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Bounds for the Number of Independent and Dominating Sets in Trees

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Abstract

In this work, we investigate bounds on the number of independent sets in a graph and its complement, along with the corresponding question for number of dominating sets. Nordhaus and Gaddum gave bounds on $\chi(G)+\chi(\overline{G})$ and $\chi(G)\ \chi(\overline{G})$, where G is any graph on n vertices and $\chi(G)$ is the chromatic number of G. Nordhaus-Gaddum-type inequalities have been studied for many other graph invariants. In this work, we concentrate on $i(G)$, the number of independent sets in G, and $\partial(G)$, the number of dominating sets in G. We focus our attention on Nordhaus-Gaddum-type inequalities over trees on a fixed number of vertices. In particular, we give sharp upper and lower bounds on $i(T)+i(\overline{T})$ where T is a tree on n vertices, improving bounds and proofs of Hu and Wei. We also give upper and lower bounds on $i(G)+i(\overline{G})$ where G is a unicyclic graph on n vertices, again improving a result of Hu and Wei. Lastly, we investigate $\partial(T)+\partial(\overline{T})$ where T is a tree on n vertices. We use a result of Wagner to give a lower bound and make a conjecture about an upper bound.

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Bounds For The Number of Independent And Dominating Sets in Trees

by

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Chapter 1

Introduction

A *graph* G is a pair consisting of a *vertex set* $V(G)$ and an *edge set* $E(G)$, a subset of pairs of $V(G)$. We will focus on *simple graphs*, a graph having no loops or multiple edges. Mathematicians have been studying graphs since at least 1736 when Leonhard Euler first studied the Konigsberg Bridge problem, which was a problem about traversing all the edges of a graph without repeating any edges.

A fundamental notion in graph theory is whether two vertices in a graph are *adjacent*. Vertices $x, y \in V(G)$ of a graph G are *adjacent*, denoted $x \sim y$, if x and y are both endpoints of the same edge, that is if $x \sim y$ then $xy \in E(G)$. Note that $x \neq y$ since we will assume throughout that G is a simple graph and there are no loops in simple graphs. The neighborhood of x, denoted $N(x)$, is the set of all vertices adjacent to x and the closed neighborhood, denoted $N[x]$, is the set of all vertices adjacent of x along with x itself ($N[x] = N(x) \cup \{x\}$).

We further characterize the vertex set by splitting $V(G)$ into *independent sets*. An *independent set* in a graph is a set of vertices, in which every pair is *not* adjacent. In other words, if $x \nsim y$ (*nonadjacent*) then $xy \notin E(G)$ and x and y can be collected into an independent set together. If x is in an independent set S, then none of the neighbors of x are in S.

A *clique*, C, in a graph G is a set of vertices such that all vertices in C are adjacent. We can think of cliques and independent sets as extremes of adjacency. The vertices in an independent set are nonadjacent to every other vertex in the set, while a vertex in a clique is adjacent to every other vertex in the set. An interesting relationship between cliques and independent sets of a graph G is the elements of each set switch when considering the *complement of* G. Given a graph G, we let the *complement* of G, denoted $\overline{G}$, have the same vertex set as G and $e \in E(\overline{G})$ if and only $e \notin E(G)$. This relationship dictates that if you understand the independent sets in a graph G you also understand the cliques in $\overline{G}$. The concept of independence lays the foundation for several areas of study of the vertex set outlined in the sections below.

1.1 Chromatic Number

We chose to explore specific invariants of a graph G. The first invariant we examine is the *chromatic number* of G. The *chromatic number* of a graph G, denoted $\chi(G)$, is the minimum number of colors needed to label the vertices so that adjacent vertices receive different colors. This means vertices in the same color set are nonadjacent and form an independent set. Another way of characterizing the chromatic number of a graph G is the minimum number k so that $V(G)$ can be partitioned into k independent sets. A graph G with chromatic number k is said to be *k-partite* since $V(G)$ can be expressed as the union of k disjoint independent sets.

Consider the graph in Figure 1.1: the vertices on the right side are all nonadjacent to one another, but each is adjacent to every vertex on the left. Thus the vertices on the right form an independent set. The vertices on the left are adjacent to every other vertex in the graph; therefore, each vertex on the left forms its own independent set as a singleton. This gives us four independent sets of the vertices, but we must insure that we cannot partition $V(G)$ into a smaller number of independent sets, which we cannot. Thus, the chromatic number in Figure 1.1 is $\chi(G) = 4$ and G is a 4-partite. Now examing the relationship betwen G and $\overline{G}$ in Figure 1.1 we see that the vertices on the left in G form a clique therefore they form an independent set in $\overline{G}$. The set of vertices on the right forms an independent set in G, hence, in $\overline{G}$ it forms a clique. We can see that $\chi(\overline{G}) = 3$.

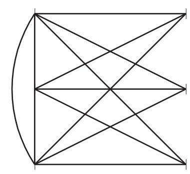

Figure 1.1: Graph of $S_{3,3}$

Our interest in the chromatic number of a graph G does not only concern finding $\chi(G)$, but identify the extremes of $\chi(G)$. Exploring the minimum and maximum values of $\chi(G)$ allows us to find relationships between the chromatic number and other properties of the graph. The chromatic number is affected by the *clique number*. The *clique number* of a graph G, denoted $\omega(G)$ is the maximum size of a clique in G. The chromatic number is also affected by the *maximum size of independent set*, denoted $\alpha(G)$, along with the number of vertices in G, denoted $n(G)$. In his book, *Introduction to Graph Theory* [1], West provides the following propositions. We include the proofs for completeness and to illustrate the application of definitions in proofs.

Proposition 1. *For every graph G, $\chi(G) \geq \omega(G)$ and $\chi(G) \geq \frac{n(G)}{\alpha(G)}$.*

Proof. We can see that no vertices in the clique may have the same color, thus the chromatic number must be at least than the clique number. The second bound holds because each color class is an independent set and has at most $\alpha(G)$ vertices. □

Clearly, an upper bound on the chromatic number must be $n(G)$, a distinct color for each vertex. This is not a great bound for all graphs, but we can prove a better bound using the maximum degree of G, denoted $\Delta(G)$, and *greedy coloring* algorithm. The *greedy coloring* relative to a vertex ordering $v_1, ..., v_n$ of $V(G)$ is obtained by coloring the vertices in the order $v_1, ..., v_n$, assigning to v_i the smallest-indexed color not already used on its lower-indexed neighbors. Through the greedy coloring algorithm we obtain the upper bound.

Proposition 2. *For every graph G,* $\chi(G) \leq \Delta(G)+1$.

Proof. In a vertex ordering, each vertex as at most $\Delta(G)$ earlier neighbors, thus we used at most $\Delta(G)$ colors leaving the current vertex uncolored. Thus $\Delta(G)+1$ colors are enought to color the vertex set. □

Given the bounds on the chromatic number, $\omega(G) \leq \chi(G) \leq \Delta(G) + 1$, we can extend them to be the bounds on the number of partition of G. Thus a graph can be partitioned between $\omega(G)$ and $\Delta(G) + 1$ independent sets, when the number of independent sets is minimized.

Propositions 1 and 2 establish a lower and an upper bound for $\chi(G)$. We now fix our gaze to the relationship between a graph and its complement. Nordhaus and Gaddum provided bounds on the sum and product of the chromatic number of a graph and its complement. They proved the following [2].

Theorem 3 (Nordhaus, Gaddum 1956)**.** *If G is a graph on n vertices, then*

$$2\sqrt{n} \leq \chi(G)+\chi(\overline{G}) \leq n+1,$$

and

$$n \leq \chi(G) \cdot \chi(G) \leq {\frac{n+1}{2}}^{2}.$$

1.2 Nordhaus and Gaddum Inequalities

Chromatic numbers have been extensively studied dating back to 1850, when Francis Guthrie proposed the Four Color Problem [3]. This problem was later solved by means of a computer in 1976 by Appel and Haken. Exploration of the chromatic number was extensive for the next hundred years, but all these studies investigating chromatic numbers considered a graph G only, not its complement. Zykov proved the lower bound $n \leq \chi(G) \cdot \chi(\overline{G})$, from Theorem 3, creating the first study of chromatic numbers on a graph G and its complement $\overline{G}$ together [3]. Nordhaus and Gaddum, in 1956, furthered the exploration of the chromatic number by proving a lower and an upper bound on the sum and on the product of $\chi(G)$ and $\chi(\overline{G})$ over various classes of graphs. One well known class is that of trees.

The relationship between a graph G and its complement $\overline{G}$ is not only a concern for chromatic numbers, but also for any invariant of the graph and its complement. The bounds found on the sum or product of an invariant in a graph and the same invariant in its complement is called a Nordhaus-Gaddum type inequality. There are currently hundreds of Nordhaus-Gaddum type of inequalities in graph theory and more being explored [3]. We use Nordhaus-Gaddum Inequalities as an inspirations for our current work and study, particularly the Nordhaus-Gaddum type inequalities pertaining to independent sets.

1.3 Trees

A graph that contains no cycle is called *acyclic* and a *tree* is a connected acyclic graph. A *leaf* or *pendant vertex* is a vertex of degree 1. We examine independent sets of trees in the following sections, but first we will give foundational knowledge on trees. One of the first characteristics we learn is that deleting a leaf from a tree results in a smaller tree.

Lemma 4. *Every tree with at least two vertices has at least 2 leaves. Deleting one leaf from an n-vertex tree results in a tree with $n-1$ vertices.*

Trees are connected and acyclic graphs, so we can draw the conclusion that a tree on n-vertices has $n-1$ edges. West [1] provides the following characterization of trees.

Theorem 5. *For an n-vertex graph G (with $n \geq 1$), the following are equivalent and characterize trees with n vertices:*

A. G is connected and has no cycles.

B. G is conneceted and has $n-1$ edges.

C. G has $n-1$ edges and no cycles.

A tree with maximum degree $\Delta \geq 2$ has at least Δ leaves. This is true since the degree of a vertex is the number of neighbors a vertex has and each neighbor corresponds to a branch and every branch must end with at least one leaf. Paths are trees that have maximum degree 2 while a *star* is a tree with maximum degree $n-1$. Let P_n be the path on n vertices and T be any tree on n vertices. Then $2 = \Delta(P_n) \leq \Delta(T) \leq \Delta(S_n) = n-1$. Paths and stars are important for our work in the later sections on independent sets in trees.

1.4 Unicyclic Graphs

Another graph that we examine is *unicyclic graph*. A *unicyclic graph* is a connected graph with exactly one cycle. We can obtain any unicyclic graph from a tree by adding one edge in the tree. Adding a one edge to a tree forms exactly one cycle.

Thus, the resulting graph is unicyclic. A tree on n vertices has $n - 1$ edges, thus, a unicyclic graph on n vertices has n edges.

A cycle is a type of unicyclic graph, because it has exactly one cycle and is connected. The other unicyclic graphs have a cycle with potential trees extending from the vertices on the the cycle. Note that the edges on the cycle are the only edges that connect the different trees. It is because if there is another edge connecting the trees but not on the cycle, we would have 2 cycles in the graph and the graph would not be unicyclic.

Chapter 2

Independent Sets in Trees

While exploring the chromatic number of graphs it naturally brings us to *independent sets*. We let $I(G)$ be the set of all independent sets in a graph G, and set $i(G) = |I(G)|$. Given $v \in V(G)$ we can calculate the number of independent sets in G by finding the number of independent sets containing v, denoted $i(G - N[v])$, and the number of independent sets not containing v, denoted $i(G - v)$. Using this approach gives us the following identity:

$$i(G) = i(G - N[v]) + i(G - v).$$

There have been many recent studies on $i(G)$ over various classes of graphs. We will first focus on the number of independent sets in a *tree*. For example, Prodinger and Tichy [4] gave bounds on the number of independent sets in a tree.

Theorem 6 (Prodinger, Tichy 1982). *If T is a tree on n vertices, then*

$$i(P_n) \leq i(T) \leq i(K_{1,n-1}).$$

Prodinger and Tichy also proved that $i(P_n) = F_{n+1}$, the $(n+1)^{st}$ Fibonacci number[1] and that $i(C_n) = L_n$, the n^{th} Lucas number[2]. If we let $S_n = K_{1,n-1}$, where S_n is an n-vertex star, then $i(S_n) = 2^{n-1} + 1$. It is quite easy to see, because an independent set either contains the central vertex of the star or it does not. There is only 1 independent set that contains the central vertex and there are 2^{n-1} sets that do not contain the central vertex, because we can take any subset of the $n-1$ leaves.

Hu and Wei [5] continued the exploration of $i(T)$ through their investigation of Nordhaus-Gaddum inequalities for the number of independent sets in specific trees. $DS_{2,n-4}$ is the double star on n vertices with one hub adjacent to two leaves and the other adjacent to $n-4$ leaves.

Theorem 7 (Hu, Wei 2018). *If T is a tree on n vertices with connected complement, then*

$$i(P_n) \leq i(T) + i(\overline{T}) \leq i(DS_{2,n-4}),$$

We are able to accomplish two objectives with Hu and Wei's result: first we can provide a simpler proof of the upper bound through compression and second we extend the proof to encompass all trees regardless of their complements.

Let G be any be any non-complete graph, and let x and y be adjacent vertices in G. The choice of x and y defines a natural partition of $V(G)$ into four parts: vertices which are adjacent only to x, vertices adjacent only to y, vertices adjacent to both, and vertices adjacent to neither. We write

$$A_{xy} = \{v \in V(G) : v \sim x, v \not\sim y\},$$

$$A_{xy} = \{v \in V(G) : v \sim x, v \sim y\},$$

[1]The *Fibonacci number* F_n is defined by $F_0 = 0$, $F_1 = 0$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.
[2]The *Lucas number* L_n is defined by $L_0 = 2$, $L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$.

$$A_{xy} = \{v \in V(G) : v \text{ ?V } x, v \sim y\}.$$

The *compression of G from x to y*, denoted $G_{x\to y}$, is the graph obtained from G by deleting all the edges between x and A_{xy} and adding all edges from y to A_{xy}. Compression along nonadjacent vertices is possible and used in other problems. However, we focus on compression only along adjacent vertices. Cutler and Radcliffe [6] established that compression on a non-complete graph does not decrease the number of independent sets in the graph.

Lemma 8. *If G is a non-complete graph, and x and y are adjacent vertices in G, then*

$$i(G) \le i(G_{x\to y}).$$

Proof. We'll show that there is an injection from $\mathbf{I}(G) \setminus \mathbf{I}(G_{x\to y})$ to $\mathbf{I}(G_{x\to y}) \setminus \mathbf{I}(G)$. If $I \in \mathbf{I}(G) \setminus \mathbf{I}(G_{x\to y})$, then $y, z \in I$ for some $z \in A_{xy}$. But since $x \sim y$ (in both graphs), we must have $x \notin I$ but then $I \setminus (\{y\} \cup \{x\})$ is an inpendendent set in $G_{x\to y}$ but not in G. □

In essence, the number of independent sets does not decrease when a graph is compressed. Using compression and Lemma 8 creates a natural avenue for simplifying Hu's and Wei's proof of the upper bound. Our goal of simplifying the proof of the upper bound is reliant on compression; however, we must establish that the compression of a tree results in a tree.

Lemma 9. *If T is a tree on n vertices and x and y are adjacent vertices, then $T_{x\to y}$ is a tree.*

Proof. To prove that $T_{x\to y}$ is a tree we must show $T_{x\to y}$ possesses two characteristics. First, we must show $T_{x\to y}$ is connected. Second, we must show $T_{x\to y}$ contains no cycles.

Let T be a tree on n vertices. The compression $T_{x \to y}$ deletes all edges from x to $A_{x\overline{y}}$ and adds them from y to $A_{x\overline{y}}$ and x becomes a leaf of y. Now we must show that $T_{x \to y}$ is connected. Given vertices $u, v \in V(T)$ we know there exists a path between u and v in T. This path either contains the vertex x or it does not in T. If the u,v-path in T does not contain the vertex x then it is unaffected in $T_{x \to y}$. If the u, v-path in T does contain the vertex x then it now travels through the vertex y in $T_{x \to y}$. Since all vertices in $A_{x\overline{y}}$ are now adjacent to y in $T_{x \to y}$. Thus, $T_{x \to y}$ is connected.

Since $T_{x \to y}$ is connected, it is equivalent to show that $T_{x \to y}$ has the same amount of edges as T, namely $n - 1$, to prove there are no cycles in $T_{x \to y}$. Because $xy \in E(T)$, $xy \in T_{x \to y}$. Now we must consider the set $A_{x\overline{y}}$ of vertices adjacent to x but not y. The edges between x and $A_{x\overline{y}}$ no longer exist in $T_{x \to y}$ but for every edge lost we gain one through the edge formed between y and $A_{x\overline{y}}$. Hence, we still have $n - 1$ edges in $T_{x \to y}$. Thus, $T_{x \to y}$ is a tree. □

We now examine the effects of repeated compressions on a tree. A single compression on a tree results in a tree, so repeated compression on a tree will also result in a tree. However, we claim that given enough compressions on a tree the result will always be a star.

Lemma 10. *Repeated compression along edges in a tree yields a star.*

Proof. Let T be a tree and let x and y be vertices. If x and y are adjacent non-leaf vertices we can compress along this edge and increase the number of leaves in the tree. Allowing $x, y \in V(T)$ to be such vertices and compressing along the edge xy we obtain $T_{x \to y}$. All other existing leaves before the compression remain being leaves and x becomes a new leaf of y. So, compression strictly increases the number of leaves. If there are not any adjacent non-leaves then the tree is a star. □

Compressions on trees do not decrease the number of independent sets and turn any tree into a star. Using these qualities provides a natural avenue for a tight upper bound on the number of independent sets in trees.

Lemma 11. *If T is a tree on n vertices then* $i(T) \leq i(S_n)$.

Proof. This is a direct consequence of Lemma 8 and Lemma 10. □

Before we explore the Nordhaus and Gaddum sum inequality on the number of independent sets in trees it is necessary to analyze the number of independent sets in a tree's complement. Hu and Wei [5] proved that $i(\overline{T}) = 2n$ for a connected complement $\overline{T}$. However, Hu's and Wei's proof extends to any complement $\overline{T}$, so we include their proof for completion.

Lemma 12. *Let T be a tree on n vertices with any complement $\overline{T}$, then*

$$i(T) + i(\overline{T}) = i(T) + 2n.$$

Proof. Let $i_k(\overline{T})$ be the number of independent sets of size k. Then $i_k(\overline{T}) = 0$ for $k \geq 3$. Assume to the contrary that $\overline{T}$ has an independent set of size 3, but that would mean T would contain a cycle C_3, a contradiction to T being a tree. Now we must consider the number of independent sets in $\overline{T}$ of sizes 0, 1, and 2, that is $i_0(\overline{T})$, $i_1(\overline{T})$ and $i_2(\overline{T})$ respectively. An independent set of size zero can only occur one way, the empty set, so $i_0(\overline{T}) = 1$. An independent set of size 1 is a set containing only one vertex thus, $i_1(\overline{T}) = n$ since there are only n distinct vertices. While looking at the number of independent sets of size two we remind ourselves that that two vertices $x, y \in V(\overline{T})$ can be in an independent set together only if $xy \in E(T)$. Since a tree T on n vertices has $n - 1$ edges this means there are $n - 1$ edges "missing" from $\overline{T}$ thus $i_2(\overline{T}) = n - 1$. Using these outcomes we have the following result.

$$i(\overline{T}) = i_0(\overline{T}) + i_1(\overline{T}) + i_2(\overline{T}).$$

$$i(\overline{T}) = 1 + n + (n - 1) = 2n.$$

Thus,

$$i(T) + i(\overline{T}) = i(T) + 2n.$$

□

Lemma 12 establishes that the number of independent sets in any tree's complement is $2n$, meaning $i(\overline{S_n}) = 2n$. Thus, if we wish to minimize or maximize $i(T)+i(\overline{T})$ we need only minimize or maximize $i(T)$. Utilizing $i(\overline{T}) = i(\overline{S_n}) = 2n$ and the effects of compressions on T we produce the following tight upper bound on the Nordhaus Gaddum inequality for the number of independent sets in a tree.

Theorem 13 (Improved result by Hu, Wei). *If T is a tree on n vertices with any complement $\overline{T}$, then*

$$i(P_n) + i(\overline{P_n}) \leq i(T) + i(\overline{T}) \leq i(S_n) + i(\overline{S_n}),$$

Proof. Let T be any tree on n vertices with any complement $\overline{T}$. From Lemma 12 we have

$$i(T) + i(\overline{T}) = i(T) + 2n.$$

Using Theorem 6 and Lemma 11 we have,

$$i(P_n) + i(\overline{P_n}) = iP(n) + 2n \leq i(T) + i(\overline{T}) \leq i(S_n) + 2n = i(S_n) + \overline{i}(S_n).$$

Therefore,

$$i(P_n) + i(\overline{P_n}) \leq i(T) + i(\overline{T}) \leq i(S_n) + \overline{i}(S_n).$$

□

Chapter 3

Independent Sets in Unicyclic Graphs

Hu and Wei also investigated Nordhaus Gaddum inequalites for the number of independent sets in a unicyclic graph and its complement. They defined the graph O_{x_1,x_2,x_3} as a unicyclic graph on n vertices created from a cycle $C_3 = v_1v_2v_3$ by attaching x_i ($i = 1, 2, 3$) pendent vertices to v_i such that $x_1 + x_2 + x_3 + 3 = n$ [5]. This means $O_{n-4,1,0}$ is a triangle with one vertex adjacent to $n-4$ leaves, the second vertex adjacent to 1 leaf and the last vertex adjacent to no leaf. Hu and Wei were able to prove the following:

Theorem 14 (Hu, Wei 2018)**.** *Let G be a unicyclic graph of order $n \geq 5$ with a connected complement $\overline{G}$, then*

$$i(C_n) + i(\overline{C_n}) \leq i(G) + i(\overline{G}) \leq i(O_{n-4,1,0}) + i(\overline{O_{n-4,1,0}}).$$

We improve upon Hu's and Wei's theorem by expanding the theorem to include a unicyclic graph with any complement, connected or disconnected, and by coming up with a simpler verison of their proof. We accomplish these tasks by first examining a result of Pedersen and Vestergaard [7], who investigated bounds for the number of independent sets in a unicyclic graph.

Theorem 15 (Pedersen, Vestergaard 2005)**.** *If G is a unicyclic graph of order n, then*

$$i(C_n) \leq i(G) \leq 3(2)^{n-3} + 1.$$

While exploring Pedersen's and Vestergaard's work, in hopes to simplify Theorem 14, we developed more concise proofs of Pedersen's and Vestergaard's bounds for the number of independent sets in a unicyclic graph. We shorten the proof of the lower bound in Theorem 15 by utilizing induction, similar to Prodinger and Tichy's proof of the lower bound on the number of independent sets in trees (Theorem 6). A key notion in our proof relies on the fact that deleting a leaf from a unicyclic graphs results in another unicyclic graph, if a leaf exists.

Lemma 16. *Let G be a uncyclic graph of on $n \geq 3$ vertices with a leaf v, then $G - v$ is unicyclic.*

Proof. Let v be a leaf on the unicyclic graph G. Consider the graph $G - v$. We want to prove that $G - v$ is unicyclic, namely, $G - v$ contains only one cycle and is connected. Let $u, w \in V(G)$ and $u, w \neq v$, then there exists a uw-path since G is connected. Vertex v cannot be a vertex on the uw-path since v is a leaf. So, deleting v leaves the uw-path unaffected in $G - v$. Thus, $G - v$ is connected. Since, v is a leaf it is not on the cycle present in G, deleting v results in the same cycle in $G - v$. Therefore $G - v$ is a unicyclic graph. □

To prove the lower bound on the number of independent sets in a unicyclic graph we follow Prodinger's and Tichy's method of calculating the number of independent sets with a leaf v and the number of independent sets without the leaf v, $i(G) = i(G - N[v]) + i(G - v)$.

Theorem 17. *Let G be a unicyclic graph on $n \geq 3$ vertices, then*

$$i(G) \geq i(C_n).$$

Proof. We use induction on the number of vertices n in G. For $n = 3$ the only unicyclic graph is C_3, $i(G) = i(C_3)$. Suppose $i(G) \geq i(C_n)$ for $n \geq 4$ vertices. Consider G on $n + 1$ vertices, we want to show that $i(G) \geq i(C_{n+1})$. If $G = C_{n+1}$ we're done. If $G \neq C_{n+1}$, then G contains a vertex v of degree one. We know that $G - v$ is uncyclic, from Lemma 16, and by induction $i(G - v) \geq i(C_n)$. Also, we know

$$i(G) = i(G - N[v]) + i(G - v).$$

Thus, by induction

$$i(G) \geq i(G - N[v]) + i(C_n).$$

The graph $G - N[v]$ may be disconnected, but we can add edges to connect the components of $G - N[v]$ to create a graph that is unicyclic, without increasing the number of independent sets. So, $G - N[v]$ is now a unicyclic graph on $n - 1$ vertices. Thus, by induction

$$i(G) \geq i(C_{n-1}) + i(C_n) = L_{n+1} + L_n = L_{n+1} = i(C_{n+1}).$$

Therefore,

$$i(G) \geq i(C_{n+1}).$$

□

We also provide a simpler the proof for the upper bound in Theorem 15 by applying compression to the graph as we did for Theorem 7. We know that compression does not decrease the number of independent sets. Compression is a natural avenue to help establish the upper bounds for the number of independent sets in a unicyclic graph. Knowing that repeated compression to a tree results in a star is going to help establish the affects of compression on a unicyclic graph. However, a unicyclic graph contains a cycle, so we must explore the effects of compresssion on a cycle.

Lemma 18. *If $n \geq 3$ and $C = C_n$ with adjacent vertices x, y, then $C_{x \to y}$ is a unicyclic graph.*

Proof. Suppose x and y are adjacent vertices on the cycle C_n. We know that $|N(x)| = 2$ and that one of these neighbors is y. Let the other neighbor of x be v. If $n = 3$ then we have a triangle, which cannot be compressed since no vertex has any unique neighbors. If $n > 3$ the compression $C_{x \to y}$ makes vertex x a leaf of y and v becomes adjacent to y. So $C_{x \to y}$ contains a cycle of length $n - 1$ and one leaf; thus, $C_{x \to y}$ is a unicyclic graph. □

Given that the compression of a cycle results in a unicyclic graph, one should ask, "How will repeated compression along a cycle affect it?" Our instincts should be telling us that repeated compression should result in a specific type of unicycle graph; similarly, how the repeated compression of a tree resulted in a star. Repeated compression on a cycle results in a triangle star, denoted TS_n. Let TS_n be a triangle with $n - 3$ leaves on one of the vertices. In other words, TS_n is the resulting graph by adding one edge connecting two leaves of the star S_n.

Lemma 19. *Repeated compressions on cycle C_n results in TS_n.*

Proof. If x and y are adjacent non-leaves we can compress along this edge and increase the number of leaves in the graph. Let $x, y \in V(C_n)$ be such vertices and compress along the edge xy. From Lemma 18 we obtain $C_{x \to y}$ is a unicyclic graph with a cycle of length $n - 1$ and x a leaf adjacent to y. Each compression along the cycle shortens the cycle in the unicyclic graph by one and adds one leaf. We can only compress along an edge if the endpoints have distinct neighbors in the cycle. Therefore, once the unicyclic graph contains a triangle we can no longer compress along the cycle since a triangle is a clique. This means one of the vertices has $n - 3$ leaves, giving us TS_n. □

Knowing the effects of compression on cycles and trees, we turn our attention to the effects of compression on a unicyclic graph. Cycles are a specific type of uncicyclic graph, so we claim compressions on a unicyclic graph have the following effect.

Lemma 20. *If G is a unicyclic graph on n vertices, then there exists a series of compressions that can be applied to G that results in TS_n.*

Proof. If G is a cycle then we can apply Lemmas 18 and 19 and we're done. Otherwise, G contains at least one leaf. If there is a leaf l that is not adjacent to a cycle vertex, then let x be the unique neighbor of l and y be the vertex adjacent to x on the path from l to the cycle in G. $G_{x \to y}$ takes all the unique neighbors of x (other than y) and makes them adjacent to y. By this compression, we have reduced the total distance from vertices off the cycle to the cycle. After repeating this, we end up with a cycle with pendant edges off some of its vertices. Now we compress along adjacent cycle vertices. When we do this, we shorten the cycle and "consolidate" pendant vertices. This yields a triangle with pendant edges. Compressing along triangle edges leaves the triangle intact, since it is a clique, but the pendant edges are consolidated to a single vertex of the triangle. Thus, we've compressed G into TS_n □

Establishing that there exists a sequence of compressions, on any unicyclic graph, that results in the TS_n graph allows us to repove Pedersen's and Vestergaard's upper bound for the number of independent sets in a unicyclic graph.

Theorem 21. *If G is a unicyclic graph on $n \geq 3$ vertices, then*

$$i(G) \leq i(TS_n).$$

Proof. This a direct result of Lemma 8 and Lemma 20. □

We now move back to Hu's and Wei's Nordhaus-Gaddum inequalities for the number of indpendent sets in a unicyclic graph and its complement. Hu and Wei obtain the minimum and maximum values of $i(G) + i(\overline{G})$, where $\overline{G}$ is a connected complement, by establishing an equality for $i(\overline{G})$ [5]. However, Hu's and Wei's proof does not rely on the fact that $\overline{G}$ is connected, so we incorporate their proof for completion.

Lemma 22. *Let G be a unicyclic graph of on n vertices with any complement $\overline{G}$, then*

$$i(G) + i(\overline{G}) = 1 + 2n + i_3(\overline{G}) + i(G).$$

Proof. Let $i_k(\overline{G})$ be the number of independent sets of size k. Then $i_k(\overline{G}) = 0$ for $k \geq 4$. Assume to the contrary then $\overline{G}$ could have an independent set of size 4, but that would mean G would contain a K_4 and this is a contradiction since G is unicyclic. Now we must consider the number of independent sets in $\overline{G}$ of sizes 0, 1, 2, and 3, $i_0(\overline{G})$, $i_1(\overline{G})$, $i_2(\overline{G})$, and $i_3(\overline{G})$ respectively. An independent set of size zero can only occur one way, the empty set, so $i_0(\overline{G}) = 1$. An independent set of size 1 is a set containing only one vertext thus, $i_1(\overline{G}) = n$, since there are only n distinct vertices. While looking at the number of independent sets of size two we remind ourselves that that two vertices $x, y \in V(\overline{G})$ can be in an independent set together only if $xy \in E(G)$. G is a unicyclic graph on n vertices with n edges, meaning there are n

edges "missing" from $\overline{G}$ thus $i_2(\overline{G}) = n$. Lastly, $i_3(\overline{G}) = 1$ or 0 if G has a triangle or is triangle free, respectively. Using these outcomes we have the following result.

$$i(\overline{G}) = i_0(\overline{G}) + i_1(\overline{G}) + i_2\overline{G}) + i_3(\overline{G}).$$

So,

$$i(\overline{G}) = 1 + n + n + i_3(\overline{G}).$$

Thus,

$$i(G) + i(\overline{G}) = i(G) + 2n + 1 + i_3(\overline{G}).$$

□

A consequence of the above lemma is the minimum or maximum of $i(G) + i(G)^{-}$ depends on the minimum and maximum of $i(G)$ and the number of complete graphs of size 3, denoted $K_3(G)$, G contains. We can now prove the following.

Proposition 23. *If G is unicyclic, then*

$$i(\overline{G}) \leq i(\overline{G_{x\to y}}).$$

Proof. If G is unicyclic, then

$$i(\overline{G}) = 1 + 2n + K_3(G).$$

Since compression on G increases the number of independent sets we have the following inequality,

$$i(\overline{G}) \leq 1 + 2n + K_3(G_{x\to y}) = i(\overline{G_{x\to y}}).$$

Thus,

$$i(\overline{G}) \leq i(\overline{G_{x\to y}}).$$

□

Knowing that $i(C_n) \leq i(G) \leq TS_n$ and $i(\overline{G}) = 1 + 2n + i_3(\overline{G})$, we can prove the following.

Theorem 24. *Let G be a unicyclic graph on n vertices with any complement $\overline{G}$, then*

$$i(C_n) + i(\overline{C_n}) \leq i(G) + i(\overline{G}) \leq i(TS_n) + i(\overline{TS_n}).$$

Proof. Let G be a unicyclic graph on n vertices with any complement $\overline{G}$. From Lemma 22 we have,

$$i(G) + i(\overline{G}) = i(G) + 2n + 1 + i_3(\overline{G}).$$

and from Theorems 17 and 21 we get,

$$i(C_n) + 2n + 1 + i_3(\overline{G}) \leq i(G) + 2n + 1 + i_3(\overline{G}) \leq i(TS_n) + 2n + 1 + i_3(\overline{G}).$$

We also know, from Lemma 22, that $2n + 1 + i_3(\overline{G})$ is the number of independent sets in any unicyclic graph's complement, so

$$i(C_n) + i(\overline{C_n}) \leq i(G) + i(\overline{G}) \leq i(TS_n) + i(\overline{TS_n}).$$

□

Chapter 4

Dominating Sets

We shift our focus from the number of independent sets in a tree to the number of dominating sets in a tree. A *dominating set* in a graph G is a set of vertices S such that every vertex of G is either in S or adjacent to a vertex in S. We let $\partial(G)$ be the number of dominating sets in a graph G. Lex Schrijver [8] proved the following theorem.

Theorem 25 (Schrijver 2009)**.** *The number of dominating sets of any graph G is always odd.*

This fascinating property means that the Nordhaus Gaddum sum of the number of dominating sets of a graph and its complement results in an even number, since adding two odd numbers is even.

Like we did with independent sets, we can count dominating sets by splitting the set into two types: the number of sets that contain a vertex x, denoted $\partial_x(G)$, and the number of sets that do not contain x, denoted $\partial_{\hat{x}}(G)$. Then the basic rule for recusively evaluating the number of dominating sets in a graph G is as follows. For any vertex x of G,

$$\partial(G) = \partial_x(G) + \partial_{\hat{x}}(G).$$

Note that a dominating set that does not contain x must contain a vertex from the neighborhood of x. If x has no neighbors then $\partial_{\bar{x}}(G) = 0$ and x must be contained in all dominating sets of G.

We have the tight lower and upper bounds for the number of dominating sets of any graph G, $1 \le \partial(G) \le 2^{n-1}$, with equality for the empty and complete graphs, respectively. Bród and Skupień studied the lower bounds on the number of dominating sets in a tree [9]. They proved the following.

Theorem 26 (Brod, Skupień 2006)**.** *If T is a tree on n vertices, then*

$$\beta_m \cdot 5^{\lfloor n/3 \rfloor} \le \partial(T) \le \partial(K_{1,n-1}),$$

where

$$\beta_m = \begin{cases} 1 & n \equiv 0 \pmod 3 \\ \frac{5}{9} & n \equiv 1 \pmod 3 \\ 3 & n \equiv 2 \pmod 3 \end{cases}.$$

There are multiple extremal graphs possessing the lower bound.

There have been many studies of $\partial(G)$ in the framework of Nordhaus-Gaddum inequalities, that is finding bounds on $\partial(G) + \partial(\overline{G})$. For example, Wagner [10] proved the following and we include Wagner's proof for completeness.

Theorem 27 (Wagner 2013)**.** *If G is a graph on n vertices, then*

$$\partial(G) + \partial(\overline{G}) \ge 2^n.$$

Proof. Consider a set S of vertices that is not a dominating set of G. Then there exists a vertex v that is not dominated by S. But this implies that v is connected to all vertices of S in the complement graph $\overline{G}$, so that set $\overline{S}$ of S is a dominating set of

$\overline{G}$. So we can conclude that $\overline{G}$ has at least as many dominating sets as the number of nondominating sets in G. Thus we have the inequality,

$$\partial(G) + \partial(\overline{G}) \geq \partial(G) + (2^n - \partial(G)) = 2^n.$$

□

Wagner's lower bound is sharp for multiple graphs. Clearly, equality holds for the complete graph; furthermore, equality also holds for the star, S_n. If the central vertex is in a dominating set we can take any combination of the pendant vertices giving us 2^{n-1} sets. If the central vertex is not in a dominating set then we must collect all pendant vertices producing only 1 set. Thus, $\partial(S_n) = 2^{n-1} + 1$. The graph of $\overline{S_n}$ is a clique of $n - 1$ vertices with an isolated vertex. To be a dominating set in $\overline{S_n}$ the set must contain the isolated vertex and a nonempty subset of the clique, so we have $\partial(\overline{S_n}) = \partial(K_{n-1} \cup E_1) = 2^{n-1} - 1$. Thus, the sum of the number of dominating sets in a star and it's complement is

$$\partial(S_n) + \partial(\overline{S_n}) = 2^{n-1} + 1 + 2^{n-1} - 1 = 2^n.$$

More recently, Keough and Shane [11] gave an upper bound on $\partial(G) + \partial(\overline{G})$

Theorem 28 (Keough, Shane 2019)**.** *If G is a graph on n vertices, then*

$$\partial(G) + \partial(\overline{G}) \leq 2^{n+1} - 2^{\lfloor \frac{n}{2} \rfloor} - 2^{\lceil \frac{n}{2} \rceil - 1}.$$

As the authors note, this bound is not sharp. The conjectured extremal example is $K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}$. The bound in Theorem 28 is correct in the lead term as

$$\partial(K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}) + \partial(\overline{K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}}) = 2(2^{\lfloor \frac{n}{2} \rfloor} - 1)(2^{\lceil \frac{n}{2} \rceil} - 1) + 2.$$

We continue the work on dominating sets by further exploring the number of dominating sets in a tree and its complement. Wagner's lower bound is sharp which is achieved by S_n and its complement, so $\partial(T)+\partial(\overline{T}) \geq 2^n$. We focus on finding the upper bound on $\partial(T)+\partial(\overline{T})$. It's interesting that the lower bound for $\partial(T)+\partial(\overline{T})$ is $\partial(S_n)+\partial(\overline{S_n})$ since through our exploration of the number of independent sets in trees the star was our upper bound. Also, in Theorem 26 Bród and Skupień proved that the upper bound for $\partial(T)$ is $\partial(S_n)$, which furthers the interest of how the star becomes the extremal graph for the lower bound of the Nordhaus-Gaddum inequality on the number of dominating sets in trees.

Our first guess for the upper bound was the path and its complement. Bród and Skupień proved that $\partial(P_n) = T_n$, where T_n is the n^{th} number in the Tribonacci sequence [9], similar to how Tichy and Prodinger showed the number of independent sets in a path followed the Fibonacci sequence. Bueno [8] has shown that Tribonacci sequence grows exponentially by a factor of approximately 1.839. We now examine the number of dominating sets in $\overline{P_n}$.

Proposition 29. *Let P_n be a path on n vertices with $\overline{P_n}$ as its complement, then*

$$\partial(\overline{P_n}) = 2^n - 2n.$$

Proof. Let vertex x be an endnpoint in P_n with vertex y as x's only neighbor. The number of dominating sets in $\overline{P_n}$ is $\partial(\overline{P_n}) = \partial_x(\overline{P_n}) + \partial_{\hat{x}}(\overline{P_n})$. We first find $\partial_x(\overline{P_n})$. We can further characterize $\partial_x(\overline{P_n})$ breaking it up to the dominating set containing x and y and the sets containing x but not y giving us

$$\partial_x(\overline{P_n}) = \partial_{xy}(\overline{P_n})+\partial_{x\hat{y}}(\overline{P_n}).$$

If x and y are in the dominating set then we can build more dominating sets of $\overline{P_n}$

using an number of the remaining $n-2$ vertices giving us $\partial_{xy}(\overline{P_n}) = 2^{n-2}$. If y is not in the dominating set then there are only two subsets of the remaming $n-2$ vertices that would not be dominating, the empty set and set containing x and y's other neighbor in P_n. Thus $\partial_{x\hat{y}}(\overline{P_n}) = 2^{n-2} - 2$. So, we have

$$\partial_x(\overline{P_n}) = 2^{n-2} + (2^{n-2} - 2) = 2^{n-1} - 2.$$

Since $\partial_{\hat{x}}(\overline{P_n}) = \partial_y(\overline{P_{n-1}}) + \partial_{\hat{y}}(\overline{P_{n-1}})$ a dominating set that contains y must also contain a vertex that is not adjacent to y in P_{n-1} in order to dominate x, so $\partial_y(\overline{P_{n-1}}) = 2^{n-2} - 2$ for $n \geq 4$. We have a recurrence for the number of dominating sets in $\overline{P_n}$ not containing an endpoint, call it $\partial*(\overline{P_n})$. So,

$$\partial*(\overline{P_n}) = \partial*(\overline{P_{n-1}}) + 2^{n-2} - 2.$$

Applying the recursion agian, we have

$$\partial*(\overline{P_n}) = \partial*(\overline{P_{n-2}}) + 2^{n-3} - 2 + 2^{n-2} - 2.$$

We can continue to apply the recursion until $\overline{P_4}$, giving

$$\partial*(\overline{P_n}) = \sum_{i=1}^{n-3} (2^{n-1-i} - 2) + \partial*(\overline{P_3}).$$

Note that $\partial*(\overline{P_3}) = 1$.Then

$$\partial*(\overline{P_n}) = \sum_{i=2}^{n-3} (2^{n-1-i}) - 2(n-2) + 1.$$

Reversing the order of the summation

$$\partial^*(\overline{P_n}) = \sum_{j=2}^{n-2}(2^j) - 2(n-2) + 1 = \sum_{j=1}^{n-2}(2^j) - 2 - 2(n-2) + 1.$$

Using the fact that $\sum_{i=1}^{k} 2^i = 2^{k+1} - 1$, we get

$$\partial^*(\overline{P_n}) = 2^{n-1} - 1 - 2 - 2n + 4 + 1 = 2^{n-1} - 2n + 2.$$

Therefore,

$$\partial(\overline{P_n}) = \partial_x(\overline{P_n}) + \partial_{x'}(\overline{P_n}) = \partial_x(\overline{P_n}) + \partial^*(\overline{P_n}) = 2^{n-1} - 2 + 2^{n-1} - 2n + 2 = 2^n - 2n.$$

□

We now have $\partial(P_n) + \partial(\overline{P_n}) = T_n + 2^n - 2n$ and as we look at the Nordhaus-Gaddum sum we clearly see $\partial(P_n) + \partial(\overline{P_n})$ is not a candidate for the upper bound for the sum of the numbers of dominating sets in a tree and its complement.

Our second graph choice is the even double star on an even number of vertices, denoted DS, with central vertices x and y. We are able to prove the following.

Proposition 30. *Let DS be an even double star on an n vertices, where n is even, with central vertices x and y then*

$$\partial(DS) = 2^{n-2} + 2^{\frac{n}{2}} + 1.$$

Proof. Let the vertices x and y be the central vertices of DS. The number of dominating sets in DS can be broken down into four subsets the dominating sets containing both x and y, containing x but not y, containing y but not x, and containing neither

x nor y. Thus,

$$\partial(DS) = \partial_{xy}(DS) + \partial_{x\hat{y}}(DS) + \partial_{\hat{x}y}(DS) + \partial_{\hat{x}\hat{y}}(DS).$$

It is important to note that the set containing x but not y and containing y but not x are equivalent since the DS is symmetric. So, we have

$$\partial(DS) = \partial_{xy}(DS) + 2 \cdot \partial_{x\hat{y}}(DS) + \partial_{\hat{x}\hat{y}}(DS).$$

Note that $\partial_{xy}(DS) = 2^{n-2}$, since x and y are adjacent to the other $n-2$ vertices and $\{x, y\}$ is dominating, we can take any subset of those $n-2$ vertices. Now we consider $\partial_{x\hat{y}}(DS)$, x is adjacent to y and $\frac{n-2}{2}$ vertices and y is adjacent to the remaining $\frac{n-2}{2}$ vertices. Since, y is not in the set then the $\frac{n-2}{2}$ neighbors of y must be in the set. We can take any subset of the $\frac{n-2}{2}$ neighbors of x meaning $\partial_{x\hat{y}}(DS) = (2^{\frac{n-2}{2}})$. Lastly, if neither x nor y is in the set then all the neighbors of x and y must be in the set, meaning $\partial_{\hat{x}\hat{y}}(DS) = 1$. Putting this together, we see

$$\partial(DS) = 2^{n-2} + 2(2^{\frac{n-2}{2}}) + 1 = 2^{n-2} + 2^{\frac{n}{2}} + 1 .$$

□

Now we consider $\overline{DS}$, which is K_{n-2} with x adjacent to half of the vertices of K_{n-2} and y adjacent to the other half of K_{n-2} and x is not adjacent to y. We are able to prove the following.

Proposition 31. *Let DS be an even double star on n vertices, where n is even, with central vertices x and y, then*

$$\partial(\overline{DS}) = 2^{n} + 2^{\frac{n}{2}+1} + 1.$$

Proof. Let DS be an even double star on an n vertices, where n is even, with central vertices x and y. We consider $\overline{DS}$, similarly to DS, the number of dominating sets in $\overline{DS}$ can be broken down into four subsets the dominating sets contain both x and y, containing x but not y, containing y but not x, and containing neither x nor y.

$$\partial(\overline{DS}) = \partial_{xy}(\overline{DS}) + \partial_{x\hat{y}}(\overline{DS}) + \partial_{\hat{x}y}(\overline{DS}) + \partial_{\hat{x}\hat{y}}(\overline{DS}).$$

It is improtant to note that the sets containing x but not y and the sets containing y but not x are equivalent since $\overline{DS}$ is symmetric. So, we have

$$\partial(\overline{DS}) = \partial_{xy}(\overline{DS}) + 2 \cdot \partial_{x\hat{y}}(\overline{DS}) + \partial_{\hat{x}\hat{y}}(\overline{DS})$$

Note that $\partial_{xy}(\overline{DS}) = 2^{n-2}$ since $\{x, y\}$ is dominating allowing us to take any subset of the $n-2$ remaing vertices.

Now we consider $\partial_{x\hat{y}}(\overline{DS})$, x is not adjacent to y but x is adjacent to $\frac{n-2}{2}$ vertices and y is adjacent to the remaining $\frac{n-2}{2}$ vertices. We can take any subset of the neighborhood of x and take any nonempty subset of the neighborhood of y to build a dominating set. Thus, $\partial_{x\hat{y}}(\overline{DS}) = 2^{\frac{n-2}{2}}(2^{\frac{n-2}{2}} - 1)$.

Lastly, we eveluate $\partial_{\hat{x}\hat{y}}(\overline{DS})$. Neither x nor y is in these sets so we need at least one of each of their neighbors. It is important to remember that $N(x) \cap N(y) = \emptyset$ and $N(x) \cup N(y) = K_{n-2}$. Meaning we need a nonempty subset of $N(x)$ and a nonempty subset of $N(y)$ to form a dominating set. It implies that $\partial_{\hat{x}\hat{y}}(\overline{DS}) = (2^{\frac{n-2}{2}} - 1)(2^{\frac{n-2}{2}} - 1)$. Putting these together, we obtain

$$\partial(\overline{DS}) = 2^{n-2} + 2(2^{\frac{n-2}{2}}(2^{\frac{n-2}{2}} - 1)) + (2^{\frac{n-2}{2}} - 1)(2^{\frac{n-2}{2}} - 1).$$

$$\partial(\overline{DS}) = 2^{n-2} + 2^{n-1} - 2^{\frac{n}{2}} + 2^{n-2} - 2^{\frac{n}{2}} + 1 = 2^{n} - 2^{\frac{n}{2}+1} + 1.$$

$\square$

Therefore the Nordhaus-Gaddum sum of the DS and $\overline{DS}$ is

$$\partial(DS)+\partial(\overline{DS}) = 2^{n-2}+2^{\frac{n}{2}}+1+2^{n}-2^{\frac{n}{2}+1}+1 = 5\cdot 2^{n-2}-2^{n/2}+2.$$

Another candidate for the upper bound on the numbers of dominating sets in a tree and its complement is the *broom* graph. The *broom*, denoted $B_{P_{k,n-k}}$, is a path of length k with $n-k$ leaves attached to an endpoint of the path. Note that $B_{2,n-2} = S_n$ and $B_{n,0} = P_n$, so we can think of the different size brooms as the varying stages of the star S_n turning into the path P_n. We will start by finding the number of dominating sets in $B_{3,n-3}$, $B_{4,n-4}$ and $B_{5,n-4}$ and their respective complements.

Proposition 32.

$$\partial(B_{P_3,n-3})+\partial(\overline{B_{P_3 n-3}}) = 2^{n}+2^{n-3}.$$

Proof. Let x, y and z be the vertices of P_3 in $B_{P_3,n-3}$ such that x is the endpoint of P_3 and x is adjacent to the $n-3$ pendant vertices and y. The dominating sets of $B_{P_3,n-3}$ can be split into two types of sets: the sets that contain x and the sets that don't. So, we have

$$\partial(B_{P_3,n-3}) = \partial_x(B_{P_3,n-3})+\partial_{\hat{x}}(B_{P_3,n-3}).$$

We first consider $\partial_x(B_{P_3,n-3})$. Since x is in the set we dominated all vertices except z meaning we need to take a nonempty subset of $\{y, z\}$, and take any subset of the $n-3$ leaves. Thus, $\partial_x(B_{P_3,n-3}) = 2^{n-3}(3)$. If x is not in the set then all $n-3$ leaves must be in the set. This leaves y and z undominated so we must have a nonempty set of $\{y, z\}$, so $\partial_{\hat{x}}(B_{P_3,n-3}) = 3$ Adding these together gives us

$$\partial(B_{P_3,n-3}) = 3 \cdot 2^{n-3} + 3.$$

The dominating sets of $\overline{B_{P_3,n-3}}$ can be broken up into two sets: the sets containing x and those that don't. Thus, we have

$$\partial(\overline{B_{P_3,n-3}}) = \partial_x(\overline{B_{P_3,n-3}}) + \partial_{\hat{x}}(\overline{B_{P_3,n-3}}).$$

Looking at $\overline{B_{P_3,n-3}}$ we note that x is only adjacent to z, while y and z are adjacent to the remaining $n - 3$ vertices but not each other. When counting $\partial_x(\overline{B_{P_3,n-3}})$ we know each set contains x so only z is dominated, so inorder for the set to be dominating we must take a nonempty set of the remaining $n - 2$ vertices. Implying, $\partial_x(\overline{B_{P_3,n-3}}) = 2(2^{n-2} - 1)$. Now counting $\partial_{\hat{x}}(\overline{B_{P_3,n-3}})$ we realize that z must be in the set in order to dominate x. This only leaves y undominated, so our sets must contain a nonempty subset of $K_{n-3} \cup \{y\}$. Meaning $\partial_{\hat{x}}(\overline{B_{P_3,n-3}}) = 2^{n-2} - 1$. Adding these together gives us

$$\partial(B_{P_3,n-3}) = 2(2^{n-2} - 1) + (2^{n-2} - 1) = 2n^{n-1} + 2^{n-2} - 3.$$

Therefore,

$$\partial(B_{P_3,n-3}) + \partial(\overline{B_{P_3,n-3}}) = 3 \cdot 2^{n-3} + 3 + 2^{n-1} + 2^{n-2} - 3 = 2 \cdot 2^{n-3} + 2^{n-3} + 2^{n-1} + 2^{n-2}.$$

Thus,

$$\partial(B_{P_3,n-3}) + \partial(\overline{B_{P_3,n-3}}) = 2^n + 2^{n-3}.$$

□

We compare the reasults of the DS and $B_{P_3,n-3}$ and we see there are more dominating sets in DS and its complement than in $B_{P_3,n-3}$ and its compement. The comparison is as follows

$$\begin{aligned}\partial(DS)+\partial(\overline{DS}) - \partial(B_{P_3,n-3}) - \partial(\overline{B_{P_3,n-3}}) &= 5\cdot 2^{n-2} - 2^{n/2}+2 - (2^n + 2^{n-3})\\ &= 10\cdot 2^{n-3} - 9\cdot 2^{n-3} - 2^{n/2}+2\\ &= 2^{n-3} - 2^{n/2} + 2.\end{aligned}$$

And $2^{n-3} - 2^{n/2} + 2 > 0$ for $n > 4$. Thus, DS and $\overline{DS}$ have more dominating but not much more so we look at the next stage of the broom, which is $B_{P_4,n-4}$.

Proposition 33.

$$\partial(B_{P_4,n-4})+\partial(\overline{B_{P_4,n-4}}) = 5\cdot 2^{n-2}.$$

Proof. Let x, y, z and w be the vertices of P_4 in $B_{P_4,n-4}$ such that x is the endpoint with $\deg(x) = n-3$ and the other vertices follow sequentially along the path. The dominating sets of $B_{P_4,n-4}$ can be split into two types of sets: the sets that contain x and the sets that don't. So, we have

$$\partial(B_{P_4,n-4}) = \partial_x(B_{P_4,n-4}) + \partial_{\hat{x}}(B_{P_4,n-4}).$$

Counting the sets that contain x, allows us to take any subset of the leaves and y since they are all dominated by x. The vertices z and w are undominated at this point, meaning we must take an unempty subset of $\{z, w\}$. Thus, $\partial_x(B_{P_4,n-4}) = 3\cdot 2^{n-3}$. The sets not containing x must contain all the leaves, meaning y, z and w are left to be dominated. Taking any nonempty subset of $\{y, z, w\}$ except $\{y\}$, $\{w\}$ will lead to a dominate set. Thus we have $\partial_{\hat{x}}(B_{P_4,n-4}) = 2^3 - 3 = 5$. Adding these together, it gives us

$$\partial(B_{P_4,n-4}) = 3 \cdot 2^{n-3} + 5.$$

We now move on to $\partial(\overline{B_{P_4,n-4}})$ which can be split into two types of sets: the sets that contain x and the sets that don't. So, we have

$$\partial(\overline{B_{P_4,n-4}}) = \partial_x(\overline{B_{P_4,n-4}}) + \partial_{\hat{x}}(\overline{B_{P_4,n-4}}).$$

If x is in the dominating set then vertices z and w are dominated leaving y and the $n-2$ vertices undominated. So we can take any nonempty set of the $n-1$ vertices except the set $\{z\}$. Thus, $\partial_x(\overline{B_{P_4,n-4}}) = 2^{n-1} - 2$. As we count the number of dominating sets without x we see that either z or w must be in the set, in order to dominate x. So, we have

$$\partial_{\hat{x}}(\overline{B_{P_4,n-4}}) = \partial_{\hat{x}zw}(\overline{B_{P_4,n-4}}) + \partial_{\hat{x}z\hat{w}}(\overline{B_{P_4,n-4}}) + \partial_{\hat{x}\hat{z}w}(\overline{B_{P_4,n-4}}).$$

If z and w are in the sets then we can take subset of the remaining $n-3$ vertices since $\{z,w\}$ is dominating, giving us $\partial_{\hat{x}zw}(\overline{B_{P_4,n-4}}) = 2^{n-3}$. If z is in the sets but not w we need to take a nonempty subset of the remaining $n-3$ vertices to ensure we dominate y and w. So, $\partial_{\hat{x}z\hat{w}}(\overline{B_{P_4,n-4}}) = 2^{n-3} - 1$. Lasly we must count the sets containing w but not z. Containing w means every vertex is dominated except z, therefore we must choose a nonempty subset of $N(z) - \{x\}$ and y can be in or out of the set. So, $\partial_{\hat{x}\hat{z}w}(\overline{B_{P_4,n-4}}) = 2 \cdot (2^{n-4} - 1) = 2^{n-3} - 2$. Altogether we have

$$\partial_{\hat{x}}(\overline{B_{P_4,n-4}}) = 2^{n-3} + (2^{n-3} - 1) + 2^{n-3} - 2 = 3 \cdot 2^{n-3} - 3.$$

So, we have

$$\partial(\overline{B_{P_4,n-4}}) = \partial_x(\overline{B_{P_4,n-4}}) + \partial_{\hat{x}}(\overline{B_{P_4,n-4}}) = 2^{n-1} - 2 + 3 \cdot 2^{n-3} - 3 = 7 \cdot 2^{n-3} - 5.$$

Therefore,

$$\partial(B_{P_4,n-4}) + \partial(\overline{B_{P_4,n-4}}) = 3 \cdot 2^{n-3} + 5 + 7 \cdot 2^{n-3} - 5 = 10 \cdot 2^{n-3} = 5 \cdot 2^{n-2}.$$

□

Looking back that the sum $\partial(DS) + \partial(\overline{DS}) = 5 \cdot 2^{n-2} - 2^{n/2} + 2$ we see that this sum is less than the sum of $\partial(B_{P_4,n-4}) + \partial(\overline{B_{P_4,n-4}}) = 5 \cdot 2^{n-2}$. So, the number of the dominating sets in a broom and its complement increased when we increased the length of the path. We continue to extend the path to see if the trend continues.

Proposition 34.

$$\partial(B_{P_5,n-5}) + \partial(\overline{B_{P_5,n-5}}) = 41 \cdot 2^{n-5} + 2.$$

Proof. Let x, y, z, v and w be the vertices of P_5 in $B_{P_5,n-5}$ such that x is an endpoint with the other vertices following sequentially along the path. The dominating sets of $B_{P_5,n-5}$ can be split into two types of sets: the sets that contain x and the sets that don't. So, we have

$$\partial(B_{P_5,n-5}) = \partial_x(B_{P_5,n-5}) + \partial_{\hat{x}}(B_{P_5,n-5}).$$

The sets containing x dominate the $n-5$ leaves and y, leaving z, v and w undominated. So, we must take subets of $\{y, z, v, w\}$ to ensure our sets dominate; however, 5 of the subsets do not result in domination, giving us $\partial_x(B_{P_5,n-5}) = 11 \cdot 2^{n-5}$. The sets not containing x must have the $n - 5$ leaves in them since x is the only vertex that dominates them and x is not in the set. Thus we must take subsets of $\{y, z, v, w\}$ to ensure the sets not contain x dominate. However, there are 7 subsets of $\{y, z, v, w\}$ that do not result in a dominating set, so $\partial_{\hat{x}}(B_{P_5,n-5}) = 9$. Add these together gives us

$$\partial(B_{P_{5,n-5}}) = 11\ \ 2^{n-5} + 9.$$

We now turn our attention to counting $\partial(\overline{B_{P_{5,n-5}}})$. Again we can split our sets into sets containing x and sets that don't. It gives

$$\partial(\overline{B_{P_{5,n-5}}}) = \partial_x(\overline{B_{P_{5,n-5}}}) + \partial_{\hat{x}}(\overline{B_{P_{5,n-5}}}).$$

If a subset contains x then we must take any nonempty subset of the remaing $n - 1$ vertices, except for $\{z\}$, to create a dominating set. This means that $\partial_x(\overline{B_{P_{5,n-5}}}) = 2^{n-1} - 2$. A subset not containing x may have w in it or not. Thus

$$\partial(\overline{B_{P_{5,n-5}}}) = \partial_{\hat{x}w}(\overline{B_{P_{5,n-5}}}) + \partial_{\hat{x}\hat{w}}(\overline{B_{P_{5,n-5}}}).$$

If w is in the set, we can build a dominating set by taking any nonempty subset of the remaining $n - 2$ vertices except $\{z\}$ so $\partial_{\hat{x}w}(\overline{B_{P_{5,n-5}}}) = 2^{n-2} - 2$. If w is not in the set then we must have a combination of z and v in our dominating sets. So,

$$\partial_{\hat{x}\hat{w}}(\overline{B_{P_{5,n-5}}}) = \partial_{\hat{x}\hat{w}zv}(\overline{B_{P_{5,n-5}}}) + \partial_{\hat{x}\hat{w}z\hat{v}}(\overline{B_{P_{5,n-5}}}) + \partial_{\hat{x}\hat{w}\hat{z}v}(\overline{B_{P_{5,n-5}}}).$$

The sets counting both z and v are dominating, so $\partial_{\hat{x}\hat{w}zv}(\overline{B_{P_{5,n-5}}}) = 2^{n-4}$. If z is in the set but not v then we can take a nonempty subset of the $n - 4$ meaning $\partial_{\hat{x}\hat{w}z\hat{v}}(\overline{B_{P_{5,n-5}}}) = 2^{n-4} - 1$. Lastly if v is in the set but z is not then we can take any nonempty subset of the $n - 4$ vertices excluding just y so we have $\partial_{\hat{x}\hat{w}\hat{z}v}(\overline{B_{P_{5,n-5}}}) = 2^{n-4} - 2$. Adding these together gives us

$$\partial_{\hat{x}\hat{w}}(\overline{B_{P_{5,n-5}}}) = 2^{n-4} + (2^{n-4} - 1) + (2^{n-4} - 2) = 3 \cdot 2^{n-4} - 3.$$

Thus,

$$\partial_{\bar{x}}(\overline{B_{P_{5,n-5}}}) = (2^{n-2} - 2) + 3 \cdot 2^{n-4} - 3 = 7 \cdot 2^{n-4} - 5.$$

So,

$$\partial(\overline{B_{P_{5,n-5}}}) = \partial_x(\overline{B_{P_{5,n-5}}}) + \partial_{\bar{x}}(\overline{B_{P_{5,n-5}}}) = 2^{n-1} - 2 + 7 \cdot 2^{n-4} - 5 = 15 \cdot 2^{n-4} - 7.$$

Therefore,

$$\partial(B_{P_{5,n-5}}) + \partial(\overline{B_{P_{5,n-5}}}) = 11 \cdot 2^{n-5} + 9 + 15 \cdot 2^{n-4} - 7 = 41 \cdot 2^{n-5} + 2.$$

$\square$

Comparing $\partial(B_{P_{5,n-5}})+\partial(\overline{B_{P_{5,n-5}}}) = 41 \cdot 2^{n-5}+2$ to the previous broom $\partial(B_{P_{4,n-4}})+\partial(\overline{B_{P_{4,n-4}}}) = 5 \cdot 2^{n-2} = 40 \cdot 2^{n-5}$, we see that $B_{P_{5,n-5}}$ and its complement produce more dominating sets as $B_{P_{4,n-4}}$ and its complement. We have established that increasing the length of the path of the broom increases the number of dominating sets in the broom and its complement. However, we know the number of dominating sets in the broom and its complement cannot keep increasing as we increase the length of the path, because eventually the broom would just be a path and we've already established that $\partial(P_n) + \partial(\overline{P_n})$ are not an upper bound for the Nordhaus-Gaddum sum inequality for trees. This means at some point the number of dominating sets must reach the peak and then start to decrease as the length of the path of the broom increases. So, our conjecture is the following

Conjecture 1. *Let T be a tree with complement $\overline{T}$ then*

$$\partial(T) + \partial(\overline{T}) \leq \partial(B_{P_{\frac{n}{2},\frac{n}{2}}}) + \partial(\overline{B_{P_{\frac{n}{2},\frac{n}{2}}}}).$$

We believe that the broom with it's vertices evenly split between the path and the leaves will yield the largest number of dominating sets for the class of trees.

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