STOCK CONTROL ANALYTICS: A DATA-DRIVEN APPROACH TO COMPUTE THE FILL RATE FOR THE (*s, S*) SYSTEM CONSIDERING UNDERSHOOTS

ABSTRACT: Inventory policies are traditionally characterized assuming several hypotheses that lead to commit important errors when are used in practical environments. This is the case when the inventory is continuously reviewed by means of the Order-Point, Order-Up-to-Level (s, S) system and undershoots, i.e. the difference between the order-point and the inventory position when it is reached, are neglected. This paper analyses conceptually and empirically the bias on the classic fill rate formula when neglecting undershoots. After that, we suggest a non-parametric approach based on a State Dependent Parameter algorithm to propose a new non-linear expression, named analytic fill rate that correct that bias. The proposed approach is developed under a data-driven perspective and is easily implementable in practice. This research is developed in a lost sales context with stochastic and i.i.d. discrete demand.

KEYWORDS: Inventory, Fill Rate, Lost sales, Undershoots, State Dependent Parameter**.**

1. INTRODUCTION

The purpose of an inventory system consists of answering two important key issues: (1) when to launch a replenishment order, and (2) the size of this replenishment order. Control parameters of inventory policies are devoted to determine these issues following a design objective, normally a cost or a customer service criterion. In practice, inventory costs are difficult to quantify or even estimate, particularly stockout costs (Larsen and Thorstenson, 2008; Liberopoulos et al., 2010; Silver, 2008). For this reason, managers set a service level to design inventory policies. The question is how to implement it. First of all, it is necessary to know the managerial context and define the type of control procedure. One of the most important issues related to the managerial context is the way the company deals with stockouts. If any unsatisfied demand is backordered and filled as soon as possible, context is named as *Backordering*. On contrary, if unsatisfied demand is lost, context is known as *Lost Sales*. Although the problem of lost sales was formulated more than 50 years ago by Karlin and Scarf (1958) traditional inventory models have focused on backordering contexts since are easier to formulate (Zipkin, 2008a, 2008b). However, interest in lost sales models have been increasing over the last years (Bijvank et al., 2014; Bijvank and Johansen, 2012; Bijvank and Vis, 2011; Cardós et al., 2017; Cardós and Babiloni, 2011a, 2011b; Guijarro et al., 2012; Johansen, 2013; Kouki et al., 2018) and this research is developed in it. Regarding the control procedure, inventory policies are split into continuous and periodic review policies. If the status of the inventory is known at any moment, the inventory is continuously reviewed. On contrary, if the status of the inventory is only known every period of time (for instance every week), the inventory is periodically reviewed. For strategic items (normally classified as A items) continuous review is preferable (Silver et al., 1998) since allows controlling stockouts more easily despite the managerial costs that transaction reporting implies.

Continuous review policies are classified into two types. In general, once the inventory position drops to the reorder point, s, (refer also as ROP further on) a replenishment order is launched and received L units of time later, where L is the lead time. Such a replenishment order may be constant, as is the case of an Order-point, Order-Quantity (s, Q) system, or variable ordering enough to raise the inventory position to the order-up-to-level, S, and thereafter managed by means of a policy Order-Point, Order-Up-to-Level (s, S) system. This paper focuses on the (s, S) due to its versatility and extended use in practice. By definition, the (s, S) policy implies that, in a stochastic demand context, both the replenishment cycle, i.e. time elapsed between two consecutive order deliveries, and the order quantity are variable and therefore simultaneous determination of parameters is necessary to guarantee the mathematical optimality of the policy once a service objective is set. However, in practice *"the values of the control parameters are set in a rather arbitrary fashion"* (Silver et al., 2017) and the most common approach consists of assuming that all demand transactions are unit sized. It implies that the inventory position always reaches exactly the ROP and therefore the order quantity is constant and equal to S-s being equivalent to the policy (s, Q) with Q=S-s (Vincent, 1985). However, the inventory position may not be exactly at ROP, but a certain amount below it. This amount is called deficit or undershoot at the reorder point (we refer to it as undershoot in the rest of the paper). Neglecting undershoots greatly reduce the mathematical complexity of the problem but can severely reduce the service level of the system (Schneider, 1979) what lead to increase stockout costs especially when managing lumpy or erratic demand items for which the unitary demand assumption is not appropriate. In practice, service levels are usually measured through the cycle service level (CSL) or the fill rate. This paper focuses on the fill rate that is commonly defined as the fraction of demand that is immediately satisfied from the stock without shortage (Brown, 1962).

Summarising, on the one hand, the (s, S) policy is widely used in practice but assuming that undershoots at ROP are negligible introduces an important bias on the fill rate. On the other hand, there is not an optimal approach to solve it and the mathematical complexity to characterize the policy makes no sense in practice (Silver et al., 2017). Indeed, as Silver and collaborators point out, practitioners are looking for understandable approaches that offer good solutions at a reasonable computational cost. The objective of this research is precisely to bridge that gap by proposing a bias analysis and correction of the undershoots assumption following a data-driven perspective.

In general, the digitalization of companies is offering a great amount and variety of data (Big Data) that is transforming the way supply chain management is working (Wang et al., 2016). In particular, Gandomi and Haider (2015) indicate the need to develop appropriate and efficient methods to leverage such data and they review areas of big data analytics as text analytics or predictive analytics. In this sense, this work extends such a vision to propose a novel area as stock control analytics. Although the nomenclature may be new, the use of data analytics tools to overcome limitations associated to theoretical models is not that new. For instance, Strijbosch et al. (1997)correct the bias introduced by using demand forecasts instead of first moments of the statistical demand distribution, which are unknown. That article proposed a heuristic method based on simulations, which reduced the bias substantially. Novel methods based on this data-driven perspective are recently published to deal with the uncertainty of demand distributions. Huber et al. (2019) revisit the newsvendor problem subject to a target CSL employing Machine Learning and Quantile Regression approaches that circumvent the need of assuming a statistical distribution for the demand. Regarding uncertainty of statistical demand distribution, Trapero et al. (2019a) investigate parametric and non-parametric methods to enhance the safety stock estimation problem for a certain target CSL. Their results show that non-parametric approaches as Kernel density estimators provide good performance in terms of cost and CSL for lower lead times and parametric approaches as GARCH outperform the rest of methods for higher values of lead time. Trapero et al. (2019b) continue with this line of research by proposing a combination scheme of GARCH and Kernel whose weight parameters are determined by minimizing the tick-loss (newsvendor) linear asymmetric cost function.

Previous articles show how data-driven approaches are offering versatile solutions to deal with limitations of traditional theoretical assumptions. In this work, following the same philosophy, we centre on the assumption of neglecting undershoots on the (s, S) inventory policy when demand is modelled by any discrete distributions for a lost sales case. To the best of authors' knowledge, this particular topic has been overlooked in the literature. Basically, this research aims at determining the extent of bias introduced by such an assumption and it also proposes a methodology to reduce it if enough data is available. The data-driven approach employed to cope with this problem is the State Dependent Parameter (SDP) estimation technique. SDP estimation involves the non-parametric identification of the state dependency using recursive methods of time variable parameter estimation which allow for rapid (state dependent) parametric change (Young et al., 2001). SDP estimation has been successfully applied in a supply chain context in (Trapero et al., 2011), particularly, to correct judgmental forecasts bias.

The results show that the fill rate bias induced by neglecting undershoots in a continuous review (s, S) policy, assuming a lost sales situation, can be substantially reduced from 7% to 1%. These results were obtained for simulated demand coming from an i.i.d. Negative Binomial distribution using a non-linear parametric approach and considering fill rates higher than 50%.

The rest of this paper is organized as follows. Section 2 describes the main assumptions and notation of the system. Section 3 introduces the classical fill rate formula and gives an intuition of the potential bias induced by neglecting undershoots. In that section, the experimental setup is also detailed. Section 4 explores the SDP approach to correct the bias associated to the classical fill rate estimation. Section 5 is devoted to discussing the results. Finally, the main conclusions are drawn in Section 6.

2. SYSTEM DESCRIPTION, NOTATION AND ASSUMPTIONS

This paper considers a single echelon, single item inventory system where demand is stochastic and modelled by a discrete distribution. The stock is controlled following a continuous review Order-Point, Order-Up-to-Level (*s, S*) system for the lost sales case. In this system, when the inventory position is at or below the ROP a sufficient amount is ordered to raise the inventory position up to the order-up-to-level. The replenishment order is received *L* unit of times after being launched. We consider that the replenishment cycle (also refer as just cycle further on) is the time between two consecutives deliveries and it starts at order delivery. Fig. 1 shows an example of the evolution of the on-hand stock and the inventory position when the system is not out of stock (a) and when it is out of stock (b).

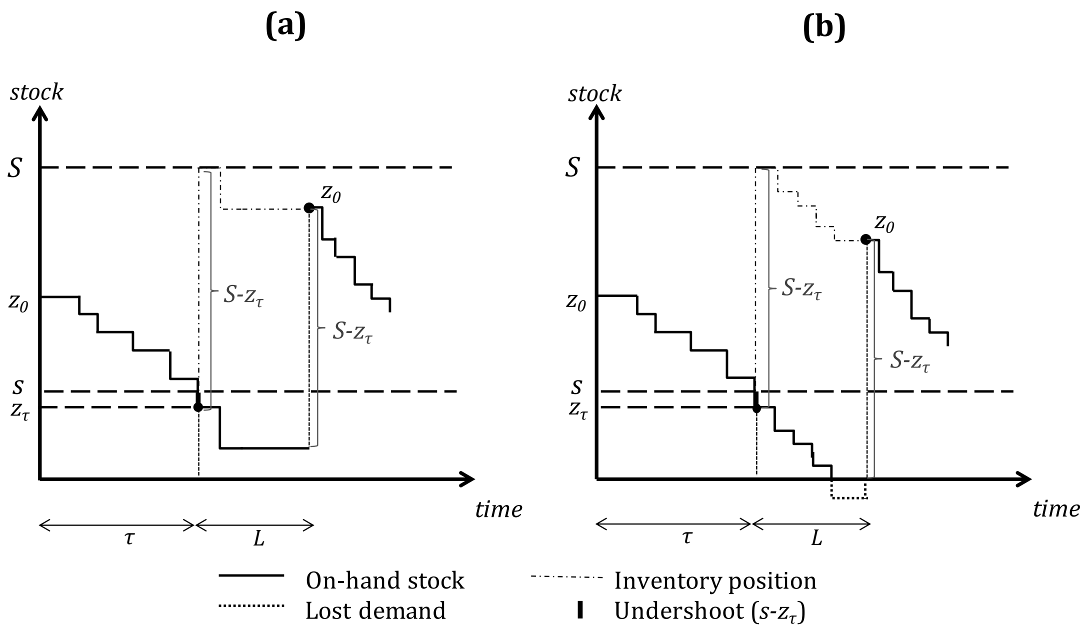

Fig. 1 Evolution of the stock in a (*s*, *S*) inventory policy and lost sales when the system is not out of stock (a) and when it is out of stock (b).

Notation in Fig. 1 and in the rest of the paper is:

s = reorder point, ROP (units),

S = order-up-to-level (units),

L = lead time (time),

τ = elapse of time from the beginning of the cycle to the ROP is reached (time),

z_t = on-hand stock at t (units),

d_t = demand at instant t (units),

D_t = accumulated demand over t (units),

$f_L(\cdot)$ = probability function of demand during L,

X^+ = maximum $[X, 0]$ for any expression X,

Q_t = order quantity at instant t (units),

$\beta_a =$ achieved fill rate,

$\beta_i =$ estimated fill rate for method i.

General assumptions of this paper are: (i) time is discrete and is organized in a numerable and infinite succession of equally spaced instants; (ii) the lead time, L, is constant and known; (iii) there is never more than one order outstanding which implies that the condition $s<S-s$ is always true; (iv) demand process is considered stationary and i.i.d., and defined by any discrete distribution function; and (v) unfilled demand is lost. Note that assumption (i) is tantamount to assuming that each transaction is updated in a specific instant of time considering a sequence of discrete and infinitely small intervals that is, indeed, the way informatics systems that are used for control continuously inventories works. On the other hand assumption (iii) is generally accepted in inventory research since if it is not the case the numerical difficulties are insurmountable (Schneider, 1981) specially for the lost sales case (Hadley and Whitin, 1963; Cohen et al., 1988; Johansen and Hill, 2000).

3. THE BIAS OF THE FILL RATE CLASSIC ESTIMATION FOR THE (s, S) POLICY

3.1 Classic approach and conceptual bias

The fill rate is defined as the fraction of demand that is immediately fulfilled from on-hand stock. The most common approach to estimate this service measure consists of computing the complement of the ratio between the expected shortage per replenishment cycle (*ESPRC*) and the total expected demand per replenishment cycle (*EDPRC*). Fill rate has been traditionally computed by assuming that undershoots at ROP are negligible and therefore order quantity is always constant. Fig. 2 shows the evolution of the stock in a (*s, S*) inventory policy where is shown that if the ROP is reached exactly, order quantity is always equal to *S-s* and therefore equivalent to the (*s, Q*) system.

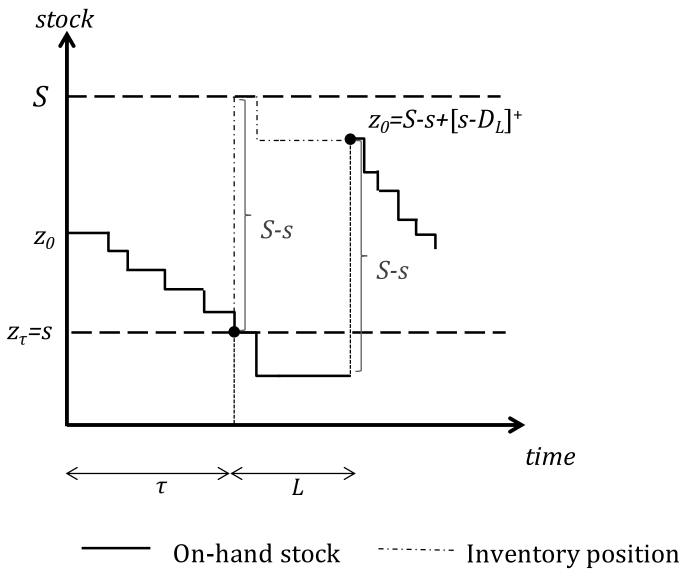

Fig. 2 Evolution of the stock in a (*s, S*) inventory policy when undershoots are neglected.

Then the expected shortage per replenishment cycle is straightforwardly computed as $ESPRC=\sum_{i=s+1}^{\infty}(i-s)\cdot f_L(i)$. The expected total demand per replenishment cycle is the accumulated demand from the beginning of the cycle until the ROP is reached, D_τ, and the accumulated demand from the moment an order is placed until its delivery, D_L. To compute D_τ, we need to know the on-hand stock balance at the beginning of the cycle, i.e. at order delivery, which, in a lost sales context is obtained as $z_0 = S - s + E\left[s - D_L\right]^+$. Therefore, in order to reach exactly the ROP, $D_\tau = z_0 - s = S - 2s + E\left[s - D_L\right]^+$. Consequently classic formula to compute the fill rate is:

$$\beta_c = 1 - \frac{\sum_{i=s+1}^{\infty} (i - s) \cdot f_L(i)}{S - 2s + \sum_{j=0}^{s} (s - j) \cdot f_L(j) + \sum_{k=0}^{\infty} k \cdot f_L(k)} \tag{1}$$

However neglecting undershoots at ROP has two important implications that directly affects the accuracy of the classic formula: (1) the system will only be out of stock during the lead time and therefore the protected period is reduced to *L*, when it is possible that the stock out occurs in the same instant the ROP is reached (Fig. 3); (2) z_τ is always equal to *s*. Despite this fact greatly reduces the mathematical complexity of the formula, since knowing the probability distribution of stock levels at τ is a challenge, it leads to overestimate systematically the real fill rate. For example, if the stock at *t* is one unit above the ROP and demand at *t+1* is 2 units, the ROP is reached and a replenishment order is placed. The real stock at launching is *s*-1 units. Therefore, the on-hand stock remaining on the shelves available to meet demand during the lead time is one unit less than what is assumed if undershoot is neglected. Thus, classic fill rate is higher than the real one and therefore the stockouts are larger than expected. This simple example shows why neglecting undershoots introduces a significant bias on classic formula of the fill rate.

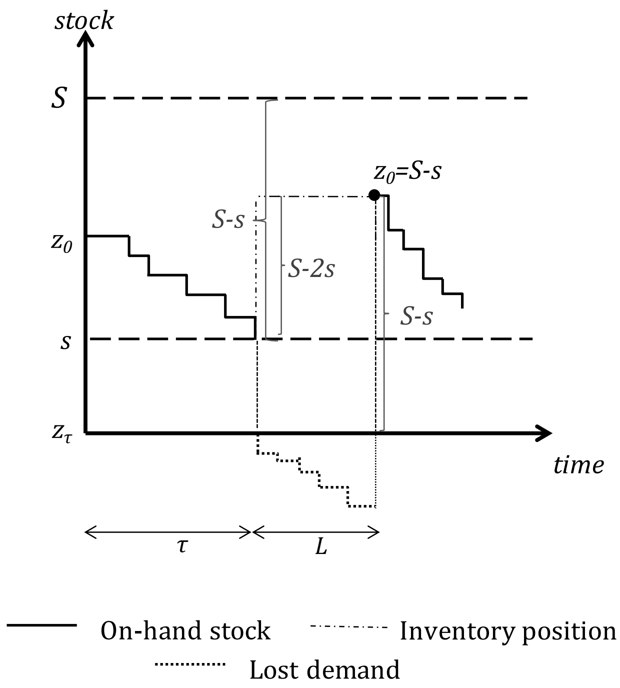

Fig. 3 Evolution of the stock in a (*s, S*) inventory policy when stockout occurs when the ROP is reached.

3.2 Empirical analysis of the bias

In order to analyse the bias of the classic fill rate, we design an experiment as follows (see Fig. 4): (1) we combine a set of input parameters that define the inventory policy and the simulated demand; (2) for each combination of the input data, we simulate a (*s, S*) inventory system using Monte Carlo method and calculate the achieved fill rate; (3) for each input data combination we also compute the classic fill rate using expression (1).

Regarding the input data, an extensive range of values for *s*, *S* and *L* is selected in order to provide realistic values of fill rate (from 0.5 to 0.99). Regarding the demand, we select a set of parameters (r, θ) for the Negative Binomial distribution based on considering the four demand categories suggested by Syntetos et al. (2005): smooth, lumpy, intermittent and erratic, where r is the number of successes and θ the probability of success. Table 1 presents the set of data used in the experiment which considers every feasible combination of these values per factor leading to 1,440 different cases (excluding cases where $s \geq S\text{-}s$). The inventory system simulation follows the diagram of Fig. 5 and for each input data combination, the achieved fill rate is obtained as the average fraction of the complement of the unfulfilled demand over the total demand in every replenishment cycle when considering 20,000 consecutive periods, i.e.:

$$\beta_a = \frac{1}{N}\sum_{n=1}^{N} 1 - \frac{unfilled\ demand_n}{total\ demand_n} \tag{2}$$

where N indicates the total number of replenishment cycles. To assure the consistency of the results, Monte Carlo simulations accomplish thirty replications to each case using the average of these replications as the final achieved fill rate.

Demand Distribution		
Negative Binomial	r	= 0.1, 0.25, 0.5, 0.75, 1, 1.5, 2, 3, 3.5, 4
	θ	= 0.1, 0.3, 0.4, 0.5, 0.6, 0.8, 0.9
Inventory system		
Reorder point	s	= 2, 3, 5, 6
Order-up-to-level	S	= 5, 7, 12, 15
Lead time	L	= 1, 2, 3, 4

Table 1: Input Data

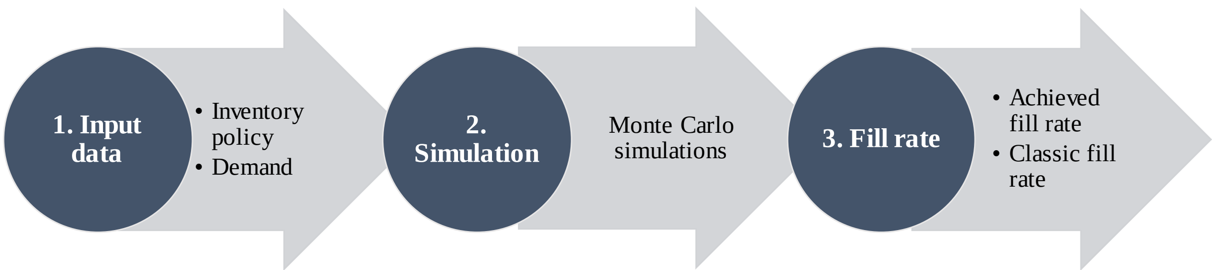

Fig. 4. Experiment design

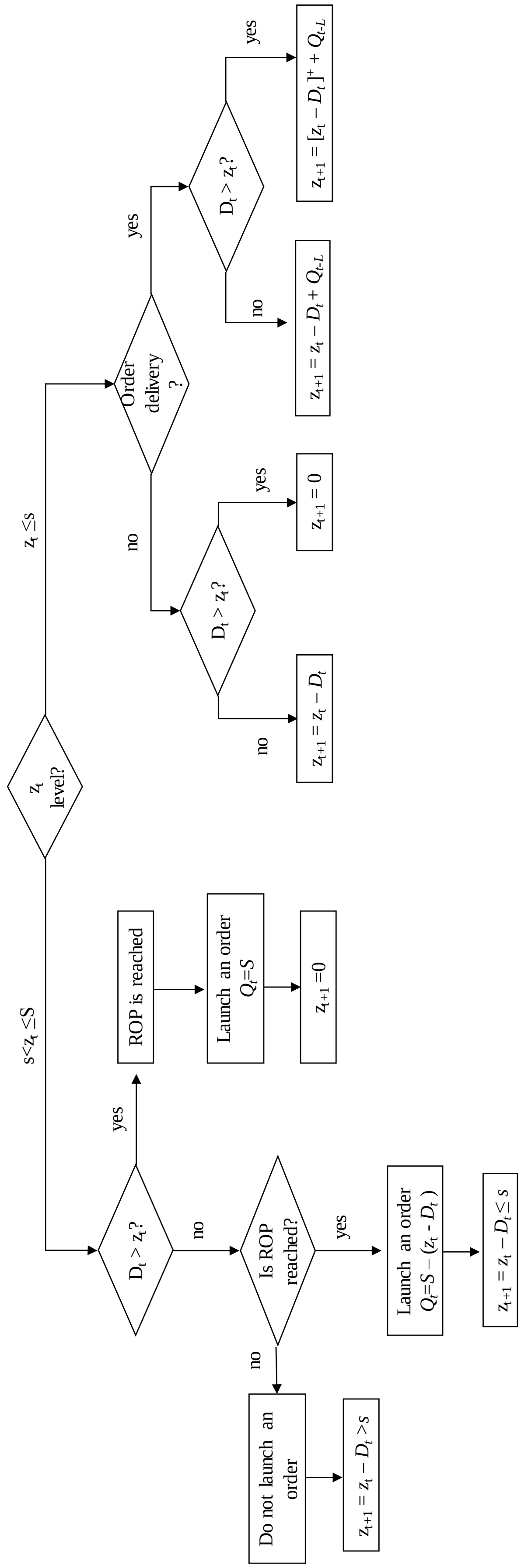

Fig. 5. Diagram of the simulation

Fig. 6 compares the classic (β_c) and the achieved (β_a) fill rate plotted as red circles and as a solid line, respectively. That figure shows the bias introduced in the calculation of the fill rate due to neglecting undershoots. The bigger the distance between red circles and the blue line, the bigger the bias. Note that the bias is positive, i.e., the classic fill rate always overestimates the achieved fill rate as explained in section 3.1. What is empirically found is that the bias size is not constant and it seems to increase over the central part of the figure.

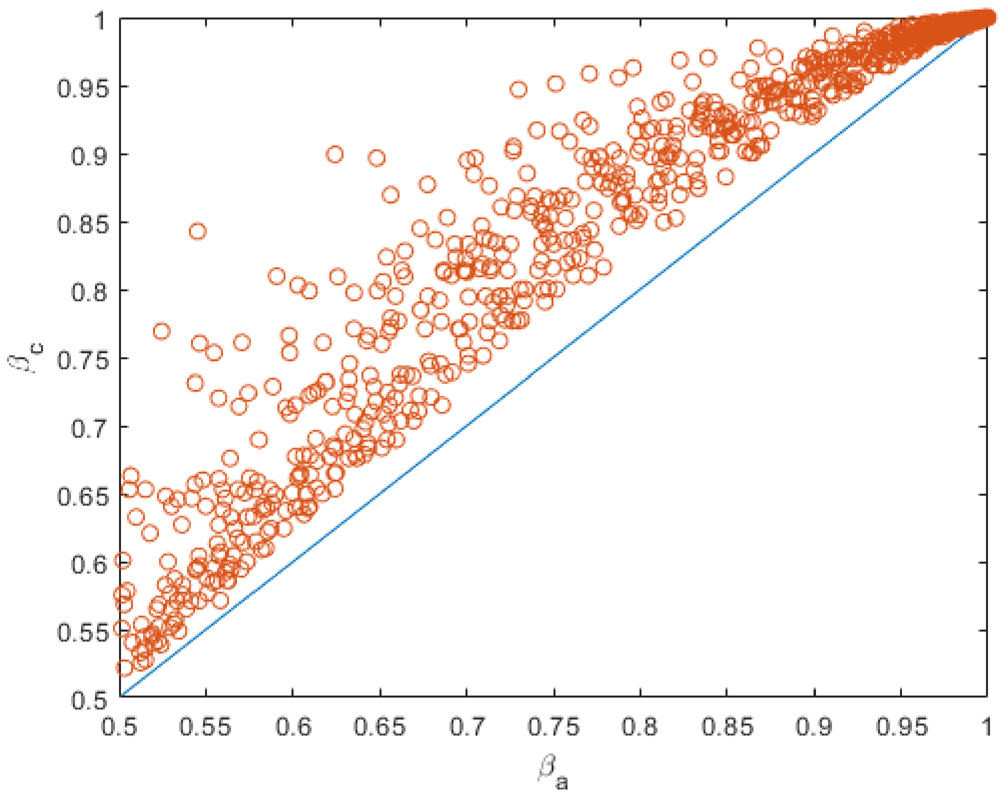

Fig. 6. Achieved fill rate (solid line) versus classic fill rate (red circles)

Therefore, the problem is to find the bias between β_c and β_a that seems to have a non-linear pattern.

4. DATA-DRIVEN BIAS CORRECTION OF THE CLASSIC FILL RATE

In the previous section, we have analysed by simulations the bias size for neglecting undershoots in the development of the classic fill rate. In this section, we will model that bias to further correct it. We assume that the data is cross-sectional, i.e., the relationship between the achieved fill rate and the classic fill rate does not depend on time. The initial number of experiments were 1,440. We focus on fill rates greater than 50 %, filtering the number of experiments to 1,091. The training and test set are divided by random sampling, where 70% of the data (764 experiments) is used for estimation purposes and the rest of data (327 experiments) for validation.

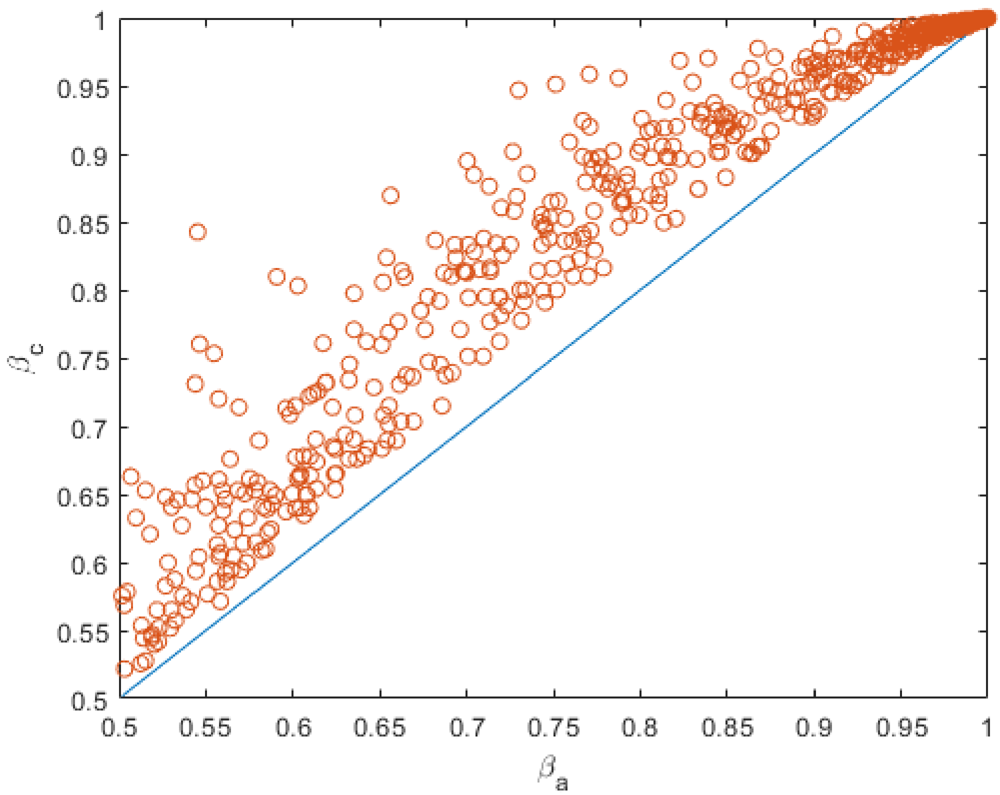

Fig. 7. Relationship between classic fill rate and achieved fill rate for the training set.

Fig. 7 shows the training set. In that figure, we can observe that the bias between fill rates is nonlinear and it is greater for values of target fill rate between 0.55 and 0.9

approximately, where the maximum bias is reached around 0.6 and 0.75. To model that bias, this work adopts an SDP approach, in which we have to define the state and the parameter that makes that state varying. An initial model could be as follows:

$$\beta_{SDP} = \alpha_1(\beta_a) \cdot \beta_c \tag{3}$$

where α_1 is the state that somehow depends on the parameter β_a. Note that, we do not know exactly the relationship between the state (α_1) and the parameter (β_a) and it will be the SDP algorithm, the one which will provide a curve to give us some insights about the pattern of such a dependency.

4.1 Non parametric approach: SDP algorithm

The SDP approach can be seen as an extension of the Time Variable Parameter estimation (TVP) (Young, 2011), where the unknown parameters are slowly variable with time. Typically, TVP are represented by a two-dimensional state (level and slope) to model its stochastic behaviour by means of a Generalized Random Walk (GRW), which is a generic model to unify different versions of the random walk as Integrated Random Walk (IRW), Random Walk (RW) and Smoothed Random Walk (SRW). More information about the use of TVPs in time series analysis and forecasting can be found in Harvey (1990) and Young (2011).

The SDP approach employs a similar procedure, but the main difference is that parameters evolve stochastically with respect to another variable instead of time. Such parameters are denoted by State Dependent Parameters (SDP). This can be done by "sorting" the data in a non-temporal order. If the new ordering provides a variation of the SDP smoother and less rapid, a GRW will be able to describe its evolution in this transformed observation space. Utilizing Fixed Interval Smoothing (FIS) estimation, the SDP estimated can be "unsorted" to the original order.

In our particular case, the SDP α_1 can be sorted with respect to β_a, this new organization of data is indexed by k, and thus, the stochastic dynamics of the SDP can be expressed as an Integrated Random Walk defined in a State Space framework as follows:

$$\begin{pmatrix} \alpha_1(k+1) \\ \alpha_1^*(k+1) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1(k) \\ \alpha_1^*(k) \end{pmatrix} + \begin{pmatrix} 0 \\ \eta^*(k) \end{pmatrix} \qquad (4)$$

Where α_1 is the level and α_1^* is the slope. The small variations with respect to the new sorting is introduced with the random Gaussian noise $\eta^*(k)$ with zero mean and constant variance σ_α. Such a variance is often referred to as *hyper-parameter* to differentiate it from the states that are the main object of the estimation analysis. The observation equation needed to complete the State Space representation is:

$$\beta_a = \alpha_1(\beta_a) \cdot \beta_c + \epsilon \qquad (5)$$

The last term ϵ is the typical error component that assumes a Gaussian distribution with zero mean and constant variance σ^2.

Fortunately, these routines are already implemented in the CAPTAIN toolbox of MATLAB available in http://captaintoolbox.co.uk. Fig. 8 shows the estimated SDP $\alpha_1(\beta_a)$ sorted with respect to the parameter β_a for the training set. That figure shows the non-linear pattern of the SDP coherently with the non-linear bias found between β_a and β_c in Fig. 7.

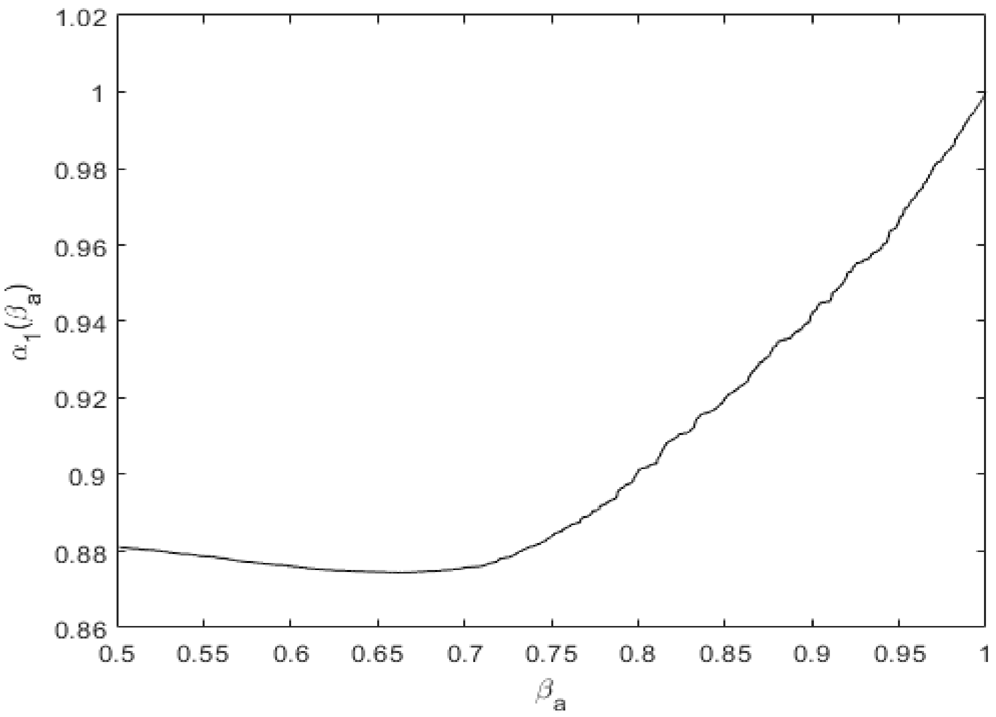

Fig. 8. SDP $\alpha_1(\beta_a)$ sorted with respect to the parameter β_a

Fig. 9 depicts the SDP estimation of the fill rate (β_{SDP}) together with the classic and achieved fill rates for the training set. That figure clearly shows how the SDP approach is capable of reducing the bias considerably. Essentially, fill rate estimates provided by β_{SDP} fluctuate around β_a randomly, instead of the systematic positive deviation found for the classic counterpart β_c.

4.2 Parameterization based on SDP graph

The SDP approach provides a non-parametric estimation of the relationship of $\alpha_1(\beta_a)$ based on a graph. Nonetheless, this approach can be complemented by parameterizing such a curve and proposing another parametric alternative. For instance, Fig. 7 and 8 show that the non-linear pattern of the bias seems to be quadratic. Therefore, a quadratic polynomial dependent of β_a can be a good candidate to model such a bias. Note that such parameterization makes the model fully self-contained and it reveals clearly the nature of the graphically identified nonlinearity with just a few parameters.

Considering that, we can propose the following model:

$$\beta_{LR} = \theta_0 + (\theta_1 + \theta_2 \cdot \beta_a + \theta_3 \cdot \beta_a^2) \cdot \beta_c \quad (6)$$

Where θ_i for i=0, 1, 2, 3 are constants that can be estimated by means of a linear regression. It is important to note that, despite the non-linearities involved in the classic fill rate bias, thanks to the SDP identification, we have arrived at a non-linear model, but linear with respect to the parameters θ_i.

Table 2 summarizes the estimation results. That table shows that all parameters are statistically significant for a significance level of 1%.

Parameter	**value (std deviation)**
θ_0	0.598 (0.005)*
θ_1	-1.07 (0.02)*
θ_2	2.25 (0.04)*
θ_3	-0.77 (0.02)*

Table 2. OLS estimation results for parameters in (8). Significant at 1% confidence level.

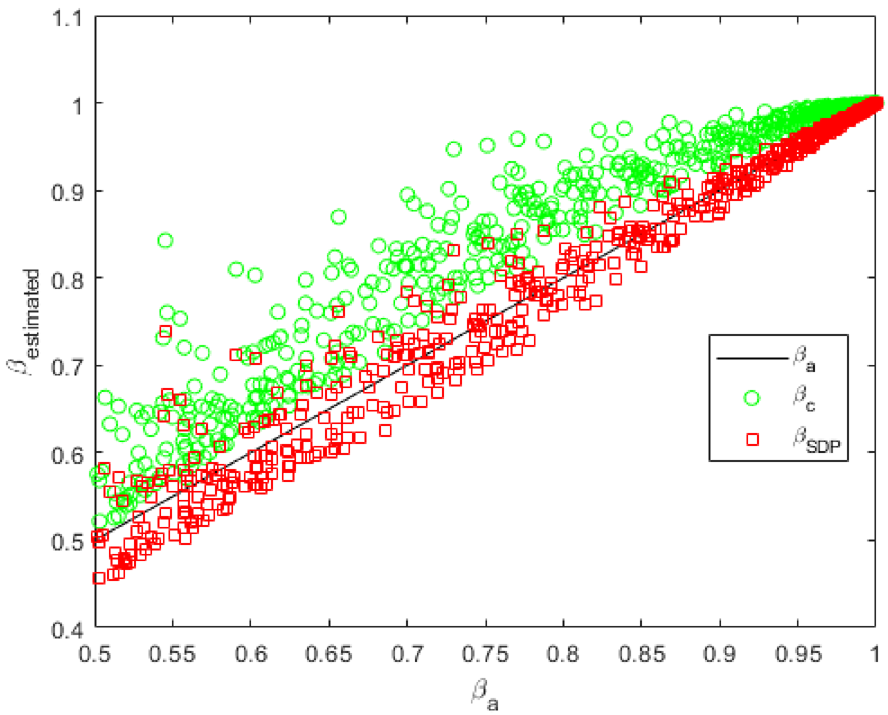

Fig. 9. Fill rate estimations provided by the classic (β_c) and the SDP (β_{SDP}) approach for the training set.

To validate that the parameters can be used to correct the bias with data that has not been used in the training set, we use those estimated parameters in the data test set. Fig. 10 shows the fill rate estimates provided by the non-parametric SDP (β_{SDP}) and the parametric model (β_{LR}) compared to the achieved and classic fill rates.

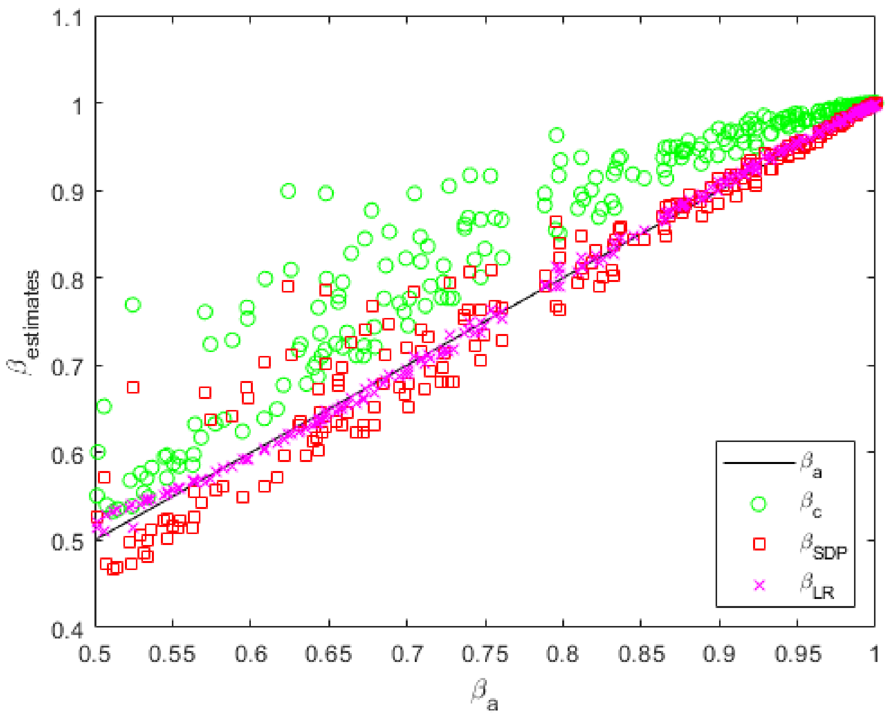

Fig. 10. Fill rate estimations provided by classic formula (β_C), SDP approach (β_{SDP}) and parametric model (β_{LR}) for the test set.

If we define the fill rate error as:

$$e_{i,j} = \beta_{i,j} - \beta_{a,j}$$

where β_{ij} for i=1, 2, 3 is the fill rate estimated by the classic equation, the non-parametric SDP and the parametric Linear Regression, respectively. The index j=1, 2,.., J is the number of fill rate observation for the entire test set J.

To summarize the performance of the different fill rate estimates, the Mean Error and the Root Mean Squared Error are chosen as metrics to offer a measure of bias and precision, respectively, such as:

$$ME_i = \frac{\sum_{j=1}^{J} e_{i,j}}{J} \quad (7)$$

$$RMSE_i = \sqrt{\frac{\sum_{j=1}^{J} e^2{}_{i,j}}{J}} \quad (8)$$

Fig. 11 shows the ME and RMSE obtained for the different fill rate estimation techniques. First, it is quantified the overestimation bias produced by the classic formula in (1). Note that, such a classic formula displays a high error variability measured by the RMSE. That figure also clearly shows how the parametric approach (β_{LR}) outperforms both the classic (β_C) and SDP (β_{SDP}) alternatives in terms of bias (ME) and error variability size (RMSE). It is important to point out that, although the parametric approach only reduces the bias slightly with regards to the SDP, the differences found in terms of RMSE are more apparent, where the (β_{LR}) improves considerably the SDP approach.

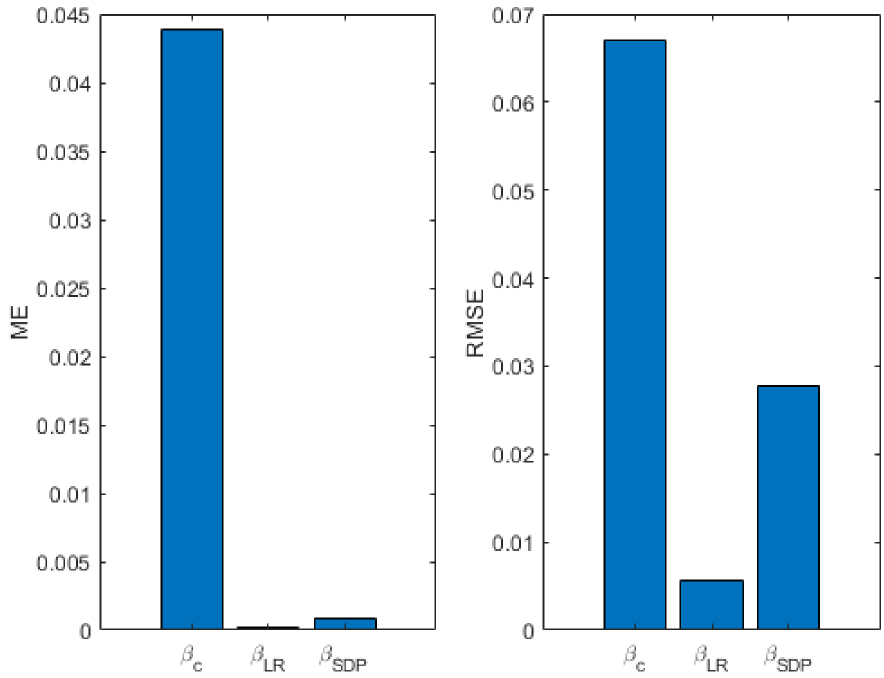

Fig. 11. ME (left panel) and RMSE (right panel) obtained for the different fill rate estimates with respect to the achieved fill rate for the test set.

5. DISCUSSION AND PRACTICAL IMPLICATIONS

The previous section shows that the parametric non-linear model obtained the best estimates of the fill rate. It is interesting to note that, observing Fig. 11, if no bias correction method is considered, the classic approach can provide an overestimate of almost 7 % in the estimation of the fill rate. In practical environments, where the fill rate is used to measure the performance in terms of customer service, number of backorders or to design inventory policies, β_a is initially unknown. Then, the question is how β_{LR} should be used to improve the fill rate estimation. The answer is straightforward. Managers set a target fill rate (β_{target}), which is the minimum fill rate the system should achieve, so that β_a is an upper bound of β_{target}. Hence, without loss of generality, $\beta_a \approx \beta_{target}$ .Therefore, $\beta_{LR} = \theta_0 + (\theta_1 + \theta_2 \cdot \beta_{target} + \theta_3 \cdot \beta_{target}^2) \cdot \beta_c$

To illustrate how β_{LR} outperforms the classic fill rate we show in Table 3 an example of how β_{LR} can help inventory managers in the decision making process. In this example, demand is Negative Binomial distributed with r=2 and θ=0.5, the order-up-to-level is set in 20 units and the lead time is 3. In this case, if the target fill rate is 0.90, with the classical approach the target is reached with s=6 whereas with the analytics fill rate (β_{LR}) it is reached with a ROP equal to 8 units. If we simulate this system (using the same data), we realised a fill rate equal to 0.90 can be only reached when s=8 whereas with s=6 the achieved fill rate is only 0.87. Therefore if managers use the classic estimation, they expect a performance in terms of fulfilling the demand that is not real indeed. Therefore, the number of demand units that are lost is higher than expected. This leads to an unexpected increase of stockout costs. As a conclusion, the replenishment parameters that results when using the classic estimation are not adequate to reach the target fill rate. This example shows the importance that neglecting undershoots has on inventory systems and how to reduce them with the analytics approach suggested in this paper.

s	β_{target}	β_c	β_{LR}
1	0.75	0.79	0.74
2	0.75	0.82	0.75
3	0.80	0.84	0.80
4	0.80	0.87	0.80
5	0.85	0.89	0.85
6	0.85	0.91	0.86
7	0.85	0.93	0.86
8	0.90	0.95	0.91
9	0.90	0.96	0.92

Table 3. ROP computed by β_c and β_{LR} with Negative Binomial demand with r=2, θ=0.5, L=3 and S=20

6. SUMMARY AND CONCLUSIONS

This paper focuses on the versatile Order-Point, Order-Up-to-Level (s, S) inventory system in the lost sales context. The control procedure of this policy consists of reviewing continuously the status of the inventory in order to know exactly when the inventory position is at or below the ROP to launch a replenishment order to raise it to the order-up-to-level, S. Despite this system is widely used in practice, normally it is implemented assuming that undershoots at ROP are negligible i.e. the inventory position always reaches exactly the ROP, which is only possible if demand transactions are unit sized. However, if demand is not unitary, the assumption regarding neglecting undershoots leads to bias the performance of the system. This paper shows both conceptually and experimentally that the classical formula to compute the fill rate is biased. It can provide an overestimate of almost 7% in the estimation of the fill rate indeed and lead to make wrong decisions that directly affects the performance and costs of the inventory system.

This paper proposes a new methodology from a data driven perspective that uses a non-parametric SDP algorithm to model the bias and, subsequently, suggests a parametric analytic fill rate that outperforms the classical approach. Note that, despite the unknown complex non-linearities involved in the classic fill rate bias, the SDP has revealed the nature of such a non-linearity, which has been subsequently parameterized by a parametric non-linear model but linear in the parameters that allows the use of the well-known linear regression. The importance of the analytic fill rate is that is unbiased and easily implemented in practice.

Although the SDP approach can be seen as a data-driven tool it is important to distinguish it from other black-box data-driven tools typically associated to machine learning alternatives. In this work, the SDP approach was employed to learn what was the nature of the non-linearities involved in the classic fill rate bias by neglecting the undershoots in the replenishment policy (*s, S*). Once those non-linearities were identified by means of a curve, the expert modeller could come with a parametric solution taking advantage of the solid theory behind parametric theory. In this case, a simple linear regression was enough to model a complex non-linearity. Note that this philosophy follows the Data Based Mechanistic approach proposed by Young (2011), where the author explains that: "*a model should not just explain the time series data well, but it should also provide a mechanistic description of the system under investigation*".

In this work, we have analysed simulated demands that follow i.i.d negative binomial distributions. Further research should extend this study to non-stationary demand scenarios, as well as replacing statistical demand distributions in the fill rate classic formula by probabilistic forecasts of real demands. In addition, further research is also needed to extend these promising results to other replenishment policies.

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REFERENCES

Bijvank, M., Huh, W.T., Janakiraman, G., Kang, W., 2014. Robustness of order-up-to policies in lost-sales inventory systems. Oper. Res. 62, 1040–1047.

Bijvank, M., Johansen, S.G., 2012. Periodic review lost-sales inventory models with compound Poisson demand and constant lead times of any length. Eur. J. Oper. Res. 220, 106–114.

Bijvank, M., Vis, I.F.A., 2011. Lost-sales inventory theory: A review. Eur. J. Oper. Res. 215, 1–13.

Brown, R.G., 1962. Smoothing Forecasting and Prediction of Discrete Time Series. Prentice-Hall, New Jersey.

Cardós, M., Babiloni, E., 2011a. Exact and approximate calculation of the cycle service level in periodic review inventory policies. Int. J. Prod. Econ. 131, 63–68.

Cardós, M., Babiloni, E., 2011b. Exact and approximated calculation of the cycle service level in a continuous review policy. Int. J. Prod. Econ. 133, 251–255. https://doi.org/10.1016/j.ijpe.2010.09.017

Cardós, M., Guijarro, E., Babiloni, E., 2017. On the estimation of on-hand stocks for base-stock policies and lost sales systems and its impact on service measures. Int. J. Prod. Res. 55, 4680–4694. https://doi.org/10.1080/00207543.2017.1279759

Cohen, M.A., Kleindorfer, P.R., Lee, H.L., 1988. Service Constrained (s, S) Inventory Systems with Priority Demand Classes and Lost Sales. Manag. Sci. 34, 482–499. https://doi.org/10.1287/mnsc.34.4.482

Gandomi, A., Haider, M., 2015. Beyond the hype: Big data concepts, methods, and analytics. Int. J. Inf. Manag. 35, 137–144.

Guijarro, E., Cardós, M., Babiloni, E., 2012. On the exact calculation of the fill rate in a periodic review inventory policy under discrete demand patterns. Eur. J. Oper. Res. 218, 442–447. https://doi.org/10.1016/j.ejor.2011.11.025

Harvey, A.C., 1990. Forecasting, Structural Time Series Models and the Kalman Filter. Cambridge University Press.

Huber, J., Müller, S., Fleischmann, M., Stuckenschmidt, H., 2019. A data-driven newsvendor problem: From data to decision. Eur. J. Oper. Res. 278, 904–915. https://doi.org/10.1016/j.ejor.2019.04.043

Johansen, S.G., 2013. Modified base-stock policies for continuous-review, lost-sales inventory models with Poisson demand and a fixed lead time. Int. J. Prod. Econ., Focusing on Inventories: Research and Applications 143, 379–384. https://doi.org/10.1016/j.ijpe.2012.02.021

Johansen, S.G., Hill, R.M., 2000. The (r, Q) control of a periodic-review inventory system with continuous demand and lost sales. Int. J. Prod. Econ. 68, 279–286.

Karlin, S., Scarf, H., 1958. Studies in the Mathematical Theory of Inventory and Production. Stanford University Press, Stanford, CA.

Kouki, C., Babai, M.Z., Jemai, Z., Minner, S., 2018. Solution procedures for lost sales base-stock inventory systems with compound Poisson demand. Int. J. Prod. Econ.

Larsen, C., Thorstenson, A., 2008. A comparison between the order and the volume fill rate for a base-stock inventory control system under a compound renewal demand process. J. Oper. Res. Soc. 59, 798–804. https://doi.org/10.1057/palgrave.jors.2602407

Liberopoulos, G., Tsikis, I., Delikouras, S., 2010. Backorder penalty cost coefficient "b": What could it be? Int. J. Prod. Econ. 123, 166–178.

Schneider, H., 1981. Effect of service-levels on order-points or order-levels in inventory models. Int. J. Prod. Res. 19, 615–631.

Schneider, H., 1979. The service level in inventory control systems. Eng. Process Econ. 4, 341–348. https://doi.org/10.1016/0377-841X(79)90046-9

Silver, E.A., 2008. Inventory Management: An Overview, Canadian Publications, Practical Applications and Suggestions for Future Research. INFOR Inf. Syst. Oper. Res. 46, 15–27. https://doi.org/10.3138/infor.46.1.15

Silver, E.A., Peterson, R., Pyke, D.F., 1998. Inventory management and production planning and scheduling, 3rd. ed. John Wiley & Sons, New York.

Silver, E.A., Pyke, D.F., Thomas, D.J., Pyke, D.F., Thomas, D.J., 2017. Inventory and Production Management in Supply Chains. CRC Press. https://doi.org/10.1201/9781315374406

Strijbosch, L.W.G., Moors, J.J.A., de Kok, A.G., 1997. On the interaction between forecasting and inventory control (No. FEW 742), Research Memorandum. Tilburg University, School of Economics and Management.

Syntetos, A.A., Boylan, J.E., Croston, J.D., 2005. On the categorization of demand patterns. J. Oper. Res. Soc. 56, 495–503.

Trapero, J.R., Cardós, M., Kourentzes, N., 2019a. Empirical safety stock estimation based on kernel and GARCH models. Omega 84, 199–211. https://doi.org/10.1016/j.omega.2018.05.004

Trapero, J.R., Cardós, M., Kourentzes, N., 2019b. Quantile forecast optimal combination to enhance safety stock estimation. Int. J. Forecast., Special Section: Supply Chain Forecasting 35, 239–250. https://doi.org/10.1016/j.ijforecast.2018.05.009

Trapero, J.R., Fildes, R., Davydenko, A., 2011. Nonlinear identification of judgmental forecasts effects at SKU level. J. Forecast. 30, 490–508. https://doi.org/10.1002/for.1184

Vincent, P., 1985. Exact fill rates for items with erratic demand patterns. INFOR Inf. Syst. Oper. Res. 23, 171–181.

Wang, G., Gunasekaran, A., Ngai, E.W.T., Papadopoulos, T., 2016. Big data analytics in logistics and supply chain management: Certain investigations for research and applications. Int. J. Prod. Econ. 176, 98–110. https://doi.org/10.1016/j.ijpe.2016.03.014

Young, P.C., 2011. Recursive Estimation and Time-Series Analysis: An Introduction for the Student and Practitioner. Springer Science & Business Media.

Young, P.C., McKenna, P., Bruun, J., 2001. Identification of non-linear stochastic systems by state dependent parameter estimation. Int. J. Control 74, 1837–1857.

Zipkin, P., 2008a. Old and New Methods for Lost-Sales Inventory Systems. Oper. Res. 56, 1256–1263.

Zipkin, P., 2008b. On the structure of lost-sales inventory models. Oper. Res. 56, 937–944.

APPENDIX A

In this appendix is shown the recursive filtering/smoothing algorithms that usually are employed to reflect the TVP evolution (Young, 2011) and how those have been modified to extend it for SDP estimation. More details about this methodology can be found in (Young, 2011; Young et al., 2001).

An overall SS model can be defined by these two equations:

Observation equation: $y_t = \boldsymbol{H}_t \boldsymbol{x}_t + e_t$

State equation: $\boldsymbol{x}_t = \boldsymbol{F}\boldsymbol{x}_{t-1} + \boldsymbol{G}\boldsymbol{\eta}_t$ (A.1)

where $\boldsymbol{x}_t$ is known as the state vector. In our case, the observation equation is defined in (5) and the state equation in (4). Then, $\boldsymbol{x}_t = [\alpha_{1,t}, \alpha^*_{1,t}]^T$. The rest of matrices and vectors are defined such as:

$$\boldsymbol{F} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{G} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{H}_t = (\beta_c \quad 0), \quad \boldsymbol{\eta}_t = (0 \quad \eta^*_t)^T \qquad \text{(A.2)}$$

The white noise inputs $\boldsymbol{\eta}_t = [0\, \eta^*(k)]^T$ are assumed to be independent of the observation noise σ^2 and has a covariance matrix **Q**. These noise inputs determine the stochastic behaviour of the states.

The SDP estimation can be divided in two steps. First, forward and backward recursive algorithms are employed. Second, a backfitting algorithm is applied with the sorted data with respect the dependent variable. Note that both the backward pass smoothing and backfitting algorithms can be applied only if it is not necessary to work in real time.

1.1 Forward pass recursive LS equations

Prediction:

$$\hat{\boldsymbol{x}}_{t|t-1} = \boldsymbol{F}\hat{\boldsymbol{x}}_{t-1}$$

$$\boldsymbol{P}_{t|t-1} = \boldsymbol{F}\boldsymbol{P}_{t-1}\boldsymbol{F}^T + \boldsymbol{G}\boldsymbol{Q}_r\boldsymbol{G}^T \quad \text{(A.3)}$$

Correction:

$$\hat{\boldsymbol{x}}_t = \hat{\boldsymbol{x}}_{t|t-1} + \boldsymbol{P}_{t|t-1}\boldsymbol{H}_t^T\left[1 + \boldsymbol{H}_t\boldsymbol{P}_{t|t-1}\boldsymbol{H}_t^T\right]^{-1}(y_t - \boldsymbol{H}_t\hat{\boldsymbol{x}}_{t|t-1})$$

$$\boldsymbol{P}_t = \boldsymbol{P}_{t|t-1} - \boldsymbol{P}_{t|t-1}\boldsymbol{H}_t^T\left[1 + \boldsymbol{H}_t\boldsymbol{P}_{t|t-1}\boldsymbol{H}_t^T\right]^{-1}\boldsymbol{H}_t\boldsymbol{P}_{t|t-1} \quad \text{(A.4)}$$

1.2. Backward pass smoothing equations

$$\hat{\boldsymbol{x}}_{t|N} = \boldsymbol{F}^{-1}\left[\hat{\boldsymbol{x}}_{t+1|N} + \boldsymbol{G}\boldsymbol{Q}_r\boldsymbol{G}^T\boldsymbol{L}_t\right]$$

$$\boldsymbol{L}_t = \left[\boldsymbol{I} - \boldsymbol{P}_{t+1}\boldsymbol{H}_{t+1}^T\boldsymbol{H}_{t+1}\right]^T[\boldsymbol{F}^T\boldsymbol{L}_{t+1} - \boldsymbol{H}_{t+1}^T(y_{t+1} - \boldsymbol{H}_{t+1}\hat{\boldsymbol{x}}_{t+1})]$$

$$\boldsymbol{P}_{t|N} = \boldsymbol{P}_t + \boldsymbol{P}_t\boldsymbol{F}^T\boldsymbol{P}_{t+1|N}^{-1}\left[\boldsymbol{P}_{t+1|N} - \boldsymbol{P}_{t+1|t}\right]\boldsymbol{P}_{t+1|t}^{-1}\boldsymbol{F}\boldsymbol{P}_t$$

$$\boldsymbol{L}_t = 0 \quad \text{(A.5)}$$

Note that $\boldsymbol{Q}_r$ and $\boldsymbol{P}_t$ are normalized with respect to the observation equation noise (σ^2) such as:

$$\boldsymbol{Q}_r = \frac{\boldsymbol{Q}}{\sigma^2}; \qquad \boldsymbol{P_t} = \frac{\boldsymbol{P}_t^*}{\sigma^2} \quad \text{(A.6)}$$

$\boldsymbol{P}_t^*$ is the error covariance matrix related to the state estimates. The parameters inside that NVR matrix are estimated by Maximum likelihood prior to applying the recursive algorithms (Young et al., 2001).

2. Backfitting algorithm for SDP models.

1. Assume that FIS estimation has yielded an initial TVP estimate of $\hat{\alpha}_{1,t|N}^0$
2. Iterate $k = 1,2, \ldots, k_c$
    a. sort both β_a and β_c according to the ascending order of β_a.

b. obtain a FIS estimate $\widehat{\alpha}^{k}_{1,t|N}$ in the relationship $\beta_a = \alpha_1(\beta_a) \cdot \beta_c$

3. Continue Step 2 until iteration k_c, that is when the SDP (which is a series of length N) remains approximately constant.