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Properties On A New Comprehensive Family Of Holomorphic Functions Associated With Ruscheweyh Derivative and Generalized Multiplier Transformations

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Abstract.

In the present paper, a new comprehensive family of holomorphic functions, which includes various new subfamilies of holomorphic functions as well as some very well-known ones, is introduced. Sharp results concerning coefficient inequalities and distortion bounds of functions belonging to these families are determined. Furthermore, functions with negative coefficients belonging to these families are also investigated.

1-Introduction and Preliminaries

Let $\triangle$ be the unit disc in the complex plane $\mathcal{C}$ and let $\mathbb{H}(\mathbb{U})$ be the space of holomorphic functions in $\triangle$. Let $\mathcal{N}$ be the set of natural numbers, $\mathbb{R}$ be the set of real numbers and $\mathcal{N}_0 = \mathcal{N} \cup \{0\}$. Let $\mathcal{A}$ denote the family of functions in $\mathbb{H}(\mathbb{U})$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1}$$

Makinde et al. in [5] have recently introduced a new generalized multiplier differential operator $I^m_{\gamma,\delta,\eta}$ as follows.

Definition 1.1. *For $f \in \mathcal{A}$, the new generalized multiplier operator $I^m_{\gamma,\delta,\eta} : \mathcal{A} \to \mathcal{A}$ is defined by*

$$I^0_{\gamma,\delta,\eta} f(z) = f(z),$$

$$I^1_{\gamma,\delta,\eta} f(z) = \frac{\gamma f(z) + \delta z f'(z) + \eta z (z f'(z))'}{\gamma + \delta + \eta}, \ldots,$$

$$I^m_{\gamma,\delta,\eta} f(z) = I_{\gamma,\delta,\eta}(I^{m-1}_{\gamma,\delta,\eta} f(z)),$$

where $m \in \mathcal{N}_0$, $\delta \geq 0, \eta \geq 0$ and $\gamma \in \mathbb{R}$ such that $\gamma + \delta + \eta > 0$.

Remark 1.1. *Observe that for $f(z)$ given by (1), we have the following representation for $I^m_{\gamma,\delta,\eta}$.*

$$I^m_{\gamma,\delta,\eta} f(z) = z + \sum_{k=2}^{\infty} \nu_k(\gamma, \delta, \eta, m) a_k z^k, \tag{2}$$

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where

$$\nu_k(\gamma,\delta,\eta,m)=\left(\frac{\gamma+k\delta+k^2\eta}{\gamma+\delta+\eta}\right)^m. \quad (3)$$

Special cases of this operator include the operator $I^m_{\gamma,\delta,0}f(z)=I^m_{\gamma,\delta}f(z)$ introduced by Swamy in [12], the Al-Oboudi operator $I^m_{1-\delta,\delta,0}f(z)=D^m_\delta f(z)$ [1] and also the generalized Al-Oboudi operator $I^m_{l+1-\delta,\delta,0}f(z)=I^m_{l,\delta}f(z), l>-1,\ \delta\geq 0$ (considered for $l\geq 0$) investigated by Catas [3]. $D^m_1 f(z)$ was introduced by Sălăgean [7] and was considered for $m\geq 0$ by Boosnurmath and Swamy in [2].

Definition 1.2. [6] *For $m\in\mathcal{N}_0$, $f\in\mathcal{A}$,the operator $\mathcal{R}^m$ is defined by $\mathcal{R}^m:\mathcal{A}\to\mathcal{A}$,*

$$\mathcal{R}^0f(z),\mathcal{R}^1f(z)=zf'(z),...,(m+1)\mathcal{R}^{m+1}f(z)=z(\mathcal{R}^mf(z))'+m\mathcal{R}^mf(z),z\in\triangle.$$

Remark 1.2. *If $f(z)=z+\sum_{k=2}^{\infty}a_kz^k\in\mathcal{A}$, then $\mathcal{R}^mf(z)=z+\sum_{k=2}^{\infty}\Psi_k(m)a_kz^k, z\in\triangle$, where*

$$\Psi_k(m)=\frac{(m+k-1)!}{m!(k-1)!}, \quad (4)$$

We now state the following new operator.

Definition 1.3. *Let $f\in\mathcal{A}$, $m\in\mathcal{N}_0$, $\rho\geq 0$, $\delta\geq 0$, $\eta\geq 0$ and $\gamma\in\mathbb{R}$ such that $\gamma+\delta+\eta>0$. The operator $\mathcal{R}I^m_{\gamma,\delta,\eta,\rho}:\mathcal{A}\to\mathcal{A}$ is defined by*

$$\mathcal{R}I^m_{\gamma,\delta,\eta,\rho}f(z)=(1-\rho)\mathcal{R}^mf(z)+\rho I^m_{\gamma,\delta,\eta}f(z),z\in\triangle.$$

Remark 1.3. *If $f(z)=z+\sum_{k=2}^{\infty}a_kz^k$,then from Remark 1.1 and Remark 1.2,we have*

$$\mathcal{R}I^m_{\gamma,\delta,\eta,\rho}f(z)=z+\sum_{k=2}^{\infty}\phi_k(\gamma,\delta,\eta,\rho,m)a_kz^k,z\in\triangle,$$

where $\phi_k(\gamma,\delta,\eta,\rho,m)=(1-\rho)\Psi_k(m)+\rho\nu_k(\gamma,\delta,\eta,m),\nu_k(\gamma,\delta,\eta,m)$ and $\Psi_k(m)$ are as defined in (3) and (4) respectively.

We now define the following by making use of the new operator $\mathcal{R}I^m_{\gamma,\delta,\eta,\rho}$.

Definition 1.4. *Let $f\in\mathcal{A}$, $m\in\mathcal{N}_0$, $\rho\geq 0$, $\sigma\in[0,1)$, $\delta\geq 0$, $\eta\geq 0$ and $\gamma\in\mathbb{R}$ such that $\gamma+\delta+\eta>0$. Then f is in the family $\mathbb{S}^m_{\gamma,\delta,\eta,\rho}(\sigma)$ if*

$$Re\left(\frac{z(\mathcal{R}I^m_{\gamma,\delta,\eta,\rho}f(z))'}{\mathcal{R}I^m_{\gamma,\delta,\eta,\rho}f(z)}\right)>\sigma,z\in\triangle.$$

Definition 1.5. *Suppose that $m\in\mathcal{N}_0,\rho\geq 0,\sigma\in[0,1),\delta\geq 0,\eta\geq 0$ and $\gamma\in\mathbb{R}$ such that $\gamma+\delta+\eta>0$, then the function $f\in\mathcal{A}$ is said to be in the family $\mathbb{K}^m_{\gamma,\delta,\eta,\rho}(\sigma)$ if*

$$Re\left(\frac{[z^2(\mathcal{R}I^m_{\gamma,\delta,\eta,\rho}f(z))']'}{(z\mathcal{R}I^m_{\gamma,\delta,\eta,\rho}f(z))'}\right)>\sigma,z\in\triangle.$$

Definition 1.6. *The function $f\in\mathcal{A}$ is said to be in the family $\mathbb{C}^m_{\gamma,\delta,\eta,\rho}(\sigma)$ if*

$$Re\left(\frac{[z(\mathcal{R}I^m_{\gamma,\delta,\eta,\rho}f(z))']'}{(\mathcal{R}I^m_{\gamma,\delta,\eta,\rho}f(z))'}\right)>\sigma,z\in\triangle,$$

where $m\in\mathcal{N}_0$, $\rho\geq 0$, $\sigma\in[0,1)$, $\delta\geq 0$, $\eta\geq 0$ and $\gamma\in\mathbb{R}$ such that $\gamma+\delta+\eta>0$.

Definition 1.7. *For* $m \in \mathcal{N}_0,\ \rho \geq 0,\ \sigma \in [0,1),\ \delta \geq 0,\ \eta \geq 0$ *and* $\gamma \in \mathbb{R}$ *such that* $\gamma+\delta+\eta > 0$. *The function* $f \in \mathcal{A}$ *is said to be in the family* $\mathfrak{J}^{m}_{\gamma,\delta,\eta,\rho}(\sigma)$ *if*

$$Re\left(\frac{[z^2(\mathcal{R}I^{m}_{\gamma,\delta,\eta,\rho}f(z))']'}{2\mathcal{R}I^{m}_{\gamma,\delta,\eta,\rho}f(z)}\right) > \sigma, z \in \triangle.$$

Definition 1.8. *Let* $f \in \mathcal{A},\ m \in \mathcal{N}_0,\ \rho \geq 0,\ \sigma \in [0,1),\ \delta \geq 0,\ \eta \geq 0$ *and* $\gamma \in \mathbb{R}$ *such that* $\gamma+\delta+\eta > 0$. *Then* f *is said to be in the family* $\mathfrak{L}^{m}_{\gamma,\delta,\eta,\rho}(\sigma)$ *if*

$$Re\left(\frac{2z[z(\mathcal{R}I^{m}_{\gamma,\delta,\eta,\rho}f(z))']'}{[z(\mathcal{R}I^{m}_{\gamma,\delta,\eta,\rho}f(z))]'}\right) > \sigma, z \in \triangle.$$

Definition 1.9. *Let* $f \in \mathcal{A},\ m \in \mathcal{N}_0,\ \tau \geq 0,\ \rho \geq 0,\ \sigma \in [0,1),\ \delta \geq 0,\ \eta \geq 0$ *and* $\gamma \in \mathbb{R}$ *such that* $\gamma+\delta+\eta > 0$. *Then* f *is in the family* $\mathfrak{R}^{m,\tau}_{\gamma,\delta,\eta,\rho}(\sigma)$ *if*

$$Re\left(\left(\frac{z(\mathcal{R}I^{m}_{\gamma,\delta,\eta,\rho}f(z))'}{\mathcal{R}I^{m}_{\gamma,\delta,\eta,\rho}f(z)}\right)\left(1+\tau\frac{z(\mathcal{R}I^{m}_{\gamma,\delta,\eta,\rho}f(z))''}{(\mathcal{R}I^{m}_{\gamma,\delta,\eta,\rho}f(z))'}\right)\right) > \sigma, z \in \triangle. \tag{5}$$

In the following definition, a new comprehensive class of holomorphic functions containing the operator $\mathcal{R}I^{m}_{\gamma,\delta,\eta,\rho}$ is introduced.

Definition 1.10. *Let* $f \in \mathcal{A},\ m \in \mathcal{N}_0,\ \tau \geq 0,\ 0 \leq \zeta \leq 1,\ \tau \geq \zeta,\ \rho \geq 0,\ \sigma \in [0,1),\ \delta \geq 0,\ \eta \geq 0,\ \gamma \in \mathbb{R}$ *such that* $\gamma+\delta+\eta > 0$. *Then* f *is in the family* $\mathscr{N}^{m,\zeta,\tau}_{\gamma,\delta,\eta,\rho}(\sigma)$ *if*

$$Re\left(\frac{z(\mathcal{R}I^{m}_{\gamma,\delta,\eta,\rho}f(z))'+\tau z^2(\mathcal{R}I^{m}_{\gamma,\delta,\eta,\rho}f(z))''}{(1-\zeta)\mathcal{R}I^{m}_{\gamma,\delta,\eta,\rho}f(z)+\zeta z(\mathcal{R}I^{m}_{\gamma,\delta,\eta,\rho}f(z))'}\right) > \sigma, z \in \triangle. \tag{6}$$

The family $\mathscr{N}^{m,\zeta,\tau}_{\gamma,\delta,\eta,\rho}(\sigma)$ is of special interest for it contains many well-known as well as new families of holomorhic functions. In view of this, we deem it worthwhile to note the relevance of the family $\mathscr{N}^{m,\zeta,\tau}_{\gamma,\delta,\eta,\rho}(\sigma)$ with families defined above. Indeed we have $i)\mathscr{N}^{m,0,0}_{\gamma,\delta,\eta,\rho}(\sigma) = \mathbb{S}^{m}_{\gamma,\delta,\eta,\rho}(\sigma)$, $ii)\mathscr{N}^{m,1/2,1/2}_{\gamma,\delta,\eta,\rho}(\sigma) = \mathbb{K}^{m}_{\gamma,\delta,\eta,\rho}(\sigma)$, $iii)\mathscr{N}^{m,1,1}_{\gamma,\delta,\eta,\rho}(\sigma) = \mathbb{C}^{m}_{\gamma,\delta,\eta,\rho}(\sigma)$, $iv)\mathscr{N}^{m,0,1/2}_{\gamma,\delta,\eta,\rho}(\sigma) = \mathfrak{J}^{m}_{\gamma,\delta,\eta,\rho}(\sigma)$, $v)\mathscr{N}^{m,1/2,1}_{\gamma,\delta,\eta,\rho}(\sigma) = \mathfrak{L}^{m}_{\gamma,\delta,\eta,\rho}(\sigma)$ and $vi)\ \mathscr{N}^{m,0,\tau}_{\gamma,\delta,\eta,\rho}(\sigma) = \mathfrak{R}^{m,\tau}_{\gamma,\delta,\eta,\rho}(\sigma)$. The families $\mathbb{S}^{m}_{\gamma,\delta,\eta,\rho}(\sigma)$, $\mathbb{K}^{m}_{\gamma,\delta,\eta,\rho}(\sigma)$ and $\mathbb{C}^{m}_{\gamma,\delta,\eta,\rho}(\sigma)$ are considered in [9], while the families $\mathfrak{J}^{m}_{\gamma,\delta,\eta,\rho}(\sigma)$ and $\mathfrak{L}^{m}_{\gamma,\delta,\eta,\rho}(\sigma)$ are defined in [10].Further we note that $\mathscr{N}^{m,\zeta,\tau}_{\gamma,\delta,0,\rho}(\sigma) = \mathfrak{N}^{m,\zeta,\tau}_{\gamma,\delta,\rho}(\sigma)$ (See [11]).

Throughout this paper, unless and otherwise mentioned we shall assume that $\phi_k(\gamma,\delta,\eta,\rho,m)$ $= (1-\rho)\Psi_k(m) + \rho\nu_k(\gamma,\delta,\eta,m), \nu_k(\gamma,\delta,\eta,m)$,where $\nu_k(\gamma,\delta,\eta,m)$ and $\Psi_k(m)$ are as defined in (3) and (4) respectively.

2. COEFFICIENT ESTIMATES

In this section, we obtain sufficient coefficient bound for $f \in \mathcal{A}$ to be in the family $\mathscr{N}^{m,\zeta,\tau}_{\gamma,\delta,\eta,\rho}(\sigma)$ following the paper of Darus and Ibrahim [4].

Theorem 2.1. *Let* $f \in \mathcal{A},\ m \in \mathcal{N}_0,\ \tau \geq 0,\ 0 \leq \zeta \leq 1,\ \tau \geq \zeta,\ \rho \geq 0,\ \sigma \in [0,1),\ \delta \geq 0,\ \eta \geq 0$ *and* $\gamma \in \mathbb{R}$ *such that* $\gamma+\delta+\eta > 0$. *If*

$$\sum_{k=2}^{\infty}[(k-\sigma)+(k-1)(k\tau-\sigma\zeta)]\phi_k(\gamma,\delta,\eta,\rho,m)|a_k| \leq (1-\sigma), \tag{7}$$

then $f \in \mathscr{N}^{m,\zeta,\tau}_{\gamma,\delta,\eta,\rho}(\sigma)$. *The result (7) is sharp.*

Proof. It suffices to show that the values of $\left(\frac{z(\mathcal{R}I^m_{\gamma,\delta,\eta,\rho}f(z))'+\tau z^2(\mathcal{R}I^m_{\gamma,\delta,\eta,\rho}f(z))''}{(1-\zeta)\mathcal{R}I^m_{\gamma,\delta,\eta,\rho}f(z)+\zeta z(\mathcal{R}I^m_{\gamma,\delta,\eta,\rho}f(z))'}\right)$ lie in a circle centred at 1 with radius $1-\sigma$. Clearly

$$\left|\left(\frac{z(\mathcal{R}I^m_{\gamma,\delta,\eta,\rho}f(z))'+\tau z^2(\mathcal{R}I^m_{\gamma,\delta,\eta,\rho}f(z))''}{(1-\zeta)\mathcal{R}I^m_{\gamma,\delta,\eta,\rho}f(z)+\zeta z(\mathcal{R}I^m_{\gamma,\delta,\eta,\rho}f(z))'}\right)-1\right|$$
$$=\left|\frac{\sum\limits_{k=2}^{\infty}(k-1)(1+\tau k-\zeta)\phi_k(\gamma,\delta,\eta,\rho,m)a_kz^k}{z+\sum\limits_{k=2}^{\infty}(1+(k-1)\zeta)\phi_k(\gamma,\delta,\eta,\rho,m)a_kz^k}\right|$$
$$\le\frac{\sum\limits_{k=2}^{\infty}(k-1)(1+\tau k-\zeta)\phi_k(\gamma,\delta,\eta,\rho,m)|a_k||z|^{k-1}}{1-\sum\limits_{k=2}^{\infty}(1+(k-1)\zeta)\phi_k(\gamma,\delta,\eta,\rho,m)|a_k||z|^{k-1}}$$
$$\le\frac{\sum\limits_{k=2}^{\infty}(k-1)(1+\tau k-\zeta)\phi_k(\gamma,\delta,\eta,\rho,m)|a_k|}{1-\sum\limits_{k=2}^{\infty}(1+(k-1)\zeta)\phi_k(\gamma,\delta,\eta,\rho,m)|a_k|}.$$

The last expression is bounded above by $1-\sigma$ if

$$\sum_{k=2}^{\infty}(k-1)(1+\tau k-\zeta)\phi_k(\gamma,\delta,\eta,\rho,m)|a_k|\le(1-\sigma)\left(1-\sum_{k=2}^{\infty}(1+(k-1)\zeta)\phi_k(\gamma,\delta,\eta,\rho,m)|a_k|\right),$$

which is equivalent to (7). Hence $\left|\left(\frac{z(\mathcal{R}I^m_{\gamma,\delta,\eta,\rho}f(z))'+\tau z^2(\mathcal{R}I^m_{\gamma,\delta,\eta,\rho}f(z))''}{(1-\zeta)\mathcal{R}I^m_{\gamma,\delta,\eta,\rho}f(z)+\zeta z(\mathcal{R}I^m_{\gamma,\delta,\eta,\rho}f(z))'}\right)-1\right|\le 1-\sigma$ and the theorem is proved. □

The assertion (7) is sharp and extremal function is given by

$$f(z)=z+\sum_{k=2}^{\infty}\frac{1-\sigma}{[(k-\sigma)+(k-1)(k\tau-\sigma\zeta)]\phi_k(\gamma,\delta,\eta,\rho,m)}z^k,\ z\in\triangle.$$

Corollary 2.1. *Let the hypothesis of (7) be satisfied.Then*

$$|a_k|\le\frac{1-\sigma}{[(k-\sigma)+(k-1)(k\tau-\sigma\zeta)]\phi_k(\gamma,\delta,\eta,\rho,m)},\ \forall k\ge 2.$$

On taking $i)\zeta=0,\tau=0, ii)\zeta=1/2,,\tau=1/2, iii)\zeta=1,\tau=1, iv)\zeta=0,\tau=1/2$ and $v)\zeta=1/2,\tau=1$ in Theorem 2.1, we obtain

Theorem 2.2. *Let $f\in\mathcal{A}, m\in\mathcal{N}_0$, $\rho\ge 0$, $\sigma\in[0,1)$, $\delta\ge 0$, $\eta\ge 0$ and $\gamma\in\mathbb{R}$ such that* $\gamma+\delta+\eta>0$.

i) *If* $\sum\limits_{k=2}^{\infty}(k-\sigma)\phi_k(\gamma,\delta,\eta,\rho,m)|a_k|\le(1-\sigma)$, *then* $f\in\mathbb{S}^m_{\gamma,\delta,\eta,\rho}(\sigma)$.

ii) *If* $\sum\limits_{k=2}^{\infty}\left(\frac{k+1}{2}\right)(k-\sigma)\phi_k(\gamma,\delta,\eta,\rho,m)|a_k|\le(1-\sigma)$, *then* $f\in\mathbb{K}^m_{\gamma,\delta,\eta,\rho}(\sigma)$.

iii) $\sum\limits_{k=2}^{\infty}k(k-\sigma)\phi_k(\gamma,\delta,\eta,\rho,m)|a_k|\le(1-\sigma)$ *then* $f\in\mathbb{C}^m_{\gamma,\delta,\eta,\rho}(\sigma)$.

iv) *If* $\sum\limits_{k=2}^{\infty}\left(\frac{k(k+1)}{2}-\sigma\right)\phi_k(\gamma,\delta,\eta,\rho,m)|a_k|\le(1-\sigma)$, *then* $f\in\mathfrak{J}^m_{\gamma,\delta,\eta,\rho}(\sigma)$.

v) *If* $\sum_{k=2}^{\infty}\left[k^2-\left(\frac{k+1}{2}\right)\sigma\right]\phi_k(\gamma,\delta,\eta,\rho,m)|a_k|\le(1-\sigma)$, *then* $f\in\mathfrak{L}^m_{\gamma,\delta,\eta,\rho}(\sigma)$.

All results are sharp.

The following inclusion theorem can be proved using Theorem 2.1.

Theorem 2.3. *Let* $0\le\sigma_1\le\sigma_2<1$. *Then* $\mathcal{N}^{m,\zeta,\tau}_{\gamma,\delta,\eta,\rho}(\sigma_2)\subseteq\mathcal{N}^{m,\zeta,\tau}_{\gamma,\delta,\eta,\rho}(\sigma_1)$.

Taking $\zeta=0$ in Theorem 2.1 we obtain

Theorem 2.4. *Let* $f\in\mathcal{A},m\in\mathcal{N}_0,\ \tau\ge0,\ \rho\ge0,\ \sigma\in[0,1),\ \delta\ge0,\ \eta\ge0$ *and* $\gamma\in\mathbb{R}$ *such that* $\gamma+\delta+\eta>0$. *If* $\sum_{k=2}^{\infty}[(k-\sigma)+(k-1)k\tau]\phi_k(\gamma,\delta,\eta,\rho,m)|a_k|\le(1-\sigma)$, *then* $f\in\mathfrak{R}^{m,\tau}_{\gamma,\delta,\eta,\rho}(\sigma)$. *The result is sharp.*

Taking $\rho=0$ and $\rho=1$ in Theorem 2.4, we get

Corollary 2.2. *Let* $f\in\mathcal{A},m\in\mathcal{N}_0,\ \tau\ge0,\ \sigma\in[0,1),\ \delta\ge0,\ \eta\ge0$ *and* $\gamma\in\mathbb{R}$ *such that* $\gamma+\delta+\eta>0$.

(i) *If* $\sum_{k=2}^{\infty}[(k-\sigma)+(k-1)k\tau]\Psi_k(m)|a_k|\le1-\sigma$, *then*

$$Re\left\{\left(\frac{z(\mathcal{R}^mf(z))'}{\mathcal{R}^mf(z)}\right)\left(1+\tau\frac{z(\mathcal{R}^mf(z))''}{(\mathcal{R}^mf(z))'}\right)\right\}>\sigma,z\in\triangle.$$

(ii) *If* $\sum_{k=2}^{\infty}[(k-\sigma)+(k-1)k\tau]\nu_k(\gamma,\delta,\eta,,m)|a_k|\le1-\sigma$, *then*

$$Re\left\{\left(\frac{z(I^m_{\gamma,\delta,\eta}f(z))'}{I^m_{\gamma,\delta,\eta}f(z)}\right)\left(1+\tau\frac{z(I^m_{\gamma,\delta,\eta}f(z)''}{(I^m_{\gamma,\delta,\eta}f(z))'}\right)\right\}>\sigma,z\in\triangle.$$

The results are sharp.

3. DISTORTION BOUNDS

In this section we obtain a distortion bound for $\mathcal{R}I^m_{\gamma,\delta,\eta,\rho}f(z)$ and $f(z)$.

Theorem 3.1. *Let* $f\in\mathcal{A},m\in\mathcal{N}_0,\ \tau\ge0,\ 0\le\zeta\le1,\ \tau\ge\zeta,\ \rho\ge0,\ \sigma\in[0,1),\ \delta\ge0,\ \eta\ge0$ *and* $\gamma\in\mathbb{R}$ *such that* $\gamma+\delta+\eta>0$. *If*

$$\sum_{k=2}^{\infty}[(k-\sigma)+(k-1)(k\tau-\sigma\zeta)]\phi_k(\gamma,\delta,\eta,\rho,m)|a_k|\le1-\sigma,$$

then

$$|z|-\frac{1-\sigma}{(2-\sigma)+(2\tau-\sigma\zeta)}|z|^2\le|\mathcal{R}I^m_{\gamma,\delta,\eta,\rho}f(z)|\le|z|+\frac{1-\sigma}{(2-\sigma)+(2\tau-\sigma\zeta)}|z|^2,z\in\triangle.$$

Proof. By Theorem 2.1,it is easy to verify that

$$\begin{aligned}&[(2-\sigma)+(2\tau-\sigma\zeta)]\sum_{k=2}^{\infty}\phi_k(\gamma,\delta,\eta,\rho,m)|a_k|\\&\qquad\le\sum_{k=2}^{\infty}[(k-\sigma)+(k-1)(2\tau-\sigma\zeta)]\phi_k(\gamma,\delta,\eta,\rho,m)|a_k|\le1-\sigma.\end{aligned}$$

So $\sum_{k=2}^{\infty} \phi_k(\gamma,\delta,\eta,\rho,m)|a_k| \leq \frac{1-\sigma}{(2-\sigma)+(2\tau-\sigma\zeta)}$. Hence we obtain

$$\begin{aligned}|\mathcal{R}I^m_{\gamma,\delta,\eta,\rho}f(z)| &\leq |z| + \sum_{k=2}^{\infty} \phi_k(\gamma,\delta,\eta,\rho,m)|a_k||z|^k \\ &\leq |z| + |z|^2 \sum_{k=2}^{\infty} \phi_k(\gamma,\delta,\eta,\rho,m)|a_k| \\ &\leq |z| + \frac{1-\sigma}{(2-\sigma)+(2\tau-\sigma\zeta)}|z|^2.\end{aligned}$$

The other assertion can be proved as follows.

$$\begin{aligned}|\mathcal{R}I^m_{\gamma,\delta,\eta,\rho}f(z)| &\geq |z| - \sum_{k=2}^{\infty} \phi_k(\gamma,\delta,\eta,\rho,m)|a_k||z|^k \\ &\geq |z| - |z|^2 \sum_{k=2}^{\infty} \phi_k(\gamma,\delta,\eta,\rho,m)|a_k| \\ &\geq |z| - \frac{1-\sigma}{(2-\sigma)+(2\tau-\sigma\zeta)}|z|^2.\end{aligned}$$

This completes the proof of the theorem. □

Taking $\zeta = 0$ in Theorem 3.1 we get

Theorem 3.2. *Let* $f \in \mathcal{A}, m \in \mathcal{N}_0,\ \tau \geq 0,\ \rho \geq 0,\ \sigma \in [0,1),\ \delta \geq 0,\ \eta \geq 0$ *and* $\gamma \in \mathbb{R}$ *such that* $\gamma+\delta+\eta > 0$. *If*

$$\sum_{k=2}^{\infty}[(k-\sigma)+(k-1)k\tau]\phi_k(\gamma,\delta,\eta,\rho,m)|a_k| \leq 1-\sigma,$$

then

$$|z| - \frac{1-\sigma}{2-\sigma+2\tau}|z|^2 \leq |\mathcal{R}^{m,\tau}_{\gamma,\delta,\eta,\rho}f(z)| \leq |z| + \frac{1-\sigma}{2-\sigma+2\tau}|z|^2, z \in \triangle.$$

Theorem 3.3. *Let* $f \in \mathcal{A}, m \in \mathcal{N}_0,\ \tau \geq 0,\ 0 \leq \zeta \leq 1,\ \tau \geq \zeta,\ \rho \geq 0,\ \sigma \in [0,1),\ \delta \geq 0,\ \eta \geq 0$ *and* $\gamma \in \mathbb{R}$ *such that* $\gamma+\delta+\eta > 0$. *If*

$$\sum_{k=2}^{\infty}[(k-\sigma)+(k-1)(k\tau-\sigma\zeta)]\phi_k(\gamma,\delta,\eta,\rho,m)|a_k| \leq 1-\sigma,$$

then

$$|f(z)| \geq |z| - \frac{1-\sigma}{[(2-\sigma)+(2\tau-\sigma\zeta)]\phi_2(\gamma,\delta,\eta,\rho,m)}|z|^2, z \in \triangle$$

and

$$|f(z)| \leq |z| + \frac{1-\sigma}{[(2-\sigma)+(2\tau-\sigma\zeta)]\phi_2(\gamma,\delta,\eta,\rho,m)}|z|^2, z \in \triangle.$$

Proof. In virtue of Theorem 2.1, we have

$$\begin{aligned}&[(2-\sigma)+(2\tau-\sigma\zeta)]\phi_2(\gamma,\delta,\eta,\rho,m)\sum_{k=2}^{\infty}|a|_k \leq \\ &\qquad \sum_{k=2}^{\infty}[(k-\sigma)+(k-1)(k\tau-\sigma\zeta)]\phi_k(\gamma,\delta,\eta,\rho,m)|a_k| \leq 1-\sigma.\end{aligned}$$

Thus $\sum_{k=2}^{\infty}|a_k| \leq \frac{1-\sigma}{[(2-\sigma)+(2\tau-\sigma\zeta)]\phi_2(\gamma,\delta,\eta,\rho,m)}$. So we get

$$|f(z)| \leq |z| + |z|^2\sum_{k=2}^{\infty}|a_k| \leq |z| + \frac{1-\sigma}{[(2-\sigma)+(2\tau-\sigma\zeta)]\phi_2(\gamma,\delta,\eta,\rho,m)}|z|^2.$$

On the otherhand

$$|f(z)| \geq |z| - |z|^2\sum_{k=2}^{\infty}|a_k| \ \geq |z| - \frac{1-\sigma}{[(2-\sigma)+(2\tau-\sigma\zeta)]\phi_2(\gamma,\delta,\eta,\rho,m)}|z|^2.$$

□

Taking $\zeta = 0$ in Theorem 3.3 we get

Theorem 3.4. *Let* $f \in \mathcal{A}, m \in \mathcal{N}_0,\ \tau \geq 0,\ \rho \geq 0,\ \sigma \in [0,1),\ \delta \geq 0,\ \eta \geq 0$ *and* $\gamma \in \mathbb{R}$ *such that* $\gamma+\delta+\eta > 0$. *If*

$$\sum_{k=2}^{\infty}[(k-\sigma)+(k-1)k\tau]\phi_2(\gamma,\delta,\eta,\rho,m)|a_k| \leq 1-\sigma,$$

then

$$|f(z)| \leq |z| + \frac{1-\sigma}{(2-\sigma+2\tau)\phi_2(\gamma,\delta,\eta,\rho,m)}|z|^2, z \in \triangle,$$

and

$$|f(z)| \geq |z| - \frac{1-\sigma}{(2-\sigma+2\tau)\phi_2(\gamma,\delta,\eta,\rho,m)}|z|^2, z \in \triangle.$$

4. FUNCTIONS WITH NEGATIVE COEFFICIENTS

Let $\mathcal{T}$ denote the subclass of $\mathcal{A}$ consisting of the form $f(z) = z - \sum_{k=2}^{\infty} a_k z^k, a_k \geq 0$.We denote by $\mathcal{T}\mathfrak{R}^{m,\tau}_{\gamma,\delta,\eta,\rho}(\sigma)$ and $\mathcal{T}\mathcal{N}^{m,\zeta,\tau}_{\gamma,\delta,\eta,\rho}(\sigma)$ the classes of functions $f(z) \in \mathcal{T}$ satisfying (5) and (6) respectively. We study the coefficient estimates, distortion theorems and other properties of the classes $\mathcal{T}\mathfrak{R}^{m,\tau}_{\gamma,\delta,\eta,\rho}(\sigma)$ and $\mathcal{T}\mathcal{N}^{m,\zeta,\tau}_{\gamma,\delta,\eta,\rho}(\sigma)$, following the paper of Silverman [8].

We now provide the necessary and sufficient coefficient bound for $f \in \mathcal{T}$ to be in $\mathcal{T}\mathcal{N}^{m,\zeta,\tau}_{\gamma,\delta,\eta,\rho}(\sigma)$.

Theorem 4.1. *A function* $f \in \mathcal{T}$ *is in* $\mathcal{T}\mathcal{N}^{m,\zeta,\tau}_{\gamma,\delta,\eta,\rho}(\sigma)$ *if and only if*

$$\sum_{k=2}^{\infty}[(k-\sigma)+(k-1)(k\tau-\sigma\zeta)]\phi_k(\gamma,\delta,\eta,\rho,m)|a_k| \leq 1-\sigma, \tag{8}$$

where $m \in \mathcal{N}_0,\ \tau \geq 0,\ 0 \leq \zeta \leq 1, \geq \zeta,\ \rho \geq 0,\ \sigma \in [0,1),\ \delta \geq 0,\ \eta \geq 0$ *and* $\gamma \in \mathbb{R}$ *such that* $\gamma+\delta+\eta > 0$. *The result is sharp.*

Proof. In view of Theorem 2.1, it suffices to prove the only if part. Assume that

$$Re\left(\frac{z(\mathcal{R}I^m_{\gamma,\delta,\eta,\rho}f(z))' + \tau z^2(\mathcal{R}I^m_{\gamma,\delta,\eta,\rho}f(z))''}{(1-\zeta)\mathcal{R}I^m_{\gamma,\delta,\eta,\rho}f(z) + \zeta z(\mathcal{R}I^m_{\gamma,\delta,\eta,\rho}f(z))'}\right) =$$

$$Re\left(\frac{z - \sum_{k=2}^{\infty}(1+\tau(k-1)\phi_k(\gamma,\delta,\eta,\rho,m)ka_kz^k}{z - \sum_{k=2}^{\infty}(1+\zeta(k-1)\phi_k(\gamma,\delta,\eta,\rho,m)a_kz^k}\right) > \sigma \tag{9}$$

Clearing the denominator in (9) and letting $z \to 1^-$ through the real values, we get

$$1 - \sum_{k=2}^{\infty} [1 + \tau(k-1)]\phi_k(\gamma, \delta, \eta, \rho, m) k a_k \geq \sigma \left(1 - \sum_{k=2}^{\infty} [1 + \zeta(k-1)]\phi_k(\gamma, \delta, \eta, \rho, m) a_k \right)$$

Hence we obtain (8), and the proof is complete. □

Finally, we note that assertion (8) of Theorem 4.1 is sharp, extremal function being

$$f(z) = z - \sum_{k=2}^{\infty} \frac{1-\sigma}{[(k-\sigma) + (k-1)(k\tau - \sigma\zeta)]\phi_k(\gamma, \delta, \eta, \rho, m)} z^k, k \geq 2, z \in \triangle.$$

Our coefficient bounds enable us the following:

Theorem 4.2. *If a function $f \in \mathcal{T}$ is in $\mathcal{T}\mathcal{N}^{m,\zeta,\tau}_{\gamma,\delta,\eta,\rho}(\sigma)$, then*

$$|f(z)| \geq |z| - \frac{1-\rho}{[(2-\sigma) + (2\tau - \sigma\zeta)]\phi_2(\gamma, \delta, \eta, \rho, m)} |z|^2, z \in \triangle$$

and

$$|f(z)| \leq |z| + \frac{1-\rho}{[(2-\sigma) + (2\tau - \sigma\zeta)]\phi_2(\gamma, \delta, \eta, \rho, m)} |z|^2, z \in \triangle,$$

where $m \in \mathcal{N}_0$, $\tau \geq 0$, $0 \leq \zeta \leq 1$, $\tau \geq \zeta$, $\rho \geq 0$, $\sigma \in [0,1)$, $\delta \geq 0$, $\eta \geq 0$ and $\gamma \in \mathbb{R}$ such that $\gamma + \delta + \eta > 0$. The result is sharp.

Taking $\zeta = 0$ in Theorem 4.1 and Theorem 4.2 , we obtain

Theorem 4.3. *If a function $f \in \mathcal{T}$ is in $\mathcal{T}\mathfrak{R}^{m,\tau}_{\gamma,\delta,\eta,\rho}(\sigma)$, then*

$$|f(z)| \geq |z| - \frac{1-\rho}{[(2-\sigma) + 2\tau]\phi_2(\gamma, \delta, \eta, \rho, m)} |z|^2, z \in \triangle$$

and

$$|f(z)| \leq |z| + \frac{1-\rho}{[(2-\sigma) + 2\tau]\phi_2(\gamma, \delta, \eta, \rho, m)} |z|^2, z \in \triangle,$$

where $m \in \mathcal{N}_0$, $\tau \geq 0$, $\rho \geq 0$, $\sigma \in [0,1)$, $\delta \geq 0$, $\eta \geq 0$ and $\gamma \in \mathbb{R}$ such that $\gamma + \delta + \eta > 0$. The result is sharp.

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