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## NEW INEQUALITIES ON LIPSCHITZ FUNCTIONS

İmdat İşcan, Cuma Altunsoy and Huriye Kadakal

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**Abstract.** In this study, some inequalities of Hermite Hadamard type obtained for  $p$ -convex functions are given for Lipschitz mappings. Also, some applications for special means have been given.

**Keywords:** Hermite Hadamard inequalities;  $p$ -convex functions; convex functions.

### 1. Preliminaries and fundamentals

**Definition 1.1.** [21] A function  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  is said to be convex if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

is valid for all  $x, y \in I$  and  $t \in [0, 1]$ . If this inequality reverses, then the function  $f$  is said to be concave on interval  $I \neq \emptyset$ .

This definition is well known in literature. Convexity theory has appeared as a powerful technique to study a wide class of unrelated problems in pure and applied sciences.

One of the most important integral inequalities for convex functions is the Hermite-Hadamard inequality. The following double inequality is well known as the Hadamard inequality in literature.

**Theorem 1.1.** Let  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a convex function defined on the interval  $I$  of real numbers and  $a, b \in I$  with  $a < b$ . Then the inequality

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}$$

is known as the Hermite-Hadamard inequality.

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Some refinements of the Hermite-Hadamard inequality on convex functions have been extensively investigated by a number of authors (e.g., [2, 3, 4, 11, 17, 20, 23, 24]).

Let us recall some definitions of several kinds of convex functions:

**Definition 1.2.** [5] Let  $I \subset \mathbb{R} \setminus \{0\}$  be a real interval. A function  $f : I \rightarrow \mathbb{R}$  is said to be harmonically convex, if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq tf(y) + (1-t)f(x)$$

for all  $x, y \in I$  and  $t \in [0, 1]$ . If this inequality is reversed, then the function  $f$  is said to be harmonically concave.

**Definition 1.3.** [7] Let  $I \subset (0, \infty)$  be a real interval and  $p \in \mathbb{R} \setminus \{0\}$ . A function  $f : I \rightarrow \mathbb{R}$  is said to be a  $p$ -convex function, if

$$f\left([tx^p + (1-t)y^p]^{\frac{1}{p}}\right) \leq tf(x) + (1-t)f(y)$$

for all  $x, y \in I$  and  $t \in [0, 1]$ . If this inequality is reversed, then the function  $f$  is said to be  $p$ -concave.

According to this definition, it can be easily seen that for  $p = 1$  and  $p = -1$ ,  $p$ -convexity is reduced to ordinary convexity and harmonical convexity of functions defined on  $I \subset (0, \infty)$ , respectively.

Hermite-Hadamard inequalities for the  $p$ -convex function are the following:

**Theorem 1.2.** [7] Let  $f : I \subset (0, \infty) \rightarrow \mathbb{R}$  be a  $p$ -convex function,  $p \in \mathbb{R} \setminus \{0\}$  and  $a, b \in I$  with  $a < b$ . If  $f \in L[a, b]$  then we have

$$(1.2) \quad f\left(\left[\frac{a^p + b^p}{2}\right]^{\frac{1}{p}}\right) \leq \frac{p}{b^p - a^p} \int_a^b \frac{f(x)}{x^{1-p}} dx \leq \frac{f(a) + f(b)}{2}.$$

These inequalities are sharp [7]. We refer the reader to the recent papers related to  $p$ -convexity (see [13, 14, 15, 16, 18, 19]) and references therein.

The purpose of this article is to obtain new inequalities on the right and left sides of the inequality (1.2) for Lipschitz functions.

**Definition 1.4.** [22]  $f : I \rightarrow \mathbb{R}$  is said to satisfy the Lipschitz condition if there is a constant  $M > 0$  such that

$$|f(x) - f(y)| \leq M|x - y|, \quad \forall x, y \in I.$$

**Theorem 1.3.** [22] If  $f : I \rightarrow \mathbb{R}$  is convex, then  $f$  satisfies a Lipschitz condition on any closed interval  $[a, b]$  contained in the interior  $I^\circ$  of  $I$ . Consequently,  $f$  is absolutely continuous on  $[a, b]$  and continuous on  $I^\circ$ .

**Lemma 1.1.** [1] *Let the function  $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$  a differentiable function on interval  $I$ ,  $a, b \in I$  with  $a < b$  and  $M = \sup_{t \in [a, b]} |f'(t)| < \infty$ . Then the function  $f$  is an  $M$ -Lipschitzian functions.*

In [1], Dragomir et al. obtained new inequalities on the right and left sides of the inequality (1.1) for Lipschitz functions as follows.

**Theorem 1.4.** [1] *Let  $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$  be an  $M$ -Lipschitzian mapping on  $I$  and  $a, b \in I$  with  $a < b$ . Then we have the inequalities:*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{M}{4}(b-a)$$

and

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{M}{3}(b-a).$$

**Corollary 1.1.** [1] *Let  $f : I \subset (0, \infty) \rightarrow \mathbb{R}$  be a convex and differentiable function on interval  $I$  and  $a, b \in I$  with  $a < b$  and  $M := \sup_{t \in [a, b]} |f'(x)| < \infty$ . Then we have the inequalities:*

$$0 \leq \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \leq \frac{M}{4}(b-a)$$

and

$$0 \leq \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{M}{3}(b-a).$$

Some results obtained in this study are reduced to the results of Theorem 1.4 and Corollary 1.1 in special cases. For more recent results connected with inequalities of the Hermite-Hadamard type on Lipschitzian functions, see [6, 8, 9, 10].

In the following part, we will give some necessary definitions and simple mathematical inequalities that will be used to achieve our main results.

**Definition 1.5.** [12] The beta function denoted by  $\beta(m, n)$  is defined as

$$\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx.$$

**Definition 1.6.** [12] The hypergeometric function denoted by  ${}_2F_1(a, b; c; z)$  is defined by the integral equality

$${}_2F_1(a, b; c; z) = \frac{1}{\beta(b, c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} dt, \quad c > b > 0, \quad |z| < 1.$$

The following Lemmas are well known in literature. Especially, Lemma 1.3 and Lemma 1.4 are easily seen from the Lagrange mean value theorem.

**Lemma 1.2.** *Let  $0 \leq x < y$ . Then the following inequality holds for  $0 < \alpha \leq 1$ :*

$$|y^\alpha - x^\alpha| \leq |y - x|^\alpha.$$

**Lemma 1.3.** *Let  $0 < x < y$ . Then the following inequality holds for  $\alpha < 0$ :*

$$|y^\alpha - x^\alpha| \leq |y - x|(-\alpha)x^{\alpha-1}.$$

**Lemma 1.4.** *Let  $0 < x < y$ . Then the following inequality holds for  $\alpha \geq 1$ :*

$$|y^\alpha - x^\alpha| \leq |y - x|\alpha y^{\alpha-1}.$$

Let  $0 < x < y$ , throughout this paper we will use

$$\begin{aligned} A &= A(x, y) = \frac{x + y}{2} \\ G &= G(x, y) = \sqrt{xy} \\ H &= H(x, y) = \frac{2xy}{x + y} \\ M_p &= M_p(x, y) = \left( \frac{x^p + y^p}{2} \right)^{\frac{1}{p}} \\ I &= I(x, y) = \frac{1}{e} \left( \frac{y^y}{x^x} \right)^{\frac{1}{y-x}} \\ L &= L(x, y) = \frac{x - y}{\ln x - \ln y} \\ L_p &= L_p(x, y) = \left( \frac{y^{p+1} - x^{p+1}}{(p+1)(y-x)} \right)^{\frac{1}{p}}, \quad p \neq -1, 0 \end{aligned}$$

for the arithmetic, geometric, harmonic, power mean of the order  $p$ , identric, logarithmic and  $p$ -logarithmic mean, respectively.

## 2. Main results

In this section, we shall establish some Hermite-Hadamard-type inequalities for Lipschitzian functions.

**Theorem 2.1.**  *$f : I \subset (0, \infty) \rightarrow \mathbb{R}$  be an  $M$ -Lipschitzian function on the interval  $I$  of real numbers and  $a, b \in I$  with  $a < b$ . Then following inequalities hold:*

a) For  $p \geq 1$ ;

$$\begin{aligned}
 i) \quad & \left| \frac{f(a)+f(b)}{2} - \frac{p}{b^p-a^p} \int_a^b \frac{f(x)}{x^{1-p}} dx \right| \leq 2M |b^p - a^p|^{\frac{1}{p}} \frac{p^2}{(p+1)(2p+1)} \\
 ii) \quad & \left| \frac{p}{b^p-a^p} \int_a^b \frac{f(x)}{x^{1-p}} dx - f \left( \left[ \frac{a^p+b^p}{2} \right]^{\frac{1}{p}} \right) \right| \leq M \left( \frac{1}{2} \right)^{\frac{1}{p}} |b^p - a^p|^{\frac{1}{p}} \frac{p}{p+1}.
 \end{aligned}$$

b) For  $p < 0$ ;

$$\begin{aligned}
 i) \quad & \left| \frac{f(a)+f(b)}{2} - \frac{p}{b^p-a^p} \int_a^b \frac{f(x)}{x^{1-p}} dx \right| \\
 & \leq M \frac{b^p-a^p}{6pb^{p-1}} \left[ {}_2F_1 \left( 1 - \frac{1}{p}, 2; 4; 1 - \left( \frac{b}{a} \right)^p \right) + 1 \right], \\
 ii) \quad & \left| \frac{p}{b^p-a^p} \int_a^b \frac{f(x)}{x^{1-p}} dx - f \left( \left[ \frac{a^p+b^p}{2} \right]^{\frac{1}{p}} \right) \right| \\
 & \leq M \frac{b^p-a^p}{2p \left( \frac{a^p+b^p}{2} \right)^{1-\frac{1}{p}}} \left[ {}_2F_1 \left( 1 - \frac{1}{p}, 1; 3; 1 - \left( \frac{b}{a} \right)^p \right) + 1 \right]
 \end{aligned}$$

c) For  $0 < p < 1$ ;

$$\begin{aligned}
 i) \quad & \left| \frac{f(a)+f(b)}{2} - \frac{p}{b^p-a^p} \int_a^b \frac{f(x)}{x^{1-p}} dx \right| \leq M \frac{b^p-a^p}{6p} [a^{1-p} + b^{1-p}], \\
 ii) \quad & \left| \frac{p}{b^p-a^p} \int_a^b \frac{f(x)}{x^{1-p}} dx - f \left( \left[ \frac{a^p+b^p}{2} \right]^{\frac{1}{p}} \right) \right| \\
 & \leq M \frac{b^p-a^p}{4p \left( \frac{a^p+b^p}{2} \right)^{1-\frac{1}{p}}} \cdot {}_2F_1 \left( 1 - \frac{1}{p}, 2; 3; \frac{a^p-b^p}{a^p+b^p} \right).
 \end{aligned}$$

*Proof.* a) For  $p \geq 1$ :

(i) Using Lemma 1.2 and taking into account that  $f$  is an  $M$ -Lipschitzian function on interval  $I$ , we have

$$\begin{aligned}
 & \left| tf(a) + (1-t)f(b) - f \left( [ta^p + (1-t)b^p]^{\frac{1}{p}} \right) \right| \\
 = & \left| t \left( f(a) - f \left( [ta^p + (1-t)b^p]^{\frac{1}{p}} \right) \right) + (1-t) \left( f(b) - f \left( [ta^p + (1-t)b^p]^{\frac{1}{p}} \right) \right) \right| \\
 \leq & t \left| f(a) - f \left( [ta^p + (1-t)b^p]^{\frac{1}{p}} \right) \right| + (1-t) \left| f(b) - f \left( [ta^p + (1-t)b^p]^{\frac{1}{p}} \right) \right| \\
 \leq & tM \left| a - (ta^p + (1-t)b^p)^{\frac{1}{p}} \right| + (1-t)M \left| b - (ta^p + (1-t)b^p)^{\frac{1}{p}} \right| \\
 \leq & M \left[ t \left| (a^p)^{\frac{1}{p}} - (ta^p + (1-t)b^p)^{\frac{1}{p}} \right| + (1-t) \left| (b^p)^{\frac{1}{p}} - (ta^p + (1-t)b^p)^{\frac{1}{p}} \right| \right] \\
 = & M \left[ t|(1-t)(b^p - a^p)|^{\frac{1}{p}} + (1-t)|t(b^p - a^p)|^{\frac{1}{p}} \right] \\
 = & M \left[ t(1-t)^{\frac{1}{p}} |b^p - a^p|^{\frac{1}{p}} + (1-t)t^{\frac{1}{p}} |b^p - a^p|^{\frac{1}{p}} \right].
 \end{aligned}$$

So, for all  $t \in [0, 1]$ , we can write the following inequality:

$$\begin{aligned}
 (2.1) \quad & \left| tf(a) + (1-t)f(b) - f \left( [ta^p + (1-t)b^p]^{\frac{1}{p}} \right) \right| \\
 & \leq M \left[ t(1-t)^{\frac{1}{p}} |b^p - a^p|^{\frac{1}{p}} + (1-t)t^{\frac{1}{p}} |b^p - a^p|^{\frac{1}{p}} \right].
 \end{aligned}$$

Integrating this inequality on  $[0, 1]$  over  $t$  we get the inequality:

$$(2.2) \quad \left| f(a) \int_0^1 t dt + f(b) \int_0^1 (1-t) dt - \int_0^1 f \left( [ta^p + (1-t)b^p]^{\frac{1}{p}} \right) dt \right| \\ \leq M |b^p - a^p|^{\frac{1}{p}} \left[ \int_0^1 t(1-t)^{\frac{1}{p}} dt + \int_0^1 t^{\frac{1}{p}}(1-t) dt \right].$$

Here, it is easy to see that

$$(2.3) \quad \int_0^1 t^{\frac{1}{p}}(1-t) dt = \int_0^1 t(1-t)^{\frac{1}{p}} dt = \frac{p^2}{(p+1)(2p+1)},$$

$$(2.4) \quad f(a) \int_0^1 t dt + f(b) \int_0^1 (1-t) dt = \frac{f(a) + f(b)}{2}$$

and

$$(2.5) \quad \int_0^1 f \left( [ta^p + (1-t)b^p]^{\frac{1}{p}} \right) dt = \frac{p}{b^p - a^p} \int_a^b \frac{f(x)}{x^{p-1}} dx.$$

Using the inequalities (2.3), (2.4) and (2.5) in the inequality (2.2), we can derive the desired inequality:

$$\left| \frac{f(a) + f(b)}{2} - \frac{p}{b^p - a^p} \int_a^b \frac{f(x)}{x^{p-1}} dx \right| \leq 2M |b^p - a^p|^{\frac{1}{p}} \frac{p^2}{(p+1)(2p+1)}$$

(ii) Putting  $t = \frac{1}{2}$  in the inequality (2.1), we have

$$(2.6) \quad \left| \frac{f(a) + f(b)}{2} - f \left( \left[ \frac{a^p + b^p}{2} \right]^{\frac{1}{p}} \right) \right| \leq M \left( \frac{1}{2} \right)^{\frac{1}{p}} |b^p - a^p|^{\frac{1}{p}}.$$

If we replace  $a$  by  $(ta^p + (1-t)b^p)^{\frac{1}{p}}$  and  $b$  by  $((1-t)a^p + tb^p)^{\frac{1}{p}}$  in the inequality (2.6), we obtain

$$(2.7) \quad \left| \frac{f \left( [ta^p + (1-t)b^p]^{\frac{1}{p}} \right) + f \left( [(1-t)a^p + tb^p]^{\frac{1}{p}} \right)}{2} - f \left( \left[ \frac{a^p + b^p}{2} \right]^{\frac{1}{p}} \right) \right| \\ \leq M \left( \frac{1}{2} \right)^{\frac{1}{p}} |(2t-1)(b^p - a^p)|^{\frac{1}{p}} \\ = M \left( \frac{1}{2} \right)^{\frac{1}{p}} |2t-1|^{\frac{1}{p}} |b^p - a^p|^{\frac{1}{p}}$$

all  $t \in [0, 1]$ . Now, if we integrate the inequality (2.7) on  $[0, 1]$  over  $t$  we can state that

$$\left| \int_0^1 \frac{f \left( [ta^p + (1-t)b^p]^{\frac{1}{p}} \right)}{2} dt + \int_0^1 \frac{f \left( [(1-t)a^p + tb^p]^{\frac{1}{p}} \right)}{2} dt - f \left( \left[ \frac{a^p + b^p}{2} \right]^{\frac{1}{p}} \right) \right|$$

$$(2.8) \quad \leq M \left(\frac{1}{2}\right)^{\frac{1}{p}} |b^p - a^p|^{\frac{1}{p}} \int_0^1 |2t - 1|^{\frac{1}{p}} dt.$$

Here, it is easy to see that

$$(2.9) \quad \int_0^1 f \left( [ta^p + (1-t)b^p]^{\frac{1}{p}} \right) dt = \int_0^1 f \left( [(1-t)a^p + tb^p]^{\frac{1}{p}} \right) dt = \frac{p}{b^p - a^p} \int_a^b \frac{f(x)}{x^{1-p}} dx$$

and

$$(2.10) \quad \int_0^1 |2t - 1|^{\frac{1}{p}} dt = \frac{p}{p+1}.$$

If we put the equalities (2.9) and (2.10) in the inequality (2.8), then we have the desired inequality:

$$\left| \frac{p}{b^p - a^p} \int_a^b \frac{f(x)}{x^{1-p}} dx - f \left( \left[ \frac{a^p + b^p}{2} \right]^{\frac{1}{p}} \right) \right| \leq M \left(\frac{1}{2}\right)^{\frac{1}{p}} |b^p - a^p|^{\frac{1}{p}} \frac{p}{p+1}.$$

b) For  $p < 0$ :

(i) Taking into account that  $f$  is an  $M$ -Lipschitzian function on interval  $I$ , we have

$$(2.11) \quad \begin{aligned} & \left| tf(a) + (1-t)f(b) - f \left( [ta^p + (1-t)b^p]^{\frac{1}{p}} \right) \right| \\ &= \left| t \left( f(a) - f \left( [ta^p + (1-t)b^p]^{\frac{1}{p}} \right) \right) + (1-t) \left( f(b) - f \left( [ta^p + (1-t)b^p]^{\frac{1}{p}} \right) \right) \right| \\ &\leq t \left| f(a) - f \left( [ta^p + (1-t)b^p]^{\frac{1}{p}} \right) \right| + (1-t) \left| f(b) - f \left( [ta^p + (1-t)b^p]^{\frac{1}{p}} \right) \right| \\ &\leq tM \left| (a^p)^{\frac{1}{p}} - (ta^p + (1-t)b^p)^{\frac{1}{p}} \right| + (1-t)M \left| (b^p)^{\frac{1}{p}} - (ta^p + (1-t)b^p)^{\frac{1}{p}} \right|. \end{aligned}$$

Also, using Lemma 1.3, we get

$$(2.12) \quad \begin{aligned} & \left| (a^p)^{\frac{1}{p}} - (ta^p + (1-t)b^p)^{\frac{1}{p}} \right| \\ &\leq |a^p - ta^p - b^p + tb^p| \left( -\frac{1}{p} \right) (ta^p + (1-t)b^p)^{\frac{1}{p}-1} \\ &= |(1-t)a^p - (1-t)b^p| \left( -\frac{1}{p} \right) (ta^p + (1-t)b^p)^{\frac{1}{p}-1} \\ &= (1-t) \frac{b^p - a^p}{p} (ta^p + (1-t)b^p)^{\frac{1}{p}-1} \end{aligned}$$

and

$$(2.13) \quad \begin{aligned} \left| (b^p)^{\frac{1}{p}} - (ta^p + (1-t)b^p)^{\frac{1}{p}} \right| &\leq |b^p - ta^p - b^p + tb^p| \left( -\frac{1}{p} \right) (b^p)^{\frac{1}{p}-1} \\ &= |t(a^p - b^p)| \left( -\frac{1}{p} \right) (b^p)^{\frac{1}{p}-1} \\ &= t \frac{b^p - a^p}{p} b^{1-p}. \end{aligned}$$

If we put the equalities (2.12) and (2.13) in the inequality (2.11), we obtain the following inequality:

$$(2.14) \quad \left| tf(a) + (1-t)f(b) - f\left([ta^p + (1-t)b^p]^{\frac{1}{p}}\right) \right| \\ \leq M \frac{b^p - a^p}{p} \left[ \frac{t(1-t)}{(ta^p + (1-t)b^p)^{1-\frac{1}{p}}} + t(1-t)b^{1-p} \right].$$

If we integrate the inequality (2.13) on  $[0, 1]$  over  $t$  we get

$$(2.15) \quad \left| f(a) \int_0^1 t dt + f(b) \int_0^1 (1-t) dt - \int_0^1 f\left([ta^p + (1-t)b^p]^{\frac{1}{p}}\right) dt \right| \\ \leq M \frac{b^p - a^p}{p} \left[ \int_0^1 \frac{t(1-t)}{(ta^p + (1-t)b^p)^{1-\frac{1}{p}}} dt + b^{1-p} \int_0^1 t(1-t) dt \right]$$

for all  $t \in [0, 1]$ . Here, we can write the following equality:

$$(2.16) \quad \int_0^1 \frac{t(1-t)}{(ta^p + (1-t)b^p)^{1-\frac{1}{p}}} dt = \int_0^1 \frac{t(1-t)}{b^{p-1} [1-t(1-(\frac{a}{b})^p)]^{1-\frac{1}{p}}} dt$$

Further, if we calculate the equality (2.16) using the definitions of the beta and hypergeometric functions, then we obtain

$$(2.17) \quad \int_0^1 \frac{t(1-t) dt}{b^{p-1} [1-t(1-(\frac{a}{b})^p)]^{1-\frac{1}{p}}} = \frac{\beta(2, 2)}{b^{p-1}} \cdot {}_2F_1\left(1 - \frac{1}{p}, 2; 4; 1 - \left(\frac{a}{b}\right)^p\right) \\ = \frac{1}{6b^{p-1}} \cdot {}_2F_1\left(1 - \frac{1}{p}, 2; 4; 1 - \left(\frac{a}{b}\right)^p\right)$$

where

$$\beta(2, 2) = \int_0^1 t(1-t) dt = \frac{1}{6}.$$

If we put the equalities (2.5) and (2.17) in the inequality (2.15), then we have the desired inequality as follows:

$$\left| \frac{f(a) + f(b)}{2} - \frac{p}{b^p - a^p} \int_a^b \frac{f(x)}{x^{1-p}} dx \right| \\ \leq M \frac{b^p - a^p}{6pb^{p-1}} \left[ {}_2F_1\left(1 - \frac{1}{p}, 2; 4; 1 - \left(\frac{b}{a}\right)^p\right) + 1 \right]$$

(ii) If we take  $t = \frac{1}{2}$  and  $\frac{b^p - a^p}{p} = \frac{|b^p - a^p|}{-p}$  in the inequality (2.14), we have

$$(2.18) \quad \left| \frac{f(a) + f(b)}{2} - f\left(\left[\frac{a^p + b^p}{2}\right]^{\frac{1}{p}}\right) \right| \leq M \frac{|b^p - a^p|}{-4p} \left[ \frac{1}{\left(\frac{a^p + b^p}{2}\right)^{1-\frac{1}{p}}} + b^{1-p} \right].$$



If we replace in the inequality (2.18)  $a$  with  $(ta^p + (1-t)b^p)^{\frac{1}{p}}$  and  $b$  with  $((1-t)a^p + tb^p)^{\frac{1}{p}}$ , we can write the following inequality for all  $t \in [0, 1]$ :

$$\begin{aligned}
 (2.19) \quad & \left| \frac{f\left([ta^p + (1-t)b^p]^{\frac{1}{p}}\right) + f\left([(1-t)a^p + tb^p]^{\frac{1}{p}}\right)}{2} - f\left(\left[\frac{a^p + b^p}{2}\right]^{\frac{1}{p}}\right) \right| \\
 & \leq M \left| \frac{(1-t)a^p + tb^p - (ta^p + (1-t)b^p)}{-4p} \right| \\
 & \quad \times \left[ \frac{1}{\left(\frac{a^p + b^p}{2}\right)^{1-\frac{1}{p}}} + \frac{1}{\left(\left[(1-t)a^p + tb^p\right]^{\frac{1}{p}}\right)^{p-1}} \right] \\
 & = M |2t - 1| \frac{b^p - a^p}{4p} \left[ \frac{1}{\left(\frac{a^p + b^p}{2}\right)^{1-\frac{1}{p}}} + \frac{1}{\left((1-t)a^p + tb^p\right)^{1-\frac{1}{p}}} \right]
 \end{aligned}$$

Integrating the last inequality (2.19) on  $[0, 1]$  over  $t$  we get

$$\begin{aligned}
 (2.20) \quad & \left| \frac{f\left([ta^p + (1-t)b^p]^{\frac{1}{p}}\right) + f\left([(1-t)a^p + tb^p]^{\frac{1}{p}}\right)}{2} - f\left(\left[\frac{a^p + b^p}{2}\right]^{\frac{1}{p}}\right) \right| \\
 & \leq M \frac{b^p - a^p}{4p} \left[ \frac{1}{\left(\frac{a^p + b^p}{2}\right)^{1-\frac{1}{p}}} \int_0^1 |2t - 1| dt + \int_0^1 \frac{|2t - 1|}{\left((1-t)a^p + tb^p\right)^{1-\frac{1}{p}}} dt \right]
 \end{aligned}$$

where

$$(2.21) \quad \int_0^1 |2t - 1| dt = \frac{1}{2}.$$

Now, let calculate the second integral in the inequality (2.22):

$$\begin{aligned}
 (2.22) \quad & \int_0^1 \frac{|2t - 1|}{\left((1-t)a^p + tb^p\right)^{1-\frac{1}{p}}} dt \\
 & = \int_0^{\frac{1}{2}} \frac{1 - 2t}{\left((1-t)a^p + tb^p\right)^{1-\frac{1}{p}}} dt + \int_{\frac{1}{2}}^1 \frac{2t - 1}{\left((1-t)a^p + tb^p\right)^{1-\frac{1}{p}}} dt.
 \end{aligned}$$

It is easy to see that

$$\begin{aligned}
 (2.23) \quad & \int_0^{\frac{1}{2}} \frac{(1 - 2t) dt}{\left((1-t)a^p + tb^p\right)^{1-\frac{1}{p}}} = \frac{\beta(2, 1)}{2 \left(\frac{a^p + b^p}{2}\right)^{1-\frac{1}{p}}} \cdot {}_2F_1\left(1 - \frac{1}{p}, 2; 3; \frac{a^p - b^p}{a^p + b^p}\right) \\
 & = \frac{1}{4 \left(\frac{a^p + b^p}{2}\right)^{1-\frac{1}{p}}} \cdot {}_2F_1\left(1 - \frac{1}{p}, 2; 3; \frac{a^p - b^p}{a^p + b^p}\right)
 \end{aligned}$$

and

$$(2.24) \quad \int_{\frac{1}{2}}^1 \frac{2t - 1}{\left((1-t)a^p + tb^p\right)^{1-\frac{1}{p}}} dt = \frac{1}{4 \left(\frac{a^p + b^p}{2}\right)^{1-\frac{1}{p}}} \cdot {}_2F_1\left(1 - \frac{1}{p}, 2; 3; \frac{a^p - b^p}{a^p + b^p}\right)$$

where

$$\beta(2, 1) = \int_0^1 t dt = \frac{1}{2}.$$

Adding the equalities (2.23) and (2.24), we get

$$(2.25) \quad \int_0^1 \frac{|2t-1|}{((1-t)a^p + tb^p)^{1-\frac{1}{p}}} dt = \frac{1}{2\left(\frac{a^p+b^p}{2}\right)^{1-\frac{1}{p}}} \cdot {}_2F_1\left(1-\frac{1}{p}, 2; 3; \frac{a^p-b^p}{a^p+b^p}\right).$$

If we put the equalities (2.21) and (2.25) in the inequality (2.20), then we obtain the following inequality for all  $t \in [0, 1]$ :

$$\begin{aligned} & \left| \frac{p}{b^p - a^p} \int_a^b \frac{f(x)}{x^{1-p}} dx - f\left(\left[\frac{a^p + b^p}{2}\right]^{\frac{1}{p}}\right) \right| \\ & \leq M \frac{b^p - a^p}{2p\left(\frac{a^p+b^p}{2}\right)^{1-\frac{1}{p}}} \left[ {}_2F_1\left(1-\frac{1}{p}, 1; 3; 1 - \left(\frac{b}{a}\right)^p\right) + 1 \right]. \end{aligned}$$

c) For  $0 < p < 1$ :

Using the inequality (2.11), we can write the inequality

$$(2.26) \quad \begin{aligned} & \left| tf(a) + (1-t)f(b) - f\left([ta^p + (1-t)b^p]^{\frac{1}{p}}\right) \right| \\ & \leq tM \left| (a^p)^{\frac{1}{p}} - (ta^p + (1-t)b^p)^{\frac{1}{p}} \right| + (1-t)M \left| (b^p)^{\frac{1}{p}} - (ta^p + (1-t)b^p)^{\frac{1}{p}} \right| \end{aligned}$$

for all  $t \in [0, 1]$ . Also, by using Lemma 1.4, we get

$$(2.27) \quad \begin{aligned} \left| (a^p)^{\frac{1}{p}} - (ta^p + (1-t)b^p)^{\frac{1}{p}} \right| & \leq |a^p - ta^p - b^p + tb^p| \left(\frac{1}{p}\right) (a^p)^{\frac{1}{p}-1} \\ & = (1-t) \frac{b^p - a^p}{p} a^{1-p} \end{aligned}$$

and

$$(2.28) \quad \begin{aligned} \left| (b^p)^{\frac{1}{p}} - (ta^p + (1-t)b^p)^{\frac{1}{p}} \right| & \leq |b^p - ta^p - b^p + tb^p| \left(\frac{1}{p}\right) (b^p)^{\frac{1}{p}-1} \\ & = t \frac{b^p - a^p}{p} b^{1-p}. \end{aligned}$$

From the inequalities (2.27) and (2.28), the inequality (2.26) can be written as

$$(2.29) \quad \begin{aligned} & \left| tf(a) + (1-t)f(b) - f\left([ta^p + (1-t)b^p]^{\frac{1}{p}}\right) \right| \\ & \leq M \frac{b^p - a^p}{p} [t(1-t)a^{1-p} + t(1-t)b^{1-p}]. \end{aligned}$$

Integrating the inequality (2.29) on  $[0, 1]$  over  $t$  we obtain

$$(2.30) \quad \left| f(a) \int_0^1 t dt + f(b) \int_0^1 (1-t) dt - \int_0^1 f \left( [ta^p + (1-t)b^p]^{\frac{1}{p}} \right) dt \right| \\ \leq M \frac{b^p - a^p}{p} \left[ a^{1-p} \int_0^1 t(1-t) dt + b^{1-p} \int_0^1 t(1-t) dt \right]$$

where

$$(2.31) \quad \int_0^1 t(1-t) dt = \frac{1}{6}.$$

If we put (2.4), (2.5) and (2.31) in the inequality (2.29), then we get the following inequality for all  $t \in [0, 1]$ :

$$(2.32) \quad \left| \frac{f(a) + f(b)}{2} - \frac{p}{b^p - a^p} \int_a^b \frac{f(x)}{x^{1-p}} dx \right| \leq M \frac{|b^p - a^p|}{6p} [a^{1-p} + b^{1-p}].$$

Putting  $t = \frac{1}{2}$  in the inequality (2.29), we have

$$(2.33) \quad \left| \frac{f(a) + f(b)}{2} - f \left( \left[ \frac{a^p + b^p}{2} \right]^{\frac{1}{p}} \right) \right| \leq M \frac{|b^p - a^p|}{4p} [a^{1-p} + b^{1-p}].$$

If we replace in the inequality (2.33)  $a$  with  $(ta^p + (1-t)b^p)^{\frac{1}{p}}$  and  $b$  with  $((1-t)a^p + tb^p)^{\frac{1}{p}}$ , we can write the following inequality for all  $t \in [0, 1]$ :

$$(2.34) \quad \left| \frac{f \left( [ta^p + (1-t)b^p]^{\frac{1}{p}} \right) + f \left( [(1-t)a^p + tb^p]^{\frac{1}{p}} \right)}{2} - f \left( \left[ \frac{a^p + b^p}{2} \right]^{\frac{1}{p}} \right) \right| \\ \leq M \frac{\left| \left( [(1-t)a^p + tb^p]^{\frac{1}{p}} \right)^p - \left( [ta^p + (1-t)b^p]^{\frac{1}{p}} \right)^p \right|}{4p} \\ \times \left[ \left( [ta^p + (1-t)b^p]^{\frac{1}{p}} \right)^{1-p} + \left( [(1-t)a^p + tb^p]^{\frac{1}{p}} \right)^{1-p} \right] \\ = M \frac{b^p - a^p}{4p} \left[ \frac{|2t-1|}{(ta^p + (1-t)b^p)^{1-\frac{1}{p}}} + \frac{|2t-1|}{((1-t)a^p + tb^p)^{1-\frac{1}{p}}} \right].$$

Integrating the inequality (2.34) on the interval  $[0, 1]$  over  $t$  we have

$$\left| \int_0^1 \frac{f \left( [ta^p + (1-t)b^p]^{\frac{1}{p}} \right)}{2} dt + \int_0^1 \frac{f \left( [(1-t)a^p + tb^p]^{\frac{1}{p}} \right)}{2} dt - f \left( \left[ \frac{a^p + b^p}{2} \right]^{\frac{1}{p}} \right) \right| \\ (2.35) \leq M \frac{b^p - a^p}{4p} \left[ \int_0^1 \frac{|2t-1|}{(ta^p + (1-t)b^p)^{1-\frac{1}{p}}} dt + \int_0^1 \frac{|2t-1|}{((1-t)a^p + tb^p)^{1-\frac{1}{p}}} dt \right].$$

Here, using (2.22) we see that the following equalities

$$\begin{aligned}
 \int_0^1 \frac{|2t-1|}{(ta^p + (1-t)b^p)^{1-\frac{1}{p}}} dt &= \int_0^1 \frac{|2t-1|}{((1-t)a^p + tb^p)^{1-\frac{1}{p}}} dt \\
 (2.36) \quad &= \frac{1}{2 \left(\frac{a^p+b^p}{2}\right)^{1-\frac{1}{p}}} \cdot {}_2F_1\left(1 - \frac{1}{p}, 2; 3; \frac{a^p - b^p}{a^p + b^p}\right)
 \end{aligned}$$

hold for all  $t \in [0, 1]$ . Finally, writing the equalities (2.9) and (2.25) in (2.35)

$$\left| \frac{p}{b^p - a^p} \int_a^b \frac{f(x)}{x^{1-p}} dx - f\left(\left[\frac{a^p + b^p}{2}\right]^{\frac{1}{p}}\right) \right| \leq M \frac{b^p - a^p}{4p \left(\frac{a^p+b^p}{2}\right)^{1-\frac{1}{p}}} \cdot {}_2F_1\left(1 - \frac{1}{p}, 2; 3; \frac{a^p - b^p}{a^p + b^p}\right)$$

Thus, the proof of the Theorem is completed.  $\square$

**Remark 2.1.** If we choose  $p = 1$  in Theorem 2.1, then the results we obtained coincide with the Theorem 1.4.

If we choose  $p = -1$  in Theorem 2.1, then we can also give the following corollary.

**Corollary 2.1.**  $f : I \subset (0, \infty) \rightarrow \mathbb{R}$  be an  $M$ -Lipschitzian function on the interval  $I$  of real numbers and  $a, b \in I$  with  $a < b$ . Then, following inequality it holds that:

$$\begin{aligned}
 i) \quad & \left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \leq M \frac{b-a}{6ab^3} \left[ {}_2F_1\left(2, 2; 4; 1 - \frac{a}{b}\right) + 1 \right], \\
 ii) \quad & \left| \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx - f\left(\frac{2ab}{a+b}\right) \right| \leq M \frac{b-a}{2ab \left(\frac{a+b}{2ab}\right)^2} \left[ {}_2F_1\left(2, 1; 3; 1 - \frac{a}{b}\right) + 1 \right].
 \end{aligned}$$

By using Theorem 1.2, Lemma 1.1 and Theorem 2.1, we can state the following corollary:

**Corollary 2.2.** Let the function  $f : I \subset (0, \infty) \rightarrow \mathbb{R}$  a differentiable  $p$ -convex function on interval  $I$ ,  $a, b \in I$  with  $a < b$  and  $M = \sup_{t \in [a, b]} |f'(t)| < \infty$ . Then:

For  $p \geq 1$ ;

$$\begin{aligned}
 i) \quad 0 &\leq \frac{f(a) + f(b)}{2} - \frac{p}{b^p - a^p} \int_a^b \frac{f(x)}{x^{1-p}} dx \leq 2M (b^p - a^p)^{\frac{1}{p}} \frac{p^2}{(p+1)(2p+1)}, \\
 ii) \quad 0 &\leq \frac{p}{b^p - a^p} \int_a^b \frac{f(x)}{x^{1-p}} dx - f\left(\left[\frac{a^p + b^p}{2}\right]^{\frac{1}{p}}\right) \leq M \left(\frac{1}{2}\right)^{\frac{1}{p}} (b^p - a^p)^{\frac{1}{p}} \frac{p}{p+1},
 \end{aligned}$$

For  $p < 0$ ;

$$\begin{aligned} i) \quad 0 &\leq \frac{f(a) + f(b)}{2} - \frac{p}{b^p - a^p} \int_a^b \frac{f(x)}{x^{1-p}} dx \\ &\leq M \frac{b^p - a^p}{6pb^{p-1}} \left[ {}_2F_1 \left( 1 - \frac{1}{p}, 2; 4; 1 - \left( \frac{b}{a} \right)^p \right) + 1 \right], \end{aligned}$$

$$\begin{aligned} ii) \quad 0 &\leq \frac{p}{b^p - a^p} \int_a^b \frac{f(x)}{x^{1-p}} dx - f \left( \left[ \frac{a^p + b^p}{2} \right]^{\frac{1}{p}} \right) \\ &\leq M \frac{b^p - a^p}{2p \left( \frac{a^p + b^p}{2} \right)^{1 - \frac{1}{p}}} \left[ {}_2F_1 \left( 1 - \frac{1}{p}, 1; 3; 1 - \left( \frac{b}{a} \right)^p \right) + 1 \right]. \end{aligned}$$

For  $0 < p < 1$ ;

$$i) \quad 0 \leq \frac{f(a) + f(b)}{2} - \frac{p}{b^p - a^p} \int_a^b \frac{f(x)}{x^{1-p}} dx \leq M \frac{|b^p - a^p|}{3p} A(a^{1-p}, b^{1-p}),$$

$$\begin{aligned} ii) \quad 0 &\leq \frac{p}{b^p - a^p} \int_a^b \frac{f(x)}{x^{1-p}} dx - f \left( \left[ \frac{a^p + b^p}{2} \right]^{\frac{1}{p}} \right) \\ &\leq M \frac{b^p - a^p}{4p \left( \frac{a^p + b^p}{2} \right)^{1 - \frac{1}{p}}} \cdot {}_2F_1 \left( 1 - \frac{1}{p}, 2; 3; \frac{a^p - b^p}{a^p + b^p} \right). \end{aligned}$$

**Remark 2.2.** If we choose  $p = 1$  in Corollary 2.2, then the results we obtained coincide with the Corollary 1.1.

**Proposition 2.1.** For  $p \geq 1$  and  $0 < a < b$ , the inequalities

$$\begin{aligned} i) \quad 0 &\leq H^{-1}(a^p, b^p) - L^{-1}(a^p, b^p) \leq \frac{(b^p - a^p)^{\frac{1}{p}}}{a^{p+1}} \frac{2p^3}{(p+1)(2p+1)} \\ ii) \quad 0 &\leq L^{-1}(a^p, b^p) - A^{-1}(a^p, b^p) \leq \frac{(b^p - a^p)^{\frac{1}{p}}}{a^{p+1}} \left( \frac{1}{2} \right)^{\frac{1}{p}} \frac{p^2}{p+1} \end{aligned}$$

hold.

*Proof.* The function  $f : (0, \infty) \rightarrow \mathbb{R}$ ,  $f(x) = x^{-p}$  ( $p \geq 1$ ) is a differentiable  $p$ -convex function. By using Corollary 2.2 for  $p \geq 1$ :

(i) Since  $|f'(x)| = |-px^{-p-1}| = px^{-p-1}$ , we obtain

$$M = \sup_{x \in [a, b]} |f'(x)| = \sup_{x \in [a, b]} px^{-p-1} = \frac{p}{a^{p+1}},$$

$$\frac{f(a) + f(b)}{2} = \frac{a^{-p} + b^{-p}}{2} = \frac{1}{2} \left( \frac{1}{a^p} + \frac{1}{b^p} \right) = \frac{1}{2} \left( \frac{a^p + b^p}{(ab)^p} \right)$$

and

$$\int_a^b \frac{x^{-p}}{x^{1-p}} dx = \int_a^b \frac{1}{x} dx = \ln b - \ln a.$$

Hence we have

$$\frac{a^p + b^p}{2(ab)^p} - \frac{p(\ln b - \ln a)}{b^p - a^p} \leq \frac{(b^p - a^p)^{\frac{1}{p}}}{a^{p+1}} \frac{2p^3}{(p+1)(2p+1)}.$$

(ii) It is easy to see that

$$f\left(\left[\frac{a^p + b^p}{2}\right]^{\frac{1}{p}}\right) = \left(\left(\frac{a^p + b^p}{2}\right)^{\frac{1}{p}}\right)^{-p} = \left(\frac{a^p + b^p}{2}\right)^{-1} = \frac{2}{a^p + b^p},$$

$$\frac{p(\ln b - \ln a)}{b^p - a^p} - \frac{2}{a^p + b^p} \leq \frac{(b^p - a^p)^{\frac{1}{p}}}{a^{p+1}} \left(\frac{1}{2}\right)^{\frac{1}{p}} \frac{p^2}{p+1}.$$

□

**Proposition 2.2.** For  $p \leq -1$  and  $0 < a < b$ , the following inequalities hold:

$$\begin{aligned} i) \quad 0 &\leq H^{-1}(a^p, b^p) - L^{-1}(a^p, b^p) \\ &\leq \frac{|b^p - a^p|}{6a^{p+1}b^{p-1}} \left[ {}_2F_1\left(1 - \frac{1}{p}, 2; 4; 1 - \left(\frac{b}{a}\right)^p\right) + 1 \right], \end{aligned}$$

$$\begin{aligned} ii) \quad 0 &\leq L^{-1}(a^p, b^p) - A^{-1}(a^p, b^p) \\ &\leq \frac{|b^p - a^p|}{2a^{p+1} \left(\frac{a^p + b^p}{2}\right)^{1 - \frac{1}{p}}} \left[ {}_2F_1\left(1 - \frac{1}{p}, 1; 3; 1 - \left(\frac{b}{a}\right)^p\right) + 1 \right]. \end{aligned}$$

*Proof.* The function  $f : (0, \infty) \rightarrow \mathbb{R}$ ,  $f(x) = x^{-p}$  ( $p \leq -1$ ) be a differentiable  $p$ -convex function. By using Corollary 2.2 for  $p < 0$ , we obtain desired inequalities. □

**Proposition 2.3.** For  $0 < p < 1$  and  $0 < a < b$ , the following inequalities hold:

$$i) \quad 0 \leq A(a, b) - L_{p-1}^{1-p}(a, b) L_p^p(a, b) \leq \frac{b^p - a^p}{3p} A(a^{1-p}, b^{1-p})$$

$$ii) \quad 0 \leq L_{p-1}^{1-p}(a, b) L_p^p(a, b) - M_p(a, b) \leq \frac{b^p - a^p}{4p \left(\frac{a^p + b^p}{2}\right)^{1 - \frac{1}{p}}} \cdot {}_2F_1\left(1 - \frac{1}{p}, 2; 3; \frac{a^p - b^p}{a^p + b^p}\right).$$

*Proof.* The function  $f : (0, \infty) \rightarrow \mathbb{R}$ ,  $f(x) = x$  ( $0 < p < 1$ ) be a differentiable  $p$ -convex function. By using the Corollary 2.2 for  $p < 1$ , we can write

$$\begin{aligned} \frac{f(a) + f(b)}{2} &= \frac{a + b}{2}, \\ \int_a^b \frac{x}{x^{1-p}} dx &= \frac{b^{p+1} - a^{p+1}}{p + 1}, \\ M &= \sup_{x \in [a, b]} |f'(x)| = 1. \end{aligned}$$

So, we obtain followings for Corollary 2.2 (i)-(ii) for  $0 < p < 1$ , respectively:

$$\left| \frac{a + b}{2} - \frac{p}{b^p - a^p} \frac{b^{p+1} - a^{p+1}}{p + 1} \right| = \frac{a + b}{2} - \frac{p}{b^p - a^p} \frac{b^{p+1} - a^{p+1}}{p + 1} \leq \frac{b^p - a^p}{3p} A(a^{1-p}, b^{1-p})$$

and

$$\frac{p}{b^p - a^p} \frac{b^{p+1} - a^{p+1}}{p + 1} - \left( \frac{a^p + b^p}{2} \right)^{\frac{1}{p}} \leq \frac{b^p - a^p}{4p \left( \frac{a^p + b^p}{2} \right)^{1 - \frac{1}{p}}} \cdot {}_2F_1 \left( 1 - \frac{1}{p}, 2; 3; \frac{a^p - b^p}{a^p + b^p} \right).$$

□

### 3. Conclusion

In this paper, using the definition of  $M$ -Lipschitzian function and some simple mathematical inequalities, we obtained new inequalities related to the right and left sides of the inequality (1.2) for Lipschitz functions. Some results obtained in this study are reduced to the results obtained in [1] in special cases.

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İmdat İşcan

Faculty of Art and Science, Department of Mathematics  
Giresun University, 28200, Giresun, Turkey  
imdati@yahoo.com

Cuma Altunsoy

Mustafa Kemal Secondary School, Mathematics Teacher  
Ministry of Education, Giresun, Turkey  
cumaltunsoy@hotmail.com

Huriye Kadakal

Hamdi Bozbağ Anatolian High School, Mathematics Teacher  
Ministry of Education, Giresun, Turkey  
huriyekadakal@hotmail.com