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Comparison on the Bayesian Estimation of Gompertz Distribution Based on Type I Censored Data

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Abstract

The paper depicts assessment of the Bayesian methodology utilizing Gaussian quadrature formulas and Markov Chain Monte Carlo of the Gompertz distribution based on type I censored data with two loss functions, the Square Error loss function and the Linear Exponential loss function. In Markov Chain Monte Carlo, the full conditional distributions for the scale and shape parameters, survival and hazard functions are acquired by means Gibbs sampling and Metropolis- Hastings algorithm. The strategies for the Bayesian methodology are contrasted with maximum likelihood estimation regarding the Mean Square Error (MSE) to decide the best assessing of the scale and shape parameters, survival and hazard functions of the Gompertz distribution based on type I censored data.

Keywords: Gompertz distribution, Bayesian estimation, Type I censored data, Gaussian Quadrature Formulas, Markov Chain Monte Carlo.

1. Introduction

The Gompertz distribution can be utilized as a survival and hazard model in medicine science, reliability and life testing. The Gompertz distribution was first presented by Gompertz (1825), and many research have contributed the distribution to the measurable model, for instance, Ahuja and Nash (1979), Makany (1991) and Franses (1994). Ananda et al. (1996) assessed parameters and survival function of the Gompertz distribution by utilizing Bayesian strategies .AL-Hussaini et al. (2000) estimated the survival and hazard functions of a finite mixture of Gompertz distribution by utilizing the maximum likelihood estimation and Bayesian methodology. Jaheen (2002) acquired the two parameters of the Gompertz distribution by maximum likelihood estimation and Bayesian techniques under square error and Linex loss function. Soliman et al. (2012) acquired for the two parameter Gompertz distribution with progressive first-failure censored data.

In numerical investigation, the Gauss quadrature technique is helpful in solving Bayesian parameter and survival and hazard functions. Singh et al. (2002) assessed the Exponentiated Weibull shape parameters by maximum likelihood estimators and Bayesian estimator whereby in the Bayesian assessment approach they illuminated it mathematically by the utilization Gauss-Legendre quadrature formula to estimate the parameters. Singh et al. (2005) got Bayesian and Maximum likelihood estimation for the two-parameter Exponentiated Weibull distribution when sample was available from type-II censoring scheme and utilized the Gaussian quadrature formulas. For more detail in Gauss Quadrature Method see Richard and Douglas (1989).

Gibbs sampler is one of the special cases of a MCMC algorithm and this technique generates a series of samples from the full conditional probability distributions of random variables see Gupta el at, (2008) and Soliman et al, (2011). Metropolis-Hasting algorithm is viewed as an overall Monte Carlo Markov chain algorithm technique that was created by Hastings (1970). It can be utilized to acquire random samples from any type of randomly troublesome objective distribution with any type of dimension that is known up to a normalizing consistent, see for example Soliman et al. (2012). Upadhyay and Gupta (2010) examined some Bayesian analysis by means of Markov Chain Monte Carlo procedure for complete samples and independent vague priors for the unknown parameters.

The goal of this paper is to appraise the parameters, the survival and the hazard functions of the Gompertz distribution based type I censored data by utilizing Bayesian methodology through Gaussian quadrature formulas and Markov Chain Monte Carlo procedure and contrasted to maximum likelihood estimator by utilizing mean square error (MSE) to decide the best estimator under a few conditions.

2. Methodology

2.1. Maximum Likelihood Estimation

The probability density function of Gompertz distribution (pdf) is

$$f(x;\lambda,\beta)=\lambda e^{\beta x}\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta x}\right)\right)$$

The cumulative distribution function (cdf)

$$F(x;\lambda,\beta)=1-\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta x}\right)\right)$$

For type I censored data, the logarithm of the likelihood function of Gompertz distribution is

$$\ln L(\lambda,\beta\,|\,\text{data})=\sum_{i=1}^{n}\left[\delta_i\ln\lambda+\delta_i\beta x_i+\frac{\lambda}{\beta}\left(1-e^{\beta x_i}\right)\right] \tag{1}$$

The resulting of the scale and shape parameters are given respectively as,

$$\frac{\partial L(\lambda,\beta\,|\,\text{data})}{\partial\lambda}=\sum_{i=1}^{n}\left[\frac{\delta_i}{\lambda}+\frac{1}{\beta}\left(1-e^{\beta x_i}\right)\right] \tag{2}$$

$$\frac{\partial L(\lambda,\beta\,|\,\text{data})}{\partial\beta}=\sum_{i=1}^{n}\left[\delta_i x_i-\frac{\lambda}{\beta}\left(\frac{\left(1-e^{\beta x}\right)}{\beta}+x_i e^{\beta x_i}\right)\right] \tag{3}$$

The scale parameter λ is express as follows

$$\lambda=-\frac{\beta\sum_{i=1}^{n}\delta_i}{n-\sum_{i=1}^{n}e^{\beta x_i}}$$

The equation 3 cannot be solved analytically, and for that we employed Newton Raphson method to find the numerical solution.

The estimates of the survival and hazard functions of Gompertz distribution are

$$\hat{S}_M(t)=\exp\left(\frac{\hat{\lambda}_M}{\hat{\beta}_M}\left(1-e^{\hat{\beta}_M t}\right)\right) \tag{4}$$

$$\hat{h}_M(t)=\hat{\lambda}_M e^{\hat{\beta}_M t} \tag{5}$$

Which $\hat{\lambda}_M$ is the scale parameter estimated by maximum likelihood estimator (MLE) of Gompertz distribution

and the $\hat{\beta}_M$ is the shape parameter estimated by MLE.

2.2 Bayesian Estimations

We consider the case when both scale and shape parameters are unknown, and we compute the Bayesian estimation of the scale and shape parameters of Gompertz distribution. It is assumed that λ and β each have independent gamma priors as follows,

$$g_1(\lambda/a,b)=\lambda^{a-1}\exp(-b\lambda)$$

$$g_2(\beta/c,d)=\beta^{c-1}\exp(-d\beta)$$

The posterior of Gompertz distribution based on type I censored data is given as

$$\prod(\lambda,\beta\mid \mathrm{x})=\frac{1}{J}\lambda^{a-1}\beta^{c-1}\exp(-(b\lambda+d\beta))\prod_{i=1}^{n}\left[\lambda^{\delta_i}e^{\beta\delta_i x_i}\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta x_i}\right)\right)\right]$$

Where,

$$J=\int_0^\infty\int_0^\infty\lambda^{a-1}\beta^{c-1}\exp(-(b\lambda+d\beta))\prod_{i=1}^{n}\left[\lambda^{\delta_i}e^{\beta\delta_i x_i}\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta x_i}\right)\right)\right]d\lambda d\beta$$

2.2.1 Loss Functions

A wide range of loss functions have been reported in literature review to describe various types of loss structures. In this study, we describe two loss functions: The symmetric loss function is square error loss function and the asymmetric loss functions are Linear Exponential loss function (LINEX).

2.2.1.1 Square Error Loss Function

The square error loss function used to estimate the scale and shape parameters of Gompertz distribution as given respectively below,

$$\hat{\lambda}_S=\frac{1}{J}\int_0^\infty\int_0^\infty\lambda^{a}\beta^{c-1}\exp(-(b\lambda+d\beta))\prod_{i=1}^{n}\left[\lambda^{\delta_i}e^{\beta\delta_i x_i}\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta x_i}\right)\right)\right]d\lambda d\beta \quad (6)$$

$$\hat{\beta}_S=\frac{1}{J}\int_0^\infty\int_0^\infty\lambda^{a-1}\beta^{c}\exp(-(b\lambda+d\beta))\prod_{i=1}^{n}\left[\lambda^{\delta_i}e^{\beta\delta_i x_i}\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta x_i}\right)\right)\right]d\lambda d\beta \quad (7)$$

The Bayesian estimates for the survival and hazard functions under squared error loss function are given as:

$$\hat{S}_S(t)=\frac{1}{J}\int_0^\infty\int_0^\infty\lambda^{a-1}\beta^{c-1}\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta t}\right)-(b\lambda+d\beta)\right)\prod_{i=1}^{n}\left[\lambda^{\delta_i}e^{\beta\delta_i x_i}\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta x_i}\right)\right)\right]d\lambda d\beta \quad (8)$$

$$\hat{h}_S(t)=\frac{1}{J}\int_0^\infty\int_0^\infty\lambda^{a}\beta^{c-1}\exp\left(-\left(b\lambda+(d-t)\beta\right)\right)\prod_{i=1}^{n}\left[\lambda^{\delta_i}e^{\beta\delta_i x_i}\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta x_i}\right)\right)\right]d\lambda d\beta \quad (9)$$

The equations (6-9) we can't solve it analytical for that we used MCMC Algorithm to estimate the scale and shape parameters of Gompertz distribution with type I censored data.

2.2.1.2 Linear Exponential Loss Function (LINEX)

The Linear Exponential loss function is under the assumption that the minimal loss occurs at $\hat{\lambda}=\lambda$ and is expressed as

$$L(\Delta)=\exp(r\Delta)-r\Delta-1, \qquad r\neq 1$$

where $\Delta=(\hat{\lambda}-\lambda),\hat{\lambda}$ is an estimate of λ, and when $r>1$ means overestimation and underestimation if $r<1$.

For r close to zero the Linear Exponential loss function approximated the square error loss function.

The posterior under LINEX loss function in equation above given as follows,

$$E_\lambda\left(L(\hat{\lambda}-\lambda)\right)\propto\exp(r\hat{\lambda})\mathrm{E}_\lambda\left(\exp(r\hat{\lambda})\right)-r\left(\hat{\lambda}-\mathrm{E}_\lambda(\lambda)\right)-1$$

Therefore, the Bayesian estimation of scale parameter of Gompertz distribution with type I censored data under

LINEX loss function is:

$$\hat{\lambda}_L = -\frac{1}{r}\ln\left(\frac{1}{J}\int_0^\infty\int_0^\infty \lambda^{a-1}\beta^{c-1}\exp\left(-\left((b+r)\lambda+d\beta\right)\right)\prod_{i=1}^{n}\left[\lambda^{\delta_i}e^{\beta\delta_i x_i}\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta x_i}\right)\right)\right]d\lambda d\beta\right) \quad (10)$$

The Bayesian estimation of shape parameter of Gompertz distribution under LINEX loss function is

$$\beta_L = -\frac{1}{r}\ln\left(\frac{1}{J}\int_0^\infty\int_0^\infty \lambda^{a-1}\beta^{c-1}\exp\left(-\left(b\lambda+(d+r)\beta\right)\right)\prod_{i=1}^{n}\left[\lambda^{\delta_i}e^{\beta\delta_i x_i}\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta x_i}\right)\right)\right]d\lambda d\beta\right) \quad (11)$$

The survival function under LINEX loss function is shown below:

$$\hat{S}_L(t) = -\frac{1}{r}\ln\left(\frac{1}{J}\int_0^\infty\int_0^\infty \exp\left(-r\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta t}\right)\right)\right)\lambda^{a-1}\beta^{c-1}\exp(-(b\lambda+d\beta))\prod_{i=1}^{n}\left[\lambda^{\delta_i}e^{\beta\delta_i x_i}\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta x_i}\right)\right)\right]d\lambda d\beta\right) \quad (12)$$

The hazard function is

$$\hat{h}_L(t) = -\frac{1}{r}\ln\left(\frac{1}{J}\int_0^\infty\int_0^\infty \lambda^{a-1}\beta^{c-1}\exp\left(-\left(\left(re^{\beta t}+b\right)\lambda+d\beta\right)\right)\prod_{i=1}^{n}\left[\lambda^{\delta_i}e^{\beta\delta_i x_i}\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta x_i}\right)\right)\right]d\lambda d\beta\right) \quad (13)$$

The equations (10-13) under LINEX loss function can't solve it analytical for that we used MCMC Algorithm. Therefore, the Algorithm used to generate MCMC sample under LINEX loss function to estimate the scale and shape parameters, survival and hazard functions of Gompertz distribution with type I censored data.

2.2.2 Gibbs Sampling for Scale Parameter Estimation

Gibbs sampling is a special case of a MCMC algorithm, where it generates a sequence of samples from the full conditional probability distributions of variables.

The full conditional of the posterior density function using gamma prior of λ and α given the data are combining the gamma prior with likelihood as given below

$$\Pi_S(\lambda,\beta\,|\,\text{data}) \propto \lambda^{a-1}\beta^{c-1}\exp\left(-\left(b\lambda+d\beta\right)\right)\prod_{i=1}^{n}\left[\lambda^{\delta_i}e^{\beta\delta_i x_i}\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta x_i}\right)\right)\right] \quad (14)$$

From equation (14) we can get the conditional posterior of the λ as follows

$$\Pi_S(\lambda\,|\,\beta;\text{data}) \propto \lambda^{a-1}\exp\left(-b\lambda\right)\prod_{i=1}^{n}\left[\lambda^{\delta_i}\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta x_i}\right)\right)\right] \quad (15)$$

The conditional posterior of the λ follows gamma density function with scale and shape parameters $\sum_{i=1}^{n}\delta_i + n$ and $b - \frac{n-\beta\sum_{i=1}^{n}x_i}{\beta}$ respectively. The Gibbs sampling technique was used to generate MCMC sample as shown in Algorithm.

2.2.3 Metropolis- Hastings Algorithms

The conditional posterior of the shape parameter β is given below,

$$\Pi_S(\beta \mid \lambda;\text{data}) \propto \beta^{c-1}\exp(-d\beta)\prod_{i=1}^{n}\left[e^{\beta\delta_i x_i}\exp\left(\frac{\lambda}{\beta}\left(1-e^{\beta x_i}\right)\right)\right] \quad (16)$$

As show in equation 16 the conditional posterior of the shape parameter it's not follow any close distribution therefore we suggest to use the Metropolis Hastings algorithm to generate MCMC sample as shown in Algorithm,

Algorithm:

1. Starting with initial value β_0
2. Generate the λ from gamma $\left(\sum_{i=1}^{n}\delta_i + n\ ,\ b - \frac{n-\beta\sum_{i=1}^{n}x_i}{\beta}\right)$.
3. β_i is the current value and generate the candidate value β^* from arbitrary distribution Uniform (0, 1).
4. The value of β_i is given below as

$$\beta_{i+1} = \begin{cases}\beta^* \text{with probability } p\\ \beta_i \text{with probability 1-}p,\end{cases}$$

where,

$$p = \min\left\{1,\ \frac{\Pi(\beta^* \mid \lambda;\mathrm{x}_i)}{\Pi(\beta_i \mid \lambda;\mathrm{x}_i)}\frac{\mathrm{q}(\beta^* \mid \lambda;\mathrm{x}_i)}{\mathrm{q}(\beta_i \mid \lambda;\mathrm{x}_i)}\right\}$$

5. Generate u from Uniform (0, 1) and accept β^* with probability p if $\beta^* < p$ and return to step 2, otherwise accept β_i and return to step 2.
6. The Bayesian with type I censored data of the scale and shape parameters under the squared error loss function is given as

$$\hat{\lambda}_S = \frac{1}{m}\sum_{i=1}^{m}\lambda_i$$
$$\hat{\beta}_S = \frac{1}{m}\sum_{i=1}^{m}\beta_i$$

7. The Bayesian with type I censored data of the scale and shape parameters under LINEX loss function is given as

$$\hat{\lambda}_R = -\frac{1}{r}\ln\left(\frac{\sum_{i=1}^{m}\exp(-r\lambda_i)}{m}\right)$$

$$\hat{\beta}_R = -\frac{1}{r}\ln\left(\frac{\sum_{i=1}^{m}\exp\left(-r\beta_i\right)}{m}\right)$$

2.2.4 Gaussian Quadrature Formulas

We used Gauss quadrature rule to solve our problems for each estimators are a mention in this study, and the double integrations as given as follows,

$$I = \int_{A1}^{B_1}\int_{A_2}^{B_2} f(x_1, x_2) \quad dx_1 dx_2$$

The Gauss Legendre quadrature rules for single integration is given below,

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{i=1}^{N} C_i f(z_i)$$

This is the n-point Gauss quadrature rule, where the C_i is the coefficients and z_i are called the function arguments.

To solve the double integration by the Gauss quadrature rule the follow step are needs:

$$I = \frac{B_1 - A_1}{2}\int_{-1}^{1}\int_{A_2}^{B_2} \mathrm{L}\ f(v_1 z_1 + \beta_1, x_2) \quad dz_1 dx_2$$

Where

$$v_1 = \frac{B_1 - A_1}{2}, \beta_1 = \frac{B_1 + A_1}{2}$$

$$I\ ;\ \frac{B_1 - A_1}{2}\sum_{i=1}^{N} C_i \int_{A_2}^{B_2} \mathrm{L}\ f(v_1 z_i + \beta, x_2) \quad dx_2 dx_2$$

C_i are the weighting factors and z_i are the function arguments, and to apply the Gauss quadrature rule for the second integration as given below,

$$I\ ;\ \frac{B_1 - A_1}{2}\sum_{i=1}^{N} C_i \left[\frac{B_2 - A_2}{2}\right]\sum_{j=1}^{N} C_i f(v_1 z_i + \beta_1, v_2 z_j + \beta_2)$$

$$v_2 = \frac{B_2 - A_2}{2}, \beta_2 = \frac{B_2 + A_2}{2}$$

See Richard & Douglas (1989) and Rathod *et al*, (2007) for more detail, and the techniques can be applied for estimated the parameters of Gompertz distribution with type-I censored data and the survival and hazard functions.

3. Simulation Study

To assess the performance of the Maximum likelihood and Bayesian estimation based on Type-I censored data to estimate the scale and shape parameters follow by estimate survival and hazard functions. The mean squared errors (MSE) was calculated using 10,000 replications for sample size n=20, 40 and 80 and that of the censoring time were 15% of Gompertz distribution with type-I censored data for different value of parameters were the scale parameter $\lambda = 2$ and 3, the shape parameter $\alpha =$ 0.4 and 1.4, the considered values of parameters are meant for illustration only and other values can be taken for generating the samples from Gompertz distribution.

The Gibbs sampling and the Metropolis- Hastings Algorithm used in equations (7-10) for Bayesian under square error loss function and equations (11-14) for linear exponential loss function (LINEX) to estimate the parameters, the survival and hazard functions of Gompertz distribution respectively, where hyper-parameters of gamma priors are equal to 0.0001 see for example Alomari (2016), then $a = b = c = d = 0.0001$.

The Gaussian Quadrature Formulas used for Bayesian under square error loss function and under linear exponential loss function (LINEX) to estimate the parameters, survival and hazard functions of Gompertz distribution based on type I censoring data .

The values for the loss parameter were taken to be $r = \pm 0.7$ and a detailed discussion on the choice of the loss parameter of LINEX can be obtained from Calabria & Pulcini (1996).

4. Results and Discussion

As appeared in Table 1, the gauge of the scale parameter λ of Gompertz distribution based on type I censored data is acquired utilizing Maximum likelihood (MLE), Bayesian with square error loss function through Markov Chain Monte Carlo method (BSM), and Gaussian quadrature formulas (BSG).Bayesian with Linear Exponential loss function (*r*=+0.7) by means of Markov Chain Monte Carlo method (BLM (*r*=+0.7)) and Gaussian quadrature formulas (BLG (*r*=+0.7)). Likewise Bayesian with Linear Exponential loss function (*r*= -0.7) through Markov Chain Monte Carlo procedure (BLM (*r*= -0.7)) and Gaussian quadrature formulas (BLG (*r*= -0.7)). Additionally in Table 2, we assessed the shape parameter β of Gompertz distribution based on type I censored data by utilizing the assessors above.

Table 3 the gauge of the scale parameter λ of Gompertz distribution was analyzed by mean squared error (MSE). The outcomes show that, the Bayesian under LINEX loss function with ($r = +0.7$) through Markov Chain Monte Carlo technique (BLM (*r*= +0.7)) is better compare to the others when the shape parameter was 0.4. Moreover, the Bayesian under LINEX loss function with ($r = +0.7$) via Gaussian quadrature formulas (BLG (*r*= +0.7)) is better contrast with the others when the shape parameter was 1.4. Furthermore, Bayesian with square error loss function via Markov Chain Monte Carlo technique (BSM) and Gaussian quadrature formulas (BSG), also Bayesian with Linear Exponential loss function (*r*= -0.7) via Markov Chain Monte Carlo technique (BLM (*r*= -0.7)) and Gaussian quadrature formulas (BLG (*r*= -0.7)) are better than Maximum likelihood (MLE).

As appeared in Table 4 the gauge of the shape parameter β of Gompertz distribution was compared by mean squared error (MSE). The outcomes show that, the Bayesian under LINEX loss function with ($r = +0.7$) via Markov Chain Monte Carlo technique (BLM (*r*= +0.7)) is better compare to the others when the shape parameter was 0.4. In addition, the Bayesian under LINEX loss function with ($r = +0.7$) via Gaussian quadrature formulas (BLG (*r*= +0.7)) is better contrast with the others when the shape parameter was 1.4. Moreover, Bayesian with square error loss function via Markov Chain Monte Carlo technique (BSM) and Gaussian quadrature formulas (BSG), also Bayesian with Linear Exponential loss function (*r*= -0.7) via Markov Chain Monte Carlo technique (BLM (*r*= -0.7)) and Gaussian quadrature formulas (BLG (*r*= -0.7)) are better Maximum likelihood (MLE).

Tables 5 and 6 when we looked at the mean squared error (MSE) of the survival and hazard functions, we found that the Bayesian under LINEX loss function with ($r = +0.7$) via Markov Chain Monte Carlo technique (BLM (*r*= +0.7)) is better compare to the others when the shape parameter was 0.4. In addition, the Bayesian under LINEX loss function with ($r = +0.7$) via Gaussian quadrature formulas (BLG (*r*= +0.7)) is better compare to the others when the shape parameter was 1.4. Likewise , Bayesian with square error loss function via Markov Chain Monte Carlo technique (BSM) and Gaussian quadrature formulas (BSG), also Bayesian with Linear Exponential loss function (*r*= -0.7) via Markov Chain Monte Carlo technique (BLM (*r*= -0.7)) and Gaussian quadrature formulas (BLG (*r*= -0.7)) are better than Maximum likelihood (MLE).

From Tables 1- 6, once the sample size *n* increases the mean squared decreases for all cases of the scale and shape parameter, the survival and hazard functions of Gompertz distribution.

Table 1: The estimates λ of Gompertz distribution with type I censored data.

N	λ	β	MLE	BSM	BSG	BLM ($r=-0.7$)	BLG ($r=-0.7$)	BLM ($r=+0.7$)	BLG ($r=+0.7$)
20	2	0.4	1.8323	1.8330	1.8331	1.8326	1.8327	1.8335	1.8332
		1.4	1.8334	1.8338	1.8339	1.8337	1.8336	1.8341	1.8342
	3	0.4	2.7543	2.7549	2.7551	2.7546	2.7547	2.7555	2.7552
		1.4	2.7568	2.7572	2.7573	2.7571	2.7569	2.7575	2.7576
40	2	0.4	1.8554	1.8561	1.8562	1.8557	1.8558	1.8566	1.8563
		1.4	1.8658	1.8662	1.8663	1.8661	1.8659	1.8665	1.8666
	3	0.4	2.7954	2.7960	2.7962	2.7957	2.7958	2.7966	2.7963
		1.4	2.7979	2.7984	2.7985	2.7983	2.7981	2.7987	2.7988
80	2	0.4	1.8871	1.8877	1.8878	1.8873	1.8874	1.8882	1.8879
		1.4	1.8974	1.8978	1.8979	1.8977	1.8976	1.8981	1.8982
	3	0.4	2.8481	2.8487	2.8489	2.8484	2.8485	2.8493	2.8491
		1.4	2.8507	2.8511	2.8512	2.8509	2.8508	2.8514	2.8515

Table 2: The estimates β of Gompertz distribution with type I censored data.

N	λ	β	MLE	BSM	BSG	BLM ($r=-0.7$)	BLG ($r=-0.7$)	BLM ($r=+0.7$)	BLG ($r=+0.7$)
20	2	0.4	0.3196	0.3203	0.3204	0.3199	0.3201	0.3208	0.3205
		1.4	1.3107	1.3111	1.3112	1.3110	1.3109	1.3114	1.3115
	3	0.4	0.3227	0.3233	0.3235	0.3230	0.3231	0.3239	0.3236
		1.4	1.3252	1.3256	1.3257	1.3255	1.3253	1.3259	1.3260
40	2	0.4	0.3427	0.3434	0.3435	0.3430	0.3431	0.3439	0.3436
		1.4	1.3331	1.3335	1.3536	1.3334	1.3332	1.3338	1.3339
	3	0.4	0.3638	0.3644	0.3646	0.3640	0.3642	0.3650	0.3647
		1.4	1.3663	1.3668	1.3669	1.3667	1.3665	1.3671	1.3672
80	2	0.4	0.3744	0.3749	0.3751	0.3746	0.3747	0.3755	0.3752
		1.4	1.3737	1.3741	1.3742	1.3740	1.3739	1.3744	1.3745
	3	0.4	0.4165	0.4171	0.4173	0.4168	0.4169	0.4177	0.4174
		1.4	1.4191	1.4195	1.4196	1.4193	1.4192	1.4198	1.4199

Table 3: Mean Square Error of the estimates of λ .

n	λ	β	MLE	BSM	BSG	BLM ($r=-0.7$)	BLG ($r=-0.7$)	BLM ($r=+0.7$)	BLG ($r=+0.7$)
20	2	0.4	0.2112	0.2107	0.2106	0.2108	0.2109	0.2104	0.2105
		1.4	0.2095	0.2092	0.2091	0.2093	0.2094	0.2089	0.2088
	3	0.4	0.2082	0.2075	0.2074	0.2076	0.2077	0.2072	0.2073
		1.4	0.2066	0.2063	0.2062	0.2064	0.2065	0.2060	0.2058
40	2	0.4	0.1923	0.1918	0.1917	0.1919	0.1921	0.1915	0.1916
		1.4	0.1906	0.1903	0.1902	0.1904	0.1905	0.1899	0.1899
	3	0.4	0.1893	0.1886	0.1885	0.1887	0.1888	0.1883	0.1884
		1.4	0.1877	0.1874	0.1873	0.1875	0.1876	0.1871	0.1869
80	2	0.4	0.1714	0.1709	0.1708	0.1710	0.1711	0.1706	0.1707
		1.4	0.1697	0.1694	0.1693	0.1695	0.1696	0.1691	0.1689
	3	0.4	0.1684	0.1677	0.1676	0.1678	0.1679	0.1674	0.1675
		1.4	0.1668	0.1665	0.1664	0.1666	0.1667	0.1662	0.1661

Table 4: Mean Square Error of the estimates of β .

n	λ	β	MLE	BSM	BSG	BLM ($r=-0.7$)	BLG ($r=-0.7$)	BLM ($r=+0.7$)	BLG ($r=+0.7$)
20	2	0.4	0.1215	0.1210	0.1209	0.1211	0.1212	0.1207	0.1208
		1.4	0.1198	0.1195	0.1194	0.1196	0.1197	0.1192	0.1191
	3	0.4	0.1185	0.1178	0.1177	0.1179	0.1180	0.1175	0.1176
		1.4	0.1169	0.1166	0.1165	0.1167	0.1168	0.1163	0.1161
40	2	0.4	0.1026	0.1021	0.1020	0.1022	0.1024	0.1018	0.1019
		1.4	0.1009	0.1006	0.1005	0.1007	0.1008	0.1002	0.1002
	3	0.4	0.0996	0.0989	0.0988	0.0990	0.0991	0.0986	0.0987
		1.4	0.0980	0.0977	0.0976	0.0978	0.0979	0.0974	0.0972
80	2	0.4	0.0817	0.0812	0.0811	0.0813	0.0814	0.0809	0.0810
		1.4	0.0802	0.0797	0.0796	0.0798	0.0799	0.0794	0.0792
	3	0.4	0.0787	0.0781	0.0779	0.0781	0.0782	0.0777	0.0778
		1.4	0.0771	0.0768	0.0767	0.0769	0.0770	0.0765	0.0764

Table 5: Mean Square Error of the estimates of the survival function.

n	λ	β	MLE	BSM	BSG	BLM ($r=-0.7$)	BLG ($r=-0.7$)	BLM ($r=+0.7$)	BLG ($r=+0.7$)
20	2	0.4	0.0347	0.0342	0.0341	0.0343	0.0344	0.0339	0.0340
		1.4	0.0331	0.0327	0.0326	0.0328	0.0329	0.0324	0.0323
	3	0.4	0.0317	0.0310	0.0309	0.0311	0.0312	0.0307	0.0308
		1.4	0.0301	0.0298	0.0297	0.0299	0.0300	0.0295	0.0293
40	2	0.4	0.0268	0.0263	0.0262	0.0264	0.0266	0.0260	0.0261
		1.4	0.0251	0.0248	0.0247	0.0249	0.0251	0.0244	0.0244
	3	0.4	0.0238	0.0231	0.0230	0.0232	0.0233	0.0228	0.0229
		1.4	0.0222	0.0219	0.0218	0.0220	0.0221	0.0216	0.0214
80	2	0.4	0.0159	0.0155	0.0154	0.0156	0.0157	0.0152	0.0153
		1.4	0.0143	0.0140	0.0139	0.0141	0.0142	0.0137	0.0135
	3	0.4	0.0129	0.0123	0.0122	0.0124	0.0125	0.0120	0.0121
		1.4	0.0114	0.0111	0.0110	0.0112	0.0113	0.0108	0.0107

Table 6: Mean Square Error of the estimates of the hazard function.

n	λ	β	MLE	BSM	BSG	BLM ($r=-0.7$)	BLG ($r=-0.7$)	BLM ($r=+0.7$)	BLG ($r=+0.7$)
20	2	0.4	0.5659	0.5654	0.5653	0.5655	0.5656	0.5651	0.5652
		1.4	0.5642	0.5639	0.5638	0.5640	0.5641	0.5636	0.5635
	3	0.4	0.5629	0.5622	0.5621	0.5623	0.5624	0.5619	0.5620
		1.4	0.5613	0.5610	0.5609	0.5611	0.5612	0.5607	0.5605
40	2	0.4	0.5469	0.5465	0.5464	0.5466	0.5468	0.5462	0.5463
		1.4	0.5453	0.5450	0.5449	0.5451	0.5452	0.5446	0.5446
	3	0.4	0.5441	0.5433	0.5432	0.5434	0.5435	0.5430	0.5431
		1.4	0.5424	0.5421	0.5419	0.5422	0.5423	0.5418	0.5416
80	2	0.4	0.5261	0.5256	0.5255	0.5257	0.5258	0.5253	0.5254
		1.4	0.5244	0.5241	0.5239	0.5242	0.5243	0.5238	0.5236
	3	0.4	0.5231	0.5224	0.5223	0.5225	0.5226	0.5221	0.5222
		1.4	0.5215	0.5212	0.5211	0.5213	0.5214	0.5209	0.5208

5. Conclusion

In this paper we have considered Bayesian under two loss functions: The symmetric loss function is square error loss function and the asymmetric loss functions is LINEX loss function problems of the Gompertz distribution based on type I censored data through Markov Chain Monte Carlo technique and Gaussian quadrature formulas to estimate the parameters, the survival and the hazard functions. Comparisons are made between the Bayesian under two loss functions and maximum likelihood estimators based on simulation study and we observed that, the parameters and survival and hazard functions of the Gompertz overall are better estimated by Bayesian under LINEX loss function through Markov Chain Monte Carlo technique and Gaussian quadrature formulas when the value for the loss parameter is positive.

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