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# Krylov's Solver Based Technique for the Cascade Connection of Multiple N-Port Multimodal Scattering Matrices

Angel Belenguer, *Member, IEEE,* Elena Diaz Caballero, *Member, IEEE,* Héctor Esteban, *Member, IEEE,* Alejandro L. Borja, and Joaquín Cascón, *Member, IEEE*

***Abstract*—In this paper a novel technique for cascading generalized scattering matrices based on a Krylov's iterative solver is presented. This new technique is fully general, since it can be applied to solve the connection of an arbitrary number of networks, each one with an arbitrary number of ports, and it is easy to implement. This technique is able to compute, not only the modal spectra at the free ports of the global network, but also the modal spectra at the connected ports, so that the field inside the full network can be computed for an arbitrary incidence. In addition, this technique can also be used to evaluate the scattering parameters of the global network.**

***Index Terms*—Cascade circuits and systems, microwave devices, scattering matrices.**

## I. Introduction

THE traditional approach for the cascade connection of generalized scattering matrices (GSM) [1], [2] was first studied by the late fifties and early sixties. First, the cascade of two two-port devices was solved and, almost at the same time, Redheffer [3] developed a special matrix product (in non-standard notation), known as "star" product, which can be used to cascade the GSMs of two 2N-port devices connected through $N$ ports. Next, Kaplan and Stock [4], [5] demonstrated that this "star" product was useful to solve the general case, that is, the cascade connection of two devices with an arbitrary number of ports, for example $N$ and $M$ respectively, and also an arbitrary number of connections, for example $K$. Besides, there are other ways of computing the cascade connection of two multiport devices other than this "star" product. For instance, one can use methods based on signal flow analysis [6], or one can force the equality between reflected and incident waves at the connected ports [7], a technique that we will use in this paper to pose the initial system of equations which we will then solve by using a Krylov's iterative solver [8], [9]. A short review of these approaches can be read in [10]. Moreover, some efforts have been recently made in order to be able to connect S-parameter networks in a multilayer printed circuit board [11].

E. Diaz Caballero and H. Esteban are with the Departamento de Comunicaciones, Universidad Politécnica de Valencia, 46022 Valencia, Spain (e-mail: eldiaca@iteam.upv.es; hesteban@dcom.upv.es).

A. Belenguer, A. L. Borja and J. Cascon are with the Departamento de Ingeniería Eléctrica, Electrónica, Automática y Comunicaciones, Escuela Universitaria Politécnica de Cuenca, Universidad de Castilla-La Mancha, Campus Universitario, 16071 Cuenca, Spain (e-mail: angel.belenguer@uclm.es; alejandro.lucas@uclm.es; joaquin.cascon@uclm.es).

The cascade connection of GSMs has been and is still widely applied to the efficient modeling of high frequency devices. These devices are divided into simple building blocks (steps, resonators, lines, etc.), and then the GSMs of each building block are obtained. Next, the cascade connection of the GSMs of all the building blocks is iteratively calculated by pairs so that the GSM of the whole structure is finally found. Recent examples of the practical application of this procedure can be read in [12]–[14]. This cascading procedure will be briefly summarized in the next section, and if we take a look at expressions (3)-(6), it has a cost of $O(N_{mod}^3)$ operations, where $N_{mod}$ the number of modes considered for each port. This cost is very low, and this is the reason why this technique is commonly employed in the characterization of a wide range of microwave devices.

Unfortunately, the traditional cascading approach ignores the connected ports. Once the cascading is complete, the modal response at the connected ports cannot be obtained. Or in other words, the connected network is completely characterized from the outside but the inside is now unknown. However, under many circumstances, it is desirable to know the internal response of the device. This could be of great interest, for example, if undesirable effects, like multipactor or corona discharge, have to be studied. In this case, if the internal response of the network was known, the fields could be recovered inside the connected network and these undesirable effects characterized.

Therefore, in this work, we propose an efficient strategy to compute the GSM of the general cascade connection of an arbitrary number of devices with an arbitrary number of ports and connections, which is, at the same time, able to provide, for an arbitrary incidence, the modal weights, not only at the free ports, but also at the connected ones. This technique is comparable, in efficiency, to the traditional cascading-by-pairs approach due to the fact that we have used a Krylov's iterative solver to find the solution and, besides, it can be used to compute the field inside the connected network for an arbitrary incidence.

## II. Cascading-by-pairs approach

At this point we will briefly summarize the traditional approach for the cascade connection of two devices with an arbitrary number of ports.

Let us suppose that we have an arbitrary device, for example device $n$, with $A^{(n)}$ ports, $c^{(n)}$ of whom will be connected to other devices. Considering $N_i^{(n)}$ modes for the i-th port of this device, the total number of modes studied for device $n$ would be $N^{(n)} = \sum_{i=1}^{A^{(n)}} N_i^{(n)}$.

This device can be completely characterized if its generalized scattering matrix (GSM), $\underline{\underline{S}}^{(n)}$, is found. This matrix, whose size is $N^{(n)} \times N^{(n)}$, relates the weights of the incident modes to the weights of the reflected modes [1]

$$\underline{b}^{(n)} = \underline{\underline{S}}^{(n)} \underline{a}^{(n)} \qquad (1)$$

where $\underline{a}^{(n)}$ and $\underline{b}^{(n)}$ are the weights of the incident and reflected modes respectively.

First of all, it is necessary to reorder the rows and columns of the GSM so that we can distinguish two sets of parameters, those belonging to the free ports and those belonging to the connected ports. After doing that, we can divide the GSM of device $n$ into four blocks and expression (1) can be rewritten:

$$\begin{pmatrix} \underline{b}_F^{(n)} \\ \underline{b}_C^{(n)} \end{pmatrix} = \begin{pmatrix} \underline{\underline{S}}_{FF}^{(n)} & \underline{\underline{S}}_{FC}^{(n)} \\ \underline{\underline{S}}_{CF}^{(n)} & \underline{\underline{S}}_{CC}^{(n)} \end{pmatrix} \begin{pmatrix} \underline{a}_F^{(n)} \\ \underline{a}_C^{(n)} \end{pmatrix} \qquad (2)$$

where $\underline{a}_F^{(n)}$ and $\underline{b}_F^{(n)}$ are, respectively, the incident and reflected modal weights at the free ports and whose size is $\sum_{i=1}^{A^{(n)}-c^{(n)}} N_i^{(n)}$; and $\underline{a}_C^{(n)}$ and $\underline{b}_C^{(n)}$ are, respectively, the incident and reflected modal weights at the connected ports and whose size is $\sum_{i=A^{(n)}-c^{(n)}}^{A^{(n)}} N_i^{(n)}$.

If, similarly, we reorder and group the parameters for a second device, for example device $m$, the blocks of the cascade connection of devices $n$ and $m$ can be computed applying [5], [10]

$$\begin{aligned} \underline{\underline{S}}_{11}^{(T)} &= \underline{\underline{S}}_{FF}^{(n)} + \underline{\underline{S}}_{FC}^{(n)} \underline{\underline{F}}_1 \underline{\underline{S}}_{CC}^{(m)} \underline{\underline{S}}_{CF}^{(n)} && (3) \\ \underline{\underline{S}}_{12}^{(T)} &= \underline{\underline{S}}_{FC}^{(n)} \underline{\underline{F}}_1 \underline{\underline{S}}_{CF}^{(m)} && (4) \\ \underline{\underline{S}}_{21}^{(T)} &= \underline{\underline{S}}_{FC}^{(m)} \underline{\underline{F}}_2 \underline{\underline{S}}_{CF}^{(n)} && (5) \\ \underline{\underline{S}}_{22}^{(T)} &= \underline{\underline{S}}_{FF}^{(m)} + \underline{\underline{S}}_{FC}^{(m)} \underline{\underline{F}}_2 \underline{\underline{S}}_{CC}^{(n)} \underline{\underline{S}}_{CF}^{(m)} && (6) \end{aligned}$$

where

- $\underline{\underline{F}}_1 = \left( \underline{\underline{I}} - \underline{\underline{S}}_{CC}^{(m)} \underline{\underline{S}}_{CC}^{(n)} \right)^{-1}$
- $\underline{\underline{F}}_2 = \left( \underline{\underline{I}} - \underline{\underline{S}}_{CC}^{(n)} \underline{\underline{S}}_{CC}^{(m)} \right)^{-1}$

And the GSM of the whole resulting network is

$$\underline{\underline{S}}^{(T)} = \begin{pmatrix} \underline{\underline{S}}_{11}^{(T)} & \underline{\underline{S}}_{12}^{(T)} \\ \underline{\underline{S}}_{21}^{(T)} & \underline{\underline{S}}_{22}^{(T)} \end{pmatrix} \qquad (7)$$

## III. Krylov's based cascading problem

Considering that we have an arbitrary device $n$, characterized by its GSM (1), some auxiliary matrices are going to be defined. If $\underline{\underline{I}}_{(N)}$, or $\underline{\underline{I}}_{(N,N)}$, is an identity matrix of size $N \times N$, and $\underline{\underline{0}}_{(M,N)}$ a null matrix of size $M \times N$, we can define the two following auxiliary matrices for that device $n$

$$\underline{\underline{I}}^{(n)} = \underline{\underline{I}}_{(N^{(n)})} \qquad (8)$$

$$\underline{\underline{I}}_i^{(n)} = \begin{pmatrix} \underline{\underline{0}}_{(N_{i-}^{(n)}, N_{i-}^{(n)})} & \underline{\underline{0}}_{(N_{i-}^{(n)}, N_i^{(n)})} & \underline{\underline{0}}_{(N_{i-}^{(n)}, N_{i+}^{(n)})} \\ \underline{\underline{0}}_{(N_i^{(n)}, N_{i-}^{(n)})} & \underline{\underline{I}}_{(N_i^{(n)}, N_i^{(n)})} & \underline{\underline{0}}_{(N_i^{(n)}, N_{i+}^{(n)})} \\ \underline{\underline{0}}_{(N_{i+}^{(n)}, N_{i-}^{(n)})} & \underline{\underline{0}}_{(N_{i+}^{(n)}, N_i^{(n)})} & \underline{\underline{0}}_{(N_{i+}^{(n)}, N_{i+}^{(n)})} \end{pmatrix} \qquad (9)$$

where $\underline{\underline{I}}_i^{(n)}$ is of size $(N^{(n)} \times N^{(n)})$ and

- $N_{i-}^{(n)} = \sum_{j=1}^{i-1} N_j^{(n)}$,
- $N_{i+}^{(n)} = \sum_{j=i+1}^{A^{(n)}} N_j^{(n)}$,
- $\underline{\underline{I}}^{(n)} = \sum_{i=1}^{A^{(n)}} \underline{\underline{I}}_i^{(n)}$.

The auxiliary matrix $\underline{\underline{I}}_i^{(n)}$ is very useful, because it can be used to get the modes of the $i$-th port easily

$$\begin{pmatrix} \underline{\underline{0}}_{(N_{i-}^{(n)},1)} \\ \underline{a}_i^{(n)} \\ \underline{\underline{0}}_{(N_{i+}^{(n)},1)} \end{pmatrix} = \underline{\underline{I}}_i^{(n)} \underline{a}^{(n)}; \begin{pmatrix} \underline{\underline{0}}_{(N_{i-}^{(n)},1)} \\ \underline{b}_i^{(n)} \\ \underline{\underline{0}}_{(N_{i+}^{(n)},1)} \end{pmatrix} = \underline{\underline{I}}_i^{(n)} \underline{b}^{(n)} \qquad (10)$$

where $\underline{a}_i^{(n)}$ y $\underline{b}_i^{(n)}$ are the weights of the incident and reflected modes through the $i$-th port respectively.

Let us suppose that another device exists, for example the device $m$, and that we have considered the same number of modes for the $i$-th port of device $m$ as for the $j$-th port of device $n$, that is $N_i^{(m)} = N_j^{(n)}$. In this case, the following auxiliary matrix will be very useful

$$\underline{\underline{I}}_{(i,j)}^{(m,n)} = \begin{pmatrix} \underline{\underline{0}}_{(N_{i-}^{(m)}, N_{j-}^{(n)})} & \underline{\underline{0}}_{(N_{i-}^{(m)}, N_j^{(n)})} & \underline{\underline{0}}_{(N_{i-}^{(m)}, N_{j+}^{(n)})} \\ \underline{\underline{0}}_{(N_i^{(m)}, N_{j-}^{(n)})} & \underline{\underline{I}}_{(N_i^{(m)}, N_j^{(n)})} & \underline{\underline{0}}_{(N_i^{(m)}, N_{j+}^{(n)})} \\ \underline{\underline{0}}_{(N_{i+}^{(m)}, N_{j-}^{(n)})} & \underline{\underline{0}}_{(N_{i+}^{(m)}, N_j^{(n)})} & \underline{\underline{0}}_{(N_{i+}^{(m)}, N_{j+}^{(n)})} \end{pmatrix} \qquad (11)$$

This matrix, whose size is $N^{(m)} \times N^{(n)}$, can be used to get the modal weights of the $j$-th port of device $n$ and place them in a vector of size $N^{(m)} \times 1$ in the position of the modal weights of the $i$-th port of device $m$.

Finally, let us suppose that we connect the $i$-th port of device $m$ to the $j$-th port of device $n$. By connecting both ports, the reflected modes of the $i$-th port of device $m$ will become the incident modes of the $j$-th port of device $n$, and vice versa. So the following equations can be written

$$\underline{\underline{I}}_i^{(m)} \underline{a}^{(m)} = \underline{\underline{I}}_{(i,j)}^{(m,n)} \underline{b}^{(n)} \qquad (12)$$

$$\underline{\underline{I}}_j^{(n)} \underline{a}^{(n)} = \underline{\underline{I}}_{(j,i)}^{(n,m)} \underline{b}^{(m)} \qquad (13)$$

However, we have to carefully analyze these equalities. We will be able to interchange the modal weights if, and only if, the reference system used to expand the modes is the same for both connected ports. If the reference systems are not the same, we must change one weight set so that we obtain new weights in terms of the proper reference system. Unfortunately, this is the most common situation.

When the GSM of a given device, for example device $m$, is computed, the most extended criterion is to choose progressive waves (towards $\hat{z}$ propagation) for the incident waves, $\underline{a}^{(m)}$, and regressive waves (towards $-\hat{z}$ propagation) for the reflected waves, $\underline{b}^{(m)}$. It is clear that when two ports are connected, e.g. the $i$-th port of device $m$ and the $j$-th port of device $n$, both waveguides become the same (see Fig. 1). In this case, we can say that $\underline{a}_i^{(m)}$ and $\underline{b}_j^{(n)}$ are modes propagating towards the same direction inside the same waveguide. However, the modes of $\underline{a}_i^{(m)}$ are formulated in terms of progressive waves, towards $\hat{z}$ propagation, while the modes of $\underline{b}_j^{(n)}$ are formulated in terms of regressive waves, towards $-\hat{z}$ propagation. The conclusion is then clear: $\hat{z}$ for the modes of $\underline{a}_i^{(m)}$ is different from $\hat{z}$ for the modes of $\underline{b}_j^{(n)}$. In short, the reference systems used to expand the modes are different, so we cannot directly compare both modal sets (see Fig. 1).

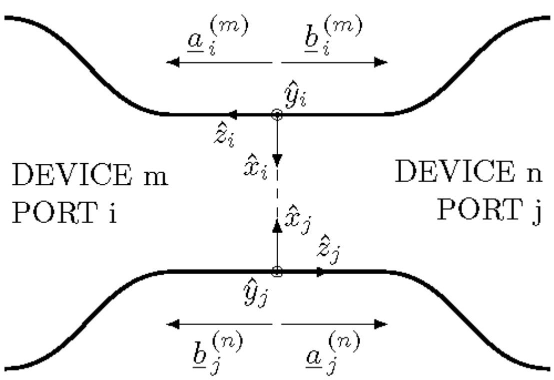


Figure 1. Connection of two ports and their reference systems.

However, if we take a look at Fig. 1, we can see that when the waveguide is symmetric along $\hat{x}$, as it is in the picture, the modes are also symmetric along this dimension. The rectangular waveguide is a good example where all the modes have even or odd symmetry with respect to $\hat{x}$. When the symmetry is even (i.e. $TE_{pq}$ and $TM_{pq}$ with odd $p$), the modes are identical no matter which reference system of Fig. 1 is used. In this case, the modal weights can be directly assigned. On the contrary, if the symmetry is odd (i.e. $TE_{pq}$ and $TM_{pq}$ with an even $p$), the modes are also identical but with opposite sign in each reference system. Therefore, equations (12) and (13) need to be rewritten for a rectangular waveguide, in order to adjust one of the modal sets, for example $\underline{b}$.

For a rectangular waveguide, a diagonal conversion matrix can be defined, whose elements will be $-1$ if the corresponding modal weight belongs to a mode with an even $p$ and 1 otherwise. Equations (12) and (13) should be then rewritten as

$$\underline{\underline{I}}_i^{(m)}\underline{a}^{(m)} = \underline{\underline{I}}_{(i,j)}^{(m,n)}\underline{\underline{D}}^{(n)}\underline{b}^{(n)} \tag{14}$$

$$\underline{\underline{I}}_j^{(n)}\underline{a}^{(n)} = \underline{\underline{I}}_{(j,i)}^{(n,m)}\underline{\underline{D}}^{(m)}\underline{b}^{(m)} \tag{15}$$

where $\underline{\underline{D}}^{(m)}$ and $\underline{\underline{D}}^{(n)}$ are the diagonal matrices already mentioned. These diagonal matrices are identity matrices in case the lines feeding the device to be connected are asymmetric and the polarity of the modes appropriately defined.

Now, since the GSMs of each device are known, we have that

$$\underline{b}^{(n)} = \underline{\underline{S}}^{(n)}\underline{a}^{(n)} \Longrightarrow \underline{a}^{(n)} = (\underline{\underline{S}}^{(n)})^{-1}\underline{b}^{(n)} \tag{16}$$

and

$$\underline{b}^{(m)} = \underline{\underline{S}}^{(m)}\underline{a}^{(m)} \Longrightarrow \underline{a}^{(m)} = (\underline{\underline{S}}^{(m)})^{-1}\underline{b}^{(m)} \tag{17}$$

Equation (16) for device $n$ can also be written as

$$(\underline{\underline{I}}^{(n)} - \underline{\underline{I}}_j^{(n)})\underline{a}^{(n)} + \underline{\underline{I}}_j^{(n)}\underline{a}^{(n)} = (\underline{\underline{S}}^{(n)})^{-1}\underline{b}^{(n)} \tag{18}$$

and then, applying (15) to (18), we obtain

$$(\underline{\underline{I}}^{(n)} - \underline{\underline{I}}_j^{(n)})\underline{a}^{(n)} = (\underline{\underline{S}}^{(n)})^{-1}\underline{b}^{(n)} - \underline{\underline{I}}_{(j,i)}^{(n,m)}\underline{\underline{D}}^{(m)}\underline{b}^{(m)} \tag{19}$$

The same can be done for device $m$ and expressions (17) and (14)

$$(\underline{\underline{I}}^{(m)} - \underline{\underline{I}}_i^{(m)})\underline{a}^{(m)} + \underline{\underline{I}}_i^{(m)}\underline{a}^{(m)} = (\underline{\underline{S}}^{(m)})^{-1}\underline{b}^{(m)} \tag{20}$$

$$(\underline{\underline{I}}^{(m)} - \underline{\underline{I}}_i^{(m)})\underline{a}^{(m)} = -\underline{\underline{I}}_{(i,j)}^{(m,n)}\underline{\underline{D}}^{(n)}\underline{b}^{(n)} + (\underline{\underline{S}}^{(m)})^{-1}\underline{b}^{(m)} \tag{21}$$

Equations (19) and (21) form a matrix system that can be used to find the weights of the reflected modes at every port, even at the connected ones, for a given incidence. Furthermore, we can simplify these equations if we multiply (19) and (21) by $\underline{\underline{S}}^{(n)}$ and $\underline{\underline{S}}^{(m)}$ respectively

$$\underline{\underline{S}}^{(n)}(\underline{\underline{I}}^{(n)} - \underline{\underline{I}}_j^{(n)})\underline{a}^{(n)} = \underline{b}^{(n)} - \underline{\underline{S}}_{(j,i)}^{(n,m)}\underline{\underline{D}}^{(m)}\underline{b}^{(m)} \tag{22}$$

$$\underline{\underline{S}}^{(m)}(\underline{\underline{I}}^{(m)} - \underline{\underline{I}}_i^{(m)})\underline{a}^{(m)} = -\underline{\underline{S}}_{(i,j)}^{(m,n)}\underline{\underline{D}}^{(n)}\underline{b}^{(n)} + \underline{b}^{(m)} \tag{23}$$

where

- $\underline{\underline{S}}_{(j,i)}^{(n,m)} = \underline{\underline{S}}^{(n)}\underline{\underline{I}}_{(j,i)}^{(n,m)}$ is a matrix of size $N^{(n)} \times N^{(m)}$ whose elements are

$$\underline{\underline{S}}_{(j,i)}^{(n,m)} = \begin{pmatrix} & \underline{\underline{S}}_{1j}^{(n)} & \\ & \vdots & \\ \underline{\underline{0}}_{(N^{(n)},N_{i-}^{(m)})} & \underline{\underline{S}}_{jj}^{(n)} & \underline{\underline{0}}_{(N^{(n)},N_{i+}^{(m)})} \\ & \vdots & \\ & \underline{\underline{S}}_{A^{(n)}j}^{(n)} & \end{pmatrix} \tag{24}$$

- $\underline{\underline{S}}_{(i,j)}^{(m,n)} = \underline{\underline{S}}^{(m)}\underline{\underline{I}}_{(i,j)}^{(m,n)}$ is a matrix of size $N^{(m)} \times N^{(n)}$ which can be computed applying (24) if $i$ and $j$ and, at the same time, $m$ and $n$, are interchanged.

Finally, at this point, a completely general situation can be analyzed. We will consider $N$ devices, $n = 1, 2, \ldots N$, every device connected to others through one or more ports. In this case, we could define $C^{(n)}$ as the set of connected ports for device $n$. We could also define the set of connections $C^{(n,m)}$, composed by the pairs $(j, i)$ which satisfy that the $j$-th port of device $n$ is connected to the $i$-th port of device $m$. By using both sets, a generalization of (22) can be written as

$$\underline{e}^{(n)} = \underline{b}^{(n)} - \sum_{m \neq n} \underline{\underline{S}}^{(n,m)}\underline{b}^{(m)} \tag{25}$$

where

$$\underline{e}^{(n)} = \underline{\underline{S}}^{(n)} \left( \underline{\underline{I}}^{(n)} - \sum_{i \in C^{(n)}} \underline{\underline{I}}_i^{(n)} \right) \underline{a}^{(n)} \tag{26}$$

$$\underline{\underline{S}}^{(n,m)} = \begin{cases} \underline{\underline{0}}_{(N^{(n)},N^{(m)})} & \text{if } C^{(n,m)} \equiv \emptyset \\ \sum\limits_{(j,i) \in C^{(n,m)}} \underline{\underline{S}}_{(j,i)}^{(n,m)} \underline{\underline{D}}^{(m)} & \text{otherwise} \end{cases} \tag{27}$$

Solving eq. (25) for all the devices will allow us to obtain not only the response of the whole connection for a given incidence, but also the response or modal weights at the connected ports.

## IV. Computational cost

The system of equations obtained from (25) can be written in matrix form as

$$\underline{e} = \underline{\underline{M}}\, \underline{b} \tag{28}$$

where the matrix of coefficients is a highly sparse matrix,

$$\underline{\underline{M}} = \begin{pmatrix} \underline{\underline{I}}^{(1)} & -\underline{\underline{S}}^{(1,2)} & \cdots & -\underline{\underline{S}}^{(1,N)} \\ -\underline{\underline{S}}^{(2,1)} & \underline{\underline{I}}^{(2)} & \cdots & -\underline{\underline{S}}^{(2,N)} \\ \vdots & \vdots & \ddots & \vdots \\ -\underline{\underline{S}}^{(N,1)} & -\underline{\underline{S}}^{(N,2)} & \cdots & \underline{\underline{I}}^{(N)} \end{pmatrix} \tag{29}$$

because the main diagonal blocks are also diagonal matrices and the rest of blocks, $\underline{\underline{S}}^{(n,m)}$, are mainly null matrices and, the few which are not null are highly sparse (see eqs. (24) and (27)).

This high sparsity is the reason why we will solve this equation system using an Krylov's iterative method [8], [9] instead of other traditional methods for solving equation systems [15]. These iterative solvers can find the solution of (28) by applying an algorithm which iteratively converges to the wanted solution, where the most time-consuming operation in every iteration is the product of the coefficients matrix ($\underline{\underline{M}}$) by an arbitrary vector ($\underline{b}$) of appropriate size. So this will be the key operation in order to estimate the cost of solving (28). In this work, a null vector has been used as initial guess for vector $\underline{b}$.

First, we will try to evaluate the cost of multiplying a row of $\underline{\underline{M}}$, for example row $n$, by a certain unknown vector $\underline{b}$. This cost will depend directly on the number of connected ports. For every connected port, for example the $j$-th port, a matrix of size $N^{(n)} \times N_j^{(n)}$ must be multiplied by a vector of size $N_j^{(n)} \times 1$ (see (24)). This product represents a cost equal to $N^{(n)} N_j^{(n)}$ operations.

Therefore, if we define $c_n$ as the number of connected ports of device $n$, the full cost of computing the product of the $n$-th row of $\underline{\underline{M}}$ by the corresponding unknown vector will be equal to $c_n N^{(n)} N_{mod}$ operations, where the number of modes of each port has been supposed to be the same, i.e. $N_{mod} = N_j^{(n)}$, $\forall n, j$.

Now, considering the rest of devices, we can say that the cost of computing the target matrix-vector product will be

$$C_{prod} = \sum_{n=1}^{N} c_n N^{(n)} N_{mod} \tag{30}$$

Then, if we replace $N^{(n)}$ by $A^{(n)} N_{mod}$, we get

$$C_{prod} = N_{mod}^2 \sum_{n=1}^{N} c_n A^{(n)} \tag{31}$$

Finally, using the biconjugate gradients stabilized method (Bi-CGSTAB) [16], [17] to solve (28), the cost of solving it is

$$C_{iter} = 2 N_{iter} N_{mod}^2 \sum_{n=1}^{N} c_n A^{(n)} \tag{32}$$

where $N_{iter}$ is the number of iterations required by the Bi-CGSTAB to converge to the wanted solution. We have also considered that this algorithm computes the analyzed matrix-vector product twice per iteration.

We can conclude that the cost remains around a reasonable value. On the one hand, the traditional approach, i.e. a recursive cascade of the GSMs by pairs [3]–[5], [10], would have a cost of $O(N_{mod}^3)$ operations (see eqs. (3)-(6)). On the other hand, the computational cost for the technique presented in this paper (see eq. (32)) is $N_{iter} O(N_{mod}^2)$. Therefore, for a reasonable value of $N_{iter}$, the technique proposed can be comparable to the multimodal implementation of the traditional approach, or even faster, but with the advantage of having the network fully characterized also in the connected ports.

## V. Results

In this section, a particular multiport network is studied so that the new method is verified. An H-diplexer, whose dimensions can be found in [18], has been chosen for this verification task. The layout of the H-diplexer is shown in Fig. 2.

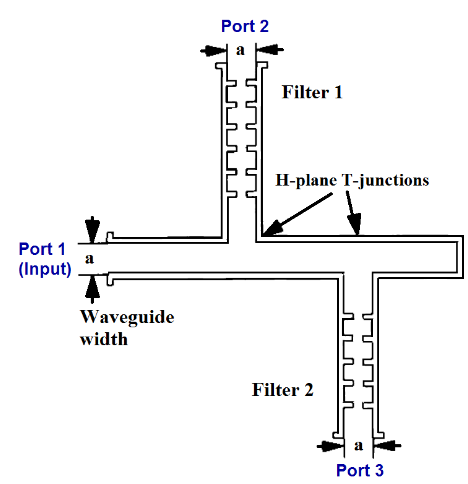


Figure 2. Layout of the studied H-diplexer.

In order to perform the whole analysis of the H-diplexer, first the GSM of each building block has been obtained, and then the whole connection has been performed with the new iterative cascading method. The individual GSMs of the waveguide sections have been analytically calculated, those of the steps have been obtained using mode-matching [19], and, finally, the GSMs of the T-junctions have been found using a mode matching analysis technique for arbitrarily shaped structures [20]. Considering 20 modes at each port and analyzing 301 frequency points, 49.8548 s have been necessary to obtain the GSMs of all the building blocks of the H-diplexer, and just 2.8222 s to perform the whole connection 301 times (one per frequency point). Since the whole network has 51 ports (3 free + 48 connected ports), we can approximate that the computational cost of the cascading method is 183.85 $\mu$s per port and per frequency point.

Furthermore, Fig. 3 shows the comparison between the global S-parameters obtained by the new iterative cascading method and the ones obtained by the FEST3D commercial software [21], which is a very efficient commercial software for the accurate analysis of passive components based on waveguide technology, and we see that both responses match almost perfectly.

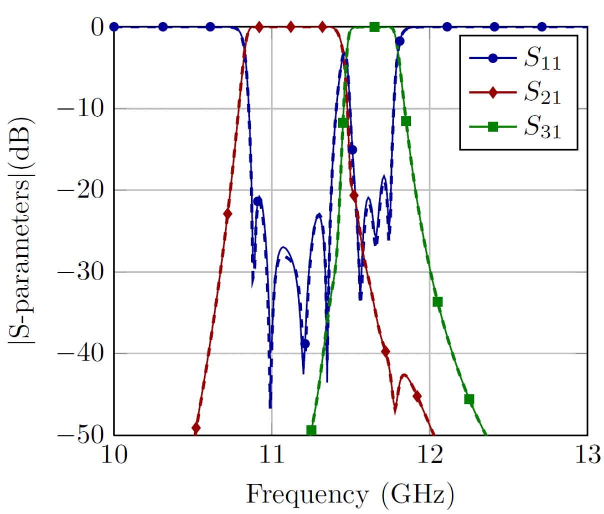


Figure 3. Scattering parameters of the H-diplexer found by the new method (solid + marks) vs. simulation with FEST3D (dashed).

But the main advantage of this new cascading method is that, being accurate and very efficient, not only provides the response of the whole structure for a given incidence, but also the modal weights at the connected ports. This fact enables us to obtain the fields inside the different blocks of the network as summations of incident and reflected modes in the different points of a mesh, as seen in Fig. 4, where the electric field magnitude has been obtained and represented at the central bandpass frequencies of both filters, respectively.

As mentioned in the introduction, obtaining the field inside the device is necessary on many occasions, for instance in order to characterize undesirable effects such as multipactor. Figure 4 is very helpful to identify the "hot spots" for multipactor [22]. We see that the three central cavities of Filter 1 and the two central ones in Filter 2 are the main candidates for multipactor discharge in their respective transmission bands. In this example, the multipactor analysis[1] shows that the elements with lowest multipactor power threshold (critical elements) are the central cavity of Filter 1, which has a power-handling capability of 9375 W at the higher frequency of its transmission band (11.37 GHz), and the second cavity of Filter 2, whose power-handling capability is 7875 W at its lower transmission frequency (11.52 GHz).

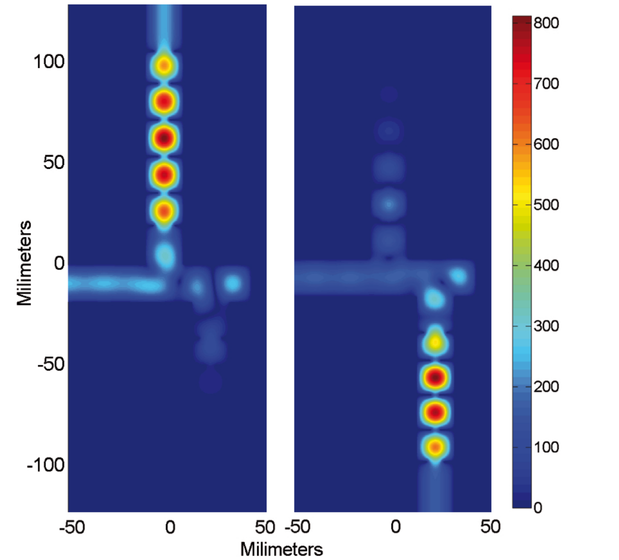


Figure 4. Representation of the electric field magnitude inside the H-diplexer at 11.15 GHz (left) and 11.65 GHz (right).

### *A. Studying other cascading possibilities*

If we ignore the simplest case of cascading two port devices [2], [14], the cascading-by-pairs approach is hard to implement for multimodal and multiport devices since, after each individual connection, successive mode reordering procedures are necessary (see Sec. II). Still, we have also performed this cascading procedure, and it has taken 1.3775 s to perform the whole connection, obtaining exactly the same global S-parameters as in Fig. 3, which demonstrates the accuracy of the new method. It also shows that the fact that the new Krylov's based cascading approach enables us to reconstruct the field inside the network does not significantly increase the cascading computational cost.

We have considered yet another cascading possibility: a modification of the intermediate expressions used in the cascading-by-pairs approach, so that we could introduce some new scattering matrices that would allow us to obtain the modal weights at the connected ports, but without needing any Krylov's iterative solver. Briefly, the definition of three types of matrices is necessary: one to compute the modes emerging through the connected ports of one device $n$ in response to an arbitrary incidence against its free ports, one to compute the emerging modes

[1]Analysis perform for aluminum.

through the connected ports of one device $m$ in response to an arbitrary incidence against the free ports of other device $n$, and one that computes the emerging modes through the connected ports of device $n$ in response to an arbitrary incidence against the free ports of device $m$. By using these matrices, we can compute the response of the network inside the connected ports in response to an arbitrary incidence at the free ports.

This modification of the cascading-by-pairs approach so that an extended GSM is obtained has been also applied to the analysis of the H-diplexer. Once again, as expected, the global response is the same as the one obtained with the method presented (see Fig. 3), but the cascading time is ten times higher: 27.5076 s (1.8 ms per port and per frequency point, against the 183.85 $\mu$s ms needed with the iterative procedure here proposed). This is to reinforce the efficiency of the procedure here presented against other possibilities, also considered by us, for obtaining the same kind of results.

## VI. Conclusion

A technique to efficiently cascade an arbitrary number of devices with an arbitrary number of ports has been presented. This technique is fully general and can be used to compute the response of the connected network to a given incidence and also to obtain its GSM if the excitations are properly chosen.

The fact that this procedure allows to obtain the modal weights also at the connected ports enables us to reconstruct the field inside the network and, for example, study undesired effects as multipactor or corona discharge.

It is worth highlighting that the cost remains around an admissible value. For a reasonable number of iterations, the technique proposed can be even faster than the multimodal implementation of the traditional approach.

Finally, the precision of the technique has been tested. In order to do so, in sec. V, an H-diplexer has been analyzed by solving the cascading of the different constituent blocks, and successfully compared with results obtained with commercial software. The efficiency and advantages of this new procedure have also been proved.

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**Angel Belenguer** (M'04) received his degree in telecommunications engineering from the Universidad Politécnica de Valencia (UPV), Spain, in 2000, and his Ph.D. degree, also from the UPV, in 2009. He joined the Universidad de Castilla-La Mancha in 2000, where he is now Profesor Titular de Universidad in the Departamento de Ingenieria Electrica, Electronica, Automatica y Comunicaciones. His research interests include methods in the frequency domain for the full-wave analysis of open-space and guided multiple scattering problems, the application of accelerated solvers or solving strategies (like grouping) to new problems or structures, EM metamaterials, and Substrate Integrated Waveguide (SIW) devices and their applications.

**Elena Díaz Caballero** (S'07-GSM'10) received the Telecommunication Engineering degree from the Universidad Politécnica de Valencia (UPV), Spain, in 2010. She is now working on her research toward the Ph.D. degree at UPV, where she was awarded Student of the Year in 2010. She won a Collaboration Fellowship at the Communications Department of UPV in 2009, has done an internship at AURORASAT Company and won an Excellence Fellowship from the UPV, as well as some other national grants and awards. She did a PhD stage in the École Politechnique Fédérale de Lausanne, Switzterland, in 2012. Since 2011 she has held several lecturing positions at the UPV. Her research interest is currently focused in the development of efficient methods for the analysis and design of substrate integrated waveguide devices and rectangular waveguide components.

**Héctor Esteban González** (S'93−M'99) received a degree in telecommunications engineering from the Universidad Politécnica de Valencia (UPV), Spain, in 1996, and a Ph.D. degree in 2002. He worked with the Joint Research Centre, European Commission, Ispra, Italy. In 1997, he was with the European Topic Centre on Soil (European Environment Agency). He rejoined the UPV in 1998. His research interests include methods for the full-wave analysis of open-space and guided multiple scattering problems, CAD design of microwave devices, electromagnetic characterization of dielectric and magnetic bodies, and the acceleration of electromagnetic analysis methods using the wavelets and the FMM.

**Alejandro L. Borja** received the M.Sc. degree in Telecommunication Engineering and PhD from the Universidad Politécnica de Valencia, Valencia, Spain, in 2004 and 2009, respectively. From 2005 to 2006, he was with the Communication Group, University of Birmingham, where he was involved with the research and development of metamaterial based antennas. He then joined, from 2007 to 2008, the Institut d'Électronique de Microélectronique et Nanotechnologies (IEMN), Université des Sciences et Technologies de Lille 1, his research activity included the design of metamaterial based structures with frequency selective properties. Since 2009, he is with the Departamento de Ingeniería Eléctrica, Electrónica, Automática y Comunicaciones, Universidad de Castilla-La Mancha, where he is an Assistant Lecturer. His current research interests include EM metamaterials, UWB antennas, Substrate Integrate Waveguide (SIW) devices, tunable structures and their applications in microwave and millimeter-wave technologies. Dr. Borja was the recipient of the 2008 CST short paper award.

**Joaquín Cascón** was born in Cuenca, Spain, in 1966. He received his degree in Telecommunications engineering from the Universidad Politécnica de Madrid (UPM), Spain, in 1991, and PhD from the Universidad Politécnica de Valencia, (UPV), Spain, in 1999. From 1991 to 1999 he was with the Optical Communication Group, UPV, where he worked in Optical Communications. In 1999 he joined the Departamento de Ingeniería Eléctrica, Electrónica, Automática y Comunicaciones, Universidad de Castilla-La Mancha (Spain), he is now Catedrático de Escuela Universitaria . His current research interest include EM metamaterials, antennas, tunable structures and their applications in microwave and millimeter-wave technologies.