geometrically spherical' if its internal and external boundaries are concentric spheres, and when this condition does not hold we say that the model is 'geometrically aspherical'. Due to topography on the Earth's surface and on internal boundaries such as the Moho or the core mantle boundary, it is geometrically aspherical models that are of the greatest practical interest.

If a model is geometrically spherical, then the completeness of the eigenfunctions of spherically symmetric earth models implies that mode coupling theory is exact, and numerical calculations can, in principle, be made as accurate as desired. When, however, we consider geometrically aspherical earth models, the wavefield we wish to expand is not defined in the same domain as the spherical earth eigenfunctions. To date, this difficulty has been overcome through the use of first-order perturbation theory to account for aspherical topography on internal or external boundaries (e.g. Woodhouse 1976; Woodhouse & Dahlen 1978; Woodhouse 1980). It should be noted that this perturbation theory is first-order accurate in the calculation of the so-called 'matrix elements' for the problem, but the wavefield depends nonlinearly on these matrix elements. It follows that there is a nonlinear dependence of the calculated seismograms on boundary topography, though the results are, nonetheless, only guaranteed to be first-order accurate. It is quite possible that this perturbation approach is sufficient for incorporating the effects of boundary topography, but it would, of course, be preferable to have a more accurate theory.

An approach for incorporating boundary topography exactly into mode coupling calculations is suggested by Woodhouse (1976). In this seminal paper, correct expressions for the first-order perturbation to the eigenfrequencies of an earth model due to boundary topography were obtained for the first time, though a later extension

was required to account for complications associated with fluid–solid boundaries (Woodhouse & Dahlen 1978). It had been observed through numerical calculations (Dziewonski & Sailor 1976) that Rayleigh's principle led to the incorrect expression when applied to boundary perturbations (Backus & Gilbert 1967). To remedy this problem, Woodhouse (1976) introduced a mapping which transformed the boundaries of the perturbed earth model onto those of the unperturbed model. The perturbed equations of motion were then transformed into equivalent equations defined on the unperturbed earth model, with these new equations depending on the mapping between the two earth models as a volumetric parameter. Rayleigh's principle could then be applied to the transformed equations of motion to determine the first-order change in the eigenfrequencies.

The key step in the method of Woodhouse (1976) is the mapping which carries the boundaries of a perturbed earth model into those of an unperturbed one. Though in its original application this approach was only used to calculate first-order perturbations, this need not be the case. Consider the elastodynamic equations in a geometrically aspherical earth model. As we noted above, we cannot simply expand the wavefield in the eigenfunctions of a spherically symmetric earth model. However, suppose that, following Woodhouse (1976), we introduce a mapping that carries the boundaries of our aspherical model onto those of a spherical earth model. We could then use this mapping to transform the elastodynamic equations into an equivalent set on the spherically symmetric model, and the solution to these new equations *could* be expanded in terms of the eigenfunctions of the spherically symmetric model. A very similar approach to incorporate boundary topography has been implemented by Takeuchi (2005) in the non-self-gravitating case and in a Cartesian geometry based on the so-called direct solution method (Geller & Ohminato 1994). The main difficulty in applying the above method is transforming the elastodynamic equations between a geometrically aspherical earth model and a spherical one, and this is the problem we wish to address.

## 1.2 Non-uniqueness results for a class of inverse problems

We are interested in transforming the elastodynamic equations defined in a given earth model into an equivalent set of equations defined in a different geometric domain. Similar problems are well known in the applied mathematics literature, and can lead to some surprising results (e.g. Stefanov & Uhlmann 1998a,b; Greenleaf *et al.* 2003). For example, consider the scalar wave equation

$$\rho \partial_t^2 u - \nabla \cdot (\mathbf{a} \cdot \nabla u) = -\nabla \cdot \mathbf{m}, \tag{1}$$

defined on a some smooth bounded domain $M \subseteq \mathbb{R}^n$ with $n \geq 2$ and boundary $\partial M$, subject to initial conditions

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}), \quad (\partial_t u)(\mathbf{x}, 0) = v_0(\mathbf{x}), \tag{2}$$

for $\mathbf{x} \in M$, and boundary conditions

$$\mathbf{n}(\mathbf{x}) \cdot (\nabla u)(\mathbf{x}, t) = 0, \tag{3}$$

for $\mathbf{x} \in \partial M$ and all times $t \geq 0$ with $\mathbf{n}$ the outward unit normal vector. Here $\rho$ is a smooth non-negative function, $\mathbf{a}$ is a smooth symmetric second-order tensor such that $\mathbf{e} \cdot \mathbf{a}(\mathbf{x}) \cdot \mathbf{e} \geq A \|\mathbf{e}\|^2$ for all vectors $\mathbf{e} \in \mathbb{R}^n$ and $\mathbf{x} \in M$ with $A$ some positive constant, and $\mathbf{m}$ is a time-dependent vector field that vanishes on $\partial M$. These equations are scalar analogues for the elastodynamic equations, but have no physical meaning. We note, however, that the form of the force term has been chosen to mimic the stress glut commonly used in seismology (Backus & Mulcahy 1976a,b).

Let $\boldsymbol{\xi} : \tilde{M} \to M$ be a smooth mapping with a smooth inverse, which we recall is known as a *diffeomorphism* (e.g. Abraham *et al.* 1988). If $u$ is a solution to the above wave equation, then we can define a new function on $\tilde{M}$ by setting

$$\tilde{u}(\mathbf{x}, t) := u[\boldsymbol{\xi}(\mathbf{x}), t], \tag{4}$$

for all $\mathbf{x} \in \tilde{M}$. It is natural to ask whether $\tilde{u}$ is also a solution of some equation defined on $\tilde{M}$. This is indeed the case, and we will show that the equation for $\tilde{u}$ takes the form

$$\tilde{\rho} \partial_t^2 \tilde{u} - \nabla \cdot (\tilde{\mathbf{a}} \cdot \nabla \tilde{u}) = -\nabla \cdot \tilde{\mathbf{m}}, \tag{5}$$

subject to initial conditions

$$\tilde{u}(\mathbf{x}, 0) = \tilde{u}_0(\mathbf{x}), \quad (\partial_t \tilde{u})(\mathbf{x}, 0) = \tilde{v}_0(\mathbf{x}), \tag{6}$$

for all $\mathbf{x} \in \tilde{M}$, and boundary conditions

$$\mathbf{n}(\mathbf{x}) \cdot (\nabla \tilde{u})(\mathbf{x}, t) = 0, \tag{7}$$

for all $\mathbf{x} \in \partial \tilde{M}$ and all times $t \geq 0$, where $\mathbf{n}$ is now the outward unit normal vector to $\partial \tilde{M}$. In this transformed wave equation, it is immediately clear that the initial conditions are given by

$$\tilde{u}_0(\mathbf{x}) = u_0[\boldsymbol{\xi}(\mathbf{x})], \quad \tilde{v}_0(\mathbf{x}) = v_0[\boldsymbol{\xi}(\mathbf{x})], \tag{8}$$

for all $\mathbf{x} \in \tilde{M}$. Expressions for the parameters $\tilde{\rho}$, $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{m}}$ are obtained below, and will be seen to depend on the original parameters $\rho$, $\mathbf{a}$ and $\mathbf{m}$ along with the diffeomorphism $\boldsymbol{\xi}$. It is notable that the equation satisfied by $\tilde{u}$ is of *exactly the same form* as the original wave equation. This 'form invariance' of a partial differential equation under such a transformation does not trivially hold. For example, were we to consider instead the rather similar wave equation

$$\rho \partial_t^2 u - \mathbf{a} : \nabla \nabla u = -\nabla \cdot \mathbf{m}, \tag{9}$$

then we would find that its form is not retained under the transformation given in eq. (4).

The equation satisfied by $\tilde{u}$ can be obtained in several ways. The most elementary is through a direct but lengthy calculation in local coordinates. A quicker and more illuminating derivation is possible using the calculus of differential forms, and depends crucially on the commutativity of the exterior derivative with the pull-back operation (e.g. Abraham *et al.* 1988), but such techniques are not very familiar within geophysics. Here we shall obtain the transformed wave equation in an elementary manner starting from a variational formulation of the wave equation; an analogous approach will be used later when transforming the elastodynamic equations. It is readily verified that the wave equation for $u$ can be obtained from Hamilton's principle for the action

$$\mathcal{S}(u) := \int_0^T \int_M \left\{ \frac{1}{2} \rho(\mathbf{x}) [(\partial_t u)(\mathbf{x}, t)]^2 - \frac{1}{2} (\nabla u)(\mathbf{x}, t) \cdot \mathbf{a}(\mathbf{x}) \right.$$
$$\left. \cdot (\nabla u)(\mathbf{x}, t) + (\nabla u)(\mathbf{x}, t) \cdot \mathbf{m}(\mathbf{x}, t) \right\} d^n \mathbf{x}, \tag{10}$$

where $[0, T]$ is some time interval of interest, and the admissible variations in $u$ satisfy

$$\delta u(\mathbf{x}, 0) = \delta u(\mathbf{x}, T) = 0. \tag{11}$$

The integration in eq. (10) can be transformed using $\boldsymbol{\xi} : \tilde{M} \to M$, and we obtain

$$\mathcal{S}(u) := \int_0^T \int_{\tilde{M}} \left\{ \frac{1}{2} \tilde{\rho}(\mathbf{x})[(\partial_t \tilde{u})(\mathbf{x}, t)]^2 - \frac{1}{2} (\nabla \tilde{u})(\mathbf{x}, t) \cdot \tilde{\mathbf{a}}(\mathbf{x}) \right.$$

$$\left. \cdot (\nabla \tilde{u})(\mathbf{x}, t) + (\nabla \tilde{u})(\mathbf{x}, t) \cdot \tilde{\mathbf{m}}(\mathbf{x}, t) \right\} \mathrm{d}^n \mathbf{x}, \tag{12}$$

where we have used the identity

$$(\nabla \tilde{u})(\mathbf{x}, t) = \mathbf{F}_\xi(\mathbf{x})^T \cdot (\nabla u)[\boldsymbol{\xi}(\mathbf{x}, t)], \tag{13}$$

with $\mathbf{F}_\xi = (\nabla \boldsymbol{\xi})^T$, and have defined the new parameters

$$\tilde{\rho}(\mathbf{x}) = J_\xi(\mathbf{x}) \rho[\boldsymbol{\xi}(\mathbf{x})], \tag{14}$$

$$\tilde{\mathbf{a}}(\mathbf{x}) = J_\xi(\mathbf{x}) \mathbf{F}_\xi(\mathbf{x})^{-1} \mathbf{a}[\boldsymbol{\xi}(\mathbf{x})] \mathbf{F}_\xi(\mathbf{x})^{-T}, \tag{15}$$

$$\tilde{\mathbf{m}}(\mathbf{x}, t) = J_\xi(\mathbf{x}) \mathbf{F}_\xi(\mathbf{x})^{-1} \mathbf{m}[\boldsymbol{\xi}(\mathbf{x}), t], \tag{16}$$

where $J_\xi(\mathbf{x}) = \det[\mathbf{F}_\xi(\mathbf{x})]$. Let us define a new action by

$$\tilde{\mathcal{S}}(\tilde{u}) := \int_0^T \int_{\tilde{M}} \left\{ \frac{1}{2} \tilde{\rho}(\mathbf{x})[(\partial_t \tilde{u})(\mathbf{x}, t)]^2 - \frac{1}{2} (\nabla \tilde{u})(\mathbf{x}, t) \cdot \tilde{\mathbf{a}}(\mathbf{x}) \right.$$

$$\left. \cdot (\nabla \tilde{u})(\mathbf{x}, t) + (\nabla \tilde{u})(\mathbf{x}, t) \cdot \tilde{\mathbf{m}}(\mathbf{x}, t) \right\} \mathrm{d}^n \mathbf{x}, \tag{17}$$

It follows that we have $\mathcal{S}(u) = \tilde{\mathcal{S}}(\tilde{u})$ whenever $u$ and $\tilde{u}$ are related through eq. (4), and we can conclude that $u$ will be a solution of the original wave equation if and only if $\tilde{u}$ is a solution of eq. (5) with the parameters given by eqs (14)–(16).

Invariance of the form of the scalar wave equation under such transformations is an elegant and non-trivial result. But it is when we consider inverse problems for this equation that things become interesting. Suppose that we have error-free observations of $u$ everywhere on $\partial M$ over some arbitrary time interval [0, $T$]. We can then regard the wave equation as implicitly defining a mapping

$$(\rho, \mathbf{a}, u_0, v_0, \mathbf{m}) \mapsto u|_{\partial M \times [0, T]}, \tag{18}$$

and can formulate an inverse problem to recover the model parameters $(\rho, \mathbf{a}, u_0, v_0, \mathbf{m})$. We now ask whether or not this inverse problem admits a unique solution. To proceed, we consider a diffeomorphism $\boldsymbol{\xi}$ from $M$ onto itself, and such that $\boldsymbol{\xi}(\mathbf{x}) = \mathbf{x}$ at all boundary points. Using a special case of the transformation defined in eq. (4) we define a new wavefield $\tilde{u}$ and note that by construction we have

$$\tilde{u}(\mathbf{x}, t) = u(\mathbf{x}, t), \tag{19}$$

for all $\mathbf{x} \in \partial M$ and $t \in [0, T]$. However, we have seen that $\tilde{u}$ is a solution of a wave equation of exactly the same form as that for $u$, but with model parameters $(\tilde{\rho}, \tilde{\mathbf{a}}, \tilde{u}_0, \tilde{v}_0, \tilde{\mathbf{m}})$ that are related to $(\rho, \mathbf{a}, u_0, v_0, \mathbf{m})$ through eqs (8) and (14)–(16). We have, therefore, constructed two sets of model parameters that lead to exactly the same surface observations, and can conclude that the given inverse problem does not have a unique solution. In fact, it is easy to see that there are infinitely many diffeomorphism from $M$ onto itself that leave the boundary points fixed, and so there is actually an infinite-fold non-uniqueness for this inverse problem (*cf.* Stefanov & Uhlmann 1998b).

An interesting feature of the above transformations is that an initially isotropic model (i.e. $\mathbf{a}$ is proportional to the identity tensor) will be transformed by a generic diffeomorphism into an anisotropic

one; were we to restrict attention to isotropic tensors, then this non-uniqueness result would not hold. Furthermore, we note that it is not only the 'material parameters' $(\rho, \mathbf{a})$ of the equation that are being transformed in the above method but also the initial conditions $(u_0, v_0)$ and the source term $\mathbf{m}$. Suppose, for example, that we know the source term $\mathbf{m}$ exactly (as is, essentially, the case in some exploration seismic problems). The non-uniqueness result above must then be modified by requiring that $\boldsymbol{\xi}$ equals the identity on both the boundary $\partial M$ and within the support of $\mathbf{m}$ in $M$ (i.e. the subset where the function is non-zero). So long as the support of $\mathbf{m}$ is not the whole of $M$ there will exist non-trivial diffeomorphisms satisfying these conditions, and it follows that we can again produce infinitely many different sets of model parameters that are mapped onto the given surface observations.

Similar methods have led to non-uniqueness results for other partial differential equations, including some with a physical basis such as the heat diffusion equation (e.g. Greenleaf *et al.* 2003) and Maxwell's equations (e.g. Pendry *et al.* 2006; Rahm *et al.* 2008). Of particular relevance to geophysics is the work of Mazzucato & Rachele (2006), who establish non-uniqueness results for both finite and linearized elasticity. Though the situation considered by these authors (the so-called Dirichlet-to-Neumann map) is not directly relevant to surface observations in seismology, it is clear that their methods could be extended in a suitable manner. At face value, such results imply that there exists an infinite-fold non-uniqueness for seismic inverse problems even with error-free data everywhere on the Earth's surface, and this clearly has significant and worrying implications for seismic tomography. The work of Mazzucato & Rachele (2006) is based on a covariant formulation of elasticity due to Marsden & Hughes (1983), but involves some lengthy calculations in local coordinates. In this paper we re-derive and extend their results using elementary methods. In particular, we incorporate seismic sources into the problem, and account for the effects of self-gravitation. We also indicate how these methods can be applied to viscoelastic materials. Importantly, however, we provide a simple explanation for the non-uniqueness in elastodynamic inverse problems, and show that while this represents a genuine mathematical result, it does not have direct physical consequences.

## 2 HAMILTON'S PRINCIPLE IN FINITE ELASTICITY

In this section, we obtain the equations of motion for a self-gravitating hyperelastic body from Hamilton's principle of stationary action. In most seismological applications it is sufficient to consider linearized motions of a body about an equilibrium state. It will, however, be conceptually simpler first to discuss finite elasticity, and then to reduce our results to the linearized case. Elasticity is most precisely and elegantly described using the language of modern differential geometry (e.g. Marsden & Hughes 1983). For our purposes, however, elementary methods will suffice, though this means we do, regrettably, lose some geometric insight. Comprehensive discussions of elasticity can be found in many places, but we mention in particular Truesdell & Noll (2004) and Antman (2005). Dahlen & Tromp (1998) and Woodhouse & Deuss (2007) also provide useful discussions from a geophysical perspective, including topics such as self-gravitation, rotating reference frames and fluid–solid interfaces. Though the material in this section is standard, we provide a fairly self-contained overview to establish notations, and recall necessary results.

## 2.1 Kinematics

We consider the motion of an *n*-dimensional body, which for the moment need not be elastic. All physical application will, of course, be in 3-D space, but the dimension will play no essential role in the theory, and we simply assume $n \geq 2$. On the most basic level, the body is represented by an abstract set that can be mapped continuously and with continuous inverse onto open subsets of Euclidean space $\mathbb{R}^n$ (e.g. Noll 1974; Antman 2005). Each such mapping defines a *configuration* of the body, and a *motion* is a time-dependent family of configurations. Elements of the body will be called *particles*, this terminology generalizing that used in classical mechanics, and is not, of course, meant to convey any links to atomic physics.

To make the description of the body more concrete, it will be necessary to select a *reference configuration* that maps the body into a subset $M \subseteq \mathbb{R}^n$ called the *reference body*. The reference configuration is often taken to be equal to the actual configuration of the body at some initial time, but this choice is not necessary and can sometimes be inconvenient. Having fixed a reference configuration, we can *label* the particles of the body by their position vectors in $M$, and describe their motion relative to the reference body. We shall write $\mathbf{x} \in M$ for such particle labels, and assume they are defined relative to a fixed Cartesian coordinate system on $\mathbb{R}^n$. The components of all vector and tensor fields introduced below will also be defined relative to this coordinate system. For simplicity we shall assume that $M$ is a connected and bounded subset of $\mathbb{R}^n$. The boundary of $M$ is denoted by $\partial M$, and $\mathbf{n}$ is the outward unit normal vector. It would be trivial to allow for internal surfaces across which material parameters are discontinuous so long as we assume that the motion is itself continuous. The inclusion of fluid–solid interfaces (e.g. Woodhouse & Dahlen 1978) into the model is more involved, and will not be considered here .

Over a time interval $I = [0, T] \subseteq \mathbb{R}$, the motion of the body relative to $M$ is given by a mapping

$$\boldsymbol{\varphi} : M \times I \to \mathbb{R}^n, \tag{20}$$

which, at time $t \in I$, takes the particle with label $\mathbf{x} \in M$ to the point $\boldsymbol{\varphi}(\mathbf{x}, t) \in \mathbb{R}^n$. Setting $\boldsymbol{\varphi}_t(\mathbf{x}) := \boldsymbol{\varphi}(\mathbf{x}, t)$, we see that

$$M_t := \boldsymbol{\varphi}_t(M) \subseteq \mathbb{R}^n, \tag{21}$$

is the subset of physical space instantaneously occupied by the body at time $t \in I$. We say that $\boldsymbol{\varphi}_t : M \to \mathbb{R}^n$ is the *configuration of the body relative to M* at this time, and assume that the mapping $\boldsymbol{\varphi}_t$ from $M$ onto $M_t$ is a diffeomorphism, which is to say it is smooth and has a smooth inverse. The *initial configuration* of the body relative to $M$ is the mapping $\mathbf{x} \mapsto \boldsymbol{\varphi}(\mathbf{x}, 0)$, and we set $\boldsymbol{\varphi}_0(\mathbf{x}) = \boldsymbol{\varphi}(\mathbf{x}, 0)$.

The velocity of the motion is defined as

$$\mathbf{v}(\mathbf{x}, t) := (\partial_t \boldsymbol{\varphi})(\mathbf{x}, t), \tag{22}$$

where $\partial_t$ denotes partial differentiation with respect to time. We write $\mathbf{v}_t(\mathbf{x}) := \mathbf{v}(\mathbf{x}, t)$, and can think of $\mathbf{v}_t$ as a vector field on $M$. More correctly, $\mathbf{v}_t$ is a vector field defined *along* the mapping $\boldsymbol{\varphi}_t : M \to \mathbb{R}^n$ (e.g. Marsden & Hughes 1983), but this distinction is not important when using Cartesian coordinates. Consider a curve $s \mapsto \boldsymbol{\gamma}(s) \in M$ defined for $s \in \mathbb{R}$ in some neighbourhood of zero. The tangent vector to this curve at the point $\mathbf{x} = \boldsymbol{\gamma}(0)$ is simply $\boldsymbol{\gamma}'(0)$, where a prime denotes ordinary differentiation. We can also form the image $s \mapsto (\boldsymbol{\varphi}_t \circ \boldsymbol{\gamma})(s)$ of this curve under the configuration $\boldsymbol{\varphi}_t$, which is seen to pass through the point $\mathbf{y} = \boldsymbol{\varphi}_t(\mathbf{x})$. Using the chain rule, the tangent vector of this new curve at the point $\mathbf{y}$ is given by

$$(\boldsymbol{\varphi}_t \circ \boldsymbol{\gamma})'(0) = \boldsymbol{\gamma}'(0) \cdot (\nabla \boldsymbol{\varphi}_t)(\mathbf{x}) := \mathbf{F}(\mathbf{x}, t) \cdot \boldsymbol{\gamma}'(0), \tag{23}$$

where we have introduced the *deformation gradient* $\mathbf{F} := (\nabla \boldsymbol{\varphi})^T$ of the motion. As we have assumed that each configuration $\boldsymbol{\varphi}_t$ is a diffeomorphism, it follows from the inverse function theorem (e.g. Abraham *et al.* 1988) that $\mathbf{F}_t(\mathbf{x}) := \mathbf{F}(\mathbf{x}, t)$ is an invertible linear mapping at each point $\mathbf{x} \in M$, and so belongs to the general linear group, $\mathbf{GL}(n)$. More formally, $\mathbf{F}_t$ is the differential of $\boldsymbol{\varphi}_t$, and maps tangent vectors in $M$ with base-point $\mathbf{x}$ into tangent vectors in $\mathbb{R}^n$ with base point $\boldsymbol{\varphi}_t(\mathbf{x})$ (e.g. Marsden & Hughes 1983), but due to our use of global Cartesian coordinates, this 'two-point' nature of the deformation gradient can be left implicit.

We now recall the *polar decomposition* of the deformation gradient (e.g. Truesdell & Noll 2004). The *right Cauchy–Green deformation tensor* is defined by

$$\mathbf{C} := \mathbf{F}^T \mathbf{F}, \tag{24}$$

and is clearly symmetric and positive-definite. From the spectral theorem (e.g. Lax 2002) we can define a positive square root $\mathbf{U} := \sqrt{\mathbf{C}}$ which is again symmetric, and we claim that

$$\mathbf{F} = \mathbf{RU}, \tag{25}$$

where $\mathbf{R}$ is an element of the orthogonal group, $\mathbf{O}(n)$. In fact, from eq. (25) it is clear that $\mathbf{R} := \mathbf{FU}^{-1}$ is well-defined, and we see trivially that $\mathbf{R}^T = \mathbf{R}^{-1}$ as required. We can also introduce the *left Cauchy–Green deformation tensor* $\mathbf{B} := \mathbf{FF}^T$, and through a similar argument obtain $\mathbf{F} = \mathbf{VR}$ where we have defined $\mathbf{V} := \sqrt{\mathbf{B}}$.

## 2.2 Mass conservation and gravitational potential

At time $t \in I$, the particles within an *n*-dimensional subset $U \subseteq M$ occupy the region $U_t := \boldsymbol{\varphi}_t(U)$ of physical space, and the mass of this sub-body can be written

$$m(U, t) := \int_{U_t} \varrho(\mathbf{y}, t) \, \mathrm{d}^n \mathbf{y}, \tag{26}$$

with $\varrho(\,\cdot\,, t)$ the instantaneous *spatial density*. Conservation of mass requires that $m(U, t)$ is independent of time, and by using $\boldsymbol{\varphi}_t$ to transform the integration in eq. (26) onto $U$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_U J(\mathbf{x}, t) \varrho[\boldsymbol{\varphi}(\mathbf{x}, t), t] \, \mathrm{d}^n \mathbf{x} = 0, \tag{27}$$

where the *Jacobian* of the motion is defined by

$$J(\mathbf{x}, t) := \det[\mathbf{F}(\mathbf{x}, t)]. \tag{28}$$

As the subset $U$ is arbitrary, it follows that there exists a *referential density* $\rho$ such that for all times $t \in I$ we have

$$\rho(\mathbf{x}) = J(\mathbf{x}, t) \varrho[\boldsymbol{\varphi}(\mathbf{x}, t), t]. \tag{29}$$

Given the spatial density $\varrho$ at a given time $t \in I$, the *gravitational potential* of the body can be written

$$\phi(\mathbf{y}, t) = \int_{M_t} \varrho(\mathbf{y}', t) \Gamma_n(\mathbf{y} - \mathbf{y}') \, \mathrm{d}^n \mathbf{y}', \tag{30}$$

where $\Gamma_n$ is the Newtonian potential in *n*-dimensional space

$$\Gamma_n(\mathbf{y}) = 4\pi G \times \begin{cases} \frac{1}{2\pi} \ln \|\mathbf{y}\| & n = 2 \\ \frac{1}{n(2-n)\omega_n} \|\mathbf{y}\|^{2-n} & n > 2 \end{cases}, \tag{31}$$

with $\omega_n$ the volume of the *n*-ball (e.g. Evans 1998), and the *gravitational acceleration* of the body is defined as

$$\mathbf{g}(\mathbf{y}, t) := -(\nabla \phi)(\mathbf{y}, t) = -\int_{M_t} \varrho(\mathbf{y}', t)(\nabla \Gamma_n)(\mathbf{y} - \mathbf{y}') \, \mathrm{d}^n \mathbf{y}'. \tag{32}$$

The gravitational potential can also be defined through the solution of Poisson's equation

$$\nabla^2 \phi = 4\pi G \varrho, \tag{33}$$

subject to appropriate boundary conditions (e.g. Dahlen & Tromp 1998), but for our purposes it is more convenient to regard $\phi$ as a functional of the motion and not an independent field. We also introduce the *referential gravitational potential* by

$$\zeta(\mathbf{x}, t) := \phi[\boldsymbol{\varphi}(\mathbf{x}, t), t], \tag{34}$$

and from eqs (29) and (30) obtain the explicit dependence of $\zeta$ on the motion

$$\zeta(\mathbf{x}, t) = \int_M \rho(\mathbf{x}') \Gamma_n[\boldsymbol{\varphi}(\mathbf{x}, t) - \boldsymbol{\varphi}(\mathbf{x}', t)] \, d^n \mathbf{x}'. \tag{35}$$

Similarly, we define the *referential gravitational acceleration* to be

$$\boldsymbol{\gamma}(\mathbf{x}, t) := \mathbf{g}[\boldsymbol{\varphi}(\mathbf{x}, t), t]$$
$$= -\int_M \rho(\mathbf{x}')(\nabla\Gamma_n)[\boldsymbol{\varphi}(\mathbf{x}, t) - \boldsymbol{\varphi}(\mathbf{x}', t)] \, d^n \mathbf{x}', \tag{36}$$

where we have again used eq. (29) to transform the domain of integration. Note here that the expression $(\nabla\Gamma_n)[\boldsymbol{\varphi}(\mathbf{x}, t) - \boldsymbol{\varphi}(\mathbf{x}', t)]$ denotes the gradient of the function $\Gamma_n(\mathbf{y})$ with respect to its argument evaluated at the point $\boldsymbol{\varphi}(\mathbf{x}, t) - \boldsymbol{\varphi}(\mathbf{x}', t)$. Finally, the total gravitational binding energy associated with the body at a given time is equal to

$$\frac{1}{2} \int_{M_t} \varrho(\mathbf{y}, t) \phi(\mathbf{y}, t) \, d^n \mathbf{y} = \frac{1}{2} \int_M \rho(\mathbf{x}) \zeta(\mathbf{x}, t) \, d^n \mathbf{x}, \tag{37}$$

where we have used eqs (29) and (34) in obtaining the second expression (e.g. Landau & Lifshitz 1975, chap. 12; Dahlen & Tromp 1998, section 2.9).

## 2.3 Constitutive relations

For a *simple elastic body* the constitutive relation takes the form

$$\mathbf{T}(\mathbf{x}, t) = \mathfrak{T}[\mathbf{x}, \mathbf{F}(\mathbf{x}, t)], \tag{38}$$

where $\mathbf{T}$ is the *first Piola–Kirchhoff stress tensor*, and $\mathfrak{T}$ is an appropriate *constitutive function* (e.g. Truesdell & Noll 2004). We recall that the first Piola–Kirchhoff stress tensor relates the normal vector of a surface within the reference body to the traction on the corresponding deformed surface in physical space. In the case of a hyperelastic body, there is a *strain-energy function* $(\mathbf{x}, \mathbf{F}) \mapsto W(\mathbf{x}, \mathbf{F})$ such that

$$\mathfrak{T}(\mathbf{x}, \mathbf{F}) = \frac{\partial W}{\partial \mathbf{F}}(\mathbf{x}, \mathbf{F}). \tag{39}$$

The constitutive functions in eqs (38) and (39) are already invariant with respect to superimposed rigid translations, and to changes in the origin time. The principle of *material frame indifference* further requires that the stress transforms correctly under superimposed rigid rotations. This means that for any $\mathbf{F} \in \mathbf{GL}(n)$ and all $\mathbf{Q} \in \mathbf{O}(n)$ we have

$$\mathfrak{T}(\mathbf{x}, \mathbf{QF}) = \mathbf{Q}\mathfrak{T}(\mathbf{x}, \mathbf{F}). \tag{40}$$

From the polar decomposition theorem, it follows that the constitutive function can be written

$$\mathfrak{T}(\mathbf{x}, \mathbf{F}) = \mathbf{R}\mathfrak{T}(\mathbf{x}, \mathbf{U}) = \mathbf{F}\mathbf{U}^{-1}\mathfrak{T}(\mathbf{x}, \mathbf{U}) := \mathbf{F}\mathfrak{S}(\mathbf{x}, \mathbf{C}), \tag{41}$$

where we recall that $\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^2$, and $(\mathbf{x}, \mathbf{C}) \mapsto \mathfrak{S}(\mathbf{x}, \mathbf{C})$ returns the value of the *second Piola–Kirchhoff stress tensor*. In the case of hyperelasticity, eq. (40) is seen to be equivalent to the condition

$$W(\mathbf{x}, \mathbf{QF}) = W(\mathbf{x}, \mathbf{F}), \tag{42}$$

for all $\mathbf{F} \in \mathbf{GL}(n)$ and $\mathbf{Q} \in \mathbf{O}(n)$. It follows that the strain energy function can be written

$$W(\mathbf{x}, \mathbf{F}) = U(\mathbf{x}, \mathbf{C}), \tag{43}$$

for some function $(\mathbf{x}, \mathbf{C}) \mapsto U(\mathbf{x}, \mathbf{C})$, and we obtain

$$\mathfrak{T}(\mathbf{x}, \mathbf{F}) = 2\mathbf{F}\frac{\partial U}{\partial \mathbf{C}}(\mathbf{x}, \mathbf{C}). \tag{44}$$

From comparison of eqs (41) and (44), we see that for a hyperelastic material

$$\mathfrak{S}(\mathbf{x}, \mathbf{C}) = 2\frac{\partial U}{\partial \mathbf{C}}(\mathbf{x}, \mathbf{C}). \tag{45}$$

We note that the symmetry of the second Piola–Kirchhoff stress tensor implied by eq. (45) actually holds in more general materials due to conservation of angular momentum (e.g. Truesdell & Noll 2004).

At a fixed point $\mathbf{x} \in M$, the *symmetry group* of a strain-energy function comprises the subgroup of $\mathbf{Q} \in \mathbf{O}(n)$ for which

$$W(\mathbf{x}, \mathbf{FQ}) = W(\mathbf{x}, \mathbf{F}), \tag{46}$$

for all $\mathbf{F} \in \mathbf{GL}(n)$. In particular, a strain-energy function is said to be *isotropic* at a given point $\mathbf{x} \in M$ if the symmetry group is the whole of the orthogonal group. From eq. (43), it follows that for an isotropic material

$$W(\mathbf{x}, \mathbf{F}) = U(\mathbf{x}, \mathbf{Q}^T\mathbf{CQ}), \tag{47}$$

for all $\mathbf{Q} \in \mathbf{O}(n)$ and $\mathbf{F} \in \mathbf{GL}(n)$, and so the strain-energy function can only depend on the eigenvalues (equivalently, the scalar invariants) of the right Cauchy–Green deformation tensor (e.g. Truesdell & Noll 2004).

## 2.4 Inclusion of internal body forces

The most general and elegant method for incorporating internal body forces into elastodynamics is through the *stress glut* described by Backus & Mulcahy (1976a,b), and here we describe an equivalent approach applicable to hyperelastic materials. To do so, we remove the constraint that the strain-energy function has no explicit dependence on time, and introduce a *time-dependent strain-energy function*

$$(\mathbf{x}, t, \mathbf{F}) \mapsto W(\mathbf{x}, t, \mathbf{F}). \tag{48}$$

This strain-energy function is invariant with respect to superimposed rigid translations, and we again require its invariance with respect to superimposed rigid rotations, meaning that

$$W(\mathbf{x}, t, \mathbf{QF}) = W(\mathbf{x}, t, \mathbf{F}), \tag{49}$$

for all $\mathbf{Q} \in \mathbf{O}(n)$ and $\mathbf{F} \in \mathbf{GL}(n)$. The time-dependence of $W$ provides a simple phenomenological description for energy being added or removed from the body due to unmodelled processes such as rupture on a fault plane. As motivation, we consider the strain-energy function

$$W(\mathbf{x}, t, \mathbf{F}) = U(\mathbf{x}, \mathbf{C}) - \frac{1}{2}\mathbf{S}_g(\mathbf{x}, t) : \mathbf{C}, \tag{50}$$

with $(\mathbf{x}, t) \mapsto \mathbf{S}_g(\mathbf{x}, t)$ a time-dependent symmetric second-order tensor field. Using eq. (44), the constitutive function is given by

$$\mathfrak{T}(\mathbf{x}, t, \mathbf{F}) = \mathbf{F}\left[2\frac{\partial U}{\partial \mathbf{C}}(\mathbf{x}, \mathbf{C}) - \mathbf{S}_g(\mathbf{x}, t)\right], \tag{51}$$

where the first term on the right-hand side is the first Piola–Kirchhoff stress tensor derived from the strain-energy function $U(\mathbf{x}, \mathbf{C})$, while the second term can be interpreted as a stress glut.

## 2.5 Equations of motion

For any motion $\boldsymbol{\varphi} : M \times I \to \mathbb{R}^n$, we define its *action* to be

$$\mathcal{S}(\boldsymbol{\varphi}) := \int_I \int_M \left\{ \frac{1}{2}\rho(\mathbf{x})\|\mathbf{v}(\mathbf{x}, t)\|^2 - W[\mathbf{x}, t, \mathbf{F}(\mathbf{x}, t)] \right.$$
$$\left. - \frac{1}{2}\rho(\mathbf{x})\zeta(\mathbf{x}, t) \right\} d^n\mathbf{x}\,dt, \tag{52}$$

where $\rho$ is the referential density, $\mathbf{v}$ is the velocity, $W$ is a time-dependent strain energy function, $\mathbf{F}$ is the deformation gradient and $\zeta$ the referential gravitational potential. The first term in the action represents the kinetic energy of the body, while (minus) the second and third are the potential energies associated with elastic and gravitational forces, respectively. Let $\boldsymbol{\varphi}_0$ be a given initial configuration, $\mathbf{v}_0$ an initial velocity, and consider the set of motions satisfying these initial conditions. Hamilton's principle requires that the motion $\boldsymbol{\varphi}$ of the body is a stationary point of the action with respect to all admissible variations. To make this precise, we consider perturbed motions $\boldsymbol{\varphi} + \delta\boldsymbol{\varphi}$ such that

$$\delta\boldsymbol{\varphi}(\mathbf{x}, 0) = \mathbf{0}, \quad \delta\boldsymbol{\varphi}(\mathbf{x}, T) = \mathbf{0}. \tag{53}$$

The first condition in eq. (53) ensures that all the motions have the same initial configuration, while the need for the latter condition will be seen below. Expanding the action to first order such that

$$\mathcal{S}(\boldsymbol{\varphi} + \delta\boldsymbol{\varphi}) = \mathcal{S}(\boldsymbol{\varphi}) + \langle\delta\mathcal{S}(\boldsymbol{\varphi}), \delta\boldsymbol{\varphi}\rangle + O(\|\delta\boldsymbol{\varphi}\|^2), \tag{54}$$

with $\langle\,\cdot\,,\cdot\,\rangle$ and $\|\cdot\|$ a suitable duality product and norm, respectively, we see that its first variation can be written

$$\langle\delta\mathcal{S}(\boldsymbol{\varphi}), \delta\boldsymbol{\varphi}\rangle = \int_I \int_M \left\{ \rho(\mathbf{x})\mathbf{v}(\mathbf{x}, t) \cdot \delta\mathbf{v}(\mathbf{x}, t) \right.$$
$$- \frac{\partial W}{\partial \mathbf{F}}[\mathbf{x}, t, \mathbf{F}(\mathbf{x}, t)] : \delta\mathbf{F}(\mathbf{x}, t)$$
$$\left. - \frac{1}{2}\rho(\mathbf{x})\delta\zeta(\mathbf{x}, t) \right\} d^n\mathbf{x}\,dt, \tag{55}$$

where $\delta\mathbf{v} := \partial_t\delta\boldsymbol{\varphi}$ and $\delta\mathbf{F} := (\nabla\delta\boldsymbol{\varphi})^T$, while from eq. (35) we have

$$\delta\zeta(\mathbf{x}, t) = \int_M \rho(\mathbf{x}')[\delta\boldsymbol{\varphi}(\mathbf{x}, t) - \delta\boldsymbol{\varphi}(\mathbf{x}', t)]$$
$$\cdot (\nabla\Gamma_n)[\boldsymbol{\varphi}(\mathbf{x}, t) - \boldsymbol{\varphi}(\mathbf{x}', t)]\,d^n\mathbf{x}'. \tag{56}$$

In the second term of the integrand of eq. (55), we identify

$$\mathbf{T}(\mathbf{x}, t) = \frac{\partial W}{\partial \mathbf{F}}[\mathbf{x}, t, \mathbf{F}(\mathbf{x}, t)], \tag{57}$$

as the first Piola–Kirchhoff stress tensor for the motion. Integrating by parts and using eq. (36), the first variation becomes

$$\langle\delta\mathcal{S}(\boldsymbol{\varphi}), \delta\boldsymbol{\varphi}\rangle = -\int_I \int_M [\rho\,\partial_t\mathbf{v} - \mathrm{Div}\mathbf{T} - \rho\boldsymbol{\gamma}] \cdot \delta\boldsymbol{\varphi}\,d^n\mathbf{x}\,dt$$
$$+ \int_M \rho\,[\mathbf{v}\cdot\delta\boldsymbol{\varphi}]_0^T\,d^n\mathbf{x} - \int_I \int_{\partial M} \delta\boldsymbol{\varphi}\cdot\mathbf{T}\cdot\mathbf{n}\,d\Sigma\,dt, \tag{58}$$

where we recall that $\boldsymbol{\gamma}$ is the referential gravitational acceleration, we have suppressed arguments of the various terms for clarity, $d\Sigma$ denotes the $(n-1)$-dimensional Euclidean surface element, and we write

$$(\mathrm{Div}\mathbf{T})_i := \partial_j T_{ij}, \tag{59}$$

for the divergence of the non-symmetric first Piola–Kirchhoff stress tensor. Using the initial and terminal conditions on the perturbation $\delta\boldsymbol{\varphi}$, the second integral on the right-hand side of eq. (58) is equal to zero. It follows that $\delta\mathcal{S}(\boldsymbol{\varphi})$ vanishes for any admissible variation if and only if the following Euler–Lagrange equation holds

$$\rho\,\partial_t\mathbf{v} - \mathrm{Div}\mathbf{T} - \rho\boldsymbol{\gamma} = \mathbf{0}, \tag{60}$$

which is subject to the boundary condition

$$\mathbf{T}(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x}) = \mathbf{0}, \tag{61}$$

for all $\mathbf{x} \in \partial M$ and $t \in I$, and the initial conditions

$$\boldsymbol{\varphi}(\mathbf{x}, 0) = \boldsymbol{\varphi}_0(\mathbf{x}), \quad \mathbf{v}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x}), \tag{62}$$

for all $\mathbf{x} \in M$. These are, of course, just the familiar equations of motion for a hyperelastic material. Note in particular that these equations can incorporate a stress glut through the time dependence of the strain-energy function. The above variational principle could be extended by allowing the motion to be specified on part or all of the outer boundary, but such boundary conditions are not typically required for seismological applications.

## 3 INVARIANCE OF THE EQUATIONS OF MOTION UNDER PARTICLE RELABELLING

### 3.1 Particle relabelling transformations

We have obtained the equations of motion for a hyperelastic body in a simple manner from Hamilton's principle. In doing so, it was necessary to label the particles of the body through their position vectors in some reference configuration, and it is on the associated reference body $M \subseteq \mathbb{R}^n$ that the material parameters $\rho$ and $W$, the initial conditions, and the equations of motion are all defined. Clearly there is no distinguished reference configuration, and so the *form* of the equations of motion should not depend on the one selected. Had we instead used a different reference configuration, the equations of motion would be defined on some other reference body $\tilde{M} \subseteq \mathbb{R}^n$, say, and on this reference body there would be associated material parameters $\tilde{\rho}$ and $\tilde{W}$, along with initial conditions $\tilde{\boldsymbol{\varphi}}_0$ and $\tilde{\mathbf{v}}_0$. A given point $\tilde{\mathbf{x}} \in \tilde{M}$ labels a unique particle in the body, and within the original reference body $M$ this particle will have label

$$\mathbf{x} = \boldsymbol{\xi}(\tilde{\mathbf{x}}), \tag{63}$$

where $\boldsymbol{\xi} : \tilde{M} \to M$ is some diffeomorphism. We will say that each such diffeomorphism, $\boldsymbol{\xi} : \tilde{M} \to M$, is associated with a *particle relabelling* for the body. If $\boldsymbol{\varphi}$ and $\tilde{\boldsymbol{\varphi}}$ denote the motion of the body relative to $M$ and $\tilde{M}$, respectively, then for $\mathbf{x} \in \tilde{M}$ it is clear that

$$\tilde{\boldsymbol{\varphi}}(\mathbf{x}, t) := \boldsymbol{\varphi}[\boldsymbol{\xi}(\mathbf{x}), t], \tag{64}$$

which simply states that within the two descriptions of the motion a given particle is always located at the same point of physical space. Eq. (64) is analogous to eq. (4) for the scalar wave equation in the introduction, though here there is a physical basis for the form of the transformation. Our aim is to obtain corresponding *particle relabelling transformations* between $(\rho, W, \boldsymbol{\varphi}_0, \mathbf{v}_0)$ and $(\tilde{\rho}, \tilde{W}, \tilde{\boldsymbol{\varphi}}_0, \tilde{\mathbf{v}}_0)$.

From eq. (64) we readily obtain the kinematic relations

$$\tilde{\mathbf{v}}(\mathbf{x}, t) = \mathbf{v}[\boldsymbol{\xi}(\mathbf{x}), t], \quad \tilde{\mathbf{F}}(\mathbf{x}, t) = \mathbf{F}[\boldsymbol{\xi}(\mathbf{x}), t]\mathbf{F}_\xi(\mathbf{x}), \tag{65}$$

where $\tilde{\mathbf{v}}$ and $\tilde{\mathbf{F}}$ are the velocity and deformation gradient relative to $\tilde{M}$, and we have introduced $\mathbf{F}_\xi := (\nabla \boldsymbol{\xi})^T$. It then follows immediately that the initial conditions relative to $\tilde{M}$ are related to those in $M$ through

$$\tilde{\boldsymbol{\varphi}}_0(\mathbf{x}) = \boldsymbol{\varphi}_0[\boldsymbol{\xi}(\mathbf{x})], \quad \tilde{\mathbf{v}}_0(\mathbf{x}) = \mathbf{v}_0[\boldsymbol{\xi}(\mathbf{x})]. \tag{66}$$

To obtain relations between the material parameters we examine the actions for the body relative to $M$ and $\tilde{M}$. We know that the action of the motion $\boldsymbol{\varphi} : M \times I \to \mathbb{R}^n$ is given by

$$\mathcal{S}(\boldsymbol{\varphi}) := \int_I \int_M \left\{ \frac{1}{2}\rho(\mathbf{x})\|\mathbf{v}(\mathbf{x}, t)\|^2 - W[\mathbf{x}, t, \mathbf{F}(\mathbf{x}, t)] \right.$$
$$\left. - \frac{1}{2}\rho(\mathbf{x})\zeta(\mathbf{x}, t) \right\} \mathrm{d}^n\mathbf{x}\,\mathrm{d}t, \tag{67}$$

and it follows that the action of the equivalent motion $\tilde{\boldsymbol{\varphi}} : \tilde{M} \times I \to \mathbb{R}^n$ takes the form

$$\tilde{\mathcal{S}}(\tilde{\boldsymbol{\varphi}}) := \int_I \int_{\tilde{M}} \left\{ \frac{1}{2}\tilde{\rho}(\mathbf{x})\|\tilde{\mathbf{v}}(\mathbf{x}, t)\|^2 - \tilde{W}[\mathbf{x}, t, \tilde{\mathbf{F}}(\mathbf{x}, t)] \right.$$
$$\left. - \frac{1}{2}\tilde{\rho}(\mathbf{x})\tilde{\zeta}(\mathbf{x}, t) \right\} \mathrm{d}^n\mathbf{x}\,\mathrm{d}t, \tag{68}$$

where the referential gravitational potential $\tilde{\zeta}$ relative to $\tilde{M}$ is given by

$$\tilde{\zeta}(\mathbf{x}, t) = \int_{\tilde{M}} \tilde{\rho}(\mathbf{x}')\Gamma_n[\tilde{\boldsymbol{\varphi}}(\mathbf{x}, t) - \tilde{\boldsymbol{\varphi}}(\mathbf{x}', t)]\,\mathrm{d}^n\mathbf{x}'. \tag{69}$$

These actions describe the same physical process, and so a motion $\boldsymbol{\varphi}$ is a stationary point of $\mathcal{S}$ if and only if $\tilde{\boldsymbol{\varphi}}$ in eq. (64) is a stationary point of $\tilde{\mathcal{S}}$. In order for this condition to hold, it is sufficient that $\mathcal{S}(\boldsymbol{\varphi}) = \tilde{\mathcal{S}}(\tilde{\boldsymbol{\varphi}})$ for all such pairs of motions. Using the mapping $\boldsymbol{\xi}$, we can transform the integration in eq. (67), and making use of the kinematic relations in eq. (65) we obtain

$$\mathcal{S}(\boldsymbol{\varphi}) = \int_I \int_{\tilde{M}} J_\xi(\mathbf{x}) \left\{ \frac{1}{2}\rho[\boldsymbol{\xi}(\mathbf{x})]\|\tilde{\mathbf{v}}(\mathbf{x}, t)\|^2 \right.$$
$$\left. - W[\boldsymbol{\xi}(\mathbf{x}), t, \tilde{\mathbf{F}}(\mathbf{x}, t)\mathbf{F}_\xi(\mathbf{x})^{-1}] - \frac{1}{2}\rho[\boldsymbol{\xi}(\mathbf{x})]\zeta[\boldsymbol{\xi}(\mathbf{x}), t] \right\} \mathrm{d}^n\mathbf{x}\,\mathrm{d}t, \tag{70}$$

where $J_\xi := \det(\mathbf{F}_\xi)$. From comparison of eqs (68) and (70), and using eqs (35) and (69), it follows that equality of the actions holds if the referential densities and strain-energy functions for the two reference bodies are related by

$$\tilde{\rho}(\mathbf{x}) = J_\xi(\mathbf{x})\rho[\boldsymbol{\xi}(\mathbf{x})], \quad \tilde{W}(\mathbf{x}, t, \tilde{\mathbf{F}}) = J_\xi(\mathbf{x})W[\boldsymbol{\xi}(\mathbf{x}), t, \tilde{\mathbf{F}}\mathbf{F}_\xi(\mathbf{x})^{-1}]. \tag{71}$$

While these are sufficient conditions, they are not necessary. For example, we can always add to the strain-energy function a term independent of the deformation gradient without altering the equations of motion. Such terms, of course, have no physical significance and can be ignored. With the transformed referential density given in eq. (71) we note from eqs (35) and (69) that

$$\tilde{\zeta}(\mathbf{x}, t) = \zeta[\boldsymbol{\xi}(\mathbf{x}), t], \tag{72}$$

for all $\mathbf{x} \in \tilde{M}$, and so the gravitational potentials $\phi$ and $\tilde{\phi}$ of the two motions are, as is physically required, equal at every spatial point. The above results generalize and extend those of Mazzucato

& Rachele (2006) for finite hyperelasticity. Our approach, however, uses only elementary methods, and does not require lengthy calculations in local coordinates. Moreover, we have seen that these transformations are associated with particle relabelling, and this will be key to understanding their implications for seismic inverse problems.

In the definition of $\tilde{W}$ in eq. (71), we see that $\mathbf{F}_\xi^{-1}$ acts on the right of $\tilde{\mathbf{F}}$, and so the strain-energy function remains material frame indifferent. However, the symmetry group of $\tilde{W}$ will, in general, differ from that of $W$. To see this, suppose for simplicity that the symmetry group of $W$ is the same at each $\mathbf{x} \in M$, and that $\mathbf{Q} \in \mathbf{O}(n)$ is an element of this group. From eq. (71), it follows that a sufficient condition for $\mathbf{Q}$ to be in the symmetry group of $\tilde{W}$ at $\mathbf{x} \in M$ is that the commutator

$$[\mathbf{Q}, \mathbf{F}_\xi(\mathbf{x})^{-1}] := \mathbf{Q}\mathbf{F}_\xi(\mathbf{x})^{-1} - \mathbf{F}_\xi(\mathbf{x})^{-1}\mathbf{Q}, \tag{73}$$

is everywhere equal to zero, but for a general $\boldsymbol{\xi}$ there is no reason to expect this to hold. For example, consider the isotropic strain-energy function for a *modified Saint-Vernant Kirchhoff* material

$$W(\mathbf{x}, \mathbf{F}) = \frac{1}{2}\lambda(\mathbf{x})\ln(J)^2 + \mu(\mathbf{x})\mathrm{tr}(\mathbf{E}^2), \tag{74}$$

where we have introduced a finite strain tensor $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{1})$, and $\lambda$ and $\mu$ are non-negative functions corresponding to the Lamé parameters for sufficiently small motions (e.g. Holzapfel 2000). Using eq. (71), we see that under the particle relabelling associated with a diffeomorphism $\boldsymbol{\xi}$, this strain-energy function transforms to become

$$\tilde{W}(\mathbf{x}, \tilde{\mathbf{F}}) = \frac{1}{2}J_\xi(\mathbf{x})\lambda[\boldsymbol{\xi}(\mathbf{x})]\ln[J_\xi(\mathbf{x})^{-1}\tilde{J}]^2$$
$$+ \frac{1}{4}J_\xi(\mathbf{x})\mu[\boldsymbol{\xi}(x)]\,\mathrm{tr}\left\{[\mathbf{F}_\xi(\mathbf{x})^{-T}\tilde{\mathbf{C}}\mathbf{F}_\xi(\mathbf{x})^{-1} - \mathbf{1}]^2\right\}, \tag{75}$$

which, for general $\boldsymbol{\xi}$, has a trivial symmetry group. If we set $\mu = 0$ in eq. (74) then the strain-energy function does, however, remain isotropic upon such a transformation. In fact, for any elastic fluid the strain-energy function depends on $\mathbf{F}$ only through $J = \det(\mathbf{F})$, and it is clear that this property is invariant under the transformation in eq. (71). These examples show that the symmetry group of a strain-energy function is dependent on the reference configuration, and is not intrinsically defined.

We have already seen how the equations of motion on $M$ can be obtained from Hamilton's principle. By an identical argument, the equations of motion in the reference body $\tilde{M}$ are given by

$$\tilde{\rho}\partial_t\tilde{\mathbf{v}} - \mathrm{Div}\tilde{\mathbf{T}} - \tilde{\rho}\tilde{\boldsymbol{\gamma}} = \mathbf{0}, \tag{76}$$

where the new constitutive equation is

$$\tilde{\mathbf{T}}(\mathbf{x}, t) = \frac{\partial \tilde{W}}{\partial \tilde{\mathbf{F}}}[\mathbf{x}, t, \tilde{\mathbf{F}}(\mathbf{x}, t)], \tag{77}$$

the transformed referential gravitational acceleration is given by

$$\tilde{\boldsymbol{\gamma}}(\mathbf{x}, t) = -\int_M \tilde{\rho}(\mathbf{x})(\nabla\Gamma_n)[\tilde{\boldsymbol{\varphi}}(\mathbf{x}, t) - \tilde{\boldsymbol{\varphi}}(\mathbf{x}', t)]\,\mathrm{d}^n\mathbf{x}', \tag{78}$$

we have the boundary condition

$$\tilde{\mathbf{T}}(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x}) = \mathbf{0}, \tag{79}$$

for all $\mathbf{x} \in \partial\tilde{M}$ and $t \in I$, and the initial conditions

$$\tilde{\boldsymbol{\varphi}}(\mathbf{x}, 0) = \tilde{\boldsymbol{\varphi}}_0(\mathbf{x}), \quad \tilde{\mathbf{v}}(\mathbf{x}, 0) = \tilde{\mathbf{v}}_0(\mathbf{x}), \tag{80}$$

for all $\mathbf{x} \in \tilde{M}$.

It follows from the above argument that the elastodynamic equations do indeed have exactly the same form relative to any suitable reference body, and in eqs (71) and (66) we have obtained simple formulae for transforming the material parameters and initial conditions. The derivation given in this section is based on Hamilton's principle for finite elasticity, and so our results are not immediately applicable to viscoelastic materials for which there is no corresponding variational principle. In Appendix A, we describe an alternative, and more general, approach that is applicable to viscoelastic materials. Within the remainder of this paper, however, we will focus only on the elastic case.

## 3.2 Particle relabelling symmetry, and non-uniqueness in seismological inverse problems

The initial-boundary value problem for the motion of a hyperelastic body defined relative to a reference body $M$ is given in eqs (60), (57), (61) and (62) and depends on four parameters: the referential density, the strain-energy function, the initial configuration and the initial velocity. We can think of this problem as defining a mapping

$$(\boldsymbol{\varphi}, \zeta) := \boldsymbol{\Phi}_M(\rho, W, \boldsymbol{\varphi}_0, \mathbf{v}_0), \tag{81}$$

which returns the motion of the body and the referential gravitational potential relative to $M$. Note that in eq. (81) the subscript on $\boldsymbol{\Phi}_M$ is used to denote the reference body relative to which the motion is defined. Given a diffeomorphism $\boldsymbol{\xi} : \tilde{M} \to M$, then following Section 3.1 we can define a particle relabelling such that the motion is described relative to a different reference body $\tilde{M} = \boldsymbol{\xi}^{-1}(M)$, and so we can instead write

$$(\tilde{\boldsymbol{\varphi}}, \tilde{\zeta}) = \boldsymbol{\Phi}_{\tilde{M}}(\tilde{\rho}, \tilde{W}, \tilde{\boldsymbol{\varphi}}_0, \tilde{\mathbf{v}}_0), \tag{82}$$

with the parameters on the right-hand side given through eqs (66) and (71). In this section, we explore some consequences of such transformations, though in doing so it will be helpful to introduce first some further notations.

Associated with a diffeomorphism $\boldsymbol{\xi} : \tilde{M} \to M$ we define a linear operator $T_\xi$ through

$$(T_\xi \boldsymbol{\varphi})(\mathbf{x}, t) = \boldsymbol{\varphi}[\boldsymbol{\xi}(\mathbf{x}, t)], \tag{83}$$

and similarly for any other (possibly time-dependent) scalar, vector, or tensor field on $M$. This operator maps fields on $M$ onto those on $\tilde{M}$, and is equal to the so-called *pull-back* under $\boldsymbol{\xi}$ in the case of scalar fields (e.g. Abraham *et al.* 1988). Let us suppose that $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ are diffeomorphisms such that the composition $\boldsymbol{\xi}_2 \circ \boldsymbol{\xi}_1$ makes sense. It is then clear that

$$T_{\xi_2 \circ \xi_1} = T_{\xi_1} \circ T_{\xi_2}, \tag{84}$$

where we note that the ordering of the composition is switched on the right-hand side. In particular, if we restrict attention to the so-called *diffeomorphism group* Diff$(M)$ of $M \subseteq \mathbb{R}^n$—which is an infinite-dimensional Lie group comprising all diffeomorphisms from $M$ onto itself—then eq. (83) defines a *right action* of Diff$(M)$ into the space of linear operators acting on tensor fields on $M$ (e.g. Ebin & Marsden 1970; Omori 1974). The subgroup Diff$_{\partial M}(M) \subseteq$ Diff$(M)$ comprises those $\boldsymbol{\xi} \in$ Diff$(M)$ such that

$$\boldsymbol{\xi}(\mathbf{x}) = \mathbf{x}, \quad \forall \mathbf{x} \in \partial M, \tag{85}$$

and will be useful below.

We also define an operator $P_\xi$ associated with $\boldsymbol{\xi} : \tilde{M} \to M$ such that

$$(\tilde{\rho}, \tilde{W}, \tilde{\boldsymbol{\varphi}}_0, \tilde{\mathbf{v}}_0) = P_\xi(\rho, W, \boldsymbol{\varphi}_0, \mathbf{v}_0), \tag{86}$$

where the parameters on the left-hand side are related to those on the right through eqs (66) and (71). The action of $P_\xi$ on the individual parameters is readily determined; for example, we find from eq. (71) that

$$(P_\xi \rho)(\mathbf{x}) = J_\xi(\mathbf{x})\rho[\boldsymbol{\xi}(\mathbf{x})],$$

$$(P_\xi W)(\mathbf{x}, t, \mathbf{F}) = J_\xi(\mathbf{x})W[\boldsymbol{\xi}(\mathbf{x}), t, \mathbf{F}\mathbf{F}_\xi(\mathbf{x})^{-1}]. \tag{87}$$

As above, if the composition $\boldsymbol{\xi}_2 \circ \boldsymbol{\xi}_1$ of two suitable diffeomorphisms makes sense, then we have

$$P_{\xi_2 \circ \xi_1} = P_{\xi_1} \circ P_{\xi_2}, \tag{88}$$

and so, in particular, $\boldsymbol{\xi} \mapsto P_\xi$ defines a right action of Diff$(M)$. Making use of these notations we can concisely write the particle relabelling transformation described in Section 3.1 as

$$T_\xi \circ \boldsymbol{\Phi}_M = \boldsymbol{\Phi}_{\tilde{M}} \circ P_\xi, \tag{89}$$

for any diffeomorphism $\boldsymbol{\xi} : \tilde{M} \to M$. In particular, for any element $\boldsymbol{\xi} \in$ Diff$(M)$ of the diffeomorphism group of $M$ we have

$$T_\xi \circ \boldsymbol{\Phi}_M = \boldsymbol{\Phi}_M \circ P_\xi, \tag{90}$$

so that $\boldsymbol{\Phi}_M$ *intertwines* with the operators $T_\xi$ and $P_\xi$ (e.g. Bump 2013). From eqs (84) and (89) we also obtain

$$\boldsymbol{\Phi}_M = T_{\xi^{-1}} \circ \boldsymbol{\Phi}_{\tilde{M}} \circ P_\xi, \tag{91}$$

showing how solutions of the elastodynamic equations in a reference body $M$ can be obtained from those calculated in $\tilde{M}$.

If we restrict the fields $(\boldsymbol{\varphi}, \zeta)$ obtained from $\boldsymbol{\Phi}_M$ to the boundary $\partial M$, we obtain an associated mapping

$$(\boldsymbol{\varphi}, \zeta)|_{\partial M} := \boldsymbol{\Phi}_{\partial M}(\rho, W, \boldsymbol{\varphi}_0, \mathbf{v}_0). \tag{92}$$

By definition, elements $\boldsymbol{\xi}$ of the subgroup Diff$_{\partial M}(M)$ of the diffeomorphism group leave boundary points fixed, and so we trivially obtain

$$[T_\xi(\boldsymbol{\varphi}, \zeta)]|_{\partial M} = (\boldsymbol{\varphi}, \zeta)|_{\partial M}, \quad \forall \boldsymbol{\xi} \in \text{Diff}_{\partial M}(M). \tag{93}$$

We then see immediately from eq. (90) that

$$\boldsymbol{\Phi}_{\partial M} = \boldsymbol{\Phi}_{\partial M} \circ P_\xi, \quad \forall \boldsymbol{\xi} \in \text{Diff}_{\partial M}(M), \tag{94}$$

which we call the *particle relabelling symmetry* for hyperelasticity. It follows that if we solve the elastodynamic equations for the parameters $(\rho, W, \boldsymbol{\varphi}_0, \mathbf{v}_0)$, then the surface motion $\boldsymbol{\varphi}|_{\partial M}$ and referential gravitational potential $\zeta|_{\partial M}$ obtained are exactly the same as would be produced from the parameters $P_\xi(\rho, W, \boldsymbol{\varphi}_0, \mathbf{v}_0)$ for any $\boldsymbol{\xi} \in$ Diff$_{\partial M}(M)$. There are infinitely many such diffeomorphisms, and so an inverse problem to recover $(\rho, W, \boldsymbol{\varphi}_0, \mathbf{v}_0)$ from even perfect knowledge of the surface motion and gravitational potential has an infinite-fold non-uniqueness. Eq. (94) is closely related to theorem 1 of Mazzucato & Rachele (2006) in the case of finite hyperelasticity, and a slight modification of our arguments in the non-gravitating case could be used to obtain an exactly equivalent result.

Eq. (94) is, at first, a rather striking result, with seemingly profound consequences for seismological inverse problems. It must, however, be recalled that $(\rho, W, \boldsymbol{\varphi}_0, \mathbf{v}_0)$ and $P_\xi(\rho, W, \boldsymbol{\varphi}_0, \mathbf{v}_0)$ are simply equivalent descriptions of the *same* elastic body corresponding to *different* reference configurations. It follows that the non-uniqueness associated with eq. (94) is a purely mathematical result

associated with an ambiguity in the formulation of elastodynamics, and *does not have direct physical significance*. This property may, nonetheless, be of some practical importance. For example, suppose that a solution of an elastodynamic inverse problem is sought using sampling methods (e.g. Sambridge 1999; de Wit *et al.* 2013). Ostensibly, the model space to be explored comprises quadruples of functions $(\rho, W, \boldsymbol{\varphi}_0, \mathbf{v}_0)$ defined on $M$, but our results show that this space is, in fact, decomposed into equivalence classes whose elements are related through the particle relabelling transformation $P_{\boldsymbol{\xi}}$ for some $\boldsymbol{\xi} \in \mathrm{Diff}_{\partial M}(M)$. Clearly, much computational effort would be wasted if equivalent models are repeatedly sampled, and so we are led to consider the following question: given two sets of model parameters $(\rho, W, \boldsymbol{\varphi}_0, \mathbf{v}_0)$ and $(\tilde{\rho}, \tilde{W}, \tilde{\boldsymbol{\varphi}}_0, \tilde{\mathbf{v}}_0)$ with $\boldsymbol{\varphi}_0|_{\partial M} = \tilde{\boldsymbol{\varphi}}_0|_{\partial M}$, is there a $\boldsymbol{\xi} \in \mathrm{Diff}_{\partial M}(M)$ such that

$$(\tilde{\rho}, \tilde{W}, \tilde{\boldsymbol{\varphi}}_0, \tilde{\mathbf{v}}_0) = P_{\boldsymbol{\xi}}(\rho, W, \boldsymbol{\varphi}_0, \mathbf{v}_0). \tag{95}$$

Using eq. (88) it is easy to see that a necessary and sufficient condition for this equality to hold is

$$P_{\boldsymbol{\varphi}_0^{-1}}(\rho, W, \boldsymbol{\varphi}_0, \mathbf{v}_0) = P_{\tilde{\boldsymbol{\varphi}}_0^{-1}}(\tilde{\rho}, \tilde{W}, \tilde{\boldsymbol{\varphi}}_0, \tilde{\mathbf{v}}_0), \tag{96}$$

in which case the desired diffeomorphism is $\boldsymbol{\xi} = \boldsymbol{\varphi}_0^{-1} \circ \tilde{\boldsymbol{\varphi}}_0 \in \mathrm{Diff}_{\partial M}(M)$.

# 4 APPLICATIONS TO LINEARIZED ELASTICITY

## 4.1 Equations of motion for linearized elastodynamics

For most seismological applications, it is sufficient to consider linearized motions about an equilibrium configuration, and we now specialize our results to this case. For simplicity, however, we shall neglect the effects of self-gravitation, leaving this for later work where we intend to discuss the complete linearized equations of motion suitable for studies of long-period seismology, solid-earth tides, orbital nutations and other similar applications (e.g. Woodhouse & Dahlen 1978; Dahlen & Tromp 1998). To proceed, we expand the motion as

$$\boldsymbol{\varphi}(\mathbf{x}, t) = \boldsymbol{\varphi}_e(\mathbf{x}) + \epsilon \, \mathbf{u}(\mathbf{x}, t), \tag{97}$$

where $\boldsymbol{\varphi}_e$ is an *equilibrium configuration* defined relative to the reference body $M$, and $\mathbf{u}$ is a time-dependent displacement vector field that is small in an appropriate sense. Here $\epsilon$ is a dimensionless parameter used to indicate the order of terms in our expansion, and whose value will eventually be set equal to one. From eq. (97) we obtain the kinematic relations

$$\mathbf{v}(\mathbf{x}, t) = \epsilon \, \partial_t \mathbf{u}(\mathbf{x}, t), \quad \mathbf{F}(\mathbf{x}, t) = \mathbf{F}_e(\mathbf{x}) + \epsilon \, \mathbf{F}_\mathbf{u}(\mathbf{x}, t), \tag{98}$$

where we have defined $\mathbf{F}_e = (\nabla \boldsymbol{\varphi}_e)^T$ and $\mathbf{F}_\mathbf{u} = (\nabla \mathbf{u})^T$, while the initial conditions become

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \partial_t \mathbf{u}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x}), \tag{99}$$

with $\mathbf{u}_0$ and $\mathbf{v}_0$ given functions. Motivated by eq. (50), we assume that the strain-energy function can be written

$$W(\mathbf{x}, t, \mathbf{F}) = W_0(\mathbf{x}, \mathbf{F}) + \epsilon \, W_1(\mathbf{x}, t, \mathbf{F}), \tag{100}$$

where we set

$$W_0(\mathbf{x}, \mathbf{F}) = U(\mathbf{x}, \mathbf{C}), \quad W_1(\mathbf{x}, t, \mathbf{F}) = -\frac{1}{2} \mathbf{S}_g(\mathbf{x}, t) : \mathbf{C}, \tag{101}$$

with $\mathbf{S}_g$ a time-dependent symmetric second-order tensor field representing a stress glut. Using eq. (57), we find that to first order in

$\epsilon$ the first Piola–Kirchhoff stress tensor for the motion is given by

$$\mathbf{T}(\mathbf{x}, t) = \frac{\partial W_0}{\partial \mathbf{F}}[\mathbf{x}, \mathbf{F}_e(\mathbf{x})] + \epsilon \left\{ \frac{\partial^2 W_0}{\partial \mathbf{F} \partial \mathbf{F}}[\mathbf{x}, \mathbf{F}_e(\mathbf{x})] : \right.$$
$$\left. \mathbf{F}_\mathbf{u}(\mathbf{x}, t) + \frac{\partial W_1}{\partial \mathbf{F}}[\mathbf{x}, t, \mathbf{F}_e(\mathbf{x})] \right\} + O(\epsilon^2). \tag{102}$$

Based on this expansion, we define the *equilibrium first Piola–Kirchhoff stress tensor* to be

$$\mathbf{T}_e(\mathbf{x}) := \frac{\partial W_0}{\partial \mathbf{F}}(\mathbf{x}, \mathbf{F}_e), \tag{103}$$

the *first Piola–Kirchhoff elastic tensor*

$$\boldsymbol{\Lambda}(\mathbf{x}) := \frac{\partial^2 W_0}{\partial \mathbf{F} \partial \mathbf{F}}(\mathbf{x}), \tag{104}$$

which possesses the important symmetry $\Lambda_{ijkl} = \Lambda_{klij}$, the *incremental first Piola–Kirchhoff stress tensor*

$$\mathbf{T}_\mathbf{u}(\mathbf{x}, t) := \boldsymbol{\Lambda}(\mathbf{x}) : \mathbf{F}_\mathbf{u}(\mathbf{x}, t), \tag{105}$$

and the *first Piola–Kirchhoff stress glut*

$$\mathbf{T}_g(\mathbf{x}, t) := -\frac{\partial W_1}{\partial \mathbf{F}}(\mathbf{x}, t, \mathbf{F}_e). \tag{106}$$

Due to the form of the strain-energy function in eq. (101), we see that

$$\mathbf{T}_e(\mathbf{x}) = \mathbf{F}_e(\mathbf{x}) \mathbf{S}_e(\mathbf{x}), \tag{107}$$

where the *equilibrium second Piola–Kirchhoff stress tensor* is given by

$$\mathbf{S}_e(\mathbf{x}) = 2 \frac{\partial U}{\partial \mathbf{C}}(\mathbf{x}, \mathbf{C}_e), \tag{108}$$

with $\mathbf{C}_e := \mathbf{F}_e^T \mathbf{F}_e$, and similarly, we can write the first Piola–Kirchhoff stress glut as

$$\mathbf{T}_g(\mathbf{x}, t) = \mathbf{F}_e(\mathbf{x}) \mathbf{S}_g(\mathbf{x}, t). \tag{109}$$

Neither the equilibrium stress tensor $\mathbf{T}_e$ nor the stress glut $\mathbf{T}_g$ are, in general, symmetric, though this is true when the equilibrium configuration equals the identity mapping. Linearizing eq. (41), we see that the incremental first Piola–Kirchhoff stress tensor can be written

$$\mathbf{T}_\mathbf{u}(\mathbf{x}, t) = \mathbf{F}_\mathbf{u}(\mathbf{x}, t) \mathbf{S}_e(\mathbf{x}) + \mathbf{F}_e(\mathbf{x}) \mathbf{S}_\mathbf{u}(\mathbf{x}, t), \tag{110}$$

where $\mathbf{S}_\mathbf{u}$ is the incremental second Piola–Kirchhoff stress. From eq. (45) we obtain

$$\mathbf{S}_\mathbf{u} = 4 \frac{\partial^2 U}{\partial \mathbf{C} \partial \mathbf{C}}[\mathbf{x}, \mathbf{C}_e(\mathbf{x})] : \mathbf{E}_\mathbf{u}(\mathbf{x}, t), \tag{111}$$

where we have defined the *incremental strain tensor* through

$$\mathbf{E}_\mathbf{u}(\mathbf{x}, t) := \frac{1}{2}[\mathbf{F}_e(\mathbf{x})^T \mathbf{F}_\mathbf{u}(\mathbf{x}, t) + \mathbf{F}_\mathbf{u}(\mathbf{x}, t)^T \mathbf{F}_e(\mathbf{x})], \tag{112}$$

and so write eq. (110) in the form

$$\mathbf{T}_\mathbf{u}(\mathbf{x}, t) = \mathbf{F}_\mathbf{u}(\mathbf{x}, t) \mathbf{S}_e(\mathbf{x}) + \mathbf{F}_e(\mathbf{x})[\mathbf{A}(\mathbf{x}) : \mathbf{E}_\mathbf{u}(\mathbf{x}, t)], \tag{113}$$

where $\mathbf{A}$ is the *second Piola–Kirchhoff elastic tensor* defined by

$$\mathbf{A}(\mathbf{x}) = 4 \frac{\partial^2 U}{\partial \mathbf{C} \partial \mathbf{C}}[\mathbf{x}, \mathbf{C}_e(\mathbf{x})], \tag{114}$$

which possesses the symmetries

$$A_{ijkl} = A_{jikl} = A_{ijlk} = A_{klij}. \tag{115}$$

From eq. (113), we see that $\mathbf{T_u}$ depends on the equilibrium state of the body through both the equilibrium second Piola–Kirchhoff stress and the deformation gradient of the equilibrium configuration. Furthermore, eq. (113) implies that $\mathbf{\Lambda}$ is not an arbitrary fourth-order tensor possessing the symmetry $\Lambda_{ijkl} = \Lambda_{klij}$, but must instead be expressible in terms of the tensor $\mathbf{A}$ introduced in eq. (114) along with $\mathbf{S}_e$ and $\mathbf{F}_e$. This relationship is most clearly written using index notation, and is given by

$$\Lambda_{ijkl}(\mathbf{x}) = [\mathbf{S}_e(\mathbf{x})]_{jl}\,\delta_{ik} + [\mathbf{F}_e(\mathbf{x})]_{im}[\mathbf{F}_e(\mathbf{x})]_{kn}\,A_{mjnl}(\mathbf{x}). \tag{116}$$

Using the above expansions, and neglecting the gravitational terms, we find that, to zeroth-order, the equations of motion given in Section 2.5 reduce to the equilibrium equation

$$\mathrm{Div}\mathbf{T}_e = \mathbf{0}, \tag{117}$$

which is subject to the boundary condition

$$\mathbf{T}_e(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) = \mathbf{0}, \tag{118}$$

for all $\mathbf{x} \in \partial M$. We observe that in this non-gravitating problem $\mathbf{T}_e = \mathbf{0}$ is a permissible solution of the equilibrium equations. Similarly, from the first-order terms in the expansion we obtain the linearized elastodynamic equation

$$\rho \partial_t^2 \mathbf{u} - \mathrm{Div}\mathbf{T_u} = -\mathrm{Div}\mathbf{T}_g, \tag{119}$$

which is subject to the boundary condition

$$\mathbf{T_u}(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x}) = \mathbf{T}_g(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x}), \tag{120}$$

for all $\mathbf{x} \in \partial M$. It is notable that the dependence of eq. (119) on the equilibrium state of the body is solely through the first Piola–Kirchhoff elastic tensor $\mathbf{\Lambda}$ and the first Piola–Kirchhoff stress glut $\mathbf{T}_g$.

## 4.2 Particle relabelling symmetry in linearized elasticity

The particle relabelling transformations and symmetry described in Sections 3.1 and 3.2 must, of course, carry over to the equations of linearized elasticity. We have seen that the linearized elastodynamic equations defined relative to a reference body $M$ depend on seven parameters: the equilibrium configuration $\boldsymbol{\varphi}_e$, the equilibrium second Piola–Kirchhoff stress tensor $\mathbf{S}_e$, the referential density $\rho$, second Piola–Kirchhoff elastic tensor $\mathbf{A}$, the second Piola–Kirchhoff stress glut $\mathbf{S}_g$, the initial displacement vector $\mathbf{u}_0$, and the initial velocity vector $\mathbf{v}_0$. Given these parameters we can, in principle, formulate and solve the equation of motion given in eq. (119) to obtain the displacement vector field over a given time interval $I = [0, T]$. The mapping from these model parameters to the displacement vector field can be written abstractly as

$$\mathbf{u} := \mathbf{U}_M(\boldsymbol{\varphi}_e, \mathbf{S}_e, \rho, \mathbf{A}, \mathbf{S}_g, \mathbf{u}_0, \mathbf{v}_0), \tag{121}$$

and it will again be useful to consider the related mapping

$$\mathbf{u}|_{\partial M} := \mathbf{U}_{\partial M}(\boldsymbol{\varphi}_e, \mathbf{S}_e, \rho, \mathbf{A}, \mathbf{S}_g, \mathbf{u}_0, \mathbf{v}_0), \tag{122}$$

which returns the displacement vector restricted to the surface of the reference body.

The linearized elastodynamic equations, in fact, depend on the seven parameters $(\boldsymbol{\varphi}_e, \mathbf{S}_e, \rho, \mathbf{A}, \mathbf{S}_g, \mathbf{u}_0, \mathbf{v}_0)$ only through the reduced set $(\rho, \mathbf{\Lambda}, \mathbf{T}_g, \mathbf{u}_0, \mathbf{v}_0)$, with the first Piola–Kirchhoff stress glut $\mathbf{T}_g$ and the first Piola–Kirchhoff elastic tensor $\mathbf{\Lambda}$ defined in eqs (109) and (116), respectively. The mapping to the displacement vector field in eq. (121) can, therefore, be factored as

$$(\boldsymbol{\varphi}_e, \mathbf{S}_e, \rho, \mathbf{A}, \mathbf{S}_g, \mathbf{u}_0, \mathbf{v}_0) \mapsto (\rho, \mathbf{\Lambda}, \mathbf{T}_g, \mathbf{u}_0, \mathbf{v}_0) \mapsto \mathbf{u}. \tag{123}$$

Considering the first step of this mapping, the only non-trivial part is

$$(\mathbf{F}_e, \mathbf{S}_e, \mathbf{A}, \mathbf{S}_g) \mapsto (\mathbf{\Lambda}, \mathbf{T}_g), \tag{124}$$

as given by eqs (109) and (116), and where we recall that $\mathbf{F}_e$ is the deformation gradient associated with the equilibrium configuration. The dependence of $\mathbf{\Lambda}$ on $\mathbf{F}_e$ is nonlinear, and so it is not possible to draw definite conclusions about the mapping in eq. (124) from dimensional arguments alone. Nonetheless, a simple calculation shows that (for fixed $\mathbf{x} \in M$) the domain of this mapping has dimension $\frac{1}{8}n\{n[n(n + 2) + 19] + 10\}$, while its image lies in a $\frac{1}{2}n^2(n^2 + 3)$-dimensional space. For all $\mathbb{N} \ni n \geq 3$ we have

$$\frac{1}{8}n\{n[n(n + 2) + 19] + 10\} \leq \frac{1}{2}n^2(n^2 + 3), \tag{125}$$

and it follows that the mapping in eq. (124) cannot be surjective. For $n = 2$ we find conversely that the domain of the mapping is larger than that of its codomain, implying that it is not injective, and so for any $n$ we conclude that the mapping in eq. (124) is not bijective. It follows that $(\boldsymbol{\varphi}_e, \mathbf{S}_e, \rho, \mathbf{A}, \mathbf{S}_g, \mathbf{u}_0, \mathbf{v}_0)$ are appropriate parameters for describing linearized elastodynamics.

It is worth noting that in the above discussion we have treated $\boldsymbol{\varphi}_e$ and $\mathbf{S}_e$ as independent parameters, but they are, of course, coupled through the equilibrium conditions in eqs (117) and (118). If, without loss of generality, we regard $\boldsymbol{\varphi}_e$ as an independent parameter, then the equilibrium equations give $n$ linearly independent partial differential equations that must be satisfied by the equilibrium second Piola Kirchhoff stress $\mathbf{S}_e$, though these equations do not, of course, determine $\mathbf{S}_e$ uniquely (e.g. Backus 1967; Al-Attar & Woodhouse 2010). However, these additional constraints act to *restrict further* the possible forms of $\mathbf{\Lambda}$ and $\mathbf{T}_g$, and so our discussion above is qualitatively unchanged.

We now specialize the particle relabelling transformations discussed above to linearized elasticity. From eq. (64) we see that the equilibrium configuration of the transformed body is given by

$$\tilde{\boldsymbol{\varphi}}_e(\mathbf{x}) = \boldsymbol{\varphi}_e[\boldsymbol{\xi}(\mathbf{x})], \tag{126}$$

and that the new initial conditions are

$$\tilde{\mathbf{u}}_0(\mathbf{x}) = \mathbf{u}_0[\boldsymbol{\xi}(\mathbf{x})], \quad \tilde{\mathbf{v}}_0(\mathbf{x}) = \mathbf{v}_0[\boldsymbol{\xi}(\mathbf{x})]. \tag{127}$$

Using eqs (71) and (103), a simple calculation shows that the new equilibrium first Piola–Kirchhoff stress tensor is

$$\tilde{\mathbf{T}}_e(\mathbf{x}) = J_{\boldsymbol{\xi}}(\mathbf{x})\mathbf{T}_e[\boldsymbol{\xi}(\mathbf{x})]\mathbf{F}_{\boldsymbol{\xi}}(\mathbf{x})^{-T}. \tag{128}$$

Similarly, from eq. (104) we find that the transformed first Piola–Kirchhoff elastic tensor has components

$$\tilde{\Lambda}_{ijkl}(\mathbf{x}) = J_{\boldsymbol{\xi}}(\mathbf{x})[\mathbf{F}_{\boldsymbol{\xi}}(\mathbf{x})^{-1}]_{jm}[\mathbf{F}_{\boldsymbol{\xi}}(\mathbf{x})^{-1}]_{ln}\,\Lambda_{imkn}[\boldsymbol{\xi}(\mathbf{x})], \tag{129}$$

while using eq. (106) we obtain

$$\tilde{\mathbf{T}}_g(\mathbf{x}) = J_{\boldsymbol{\xi}}(\mathbf{x})\mathbf{T}_g[\boldsymbol{\xi}(\mathbf{x}), t]\mathbf{F}_{\boldsymbol{\xi}}(\mathbf{x})^{-T}, \tag{130}$$

for the new first Piola–Kirchhoff stress glut. From eqs (71) and (101) it follows that the strain-energy function $U$ transforms as

$$\tilde{U}(\mathbf{x}, \tilde{\mathbf{C}}) = J_{\boldsymbol{\xi}}(\mathbf{x})U[\boldsymbol{\xi}(\mathbf{x}), \mathbf{F}_{\boldsymbol{\xi}}(\mathbf{x})^{-T}\tilde{\mathbf{C}}\mathbf{F}_{\boldsymbol{\xi}}(\mathbf{x})^{-1}], \tag{131}$$

which leads immediately to the expression

$$\tilde{\mathbf{S}}_e(\mathbf{x}) = J_{\boldsymbol{\xi}}(\mathbf{x})\mathbf{F}_{\boldsymbol{\xi}}(\mathbf{x})^{-1}\mathbf{S}_e[\boldsymbol{\xi}(\mathbf{x})]\mathbf{F}_{\boldsymbol{\xi}}(\mathbf{x})^{-T}, \tag{132}$$

for the second Piola–Kirchhoff equilibrium stress, and by an identical calculation

$$\tilde{\mathbf{S}}_g(\mathbf{x}) = J_{\boldsymbol{\xi}}(\mathbf{x})\mathbf{F}_{\boldsymbol{\xi}}(\mathbf{x})^{-1}\mathbf{S}_g[\boldsymbol{\xi}(\mathbf{x})]\mathbf{F}_{\boldsymbol{\xi}}(\mathbf{x})^{-T}. \tag{133}$$

Similarly, we find that the transformed second Piola–Kirchhoff elastic tensor is given by

$$\tilde{A}_{ijkl}(\mathbf{x}) = J_\xi(\mathbf{x})[\mathbf{F}_\xi(\mathbf{x})^{-1}]_{im}[\mathbf{F}_\xi(\mathbf{x})^{-1}]_{jn}$$
$$\times [\mathbf{F}_\xi(\mathbf{x})^{-1}]_{kp}[\mathbf{F}_\xi(\mathbf{x})^{-1}]_{lq} A_{mnpq}[\boldsymbol{\xi}(\mathbf{x})], \quad (134)$$

and can verify that the transformed elastic tensors $\tilde{\boldsymbol{\Lambda}}$ and $\tilde{\mathbf{A}}$ are related in the correct manner. These various formulae, along with the transformation for the referential density given in eq. (71), provide a complete recipe for transforming the linearized elastodynamic equations under a particle relabelling. As with finite elasticity, the form of the equations of motion is invariant under such a transformation, and it is only the material parameters and initial conditions that are altered.

Following our approach in Section 3.2, we can formulate these results concisely using the operator $T_\xi$ defined in eq. (83) for any diffeomorphism $\boldsymbol{\xi} : \tilde{M} \to M$, and by modifying the definition of $P_\xi$ in eq. (86) such that

$$(\tilde{\boldsymbol{\varphi}}_e, \tilde{\mathbf{S}}_e, \tilde{\rho}, \tilde{\mathbf{A}}, \tilde{\mathbf{S}}_g, \tilde{\mathbf{u}}_0, \tilde{\mathbf{v}}_0) := P_\xi(\boldsymbol{\varphi}_e, \mathbf{S}_e, \rho, \mathbf{A}, \mathbf{S}_g, \mathbf{u}_0, \mathbf{v}_0), \quad (135)$$

where the parameters on the left-hand side are related to those on the right through the various transformations given above. Having done this, we can write the general particle labelling transformation as

$$T_\xi \circ \mathbf{U}_M = \mathbf{U}_{\tilde{M}} \circ P_\xi, \quad (136)$$

for any diffeomorphism $\boldsymbol{\xi} : \tilde{M} \to M$. Using eq. (88) we then immediately obtain

$$\mathbf{U}_M = T_{\xi^{-1}} \circ \mathbf{U}_{\tilde{M}} \circ P_\xi, \quad (137)$$

which gives the theoretical basis for mapping topography into volumetric heterogeneity within numerical simulations of linearized elastic wave propagation. If we restrict attention to $\boldsymbol{\xi} \in \mathrm{Diff}(M)$ or $\boldsymbol{\xi} \in \mathrm{Diff}_{\partial M}(M)$, then eq. (89) reduces to

$$T_\xi \circ \mathbf{U}_M = \mathbf{U}_M \circ P_\xi, \quad (138)$$

showing that $\mathbf{U}_M$ intertwines with $T_\xi$ and $P_\xi$. Finally, for any $\boldsymbol{\xi} \in \mathrm{Diff}_{\partial M}(M)$ we readily obtain the identity

$$\mathbf{U}_{\partial M} = \mathbf{U}_{\partial M} \circ P_\xi, \quad (139)$$

which is the particle relabelling symmetry in the case of linearized elasticity, and can be interpreted as a non-uniqueness result for linearized elastodynamic inverse problems. As with eq. (94), the non-uniqueness implied by this equation does not have direct physical consequences, but instead results from the arbitrariness in the choice of reference configuration. Eq. (139) is closely related to Theorem 1 of Mazzucato & Rachele (2006) for the case of linearized elasticity.

### 4.3 Classical linear elasticity

The equations of motion for linearized elasticity obtained in Section 4.1 are more general than those usually considered in seismology (e.g. Dahlen & Tromp 1998; Chapman 2004). To obtain the more familiar form of the equations we simply assume that the equilibrium configuration defined in eq. (97) equals the identity mapping. This amounts to choosing the reference configuration such that a particle's label $\mathbf{x} \in M$ equals its spatial location when the body is in its equilibrium state. Such a reference configuration will be called *natural*, and from eq. (29) it follows that the referential density $\rho$

equals the spatial density $\varrho_e$ of the body at equilibrium, while eq. (116) reduces to

$$\Lambda_{ijkl}(\mathbf{x}) = [\mathbf{S}_e(\mathbf{x})]_{jl} \delta_{ik} + A_{ijkl}(\mathbf{x}). \quad (140)$$

If the linearized motion of an elastic body is described through the parameters $(\boldsymbol{\varphi}_e, \mathbf{S}_e, \rho, \mathbf{A}, \mathbf{S}_g, \mathbf{u}_0, \mathbf{v}_0)$ relative to a reference body $M$, then from eq. (126) it is clear that a natural reference configuration for this body can be obtained through the particle relabelling transformation associated with $\boldsymbol{\varphi}_e^{-1} : \boldsymbol{\varphi}_e(M) \to M$. Furthermore, when the equilibrium second Piola–Kirchhoff stress tensor vanishes, then, relative to a natural reference configuration, the first Piola–Kirchhoff elastic tensor $\boldsymbol{\Lambda}$ possesses the symmetries

$$\Lambda_{ijkl} = \Lambda_{jikl} = \Lambda_{ijlk} = \Lambda_{klij}, \quad (141)$$

and the equations of motion reduce to those of classical linear elasticity (e.g. Chapman 2004).

It is only relative to a natural reference configuration that the material parameters of the elastic body have their familiar physical interpretations. For example, it is well known that an isotropic elastic tensor takes the form

$$\left(\kappa - \frac{2}{3}\mu\right) \delta_{ij}\delta_{kl} + \mu \left(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}\right), \quad (142)$$

where $\kappa$ and $\mu$ are, respectively, the bulk and shear moduli (e.g. Chapman 2004). However, this form of an isotropic elastic tensor does not hold if an arbitrary reference configuration is used to describe the elastic body. Let $\boldsymbol{\Lambda}$ be the first Piola–Kirchhoff elastic tensor relative to an arbitrary reference body $M$, and $\boldsymbol{\varphi}_e$ the corresponding equilibrium configuration. If we wish $\boldsymbol{\Lambda}$ to represent an isotropic tensor with respect to its natural reference configuration, then using eq. (129) and the particle relabelling transformation associated with $\boldsymbol{\varphi}_e^{-1}$, we find that relative to $M$ it must take the form

$$\Lambda_{ijkl} = J_e \left(\kappa - \frac{2}{3}\mu\right) [\mathbf{F}_e^{-1}]_{ji}[\mathbf{F}_e^{-1}]_{lk}$$
$$+ J_e\mu \left\{[\mathbf{C}_e^{-1}]_{jl}\delta_{ik} + [\mathbf{F}_e^{-1}]_{jk}[\mathbf{F}_e^{-1}]_{li}\right\}, \quad (143)$$

where $J_e$ and $\mathbf{C}_e$ are, respectively the Jacobian and right Cauchy-Green deformation tensor of the equilibrium configuration, and $\kappa$ and $\mu$ are two scalar-valued functions defined on $M$. If $\boldsymbol{\varphi}_e$ is the identity mapping, then this expression reduces to the familiar form given above, and $\kappa$ and $\mu$ have their usual significance.

### 4.4 Asymptotic ray theory

To illustrate some consequences of using different reference bodies in linearized elastodynamics, we now consider the propagation of high-frequency body waves using asymptotic ray theory. Let $M$ denote some reference body, with material parameters $\rho$ and $\boldsymbol{\Lambda}$. The ray-theoretic ansatz for the displacement vector field $\mathbf{u}$ can be written

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{a}(\mathbf{x})e^{i\omega[t - T(\mathbf{x})]}, \quad (144)$$

for $\mathbf{x} \in M$, where $\mathbf{a}(\mathbf{x})$ is vector field describing the amplitude and polarization of the displacement, $\omega$ its frequency and $T(\mathbf{x})$ is the traveltime (e.g. Chapman 2004). Inserting this displacement into eq. (119) in the absence of any stress glut and retaining only terms to highest order in $\omega$, we obtain the so-called Christoffel equation

$$\boldsymbol{\Gamma}[\mathbf{x}, \mathbf{p}(\mathbf{x})] \cdot \mathbf{a}(\mathbf{x}) = \mathbf{a}(\mathbf{x}), \quad (145)$$

where we have defined the slowness vector $\mathbf{p} = \nabla T$, and the Christoffel matrix $\mathbf{\Gamma}$ has components

$$\Gamma_{ik}(\mathbf{x}, \mathbf{p}) = \frac{1}{\rho(\mathbf{x})} \Lambda_{ijkl}(\mathbf{x}) p_j p_l. \tag{146}$$

The slowness vector $\mathbf{p}$ lies orthogonal to the wave front, and so we can write

$$\mathbf{p}(\mathbf{x}) = \frac{1}{c(\mathbf{x})} \hat{\mathbf{p}}(\mathbf{x}), \tag{147}$$

with $\hat{\mathbf{p}}(\mathbf{x})$ a unit vector giving the direction of propagation, and $c(\mathbf{x})$ the wave speed. From eq. (145), we then obtain the eigenvalue problem

$$\mathbf{\Gamma}[\mathbf{x}, \hat{\mathbf{p}}(\mathbf{x})] \cdot \mathbf{a}(\mathbf{x}) = c(\mathbf{x})^2 \mathbf{a}(\mathbf{x}), \tag{148}$$

and as we are working in $\mathbb{R}^n$ with $n \geq 2$, this problem will have $n$ eigenvalues (including degeneracies) that give the possible wave speeds, while the corresponding eigenvectors $\mathbf{a}(\mathbf{x})$ give the polarization directions of each such ray. In the case of an isotropic elastic tensor as given in eq. (142), it is well known that the solutions of this eigenvalue problem take the form of a single *p*-wave whose polarization vector is parallel to the propagation direction, and $n - 1$ degenerate *s*-waves with polarization vectors lying in the plane orthogonal to the propagation direction.

Suppose that we were to describe the same problem using a different reference body $\tilde{M}$, and hence consider the effects of particle relabelling associated with some diffeomorphism $\boldsymbol{\xi} : \tilde{M} \to M$. With respect to the new reference body, the material parameters $\tilde{\rho}$ and $\tilde{\mathbf{\Lambda}}$ are given by eqs (71) and (129), and we could form the associated Christoffel equation. Without detailed calculations it is, however, possible to determine the qualitative properties of the ray theoretic solution relative to the new reference body. By definition, the displacement vector $\mathbf{u}$ in $M$ and $\tilde{\mathbf{u}}$ in $\tilde{M}$ are related by

$$\tilde{\mathbf{u}}(\mathbf{x}, t) = \mathbf{u}[\boldsymbol{\xi}(\mathbf{x}), t], \tag{149}$$

and the ray theoretic ansatz in $\tilde{M}$ corresponding to eq. (144) is given by

$$\tilde{\mathbf{u}}(\mathbf{x}, t) = \mathbf{a}[\boldsymbol{\xi}(\mathbf{x})] e^{i\omega\{t - T[\boldsymbol{\xi}(\mathbf{x})]\}} := \tilde{\mathbf{a}}(\mathbf{x}) e^{i\omega[t - \tilde{T}(\mathbf{x})]}, \tag{150}$$

where we have defined

$$\tilde{\mathbf{a}}(\mathbf{x}) = \mathbf{a}[\boldsymbol{\xi}(\mathbf{x})], \quad \tilde{T}(\mathbf{x}) = T[\boldsymbol{\xi}(\mathbf{x})]. \tag{151}$$

It is clear that $\tilde{T}$ represents the traveltime relative to $\tilde{M}$, and it follows that if $S \subseteq M$ is a wave front of the traveltime $T$ then the corresponding wave front of $\tilde{T}$ is given by the inverse image $\boldsymbol{\xi}^{-1}(S) \subseteq \tilde{M}$. Furthermore, we see that the polarization vectors of the two solutions $\tilde{\mathbf{a}}$ and $\mathbf{a}$ are equal when evaluated at $\mathbf{x} \in \tilde{M}$ and $\boldsymbol{\xi}(\mathbf{x}) \in M$, respectively, with these two labels being, of course, associated with the same particle. These properties of the wave fronts and polarization vectors reflect the fact that in transforming the reference body we are not altering the physics of the problem, but merely changing the way it is described.

Within Hamiltonian ray theory, we recall that the particle label $\mathbf{x}$ and the slowness vector $\mathbf{p}$ act as coordinates on a phase space for the ray tracing equations (e.g. Chapman 2004). We have already seen that the form of the elastodynamic equations is invariant under a particle relabelling transformation, and it follows that the same must be true of the associated ray tracing equations in Hamiltonian form. We can, therefore, conclude that a particle relabelling transformation acts on the phase space of Hamiltonian ray theory as a *canonical transformation*, and in particular as a so-called *point transformation*, this being a type of canonical transformation

induced by a diffeomorphism on the configuration space of the system (e.g. Abraham & Marsden 1978, section 3.2). This property suggests a potentially interesting link with the work of Virieux & Ekström (1991) who employ the method of *Lie series* to generate canonical transformations which simplify the solution of the surface wave ray tracing equations in laterally heterogeneous earth models. Though we have not investigated the relation between these two methods in any detail, it may prove fruitful to do so in later work. Indeed, the form invariance of Hamilton's equations under point transformations provides a useful analogue for the properties of the elastodynamic equations described in this work. This similarity may, in fact, be more than a formal one, because it is known that the elastodynamic equations can be expressed as Hamilton's equations on an infinite-dimensional phase space (e.g. Marsden & Hughes 1983).

A further interesting aspect of the particle relabelling transformations is brought to light if we consider a body that is isotropic when described relative to a reference body $M$. As noted above, in such a medium we know that, asymptotically, there exist *p*-waves whose polarization vectors are orthogonal to their wave front, and degenerate *s*-waves with polarization vectors tangential to their common wave front. If we perform a particle relabelling on such a body associated with a diffeomorphism $\boldsymbol{\xi} : \tilde{M} \to M$, then we have seen above that the wave fronts in $M$ are mapped onto their inverse images in $\tilde{M}$. Moreover, if we take two points $\mathbf{x} \in \tilde{M}$ and $\boldsymbol{\xi}(\mathbf{x}) \in M$ that lie on corresponding wave fronts in the two reference bodies we know that their polarization vectors are equal. It follows that under a generic particle relabelling a *p*-wave wave front in $M$ will be mapped into a wave front in $\tilde{M}$ whose polarization vectors are no longer orthogonal to the wave front. Similarly, such a particle relabelling will map an *s*-wave wave front into one in which the polarization vectors are no longer tangential to the wave front. We can conclude that a medium that is isotropic with respect to a reference body $M$ will not, in general, be isotropic when described relative to another reference body. These results are consistent with the observation in Section 3.1 that particle relabelling transformations do not, in general, preserve the symmetry group of a strain-energy function. Importantly, however, the anisotropy introduced by such transformations does not break the degeneracy of *s*-wave wave speeds, and so does not produce shear wave splitting. In fact, it is easy to see using the results of this section that shear wave splitting can only arise if a material is anisotropic when described relative to a natural reference configuration.

## 5 NUMERICAL EXAMPLES

### 5.1 Spectral element solution of the elastodynamic equations

Following Komatitsch & Tromp (1999), we obtain numerical solutions of the linearized elastodynamic equations using the spectral element method, but for simplicity we work only in 2-D space. This approach is based on the weak-form of the linearized elastodynamic equations, which can be derived from eqs (119) and (120) using standard methods, and requires that

$$\int_M \rho \partial_t^2 \mathbf{u} \cdot \mathbf{u}' \, d^n\mathbf{x} + \int_M \mathbf{T_u} : \mathbf{F_{u'}} \, d^n\mathbf{x} = \int_M \mathbf{T_g} : \mathbf{F_{u'}} \, d^n\mathbf{x}, \tag{152}$$

for all time-independent test functions $\mathbf{u}'$, and where $\mathbf{F_{u'}} = (\nabla \mathbf{u}')^T$. Spatial discretization is done using Lagrange polynomial interpolation on a quadrilateral mesh, and the system is time-stepped using a second-order Newmark scheme. Our approach differs very slightly

from that of Komatitsch & Tromp (1999) due to the occurrence of the non-symmetric first Piola–Kirchhoff stress tensor in the equations of motion above, but this extension is trivial.

## 5.2 Practical generation of diffeomorphisms

To perform particle relabelling transformations practically, we need a numerical method for generating diffeomorphisms $\boldsymbol{\xi} : \tilde{M} \to M$ between two reference bodies $\tilde{M}$ and $M$. It is only in simple cases that such diffeomorphisms can be defined analytically. Here we describe a more general method based on computing the *flow* of a suitable vector field.

Consider a one-parameter family of diffeomorphisms $s \mapsto \boldsymbol{\xi}_s$ defined for $s$ in some neighbourhood of zero, such that each $\boldsymbol{\xi}_s$ has a fixed domain $\tilde{M}$, image $M_s := \boldsymbol{\xi}_s(M) \subseteq \mathbb{R}^n$, and is equal to the identity mapping for $s = 0$. Associated with such a one-parameter family, we can define a vector field on $\tilde{M}$ by

$$\mathbf{q} := \partial_s \boldsymbol{\xi}_s |_{s=0}, \tag{153}$$

which we call its *infinitesimal generator*. Restricting attention to elements of the diffeomorphism group Diff($M$), it is clear that the corresponding infinitesimal generator is tangential to the boundary

$$\mathbf{q} \cdot \mathbf{n} = 0, \tag{154}$$

for all $\mathbf{x} \in \partial M$, while for $\boldsymbol{\xi}_s$ in the sub-group Diff$_{\partial M}(M)$ we have

$$\mathbf{q} = \mathbf{0}, \tag{155}$$

everywhere on the boundary.

Conversely, suppose we are given a smooth vector field $\mathbf{q}$ on $\tilde{M} \subseteq \mathbb{R}^n$. We define the flow $(\mathbf{x}, s) \mapsto \mathbf{X_q}(\mathbf{x}, s)$ of $\mathbf{q}$ through the solution of the differential equation

$$\frac{d}{ds}\mathbf{X_q}(\mathbf{x}, s) = \mathbf{q}[\mathbf{X_q}(\mathbf{x}, s)], \quad \mathbf{X_q}(\mathbf{x}, 0) = \mathbf{x}, \quad \forall \mathbf{x} \in \tilde{M}, \tag{156}$$

and it may be shown that the mapping $\mathbf{x} \mapsto \mathbf{X_q}(\mathbf{x}, s)$ is a diffeomorphism from $\tilde{M}$ onto $M_s := \mathbf{X_q}(\tilde{M}, s)$ (e.g. Abraham *et al.* 1988). If we choose a vector field $\mathbf{q}$ on $M$ satisfying eq. (154) it is clear that for sufficiently small $s$ the flow is an element of Diff($M$), while the flow of $\mathbf{q}$ satisfying eq. (155) produces elements of Diff$_{\partial M}(M)$.

It follows that diffeomorphisms required for particle relabelling transformations can be generated from the flow of suitable vector fields, and this can be done numerically by integrating the ordinary differential equation given in eq. (156). To perform the particle relabelling transformations on the material parameters we also require the deformation gradient $\mathbf{F}_\xi$ of the diffeomorphism. This could be done by first determining $\boldsymbol{\xi}$ at the nodes of a numerical mesh, and then using Lagrange interpolation to obtain the necessary derivatives. A nicer method comes from differentiating eq. (156) with respect to $\mathbf{x}$ to obtain

$$\frac{d}{ds}\mathbf{F}_{\mathbf{X_q}}(\mathbf{x}, s) = \mathbf{F_q}[\mathbf{X_q}(\mathbf{x}, t)]\mathbf{F}_{\mathbf{X_q}}(\mathbf{x}, s),$$

$$\mathbf{F}_{\mathbf{X_q}}(\mathbf{x}, 0) = \mathbf{1}, \quad \forall \mathbf{x} \in \tilde{M}, \tag{157}$$

where $\mathbf{F_q} = [\nabla \mathbf{q}]^T$ and $\mathbf{F}_{\mathbf{X_q}} = [\nabla \mathbf{X_q}]^T$, and $\mathbf{1}$ is the identity matrix. This equation can be numerically integrated simultaneously with eq. (156), and is particularly useful when $\mathbf{q}$ is given analytically, as differentiation of an interpolated function can be avoided.

## 5.3 Example 1: plane wave propagation

In our first example, we consider an elastic body whose equilibrium stress vanishes, and is homogeneous and isotropic relative

to its natural reference configuration. The reference body $M$ is taken as a rectangle $\{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 2, \ 0 \leq x_2 \leq 1\}$ in non-dimensional units, and we apply periodic boundary conditions on the top and bottom of the domain. A solution of the elastodynamic equations is given by a horizontally travelling plane $p$-wave

$$\mathbf{u}(\mathbf{x}, t) = f(t - \mathbf{x} \cdot \hat{\mathbf{x}}_1/c_p)\hat{\mathbf{x}}_1, \tag{158}$$

where $f$ is an arbitrary smooth function, $c_p$ is the $p$-wave speed of the body, and $\hat{\mathbf{x}}_1$ is a horizontal unit vector. To simulate such a plane wave numerically, we use eq. (158) to obtain initial values for the displacement and velocity vector fields, and then time step the unforced system. In the upper panel of Fig. 1 we show the propagation of such a plane wave at various times as it travels across the domain.

Suppose we now define a diffeomorphism $\boldsymbol{\xi} \in \text{Diff}_{\partial M}(M)$ that differs from the identity mapping only in the central region of the reference body. We can then perform a particle relabelling transformation that leaves the material parameters and initial conditions unchanged at the edges of the body. Having done so, we can again start with a horizontally travelling plane wave near to the left-hand boundary, simulate its propagation through the central region where it will be suitably modified, and watch it emerge as an undistorted plane wave at the other side. Such a calculation is shown in the lower panel of Fig. 1. In particular, we note that the transformed body is strongly anisotropic, with the polarization vector of this '$p$-wave' being sometimes almost parallel to the wave front. The behaviour seen in this example is in accordance with the discussion in Section 4.4. In particular, we note that the polarization vectors in the transformed body are everywhere horizontal, and that the wave fronts in the two simulations are related through the given particle relabelling transformation.

The specific diffeomorphism used in this example was generated by numerical integration of the flow of the vector field

$$\mathbf{q}(\mathbf{x}) = \exp[-0.4(x_1 - 2x_2)^2/k_1^2] \exp[-2(0.2x_2^2 + 0.8x_1^2$$
$$+ 0.8x_1x_2 + 1.25 - x_2 - 2x_1)/k_2^2]\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \tag{159}$$

where $k_1 = 0.05$ and $k_2 = 0.6$. This vector field was chosen arbitrarily, but subject to the constraint that it is (effectively) equal to zero away from the centre of the reference body. The resulting inhomogeneous elastic body, shown in Fig. 2, is strongly anisotropic and so it is difficult to define a meaningful average velocity. We have, therefore, simply plotted the arithmetic average of the fastest and slowest $p$-wave velocities at each location. The orientation and relative magnitudes of the fast and slow directions in the model are also shown. Note that values are normalized such that the density and $p$-wave velocity in the original model equal one. Importantly, the elastic tensor $\tilde{\boldsymbol{\Lambda}}$ for the transformed body does not possess the symmetries of the elastic tensor in classical linear elasticity, but instead takes the form given in eq. (143). Consequently, it can be recognized that this body is not defined relative to a natural reference configuration.

By construction, the two wave propagation simulations shown in Fig. 1 represent the same physical process. This can be verified by using the appropriate diffeomorphism $\boldsymbol{\xi}$ to transform the complicated wavefield into the original plane wavefield. From a mathematical and numerical point of view, however, this is still a rather interesting result as the two models are very different, and the numerical wave propagation code does not 'know' that they are related in any way. In Fig. 3, we repeat the calculations used to generate the lower panel in Fig. 1, but now consider the effects of
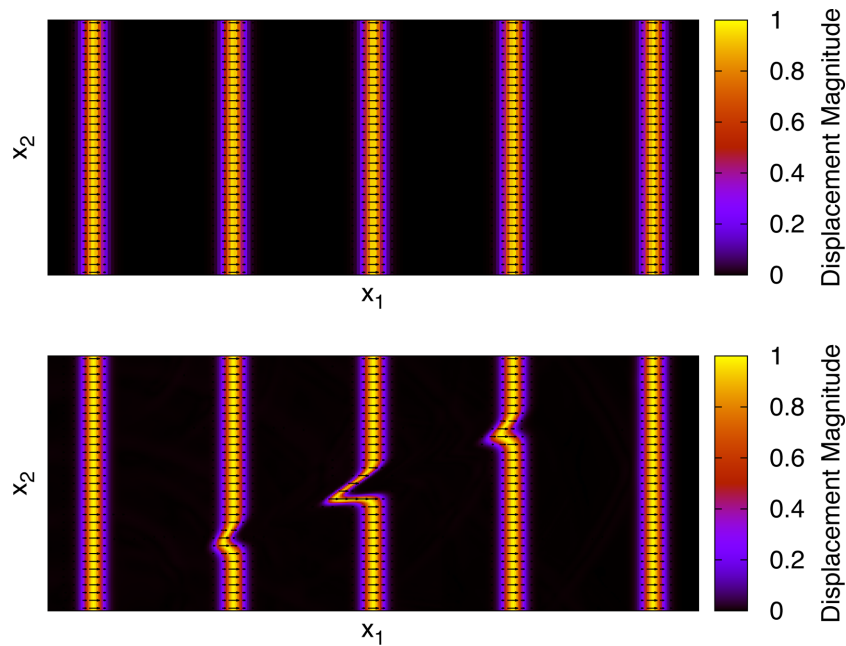
**Figure 1.** In the upper figure, we show a plane *p*-wave propagating horizontally through an isotropic and homogeneous elastic body. The computational domain was the rectangle $\{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 2, \, 0 \leq x_2 \leq 1\}$ in non-dimensional units, and we apply periodic boundary conditions on the top and bottom boundaries. Images of the wavefield at five equally spaced times have been superimposed, with the earliest time being represented by the wave front on the far left. The colour plotted denotes the magnitude of the displacement vector field, while arrows are used to show its relative size and orientation. In the lower figure, we display in the same manner wave propagation through an equivalent body obtained through a particle relabelling transformation. Here we see that the wavefield is strongly distorted and anisotropic where the material parameters have been changed, but outside these regions it returns to a simple plane wave. Note that in accordance with eq. (64) the displacement vector in the lower simulation remains everywhere and always horizontal.
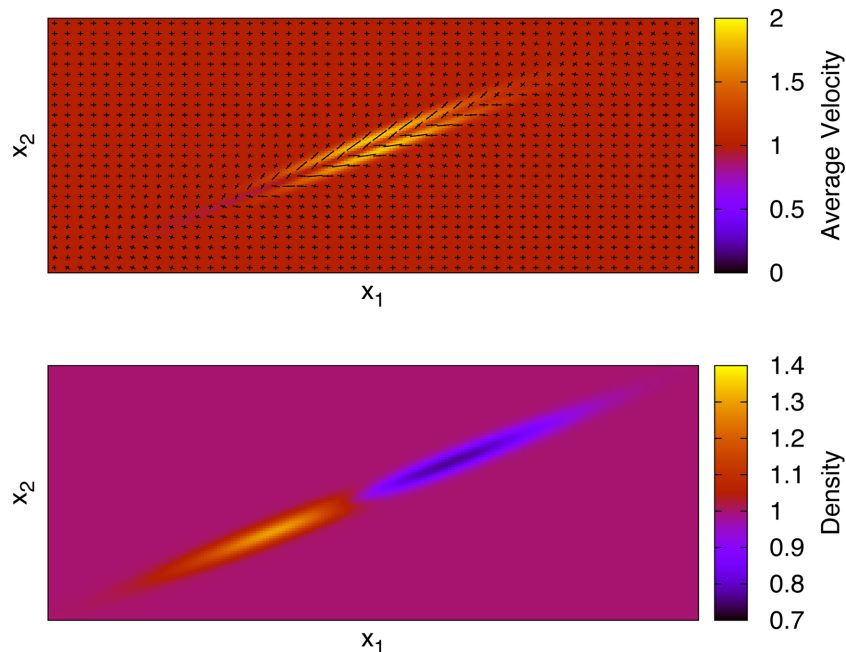




**Figure 2.** The inhomogeneous elastic body used in generating the lower plot in Fig. 1. In the upper panel, we show the average wave speed in the model (as defined in the main text) along with crosses indicating the orientation and relative magnitude of anisotropy; for each cross the longer line gives the direction with the fastest *p*-wave speed, while the shorter line gives the direction of the slowest, and the relative lengths of these lines indicate the magnitude of anisotropy. In locations where the material is isotropic we have fixed the orientation of these directions to be aligned with the vertical and horizontal. In the lower figure, we plot the spatial variations in the referential density.
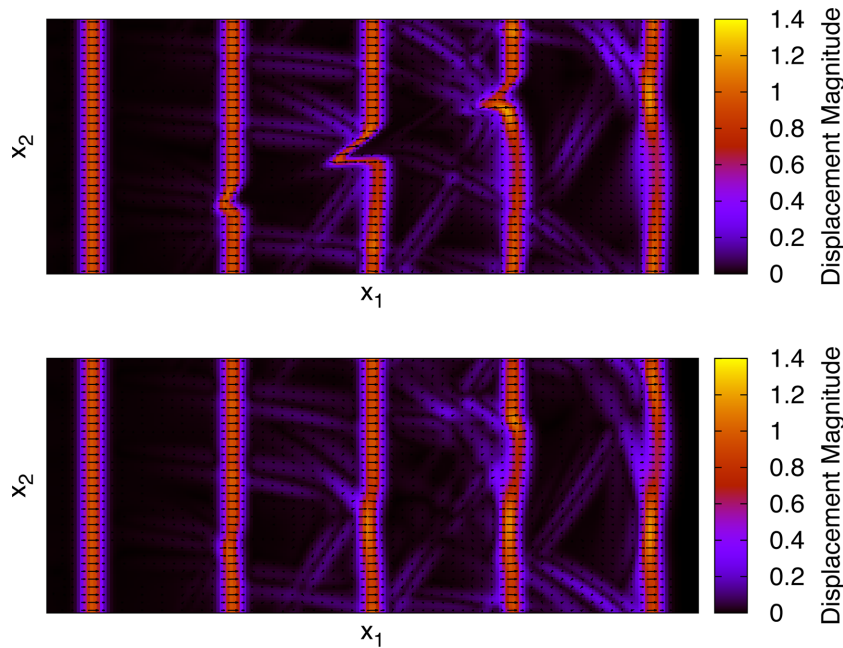
**Figure 3.** Two wavefield simulations plotted in the same manner as Fig. 1. In the upper panel, we show the wavefield generated by considering just the elastic tensor changes associated with the particle relabelling transformation, while in the lower figure we plot the corresponding results for the density changes alone. In both cases, there is very strong scattering and distortion of the wavefield. Comparison of these simulations with the lower panel of Fig. 1 shows that the magnitudes of these perturbations are well beyond the regime of linear scattering. It is, therefore, quite remarkable that their combined effects lead to the relatively simple wavefield in Fig. 1 that emerges from the heterogeneous central region as an unperturbed plane wave.

the elastic tensor and density changes separately. In both cases the wavefield is strongly scattered by the heterogeneity, and the final time slice of the wavefield differs greatly from a simple plane wave.

### 5.4 Example 2: mapping topography into volumetric heterogeneity

In our second example, we show how particle relabelling transformations can be used to map boundary topography into volumetric heterogeneity. Such applications could be practically useful because the inclusion of boundaries with complex geometry can be challenging when using numerical methods such as finite-differences (e.g. Levander 1988). The basis for the method is eq. (137) which shows how we can start with a model possessing complex topography, use a particle relabelling transformation to flatten this topography, solve the elastodynamic equations in the simplified model and then obtain the original wavefield through an inverse particle relabelling transformation. It is notable that this approach requires only modification to the material parameters describing the model, but involves no changes to the actual wave propagation code. This is in contrast to methods based on 'tensorial formulations' of the elastic wave equation in general curvilinear coordinates that lead to the introduction of additional terms into the equations of motion (e.g. Hestholm & Ruud 1994; Komatitsch *et al.* 1996; Zhang & Chen 2006; Tarrass *et al.* 2011; Zhang *et al.* 2012).

To illustrate this method we show in Fig. 4 a time slice of the elastic wavefield generated in a homogeneous and isotropic model possessing pronounced sinusoidal boundary topography. To construct the domain in this example we started with the square $\{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 1, \ 0 \leq x_2 \leq 1\}$ in dimensionless units and deformed the upper boundary into the curve
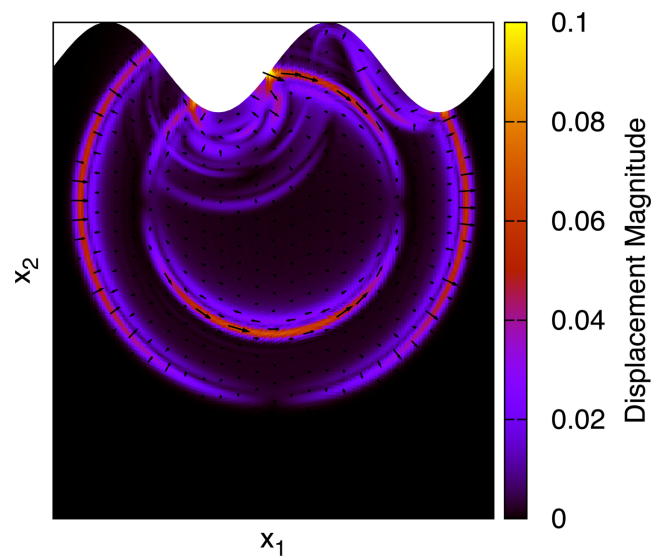
$$x_2 = 1 + 0.1 \sin(4\pi x_1). \tag{160}$$



**Figure 4.** A time slice of a wavefield simulated in a homogeneous and isotropic elastic body with significant surface topography as described in Section 5.4. As with Fig. 1, the colour indicates the magnitude of the displacement vector field, while arrows are used to denote its orientation and relative magnitude.

The wavefield shown in Fig. 4 was then generated by specifying an initial displacement and velocity within the model which roughly mimics a point source and time-stepping the discretized equations of motion.

The same wave propagation problem was also solved by first using a particle relabelling transformation to 'flatten' the boundary topography. Specifically we defined a diffeomorphism from
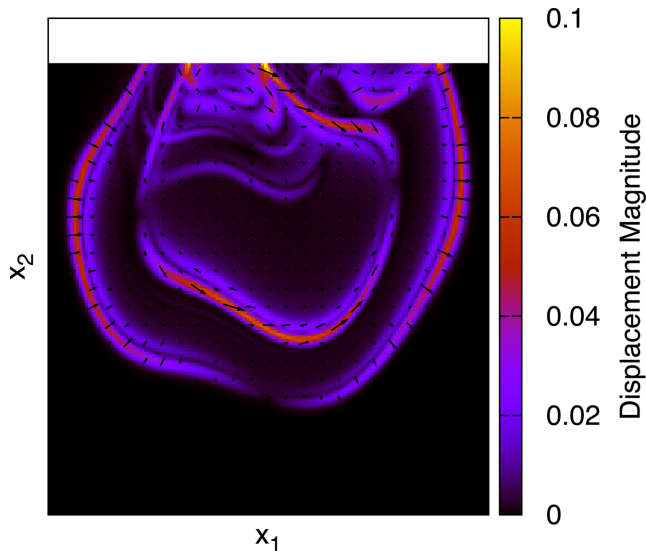
**Figure 5.** Here we show the equivalent wavefield to that in Fig. 4 calculated in an inhomogeneous and anisotropic body obtained using a particle relabelling transformation to 'flatten' the boundary topography of the original model.

$\{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1\}$ into the domain shown in Fig. 4 by setting

$$\xi_1(x_1, x_2) = x_1, \quad \xi_2(x_1, x_2) = x_2[1 + 0.1\sin(4\pi x_1)]. \tag{161}$$

Using the associated particle relabelling transformations we could then obtain analytically the density and elastic tensor for the prob-

lem with respect to the new reference body. The time slice of the wavefield in the transformed model equivalent to that in Fig. 4 is shown in Fig. 5, and the effects of heterogeneity and anisotropy can be clearly seen. According to eq. (137) these two wavefields are related through the particle relabelling transformation used to obtain the second model, and this has been directly verified. In particular, we plot horizontal component seismograms recorded at equivalent surface locations in the two models in Fig. 6, and they are seen to agree to a high precision.

## 6 CONCLUSIONS

In this work, we have shown that the form of the elastodynamic equations is invariant under particle relabelling transformations. The introduction of such transformations was motivated by a computational problem in normal-mode seismology, and the results of this paper make a substantial contribution to its solution. In particular, we have obtained formulae that can be used to transform the material parameters of a geometrically aspherical earth model into a geometrically spherical one. We have not yet, however, incorporated fluid–solid boundaries into the theory (Woodhouse & Dahlen 1978), allowed for rotating reference frames, nor discussed in any detail viscoelasticity. In a later paper we shall extend our present results to include such features, and will provide a full discussion of the exact incorporation of boundary topography into mode coupling calculations.

The applications of this present work do, however, extend well beyond normal-mode seismology. For example, we have generalized the non-uniqueness results of Mazzucato & Rachele (2006) in
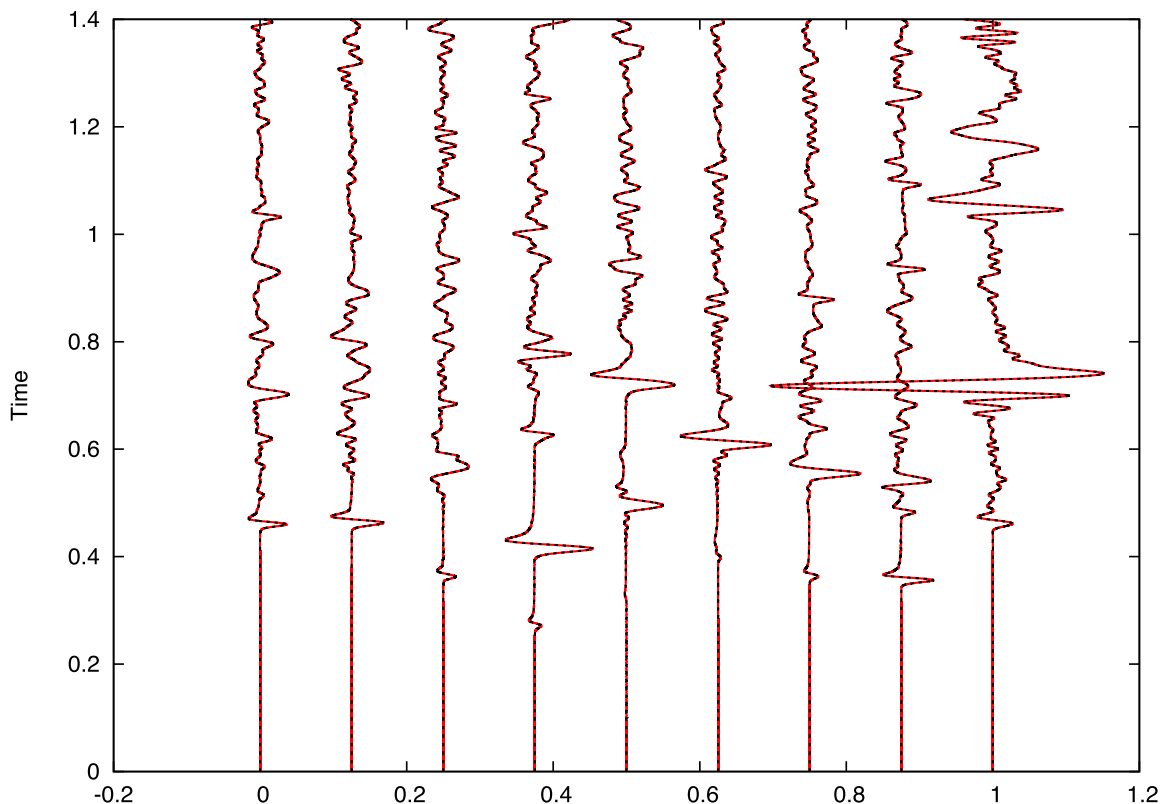


**Figure 6.** Horizontal component seismograms recorded at the surface of the two models shown in Figs 4 and 5. The horizontal axis in the plot denotes the distance of each recording point from the left most edge of the domain (of total length equal to one). Seismograms for the original model with boundary topography are plotted in a solid red line, while those for the transformed model are shown in a dashed black line. The two sets of seismograms agree to a very high precision, in accordance with eq. (137).

a form directly relevant to seismology, and have provided a physical interpretation in terms of particle relabelling. In particular, we have shown that while there is an infinite-fold non-uniqueness in elastodynamic problems due to the arbitrariness in assigning particle labels, this non-uniqueness does not have direct physical consequences. Indeed, within linearized elasticity this non-uniqueness can be completely removed by always describing the earth model with respect to a natural reference configuration, and this is already standard practice in seismology.

There may, however, be some advantages to employing non-natural reference configurations. For example, consider a tomographic inversion in which we wish to estimate simultaneously volumetric heterogeneity and topography on internal boundaries. For definiteness suppose that this problem is to be solved in an iterative manner using 'adjoint tomography' as pioneered by Tarantola (1984), Tromp *et al.* (2005) and others. Within such a scheme it is possible to include perturbations to both volumetric parameters and boundary topography (e.g. Liu & Tromp 2008). If, however, the effects of boundary topography are to be included, it is necessary to modify the numerical mesh used in the wave propagation calculations at each iteration, and this could, potentially, cause difficulties such as elements near the boundaries becoming overly deformed. This issue could, however, be circumvented by using a fixed non-natural reference configuration during the inversion. Variations to the boundary topography would then be contained within the equilibrium configuration $\varphi_e$ which is a volumetric parameter. In later work we will obtain sensitivity kernels for perturbations to $\varphi_e$ along with the other parameters defined on the fixed reference configuration, and investigate whether this method offers any practical advantages over those currently employed.

The above application also points to a possibly significant consequence of the particle relabelling symmetry described in Section 4.2. Suppose again that we are performing a tomographic inversion, but in this case we only invert for volumetric heterogeneity. To do this we must fix the location of any internal discontinuities within the earth model. Invariably the assumed locations of such discontinuities will be incorrect, and so we would, in effect, be using a non-natural reference configuration to describe the earth model. With respect to such a reference configuration, we know that the elastic tensor occurring in the linearized elastodynamic equations does not have the usual symmetries. In particular, eq. (143) shows that an isotropic elastic tensor relative to a non-natural reference configuration takes a more complicated and ostensibly anisotropic form. If, in such a situation, we performed a tomographic inversion based on the equations of classical linear elasticity we would actually be using the *wrong physics* to describe our observations, and errors would necessarily be introduced into the solution. For example, suppose that the real Earth is everywhere isotropic, but has unknown topography on internal boundaries. It is then quite conceivable that a tomographic inversion which fixed the boundary topography could lead to artificial anisotropic in the earth model obtained. Such effects may, in particular, be important for surface wave tomography if variations in the depth of discontinuities in the crust and upper mantle are neglected. The generation of such non-physical anisotropy is somewhat similar to homogenization effects (e.g. Backus 1962), but it is notable that the mechanism described here does not depend on a long-wavelength approximation, and instead follows from the exact particle relabelling symmetry in eq. (139). Future work will investigate quantitatively whether such effects could be important.

Finally, the particle relabelling transformations have potential applications to numerical simulation of elastic wave propagation, with particular relevance to the incorporation of complex topography on internal or external boundaries. While this approach is broadly similar to existing methods based on curvilinear coordinates (e.g. Hestholm & Ruud 1994; Komatitsch *et al.* 1996; Zhang & Chen 2006; Tarrass *et al.* 2011; Zhang *et al.* 2012), it is distinct. In these latter methods the various terms in the equations of motion are regarded as tensor fields defined on the equilibrium body. The behaviour of the equations of motion under a general curvilinear coordinate transformation is given by the rules of tensor calculus, and various terms involving the metric tensor and Christoffel symbols must be introduced. In this paper we have, however, taken a more general view of elastodynamics for which there are two geometric domains of interest: (i) the reference body and (ii) physical space. An object such as the displacement vector $\mathbf{u}(\mathbf{x}, t)$ is regarded a function on the reference body whose values are vectors attached to the point $\varphi_e(\mathbf{x})$ of physical space (where we recall that $\varphi_e$ denotes the equilibrium configuration). A particle relabelling transformation is, essentially, a change of coordinates within the reference body, but such transformations do not affect the vector part of the displacement vector which 'lives' in physical space. This property is seen clearly in the transformation of the displacement vector under a given diffeomorphism $\boldsymbol{\xi} : \tilde{M} \to M$

$$\tilde{\mathbf{u}}(\mathbf{x}, t) = \mathbf{u}[\boldsymbol{\xi}(\mathbf{x}), t], \tag{162}$$

and speaking roughly we could say that under such a change of reference configuration the displacement vector 'transforms like a scalar'. A full discussion of the geometric significance of particle relabelling transformations is, however, beyond the scope of this work, and would be most clearly expressed using the language of vector-valued differential forms (e.g. Kanso *et al.* 2007). From a practical perspective, however, the advantage of our method is that the form of the elastodynamic equations is unchanged, and so any necessary modifications to wave propagation codes are minimal.

## ACKNOWLEDGEMENTS

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# APPENDIX A: EXTENSION TO VISCOELASTIC MATERIALS

In this appendix, we show how the particle relabelling transformations described in Section 3.1 can be extended to include viscoelastic materials. Starting from eqs (71) and (77) we can use the chain rule to obtain the identity

$$\tilde{\mathbf{T}}(\mathbf{x}, t) = J_\xi(\mathbf{x})\mathbf{T}[\boldsymbol{\xi}(\mathbf{x}), t]\mathbf{F}_\xi(\mathbf{x})^{-T}, \tag{A1}$$

showing that the stresses $\mathbf{T}$ and $\tilde{\mathbf{T}}$ are related through the 'Piola-transform' associated with $\boldsymbol{\xi}$ (e.g. Marsden & Hughes 1983). This suggests an alternative method for obtaining, and generalizing, the preceding results. Consider an $n$-dimensional subset $U \subseteq M$, and its image $U_t = \boldsymbol{\varphi}_t(U)$ in physical space at a given instant of time. We recall that the first Piola–Kirchhoff stress tensor $\mathbf{T}$ is defined such that the total surface force acting on this sub-body is given by

$$\mathbf{f}_s(t) = \int_{\partial U} \mathbf{T}(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x}) \, d\Sigma(\mathbf{x}), \tag{A2}$$

where **n** is the outward normal vector to $U$. As we have seen, the same motion can be described with respect to another reference body $\tilde{M}$, with these two descriptions related through the particle relabelling associated with some diffeomorphism $\boldsymbol{\xi} : \tilde{M} \to M$. Relative to the new reference body $\tilde{M}$, the particles in $U_t \subseteq \mathbb{R}^n$ lie in the subset $\tilde{U} := \tilde{\boldsymbol{\varphi}}_t^{-1}(U_t) = \boldsymbol{\xi}^{-1}(U) \subseteq \tilde{M}$, and the total surface force can be written

$$\mathbf{f}_s(t) = \int_{\partial \tilde{U}} \tilde{\mathbf{T}}(\mathbf{x}, t) \cdot \tilde{\mathbf{n}}(\mathbf{x}) \, d\tilde{\Sigma}(\mathbf{x}), \tag{A3}$$

where $\tilde{\mathbf{T}}$ is the first Piola–Kirchhoff stress tensor defined relative to $\tilde{M}$, we have written $\tilde{\mathbf{n}}$ for the outward unit normal vector to $\tilde{U}$, and $d\tilde{\Sigma}$ is the surface element on $\partial \tilde{U}$. By a simple geometric argument (e.g. Dahlen & Tromp 1998; Antman 2005) we can obtain the identity

$$\mathbf{n}[\boldsymbol{\xi}(\mathbf{x})] \, d\Sigma[\boldsymbol{\xi}(\mathbf{x})] = J_{\xi}(\mathbf{x}) \mathbf{F}_{\xi}(\mathbf{x})^{-T} \cdot \tilde{\mathbf{n}}(\mathbf{x}) \, d\tilde{\Sigma}(\mathbf{x}), \tag{A4}$$

and it follows that the stress tensors **T** and $\tilde{\mathbf{T}}$ must be related through eq. (A1) in order for the above two expressions for the total surface force to be equal for an arbitrary subset.

Building on this idea, suppose that the first Piola–Kirchhoff stress tensor is related to the motion through the constitutive relation

$$\mathbf{T}(\mathbf{x}, t) := \mathfrak{T}[\mathbf{x}, \mathbf{F}^t(\mathbf{x}, \cdot)], \tag{A5}$$

where we have defined the *deformation gradient history* up to the time $t \in \mathbb{R}_+$ by

$$\mathbf{F}^t(\mathbf{x}, t') = \begin{cases} \mathbf{F}(\mathbf{x}, t - t') & 0 \leq t' \leq t \\ \mathbf{F}(\mathbf{x}, 0) & t' \geq t \end{cases}, \tag{A6}$$

and $\mathfrak{T}$ is now a *nonlinear operator* relating the deformation gradient history to the stress at a given time. Such a constitutive relation is quite general, and can, for example, describe both elastic and viscoelastic materials (e.g. Coleman & Noll 1961; Truesdell & Noll 2004). There are, of course, constraints placed on the form of the constitutive operator due to material frame indifference and the *principle of fading memory* (e.g. Coleman 1964; Wang 1965; Gurtin 1968), but these are not crucial to the present discussion. Furthermore, a stress glut could be incorporated into the theory in a simple manner. With the above notation, we can write the balance

of linear momentum within an $n$-dimensional subset $U \subseteq M$ as

$$\frac{d}{dt} \int_U \rho(\mathbf{x}) \mathbf{v}(\mathbf{x}, t) \, d^n\mathbf{x} = \int_{\partial U} \mathfrak{T}[\mathbf{x}, \mathbf{F}^t(\mathbf{x}, \cdot)] \cdot \mathbf{n}(\mathbf{x}) \, d\Sigma(\mathbf{x})$$
$$+ \int_U \rho(\mathbf{x}) \boldsymbol{\gamma}(\mathbf{x}, t) \, d^n\mathbf{x}, \tag{A7}$$

where the integral on the left-hand side represents the total linear momentum, the first term on the right-hand side is the total surface force, and the final term gives the gravitational force acting on the sub-body. Each of these terms can, of course, be written relative to a different reference body $\tilde{M}$, and we obtain

$$\frac{d}{dt} \int_{\tilde{U}} \tilde{\rho}(\mathbf{x}) \tilde{\mathbf{v}}(\mathbf{x}, t) \, d^n\mathbf{x} = \int_{\partial \tilde{U}} \tilde{\mathfrak{T}}[\mathbf{x}, \tilde{\mathbf{F}}^t(\mathbf{x}, \cdot)] \cdot \tilde{\mathbf{n}}(\mathbf{x}) \, d\tilde{\Sigma}(\mathbf{x})$$
$$+ \int_{\tilde{U}} \tilde{\rho}(\mathbf{x}) \tilde{\boldsymbol{\gamma}}(\mathbf{x}, t) \, d^n\mathbf{x}, \tag{A8}$$

where, as above $\tilde{U} = \boldsymbol{\xi}^{-1}(U)$, and we have written $\tilde{\mathfrak{T}}$ for the constitutive functional defined with respect to $\tilde{M}$. Transforming the integration in eq. (A7) using $\boldsymbol{\xi}$, and using the kinematic relations in eqs (65) and (A4), we find

$$\frac{d}{dt} \int_{\tilde{U}} J_{\xi}(\mathbf{x}) \rho[\boldsymbol{\xi}(\mathbf{x})] \tilde{\mathbf{v}}(\mathbf{x}, t) \, d^n\mathbf{x}$$

$$= \int_{\partial \tilde{U}} J_{\xi}(\mathbf{x}) \mathfrak{T}[\boldsymbol{\xi}(\mathbf{x}), \tilde{\mathbf{F}}^t(\mathbf{x}, \cdot) \mathbf{F}_{\xi}(\mathbf{x})^{-1}] \mathbf{F}_{\xi}(\mathbf{x})^{-T} \cdot \tilde{\mathbf{n}}(\mathbf{x}) \, d\tilde{\Sigma}(\mathbf{x})$$
$$+ \int_{\tilde{U}} J_{\xi}(\mathbf{x}) \rho[\boldsymbol{\xi}(\mathbf{x})] \boldsymbol{\gamma}[\boldsymbol{\xi}(\mathbf{x}), t] \, d^n\mathbf{x}. \tag{A9}$$

The three terms in this equation can be identified with the corresponding parts of eq. (A8), and as these equalities must hold for arbitrary subbodies, we obtain the relations

$$\tilde{\rho}(\mathbf{x}) = J_{\xi}(\mathbf{x}) \rho[\boldsymbol{\xi}(\mathbf{x})],$$

$$\tilde{\mathfrak{T}}[\mathbf{x}, \tilde{\mathbf{F}}^t(\mathbf{x}, \cdot)] = J_{\xi}(\mathbf{x}) \mathfrak{T}[\boldsymbol{\xi}(\mathbf{x}), \tilde{\mathbf{F}}^t(\mathbf{x}, \cdot) \mathbf{F}_{\xi}(\mathbf{x})^{-1}] \mathbf{F}_{\xi}(\mathbf{x})^{-T}, \tag{A10}$$

which generalize those in eq. (71). A more complete discussion of the viscoelastic problem is intended for a later work.