Path integral calculation of heat kernel traces with first order operator insertions

Fiorenzo Bastianelli [a,b,*], Francesco Comberiati [a]

[a] *Dipartimento di Fisica e Astronomia, Università di Bologna, via Irnerio 46, I-40126 Bologna, Italy*
[b] *INFN, Sezione di Bologna, via Irnerio 46, I-40126 Bologna, Italy*

Abstract

We study generalized heat kernel coefficients, which appear in the trace of the heat kernel with an insertion of a first-order differential operator, by using a path integral representation. These coefficients may be used to study gravitational anomalies, i.e. anomalies in the conservation of the stress tensor. We use the path integral method to compute the coefficients related to the gravitational anomalies of theories in a non-abelian gauge background and flat space of dimensions 2, 4, and 6. In 4 dimensions one does not expect to have genuine gravitational anomalies. However, they may be induced at intermediate stages by regularization schemes that fail to preserve the corresponding symmetry. A case of interest has recently appeared in the study of the trace anomalies of Weyl fermions.

1. Introduction

Heat kernel methods provide a useful tool for investigating QFTs. They were introduced by Schwinger for studying QED processes [1] and extended to curved spaces and non-abelian gauge fields by DeWitt [2]. There are many reviews and books dedicated to them, as [3–5].

One application of the heat kernel finds its place in the study of anomalies. The connection is most easily seen by recalling Fujikawa's method [6], which identifies the anomalies as arising

---

from the non-invariance of the path integral measure under the symmetry transformations. In that approach the anomalies are cast as regulated infinitesimal jacobians, $\lim_{\beta \to 0} \text{Tr}\, J\, e^{-\beta \mathcal{R}}$, where J is the generator of the anomalous symmetry and $\mathcal{R}$ a regulator, usually a second-order differential operator. Once Wick-rotated to euclidean space the regulator $\mathcal{R}$ becomes an elliptic operator and $e^{-\beta \mathcal{R}}$ defines the associated heat kernel. Often the operator J depends only on the spacetime coordinates, and it does not contain any differential operator. This is the case of the usual chiral and trace anomalies.

Our interest in this paper is in traces that contain a first-order differential operator. This situation arises when one considers gravitational anomalies [7]. The latter are anomalies in the conservation of the stress tensor, and the corresponding symmetry is the arbitrary change of coordinates (diffeomorphisms). Diffeomorphisms are generated by the Lie derivative of the quantum fields, and on scalars and Dirac spinors the Lie derivative $\mathcal{L}_\xi$ takes the simple form

$$\mathcal{L}_\xi = \xi^\mu(x)\partial_\mu \tag{1}$$

where $\xi^\mu(x)$ is the vector field due to an infinitesimal change of coordinates ($x^\mu \to x'^\mu = x^\mu - \xi^\mu(x)$). The corresponding anomaly is then related to the regularization of an infinitesimal Fujikawa jacobian of the form

$$J = \left[\xi^\mu(x)\partial_\mu + \sigma(x)\right]\delta^D(x-y) \tag{2}$$

where $\sigma(x)$ is a function that depends on the regularization scheme adopted (often one takes $\sigma(x) = \frac{1}{2}\partial_\mu \xi^\mu(x)$ as a convenient choice, see [8,9]). An application of this type of traces has recently appeared in [10].

In this paper, we shall study traces of the heat kernel with an insertion of a first-order differential operator of the form given in (2) by using a quantum mechanical path integral. After making explicit the relationship between the heat kernel traces and their path integral representation, we use the latter to evaluate the first three heat kernel coefficients for an elliptic operator $\mathcal{R}$ containing a non-abelian gauge field $A_\mu(x)$ and an arbitrary matrix-valued scalar potential $V(x)$. These coefficients are a generalization of the standard heat kernel coefficients, also known as Seeley-DeWitt coefficients, as they contain the insertion of a first-order differential operator. We shall call them generalized heat kernel coefficients, for simplicity. Some of these coefficients have been computed before in [11] and [12]. Here we shall reproduce some of those results with the path integral method, and compute an additional one.

Our motivation for investigating these coefficients stems from a desire of addressing the anomalies of a Weyl fermion in four dimensions by using a regularization scheme that induces gravitational anomalies as well. This situation appeared in [13], where the trace anomaly of a Weyl fermion in an abelian gauge background was computed to verify the absence of a parity violating term, conjectured in [14] to be a possibility for CP violating theories. The use of a Pauli-Villars (PV) regularization with a Majorana mass showed the absence of such a term, as the PV Majorana mass preserves CP and diffeomorphism invariance. On the other hand, the verification of the same result with a PV Dirac mass could not be completed, as the latter induces gravitational anomalies, which can be computed by using a generalized heat kernel coefficient in the background of a non-abelian gauge field and flat spacetime. This justifies the use of flat space that we consider in our analysis. The non-abelian background is however needed as the regulator $\mathcal{R}$ contains gamma matrices, making the connection contained in $\mathcal{R}$ effectively non-abelian, even for the simple case of a Weyl fermion coupled to a $U(1)$ gauge field.

Thus, the problem we face in this paper is to study the path integral method to compute generalized heat kernel coefficients, verifying the ones previously known and producing a new

one. In section 2 we review the path integral representation of the heat kernel and its traces. We start by considering a simple elliptic operator $\mathcal{R}$, interpreted as the quantum hamiltonian of a non-relativistic particle in a scalar potential, and study how to insert an arbitrary function of the particle coordinates inside the heat kernel trace. We discuss the role played by the propagators defined either by the Dirichlet boundary conditions (DBC) or by the string inspired (SI) method, which can be used equivalently for generating the perturbative expansion of the path integral. Section 3 extends the previous set-up to include the insertion of a first-order differential operator inside the heat kernel trace, and uses a more general hamiltonian $\mathcal{R}$ with a non-abelian gauge potential A_μ and a matrix-valued scalar potential V. The corresponding particle action is also matrix-valued, and the path integral contains a time ordering prescription to maintain gauge covariance. In section 4 we present the first three generalized heat kernel coefficients, and in section 5 we describe the calculation of the simplest one with the path integral method, reproducing the result of [11]. Having verified the consistency of the method, in section 6 we proceed with the calculation of two more heat kernel coefficients. The first one is the flat space limit of a more general result originally obtained in [12], which is relevant for the gravitational anomalies in a flat four-dimensional space. The last coefficient is new, and may be useful for the gravitational anomalies of gauge theories in a flat six-dimensional space. After our conclusions, we report in appendix A the worldline propagators defined by the Dirichlet boundary conditions and by the string inspired method, in appendix B we use them for computing some simple Seeley-DeWitt coefficients as a simple review of the path integral method, and in appendix C we report further calculational details.

2. Path integral representation of heat kernel traces

The path integral representation was used to study the trace anomalies in [15] and [16], where the object of interest was represented by a heat kernel trace of the form

$$\text{Tr}\left[\sigma(x)\, e^{-\beta\mathcal{R}}\right] \tag{3}$$

with $\sigma(x)$ an arbitrary function and $\mathcal{R}$ an elliptic differential operator. In this section we take as guiding example the operator

$$\mathcal{R} = -\frac{1}{2}\partial^2 + V(x) = \frac{1}{2}p^2 + V(x) \tag{4}$$

where $\partial^2 = \partial^\mu \partial_\mu$ is the laplacian and $p_\mu = -i\partial_\mu$ the momentum operator in the coordinate representation of quantum mechanics. $\mathcal{R}$ is directly interpreted as the hamiltonian of a non-relativistic particle of unit mass in D dimensions, and the functional trace is understood as a trace on the Hilbert space of the particle

$$\text{Tr}\left[\sigma(x)\, e^{-\beta\mathcal{R}}\right] = \int d^D x\, \sigma(x)\langle x|e^{-\beta\mathcal{R}}|x\rangle = \int \frac{d^D x}{(2\pi\beta)^{\frac{D}{2}}}\sigma(x)\sum_{n=0}^{\infty} a_n(x)\beta^n \tag{5}$$

where $\langle x|e^{-\beta\mathcal{R}}|x\rangle$ is the transition amplitude for an euclidean time β with coinciding initial and final points, i.e. the heat kernel at coinciding points. The evaluation of the latter for an arbitrary potential $V(x)$ is not known in closed form, but often one needs only its perturbative expansion for small values of β, which gives rise to the Seeley-DeWitt coefficients $a_n(x)$. The first few ones are

$$a_0(x) = 1$$

$$a_1(x) = -V$$

$$a_2(x) = \frac{1}{2}V^2 - \frac{1}{12}\partial^2 V$$

$$a_3(x) = -\frac{1}{6}V^3 + \frac{1}{12}V\partial^2 V + \frac{1}{24}\partial_\mu V \partial^\mu V - \frac{1}{240}\partial^4 V \ . \tag{6}$$

A way of computing them is to use the path integral representation of the transition amplitude for the quantum mechanical model with hamiltonian $\mathcal{R}$ and euclidean action

$$S[x(t)] = \int\limits_0^\beta dt \left(\frac{1}{2}\dot{x}^\mu \dot{x}_\mu + V(x) \right) \ . \tag{7}$$

Then, using the equivalence of path integrals with canonical quantization, one may write

$$\text{Tr}\left[\sigma(x) e^{-\beta\mathcal{R}} \right] = \int d^D x_0\, \sigma(x_0) \langle x_0 | e^{-\beta\mathcal{R}} | x_0 \rangle$$

$$= \int d^D x_0\, \sigma(x_0) \int\limits_{x(0)=x_0}^{x(\beta)=x_0} Dx(t)\, e^{-S[x(t)]} = \int\limits_{PBC} Dx(t)\, \sigma(x(0))\, e^{-S[x(t)]} \tag{8}$$

where in the second line we have used the path integral representation of the transition amplitude at coinciding points, and recognized that the additional integration over the point x_0, which creates the trace, implements periodic boundary conditions (PBC). Thus one finds a path integral on all loops with an insertion of the function $\sigma(x(t))$. The argument $x^\mu(t)$ of the function σ is evaluated at $t = 0$, which corresponds to the base point $x^\mu(0) = x_0^\mu$ of the parametrized loop, but it could be anywhere on the loop described by the function $x^\mu(t)$ as a consequence of time translational invariance. Now, given the relation (8), one can use the perturbative expansion of the path integral in the euclidean time β to evaluate the heat kernel coefficients in (5) with worldline propagators and Feynman diagrams.

This set-up was discussed in [15,16], where it was extended to curved space and non-abelian gauge fields and used to rederive the trace anomalies of many field theories. Actually, the precise observable available (the trace anomalies) could be used as a benchmark to construct well-defined path integrals for particles in curved spaces, stressing the necessity of using precise regularization schemes on the worldline, which must include well-defined but scheme dependent counterterms [15–18,9]. A list of counterterms needed for sigma models with N supersymmetries in various regularization schemes is given in [19], with the $N = 4$ case that has been applied in the recent construction of the path integral for the graviton in first quantization [20–22].

A direct extension of the above construction to the case of an insertion of a first-order differential operator in the trace of the heat kernel may not seem immediate. A simple way of obtaining the insertion is to exponentiate the corresponding operator, and view it as a source added to the action. Then a derivative creates the required insertion. Let us check the formulae we get this way for a scalar insertion and compare them with the set-up described above.

To start with, let us consider

$$\text{Tr}\left[\sigma(x) e^{-\beta\mathcal{R}} \right] = \frac{\partial}{\partial\lambda} \text{Tr}\left[e^{-\beta\mathcal{R} + \lambda\sigma(x)} \right] \Big|_{\lambda=0} \tag{9}$$
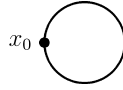
Fig. 1. Loop with Dirichlet boundary conditions (DBC) at x_0.

where the trace guarantees that the insertion arising by acting with a derivative can be placed on the left of the exponential. The exponentiation can be viewed as a deformation of the hamiltonian, which in turns generates a modified euclidean action with $V(x) \to V(x) - \frac{\lambda}{\beta}\sigma(x)$, so that

$$S[x] \;\to\; S_\lambda[x] = \int_0^\beta dt \left(\frac{1}{2}\dot{x}^\mu \dot{x}_\mu + V(x) - \frac{\lambda}{\beta}\sigma(x) \right) . \tag{10}$$

Using the path integral representation one finds

$$\mathrm{Tr}\left[\sigma(x) e^{-\beta \mathcal{R}} \right] = \int_{PBC} Dx \left(\frac{1}{\beta} \int_0^\beta dt\, \sigma(x(t)) \right) e^{-S[x]} . \tag{11}$$

This formula is equivalent to the one obtained earlier in (8). The equivalence between the two expressions is understood by invoking the time translational invariance of the one-point function of the operator $\sigma(x(t))$, which may be substituted by its time average.

As a side result, this reformulation makes it clear how to use different worldline propagators for obtaining the same heat kernel coefficients. In the set-up described by eq. (8), it is natural to parametrize the quantum integration variables by

$$x(t) = x_0 + q(t) \tag{12}$$

with $q(0) = q(\beta) = 0$, thus defining Dirichlet boundary conditions (DBC) on the quantum fluctuations $q(t)$. They parametrize loops with a fixed base point x_0. The final integration over x_0 produces all possible loops in target space, thus implementing the full periodic boundary condition (PBC) prescription, see Fig. 1. The emerging quantum integration variables $q(t)$ have a perturbatively well-defined propagator, as fixed by Dirichlet boundary conditions. This was the approach used in [15,16].

Alternatively, one may find it useful to employ the so-called "string-inspired" (SI) propagator [23], obtained by setting again

$$x(t) = x_0 + q(t) \tag{13}$$

but now with the condition

$$x_0 = \frac{1}{\beta} \int_0^\beta dt\, x(t) \quad \Rightarrow \quad \int_0^\beta dt\, q(t) = 0 \tag{14}$$

where the zero mode x_0 is the average position of the loop, see Fig. 2. The non-local constraint on $q(t)$ defines the SI propagator. Again, the final integration over x_0 creates all loops in target space.

As a preparation for our worldline calculations, we collect these propagators in appendix A, and use them in appendix B for obtaining the Seeley-DeWitt coefficients of eq. (6) with a simple perturbative path integral calculation.

Fig. 2. Loop with average position x_0 (SI).

The previous set-up is easily generalized by coupling the model to curved space and to non-abelian gauge fields. For the latter, the simplest strategy requires the use of a time ordering prescription to exponentiate the action with the matrix-valued gauge field, a method already employed in [11]. More elaborate methods that avoid the time ordering are also available [24], and could be used as well. More general ways of factorizing the zero mode x_0 of the periodic functions $x(t)$ can be found in [25] and [26].

3. Insertion of a first-order operator

In this section, we consider the insertion of a first-order differential operator inside the trace of the heat kernel and construct a path integral representation for it.

To start with, let us consider a more general hamiltonian $\mathcal{R}$, with a non-abelian connection A_μ and a matrix-valued scalar potential V

$$\mathcal{R} = -\frac{1}{2}\nabla^2 + V, \qquad \nabla_\mu = \partial_\mu + A_\mu \ . \tag{15}$$

The corresponding matrix-valued euclidean action for the point particle of coordinates $x^\mu(t)$ reads

$$S[x] = \int_0^\beta dt \left(\frac{1}{2}\dot{x}^\mu \dot{x}_\mu + \dot{x}^\mu A_\mu(x) + V(x) \right) \tag{16}$$

and its exponential appears in the path integral with a time ordering. The latter guarantees gauge covariance as in the standard construction of Wilson lines. The heat kernel is thus computed by the path integral on the particle coordinates $x^\mu(t)$ as

$$e^{-\beta\mathcal{R}} = \int Dx(t)\,\mathrm{T}\,e^{-S[x(t)]} \tag{17}$$

where T denotes the time ordering along the worldline parametrized by t: upon the expansion of the exponential one should place the matrices associated with earlier times on the right of those associated with later times. The trace of the heat kernel is computed by periodic boundary conditions with period β, $x^\mu(\beta) = x^\mu(0)$, and further implementing a finite dimensional trace (denoted by "tr") on the vector space where the matrix-valued potentials A_μ and V act upon

$$\mathrm{Tr}\left[e^{-\beta\mathcal{R}}\right] = \mathrm{tr} \int_{PBC} Dx(t)\,\mathrm{T}\,e^{-S[x(t)]} \ . \tag{18}$$

Next, we would like to insert on the left-hand side an operator of the form

$$J = \left[\xi^\mu(x)\partial_\mu + \sigma(x)\right] \tag{19}$$

where $\xi^\mu(x)$ is an arbitrary vector field (we have in mind applications to diffeomorphism anomalies) and $\sigma(x)$ a matrix-valued function that we will choose appropriately to simplify the relation

with the path integral and keep gauge invariance manifest. The last contribution can be modified at will by adding a standard heat kernel trace with the insertion of a matrix-valued scalar function.

As in the previous section, we modify the action and the hamiltonian by adding a source so that a derivative on its coupling constant creates an insertion. The source term in the action must have a coupling to $\xi^\mu(x)$, which can be considered as an abelian gauge field, so we deform the action as

$$S[x] \to S_\lambda[x] = \int_0^\beta dt \left(\frac{1}{2} \dot{x}^\mu \dot{x}_\mu + \dot{x}^\mu A_\mu(x) + V(x) + \lambda \dot{x}^\mu \xi_\mu(x) \right) \tag{20}$$

where λ is a coupling constant. By going through the canonical formalism, one finds that the hamiltonian corresponding to the previous euclidean action is given by

$$H = \frac{1}{2} \pi^2 + V(x) \tag{21}$$

where the covariant momentum

$$\pi_\mu = p_\mu - i A_\mu(x) - i\lambda \xi_\mu(x) \tag{22}$$

becomes a covariant derivative $\overline{\nabla}_\mu$ upon quantization

$$\pi_\mu \quad \to \quad -i\overline{\nabla}_\mu = -i(\partial_\mu + A_\mu(x) + \lambda \xi_\mu(x)) . \tag{23}$$

Fixing the ordering ambiguities to maintain gauge covariance, one finds a corresponding quantum hamiltonian

$$\mathcal{R}_\lambda = -\frac{1}{2} \overline{\nabla}^2 + V = \mathcal{R} - \lambda \left(\xi^\mu(x)\nabla_\mu + \frac{1}{2}(\partial_\mu \xi^\mu(x)) \right) - \frac{\lambda^2}{2} \xi^2(x) \tag{24}$$

and a deformed version of (18) may be written down

$$\mathrm{Tr}\left[e^{-\beta \mathcal{R}_\lambda} \right] = \mathrm{tr} \int_{PBC} Dx \, \mathrm{T} e^{-S_\lambda[x]} . \tag{25}$$

Taking a λ-derivative on both sides, and setting $\lambda = 0$, one finds on the left-hand side the insertion of the operator

$$\xi^\mu \nabla_\mu + \frac{1}{2}(\partial_\mu \xi^\mu) \tag{26}$$

and on the right-hand side its path integral realization

$$\mathrm{Tr}\left[\left(\xi^\mu \nabla_\mu + \frac{1}{2}(\partial_\mu \xi^\mu) \right) e^{-\beta \mathcal{R}} \right] = \mathrm{tr} \int_{PBC} Dx \left(-\frac{1}{\beta} \int_0^\beta dt \, \dot{x}^\mu \xi_\mu(x) \right) \mathrm{T} e^{-S[x]} \tag{27}$$

which is the formula we were looking for.

The insertion on the path integral side may be simplified by using time translation invariance on the worldline, and one may substitute

$$-\frac{1}{\beta} \int_0^\beta dt \, \frac{dx^\mu(t)}{dt} \xi_\mu(x(t)) \qquad \to \qquad -\frac{dx^\mu(0)}{dt} \xi_\mu(x(0)) \tag{28}$$

with the insertion evaluated at a point of the loop, chosen here as the initial point. In the DBC method for evaluating the path integral one can use $x^\mu(0) = x_0^\mu$, see Fig. 1. In the SI method one will have to set $x^\mu(0) = x_0^\mu + q^\mu(0)$ instead, see Fig. 2.

To summarize, we have found that computing with the path integral the expectation value of

$$-\frac{1}{\beta} \int\limits_0^\beta dt \, \dot{x}^\mu \xi_\mu(x) \qquad \text{or} \qquad -\frac{dx^\mu(0)}{dt} \xi_\mu(x(0)) \tag{29}$$

creates an insertion of the operator (19) with the matrix-valued function $\sigma(x)$ fixed to be

$$\sigma(x) = \xi^\mu A_\mu + \frac{1}{2}(\partial_\mu \xi^\mu) \,. \tag{30}$$

We will study generalized heat kernel coefficients corresponding to this particular insertion. Other forms of $\sigma(x)$ can be easily worked out.

4. Generalized heat kernel coefficients

Having found a path integral representation of the trace of the heat kernel with the insertion of a first-order differential operator, we evaluate the corresponding heat kernel coefficients by using the perturbative expansion in β of the path integral. It takes the form

$$\text{Tr}\left[\left(\xi^\mu \nabla_\mu + \frac{1}{2}(\partial_\mu \xi^\mu)\right) e^{-\beta \mathcal{R}}\right] = \int \frac{d^D x}{(2\pi\beta)^{\frac{D}{2}}} \sum_{n=0}^\infty b_n(x)\beta^n \tag{31}$$

where the $b_n(x)$ are the generalized heat kernel coefficients which include at the linear order the abelian vector field ξ^μ. For the operator $\mathcal{R}$ in (15) we use the action in (16) and compute up to order β^3 to find

$$b_0 = 0$$

$$b_1 = \frac{1}{24} \text{tr} \, G^{\mu\nu} F_{\mu\nu}$$

$$b_2 = \frac{1}{480} \text{tr}\left[\partial^2 G^{\mu\nu} F_{\mu\nu} - 20 G^{\mu\nu} F_{\mu\nu} V\right] \tag{32}$$

$$b_3 = \frac{1}{1440} \text{tr}\left[\frac{3}{28} \partial^4 G^{\mu\nu} F_{\mu\nu} + \frac{5}{4} G^{\mu\nu} F_{\mu\nu} F^2 + G^{\mu\nu} F_{\nu\rho} F^{\rho\lambda} F_{\lambda\mu} + 30 G^{\mu\nu} F_{\mu\nu} V^2 \right.$$
$$\left. - \left(6 G^{\mu\nu} \nabla^2 F_{\mu\nu} + 6 \partial^2 G^{\mu\nu} F_{\mu\nu} + 8 \partial^\lambda G^{\mu\nu} \nabla_\lambda F_{\mu\nu} + 2 \partial_\mu G^{\mu\nu} \nabla^\lambda F_{\lambda\nu} + 6 G^{\mu\nu} F_{\mu\lambda} F^\lambda{}_\nu\right) V\right]$$

where $G_{\mu\nu} = \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$ is the abelian field strength, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ the non-abelian field strength, and ∇_μ the covariant derivative of A_μ. Of course, $G_{\mu\nu}$ could be taken out of the color trace "tr". These coefficients are up to total derivatives, and we have freed V from derivatives.

The coefficient b_1 was given in [11] and b_2 in [12], both including their coupling to gravity. The coefficient b_3 is new, as far as we know. In the next sections, we describe their explicit evaluation through the perturbative expansion of the path integral.

We have used an abelian vector field ξ^μ, which allows for simplifications in the above formulae. For example, in b_1 the term $G_{\mu\nu}$ can be taken out of the color trace so that only the abelian part of $F_{\mu\nu}$ survives the trace. Similarly, one may simplify the other coefficients, or write them in

equivalent ways. These coefficients may also be generalized by considering a non-abelian vector field ξ^μ, as in [12], but we have chosen to keep it abelian for a direct application to the anomalies in the conservation of the stress tensor.

5. Perturbative expansion

We now study the perturbative expansion of the path integral with PBC, i.e. considering worldlines with the topology of a circle. Since the kinetic operator cannot be inverted on the circle, one has to factor out a zero mode x_0^μ and split the path integration variable as

$$x^\mu(t) = x_0^\mu + q^\mu(t). \tag{33}$$

This can be done using either the DBC method or the SI one, as explained earlier.

To start with we rescale the time $t \to \tau = \frac{t}{\beta}$, so that $\tau \in [0, 1]$, and we find the following path integral representation of the trace

$$
\begin{aligned}
&\mathrm{Tr}\left[\left(\xi^\mu \nabla_\mu + \frac{1}{2}(\partial_\mu \xi^\mu)\right) e^{-\beta \mathcal{R}}\right] \\
&= \int d^D x_0 \, \mathrm{tr} \int_{DBC} Dq \left(-\frac{1}{\beta}\int_0^1 d\tau \, \dot{q}^\mu \xi_\mu(x_0+q)\right) \mathrm{T}\, e^{-S[x_0+q]} \\
&= \int \frac{d^D x_0}{(2\pi\beta)^{\frac{D}{2}}} \, \mathrm{tr}\left\langle \left(-\frac{1}{\beta}\int_0^1 d\tau \, \dot{q}^\mu \xi_\mu(x_0+q)\right) \mathrm{T}\, e^{-S_{int}[x_0+q]}\right\rangle
\end{aligned}
\tag{34}
$$

where dots indicate derivative with respect to τ, and angle brackets denote normalized averages with the free action, $\langle 1 \rangle = 1$. The expectation values are to be computed by Wick contracting with the propagators in appendix A, and with the interaction vertex taking the form

$$S_{int}[x_0+q] = \int_0^1 d\tau \left(\dot{q}^\mu A_\mu(x_0+q) + \beta V(x_0+q)\right). \tag{35}$$

We now start computing at order β to get b_1. There are two contributions. The first one comes from Taylor expanding ξ_μ and A_μ to first order in q^μ, and gives for the right-hand side of (34)

$$\int \frac{d^D x_0}{(2\pi\beta)^{\frac{D}{2}}} \, \mathrm{tr}\left\langle \left(-\frac{1}{\beta}\partial_\nu \xi_\mu(x_0)\int_0^1 d\tau \, q^\nu(\tau)\dot{q}^\mu(\tau)\right)\left(-\partial_\beta A_\alpha(x_0)\int_0^1 d\tau' \, q^\beta(\tau')\dot{q}^\alpha(\tau')\right)\right\rangle. \tag{36}$$

Time ordering is not needed and the disconnected Wick contractions vanish

$$\mathbf{F}_1 = \bigcirc = \int_0^1 d\tau \, \Delta^\bullet(\tau, \tau) = 0. \tag{37}$$

The remaining connected correlation function gives

$$\langle q^\nu(\tau)\dot{q}^\mu(\tau)q^\beta(\tau')\dot{q}^\alpha(\tau')\rangle_c = \beta^2\left(\delta^{\nu\beta}\delta^{\mu\alpha}\,\Delta(\tau,\tau')\,{}^\bullet\Delta^\bullet(\tau,\tau') + \delta^{\nu\alpha}\delta^{\mu\beta}\,\Delta^\bullet(\tau,\tau')\,{}^\bullet\Delta(\tau,\tau')\right) \tag{38}$$

where the first term corresponds to a worldline Feynman diagram of the form

$$\mathbf{F}_2 = \begin{gathered}\includegraphics{fig}\end{gathered} = \int_0^1 d\tau \int_0^1 d\tau'\,\Delta(\tau,\tau')\,{}^\bullet\Delta^\bullet(\tau,\tau') = \frac{1}{12} \tag{39}$$

and the second one to a diagram of the form

$$\mathbf{F}_3 = \begin{gathered}\includegraphics{fig}\end{gathered} = \int_0^1 d\tau \int_0^1 d\tau'\,\Delta^\bullet(\tau,\tau')\,{}^\bullet\Delta(\tau,\tau') = -\frac{1}{12}\,. \tag{40}$$

In drawing Feynman diagram we denote vertices by black dots and derivatives by white circles on the legs. Integration by parts relates the two integrals, $\mathbf{F}_2 = -\mathbf{F}_3$, hinting at gauge invariance. The above values are obtained using equivalently the DBC or the SI propagators. In the latter case one may use translational invariance to eliminate one integration. Thus, the trace inside (36) reduces to

$$\beta\,\mathrm{tr}\,\partial^\nu\xi^\mu(x_0)\left(\partial_\nu A_\mu(x_0)\mathbf{F}_2 + \partial_\mu A_\nu(x_0)\mathbf{F}_3\right) = \frac{\beta}{12}\mathrm{tr}\,\partial^\nu\xi^\mu(x_0)\left(\partial_\nu A_\mu(x_0) - \partial_\mu A_\nu(x_0)\right). \tag{41}$$

A second term of the same order in β arises from considering two interaction vertices and has the effect of completing the non-abelian gauge invariance. Keeping the leading term of the Taylor expansion of the non-abelian potential inside $\frac{1}{2}S_{int}^2[x_0+q]$ one finds

$$\frac{1}{2}\,\mathrm{T}\int_0^1 d\tau_1\dot{q}^\alpha(\tau_1)A_\alpha(x_0)\int_0^1 d\tau_2\,\dot{q}^\beta(\tau_2)A_\beta(x_0)\,. \tag{42}$$

The time ordering is implemented explicitly as

$$\frac{1}{2}A_\alpha(x_0)A_\beta(x_0)\int_0^1 d\tau_1\int_0^{\tau_1} d\tau_2\,\dot{q}^\alpha(\tau_1)\dot{q}^\beta(\tau_2) + \frac{1}{2}A_\beta(x_0)A_\alpha(x_0)\int_0^1 d\tau_1\int_{\tau_1}^1 d\tau_2\,\dot{q}^\alpha(\tau_1)\dot{q}^\beta(\tau_2) \tag{43}$$

and simplified using a Heaviside step function (and renaming integration variables)

$$A_\alpha(x_0)A_\beta(x_0)\int_0^1 d\tau_1\int_0^1 d\tau_2\,\dot{q}^\alpha(\tau_1)\dot{q}^\beta(\tau_2)\theta(\tau_1 - \tau_2)\,. \tag{44}$$

Inserted into the right-hand side of (34) it leads to

$$\frac{(-1)}{\beta}\mathrm{tr}\,\partial_\nu\xi_\mu(x_0)A_\alpha(x_0)A_\beta(x_0)\int_0^1 d\tau\int_0^1 d\tau_1\int_0^1 d\tau_2\,\langle\dot{q}^\mu(\tau)q^\nu(\tau)\dot{q}^\alpha(\tau_1)\dot{q}^\beta(\tau_2)\rangle\,\theta(\tau_1 - \tau_2) \tag{45}$$

with nonvanishing contractions that produce the following integrals

$$
\int_0^1 d\tau \int_0^1 d\tau_1 \int_0^1 d\tau_2 \, {}^{\bullet}\Delta^{\bullet}(\tau, \tau_1) \Delta^{\bullet}(\tau, \tau_2) \theta(\tau_1 - \tau_2) = \frac{1}{12}
$$

$$
\int_0^1 d\tau \int_0^1 d\tau_1 \int_0^1 d\tau_2 \, {}^{\bullet}\Delta^{\bullet}(\tau, \tau_2) \Delta^{\bullet}(\tau, \tau_1) \theta(\tau_1 - \tau_2) = -\frac{1}{12}
$$

(46)

independently of the propagator used. At the end one finds a commutator term

$$
\frac{\beta}{12} \mathrm{tr} \, \partial^\nu \xi^\mu(x_0) [A_\nu(x_0), A_\mu(x_0)] .
$$

(47)

These are all the terms of order β. Summing (41) and (47) one finds for (34)

$$
\int \frac{d^D x_0}{(2\pi\beta)^{\frac{D}{2}}} \, \mathrm{tr} \, \frac{\beta}{12} \partial^\nu \xi^\mu(x_0) F_{\nu\mu}(x_0) = \int \frac{d^D x_0}{(2\pi\beta)^{\frac{D}{2}}} \, \mathrm{tr} \, \frac{\beta}{24} G^{\mu\nu}(x_0) F_{\mu\nu}(x_0)
$$

(48)

which delivers the coefficient b_1 of eq. (32). $G_{\mu\nu}$ is the abelian field strength of ξ_μ and can be taken out of the trace, showing that only the abelian part of $F_{\mu\nu}$ contributes. The time ordered diagram that leads to the commutator in (47) does not survive the trace, but we have presented it to exemplify the role of the time ordering prescription.

As noted, the SI propagators are explicitly translational invariant and in the perturbative expansion one may eliminate an integration of the Feynman diagrams, fixing for example the insertion at $\tau = 0$. In the DBC method, translational invariance on the circle can be used as well. However, the calculation proceeds somewhat differently. One uses translational invariance to fix the insertion at $\tau = 0$ and identifies $x^\mu(0) = x_0^\mu$ (since $q^\mu(0) = 0$ by DBC). Then, eq. (34) simplifies to

$$
\mathrm{Tr}\left[\left(\xi^\mu \nabla_\mu + \frac{1}{2}(\partial_\mu \xi^\mu)\right) e^{-\beta\mathcal{R}}\right] = \int \frac{d^D x_0}{(2\pi\beta)^{\frac{D}{2}}} \, \xi_\mu(x_0) \frac{(-1)}{\beta} \mathrm{tr} \left\langle \dot{q}^\mu(0) \mathrm{T} \, e^{-S_{int}[x_0+q]} \right\rangle_{DBC}
$$

(49)

which delivers the result with the vector field $\xi_\mu(x_0)$ explicitly factored out. To evaluate the same coefficient b_1 in this set-up, one needs to expand A_μ to higher orders. The calculation is simplified by using the Fock-Schwinger gauge (more on this later)

$$
A_\nu(x_0 + q) = A_\nu(x_0) + \frac{1}{2} q^\rho F_{\rho\nu}(x_0) + \frac{1}{3} q^\rho q^\sigma \nabla_\sigma F_{\rho\nu}(x_0) + ...
$$

(50)

to find

$$
\frac{(-1)}{\beta} \left\langle \dot{q}^\mu(0) \mathrm{T} \, e^{-S_{int}[x_0+q]} \right\rangle_{DBC} = \frac{1}{\beta} \left\langle \dot{q}^\mu(0) S_{int}[x_0+q] + \cdots \right\rangle_{DBC}
$$

$$
= \frac{1}{3\beta} \nabla_\sigma F_{\rho\nu}(x_0) \int_0^1 d\tau \, \langle \dot{q}^\mu(0) \dot{q}^\nu(\tau) q^\rho(\tau) q^\sigma(\tau) \rangle_{DBC}
$$

(51)

$$
= \frac{\beta}{3} \nabla_\nu F^{\nu\mu}(x_0)(\mathbf{G}_1 - \mathbf{G}_2) = -\frac{\beta}{12} \nabla_\nu F^{\nu\mu}(x_0)
$$

where the integrals with the DBC propagators give

$$\mathbf{G}_1 = \text{⨯⊶⊶}\bigcirc = \int_0^1 d\tau \, {}^\bullet\Delta^\bullet(0,\tau)\Delta(\tau,\tau) = -\frac{1}{6} \qquad (52)$$

$$\mathbf{G}_2 = \text{⨯⊶}\bigcirc = \int_0^1 d\tau \, {}^\bullet\Delta(0,\tau)\Delta^\bullet(\tau,\tau) = \frac{1}{12}$$

(the cross in the vertex singles out the vertex without integration). Considering the color trace in (49) one may substitute the covariant derivative with the standard derivative and notice that only the abelian part of $F^{\mu\nu}$ survives the trace, so that by setting $b_1 = b_{1\mu}\xi^\mu$ one finds

$$b_{1\mu} = -\frac{1}{12}\,\mathrm{tr}\,\partial^\nu F_{\nu\mu} \qquad (53)$$

that indeed reproduces b_1 after integrating by part inside (49).

The coefficient b_1 was known from ref. [11], where it was obtained by computing the heat kernel trace with plane waves, but presented in a form that did not show manifest gauge invariance. The previous calculation verifies the consistency of the path integral method, and one may proceed with confidence to evaluate higher order coefficients.

6. Higher order coefficients

To get additional coefficients one needs to push the perturbative expansion in β to higher orders. To proceed faster, we use gauge invariance and select the Fock-Schwinger (FS) gauge

$$q^\mu(\tau)A_\mu(x_0 + q(\tau)) = 0 \qquad (54)$$

which allows to expand the gauge potential in terms of its curvature (and derivatives thereof) evaluated at x_0 (see for example [27])

$$A_\mu(x_0+q) = A_\mu(x_0) + \frac{1}{2}q^\nu F_{\nu\mu}(x_0) + \frac{1}{3}q^\nu q^\rho \nabla_\rho F_{\nu\mu}(x_0) + \frac{1}{8}q^\nu q^\rho q^\sigma \nabla_\sigma \nabla_\rho F_{\nu\mu}(x_0) + \dots \,. \qquad (55)$$

A similar gauge holds also for $\xi_\mu(x)$

$$\xi_\mu(x_0+q) = \xi_\mu(x_0) + \frac{1}{2}q^\nu G_{\nu\mu}(x_0) + \frac{1}{3}q^\nu q^\rho \partial_\rho G_{\nu\mu}(x_0) + \frac{1}{8}q^\nu q^\rho q^\sigma \partial_\sigma \partial_\rho G_{\nu\mu}(x_0) + \dots \,. \qquad (56)$$

Next, we give some details on the calculation of the higher order coefficients b_2 and b_3.

To identify b_2 we need terms with one vertex (from S_{int}) and two vertices (from S_{int}^2). Substituting the potentials in the FS gauge, both in the insertion and in the vertices, we get the following contribution of order β^2 to eq. (34)

$$\int \frac{d^D x_0}{(2\pi\beta)^{\frac{D}{2}}}\,\mathrm{tr}\Big(A_1(x_0) + A_2(x_0) + A_3(x_0) + A_4(x_0)\Big) \qquad (57)$$

where the single vertex produces[1]

---

[1] We now use a compact notation, indicating $q_0 \equiv q(\tau_0)$, $\int_{01} \equiv \int_0^1 d\tau_0 \int_0^1 d\tau_1$, etc.

$$A_1(x_0) = \frac{1}{16\beta} \int\limits_{01} \langle \dot{q}_0^\mu q_0^\nu \dot{q}_1^\alpha q_1^\beta q_1^\gamma q_1^\delta \rangle G_{\nu\mu}(x_0) \nabla_\delta \nabla_\gamma F_{\beta\alpha}(x_0) \tag{58}$$

$$A_2(x_0) = \frac{1}{9\beta} \int\limits_{01} \langle \dot{q}_0^\mu q_0^\nu q_0^\rho \dot{q}_1^\alpha q_1^\beta q_1^\gamma \rangle \partial_\rho G_{\nu\mu}(x_0) \nabla_\gamma F_{\beta\alpha}(x_0) \tag{59}$$

$$A_3(x_0) = \frac{1}{16\beta} \int\limits_{01} \langle \dot{q}_0^\mu q_0^\nu q_0^\rho q_0^\sigma \dot{q}_1^\alpha q_1^\beta \rangle \partial_\sigma \partial_\rho G_{\nu\mu}(x_0) F_{\beta\alpha}(x_0) \tag{60}$$

while the two interaction vertices contribute with

$$A_4(x_0) = -\frac{1}{4} \int\limits_{01} \langle \dot{q}_0^\mu q_0^\nu \dot{q}_1^\alpha q_1^\beta \rangle G_{\nu\mu}(x_0) F_{\beta\alpha}(x_0) V(x_0) . \tag{61}$$

In this last term we have used cyclicity of the trace in (57) to eliminate the time ordering. Then, it describes a disconnected contribution that embeds b_1, so it is immediately evaluated to

$$A_4 = -\frac{\beta^2}{24} G^{\mu\nu} F_{\mu\nu} V . \tag{62}$$

As for the remaining terms, since they enter eq. (57), we simplify them with integration by parts (covariant derivatives acting on $F_{\mu\nu}$ become usual derivatives acting on $G_{\mu\nu}$). Then, renaming the time variables we get

$$A_1 + A_3 = \frac{1}{16\beta} \int\limits_{01} \langle \dot{q}_0^\mu q_0^\nu \dot{q}_1^\alpha q_1^\beta q_1^\gamma q_1^\delta \rangle \left(\partial_\gamma \partial_\delta G_{\nu\mu} F_{\beta\alpha} + \partial_\gamma \partial_\delta G_{\beta\alpha} F_{\nu\mu} \right) \tag{63}$$

and performing the Wick contractions (and also integrating by parts on the worldline to get rid of ${}^\bullet\Delta_{01}^\bullet$ in the Feynman diagrams) we get

$$A_1 + A_3 = \frac{\beta^2}{4} \mathbf{H}_1 (\partial^2 G^{\mu\nu} F_{\mu\nu} + 2\partial^\alpha \partial_\beta G^{\beta\nu} F_{\alpha\nu}) = \frac{\beta^2}{2} \mathbf{H}_1 \partial^2 G^{\mu\nu} F_{\mu\nu} \tag{64}$$

where $\mathbf{H}_1$ is the Feynman diagram

$$\mathbf{H}_1 = \;\;\;\;\;\; = \int\limits_{01} {}^\bullet\Delta_{01} \Delta_{01}^\bullet \Delta_{11} = \begin{cases} \frac{1}{60} & DBC \\ \frac{1}{144} & SI . \end{cases} \tag{65}$$

We proceed similarly with A_2 to get

$$A_2 = -\frac{1}{9\beta} \int\limits_{01} \langle \dot{q}_0^\mu q_0^\nu q_0^\rho \dot{q}_1^\alpha q_1^\beta q_1^\gamma \rangle \partial_\gamma \partial_\rho G_{\nu\mu} F_{\beta\alpha}$$

$$= \frac{\beta^2}{9} \partial^2 G^{\mu\nu} F_{\mu\nu} \left(-\frac{9}{2} \mathbf{H}_2 + \frac{1}{2} \mathbf{H}_3 - \frac{1}{4} \mathbf{H}_4 + 2\mathbf{H}_5 \right) \tag{66}$$

where

$$\mathbf{H}_2 = \text{} = \int_{01} {}^\bullet\!\Delta_{01}\Delta_{01}\Delta_{01}^\bullet = \begin{cases} \frac{1}{90} & DBC \\ \frac{1}{360} & SI \end{cases}$$

$$\mathbf{H}_3 = \text{} = \int_{01} \Delta_{00}{}^\bullet\!\Delta_{01}^\bullet\Delta_{11} = \begin{cases} -\frac{1}{180} & DBC \\ 0 & SI \end{cases}$$

$$\mathbf{H}_4 = \text{} = \int_{01} \Delta_{00}{}^\bullet\!\Delta_{01}\Delta_{11}^\bullet = \begin{cases} \frac{1}{360} & DBC \\ 0 & SI \end{cases} \tag{67}$$

$$\mathbf{H}_5 = \text{} = \int_{01} {}^\bullet\!\Delta_{00}\Delta_{01}\Delta_{11}^\bullet = \begin{cases} -\frac{1}{720} & DBC \\ 0 & SI \, . \end{cases}$$

Adding all terms up we get

$$A_1 + A_2 + A_3 = \beta^2 \partial^2 G^{\mu\nu} F_{\mu\nu} \Big(\frac{1}{2}\mathbf{H}_1 - \frac{1}{2}\mathbf{H}_2 + \frac{1}{18}\mathbf{H}_3 - \frac{1}{36}\mathbf{H}_4 + \frac{2}{9}\mathbf{H}_5 \Big) = \frac{\beta^2}{480} \partial^2 G^{\mu\nu} F_{\mu\nu}$$

independently of the worldline propagators used. Including A_4 we obtain the generalized coefficient b_2

$$b_2 = \frac{1}{480} \operatorname{tr} \Big[\partial^2 G^{\mu\nu} F_{\mu\nu} - 20 G^{\mu\nu} F_{\mu\nu} V \Big] \tag{68}$$

which correctly reproduces the one reported in [12] (with abelian ξ^μ and in flat space).

Finally, we wish to compute the coefficient b_3, which did not appear in the literature so far. It is more laborious, so we just present the calculation of a single term, i.e. the first one inside b_3 of eq. (32), dumping details on the calculation of the remaining part into appendix C. This term receives contributions from the $F_{\mu\nu}$ dependence of a single S_{int} vertex insertion, which read

$$B_1(x_0) = \frac{1}{288\beta} \int_{01} \langle \dot{q}_0^\mu q_0^\nu \dot{q}_1^\alpha q_1^\beta q_1^\gamma q_1^\delta q_1^\epsilon q_1^\eta \rangle \, G_{\nu\mu} \nabla_\eta \nabla_\epsilon \nabla_\delta \nabla_\gamma F_{\beta\alpha} \tag{69}$$

$$B_2(x_0) = \frac{1}{90\beta} \int_{01} \langle \dot{q}_0^\mu q_0^\nu q_0^\rho \dot{q}_1^\alpha q_1^\beta q_1^\gamma q_1^\delta q_1^\epsilon \rangle \, \partial_\rho G_{\nu\mu} \nabla_\epsilon \nabla_\delta \nabla_\gamma F_{\beta\alpha} \tag{70}$$

$$B_3(x_0) = \frac{1}{64\beta} \int_{01} \langle \dot{q}_0^\mu q_0^\nu q_0^\rho q_0^\sigma \dot{q}_1^\alpha q_1^\beta q_1^\gamma q_1^\delta \rangle \, \partial_\sigma \partial_\rho G_{\nu\mu} \nabla_\delta \nabla_\gamma F_{\beta\alpha} \tag{71}$$

$$B_4(x_0) = \frac{1}{90\beta} \int_{01} \langle \dot{q}_0^\mu q_0^\nu q_0^\rho q_0^\sigma q_0^\tau \dot{q}_1^\alpha q_1^\beta q_1^\gamma \rangle \, \partial_\tau \partial_\sigma \partial_\rho G_{\nu\mu} \nabla_\gamma F_{\beta\alpha} \tag{72}$$

$$B_5(x_0) = \frac{1}{288\beta} \int_{01} \langle \dot{q}_0^\mu q_0^\nu q_0^\rho q_0^\sigma q_0^\tau q_0^\lambda \dot{q}_1^\alpha q_1^\beta \rangle \, \partial_\lambda \partial_\tau \partial_\sigma \partial_\rho G_{\nu\mu} F_{\beta\alpha} \, . \tag{73}$$

As they are integrated in spacetime, see eq. (34), we integrate by parts the covariant derivatives from $F_{\beta\alpha}$ to $G_{\nu\mu}$, where they become standard derivatives. Then collecting identical Wick contractions we get

$$B_1 + B_5 = -\frac{\beta^3}{8} \partial^4 G^{\mu\nu} F_{\mu\nu} \mathbf{I}_1$$

$$B_2 + B_4 = -\frac{\beta^3}{30} \partial^4 G^{\mu\nu} F_{\mu\nu} \left(-15\mathbf{I}_2 - 3\mathbf{I}_3 + \mathbf{I}_4 - \mathbf{I}_5\right) \tag{74}$$

$$B_3 = -\frac{\beta^3}{8} \partial^4 G^{\mu\nu} F_{\mu\nu} \left(2\mathbf{I}_6 + \mathbf{I}_7 + 2\mathbf{I}_8\right)$$

where the integrals corresponding to the worldline Feynman diagrams are listed in appendix C. Adding these terms together we get the following contribution to b_3

$$\frac{1}{1440} \int \frac{d^D x}{(2\pi\beta)^{\frac{D}{2}}} \mathrm{tr} \frac{3}{28} \partial^4 G^{\mu\nu} F_{\mu\nu} \,. \tag{75}$$

In appendix C we report details on the calculation of the other terms contributing to b_3.

7. Conclusions

We have studied path integral methods to compute heat kernel traces with insertion of a first-order differential operator. We have considered hamiltonians $\mathcal{R}$ with couplings to non-abelian gauge fields and matrix-valued potentials only. The coupling to a curved space metric is however straightforward, as path integrals on curved spaces are well-studied by now [9]. The insertion of a first-order differential operator into the trace of the heat kernel has been obtained by modifying the hamiltonian with a source coupled to the first-order operator, and then varying the source. This procedure translates then into the path integral representation of the desired trace, which we have used to calculate the first three generalized heat kernel coefficients. Alternatively, one could have applied the variational procedure directly to the standard heat kernel coefficients, as the source term is structurally similar to the gauge coupling already present in the hamiltonian. Indeed, this was the method followed in [12]. In the present case our results can be checked, and further extended to reach b_4 and b_5, by a gauge variation of the coefficients already calculated with worldline methods in [25]. We have verified the correctness of our calculation this way as well.

Our interest in these particular traces stems from a desire to compute the anomalies in the conservation of the stress tensor, which appear in four dimensions if one uses regularization schemes that are not symmetric enough. Such a situation emerged in the study of a Weyl fermion in a U(1) background once regulated with Pauli-Villars fields with Dirac mass [13]. The gravitational anomaly emerging in this scheme was calculated using generalized heat kernel coefficients in [28]. The study of the anomaly structure of chiral fermions in four dimensions has become recently of renewed interest, in particular regarding the trace anomaly. The latter has been scrutinized from various perspectives [29,13,30–33] to verify the absence of the Pontryagin topological density (in curved space) or Chern-Pontryagin topological density (for couplings to gauge fields). The presence of these topological densities was conjectured to be a possibility in [14] (see also [34] for a supersymmetric extension of the conjecture), and the analyses of refs. [35–37] claimed their existence in the trace anomaly of a Weyl fermion in curved space. It seems useful to consider these issues even within regularization schemes that induce anomalies in the conservation of the stress tensor.

The methods presented here may be considered as part of a general strategy of using worldline path integrals to obtain field theoretical results in flat [23] and curved space [38], a strategy often referred to as the worldline formalism. These methods are quite efficient from a calculational point of view, and it seems worthwhile to extend their development and applications.

CRediT authorship contribution statement

All authors have contributed with equal weight on the various steps involved in the research reported in the manuscript.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A. Worldline propagators

For perturbative computations in β we find it useful to rescale the time $t \to \tau = \frac{t}{\beta}$, so that $\tau \in [0, 1]$. Then, β appear explicitly as a perturbative parameter multiplying suitably the various terms of the action (7), which takes the form

$$S[x(\tau)] = \int_0^1 d\tau \left(\frac{1}{2\beta} \dot{x}^\mu \dot{x}_\mu + \beta V(x) \right) \tag{76}$$

where now $\dot{x}^\mu \equiv \frac{dx^\mu}{d\tau}$. We consider periodic boundary conditions for $x^\mu(\tau)$, appropriate for creating a functional trace in the path integral. The kinetic term identifies the propagator, which then carries a power of β, while the potential term is treated perturbatively. Setting

$$x^\mu(\tau) = x_0^\mu + q^\mu(\tau) \tag{77}$$

one finds a perturbative propagator for the quantum field $q^\mu(\tau)$ of the form

$$\langle q^\mu(\tau) q^\nu(\tau') \rangle = -\beta \delta^{\mu\nu} \Delta(\tau, \tau') \tag{78}$$

where $\Delta(\tau, \tau')$ is the Green function of the operator $\frac{d^2}{d\tau^2}$ that depends on the boundary conditions and the way the zero mode x_0^μ is factored out (recall that the differential operator $\frac{d^2}{d\tau^2}$ is not invertible on the space of periodic functions, the constant function has zero eigenvalue and constitutes a zero mode of the operator).

Dirichlet boundary conditions
Using the Dirichlet boundary conditions (DBC), $q^\mu(0) = q^\mu(1) = 0$, one finds for $\Delta(\tau, \tau')$

$$\begin{aligned} \Delta_D(\tau, \tau') &= (\tau - 1)\tau' \theta(\tau - \tau') + (\tau' - 1)\tau \, \theta(\tau' - \tau) \\ &= \frac{1}{2}|\tau - \tau'| - \frac{1}{2}(\tau + \tau') + \tau\tau' \end{aligned} \tag{79}$$

with the step function $\theta(\tau)$ defined such that $\theta(0) = \frac{1}{2}$. It satisfies

$$\frac{d^2}{d\tau^2} \Delta_D(\tau, \tau') = \delta(\tau - \tau') \tag{80}$$

where the Dirac delta is the one appropriate for functions with vanishing boundary conditions. For later use it is convenient to list the derivatives of the worldline propagator in DBC, where a left/right dot indicates a derivative with respect to the first/second argument

$$
\begin{aligned}
{}^{\bullet}\Delta_D(\tau, \tau') &= \tau' - \theta(\tau' - \tau) \\
\Delta_D^{\bullet}(\tau, \tau') &= \tau - \theta(\tau - \tau') \\
{}^{\bullet}\Delta_D^{\bullet}(\tau, \tau') &= 1 - \delta(\tau - \tau') \\
{}^{\bullet\bullet}\Delta_D(\tau, \tau') &= \delta(\tau - \tau')
\end{aligned}
\tag{81}
$$

with coincident points limits

$$
\begin{aligned}
\Delta_D(\tau, \tau) &= \tau^2 - \tau \\
{}^{\bullet}\Delta_D(\tau, \tau) &= \tau - \frac{1}{2} .
\end{aligned}
\tag{82}
$$

String inspired propagator

The "string inspired" (SI) propagator for the quantum fluctuations $q^\mu(\tau)$ satisfies periodic boundary conditions but with the constraint $\int_0^1 d\tau \, q^\mu(\tau) = 0$. Then, one finds for $\Delta(\tau, \tau')$

$$
\Delta_{SI}(\tau, \tau') = \Delta_{SI}(\tau - \tau') = \frac{1}{2}|\tau - \tau'| - \frac{1}{2}(\tau - \tau')^2 - \frac{1}{12}
\tag{83}
$$

which satisfies

$$
\frac{d^2}{d\tau^2}\Delta_{SI}(\tau - \tau') = \delta(\tau - \tau') - 1 .
\tag{84}
$$

It has the useful property of being translational invariant. It is an even function of $\tau - \tau'$, and its first derivative is odd which implies that its coincident points limit vanishes. Here is a list of its properties

$$
\begin{aligned}
{}^{\bullet}\Delta_{SI}(\tau - \tau') &= \frac{1}{2}\mathrm{sgn}(\tau - \tau') - (\tau - \tau') \\
{}^{\bullet\bullet}\Delta_{SI}(\tau - \tau') &= \delta(\tau - \tau') - 1 \\
\Delta_{SI}(0) &= -\frac{1}{12} \\
{}^{\bullet}\Delta_{SI}(0) &= 0
\end{aligned}
\tag{85}
$$

where by $\mathrm{sgn}(x)$ we denote the sign function. Of course ${}^{\bullet}\Delta_{SI}(\tau, \tau') = -\Delta_{SI}^{\bullet}(\tau, \tau')$.

Appendix B. Perturbative expansion and heat kernel coefficients

Here we compute the heat kernel coefficients given in (6) by evaluating the path integral in (11). We present it as a review of worldline methods and to exemplify the equivalence of the DBC and SI methods for treating the zero mode on the circle [25].

Let us first rewrite (11) by factoring out the zero mode integration, and set up the perturbative expansion

$$
\begin{aligned}
\mathrm{Tr}\left[\sigma(x)e^{-\beta\mathcal{R}}\right] &= \int\limits_{PBC} Dx \left(\frac{1}{\beta}\int\limits_0^\beta dt\, \sigma(x(t))\right) e^{-S[x]} \\
&= \int d^D x_0 \int Dq \left(\int\limits_0^1 d\tau\, \sigma(x_0 + q(\tau))\right) e^{-S[x_0 + q]}
\end{aligned}
\tag{86}
$$

$$= \int \frac{d^D x_0}{(2\pi\beta)^{\frac{D}{2}}} \left\langle \int_0^1 d\tau \, \sigma(x_0 + q(\tau)) \, e^{-S_{int}[x_0 + q]} \right\rangle$$

where we have rescaled the time in the insertion and action, with the latter taking the form

$$S[x_0 + q] = S_0[q] + S_{int}[x_0 + q] = \frac{1}{\beta} \int_0^1 d\tau \, \frac{1}{2} \dot{q}^2(\tau) + \beta \int_0^1 d\tau \, V(x_0 + q(\tau)) \,. \qquad (87)$$

Normalized averages with the free path integral are denoted by angle brackets, $\langle 1 \rangle = 1$, and we have extracted the overall normalization constant $\int Dq \, e^{-S_0[q]} = (2\pi\beta)^{-\frac{D}{2}}$.

The perturbative expansion is implemented by Taylor expanding about x_0 the function $\sigma(x)$ and the potential $V(x)$, and further expanding the exponential of the interaction term. Next one computes the correlation functions by Wick contractions. Keeping exponentiated the terms that generate disconnected diagrams, and recalling that the propagators carry a factor of β, we find at order β^3

$$\left\langle \int_0^1 d\tau \, \sigma(x_0 + q) \, e^{-S_{int}[x_0 + q]} \right\rangle$$

$$= e^{-\beta V(x_0)} \left[\sigma(x_0) \left(1 + \frac{\beta^2}{2} \partial^2 V(x_0) \mathbf{J}_1 - \frac{\beta^3}{8} \partial^4 V(x_0) \mathbf{J}_2 - \frac{\beta^3}{2} \partial_\mu V(x_0) \partial^\mu V(x_0) \mathbf{J}_3 \right) \right.$$

$$+ \partial_\mu \sigma(x_0) \left(\beta^2 \partial^\mu V(x_0) \mathbf{J}_3 - \frac{\beta^3}{2} \partial^\mu \partial^2 V(x_0) \mathbf{J}_4 \right)$$

$$+ \partial^2 \sigma(x_0) \left(-\frac{\beta}{2} \mathbf{J}_1 \right) \left(1 + \frac{\beta^2}{2} \partial^2 V(x_0) \mathbf{J}_1 \right)$$

$$+ \partial_\mu \partial_\nu \sigma(x_0) \partial^\mu \partial^\nu V(x_0) \left(-\frac{\beta^3}{2} \mathbf{J}_5 \right)$$

$$+ \partial_\mu \partial^2 \sigma(x_0) \partial^\mu V(x_0) \left(-\frac{\beta^3}{2} \mathbf{J}_4 \right)$$

$$\left. + \partial^4 \sigma(x_0) \frac{\beta^2}{8} \mathbf{J}_2 \right] + O(\beta^4)$$

$$(88)$$

where one should expand the exponential in front by keeping only the powers of β needed to match the chosen perturbative order. The $\mathbf{J}$'s denote the worldline Feynman diagrams, where lines depict propagators and dots denote vertices which include an integration over the time $\tau \in [0, 1]$. They are as follows

$$\mathbf{J}_1 = \bigcirc = \int_0^1 \Delta_{00} = \begin{cases} -\frac{1}{6} & DBC \\ -\frac{1}{12} & SI \end{cases}$$

$$\mathbf{J}_2 = \infty = \int_0^1 \Delta_{00}^2 = \begin{cases} \frac{1}{30} & DBC \\ \frac{1}{144} & SI \end{cases}$$

$$\mathbf{J}_3 = \bullet\!\!-\!\!-\!\!-\!\!\bullet = \int\limits_{01} \Delta_{01} = \begin{cases} -\frac{1}{12} & DBC \\ 0 & SI \end{cases} \tag{89}$$

$$\mathbf{J}_4 = \bullet\!\!-\!\!\bigcirc = \int\limits_{01} \Delta_{01}\Delta_{11} = \begin{cases} \frac{1}{60} & DBC \\ 0 & SI \end{cases}$$

$$\mathbf{J}_5 = \bigcirc = \int\limits_{01} \Delta_{01}^2 = \begin{cases} \frac{1}{90} & DBC \\ \frac{1}{720} & SI\,. \end{cases}$$

We have given their values both in DBC and SI. To verify explicitly the equivalence between DBC and SI, we first plug in (88) in (86), perform an integration by parts to free the function σ from derivatives, and drop total derivative terms. Comparing with (5) we recognize the following Seeley-DeWitt coefficients

$$a_0(x) = 1$$
$$a_1(x) = -V$$
$$a_2(x) = \frac{1}{2}V^2 + (\mathbf{J}_1 - \mathbf{J}_3)\partial^2 V$$
$$a_3(x) = -\frac{1}{6}V^3 + (\mathbf{J}_3 - \mathbf{J}_1)V\partial^2 V + \frac{1}{2}(\mathbf{J}_3 - \mathbf{J}_1)\partial_\mu V\partial^\mu V - \frac{1}{4}(\mathbf{J}_2 + \mathbf{J}_1^2 - 4\mathbf{J}_4 + 2\mathbf{J}_5)\partial^4 V \tag{90}$$

which reproduce those quoted in (6), independently of the propagator used. Note that the manifest translational invariance of the SI method allows to get rid of one of the time integrations in Feynman diagrams. One may use it to fix $\tau = 0$ in the insertion, thus relating (11) to (8).

Alternatively, one could compute eq. (8) directly with the DBC method. The answer is encoded in the second line of eq. (88) (the one proportional to $\sigma(x_0)$), from which one extracts the expected answer.

Appendix C. Evaluation of b_3

Here we give additional details on the evaluation of b_3. First we list the diagrams needed for evaluating the leading term discussed in section 6. The worldline diagrams have been computed both in DBC and SI, which serves as a check on the final result

$$\mathbf{I}_1 = \int\limits_{01} \Delta_{00}^2 {}^{\bullet}\Delta_{01}\Delta_{01}^{\bullet} = \begin{cases} -\frac{1}{280}, DBC \\ -\frac{1}{1728}, SI \end{cases}$$

$$\mathbf{I}_2 = \int\limits_{01} {}^{\bullet}\Delta_{01}\Delta_{01}^{\bullet}\Delta_{01}\Delta_{11} = \begin{cases} -\frac{1}{420}, DBC \\ -\frac{1}{4320}, SI \end{cases}$$

$$\mathbf{I}_3 = \int\limits_{01} {}^{\bullet}\Delta_{00}\Delta_{01}^{\bullet}\Delta_{11}^2 = \begin{cases} -\frac{1}{1260}, DBC \\ 0, SI \end{cases}$$

$$\mathbf{I}_4 = \int\limits_{01} {}^{\bullet}\Delta_{00}\Delta_{01}{}^{\bullet}\Delta_{11}\Delta_{11} = \begin{cases} \frac{1}{5040}, DBC \\ 0, SI \end{cases}$$

$$\mathbf{I}_5 = \int_{01} \Delta_{00}{}^\bullet \Delta_{01}{}^\bullet \Delta_{11} \Delta_{11} = \begin{cases} -\frac{1}{2520}, DBC \\ 0, SI \end{cases} \tag{91}$$

$$\mathbf{I}_6 = \int_{01} {}^\bullet\Delta_{00} \Delta_{01}^\bullet \Delta_{11} \Delta_{01} = \begin{cases} -\frac{1}{5040}, DBC \\ 0, SI \end{cases} \tag{92}$$

$$\mathbf{I}_7 = \int_{01} \Delta_{00}{}^\bullet \Delta_{01} \Delta_{01}^\bullet \Delta_{11} = \begin{cases} -\frac{17}{5040}, DBC \\ -\frac{1}{1728}, SI \end{cases} \tag{93}$$

$$\mathbf{I}_8 = \int_{01} {}^\bullet\Delta_{01} \Delta_{01}^\bullet \Delta_{01}^2 = \begin{cases} -\frac{1}{560}, DBC \\ -\frac{11}{60480}, SI. \end{cases} \tag{94}$$

Let us now consider the other contributions. We made extensively use of the color trace and partial integration (both in spacetime and on the worldline), as they allow to collect identical Wick contractions. We start by considering the terms arising from the expansion of two vertices S_{int} coupling $F_{\mu\nu}$ to V

$$C_1(x_0) = -\frac{1}{16} \int_{01} \langle \dot{q}_0^\mu q_0^\nu \dot{q}_1^\alpha q_1^\beta q_1^\gamma q_1^\delta \rangle G_{\nu\mu}(x_0) \nabla_\delta \nabla_\gamma F_{\beta\alpha}(x_0) V(x_0) \tag{92}$$

$$C_2(x_0) = -\frac{1}{8} \int_{012} \langle \dot{q}_0^\mu q_0^\nu \dot{q}_1^\alpha q_1^\beta \dot{q}_2^\gamma q_2^\delta \rangle G_{\nu\mu}(x_0) F_{\beta\alpha}(x_0) \nabla_\delta \nabla_\gamma V(x_0) \tag{93}$$

$$C_3(x_0) = -\frac{1}{6} \int_{012} \langle \dot{q}_0^\mu q_0^\nu \dot{q}_1^\alpha q_1^\beta q_1^\gamma \dot{q}_2^\delta \rangle G_{\nu\mu}(x_0) \nabla_\gamma F_{\beta\alpha}(x_0) \nabla_\delta V(x_0) \tag{94}$$

$$C_4(x_0) = -\frac{1}{9} \int_{01} \langle \dot{q}_0^\mu q_0^\nu q_0^\rho \dot{q}_1^\alpha q_1^\beta q_1^\gamma \rangle \partial_\rho G_{\nu\mu}(x_0) \nabla_\gamma F_{\beta\alpha}(x_0) V(x_0) \tag{95}$$

$$C_5(x_0) = -\frac{1}{16} \int_{01} \langle \dot{q}_0^\mu q_0^\nu q_0^\rho q_0^\sigma \dot{q}_1^\alpha q_1^\beta \rangle \partial_\rho \partial_\sigma G_{\nu\mu}(x_0) F_{\beta\alpha}(x_0) V(x_0) \tag{96}$$

$$C_6(x_0) = -\frac{1}{6} \int_{012} \langle \dot{q}_0^\mu q_0^\nu q_0^\rho \dot{q}_1^\alpha q_1^\beta q_2^\gamma \rangle \partial_\rho G_{\nu\mu}(x_0) F_{\beta\alpha}(x_0) \nabla_\gamma V(x_0) \tag{97}$$

where, strictly speaking, the derivatives in the Taylor expansion of the potential are standard derivative, but we have covariantized them anticipating the effect of the time ordering with insertions of vertices with bare A_μ, as discussed in sec. 5 (see the example in (47)). In addition, we have the terms with three vertices and containing the scalar potential

$$C_7(x_0) = \frac{1}{8} \int_{0123} \langle \dot{q}_0^\alpha q_0^\beta \dot{q}_1^\gamma q_1^\delta \dot{q}_3^\mu q_3^\nu \rangle g_{012} \, G_{\nu\mu}(x_0) F_{\beta\alpha}(x_0) F_{\delta\gamma}(x_0) V(x_0) \tag{98}$$

$$C_8(x_0) = \frac{\beta}{8} \int_{01} \langle \dot{q}_0^\mu q_0^\nu \dot{q}_1^\alpha q_1^\beta \rangle \, G_{\nu\mu}(x_0) F_{\beta\alpha}(x_0) V^2(x_0) \tag{99}$$

where the function g_{012} contains step functions that take care of the time ordering

$$g_{012} = \theta_{01}\theta_{12} + \theta_{12}\theta_{20} + \theta_{20}\theta_{01} \; . \tag{100}$$

In C_8 the time ordering is not necessary, thanks to the color trace. It is then recognized as a disconnected diagram that is calculated straightforwardly from previous results. Moving on to the term with three non-abelian field strengths we find

$$C_9(x_0) = \frac{1}{16\beta} \int\limits_{0123} \langle \dot{q}_0^\mu q_0^\nu \dot{q}_1^\alpha q_1^\beta \dot{q}_2^\gamma q_2^\delta \dot{q}_3^\epsilon q_3^\eta \rangle \theta_{12}\theta_{23} G_{\nu\mu}(x_0) F_{\beta\alpha}(x_0) F_{\delta\gamma}(x_0) F_{\eta\epsilon}(x_0) \; . \tag{101}$$

Let us now evaluate these terms. For C_1 we get

$$C_1 = -\frac{\beta^3}{4} G^{\mu\nu} \nabla^2 F_{\mu\nu} V \mathbf{H}_1 + \frac{\beta^3}{4} G^{\mu\nu} F_{\nu\alpha} F^\alpha{}_\mu V \mathbf{H}_1 \tag{102}$$

For C_2 we get

$$\begin{aligned}
C_2 = \frac{\beta^3}{2} \Big(& -\partial^2 G^{\mu\nu} F_{\mu\nu} - G^{\mu\nu} \nabla^2 F_{\mu\nu} - \partial^\alpha G^{\mu\nu} \nabla_\alpha F_{\mu\nu} - 2\partial_\alpha G^{\alpha\nu} \nabla^\mu F_{\mu\nu} \\
& + 2 G^{\alpha\nu} F_{\nu\mu} F^\mu{}_\alpha \Big) V \mathbf{K}_1 \\
& - \frac{\beta^3}{4} \Big(2\partial^\alpha G^{\mu\nu} \nabla_\alpha F_{\mu\nu} + \partial^2 G^{\mu\nu} F_{\mu\nu} + G^{\mu\nu} \nabla^2 F_{\mu\nu} \Big) V \mathbf{F}_3 \mathbf{J}_1
\end{aligned} \tag{103}$$

with the new diagram

$$\mathbf{K}_1 = \int\limits_{012} {}^\bullet\Delta_{01} {}^\bullet\Delta_{12} \Delta_{02} = \begin{cases} \frac{1}{360}, DBC \\ \frac{1}{720}, SI \; . \end{cases} \tag{104}$$

Next, C_3 evaluates to

$$\begin{aligned}
C_3 = & \frac{\beta^3}{2} \Big(G^{\mu\nu} \nabla^2 F_{\mu\nu} + \partial^\sigma G^{\mu\nu} \nabla_\sigma F_{\mu\nu} \Big) V \mathbf{K}_2 \\
& + \frac{\beta^3}{2} \Big(G^{\mu\nu} \nabla^2 F_{\mu\nu} + 2\partial_\alpha G^{\alpha\mu} \nabla^\sigma F_{\sigma\mu} - 4 G^{\alpha\mu} F_{\mu\sigma} F^\sigma{}_\alpha \Big) V \mathbf{K}_3
\end{aligned} \tag{105}$$

with

$$\mathbf{K}_2 = \int\limits_{012} {}^\bullet\Delta_{01} \Delta^\bullet_{01} \Delta_{12} = \begin{cases} \frac{1}{120}, DBC \\ 0, SI \end{cases} \tag{106}$$

$$\mathbf{K}_3 = \int\limits_{012} {}^\bullet\Delta_{01} {}^\bullet\Delta_{11} \Delta_{02} = \begin{cases} \frac{1}{720}, DBC \\ 0, SI \end{cases}$$

while (95) and (96) produce

$$\begin{aligned}
C_4 = & -\frac{\beta^3}{2} \partial^\nu G_{\nu\mu} \nabla_\alpha F^{\alpha\mu} V \mathbf{H}_4 - \frac{\beta^3}{2} \partial^\lambda G^{\mu\nu} \nabla_\lambda F_{\mu\nu} V \mathbf{H}_2 \\
C_5 = & -\frac{\beta^3}{4} \partial^2 G^{\mu\nu} F_{\mu\nu} V \mathbf{H}_1 \; .
\end{aligned} \tag{107}$$

For C_6 we have

$$C_6(x_0) = \frac{\beta^3}{2} \left(\partial^2 G^{\mu\nu} F_{\mu\nu} + \partial^\sigma G^{\mu\nu} \nabla_\sigma F_{\mu\nu} \right) V \mathbf{K}_2$$
$$+ \frac{\beta^3}{2} \left(\partial^2 G^{\mu\nu} F_{\mu\nu} + 2\partial_\lambda G^{\lambda\mu} \nabla^\alpha F_{\alpha\mu} \right) V \mathbf{K}_3 . \tag{108}$$

Finally, we consider C_7 that mixes with the above terms

$$C_7 = \frac{\beta^3}{4} G^{\rho\nu} F_{\nu\mu} F^\mu{}_\rho V \left(\mathbf{K}_4 - \mathbf{K}_5 + \mathbf{K}_6 - \mathbf{K}_7 \right) \tag{109}$$

and contains the following integrals

$$\mathbf{K}_4 = \int_{0123} {}^\bullet\Delta_{03}\Delta^\bullet_{01}\Delta_{13} g_{012} = \begin{cases} \frac{1}{240}, & DBC \\ \frac{1}{360}, & SI \end{cases}$$

$$\mathbf{K}_5 = \int_{0123} {}^\bullet\Delta^\bullet_{13}{}^\bullet\Delta_{03}\Delta_{01} g_{012} = \begin{cases} \frac{7}{720}, & DBC \\ \frac{1}{360}, & SI \end{cases}$$

$$\mathbf{K}_6 = \int_{0123} {}^\bullet\Delta^\bullet_{01}\Delta^\bullet_{03}\Delta_{13} g_{012} = \begin{cases} \frac{1}{720}, & DBC \\ \frac{1}{720}, & SI \end{cases} \tag{110}$$

$$\mathbf{K}_7 = \int_{0123} {}^\bullet\Delta_{01}\Delta^\bullet_{03}{}^\bullet\Delta_{13} g_{012} = \begin{cases} -\frac{1}{240}, & DBC \\ -\frac{1}{360}, & SI . \end{cases}$$

Collecting all the terms from C_1 to C_7 we find that they sum up to the entire second line of b_3 in (32), independently of the propagator used. We have performed integrations by parts to reduce to a set of independent terms (in particular, we have left V free from derivatives).

As for C_8, since the time ordering can be neglected, we find that it is given by a disconnected correlation functions that embeds the first piece of b_2 and produces

$$C_8 = -\frac{\beta^3}{4} G^{\mu\nu} F_{\mu\nu} V^2 \mathbf{F}_3 = \frac{\beta^3}{48} G^{\mu\nu} F_{\mu\nu} V^2 \tag{111}$$

which sits inside (32) as the last term of the first row of b_3.

Finally, we consider (101), that contains three non-abelian field strengths $F_{\mu\nu}$. In (101) we kept the time ordering encoded in the step functions. However, one may note that the Wick contractions produce terms that have no ordering ambiguities under the trace, implying that the abelian limit contains precisely the same information. Thus, we are allowed to drop the step functions and consider the equivalent (under the color trace) form

$$\tilde{C}_9(x_0) = \frac{1}{96\beta} \int_{0123} \langle \dot{q}_0^\mu q_0^\nu \dot{q}_1^\alpha q_1^\beta \dot{q}_2^\gamma q_2^\delta \dot{q}_3^\epsilon q_3^\eta \rangle G_{\nu\mu}(x_0) F_{\beta\alpha}(x_0) F_{\delta\gamma}(x_0) F_{\eta\epsilon}(x_0) . \tag{112}$$

We compute it as

$$\tilde{C}_9 = \frac{\beta^3}{8} G^{\mu\nu} F_{\mu\nu} F^2 (\mathbf{F}_3)^2 + \frac{\beta^3}{2} G^{\mu\nu} F_{\nu\rho} F^{\rho\lambda} F_{\lambda\mu} \mathbf{K}_8 \tag{113}$$

where the first term arises from disconnected diagrams with $\mathbf{F}_3 = -\frac{1}{12}$ already given in (40), and

$$\mathbf{K}_8 = \int\limits_{0123} {}^{\bullet}\Delta_{01}{}^{\bullet}\Delta_{12}{}^{\bullet}\Delta_{23}{}^{\bullet}\Delta_{30} = \frac{1}{720} \tag{114}$$

valid in DBC and SI, thus producing the second and third term of b_3 in (32).

References

[1] J.S. Schwinger, On gauge invariance and vacuum polarization, Phys. Rev. 82 (1951) 664, https://doi.org/10.1103/PhysRev.82.664.

[2] B.S. DeWitt, Dynamical theory of groups and fields, Conf. Proc. C 630701 (1964) 585; Les Houches Lect. Notes 13 (1964) 585.

[3] A.O. Barvinsky, G.A. Vilkovisky, The generalized Schwinger-DeWitt technique in gauge theories and quantum gravity, Phys. Rep. 119 (1985) 1, https://doi.org/10.1016/0370-1573(85)90148-6.

[4] B.S. DeWitt, The global approach to quantum field theory. Vol. 1, 2, Int. Ser. Monogr. Phys. 114 (2003) 1.

[5] D.V. Vassilevich, Heat kernel expansion: user's manual, Phys. Rep. 388 (2003) 279, https://doi.org/10.1016/j.physrep.2003.09.002, arXiv:hep-th/0306138.

[6] K. Fujikawa, Path integral measure for gauge invariant fermion theories, Phys. Rev. Lett. 42 (1979) 1195, https://doi.org/10.1103/PhysRevLett.42.1195.

[7] L. Alvarez-Gaume, E. Witten, Gravitational anomalies, Nucl. Phys. B 234 (1984) 269, https://doi.org/10.1016/0550-3213(84)90066-X.

[8] K. Fujikawa, H. Suzuki, Path Integrals and Quantum Anomalies, Clarendon Press, Oxford UK, 2004.

[9] F. Bastianelli, P. van Nieuwenhuizen, Path Integrals and Anomalies in Curved Space, Cambridge University Press, Cambridge UK, 2006.

[10] M. Kurkov, D. Vassilevich, Gravitational parity anomaly with and without boundaries, J. High Energy Phys. 1803 (2018) 072, https://doi.org/10.1007/JHEP03(2018)072, arXiv:1801.02049 [hep-th].

[11] F. Bastianelli, P. van Nieuwenhuizen, A. Van Proeyen, Superstring anomalies in the semilight cone gauge, Phys. Lett. B 253 (1991) 67, https://doi.org/10.1016/0370-2693(91)91365-3.

[12] T.P. Branson, P.B. Gilkey, D.V. Vassilevich, Vacuum expectation value asymptotics for second order differential operators on manifolds with boundary, J. Math. Phys. 39 (1998) 1040, https://doi.org/10.1063/1.532369, arXiv:hep-th/9702178; Erratum: J. Math. Phys. 41 (2000) 3301.

[13] F. Bastianelli, M. Broccoli, On the trace anomaly of a Weyl fermion in a gauge background, Eur. Phys. J. C 79 (4) (2019) 292, https://doi.org/10.1140/epjc/s10052-019-6799-z, arXiv:1808.03489 [hep-th].

[14] Y. Nakayama, CP-violating CFT and trace anomaly, Nucl. Phys. B 859 (2012) 288, https://doi.org/10.1016/j.nuclphysb.2012.02.006, arXiv:1201.3428 [hep-th].

[15] F. Bastianelli, The path integral for a particle in curved spaces and Weyl anomalies, Nucl. Phys. B 376 (1992) 113, https://doi.org/10.1016/0550-3213(92)90070-R, arXiv:hep-th/9112035.

[16] F. Bastianelli, P. van Nieuwenhuizen, Trace anomalies from quantum mechanics, Nucl. Phys. B 389 (1993) 53, https://doi.org/10.1016/0550-3213(93)90285-W, arXiv:hep-th/9208059.

[17] J. de Boer, B. Peeters, K. Skenderis, P. van Nieuwenhuizen, Loop calculations in quantum mechanical nonlinear sigma models with fermions and applications to anomalies, Nucl. Phys. B 459 (1996) 631, https://doi.org/10.1016/0550-3213(95)00593-5, arXiv:hep-th/9509158.

[18] F. Bastianelli, K. Schalm, P. van Nieuwenhuizen, Mode regularization, time slicing, Weyl ordering and phase space path integrals for quantum mechanical nonlinear sigma models, Phys. Rev. D 58 (1998) 044002, https://doi.org/10.1103/PhysRevD.58.044002, arXiv:hep-th/9801105.

[19] F. Bastianelli, R. Bonezzi, O. Corradini, E. Latini, Extended SUSY quantum mechanics: transition amplitudes and path integrals, J. High Energy Phys. 1106 (2011) 023, https://doi.org/10.1007/JHEP06(2011)023, arXiv:1103.3993 [hep-th].

[20] R. Bonezzi, A. Meyer, I. Sachs, Einstein gravity from the $\mathcal{N} = 4$ spinning particle, J. High Energy Phys. 1810 (2018) 025, https://doi.org/10.1007/JHEP10(2018)025, arXiv:1807.07989 [hep-th].

[21] F. Bastianelli, R. Bonezzi, O. Corradini, E. Latini, One-loop quantum gravity from the $\mathcal{N} = 4$ spinning particle, J. High Energy Phys. 1911 (2019) 124, https://doi.org/10.1007/JHEP11(2019)124, arXiv:1909.05750 [hep-th].

[22] R. Bonezzi, A. Meyer, I. Sachs, A worldline theory for supergravity, arXiv:2004.06129 [hep-th].

[23] C. Schubert, Perturbative quantum field theory in the string inspired formalism, Phys. Rep. 355 (2001) 73, https://doi.org/10.1016/S0370-1573(01)00013-8, arXiv:hep-th/0101036.

[24] F. Bastianelli, R. Bonezzi, O. Corradini, E. Latini, Particles with non Abelian charges, J. High Energy Phys. 10 (2013) 098, https://doi.org/10.1007/JHEP10(2013)098, arXiv:1309.1608 [hep-th].

[25] D. Fliegner, P. Haberl, M.G. Schmidt, C. Schubert, The higher derivative expansion of the effective action by the string inspired method. Part 2, Ann. Phys. 264 (1998) 51, https://doi.org/10.1006/aphy.1997.5778, arXiv:hep-th/9707189.

[26] F. Bastianelli, O. Corradini, A. Zirotti, BRST treatment of zero modes for the worldline formalism in curved space, J. High Energy Phys. 01 (2004) 023, https://doi.org/10.1088/1126-6708/2004/01/023, arXiv:hep-th/0312064 [hep-th].

[27] U. Muller, C. Schubert, A.M. van de Ven, A closed formula for the Riemann normal coordinate expansion, Gen. Relativ. Gravit. 31 (1999) 1759–1768, https://doi.org/10.1023/A:1026718301634, arXiv:gr-qc/9712092 [gr-qc].

[28] S. Montefiori, Anomalies in the stress tensor of a chiral fermion in a gauge background, tesi di Laurea Magistrale, Università di Bologna, 2019.

[29] F. Bastianelli, R. Martelli, On the trace anomaly of a Weyl fermion, J. High Energy Phys. 11 (2016) 178, https://doi.org/10.1007/JHEP11(2016)178, arXiv:1610.02304 [hep-th].

[30] H. Godazgar, H. Nicolai, A rederivation of the conformal anomaly for spin-$\frac{1}{2}$, Class. Quantum Gravity 35 (10) (2018) 105013, https://doi.org/10.1088/1361-6382/aaba97, arXiv:1801.01728 [hep-th].

[31] M.B. Fröb, J. Zahn, Trace anomaly for chiral fermions via Hadamard subtraction, J. High Energy Phys. 10 (2019) 223, https://doi.org/10.1007/JHEP10(2019)223, arXiv:1904.10982 [hep-th].

[32] F. Bastianelli, M. Broccoli, Weyl fermions in a non-Abelian gauge background and trace anomalies, J. High Energy Phys. 10 (2019) 241, https://doi.org/10.1007/JHEP10(2019)241, arXiv:1908.03750 [hep-th].

[33] F. Bastianelli, M. Broccoli, Axial gravity and anomalies of fermions, Eur. Phys. J. C 80 (3) (2020) 276, https://doi.org/10.1140/epjc/s10052-020-7782-4, arXiv:1911.02271 [hep-th].

[34] K. Nakagawa, Y. Nakayama, CP-violating super Weyl anomaly, arXiv:2002.01128 [hep-th].

[35] L. Bonora, S. Giaccari, B. Lima de Souza, Trace anomalies in chiral theories revisited, J. High Energy Phys. 07 (2014) 117, https://doi.org/10.1007/JHEP07(2014)117, arXiv:1403.2606 [hep-th].

[36] L. Bonora, M. Cvitan, P. Dominis Prester, A. Duarte Pereira, S. Giaccari, T. Štemberga, Axial gravity, massless fermions and trace anomalies, Eur. Phys. J. C 77 (8) (2017) 511, https://doi.org/10.1140/epjc/s10052-017-5071-7, arXiv:1703.10473 [hep-th].

[37] L. Bonora, M. Cvitan, P. Dominis Prester, S. Giaccari, M. Paulišic, T. Štemberga, Axial gravity: a non-perturbative approach to split anomalies, Eur. Phys. J. C 78 (8) (2018) 652, https://doi.org/10.1140/epjc/s10052-018-6141-1, arXiv:1807.01249 [hep-th].

[38] F. Bastianelli, A. Zirotti, Worldline formalism in a gravitational background, Nucl. Phys. B 642 (2002) 372–388, https://doi.org/10.1016/S0550-3213(02)00683-1, arXiv:hep-th/0205182 [hep-th].