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WEIGHTED STATISTICAL CONVERGENCE OF REAL VALUED SEQUENCES

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Abstract. Functions defined in the form "$g : \mathbb{N} \to [0, \infty)$ such that $\lim_{n\to\infty} g(n) = \infty$ and $\lim_{n\to\infty} \frac{n}{g(n)} = 0$" are called weight functions. Using the weight function, the concept of weighted density, which is a generalization of natural density, was defined by Balcerzak, Das, Filipczak and Swaczyna in the paper "Generalized kinsd of density and the associated ideals", Acta Mathematica Hungarica 147(1) (2015), 97-115.

In this study, the definitions of g-statistical convergence and g-statistical Cauchy sequence for any weight function g are given and it is proved that these two concepts are equivalent. Also, some inclusions of the sets of all weight g_1-statistical convergent and weight g_2-statistical convergent sequences for g_1, g_2 which have the initial conditions are given.
Keywords: weight functions; natural density; statistical convergent sequences.

1. Introduction

In [5], Fast introduced the concept of statistical convergence. In [15], Schoenberg gave some basic properties of statistically convergence and also studied the concept as a summability method. After this works many Mathematician have used these concept in their studies [8, 9, 10, 11]. In [2, 3], the authors proposed a modified version of density by replacing n by n^α where $0 < \alpha \leq 1$. In [1], the authors defined a more general kind of density by replacing n^α by a function $g : \mathbb{N} \to [0, \infty)$ with $\lim_{n\to\infty} g(n) = \infty$. In this paper, we will study the weighted g-statistically convergence concept.

Let K be a subset of natural numbers. Natural density of K is defined by

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} |K(n)|$$

where $K(n) = \{k \leq n : k \in K\}$ and the vertical bars denotes the number of elements of $K(n)$.

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Let $g : \mathbb{N} \to [0, \infty)$ be a function with $\lim_{n\to\infty} g(n) = \infty$. Let us remember that the definition of density of weight $g(n)$.

Definition 1.1. The density of weight g defined by the formula

$$d_g(A) = \lim_{n\to\infty} \frac{|A(n)|}{g(n)}$$

for $A \subset \mathbb{N}$ [1, 4].

After the study [1], the concept of g-density was applied to various problems related to sequences and interesting results were obtained in [4, 7, 12, 13, 14].

Basically in this study, it will be shown that the results given in [6] can be re-examined by using g-density.

In this paper, we are concerned with the subsets of natural numbers having weight $g(n)$ density zero. To facilitate this, we have introduced the following notation: If x is a sequence such that x_k satisfies property P for all k except a set of weight $g(n)$ density zero, then we say that x_k satisfies P for (weight g almost all k) and it is denoted by $(g - a.a.k)$ for simplicity.

Definition 1.2. Let $x = (x_k)$ be a real valued sequence. x is weight g-statistical convergent to the number L if for each $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{g(n)} = 0,$$

i.e., $|x_k - L| < \varepsilon \quad (g - a.a.k)$. In this case we write $g - st - \lim x_k = L$.

C_g^{st} denotes the set of all weight g-statistical convergent sequences.

If we take the function $g(n) = n$ we obtain the usual statistical convergence.

It is clear that every convergent sequence is also weight g-statistical convergent. But the converse is not true in general.

Example 1.1. Let us define the function $g(n) = 2n$ and the sequence as

$$x_k = \begin{cases} 3, & k = m^2, \quad m \in \mathbb{N}, \\ 0, & k \neq 0. \end{cases}$$

Then $|k \leq n : x_k \neq 0| \leq \sqrt{n}$. So, $g - st - lim x_k = 0$.

Theorem 1.1. *If the sequence (x_n) is weight-g-statistical convergent to L then there is a set $K = \{k_1 < k_2 < ...\}$ such that $d_g(K) = d_g(\mathbb{N})$ and $\lim_{n\to\infty} x_{k_n} = L$.*

Proof. Let us assume that $g - st - lim x_k = L$. Take $K_i := \{n \in \mathbb{N} : |x_n - L| < \frac{1}{i}\}$, $(i = 1, 2, ...)$. Then by definition we have $d_g(K_i^c) = 0$ and it is clear that $d_g(K_i) = d_g(\mathbb{N})$, $(i = 1, 2, ...)$. Also it is easy to control that

$$... \subset K_{i+1} \subset K_i \subset ... \subset K_2 \subset K_1 \tag{1.1}$$

Let $\{T_j\}_{j\in\mathbb{N}}$ be a strictly increasing sequence of positive real numbers. Let choose an arbitrary number $a_1 \in K_1$. By (1.1) we can choose an element $a_2 \in K_2$, $a_2 > a_1$ such that for each $n \geq a_2$ we have $\frac{K_2(n)}{g(n)} > T_2$. Moreover choose $a_3 > a_2$, $a_3 \in K_3$ such that for each $n \geq a_3$ we have $\frac{K_3(n)}{g(n)} > T_3$. If we proceed in this way we obtain a sequence $a_1 < a_2 ... < a_i < ...$ of positive integers such that

$$a_i \in K_i,\ (i = 1, 2, ...) \text{ and } \frac{K_i(n)}{g(n)} > T_i \tag{1.2}$$

for each $n \geq a_i$, $i = 1, 2, ...$

Let us establish the set K as follows: each natural number of the interval $[1, a_1]$ belong to K, moreover, any natural number of the interval $[a_i, a_{i+1}]$ belongs to K if and only if it belongs to K_i $(i = 1, 2, ...)$. From (1.1) and (1.2) we have

$$\frac{K(n)}{g(n)} \geq \frac{K_i(n)}{g(n)} > T_i$$

for each n, $a_i \leq n < a_{i+1}$. By last inequality it is clear that $\overline{d}_g(K) = \infty$.

Let $\varepsilon > 0$, and choose i such that $\frac{1}{i} < \varepsilon$. Let $n \geq a_i$, $n \in K$. There exists a number $t \geq i$ such that $a_t \leq n < a_{t+1}$. But from the definition of K, $n \in K_t$. Thus $|x_n - L| < \frac{1}{t} \leq \frac{1}{i} < \varepsilon$. Hence, $\lim_{n\to\infty} x_{k_n} = L$. □

Remark 1.1. The converse of Theorem 1.1 is not true.

Example 1.2. Let us consider the sequence

$$(x_k) := \begin{cases} 1, & k = n^2, \\ 0, & k \neq n^2, \end{cases}$$

and $g(n) = n^{1/4}$. It is clear that the set $K = \{k : k = n^2, n \in \mathbb{N}\} \subset \mathbb{N}$ has the property $\overline{d}_g(K) = \infty$. But $g - st - \lim x_k \neq 1$.

Let us note that every statistical convergent sequence is also weight-g-statistical convergent to the same number. But the converse of this situation is not true.

Example 1.3. Let $a_k = 2^{2^k}$, and

$$g(n) := \begin{cases} a_{2k}, & n \in [a_k, a_{k+1}),\ k = 1, 2, ... \\ 1, & n < 4. \end{cases}$$

Let $A_k := \{n \in \mathbb{N} : a_k \leq n < 2a_k\}$ and $A := \cup_{k\geq 1} A_k$. Let us take account the sequence

$$x_n := \begin{cases} 1, & n \in A, \\ 0, & n \notin A. \end{cases}$$

It is clear that $\frac{1}{2}a_k \leq |A_k| \leq a_k$. Let us check that $x_n \nrightarrow 0(st)$. If we put $m_k = \max A_k$, we obtain

$$\frac{|\{k \leq n : |x_k - 0| \geq \varepsilon\}|}{n} = \frac{|\{k \leq n : x_k \in A\}|}{n} = \frac{|A|}{m_k} \geq \frac{|A_k|}{m_k} \geq \frac{\frac{1}{2}a_k}{2a_k} = \frac{1}{4}$$

for all $k \geq 1$.

Moreover, $g - st - \lim x_k = 0$. For sufficiently large n, we have

$$\begin{aligned} \frac{|\{k \leq n : |x_k - 0| \geq \varepsilon\}|}{g(n)} &= \frac{|\{k \leq n : x_k \in A\}|}{g(n)} = \frac{|A|}{g(n)} \\ &= \frac{|\{k \leq m_k : x_k \in A\}|}{g(m_k)} \\ &\leq \frac{|A_k|}{a_{2k}} \leq \frac{a_k}{a_{2k}} \to 0. \end{aligned}$$

Definition 1.3. Let $x = (x_k)$ be a real valued sequence. x is weight g-statistical Cauchy sequence if for each $\varepsilon > 0$ there exists a natural number $N = N(\varepsilon)$ such that

$$\lim_{n\to\infty} \frac{|\{k \leq n : |x_k - x_N| \geq \varepsilon\}|}{g(n)} = 0,$$

i.e., $|x_k - x_N| < \varepsilon \quad (g - a.a.k)$. In this case we write x is weight g-Cauchy sequence.

Lemma 1.1. *The following statements are equivalent:*

(i) x is a weight g-statistically convergent sequence,

(ii) x is a weight g-statistically Cauchy sequence,

(iii) x is a sequence for which there is a convergent sequence y such that $x_k = y_k$ $(g - a.a.k)$.

Proof. $(i) \Rightarrow (ii)$ Let us assume that x is a weight g-statistical convergent sequence. Suppose $\varepsilon > 0$ and $g - st - \lim x = L$. Then $|x_k - L| < \frac{\varepsilon}{2}$ $(g - a.a.k)$ holds.

If we choose a natural number N such that $|x_N - L| < \frac{\varepsilon}{2}$, then we have

$$|x_k - x_N| < |x_k - L| + |x_N - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \ (g - a.a.k).$$

Hence, x is a weight g-statistical Cauchy sequence.

$(ii) \Rightarrow (iii)$ Let us assume that x is a weight g-statistical Cauchy sequence. Choose $N(1)$ such that the interval $I = [x_{N(1)} - 1, x_{N(1)} + 1]$ contains x_k $(g - a.a.k)$. Also apply (ii) to choose M such that $I' = [x_M - \frac{1}{2}, x_M + \frac{1}{2}]$ contains x_k $(g - a.a.k)$. We claim that

$$I_1 = I \cap I' \text{ contains } x_k \ (g - a.a.k),$$

for

$$\{k \leq n : x_k \notin I \cap I'\} = \{k \leq n : x_k \notin I\} \cup \{k \leq n : x_k \notin I'\}.$$

Thus,

$$\begin{aligned} & \lim_{n\to\infty} \frac{1}{g(n)} |\{k \leq n : x_k \notin I \cap I'\}| \leq \\ \leq \quad & \lim_{n\to\infty} \frac{1}{g(n)} |\{k \leq n : x_k \notin I\}| + \lim_{n\to\infty} \frac{1}{g(n)} |\{k \leq n : x_k \notin I'\}| = 0. \end{aligned}$$

So, I_1 is closed interval of length less than or equal to 1 and contains x_k $(g - a.a.k)$. Now we continue by choosing $N(2)$ such that $I'' = [x_{N(2)} - \frac{1}{4}, x_{N(2)} + \frac{1}{4}]$ contains x_k $(g - a.a.k)$, by the previously argument $I_2 = I_1 \cap I''$ contains x_k $(g - a.a.k)$, and I_2 has length less than or equal to $\frac{1}{2}$. Proceeding inductively we construct a sequence$\{I_m\}_{m=1}^{\infty}$ of closed intervals such that for each m, $I_{m+1} \subseteq I_m$, and the length of I_m is not greater than 2^{1-m}, and $x_k \in I_m$ $(g - a.a.k)$. From the Nested Interval Theorem there is a number α such that $\alpha = \cap_{m=1}^{\infty} I_m$. If we use $x_k \in I_m$ $(g - a.a.k)$, we can choose an increasing positive sequence $\{T_m\}_{m=1}^{\infty}$ such that

$$\frac{1}{g(n)} |\{k \leq n : x_k \notin I_m\}| < \frac{1}{g(m)} \; if \; n > \; T_m. \tag{1.3}$$

Next define a subsequence z of x consisting of all terms x_k such that $k > T_1$ and if $T_m < k \leq T_{m+1}$ then $x_k \notin I_m$.

Now define the sequence y by

$$y_k = \begin{cases} \alpha, & \text{if } x_k \text{ is a term of } z, \\ x_k, & \text{otherwise.} \end{cases}$$

Then $\lim y_k = \alpha$; for , if $\varepsilon > \frac{1}{g(m)} > 0$ and $k > T_m$ then either x_k is a term of z, which means $y_k = \alpha$ or $y_k = x_k \in I_m$ and $|y_k - \alpha| \leq$ length of $I_m < 2^{1-m}$. We also assert that $x_k = y_k$ $(g - a.a.k)$. To confirm this we observe that if $T_m < n < T_{m+1}$ then

$$\{k \leq n : y_k \neq x_k\} \subseteq \{k \leq n : x_k \notin I_m\}$$

so from (1.3)

$$\frac{1}{g(n)} |\{k \leq n : y_k \neq x_k\}| \leq \frac{1}{g(n)} |\{k \leq n : x_k \notin I_m\}| < \frac{1}{g(m)}$$

is obtained. Thus, the limit as $n \to \infty$ is 0 and $x_k = y_k$ $(g - a.a.k)$.

$(iii) \Rightarrow (i)$ Let us assume that $x_k = y_k$ $(g - a.a.k)$ and $\lim y_k = L$. Suppose $\varepsilon > 0$. Then for each n,

$$\{k \leq n : |x_k - L| > \varepsilon\} \subseteq \{k \leq n : x_k \neq y_k\} \cup \{k \leq n : |y_k - L| > \varepsilon\}$$

from the assumption $\lim y_k = L$, the second set contains a fixed number of integers, say $l = l(\varepsilon)$. So,

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{g(n)}|\{k \le n : |x_k - L| > \varepsilon\}| &\le \lim_{n\to\infty} \frac{1}{g(n)}|\{k \le n : x_k \ne y_k\}| + \\ &+ \lim_{n\to\infty} \frac{l}{g(n)} = 0 \end{aligned}$$

because $x_k = y_k$ $(g - a.a.k)$. Hence, $|x_k - L| \le \varepsilon$ $(g - a.a.k)$. So, the proof is complete. □

Corollary 1.1. *Let x be a real valued sequence. If $g - st - \lim x_k = L$, then x has a subsequence y such that $\lim y_k = L$.*

2. Inclusion Between Two $g - st-$Convergence

Let G denotes the set of all functions $g : \mathbb{N} \to [0, \infty)$ satisfying the condition $g(n) \to \infty$ and $\frac{n}{g(n)} \nrightarrow 0$. In this section, we will introduce some inclusions between various $g \in G$.

Lemma 2.1. *Let $g_1, g_2 \in G$ such that there exist $M,\ m > 0$ and $k_0 \in \mathbb{N}$ such that $m \le \frac{g_1(n)}{g_2(n)} \le M$ for all $n \ge k_0$. Then $C^{st}_{g_1}(x) = C^{st}_{g_2}(x)$.*

Proof. Suppose the sequence x is weight g_1-statistical convergence to L. This implies that for each $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{|\{k \le n : |x_k - L| \ge \varepsilon\}|}{g_1(n)} = 0.$$

Together with the fact that $\frac{g_1(n)}{g_2(n)} \le M$, this implies that

$$\frac{|\{k \le n : |x_k - L| \ge \varepsilon\}|}{M g_2(n)} \le \frac{|\{k \le n : |x_k - L| \ge \varepsilon\}|}{g_1(n)}.$$

for all $n \ge k_0$. This implies

$$\lim_{n\to\infty} \frac{|\{k \le n : |x_k - L| \ge \varepsilon\}|}{M g_2(n)} \le \lim_{n\to\infty} \frac{|\{k \le n : |x_k - L| \ge \varepsilon\}|}{g_1(n)} = 0.$$

From the hypothesis we obtain

$$\lim_{n\to\infty} \frac{|\{k \le n : |x_k - L| \ge \varepsilon\}|}{g_2(n)} = 0.$$

Thus, the sequence x is weight g_2-statistical convergent to L. So, $C^{st}_{g_1}(x) \subset C^{st}_{g_2}(x)$. We can prove the iclusion $C^{st}_{g_2}(x) \subset C^{st}_{g_1}(x)$ by similar way. □

Lemma 2.2. *For each function $f \in G$ there exists a nondecreasing function $g \in G$ such that $C_f^{st}(x) = C_g^{st}(x)$. Moreover,*

$$g(n) \leq f(n) \tag{2.1}$$

for all $n \in \mathbb{N}$.

Proof. If f is nondecreasing, it is nclear. Otherwise, define the related function $g : \mathbb{N} \to [0, \infty)$ as follows. Let $a_1 = \min\{f(n) : n \in \mathbb{N}\}$, $i_1 = \max\{i \in \mathbb{N} : f(i) = a_1\}$ and $g(i) = a_1$ for $0 \leq i \leq i_1$. Next, let $a_2 = \min\{f(n) : n > i_1\}$, $i_2 = \max\{i \in N : f(i) = a_2\}$ and $g(i) = a_2$ for $i_1 < i \leq i_2$. Rest of the function g is established by induction.

Obviously, the function g is nondecreasing and $g(n) \to \infty$. By the construction, $g(n) \leq f(n)$, for all $n \in \mathbb{N}$. Hence $\frac{n}{f(n)} \leq \frac{n}{g(n)}$ for all n which implies that $\frac{n}{g(n)} \nrightarrow 0$. Thus $g \in G$.

Let (x_n) be a weight g-statistical convergent sequence to L. So, for each $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{g(n)} = 0$$

holds. From (2.1) we have following inequality

$$\frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{f(n)} \leq \frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{g(n)}.$$

If we take limit when $n \to \infty$ we obtain $f - st - \lim x_k = L$. Thus, the inclusion $C_g^{st} \subset C_f^{st}$.

By construction, for each $n \in \mathbb{N}$ there exist $m \geq n$ such that $g(n) = g(m) = f(m)$. Suppose that $x_n \nrightarrow L\ (g - st)$. Then there exists a, where $a \in \mathbb{R}^+ \cup \{+\infty\}$ and an inreasing sequence (n_i) of indices such that

$$\lim_{i\to\infty} \frac{|\{k \leq n_i : |x_k - L| \geq \varepsilon\}|}{g(n_i)} = a > 0.$$

For each $i \in \mathbb{N}$ we can find $m_i \geq n_i$ such that $g(n_i) = g(m_i) = f(m_i)$. Hence

$$\frac{|\{k \leq n_i : |x_k - L| \geq \varepsilon\}|}{g(n_i)} \leq \frac{|\{k \leq m_i : |x_k - L| \geq \varepsilon\}|}{f(m_i)}$$

holds. So, $x_n \nrightarrow L\ (f - st)$. $\square$

Lemma 2.3. *Let $f \in G$ be such that $\frac{n}{f(n)} \to \infty$, L, ε real numbers with $\varepsilon > 0$. Then there exists a sequence (x_n) such that $\left(\frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{f(n)}\right)$ is bounded but not convergent to zero.*

Proof. Firstly, let us assume that f is nondecreasing. Take to the smallest non negative integer, k_0, such that for $n \geq k_0$, $f(n) > 2$. Let us define a set $A \subset \mathbb{N} \backslash \{0, 1, 2, ...k_0 - 1\}$ inductively, deciding whether $n \geq k_0$ should belong to A or not. Let $n \notin A$ for all $n < k_0$. Suppose that $n \geq k_0$ and then we have defined $A(n)$. If $\frac{|A(n)|}{f(n+1)} < 1$ then let $n \in A$. Otherwise, let $n \notin A$. So, we construct the set A. From this construction and the condition $f(n) \to \infty$, A is infinite.

We assert that $\mathbb{N} \backslash A$ is also infinite. Let us assume that it is finite and choose $n_0 \in \mathbb{N}$ such that $n \in A$ for all $n \geq n_0$. Then, we have

$$\frac{n - n_0}{f(n+1)} \leq \frac{|A(n)|}{f(n+1)} < 1$$

for all $n \geq n_0$. But this is impossible because of the assumption, $\frac{n-n_0}{f(n+1)} \to \infty$. Now, we will show that $\frac{|A(n)|}{f(n)} < 2$ for each $n \geq k_0$. It is clear that if $n = k_0$ it is true. Suppose that $\frac{|A(n)|}{f(n)} < 2$ for a fixed $n \geq k_0$.

If $\frac{|A(n)|}{f(n+1)} < 1$, we have

$$\begin{aligned} \frac{|A(n+1)|}{f(n+1)} &= \frac{|A(n)|}{f(n+1)} + \frac{1}{f(n+1)} \\ &\leq \frac{|A(n)|}{f(n+1)} + \frac{1}{f(n)} \\ &\leq 1 + \frac{1}{2} < 2. \end{aligned}$$

If $\frac{|A(n)|}{f(n+1)} > 1$, then $n \notin A$ and so,

$$\frac{|A(n+1)|}{f(n+1)} = \frac{|A(n)|}{f(n+1)} \leq \frac{|A(n)|}{f(n)} < 2.$$

Now, let us define a sequence (x_n) as follows:

$$x_n := \begin{cases} n & n \in A \\ L & n \notin A \end{cases}$$

where $L \in \mathbb{R}$ is a fixed number. It is clear that the sequence $\left(\frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{f(n)}\right)$ is bounded from the first part of this proof.

Now, we will show that the sequence $\left(\frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{f(n)}\right)$ is not convergent to 0. For this aim consider any $n \geq k_0$. We will find $m \geq n$ such that $\frac{|A(m)|}{f(m)} \geq 1$. If $\frac{|A(n)|}{f(n)} \geq 1$, put $m := n$. Otherwise, choose the smallest $m \geq n$ such that $m \in \mathbb{N} \backslash A$. Then $\frac{|A(m)|}{f(m+1)} \geq 1$ and so, $\frac{|A(m)|}{f(m)} \geq 1$. Thus, the sequence $\left(\frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{f(n)}\right)$ is not convergent to 0.

Now, let us back to the general case where $f \in G$ need not be nondecreasing. Then we assume the associated function $g \in G$ from Lemma 2.2. Note that $\frac{n}{g(n)} \to \infty$ since $\frac{n}{g(n)} \geq \frac{n}{f(n)}$ for all n and $\frac{n}{f(n)} \to \infty$. By the above reasons we obtain the respective set A for g. Thus, $\frac{|A(n)|}{g(n)} \nrightarrow 0$ and the sequence $(\frac{|A(n)|}{g(n)})$ is bounded. Then $\frac{|A(n)|}{f(n)} \nrightarrow 0$, and the sequence $\left(\frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{f(n)}\right)$ is bounded since $g(n) \leq f(n)$ for all $n \in \mathbb{N}$. □

Theorem 2.1. *If g_1, g_2 belong to G such that $\frac{g_2(n)}{g_1(n)} \to \infty$ then $C^{st}_{g_1}(x) \subsetneq C^{st}_{g_2}(x)$. If $g \in G$ and $\frac{n}{g(n)} \to \infty$ then $C^{st}_g(x) \subsetneq C^{st}(x)$.*

Proof. To prove the first claim note that the inclusion $C^{st}_{g_1}(x) \subset C^{st}_{g_2}(x)$ follows from Lemma 2.1. Set $f := \sqrt{g_1.g_2}$. Then

$$\lim_{n\to\infty} \frac{f(n)}{g_1(n)} = \lim_{n\to\infty} \frac{g_2(n)}{f(n)} = \infty. \tag{2.2}$$

Also we have

$$\frac{n}{g_1(n)} = \frac{n}{g_2(n)}.\frac{g_2(n)}{g_1(n)} \to \infty.$$

So $\frac{n}{f(n)} = \sqrt{\frac{n^2}{g_1(n)g_2(n)}} \to \infty$. Hence f have the assumption of Lemma 2.3. Take the sequence (x_n) obtained in this lemma. Then $x_n \in C^{st}_{g_2}(x)$ but $x_n \notin C^{st}_{g_1}(x)$. Indeed, using (2.2) we have

$$\frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{g_2(n)} = \frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{f(n)}.\frac{f(n)}{g_2(n)} \to 0$$

because $\left(\frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{f(n)}\right)_{n\in\mathbb{N}}$ is bounded from Lemma 2.3. Thus, $x_n \in C^{st}_{g_2}(x)$. To prove that $x_n \notin C^{st}_{g_1}(x)$ observe that

$$\frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{g_1(n)} = \frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{f(n)} \frac{f(n)}{g_1(n)}.$$

So, $x_n \notin C^{st}_{g_1}(x)$ because $\frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{f(n)} \nrightarrow 0$, and $\frac{f(n)}{g_1(n)} \to \infty$ from (2.2).

If we take $g_2(n) = n$, for all $n \in \mathbb{N}$, second assertion proved easily from the same way. □

Corollary 2.1. *Let $0 < \alpha < \beta \leq 1$ and $g_1(n) = n^\alpha$, $g_2 = n^\beta$ for $n \in \mathbb{N}$. Then $C^{st}_{g_1}(x) \subsetneq C^{st}_{g_2}(x)$.*

Example 2.1. Let

$$g_1(n) = \begin{cases} n, & \text{for even } n \in \mathbb{N} \\ \sqrt{n}, & \text{for odd } n \in \mathbb{N} \end{cases}$$

and $g_2(n) = \sqrt{n}$ for $n \in \mathbb{N}$. It is clear that, $\limsup_{n\to\infty} \frac{g_1(n)}{g_2(n)} = \infty$. However, $C^{st}_{g_1}(x) = C^{st}_{g_2}(x)$. Indeed, construct a nondecreasing function $g \in G$ such that $C^{st}_g(x) = C^{st}_{g_1}(x)$, according to the method used in the proof of Lemma 2.1. Then it follows from simple calculations that g is given by

$$g(n) = \begin{cases} \sqrt{n+1} & \text{for even } n \in \mathbb{N} \\ \sqrt{n} & \text{for odd } n \in \mathbb{N}. \end{cases}$$

Obviously, $\frac{1}{2} \leq \frac{g(n)}{g_2(n)} \leq 2$ for all $n \geq 1$. Therefore, by Lemma 2.1 we have $C^{st}_g(x) = C^{st}_{g_1}(x)$.

Theorem 2.2. *There exists a function $g \in G$ such that C^{st}_g is different from $C^{st}_{n^\alpha}$ with $0 < \alpha < 1$.*

Proof. Let a_k and $g(n)$ defined as in Example 1.3. Let $A_k := \{n \in \mathbb{N} : a_{k+1} - (a_{k+1})^{1/4} \leq n < a_{k+1}\}$ and $A = \cup_{k\geq 2} A_k$. Let us take account the sequence

$$x_n = \begin{cases} n, & n \in A \\ 0, & n \notin A. \end{cases}$$

It is clear that $\frac{1}{2}(a_{k+1})^{1/4} \leq |B_k| \leq (a_{k+1})^{1/4}$. Let us check that $g{-}st{-}\lim x_k \neq 0$. For $k > 0$ we have

$$\frac{|\{k \leq a_{k+1} - 1 : |x_k - 0| \geq \varepsilon\}|}{g(a_{k+1} - 1)} \geq \frac{\frac{1}{2}|B_k|}{g(a_k)} \geq \frac{\frac{1}{4}(a_{k+1})^{1/4}}{(a_{k+1})^{1/4}} = \frac{1}{4},$$

so, $g - st - \lim x_k \neq 0$. Furthermore,

$$|\{k \leq a_{k+1} : |x_k - 0| \geq \varepsilon\}| \leq (a_k)^{1/4} + (a_{k+1})^{1/4} \leq 2(a_{k+1})^{1/4}$$

and so,

$$\frac{|\{k \leq a_{k+1} : |x_k - 0| \geq \varepsilon\}|}{(a_{k+1})^{1/3}} \leq \frac{2(a_{k+1})^{1/4}}{(a_{k+1})^{1/3}} = 2(a_{k+1})^{-1/12} \to 0, \ (k \to \infty)$$

holds.

Now, fix any $n \geq 4$ and choose a unique $k \in \mathbb{N}$ such that $n \in [a_k, a_{k+1})$. If $n < a_{k+1} - (a_{k+1})^{1/4}$ then

$$\begin{aligned} \frac{|\{k \leq n : |x_k - 0| \geq \varepsilon\}|}{n^{1/3}} &= \frac{|\{k \leq a_k : |x_k - 0| \geq \varepsilon\}|}{n^{1/3}} \\ &\leq \frac{|\{k \leq n : |x_k - 0| \geq \varepsilon\}|}{(a_k)^{1/3}} \leq 2(a_k)^{-1/12}. \end{aligned}$$

If $a_{k+1} - (a_{k+1})^{1/4} \leq n < a_{k+1}$ then for $b > a > 0$, the function

$$f(x) := \frac{a + x}{(b + x)^{1/3}}, \ x \geq 0$$

is increasing, thus

$$\frac{|\{k \le n : |x_k - 0| \ge \varepsilon\}|}{n^{1/3}} \le \frac{|\{k \le a_{k+1} : |x_k - 0| \ge \varepsilon\}|}{(a_{k+1})^{1/3}}.$$

So, $x_n \in C^{st}_{n^{1/3}}(x)$.

Now, let $0 < \alpha < 1$, $\alpha \neq \frac{1}{3}$. If $\alpha < \frac{1}{3}$ then from Corollary 2.1 $C^{st}_{n^\alpha} \subsetneq C^{st}_{n^{1/3}}$ and $C^{st}_g \backslash C^{st}_{n^\alpha} \neq \emptyset$ because $C^{st}_g \backslash C^{st}_{n^{1/3}} \neq \emptyset$. If $\alpha > \frac{1}{3}$ then $C^{st}_{n^\alpha} \backslash C^{st}_g \neq \emptyset$. By the same way we can show that $x_n \in C^{st} \backslash C^{st}_g$. So $C^{st}_g \subsetneq C^{st}$. □

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