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SOME RESULT OF NON-COPRIME GRAPH OF INTEGERS MODULO n GROUP FOR n A PRIME POWER

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Abstract. One interesting topic in algebra and graph theory is a graph representation of a group, especially the representation of a group using a non-coprime graph. In this paper, we describe the non-coprime graph of integers modulo n group and its subgroups, for n is a prime power or n is a product of two distinct primes.
Keywords: group, integer module, non-coprime.

I. INTRODUCTION

The non-coprime graph of a finite group was introduced by Mansoori *et al.* [1]. In [1], the authors determined some numerical invariants of the non-coprime graph of a finite group, such as its diameter, girth, dominating number, independence, and chromatic number. Moreover, they characterize the planar non-coprime graph of a group and the regular non-coprime graph of a nilpotent group. Furthermore, they also stated a connection between the non-coprime graphs and some prime graphs.

Aghababaei-Beni and Jafarzadeh [2] investigated the properties of Cartesian and tensor products of non-coprime graphs of finite groups such as the dihedral and semi-dihedral groups. They considered the properties such as the independence, clique, chromatic number, covering number, diameter, connectedness, and the existence of the Eulerian spanning subgraph. They also gave a characterization for such graphs to be an Eulerian graph and to be a planar graph. Recently, Aghababaei *et al.* [3] extended some results in [2]. They studied the non-coprime of a finite group with respect to a subgroup and investigated some properties of such a graph, including its diameter, chromatic number, clique, and the number of connected components. They also investigated some properties of the non-coprime graph of the nilpotent group.

Some authors give some properties of the non-coprime graph and the coprime graph to more specific groups. Rilwan et al. give some properties of the non-coprime graph of integer [4], Juliana et al. give some properties of the non-coprime graph of an integer modulo [7], and Syarifudin et al. give some properties of the non-coprime graph of dihedral groups [8].

In this paper, we describe the non-coprime graph of the group $\mathbb{Z}_n$ and that of its subgroups, where n is a prime power or n is a product of two distinct primes. We used the result of the coprime graph of the group $\mathbb{Z}_n$ as the non-coprime graph is the duality of the coprime graph [6]. This paper is organized as follows. Section 2 (Some Basic Notions) collects some basic notions related to group and graph. We give our main results in Section 3 (Main Results). Some concluding remarks are collected in Section 4 (Conclusions). Finally, we give some related references in the References section.

II. SOME BASIC NOTION

Let G be a finite group and $|G|$ be the number of elements in G or the order of G. The definition of the order of an element in G is as follows.

Definition 1. *Let G be a finite group with the identity element e. The order of $g \in G$, denoted by $|g|$, is the smallest positive integer n such that $g^n = e$.*

Let H be any subgroup of G. In the rest of the paper, if H is a subgroup of G, then we denote it by $H \leq G$. Also, let a be an element in G. A subgroup $< a >= \{a^n |\ n \in \mathbb{Z}\}$ is called a cyclic subgroup of G generated by an element a. The following theorem states a relation between $|H|$ and $|G|$.

Theorem 1. (Lagrange's Theorem [4]). *If G is a finite group and $H \leq G$, then $|H|$ is a divisor of $|G|$.*

As a consequence of Theorem 1, we have that $| < g > |$ divides $|G|$.

A graph is one crucial object in mathematics, especially in discrete mathematics and its applications. The definition of a graph is as follows.

Definition 2. [5]. *A graph is a pair $\Gamma = (V, E)$, where V is a non-empty set of vertices, and $E \subseteq V \times V$ is a set of edges.*

We have to note that, in the rest of the paper, we only use a *simple undirected graph*, *i.e.*, we assume that $(v_i, v_j) = (v_j, v_i)$ for all $(v_i, v_j) \in E$.

Definition 3. *An undirected graph Γ is complete if for any $v_i, v_j \in V$, we have that $(v_i, v_j) \in E$. If $|V| = m$, then we denote an undirected complete graph Γ as K_m.*

Let a and b be two integers. The greatest common divisor of a and b usually denoted by (a, b). The following definition defines the non-coprime graph of a finite group.

Definition 4. [1]. *Let G be a finite group. The non-coprime graph of G denoted by $\bar{\Gamma}_G$, is a graph whose vertices are all elements of $G\backslash\{0\}$. Two different vertices x and y in $\bar{\Gamma}_G$ are adjacent if $(|x|, |y|) \neq 1$.*

III. MAIN RESULT

Let $\mathbb{Z}_n = \{0,1, \dots, n-1\}$ be the group of integers modulo n with addition (mod n) operation. The following proposition gives the non-coprime graph of $\mathbb{Z}_n$ when n is a prime number.

Proposition 1. *If n is a prime number, then the non-coprime graph of $\mathbb{Z}_n$ is a complete graph.*

Proof. Since n is a prime number, we have that $|i| = n$, for all $i = 1,2,\dots,n-1$. So,$(|a|,|b|) \neq 1$, for all, $b \in \mathbb{Z}_n$. These imply, a and b are adjacent in $\bar{\Gamma}_{\mathbb{Z}_n}$for all, $b \in \mathbb{Z}_n - \{0\}$. Therefore, the non-coprime graph of $\mathbb{Z}_n$ is a complete graph K_{n-1}.□

Here is an example of Proposition 1.

Example 1. Let $\mathbb{Z}_7 = \{0,1,2,3,4,5,6\}$. As we can see, $|1| = 7, |2| = 7, |3| = 7, |4| = 7, |5| = 7, |6| = 7$. So, we have that $(|a|,|b|) \neq 1, \forall\, a, b \in \mathbb{Z}_7$. Therefore, every non-zero element of $\mathbb{Z}_7$ are adjacent to each other. We can see $\bar{\Gamma}_{\mathbb{Z}_7}$in Figure1.

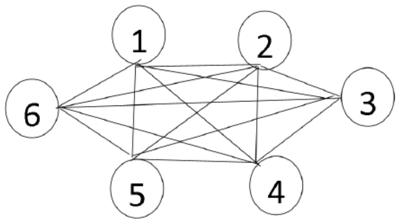

Figure 1. Non-coprime graph of $\mathbb{Z}_7$

Let $n = p^s$ for some prime number p and a natural number $s \geq 2$. The following theorem describes the non-coprime graph of $\mathbb{Z}_n$, when $n = p^s$.

Theorem 2. *If $n = p^s$, for some prime number p and natural number $s \geq 2$, then the non-coprime graph of $\mathbb{Z}_n$ is a complete graph.*

Proof. Let a be an element in $\mathbb{Z}_{p^s}$ with $(p^s, a) \neq 1$.The element a can be written as $a = p^k q$, for some $1 \leq k < s$ and an integer q, where $(p,q) = 1$. As a consequence, we have that $|a| = p^{s-k}$. Also, for any $b \in \mathbb{Z}_{p^s}$ with $(p^s, b) = 1$, we have that $|b| = p^s$. These imply $(|a|,|b|) \neq 1$, for all, $b \in \mathbb{Z}_{p^s} - \{0\}$. So, a and b are adjacent in $\bar{\Gamma}_{\mathbb{Z}_{p^s}}$ for all, $b \in \mathbb{Z}_{p^s} - \{0\}$. Therefore, the non-coprime graph of $\mathbb{Z}_{p^s}$ is a complete graph K_{p^s-1}.□

Here is an example of Theorem 2

Example 2. Let $\mathbb{Z}_{3^2} = \{0,1,2,3,4,5,\dots,8\}$. As we can see, $|1| = 9$, $|2| = 9, |3| = 3$, $|4| = 9, |5| = 9, |6| = 3, |7| = 9, |8| = 9$. Consequently, we have that a and b are adjacent in $\bar{\Gamma}_{\mathbb{Z}_{3^2}}$for all $a, b \in \mathbb{Z}_{3^2} - \{0\}$. The non-coprime graph of $\mathbb{Z}_{3^2}$ is shown in Figure 2.

Figure 2. Non-coprime graph of$\mathbb{Z}_9$

Let n be a product of two distinct primes. The following theorem describes the non-coprime graph of $\mathbb{Z}_n$, when n is a product of two distinct primes.

Theorem 3. *Let $= p_1p_2$, where p_1, p_2 are two distinct primes. If H is a proper subgroup of $\mathbb{Z}_n$, then the non-coprime graph of H is complete.*

Proof. Let H be any proper subgroup of $\mathbb{Z}_n$. By Theorem 1 (Lagrange's Theorem), we have that $|H| = p_1$ or $|H| = p_2$. Therefore, by Proposition 1, we have that Γ_H is a complete graph.

Here is an example of Theorem 3.

Example 3. Let $\mathbb{Z}_{15} = \{0, 1, 2, \dots, 14\}$. We can check that non-trivial subgroups of $\mathbb{Z}_{15}$ are $\langle 3 \rangle$ and $\langle 5 \rangle$. Moreover, we can see that $\langle 3 \rangle = \{0,3,6,9,12\}$ and $\langle 5 \rangle = \{0,5,10\}$. The non-coprime graphs of $\langle 3 \rangle$ and $\langle 5 \rangle$ are shown in Figure3.

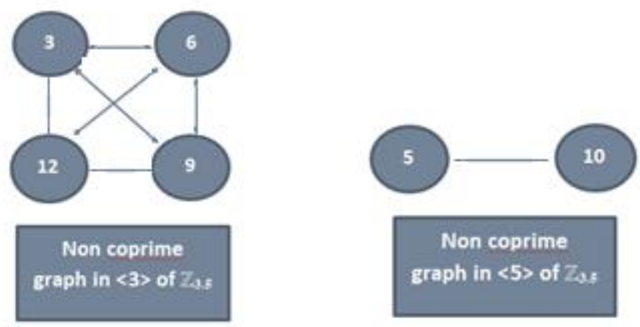

Figure 3. Non-coprime graph of subgroups in$\mathbb{Z}_{15}$

IV. CONCLUSIONS

We have shown that the non-coprime graph of $\mathbb{Z}_n$, when n is a prime power, is a complete graph K_{n-1}. Moreover, when n is a product of two distinct primes, the non-coprime graphs of its non-trivial subgroups are complete graphs.

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