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CHARACTERISTICS OF STUDENTS' ABDUCTIVE REASONING IN SOLVING ALGEBRA PROBLEMS

Indriati Nurul Hidayah, Cholis Sa'dijah, Subanji, Sudirman

Universitas Negeri Malang, Jl. Semarang 5, Malang, Jawa Timur, Indonesia
Email: sudirman.fmipa@um.ac.id

Abstract

When students solve an algebra problem, students try to deduce the facts in the problem. This step is imperative, students can draw conclusions from the facts and devise a plan to solve the problem. Drawing conclusions from facts is called reasoning. Some kinds of reasoning are deductive, inductive, and abductive. This article explores the characteristics of some types of abductive reasoning used by mathematics education students in problem-solving related to using facts on the problems. Fifty-eight students were asked to solve an algebra problem. It was found that the student's solutions could be grouped into four types of abductive reasoning. From each group, one student was interviewed to have more details on the types. First, the creative conjectures type, the students can solve the problems and develop new ideas related to the problems; second, fact optimization type, the students make conjecture of the answer, then confirm it by deductive reasoning; third, factual error type, students use facts outside of the problems to solve it, but the facts are wrong; and fourth, mistaken fact type, the students assume the questionable thing as a given fact. Therefore, teachers should encourage the students to use creative conjectures and fact optimization when learning mathematics.

Keywords: Characteristics, Abductive reasoning, Algebra problems

Abstrak

Ketika mahasiswa menyelesaikan masalah aljabar, mahasiswa berusaha untuk menyimpulkan fakta-fakta yang terdapat pada masalah. Langkah ini sangat penting karena mahasiswa dapat menarik kesimpulan dari fakta-fakta dan merancang rencana untuk menyelesaikan masalah. Penarikan kesimpulan dari fakta-fakta disebut penalaran. Beberapa jenis penalaran di antaranya adalah penalaran deduktif, induktif dan abduktif. Artikel ini mendiskusikan karakteristik dari beberapa tipe penalaran abduktif yang digunakan oleh mahasiswa pendidikan matematika dalam penyelesaian masalah yang dikaitkan dengan penggunaan fakta pada masalah. Lima puluh delapan mahasiswa diminta untuk menyelesaikan masalah aljabar. Ditemukan bahwa jawaban mahasiswa dapat dikelompokkan ke dalam empat tipe penalaran abduktif. Dari masing-masing kelompok, diambil satu mahasiswa untuk diwawancara dengan tujuan mendapatkan penjelasan yang lebih detil dari masing-masing tipe tersebut. Empat tipe tersebut dapat dijelaskan sebagai berikut, pertama, tipe kreatif konjektur, mahasiswa dapat menyelesaikan masalah dan mengembangkan ide baru yang berkaitan dengan masalah; kedua, tipe optimalisasi fakta, mahasiswa membuat konjektur tentang jawaban masalah, kemudian mengonfirmasi konjektur dengan melakukan penalaran deduktif; ketiga, tipe fakta salah, mahasiswa mengambil fakta di luar soal untuk menyelesaikan masalah, namun fakta yang diambil salah; dan keempat, tipe kesalahan fakta, mahasiswa menganggap hal yang ditanyakan pada soal sebagai fakta yang diberikan. Oleh karena itu, dosen seharusnya mendorong mahasiswa untuk menggunakan kreatif konjektur dan optimalisasi fakta ketika mempelajari matematika.

Kata kunci: Karakteristik, Penalaran abduktif, Masalah aljabar

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Problem-solving has been regarded as an essential domain in the teaching and learning of mathematics, though it takes several requirements to understand the process of doing problem-solving. In recent years, several studies have been carried out to explore the essence of problem-solving in mathematics learning (Gurat, 2018; Jäder et al., 2019; Chew et al., 2019; Reiss & Törner, 2007; Ekawati et al., 2019). For instance, Gurat (2018) discussed problem-solving strategies in teacher-student interaction, and

Reiss and Törner (2007) investigated cognitive psychologists influencing students to solve mathematics problems in a classroom. A recent report also informed that problem-solving learning must be supported by the availability of books (Jäder et al., 2019).

Based on our classroom observations, some students encountered difficulties in problem-solving activities. Their inability may be caused by the complexities of making conclusions of the given facts in the problem-solving. Students' failure in problem-solving shows that the importance of problem-solving has not been well-taught in mathematics learning.

One of the prominent factors that can support problem-solving in practice is reasoning. Therefore, NCTM (2000) emphasizes the importance of reasoning and evidence as fundamental aspects of mathematics learning. Theoretically, the reasoning is defined as the process of drawing conclusions (Leighton & Sternberg, 2003; Sternberg & Sternberg, 2012). Reasoning and evidence are needed in building up a reasonable argument to prove the truth of a statement.

Reasoning in mathematics learning has been found to be varied such as quantitative reasoning (Moore, 2014), covariational reasoning (Subanji & Supratman, 2015), proportional reasoning (Im & Jitendra, 2020), analogy reasoning (Lailiyah, et al., 2018), algebra reasoning (Otten et al., 2019), and many more. The nature of mathematical thinking serves as a character in reasoning. In this research, one aspect of thinking in mathematics learning that is drawing a conclusion from an abductive perspective is discussed. In particular, the term abductive reasoning is used in this paper.

Abductive reasoning is closely related to human lives in the form of conjecture that is made based on some facts. Abductive reasoning is also widely explored in several fields, such as in medical diagnosis (Alirezaie & Loutfi, 2014; Aliseda, 2006), computer and programing (Dong et al., 2015; Ma et al., 2008), and critical thinking-related activities (O'Reilly, 2016).

In the context of mathematics learning, abductive reasoning is adopted to improve discovery making and develop students' creative reasoning (Ferguson, 2019). Other researchers have developed a learning model by using abductive reasoning and drawing the conclusion process to generate conjectures in geometry (Baccaglini-Frank, 2019), explore the effect of learning with abductive-deductive reasoning (Shodikin, 2017), and facilitate students' inquiries in calculus (Park & Lee, 2016). The importance of abductive reasoning in problem-solving is discussed further by Cifarelli (2016).

Abductive reasoning is another type of reasoning, besides deductive and inductive reasoning (Peirce, 1958). Deductive reasoning is defined as the process of reasoning from general principles to particular cases (Cohen & Stemmer, 2007). It involves reasoning from one or more general statements that are known to a specific case of the general statement. In contrast, deductive reasoning involves reaching conclusions from a set of conditional propositions or a syllogistic pair of premises (Sternberg & Sternberg, 2012).

Inductive reasoning is the process of reasoning from specific facts or observations to reach a conclusion that may explain the facts (Johnson-Laird, 1999). Inductive reasoning is a type of reasoning which takes a conclusion as approximate as possible because it is derived from a method of inference that generally leads to the truth in the long run (Peirce, 1958). Although we cannot reach logically certain conclusions through inductive reasoning, we can still at least reach highly probable conclusions through careful reasoning (Sternberg & Sternberg, 2012).

Abductive reasoning is defined as the process of forming an explanatory hypothesis from an observation that requires an explanation. Abductive reasoning is started from the facts, uses any particular theory in view, and it is motived by the feeling that a theory is needed to explain the surprising facts. Abductive reasoning is a data-driven process and also dependent on the knowledge domain. This process is not algorithmic, which is less on its logical form and an act of insight, although it is extremely fallible insight (Peirce, 1958). Abductive reasoning is often portrayed as explanatory reasoning that leads back from facts to a proposed explanation of those facts (Seel, 2012).

Deductive, inductive, and abductive reasoning plays an important role together in scientific inquiry to find new knowledge (Bellucci, 2018; Moscoso & Palacios, 2019; Peirce, 1958; Shodikin, 2017). Abduction plays a role as the track from facts towards ideas and theories, while induction is the path from concepts and theories towards facts, and deductive plays a role to prove a theory.

Students can form inferences using abductive reasoning when given facts in the problem-solving activities. The inference is indeed unnecessary and logically correct, so it needs to be investigated for its truth by using deductive reasoning. According to some experts, abductive reasoning proposes a plausible conclusion, inductive reasoning proposes a probable conclusion, and deductive reasoning produces a certain conclusion (Hwang et al., 2019; Mirabile & Douven, 2020; Moscoso & Palacios, 2019; Peirce, 1958). From this explanation, it appears that what plays a role in the forming of knowledge, ideas, or new ideas is abductive reasoning. We need abductive reasoning if we wish to "learn anything" new (Niiniluoto, 2018), and abductive reasoning can produce a piece of new knowledge (Rapanta, 2018).

We will understand abductive reasoning through an example of deductive, inductive, and abductive reasoning. The following is an example of deductive, inductive, and abductive reasoning made by Peirce (1958). The examples contain some facts and inferences. The example of deductive reasoning is: *"Suppose there is a bag that only contains red marbles, and you take one (as facts). You can conclude that the marble is red."* From the example, *"there is a bag that only contains red marbles" is a general fact, and "we take one marble"* is a fact. The conclusion, that is, *"the marble has taken is red",* is a particular case. In the example, we conclude by modus ponens rule. Thus, the drawing conclusion is valid by deductive reasoning. We can use modus tollens, syllogism or another rule to draw conclusions provided such that the argument is valid. The following is an example of inductive reasoning: *"Suppose there is a bag on the table, but you don't know the color of the marbles in the bag. You take one by one marble, and it's red. You may infer that all the marbles in the bag are red."* From the example of inductive reasoning, the proposition, *"We take one marble from a bag"* is a fact, and *"It's red"* is a specific fact. If we take one by one marble from a bag, although we do not take all marbles in the bag, we can conclude that all marbles in the bag are red. Even though the drawing conclusion by inductive is not mathematically valid but sometimes inductive reasoning used in science.

The example of abductive reasoning is: *"Suppose there is a bag on the table. All marbles in the bag are red, and you find a red marble around a bag. You can conclude that the red marble is from the bag."* From the example above, *"All marbles in the bag are red", and "you find a red marble around a bag"* is some facts. The conclusion *"the red marble is from the bag"* could serve as part of an explanation for the fact. The conclusion *"the red marble is from the bag"* may be true or false. We must find explanations so that we get the conclusion which true. Finding explaining hypotheses is one of the most challenging creative tasks (Preyer & Mans, 1999).

Thus, we can draw a conclusion based on some facts. In problem-solving, some facts can be founded from the problems. If people who do a reason think that something is true, then something is called a fact (Ferrando, 2006). It means that a matter is considered a fact or not depends on each individual as a problem solver. Some students can find a fact easily, but others cannot. Therefore, it is fascinating to explore the students' abductive reasoning in making conclusions based on the use of the fact.

Fact is a piece of information that can be found in a problem. Something is called fact if people think that something is true. From some facts, we can draw a conclusion. By abductive reasoning, the conclusion is stated as a conjecture. A conjecture is a logical statement, but its truth has not been confirmed (Cañadas & Castro, 2007). A conjecture can play a role as a "hypothesis," which is an idea suggested as a possible explanation for a particular situation or condition and as a "fact-conjecture" that is the final answer to a problem or solution to specific steps of the settlement process. Facts and conjectures can be expressed in the statement form (Ferrando, 2006). In a problem-solving process, we can make a conjecture many times.

Conclusions by using deductive reasoning are valid because their arguments form a tautology. In contrast, abductive reasoning produces findings that are 'not always true' so that further clarification is needed to check the truth of the conclusions. Therefore, abductive reasoning can arise an opportunity to explore new ideas that are indispensable for the development of knowledge in overcoming problems. One exciting part of abductive reasoning is it can produce new knowledge for someone who has a reason. The new knowledge in mathematics can be in the form of a new concept or unique nature (Gonzalez & Haselager, 2005; Niiniluoto, 2018).

Several studies have been carried out to investigate abductive reasoning enacted by students in high school algebra contexts. These studies have informed that generalization processes figural patterns in algebra (Rivera, 2013; Rivera, 2010; Rivera & Becker, 2007). While research of abductive reasoning in high school contexts has been widely explored, little attention has been directed to uncovering abductive reasoning enacted by university students majoring in mathematics education. Thereby, to fill such a void, the present study was designed to explore some types of abductive reasoning employed by mathematics education students in higher education contexts to solve algebra problems related to using facts on the problems. During the enactment of this study, the students were taking algebra courses. Some of the topics were Group Theory, as in Gallian (2016). Thus, this study was set specifically on algebra for undergraduate students, which discerned the Group Theory.

METHOD

The participants in this study were 58 second-year students of a public university based in Malang, East Java, Indonesia. On 12 March 2020, they were asked to solve algebra problems. Their solutions were included in four groups based on abductive reasoning of using facts. Furthermore, to explore the characteristics of each group, we carried out task-based interviews with one student from each group. Thus, there were four students to be interviewed. They were chosen because (1) they fulfill the indicators of abductive reasoning of using facts, (2) they have good communication skills, and (3) they are willing to participate in the study.

To observe the process of students' abductive reasoning and record the interviews, an observation sheet was employed. It consists of one open-ended question that allows students to do reasoning in the problem-solving process. The instruments have been tested for validity and reliability before it was used. Content validation of questions and interview sheets was conducted by two mathematics experts and two education experts. The item of the validity of the instrument included the eligibility of the item test, the truth of concept, multiple interpretations, and appropriate instructions to do abductive reasoning. The algebra problems provided several facts, and the participants were asked to check whether a property is true or not based on the fact. The problem can be seen in Figure 1.

> Suppose $\mathbb{Z}_8$ is a set of the integers modulo 8 under multiplication.
> If $\forall a, b, c \in \mathbb{Z}_8$, $ac = bc$, is $a = b$? Please give some reasons for your answer

Figure 1. Algebra Problem to Observe Students' Abductive Reasoning

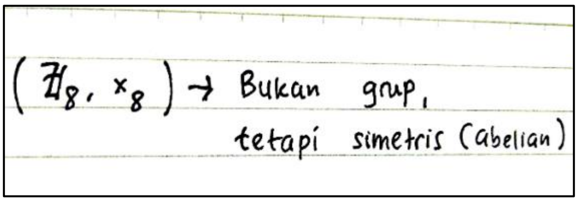
To analyze the data, first, we administered a problem to the students, and we sorted it based on participants' reasoning when doing the problem-solving process. Their solution can be grouped into four groups based on abductive reasoning related to using facts. From each group, one student was observed in-depth related to the reasoning aspect. The triangulation process was done to verify the data collected from the interview. The triangulation was also carried out to confirm the finding from the students' answers. We code students as S and researchers as R. Finally, we summarized abductive reasoning from four students in solving the problem.

RESULTS AND DISCUSSION

From four groups with different kinds of answers, the first group involves one student who can bring up new conjecture beyond the questions related to the problems. The second group that involves twenty students use facts that are "true" outside the problem to solve the problem. The third group that involves twelve students, use facts that are "false" outside the problem to solve the problem. The fourth group that involves twenty-five students assume the questionable thing as a given fact. To know deeply how students' reasoning is, the following are the results of the interview with the four students, namely Student 1 (S1), Student 2 (S2), Student 3 (S3), and Student 4 (S4).

Abductive Reasoning in Problem Solving Process of S1

S1 has written down all the facts in the problem. It means that S1 understands what is given and what has to be found in the problem. From the student sheet, it was found that S1 also wrote, as seen in Figure 2.



Translate Version

$\mathbb{Z}_8, x_8 \rightarrow$ not a group but symmetrical (abelian?)

Figure 2. Conjecture by S1

From the results of the interview with S1, she made the conjecture that built based on Table Cayley made by S1. Following interviews with S1.

R	:	Why do you write question mark "?" in a sentence "$\mathbb{Z}_8, x_8 \rightarrow$ not a group but symmetrical (abelian?)"
S1	:	Because I am not sure about my answer, ma'am
R	:	What part are you not sure?
S1	:	On the symmetric (she means abelian)

S1 makes a conjecture that "$\mathbb{Z}_8$ is not a group under multiplication" but she still doubts the commutative property under multiplication. S1 arranges a table of a multiplication modulo 8 to persuade her conjecture. So, by abductive reasoning, S1 makes a conjecture and explains it based Cayley Table (Thagard & Shelley, 1997). Although a commutative property in $\mathbb{Z}_8$ is not used in problem-solving, but a commutative property is appeared as part of the abductive reasoning process to find new ideas of S1.

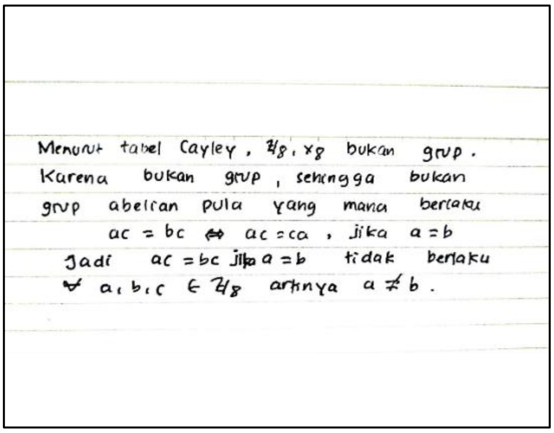
The action taken by S1 by making a multiplication table to convince the conjecture is appropriate that the activity of deciding conclusions obtained from feeling, seeing, and hearing, is useful when used to build a conjecture (Farah, 1988; Ferrando, 2006) and pictures (in this case the modulo 8 multiplication table) can be made in a creative way to explain the conjecture (Finke & Slayton, 1988). The making conclusion by abductive reasoning of S1 can be described as follows.

(Fact) The set $\mathbb{Z}_8$ with multiplication modulo 8

multiplication table modulo 8 of $\mathbb{Z}_8$

$\therefore$ $\mathbb{Z}_8$ is not a group but $\mathbb{Z}_8$ is symmetric (it's mean abelian)

The fact, " multiplication modulo 8 table of $\mathbb{Z}_8$ " is not clearly stated in the question, but S1 raised it to help to solve algebra problems. The conjecture stated in the statement "$\mathbb{Z}_8$ is not a group but symmetrical (abelian?)" is a temporary conclusion that decided from the facts. It means that at that time, S1 made a conclusion that was not based on deductive or inductive reasoning. The conclusion made by S1 is something new. We can see that abductive reasoning can build a new idea (Eco, 1984; Moscoso & Palacios, 2019; Niiniluoto, 2018; Peirce, 1958). The next step taken by S1 is writing the statement as shown in Figure 3.


Menurut tabel Cayley, $\mathbb{Z}_8, \times_8$ bukan grup.
Karena bukan grup, sehingga bukan
grup abelian pula yang mana berlaku
$ac = bc \Leftrightarrow ac = ca$, jika $a = b$
Jadi $ac = bc$ jika $a = b$ tidak berlaku
$\forall a, b, c \in \mathbb{Z}_8$ artinya $a \neq b$.

Translate Version

According to the Cayley Table, $\mathbb{Z}_8, \times_8$ is not a group, so it is not an abelian group, that applies $ac = bc \leftrightarrow ac = ca$ if $a = b$. So $ac = bc, a = b$ does not hold $\forall a, b, c \in \mathbb{Z}_8$ meaning $a \neq b$ ".

Figure 3. Abductive Statement and Its Explanation

S1 concluded that " $\forall a, b, c \in \mathbb{Z}_8, ac = bc$ but $a \neq b$ " based on the Cayley Table. So actually S1 understands that to show that the statement is false, she must make one counterexample such as $a = 4$ and $c = 3$, $ac = 4$. From the interviews, it was seen that to get $b \in \mathbb{Z}_8$, S1 used trial and error as seen in the following interview.

R	:	Why are a, b, c in $\mathbb{Z}_8$? try to check again, is the sentence "If $ac = bc$ for any $a, b, c \in \mathbb{Z}_8$, then $a \neq b$ " true?
S1	:	I was trying like this, can I, ma'am? I take $a = 4$ and $c = 3, ac = 4$, Because $ac = bc, bc = 4$. Because $c = 3$, then the possible is b= 4. So $a = b$

The drawing conclusion by abductive reasoning of S1 can be described as follows.

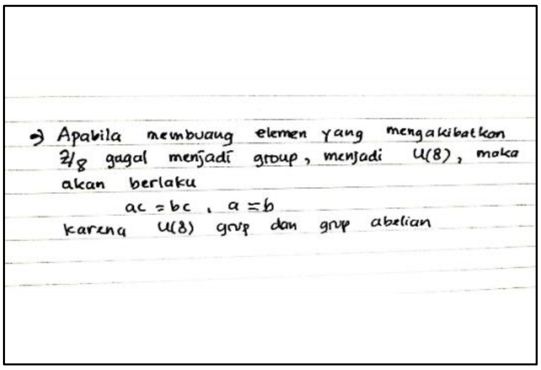
(Fact) $(\mathbb{Z}_8, x_8)$ not a group → $(\mathbb{Z}_8, x_8)$ is not an abelian group anyway

$ac = bc \leftrightarrow ac = ca\ jika\ a = b.$

$\therefore\ ac = bc \rightarrow a = b$ is not held.

The drawing conclusion made by S1 is not deductive nor inductive. However, the conclusion decided is true that " $ac = bc \rightarrow a = b$ is not held". It should be $\exists a, b, c \in \mathbb{Z}_8$ with $ac = bc$ but $a \neq b$.

From this step, the problem should have been answered even though the answer to S1 was not complete because there was still a slight error, but in fact, S1 continued the answer also though the problem did not ask it. S1 had a desire to find another set with the multiplication of modulo 8 so that it applies "$ac = bc \rightarrow \ a \ = b$". The statement that appears is "if you discard some elements of $\mathbb{Z}_8$ that causes $\mathbb{Z}_8$ is failed to become a group then a new set, that is $U(8)$ will become group under multiplication modulo 8". A set $U(8)$ is a set of all positive integers less than 8 and relatively prime to 8. So, proposition $\forall a, b, c \in U(8), ac = bc \rightarrow a = b$ is true, as shown in Figure 4.


→ Apabila membuang elemen yang mengakibatkan $\mathbb{Z}_8$ gagal menjadi group, menjadi U(8), maka akan berlaku
ac = bc, a = b
karena U(8) grup dan grup abelian

Translate Version

If we discard element that causes $\mathbb{Z}_8$ fails to be group, be U(8), then will be applied ac=bc, a=b because U(8) is a group and Abelian group.

Figure 4. New Ideas by S1

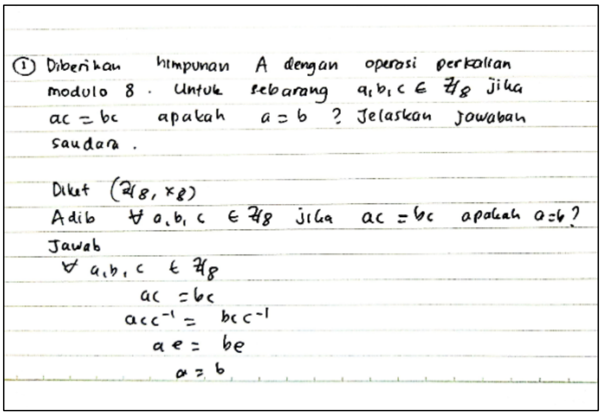
S1 finds new ideas, conception, and knowledge. This is consistent with Niiniluoto (2018) that abductive reasoning is done when we are learning new things (Niiniluoto, 2018). It was also stated that abductive reasoning is bringing up new ideas that are tentative and related to the context being discussed (Duarte, 2019; O'Reilly, 2016; Tomiyama et al., 2010).

Abductive Reasoning in Problem Solving Process of S2

S2 can solve problems very precisely. They are using all the facts inside the problems and knowing that asking about it. The student uses facts that "true" outside the problems to solve the problems, that is, a set $\mathbb{Z}_8$ is not group under multiplication modulo 8. Students in this group made appropriate conjectures. The conjecture is "$\forall a, b, c \in \mathbb{Z}_8$ if $ac = bc$, then $a = b$ is false." The action taken by students of this type is finding out "$\forall a, b, c \in \mathbb{Z}_8$ who fulfills $ac = bc$, but $a \neq b$. This action is very appropriate for solving problems.

Abductive Reasoning in Problem Solving Process of S3

S3 has written down all the facts contained in the problem. It means S3 understands what given and what has to be found in the problem. This is seen in the answer sheet in Figure 5.


① Diberikan himpunan A dengan operasi perkalian modulo 8 . Untuk sebarang $a, b, c \in \mathbb{Z}_8$ jika $ac = bc$ apakah $a = b$? Jelaskan jawaban saudara .

Diket $(\mathbb{Z}_8, \times_8)$

Adib $\forall a, b, c \in \mathbb{Z}_8$ jika $ac = bc$ apakah $a = b$?

Jawab

$\forall a, b, c \in \mathbb{Z}_8$

$ac = bc$

$acc^{-1} = bcc^{-1}$

$ae = be$

$a = b$

Translate Version

Suppose that a set $\mathbb{Z}_8$ under multiplication modulo 8. For arbitrary $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{Z}_8$ if $\boldsymbol{ac} = \boldsymbol{b}$c, is $\boldsymbol{a} = \boldsymbol{b}$? Explain your answer!

Information: ($\mathbb{Z}_8$, x)

Question: $\forall a, b, c \in \mathbb{Z}_8$ If $ac = bc$, is $a = b$?

Answer:

$\forall a, b, c \in \mathbb{Z}_8$

- $ac = bc$
- $\mathrm{acc}^{-1} = \mathrm{acc}^{-1}$
- $\mathrm{ae} = \mathrm{be} \rightarrow a = b$

Figure 5. Answer of S3

After S3 understood the problem, S3 made the conjecture that is "if $ac = bc$ then $a = b$ is true" by doing the reasoning as in the interview below.

R : Why did you write the words "inverse" on the answer sheet? Do you feel a statement "if $ac = bc$, then $a = b$" is true?

S3 : Because the two right and left sides have the same element of $\mathbb{Z}_8$, namely c, Ma'am.

The making conclusion by abductive reasoning of S3 can be described as follows.

(Fact) $\mathbb{Z}_8$ with modulo multiplication 8

$\forall a, b, c \in \mathbb{Z}_8, ac = bc$

Inverse multiplication on $\mathbb{Z}_8$

$\therefore ac = bc \rightarrow a = b$

S3 performed abductive reasoning by use inverse element concept to the fact, even though there are elements $\mathbb{Z}_8$ that does not have an inverse. Thus, S3 added the wrong concept to the facts. From the set of facts chosen by S3, the conjecture is written in the statement form "$ac = bc \rightarrow a = b$." The making conclusion by abductive reasoning of S3 can be described as follows.

(Fact) $\forall a, b, c \in \mathbb{Z}_8, ac = bc$

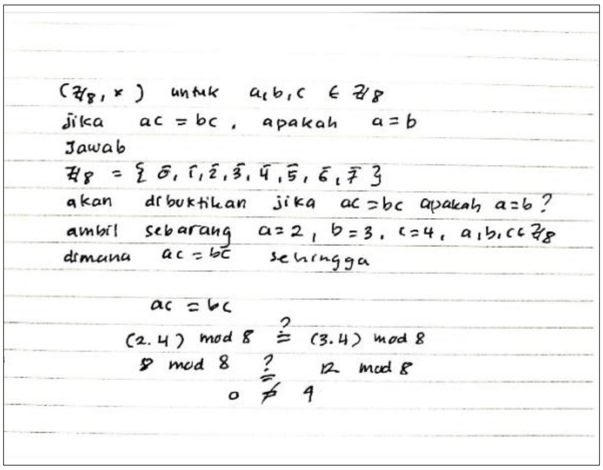
$acc^{-1} = bcc^{-1}$

$\therefore a = b$

The conclusion decided by S3 appears to be valid because it used deductive reasoning, but due to the addition of the fact that there is an inverse of c in $\mathbb{Z}_8$, the conclusion $a = b$ is false.

Abductive Reasoning in Problem Solving Process of S4

S4 has written all the facts contained in the problem by writing "$\mathbb{Z}_8 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}\}$, will be proven if $ac = bc$ whether $a = b$". This is seen in the answer sheet in Figure 6.


$(\mathbb{Z}_8, \times)$ untuk $a, b, c \in \mathbb{Z}_8$
jika $ac = bc$, apakah $a = b$
Jawab
$\mathbb{Z}_8 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}\}$
akan dibuktikan jika $ac = bc$ apakah $a = b$?
ambil sebarang $a = 2$, $b = 3$, $c = 4$, $a, b, c \in \mathbb{Z}_8$
dimana $ac = bc$ sehingga
$ac = bc$
$(2.4) \bmod 8 \overset{?}{=} (3.4) \bmod 8$
$8 \bmod 8 \overset{?}{=} 12 \bmod 8$
$0 \neq 4$

Translate Version

$(\mathbb{Z}_8, \times)$, for $a, b, c \in \mathbb{Z}_8$

If $ac = bc$, is $a = b$?

Answer:

$\mathbb{Z}_8 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, 5, \bar{6}, \bar{7}\}$ will be proved if

$\mathrm{ac} = \mathrm{bc}$, is $\mathrm{a} = \mathrm{b}$?

If $\mathrm{a} = 2, \mathrm{b} = 3, \mathrm{c} = 4, \mathrm{a}, \mathrm{b}, \mathrm{c} \in \mathbb{Z}_8$ that

$\mathrm{ac} = \mathrm{bc}$, so that

$ac = bc$

$(2.4)\ mod\ 8\ \ ?\ \ (3.4)\ mod\ 8$

$8\ mod\ 8\ \ ?\ \ 12\ mod\ 8 \leftrightarrow 0 \neq 4$

Figure 6. Answer of S4

After S4 understood the problem by writing down what is known and asked, then S4 plan to solve the problem by making a conjecture that is if $ac = bc$, then $a = b$ is true as seen from the following interview.

R	:	Why did you immediately take $\mathrm{a} = 2, \mathrm{b} = 3, \mathrm{c} = 4, \mathrm{a}, \mathrm{b}, \mathrm{c} \in \mathbb{Z}_8$?
S4	:	So, first, I look for an element of $\mathbb{Z}_8$ then to prove the statement, I take any a, b, c, elements of $\mathbb{Z}_8$, because in the problem it is known "if $ac = bc$ *then* $a = b$," then I want to prove whether if a, b, c are different, then $ac \neq \mathrm{bc}$ In my answer, I give a "question mark" because I still want to prove whether ac = bc and it was found that $ac \neq bc$.
R	:	Which statement do you want to prove?
S4	:	The statement "if ac = bc then a = ba"
R	:	Oh yes ... So, is that statement true or not?
S4	:	True, Ma'am

So, S4 made the conjecture and statement that "if ac = bc then a = b" is true for the following reasons (as in the interview).

R : Does that mean you guess that statement is correct? Why?
S4 : Yes Ma'am
Because of the property of multiplication Ma'am
R : Could you explain it?
S4 : If $ac = bc$, then $a = b$. So if I multiply two same numbers to each side, then the result will be the same.
Take any a, b, c member of $\mathbb{Z}_8$ where $a = b$ is consequently $ac = bc$

$$a = b = 3, c = 2$$
$$ac \ ? \ bc$$
$$(3x2) \bmod 8 \ ? \ (3 \times 2) \bmod 8$$
$$6 \bmod 8 \ ? \ 6 \bmod 8$$
$$6 = 6$$

So, it is proven that if $ac = bc$, then $a = b$

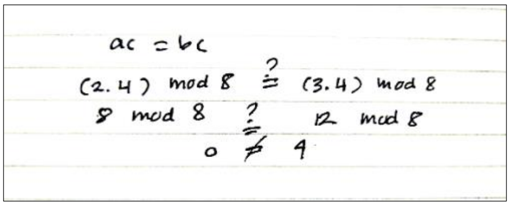
From the answer of S 4 and interview with S4, the following conclusions can be analyzed

(Fact) $\mathbb{Z}_8 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}\}$

Properties of multiplication is that if $a = b$ then $ac = bc$

$\therefore$ if ac = bc then a = b

The using reasoning in here is not deductive or inductive but abductive, which is following Peirce's abductive theory (Niiniluoto, 2018; Peirce, 1958). After making the conjecture, S4 finds a, b, c so $ac = bc$ results in $a = b$ seen in Figure 7.



Translate Version

$$ac = bc$$
$$(2.4) \bmod 8 \ ? \ (3.4) \bmod 8$$
$$8 \bmod 8 \ ? \ 12 \bmod 8$$
$$0 \neq 4$$

Figure 7. Answer the Problem of S4

From the answer of S4, it shows that the conjecture carried out when understanding the problem makes the problem-solving plan wrong. When teachers already know their mistakes, it will be easy to correct students' mistakes.

The analysis of data in this study documented four types of abductive reasoning based on using the facts of the problem, such as creative conjecture type, the fact optimization type, the factual error type, and the mistaken fact type. Explanation about the characteristics of each type is displayed in Table 1.

Table 1. Characteristics of Students in Abductive Reasoning

Types of Abductive Reasoning based on Using The Fact	Description of Characteristic
First Type: Creative Conjecture	• using all the facts inside the problem to solve the problem • knowing what the questions are asking about, • using facts that "true" outside the problem to solve the problem • Making a conjecture from facts by writing or describing or sketching to design problem-solving • writing a new conjecture outside the question on the problem related to the question
Second Type : Fact Optimization	• using all the facts inside the problem to solve the problem • knowing what the questions are asking about, • using facts that "true" outside the problem to solve the problem
Third Type: Factual Error	• using all the facts inside the problem to solve the problem • knowing what the questions are asking about, • using all the fact to solve the problem • using facts that "false" outside the problem to solve the problem
Fourth Type : Mistaken Fact	• not using the facts inside the problem to solve the problem • not knowing what the questions are asking about, • assuming the questionable thing as a given fact

The uniqueness of the student with "creative conjectures," is writing a new conjecture. The new conjecture is outside the question on the problem related to the question. The fact optimization students solved problems by using all the facts inside the problem, know what the questions are asking about, and using facts that "true" outside the problem to solve the problem knowledge that is not written in the problem (i.e $\mathbb{Z}_8$ is not a group). Because other knowledge taken is the right knowledge and all the facts are known, then the conclusion of the facts is right, so that the planning of the solution is also right.

In the group with "factual error", they used all the facts. When they made a conjecture, this group added other knowledge, that is not contained in the problem, that is the inverse of element concept. The concept that taken is false because not all elements in $\mathbb{Z}_8$ have a multiplicative inverse. Adding error facts to the problem will be the product of a false conjecture so that the planning of the problem is wrong.

In the group with "Mistaken Fact," The students concluded $\boldsymbol{a} = \boldsymbol{b}$ by using the questionable thing as a given fact. The answer is indeed wrong, but it is interesting to analyze because many students do it.

CONCLUSION

This study captures the types of abductive reasoning enacted by students in solving algebra problems related to using facts on the problems. The results showcased some types such as creative conjectures type, fact optimization type, factual error type, and mistaken fact type. In the creative conjecture type, the students can solve the problem, but they were unsatisfied with the solution. The students develop new ideas related to the questions carried out using abductive reasoning. In fact optimization type, students make conjectures about the answers to problems, then confirm the conjectures by making deductive reasoning. In the factual error types, students add facts outside of the problem to help them solve the problem, but the facts taken are incorrect, leading to the wrong conclusion. In the mistaken fact type, students assume that the question is a true value so that it is appointed as a fact. As a result, the actions taken to make conclusions are also wrong.

The creative conjectures type and the fact optimization type are very useful in the problem-solving process. Therefore, teachers should encourage students to use creative conjectures and fact optimization when learning mathematics. Abductive reasoning can encourage students to find new knowledge. Meanwhile, abductive reasoning can also lead students to make mistakes in the conclusion. By knowing the types of abductive reasoning of the students in solving algebra problems, attentive actions can be taken to correct the errors made and can also provide more activities for student learning so that new knowledge continues to emerge.

This study of abductive reasoning was carried out on mathematics education students, who, at the time of this study enactment, learn both mathematics and pedagogy materials for teaching and learning. The results may be slightly different if it is carried out on students who enroll in the pure mathematics program. Therefore, future investigation is encouraged to examine the abductive reasoning of students in the non-educational programs.

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