University of Castilla-La Mancha

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Programación Lógica Difusa para la Gestión Flexible de Documentos XML

Fuzzy Logic Programming for the Flexible Management of XML Documents

TESIS DOCTORAL

Presentada por:

Alejandro Luna Tedesqui

Dirigida por:

Ginés Moreno Valverde (UCLM)

Jesús Manuel Almendros Jiménez (U. Almería)

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Alejandro Luna Tedesqui

Departamento de Sistemas Informáticos
Universidad de Castilla-La Mancha

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Resumen

Esta tesis presenta una extensión del popular lenguaje XPath, que ofrece respuestas una lista de respuestas ordenadas a una consulta flexible aprovechando las variantes difusas de los operadores *and*, *or* y *avg* para las condiciones XPath, así como dos restricciones estructurales, llamadas *down* y *deep*, para el que se asocia un cierto grado de relevancia. En la práctica, este grado es muy bajo para algunas respuestas obtenidas con la consulta original, y por lo tanto, no deberian ser calculadas, con el fin de aliviar la complejidad computacional del proceso de recuperación de información. Con el fin de mejorar la escalabilidad de nuestro intérprete para hacer frente a archivos XML grandes, hacemos uso de la capacidad de la programación lógica difusa para descartar de forma anticipada los cálculos que conducen a soluciones poco significativas (es decir, con un pobre grado de relevancia según las preferencias expresadas por los usuarios cuando usan el nuevo comando FILTER). Nuestra propuesta se ha implementado en un lenguaje lógico difuso, aprovechando los altos recursos expresivos de este paradigma declarativo para la gestion de "umbrales dinámicos" de una manera natural y eficiente. Además de utilizar nuestro entorno FLOPER para desarrollar el intérprete, también proponemos su implementación con el lenguaje estándar XQuery. Básicamente, definimos una biblioteca XQuery capaz de gestionar de forma difusa expresiones XPath, de tal manera que nuestro FuzzyXPath puede ser codificado como expresiones XQuery. Las ventajas de nuestro enfoque es que cualquier interprete XQuery puede manipular una versión borrosa de XPath mediante el uso de la biblioteca que hemos implementado.

Por otro lado, se presenta un método para depurar consultas XPath, describiendo cómo las expresiones XPath puede manipularse para obtener un conjunto de consultas alternativas que coincidan con un documento XML determinado. Para cada nueva consulta, damos un "chance degree" que representa una estimación de su desviación con respecto a la expresión inicial. Nuestro trabajo se centra en pro-

porcionar a los programadores un repertorio de alternativas (que contienen nuevos comandos como las etiquetas "JUMP/DELETE/SWAP") que se pueden utilizar para obtener mas respuestas. Nuestro depurador es capaz, de la misma manera que el intérprete, de gestionar grandes documentos XML haciendo uso del comando FILTER que ignora de forma anticipada cálculos que conducen a soluciones no significativas (es decir, con un "chance degree" muy rebajado, según las preferencias del usuario). El punto clave, nuevamente, es la capacidad natural para realizar "umbralizacion dinámica" que ofrece el lenguaje lógico difuso usado para implementar la herramienta, conectando asi de alguna manera con el llamado «top-k answering problem» muy conocido en la lógica difusa y el soft-computing (o computación flexible).

En cuanto a nuevas aplicaciones no estandares, en el último bloque de esta tesis reforzamos las sinergias bilaterales entre FuzzyXPath y FLOPER. En particular, nos ocupamos de fórmulas proposicionales difusas que contienen varios símbolos proposicionales vinculados con conectivos definidos en un retículo de grados de verdad más complejos que *Bool*. En primer lugar, recordamos un método basado en SMT (*Satisfiability Modulo Theories*) difuso para demostrar automáticamente teoremas en relevantes logicas con infinitos valores (incluyendo a las de *Łukasiewicz* y *Gödel*). A continuación, en lugar de centrarnos en cuestiones de satisfactibilidad (es decir, demostrar la existencia de al menos un modelo) como normalmente se hace en un entorno SAT/SMT, nuestro interés se traslada al problema de encontrar un conjunto de modelos (sobre un dominio finito) para una fórmula difusa dada. Reutilizaremos un método anterior basado en la programación lógica difusa donde la fórmula se concibe como un objetivo de un árbol de derivación, proporcionado por nuestra herramienta FLOPER, que contiene en sus ramas todos los modelos de la fórmula original, junto con otras interpretaciones (obtenidas tras interpretar de forma exsaustiva cada simbolo proposicional de todas las formas posibles con respecto a un conjunto de valores recogidos en un reticulo subyacente de grados-de-verdad). A continuación utilizamos la capacidad de la herramienta FuzzyXPath para explorar estos árboles de derivación una vez exportados a formato XML, con el fin de detectar automáticamente si la fórmula es una tautología, satisfactible o una contradicción.

Summary

This thesis presents an extension of the popular XPath language which provides ranked answers to flexible queries taking profit of fuzzy variants of *and*, *or* and *avg* operators for XPath conditions, as well as two structural constraints, called *down* and *deep*, for which a certain degree of relevance is associated. In practice, this degree is very low for some answers weakly accomplishing with the original query, and hence, they should not be computed in order to alleviate the computational complexity of the information retrieval process. In order to improve the scalability of our interpreter for dealing with massive XML files, we make use of the ability of fuzzy logic programming for prematurely disregarding those computations leading to non significant solutions (i.e., with a poor degree of relevance according the preferences expressed by users when using the new command FILTER). Since our proposal has been implemented with a fuzzy logic language, we have exploited the high expressive resources of this declarative paradigm for performing "dynamic thresholding" in a very natural and efficient way. But apart from using our FLOPER environment for developing the interpreter, we also propose an implementation coded with the standard XQuery language. Basically, we have defined an XQuery library able to diffusely handle XPath expressions in such a way that our proposed FuzzyXPath can be encoded as XQuery expressions. The advantages of our approach is that any XQuery processor can handle a fuzzy version of XPath by using the library we have implemented.

On the other hand, we present a method for debugging XPath queries by describing how XPath expressions can be manipulated for obtaining a set of alternative queries matching a given XML document. For each new proposed query, we give a "chance degree" that represents an estimation on its deviation w.r.t. the initial expression. Our work is focused on providing to the programmers a repertoire of paths (containing new commands for "JUMP/DELETE/SWAP" tags) which can be used

to retrieve answers. Our debugger is able to manage big XML documents by making use of the new command FILTER which is intended to prematurely disregard those computations leading to non significant solutions (i.e., with a poor "chance degree" according to the user's preferences). The key point again is the natural capability for performing "dynamic thresholding" enjoyed by the fuzzy logic language used for implementing the tool, which somehow connects with the so-called «top-k answering problem» very well-known in the fuzzy logic and soft computing.

Regarding non standard applications, in the last block of this thesis we reinforce the bi-lateral synergies between FuzzyXPath and FLOPER. In particular, we deal with propositional fuzzy formulae containing several propositional symbols linked with connectives defined in a lattice of truth degrees more complex than *Bool*. We firstly recall a fuzzy SMT (*Satisfiability Modulo Theories*) based method for automatically proving theorems in relevant infinitely-valued (including *Łukasiewicz* and *Gödel*) logics. Next, instead of focusing on satisfiability (i.e., proving the existence of at least one model) as usually done in a SAT/SMT setting, our interest moves to the problem of finding the whole set of models (with a finite domain) for a given fuzzy formula. We re-use a previous method based on fuzzy logic programming where the formula is conceived as a goal whose derivation tree, provided by our FLOPER tool, contains on its leaves all the models of the original formula, together with other interpretations (by exhaustively interpreting each propositional symbol in all the possible forms according the whole set of values collected on the underlying lattice of truth-degrees). Next, we use the ability of the FuzzyXPath tool for exploring these derivation trees once exported in XML format, in order to automatically discover whether the formula is a tautology, satisfiable, or a contradiction.

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Contents

Chapter 1

Introduction

Now, we will talk first about the objectives and structure of the thesis, where a brief overview of the most important points will be given. Then take a brief look at important points on which this thesis is based concepts; firstly we describe the field of fuzzy logic, that was originated by the works [Zad65b, Zad65a, Gog69, Pav79] and we use as an extension of the polyvalent logic systems [Pav79, H98́, NPM99]. In this logic theory it is possible to define a wide variety of operators, like t-norms, t-conorms and aggregators [DP84, DP85, DP86, FY94, FR92, Yag93a, Yag93b, Yag94b, Yag94a, CBM99, CKKM02], and there are many definitions for implications [TCC00, CF95]. After its description, a historical background is provided [Zad96].

Finally, we detail the main notions of *logic programming* [Her30, Rob65, Kow74, War83, Llo87, Apt90, Apt97, JA07], before addressing the area of *fuzzy logic programming* [Hin86, MBP87, LL90, IK85]. We review the more prominent languages in this field; in particular, those based on weighted rules (Prolog-Elf [IK85], FProlog [MBP87], Fuzzy Prolog [MSD89], f-Prolog [LL90], RFuzzy and the multi-adjoint logic programming language MALP [MOV01d, MOV01c, MOV01b, MOV01a]), and the other based on similarities (LIKELOG[FGS00], SiLog[LSS01] based on [Ses02], and BOUSI∼PROLOG [JRG08, JRG09a, JR09b, JR10a]).

1.1 Objectives and structure of the Thesis

After introducing in the first pair of chapters some preliminary concepts regarding fuzzy logic, logic programming and the "Fuzzy LOgic Programming Environment for Research" FLOPER, in Chapter 3 we detail the design of our FuzzyXPath interpreter –which represents a fuzzy variant of the popular XPath query language for the flexible information retrieval on XML documents– thus providing a repertoire of operators that offer the possibility of managing satisfaction degrees by adding structural constraints and fuzzy operators inside conditions, in order to produce a ranked sorted list of answers according to user's preferences when composing queries [ALM11a, ALM11b, ALM12c].

By using the FLOPER system, our proposal has been implemented with a fuzzy logic language to take profit of the clear synergies between both target and source fuzzy languages [ALM15a]. In Chapter 4 we discuss the advantages of exploiting the high expressive resources of this declarative paradigm for performing "dynamic thresholding" when evaluating queries [ALM14a]. Moreover, in Section 4.3 we also provide an alternative implementation based on XQuery which increases the portability of the FuzzyXPath interpreter [ALM14b].

In Chapter 5 we recast from [ALM12a, ALM12b, ALM13] our recently designed method for debugging XPath queries which produces a set of alternative XPath expressions (where some tags have been "jumped", "deleted" or "swapped") with higher chances for retrieving answers from XML files. The use of filtering techniques in the FLOPER-based implemention of the tool represents once again the key point for gaining efficiency and increasing its scalability when managing very large XML documents [ALM15b].

Regarding applications, in Chapter 6 we describe a new feedback between FLOPER and FuzzyXPath. In [ALMV13, ALMV15] we focus on the ability of our interpreter for exploring derivation trees generated by FLOPER once they are exported in XML format, which somehow serves as a debugging tool for analyzing computational details such as discovering the set of fuzzy computed answers for a given goal, performing depth/breadth-first traversals of its associated derivation tree, finding non fully evaluated branches, etc. Such relationship grows through the connections we establish with recent (fuzzy) SAT/SMT techniques as explained in [ABL$^+$15].

Finally, this thesis concludes in Chapter 7 by collecting a brief summary of the achieved results and by proposing too some lines for future work.

1.2 Fuzzy logic

Fuzzy logic applies to the field of imprecise or vague statements we use to describe complex systems with unclear boundaries. In this sense, if classical logic was defined as the science that studies the laws, ways and types of reasoning, fuzzy logic could be defined in the same way as the science that studies the laws, ways and types of approximated reasoning.

Fuzzy logic was first formulated by Lofti Zadeh [Zad65b, Zad65a] and widen by Goguen [Gog69] and Pavelka [Pav79], in order to incorporate to formal logic the imprecise predicates of the common language, and build an approximated type of reasoning.

For [Zad96], creator of this discipline and the one who introduced the term "fuzzy", there are two meanings for the concept of fuzzy logic. In a broad sense, as generally understood, fuzzy logic refers to the use of fuzzy sets for manipulating imprecise knowledge. Therefore, it defines a theory of classes with non-sharp boundaries. This approximation differs clearly from ordinary logic in the use of fuzzy relations, the generalisation of traditional logic connectives such as '$\neg$', '$\wedge$' y '$\vee$' by their fuzzy counterparts, the fuzzy negations, t-norms and t-conorms, as well as other concepts like truth values, the presence of linguistic modifiers, etc. In this wider conception of the fuzzy logic, truth values can be fuzzy themselves, independently of the vagueness of the predicates. In practice, this means that the evaluation of a relation does not give a value (e.g., a number between 0 and 1), but the characteristic function that defines the relation itself.

In the other hand, strictly speaking, fuzzy logic refers to a logic system that formalizes approximated reasoning. In its simplest formulation, it constitutes an extension of classical bivalent logic to a logic with infinite truth values in the closed interval $[0, 1]$ with the usual ordering relation. It is, therefore, an extension of the polyvalent logic systems that shares with classical logic the search for soundness and completeness of the systems it studies [Haj06], although with different aims. This orientation of logic is relatively recent. It dates back to the works of [Pav79], who, together with [H98́, NPM99], constitutes the fundamental references in this second definition.

Traditionally, imprecise reasoning has not been appropriately addressed. According to [TAT95], since the beginnings of classical logic there have been only insufficient solutions: some has tried to gain precision against the imprecise (Frege or Russel), and others tried to isolate the imprecise to carefully avoid it (Plato, Hume or some contemporary strategies). This limitation in the traditional tools motivates the investigation on fuzzy sets and, in parallel, fuzzy logic. Classical logic, classical set theory and probability are not well suited to address the vagueness, imprecision, uncertainty, lack of specificity, inconsistency and complexity of the real world.

In the crisp (bivalent) ambit of classical mathematics there is no room for vagueness and partial truth. In this framework, all statement has to admit a precise definition that divides the objects of the considered universe into to subsets: one for those that satisfy it, and the other for those that do not satisfy it; and there is no possibility of dubious cases.

In frontal opposition to this world of clear borders, the perception of reality is full of concepts that do not admit a strict categorisation [TT89], as *tall*, *big*, *many*, *slowly*, *young*, *healthy*, *relevant*, *much greater than*, *kind*, among others. In the framework of fuzzy logic such concepts determine fuzzy sets, i.e., identify classes of objects for which the transition of membership to non-membership is gradual and not crisp.

Furthermore, the effort to acquire knowledge relating the real world is giving way to the effort to know aspects of the knowledge itself. Currently the delimitation of the scope and soundness of the information is as relevant (if not more) as its mere acquisition: it is necessary to know to which extent do we know something, i.e., assign a truth degree to it. Uncertainty is inextricably bounded to information. Even though there are different types of uncertainty, the one produced by the imprecision and subjectivity of human thinking is the most relevant [Zad08]. In many occasions it is convenient to sacrifice part of the precise information available to have a more vague but useful information in order to cope efficiently with the complexity of real world. Many of the usual concepts share an imprecise nature, i.e., they are not clearly delimited but they are significant. Fuzzy logic and fuzzy set theory offer a natural method to deal with vagueness and imprecision.

In [Zad65b] the author introduces for the first time a theory of fuzzy sets, that are sets with non precise borders and whose membership function gives a degree. One of the main goals of this theory is to provide a basis for approximate reasoning that uses vague hypotheses as a tool for formulating knowledge. Although its nature is different from other logics, in opinion of [TAT95], this logic of infinite truth values can be seen as an extension of the bivalent logic, of the trivalent logic defined by Łukasiewicz in 1922 and, in general, of the multivalued logic [Ack67, Cha58].

In other words, fuzzy logic can be seen as a reasoning model that takes into account qualitative or approximated aspects and that has a great ability to manage very complex or poorly defined problems. Its natural basis is conformed by the above mentioned fuzzy sets, that are the mathematical columns that sustain fuzzy predicates and allow to perform the logic calculations needed to perform inferences.

This logic is one of the most interesting and recent theories to model an abundant number of systems for which classical logic, multivalent logics and the probability framework are insufficient or inappropriate. For this problems, fuzzy logic offers symbols and operators that operates with the notion of vagueness, and inference rules that preserve, delimit and transmit the truth values from the hypothesis to the thesis. Next, we define some of the basic notions of fuzzy logic as collected in the thesis of Jaime Penabad [Pen10].

1.2.1 Fuzzy sets, aggregators and fuzzy implications

[Zad65b] introduces a notion of fuzzy set through which the concepts of fuzzy interpretation, fuzzy logic operations and linguistic modifiers, among others ([PG98]) are formalized. Consider ordinary sets like

$$A = \{x \in \mathbb{Z} : x \text{ is prime}\},\ A = \{x \in \mathbb{N} : x \text{ is even}\},\ A = \{x : x \text{ is mortal}\}$$

for which the membership relation is discrete, i.e., an element (of the corresponding universe) belongs or does not belong to the set:

$$\forall x \in \mathcal{U}, \quad x \in A \ \vee \ x \notin A; \quad A \subset \mathcal{U},$$

In the case of fuzzy sets, the membership is associated to a degree; is the case of sets like:

$$A = \{x \in \mathbb{Z} : x \text{ is big}\}$$

$$A = \{x : x \text{ is a warm day}\}$$

$$A = \{x : x \text{ is a developed country}\}$$

in which the nature of the property (predicate) that characterizes them is not clear (as it is in ordinary sets), but fuzzy. Therefore, we cannot say that the elements of the universe satisfy or not a certain predicate, but they satisfy it at some degree. These sets are formalised providing this new notion of membership, as we do next.

A fuzzy set A, in a universe $\mathcal{U}$, is expressed as:

$$A = \{x|\mu_A(x) : \mu_A(x) \neq 0, x \in \mathcal{U}\},$$

where the application

$$\mu_A : \mathcal{U} \to [0,1]$$

is the membership degree function.

In other words, a fuzzy set is determined by a function μ_A. For each $x \in \mathcal{U}$, $\mu_A(x) \in [0,1]$ is a real number that indicates the compatibility of x with the characteristic (predicate) that defines the set A.

It is also possible to consider that the ordinary membership is determined by the characteristic function

$$\chi_A : \mathcal{U} \to \{0,1\}, \quad \chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$$

that is,

$$x \in A \Leftrightarrow \chi_A(x) = 1, \quad x \notin A \Leftrightarrow \chi_A(x) = 0, \quad \forall x \in \mathcal{U}$$

The fuzzy membership, given by μ_A, is a generalisation of the classical one, since χ_A is a particular case of μ_A.

From this easy but relevant observation follows that the notion of fuzzy set extends the one of classical set. That is, an ordinary set is a fuzzy set. Particularly, the universe $\mathcal{U}$ (that we take as ordinary in the definition of A) and the empty set $\varnothing$ are fuzzy. Indeed,

$$\mu_\varnothing = \chi_\varnothing \text{ such that } \mu_\varnothing(x) = \chi_\varnothing(x) = 0, \ \forall x \in \mathcal{U}$$

$$\mu_\mathcal{U} = \chi_\mathcal{U} \text{ such that } \mu_\mathcal{U}(x) = \chi_\mathcal{U}(x) = 1, \ \forall x \in \mathcal{U}$$

Once characterized a fuzzy set by its membership degree function, μ_A, all its properties are referred to this, so the content, complementary, operations, etc., are expressed in terms of the corresponding membership functions.

From a semantic point of view, the essential definition of fuzzy logic is the one of interpretation, which associates to each (atomic) formula an element usually taken[1] in the real interval $[0,1]$. We detail now how an interpretation gives a truth degree to a fuzzy proposition through the concept of fuzzy set.

Given a predicate $A(x)$ in a universe $\mathcal{U}$ and an element $x_0 \in \mathcal{U}$, the formula $A(x_0)$ is interpreted as true with truth degree $\mu_A(x_0)$. In that case we write:

$$\mathcal{I}(A(x_0)) = \mu_A(x_0)$$

[1]In a more general way it can be taken from a certain ordered set [Zad08], as we consider later in this chapter.

Then we say that the proposition $A(x_0)$ holds with degree $\mu_A(x_0)$, that is the membership degree of x_0 to the fuzzy set A. And that set is, necessary, the set associated to predicate $A(x)$:

$$A = \{x \in \mathcal{U} : A(x)\},$$

We can assume that the previous definition is the formalisation (interpretation) of predicate $A(x)$ through the fuzzy set A. Indeed, it is legitimate to define it in these terms: "x_0 satisfies predicate $A(x)$ with degree $\mu_A(x_0)$, that is, the membership degree of x_0 to the fuzzy set $A = \{x \in \mathcal{U} : A(x)\}$."

Later in this chapter, by means of linguistic modifiers, we provide a meaning to consider fuzzy propositions as *false*, *very true*, *very false*, *more or less true*, etc. This way we also incorporate the fuzzy nature to the concept of interpretation: particularly, it is possible to associate a different interpretation to each predicate modifier (like *no*, *too*, *a little*, *approximately*, etc.) to be formalized. This possibility provides another differentiating element of fuzzy logic.

Before proceeding, it is mandatory to distinguish the vagueness of a statement (that affects to the statement itself) and uncertainty (that affects its compliance). In other words, this is not a possibilistic logic that considers the nonrandom uncertainty of non fuzzy propositions.

We detail now the syntax of this paradigm beyond fuzzy sets. The syntax of fuzzy logic has not too many novelties with respect to the interpretation of connectives. Once an elemental expression has been interpreted, the compounded expressions takes their values using ad hoc formulae [Lee72]. Thus, for instance, the conjunction is usually defined by the formula

$$\mathcal{I}(A(x_0) \wedge B(x_0)) = min\{\mathcal{I}(A(x_0)), \mathcal{I}(B(y_0))\},$$

where $A(x), B(y)$ are some predicates in universes $\mathcal{U}$, $\mathcal{V}$, respectively, and $x_0 \in \mathcal{U}, y_0 \in \mathcal{V}$.

If we take predicates $A(x), B(x)$ over the same universe $\mathcal{U}$ and they define, respectively, fuzzy sets $A, B \subset \mathcal{U}$, it also follows

$$\mathcal{I}(A(x_0) \wedge B(x_0)) = \mu_{A\cap B}(x_0),$$

where $\mu_{A\cap B}(x_0)$ is the membership degree of x_0 to the intersection set $A \cap B$. That is, it is allowed to define the fuzzy conjunction by means of the corresponding intersection of sets.

Generally speaking, the truth function of the fuzzy conjunction can be defined also by the wide range of functions known as triangular norms, introduced by [SS83] to model distances in probabilistic metric spaces (defined by K. Menger in 1942) and the semigroups of distribution functions.

We define now these functions in the interval $[0, 1]$.

Definition 1.2.1 ([NW06]). *An operation $T : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is a triangular norm or t-norm if, and only if, it verifies*

i) *is commutative, i.e., $T(x, y) = T(y, x), \forall x, y \in [0, 1]$.*

ii) *is associative, i.e., $T(x, T(y, z)) = T(T(x, y), z), \forall x, y, z \in [0, 1]$.*

iii) *$T(x, 1) = x, \forall x \in [0, 1]$.*

iv) *is monotonic in each component, i.e.*[2]*, if $x_1 \leq x_2$, then $T(x_1, y) \leq T(x_2, y)$, $\forall x_1, x_2, y \in [0, 1]$.*

Analogously, disjunction is usually characterized by the expression

$$\mathcal{I}(A(x_0) \vee B(x_0)) = max\{\mathcal{I}(A(x_0)), \mathcal{I}(B(y_0))\},$$

and if we consider predicates $A(x), B(x)$ on the same universe $\mathcal{U}$ defining, respectively, the fuzzy sets $A, B \subset \mathcal{U}$, we also have

$$\mathcal{I}(A(x_0) \vee B(x_0)) = \mu_{A \cup B}(x_0),$$

where $\mu_{A \cup B}(x_0)$ is the membership degree of x_0 to the union set $A \cup B$. Consequently, this logic operation is associated to the union of sets. More precisely, it is possible to formalize fuzzy disjunction (of propositions and also of predicates) by the union of fuzzy sets.

Furthermore, as in the conjunction, the (truth function of the) fuzzy disjunction can be defined by the wide range of functions called t-conorms, characterized in the following way in the interval $[0, 1]$.

Definition 1.2.2 ([NW06]). *An operation $S : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is a triangular conorm, or t-conorm, if, and only if, it verifies*

i) *is commutative, i.e., $S(x, y) = S(y, x), \forall x, y \in [0, 1]$.*

[2]From the given characterization (only for the first component) follows also the monotonicity in the second one using conditions *i)* and *iv)*.

ii) *is associative, i.e.,* $S(x, S(y,z)) = S(S(x,y),z), \forall x,y,z \in [0,1]$.

iii) $S(x,0) = x, \forall x \in [0,1]$.

iv) *is monotonic in each component, i.e.*[3]*, if* $x_1 \leq x_2$*, then* $S(x_1,y) \leq S(x_2,y)$, $\forall x_1,x_2,y \in [0,1]$.

If T is a t-norm in $[0,1]$, then $S(x,y) = 1 - T(1-x, 1-y)$ defines a t-conorm and S is said to derive from T. More generally, given a t-norm T and a strong negation[4] N, then function $S_N : [0,1] \times [0,1] \longrightarrow [0,1]$, defined as $S_N(x,y) = N(T(N(x),N(y)))$, is a t-conorm called N-dual of T.

By the elemental properties of negation we have $T(x,y) = N(S_N(N(x),N(y)))$, that is, T is the N-dual t-norm of S_N. Given a t-conorm S and a strong negation N, the function $T_N : [0,1] \times [0,1] \longrightarrow [0,1]$, defined as $T_N(x,y) = N(S(N(x),N(y)))$, is a t-norm called N-dual t-norm of S.

Again, since N is a negation, $S(x,y) = N(T_N(N(x),N(y)))$, that is, S is the N-dual t-conorm of T_N.

Concluding, we say that T and S are N-dual if $\forall x,y \in [0,1]$ it holds:

$$T(x,y) = N(S(N(x),N(y))) \qquad S(x,y) = N(T(N(x),N(y)))$$

Particularly, taking the usual negation $N(x) = 1-x$, T and S are dual if $\forall x \in [0,1]$ it holds:

$$T(x,y) = 1 - S(1-x,1-y) \qquad S(x,y) = 1 - T(1-x,1-y)$$

We present below basic pairs of basic t-norms and t-conorms (dual) ([CFF97]):

- Zadeh's (or the Minimum/Maximum) defined by

$$T(x,y) = min\{x,y\} \qquad S(x,y) = max\{x,y\}$$

- Łukasiewicz's defined by

$$T(x,y) = max\{x+y-1,0\} \qquad S(x,y) = min\{x+y,1\}$$

[3]The monotonicity in the second component follows also from *i*) and *iv*).

[4]A strong negation in $[0,1]$ is a function $N : [0,1] \longrightarrow [0,1]$ that is continuous, strictly decreasing and $N(0) = 1, N(N(x)) = x$.

- Of the Product, defined by

$$T(x,y) = xy \qquad S(x,y) = x + y - xy$$

- Weak/Strong, defined by

$$T(x,y) = \begin{cases} min\{x,y\}, & \text{if } max\{x,y\} = 1 \\ 0, & \text{otherwise} \end{cases} \qquad S(x,y) = x + y - xy$$

- Hamacher's, defined for each $\gamma \geq 0$ by

$$T_\gamma(x,y) = \frac{xy}{\gamma + (1-\gamma)(x+y-xy)} \qquad S(x,y) = \frac{x+y-(2-\gamma)xy}{1-(1-\gamma)xy}$$

- Yager's, defined for each $p > 0$ by

$$T_p(x,y) = 1 - min\{1, \sqrt[p]{(1-x)^p + (1-y)^p}\} \quad S_p(x,y) = min\{1, \sqrt[p]{x^p + y^p}\}$$

It is common to use t-norms and t-conorms to produce new connectives [Miz89a, Miz89b, Tur92, FC98, DSMK07, KMP04]. T-norms and t-conorms are particular cases of aggregation operators[5] (studied by [DP84, DP85, DP86], [FY94, FR92] and [Yag93a, Yag93b, Yag94b, Yag94a]) and, also, certain combinations of them originate new aggregation operators [CBM99, CKKM02].

It is possible to produce aggregators (see [Lin65, Miz89b, Tur92, MTK99, JM03, Jen04, Jen06]) by convex combinations of a t-norm T and a t-conorm S, that is, produce the aggregator $@(x,y) = \alpha T(x,y) + (1-\alpha)S(x,y)$, that preserves symmetry and idempotence.

Aggregators are common in the development of multiple intelligent systems, as is the case of neuronal networks, fuzzy controllers, expert systems and, specially, in decision theory. Aggregators allow the efficient and flexible combination of information [HHV96], which has become a main task in multiple-criteria decision problems where it is necessary to process a great deal of information of different quality and precision.

The most general definition for the aggregation operator, in the interval $[0,1]$, is the one given in [KK99], that we reproduce here.

Definition 1.2.3. *An aggregation operator* $@$ *is an application* $@ : [0,1]^n \longrightarrow [0,1]$ *that fulfils:*

[5]See Definition 1.2.3 for a characterization of the former.

i) $@(0,\dots,0)=0, @(1,\dots,1)=1$ *(boundary conditions)*

ii) $\forall(x_1,\dots,x_n),(y_1,\dots,y_n)\in[0,1]^n$,
$(x_1,\dots,x_n)\leq(y_1,\dots,y_n)^6 \Longrightarrow @(x_1,\dots,x_n)\leq @(y_1,\dots,y_n)$ *(monotonicity)*

Sometimes other conditions are required together with the ones mentioned above, such as continuity, symmetry and idempotence. Particularly, @ is symmetric if, and only if, for all permutation σ of $\{1,\dots,n\}$ and all n-uple $(x_1,\dots,x_n)\in[0,1]^n$ the next holds: $@(x_1,\dots,x_n)=@(x_{\sigma(1)},\dots,x_{\sigma(n)})$; also, @ is idempotent (i.e., $@(x,\dots,x)=x$) if and only if for all n-uple $(x_1,\dots,x_n)\in[0,1]^n$, $min\{x_1,\dots,x_n\}\leq @(x_1,\dots,x_n)\leq max\{x_1,\dots,x_n\}$ holds.

Some well known examples of aggregation operators are t-norms and t-conorms (previously detailed), Quasi-Linear Weighted Means [Acz48, Yag94b] (if they are, also, symmetric, they give the quasi-arithmetic mean, like the arithmetical average, the geometric average, and harmonic and quadratic means), OWA operators [Yag88] (arithmetic average is also a particular case of these operators), the extended aggregation functions [MC97], and the γ-operators of [ZZ80], among others.

We end this section addressing a fundamental element in (fuzzy) logic: implication. Fuzzy implication constitutes the most interesting composed operation of fuzzy logic (as well as in classical logic), since it allows to perform logical inferences and deduce theorems from axioms. Its truth function allows different non-equivalent formulations; there are, thus, many different fuzzy implications that not always extend the usual (classical) implication [TCC00, CF95].

The usual way to interpret fuzzy implication

$$A(x_0)\Rightarrow B(y_0)$$

is given by the formula

$$\mathcal{I}(A(x_0)\Rightarrow B(y_0)) = max\{min\{\mathcal{I}(A(x_0)),\mathcal{I}(B(y_0))\},1-\mathcal{I}(A(x_0))\},$$

where $A(x),B(y)$ are arbitrary predicates in universes $\mathcal{U}$, $\mathcal{V}$ respectively and $x_0\in\mathcal{U}, y_0\in\mathcal{V}$. If predicates $A(x),B(y)$ define fuzzy sets $A\subset\mathcal{U}, B\subset\mathcal{V}$, it follows

$$\mathcal{I}(A(x_0)\Rightarrow B(y_0)) = max\{min\{\mu_A(x_0),\mu_B(y_0)\},1-\mu_A(x_0)\};\quad x_0\in\mathcal{U}, y_0\in\mathcal{V}$$

This truth function for fuzzy implication, provided by Zadeh, generalizes the classical implication.

[6]Where $(x_1,\dots,x_n)\leq(y_1,\dots,y_n)$ if, and only if, $x_i\leq y_i, i=1,\dots,n$.

Mamdani and Larsen provide other interesting examples of fuzzy implication, whose interpretations we present here[7]

$$\text{Mamdani}: \quad \mathcal{I}(A(x_0) \Rightarrow B(y_0)) = min\{\mathcal{I}(A(x_0)), \mathcal{I}(B(y_0))\}$$

$$\text{Larsen}: \quad \mathcal{I}(A(x_0) \Rightarrow B(y_0)) = \mathcal{I}(A(x_0)) \cdot \mathcal{I}(B(y_0))$$

We present now the fuzzy implication in the most general way. First, we consider that, given a Boole algebra $(A, \wedge, \vee, ', 0, I)$, an operation $\rightarrow : A \times A \longrightarrow A$ is an implication if for each $x, y \in A$, it holds $x \wedge (x \rightarrow y) \leq y$. It is well known that $p \rightarrow q = (p \wedge q')'$, $p \rightarrow q = (p \wedge q) \vee (p' \wedge q) \vee (p \wedge q')$ are implications and, by the properties of the Boole algebra, we have that $(p \wedge q) \vee (p' \wedge q) \vee (p \wedge q') = (p' \wedge q') \vee q = p' \vee (p \wedge q) = p' \vee q$, and also $(p \wedge q')' = p' \vee q$.

If, in the fuzzy context, we choose a t-norm T instead of a conjunction $\wedge$, a t-conorm S instead of a disjunction $\vee$ and a strong negation N instead of a negation $'$, we obtain the following models of fuzzy implication [TCC00]:

$$\begin{array}{rcl} J_1(x,y) & = & N(T(x, N(y))) \\ J_2(x,y) & = & S(N(x), y) \\ J_3(x,y) & = & S(N(x), T(x,y)) \\ J_4(x,y) & = & S(T(N(x), N(y)), y) \end{array}$$

With respect to its characterization, a fuzzy implication in the real interval $[0,1]$ is defined by the following truth function[8] [TCC00, TV85].

Definition 1.2.4. *An implication function J is an application $J : [0,1]^n \longrightarrow [0,1]$ that fulfils:*

(1) If $x_1 \leq x_2$, then $J(x_1, y) \geq J(x_2, y)$

(2) If $y_1 \leq y_2$, then $J(x, y_1) \leq J(x, y_2)$

(3) $J(0, y) = 1$

(4) $J(1, y) = y$

(5) $J(y, J(x, z)) = J(x, J(y, z)), \forall x, y, z \in [0,1]$

[7]Both implications are very used in the field of fuzzy control.

[8]For convenience, we omit from now on all mention to the logic expressions involved in the hypotheses and the theses of the implication, and we refer only to the truth function of the implication connective.

It is also frequent to require some of the next conditions

(6) $J(x, 0) = N(x)$

(7) J is continuous

(8) $J(x, y) = J(N(y), N(x))$, for some strong negation N

(9) $J(x, x) = 1$

(10) $x \leq y$ if, and only if, $J(x, y) = 1$

There are three main classes of (truth functions of) implications [CFF97, ACT95]:

- S-implications defined by

$$x \longrightarrow y = S(N(x), y)$$

  where S is a t-conorm and N is a negation in $[0, 1]$. These implications come from the equivalence, in binary logic, of formulae $p \to q$ and $p' \vee q$. Some of these S-implications are given by

  - Łukasiewicz, defined by $x \longrightarrow y = min\{1 - x + y, 1\}$
  - Kleene-Dienes, defined by $x \longrightarrow y = max\{1 - x, y\}$

- R-implications defined by residuation of a continuous t-norm T, like

$$x \longrightarrow y = sup\{z \in [0, 1] : T(x, z) \leq y\}$$

  These implications come from Gödel logic and, among them, we have the ones given by:

  - Gödel, defined by $x \longrightarrow y = \begin{cases} 1, & \text{if } x \leq y \\ y, & \text{if } x > y \end{cases}$
  - Łukasiewicz, defined by $x \longrightarrow y = min\{1 - x + y, 1\}$

Also, the following are admitted as implications:

- T-norm implications, defined through a t-norm T as

$$x \longrightarrow y = T(x, y)$$

  This group includes the Mamdani implication, used in theory of fuzzy control, and the implication from the product t-norm.

- QM-implications [TCC00, Yin02], characterized by $Q : [0,1]^2 \longrightarrow [0,1]$ given by $Q(x,y) = S(N(x), T(x,y))$ from a t-norm T, a t-conorm S and a strong negation N.

  Some QM-operators (that is the name that we should use in this case since these do not determine, in general, an implication) are

  - $Q_1(x,y) = S(1-x, T(x,y)) = max\{1-x, y\}$, where T is the t-norm of Łukasiewicz and S its dual t-conorm.
  - $Q_2(x,y) = S(1-x, T(x,y)) = 1-x+x^2y$, where T is the Product t-norm and S its dual t-conorm.
  - $Q_3(x,y) = \begin{cases} 1, & \text{if } y = 1 \\ y, & \text{if } x = 1 \\ 1-x, & \text{otherwise} \end{cases}$

Note that in classical logic the S-implication $p' \vee q$ and the QM-implication $p' \vee (p \wedge q)$ are equivalent and define the ordinary logic implication, although they are different as fuzzy operators.

In order to perform fuzzy inference, it is essential the property of the fuzzy (or generalized) modus ponens, that was first proposed by L. A. Zadeh and that propagates the truth degrees of the premises to the conclusion by means of a composition of fuzzy relations. ^{o}:

$$\left.\begin{array}{c} \text{If } A(x) \text{ then } B(y) \\ \text{and} \\ A'(x) \end{array}\right\} \text{ then } B'(y)$$

where $A(x)$, $A'(x)$ are arbitrary fuzzy predicates (in an arbitrary universe $\mathcal{U}$) as well as $B(y)$, $B'(y)$ (in a universe $\mathcal{V}$). Such predicates are associated to the corresponding fuzzy sets A, A', B, B'.

The truth degree for this expression makes use of the composition rule

$$\left.\begin{array}{c} \text{If } A(x) \\ \text{and} \\ R(x,y) \end{array}\right\} \text{ then } (A \circ R)(y)$$

being:

$$\mu_{A \circ R}(y) = max\{min\{\mu_A(x), \mu_R(x,y)\}\}, x \in \mathcal{U}$$

where R is a binary fuzzy relation over $\mathcal{U} \times \mathcal{U}$, and $\circ$ denotes the (unary) composition of the fuzzy set A –characterized by predicate $A(x)$ in the universe $\mathcal{U}$– and the fuzzy relation R.

In contrast to ordinary logic, the antecedent $A(x)$ is not required to coincide with the previous $A'(x)$, and this fuzzy modus ponens can be seen as a particular case of the composition rule, where the relation R is the fuzzy cartesian product $A \times B$.

A fuzzy modus ponens appears, for instance, in [VP96], in the language f-Prolog that we describe in Section 1.4, although in its formalisation fuzzy relations are not involved.

While negation is the only modifier for classical predicates, a fuzzy predicate $A(x)$ can also be modified by

not $A(x)$, very $A(x)$, a little $A(x)$, more or less $A(x)$, approximately $A(x)$,...

Indeed, it is possible to formalize these so-called predicate modifiers, that correspond to adverbs and shapes the use of property $A(x)$. For instance, we can define modifier *very* in this way (supposing that $A(x)$ is defined in $\mathcal{U}$ and $x_0 \in \mathcal{U}$).

$$\mathcal{I}(very\ A(x_0)) = [\mathcal{I}(A(x_0))]^2 \quad \text{or equivalently} \quad \mathcal{I}(very\ A(x_0)) = [\mu_A(x_0)]^2$$

and for the modifier approximately

$$\mathcal{I}(approx\ A(x_0)) = [\mathcal{I}(A(x_0))]^{1/2} \quad \text{or equivalently} \quad \mathcal{I}(approx\ A(x_0)) = [\mu_A(x_0)]^{1/2}$$

It is very useful, for approximate reasoning, the logic concept of linguistic variable, developed by [Zad75]. A linguistic variable is a set of terms of natural (or formal) language expressions that can be taken as linguistic labels in the considered context. These labels are fuzzy sets over a domain. As an example of linguistic variable, consider variable *speed* with values: low, medium, high, among others, defined over the domain of kilometers per hour. Other example is the *truth*, with values: very true, a little true, false, very false, etc., that is, the values associated to the corresponding modifiers previously formalized.

Finally, aside from the already observed differences with classical logic, as the vague character of predicates, the infinite truth values, the presence of linguistic modifiers and the different interpretations, it is noteworthy the presence of specific quantifiers (see [DP80]) like: nearly all, some, the majority, quite a little.

1.2.2 Applications

Fuzzy logic (FL) constitutes a model for reasoning that allows to deal with complex problems, poorly defined problems or problems for which there are no precise mathematical model. Thanks to this kind of logic it has been possible to model a solve situations traditionally considered untreatable from the point of view of classical logic. In the last decades fuzzy logic has been used in a growing variety of instruments, machines, software and diverse fields of daily life. This proliferation of applications has diminished the initial distrust to this kind of logic.

In fact, since the basic notions of fuzzy logic where stablished, by the first time, by the paper of [Zad65b] on fuzzy sets; and in spite of the enthusiasm of some researchers, like the mathematicians R. Bellman and G. Moisil, to adopt the new ideas; the main tendency was skeptical, even hostile, towards the new theory. Currently, while some controversy still remains, the value of its contribution to multiple applications has consolidated this paradigm.

Professor Zadeh's original intention was to provide a formalism to handle the imprecision and vagueness of human thinking, expressed linguistically, although afterwards much of the merit of fuzzy logic has focused on the field of automatic control of processes. This is due to the fuzzy "boom" in Japan, that began in 1987 and it reached its peak at the beginnings of the nineties. Indeed, aside from the relevant seminary EE.UU.-Japan on fuzzy sets and its applications held in Berkeley in 1974, other important milestone for the development of this logic was the congress IFSA (International Fuzzy Systems Association) of Tokyo that year (1987). In that congress, Matsushita announced the first consumer product based on fuzzy logic (a showerhead). Simultaneously, in other field, the Sendai underground was launched. It used a controller based on fuzzy logic, and is considered as one of the most successful applications of this logic [VZ96].

Since then, a large amount of consumer products use fuzzy technology, many of them using the label "fuzzy" as a symbol of quality and high performance. As early as in 1974, professor Mamdani experimented successfully with a fuzzy controller in a steam machine, while the first real implantation of such a controller was in performed in 1980 by F. L. Smidth & Co. in a cement plant in Denmark. In 1983, Fuji applied fuzzy logic to the control of chemical injection for a water treatment plant, for the first time in Japan. In 1987, OOMRON developed the first commercial fuzzy controllers with the professor Yamakawa. From then on, fuzzy control has been successfully applied to many branches of technology, as we see with examples of specific applications at the end of this section. Its success lies in the conceptual and developmental simplicity of these control applications.

Worldwide, Japan is, as we have seen, the country where fuzzy logic and its applications has been best welcome. Professors K. Asai, J. Tanaka and T. Terano were precursors in 1968 with their works on fuzzy automata and learning systems. In Europe, the interest for fuzzy logic began in the seventies and the most significant contributions focuses on theoretical developments. With respect to Spain, professor E. Trillas began a research on fuzzy logic and its applications and, in opinion of [Zad96], thanks to his contributions Spain is a leading country in Europe in this field.

With respect to its application fields, one of the most important ones is control theory. Indeed, the application of fuzzy logic to control has been natural, and the "fuzzy" label was introduced initially linked to this area. The evolution of fuzzy control has been spectacular. Its growth has been quick because fuzzy control application are easy to make since they only require fuzzy rules of the form "if then" to handle commands. Fuzzy rules, usually fine-tuned by experts, are fuzzy implications involving fuzzy propositions (simple of compound) [DHR96, PDH97]. Also, fuzzy controllers are simple and sound and, in mane cases, there are no possibility to use traditional controllers since there are no mathematical model (or it is non practical), as states Ebrahim Mamdani (pioneer in fuzzy control) in his work [Mam93].

L. A. Zadeh creates also the theory of approximate reasoning of fuzzy logic in the context of artificial intelligence in his search for more efficient tools for building expert systems (other main field of FL).

In [Zad73] the principle of incompatibility is enunciated. It states that complexity and precision are antagonistic when describing the behaviour of a system, so conventional programs have little effectiveness to model human behaviour. Addressing this problem, it suggests, in one hand, to represent (imprecise) information by means of fuzzy sets; and in the other, the inference over imprecise information, based on the use of fuzzy implication and the most relevant property: generalized modus ponens, formalized as the unary composition of a fuzzy set in a fuzzy relation (the so-called inference compositional rule). This composition property of two fuzzy relations allows to apply fuzzy logic to fuzzy control and, then, the development of reasoning systems and their implementation. The concrete implication is to be chosen carefully since it is essential to the system. The effectiveness of different implication functions for reproducing human reasoning and ease inference methods has been quite studied in the literature.

Fuzzy logic emerges in the search of professor Zadeh, as an answer to fuzzy logic because of two main aspects: it represents formally the imprecise knowledge and manages adequately the uncertainty in some expert systems. These characteristics gives to FL great relevancy in the field of knowledge engineering.

So, an expert system is a system based on knowledge plus information from the experts on the domain [PS05]. Its goal is the resolution of problems in this domain, applying reasoning techniques over the information in their base of knowledge. Despite probability theory is the classical formal model to represent uncertainty, it has not been universally accepted in the design of expert systems to address uncertainty. It is well known that it requires a large collection of data and operations to be appropriate [SB75, Ada76]. Many methods and specialized extensions of probability calculus has been developed to overcome these limitations, suchs as certainty factors of MYCIN[9] [SB75], the subjective Bayesian [DHN90], the theory of evidence of Dempster-Shafer [Sha76] and the theory of *endorsements* [Coh85]. In contrast to the previous methods, that lack a well known semantic framework, L. A. Zadeh proposed a formal logic based on theory of fuzzy sets that is very adequate for dealing with uncertainty.

Fuzzy Rule-Based Systems (FRBSs) are an extension of classical systems for representing knowledge based on rules [CHHM01]. As those, FRBSs are composed of conditional rules of the form “if then”, with the particularity that the antecedent and consequent are fuzzy expressions. We list next some advantages of fuzzy expert systems:

- They are an easy way of codifying a non-linear system.
- They correspond well with the schemes of human thinking over a large amount of mathematical problems.
- They are efficient (the run quickly) on conventional computers.
- They run extremely quick on specialized hardware.

The application of fuzzy logic to rule-based systems has focused mainly in, in one hand, generalize the model of certainty factors and, in the other, the use of fuzzy predicates in the description of rules and reality.

[9]MYCIN is an expert system written in Prolog.

Other field of application to highlight is, finally, the contribution of fuzzy logic to *soft computing*. The emergence of neurocomputation and genetic algorithms (in the mid 80s) had a significant impact on the development of fuzzy logic. Probability and fuzzy logic can be used together in the methodologies of neurocomputation and genetic algorithms. This suggests to [Zad96] the concept of *soft computing*, understood as “a kind of society of fuzzy logic, neurocomputation and probabilistic reasoning”. In this scope, fuzzy logic provides a methodology to deal with imprecision, approximate reasoning and computation with words. The most important of *soft computing* is that it suggests the possibility of using fuzzy logic, neurocomputation and genetic algorithms combined instead of isolated. One of the most relevant combinations currently is the “neuro-fuzzy” systems (for an introduction, see [Ngu02]). The growing use of *soft computing* has brought an important contribution for the conception, design and development of intelligent systems.

A part of this field is considered by many the new challenge for fuzzy logic: the Internet. As stated by professor José Ángel Olivas from the University of Castilla-La Mancha, the use of fuzzy technologies is mandatory to address the massive amount of data, retrieve information, and control and manage the net. This intuition coincides also with the new path that, according to professor Zadeh, should follow fuzzy logic. The first encounter on fuzzy logic and the Internet (FLINT 2001) held at the University of Berkely on summer 2001 and organized by Zadeh itself is proof of this. The main idea that arose is the tendency towards *Computing with words*, by means of techniques of *soft computing* (that includes fuzzy logic, neuronal networks and evolutionary computation). These terms, coined by professor Zadeh, materializes in many research lines like:

- A new generation of search engines on the Internet, using techniques of *soft computing* to enhance the current (lexicographic) search to a conceptual search.

- Advanced techniques to describe user profiles that allow a more intelligent use of the Internet.

- Semantic web, where users could delegate tasks on the software, that will be able to process, reason, combine information and perform logic deductions to solve daily problems.

And much more new fields of application of *soft computing*which already are producing promising results.

To end this section we relate some of the multiple specific applications of this logic (in the field of fuzzy control and expert systems, mainly).

- Consumer electronic products: intelligent washing machines of Panasonic or Bosch (Matsuhita Electronic Industrial), microwave ovens, termic systems, video recorders, televisions, image stabilizing systems in photographic and video cameras of Sony, Sanyo, Cannon (Matsuhita) and automatic focus systems in photographic cameras.

- Systems: automatic pilotage systems for airplanes, maneuvering control for lift or trains (underground of Senadi, Japan, 1987), water treatment systems, automotive systems (ABS of Mazda and Nissan, automatic speed control, climate control, automatic driving systems), industrial combustion control systems, traffic controllers, heating/cooling systems (Mitsubishi air conditioning, *rice-cooker*), climate prediction systems, atmospheric prediction systems and writing recognition systems.

- Software for: clinical diagnosis (CADAG, Adlssnig, Arita, OMRON), security (Yamaichi, Hitachi), linguistic translation, data understanding, informatics technology and fuzzy data bases for storing and querying imprecise information (use of language FSQL).

To summarize, and attending the opinion of [VZ96], fuzzy logic has applications, mainly, in two very different fields: the first, control theory applications, and the other, expert system development. Internet and *soft computing* could be third and fourth fields. According to him, we are entering an era of intelligent systems that will have a deep impact on the way we communicate, take decisions and use machines, and fuzzy logic (together with *soft computing* and fuzzy declarative languages, in our opinion), will play an important role in bringing the era of intelligent systems.

With respect to the work performed in this thesis, its practical applications are part of other promising line of application of fuzzy logic: the design and enhancement of declarative languages that allow to codify easily applications with fuzzy taste in the referred fields. Concretely, we focused on enhancements related to the procedural semantics of one of the most interesting paradigms in fuzzy logic programming, from our point of view, that is the multi-adjoint logic. In the next section we provide a brief view of logic programming to expose, afterwards, in a more detailed way, the most relevant aspects of fuzzy logic programming.

1.3 Logic programming

Logic programming (LP) was originated in the research on Automatic Theorem Proving. Since the works of [Her30, Rob65, Kow74, War83], logic programming reaches maturity at the beginnings of the eighties. In essence, logic programming (for which [Llo87, Apt90, Apt97, JA07]) are fundamental references) is based on a subset of predicate logic, concretely in Horn clauses, that are used as the core of a programming language together with an operational semantics, SLD-resolution, for which there are an efficient implementation.

Its main feature is, indeed, the use of logic as programming language. More precisely, a program in LP is conceived as a formal theory in a certain logic, and computation is understood as a logic deduction in this logic.

The base logic has to include the following elements (see [Jul00]):

- A language expressive enough to address an interesting field of application,
- An operational semantics, that is, a calculation mechanism to execute programs,
- A declarative semantics to provide a meaning to programs independently of their possible execution, and
- Results of soundness and completion to assure that the computed result coincides with what is considered true according to the notion of truth given by the declarative semantics.

Also, this declarative semantics specifies the meaning of the syntactic objects of the language by means of its translation to elements and structures in a known (generally mathematical) domain.

The operational semantics in LP is based on a method of proof by refutation called SLD-resolution, that is an instance of the resolution strategy. SLD-resolution is based on the unification algorithm and allows the retrieval of answers, i.e., the link of a value to a logical variable. It is a refinement of Robinson's resolution, that was first described by [Kow74], and whose name comes from "Selective Linear Definite clause resolution". Besides, it is a sound and complete method for the referred logic.

The declarative semantics of LP can be defined in many ways. An illustrative example is model theory, whose domain is a purely syntactic universe: the Herbrand universe.

In essence, a logic program is a set of Horn clauses. A clause has the form $A \leftarrow B_1, \dots, B_n$, and can be considered as a part of the definition of a routine. A clause of the form $\leftarrow C_1, \dots, C_k$ is a goal, and each C_k can be understood as a call to a routine. To execute a program is to query a goal. If the goal is $\leftarrow C_1, \dots, C_k$, a computation step implies unifying some C_j with the head A of a clause $A \leftarrow B_1, \dots, B_n$, thus obtaining:

$\leftarrow (C_1, \dots, C_{j-1}, B_1, \dots, B_n, C_{j+1}, \dots, C_k)\theta$

where θ is a unifier substitution. Unification is, then, a mechanism for the argument passing, data selection and data building. The computation ends when the goal transits to an empty clause and there are no more literals inside of it to solve.

We introduce now some basic notions of logic programming that we use in further chapters. These concepts are addressed with more detail in [Llo87, JA07].

Let $\mathcal{V}$ be an infinite set of variables and Σ a set of function symbols f/n, each one of them with an arity n associated. $\mathcal{T}(\Sigma, \mathcal{V})$[10] and $\mathcal{T}(\Sigma)$ stand, respectively, for the set of terms and ground terms (terms with no variables) built upon $\Sigma \cup \mathcal{V}$ and Σ. The set of variables in an expression E is denoted by $\mathcal{V}ar(E)$. A term, then, is said to be ground if $\mathcal{V}ar(t) = \varnothing$.

Definition 1.3.1 ([JA07])**.** *A substitution σ is an application $\sigma : \mathcal{V} \longrightarrow \mathcal{T}$ that assigns to each variable x the set of variables $\mathcal{V}$ of a first order language $\mathcal{L}$, a term $\sigma(x)$ of the set of terms $\mathcal{T}$.*

It is usual to require $\sigma(x) \neq x$ only for a finite number of variables and, also, to express the substitution in terms of sets, identifying (in some sense) the application σ to the set of images. That is, we write $\sigma = \{x_1/t_1, \dots, x_n/t_n\}$, where $t_i = \sigma(x_i)$ is different from x_i and each pair x_i/t_i is called "binding" or substitution element.

The set $\mathcal{D}om(\sigma) = \{x \in \mathcal{V} : \sigma(x) \neq x\} = \{x_1, \dots, x_n\}$ is said to be the domain of σ, and its range is $\mathcal{R}an(\sigma) = \{\sigma(x) : x \in \mathcal{D}om(\sigma)\} = \{t_1, \dots, t_n\}$. Additionally, we represent by *id* the identity substitution, that can be understood as the set of empty bindings, so $\mathcal{D}om(id) = \varnothing$, that is, $id(x) = x$ for all $x \in \mathcal{V}$. Also, σ is said to be ground if the terms t_i are ground (the include no variables).

Definition 1.3.2. *Given an expression E and a substitution σ, $\sigma(E)$ is called* instance *and is the result of applying σ over E, replacing simultaneously all instance of x_i in E by the corresponding term t_i, being x_i/t_i an element of substitution σ.*

[10]Occasionally we only write $\mathcal{T}$.

Usually the previous instance is written $E\sigma$ instead of $\sigma(E)$. Whenever a substitution applies to the more general formulae of language $\mathcal{L}$, and not only to expressions in a clausal language, it is convenient to rename the bound variables before applying the substitution (see, [Jul04]).

Definition 1.3.3. *Given the substitutions $\sigma = \{x_1/t_1, \ldots, x_n/t_n\}$, $\theta = \{y_1/s_1, \ldots, y_m/s_m\}$, the composition $\sigma \circ \theta$*[11] *is the substitution determined from the set $\sigma \circ \theta = \{x_1/\theta(t_1), \ldots, x_n/\theta(t_n), y_1/s_1, \ldots, y_m/s_m\}$, removing the bindings $x_i/\theta(t_i)$ such that $x_i = \theta(t_i)$ and removing from θ the bindings y_j/s_j such that $y_j \in \{x_1, \ldots, x_n\}$.*

This composition verifies, over an expression E, that $(\sigma \circ \theta)(E) = \sigma(\theta(E))$ is associative and the identity substitution is the (two-sided) identity element. In the other hand, given σ, θ with $\mathcal{V}ar(\sigma) \cap \mathcal{V}ar(\theta) = \varnothing$, the union $\sigma \cup \theta$ is defined by the union set of both, that is, $(\sigma \cup \theta)(x) = \sigma(x), x \in \mathcal{D}om(\sigma)$ and $(\sigma \cup \theta)(x) = \theta(x), x \in \mathcal{D}om(\theta)$.

A substitution ρ is called renaming substitution or, simply, renaming, if there is ρ^{-1} (inverse substitution) such that $\rho \circ \rho^{-1} = \rho^{-1} \circ \rho = id$. Two expressions E_1, E_2 are variant if there are renaming substitutions ρ, ρ' such that $E_1 = \rho(E_2)$ and $E_2 = \rho'(E_1)$.

Composition of substitutions induces the usual preorder among substitutions: $\theta \leq \sigma$ if, and only if, there is γ that $\sigma = \theta \circ \gamma$, and we say that θ is a more general substitution than σ. This preorder induces a partial preorder over terms given by $t \leq t'$ if there is γ that $t' = t\gamma$.

Two terms t and t' are variants (one another) if there is a renaming ρ that $t\rho = t'$. Given a substitution θ and a set of variables $W \subseteq \mathcal{V}$, we denote by $\theta_{\restriction W}$ the substitution obtained from θ by restricting $Dom(\theta)$ only to the variables W. We write $\theta = \sigma\ [W]$ if $\theta_{\restriction W} = \sigma_{\restriction W}$, and $\theta \leq \sigma\ [W]$ denotes the existence of a substitution γ such that $\theta \circ \gamma = \sigma\ [W]$.

In the next definition, we address the concept of unification, that is fundamental in logic programing and automatic proving ([JA07]). In an intuitive way, to unify two expressions is to make them syntactically equal by applying over them a substitution called unifier (i.e., both expressions become equal to the instances resulting from them trough some substitution).

Definition 1.3.4. *A substitution θ is a unifier of the expressions E_1, E_2 if, and only if, $\theta(E_1) = \theta(E_2)$.*

[11] Occasionally we write only $\sigma\theta$ instead of $\sigma \circ \theta$ to abbreviate.

We can extend this definition in a very natural way to an infinite number of expressions $E_1, \ldots, E_n$, and we use, then, the unifier of the set $S = \{E_1, \ldots, E_n\}$.

Definition 1.3.5 ([JA07]). *A unifier σ of a set of expressions S is the most general unifier for S if, and only if, any other unifier θ is such that $\sigma \leq \theta$.*

We write *mgu* for the *most general unifier* of a set of expressions. The *mgu* always exists and is unique (not taking renaming into account, see [LMM88]).

To end this brief summary to logic programming, we state that the strength of this paradigm reside in its declarative component, that allows the construction of software by specifying "what" to compute instead of "how" to compute it, task delegated to the control system. Furthermore, since LP is based upon logic, it is well suited for representing knowledge and to obtain new information from the represented information. By contrast, it is not possible to represent vagueness or imprecise knowledge in LP, in principle, due to its rigid way to answer queries. This characteristic can be considered a limitation when modelling certain problems. So, in order to extend a framework as rich as logic programming to overcome this limitation, we have to exploit methods, techniques or tools to handle imprecision in an efficient way through a computer; these tools have also to adapt to the framework to be extended, that is, the extension of the framework has to be natural and, in absence of imprecision, it has to preserve the properties of the original framework. As seen in the previous section, fuzzy logic is a mathematical tool that fulfils those constraints.

Then, from the fusion of logic programming and fuzzy logic comes fuzzy logic programming, that handles imprecision and vagueness in a natural way, thus addressing that limitation of LP by integrating the well-established concepts of fuzzy logic. As we see in the next section, there are two main approaches in this paradigm. One possibility consists on the implementation of the same language enhanced to deal with imprecision [Hin86]; The other consists on extending the original language to allow the aforementioned goal [MBP87, LL90, IK85].

We focus in the next section on detailing in depth the most important notions of fuzzy logic programming, studying also the main approaches to this paradigm and classifying them by different criteria.

1.4 Fuzzy logic programming

Fuzzy logic programming (FLP) arose as an extension of LP in the same sense that fuzzy logic extends classical logic. FLP is defined formally as a part of fuzzy logic focused on the study of fuzzy theories or fuzzy programs, that are a set of fuzzy logic expressions in a first order language directly executables in a computer.

This style of programming applies to areas where the high level of abstraction and expressiveness of traditional declarative languages is required, but those are not able to neither model vague or imprecise scenarios nor formulate approximate reasoning. To this end, new expressive resources from fuzzy logic are incorporated, as the ones mentioned in 1.2.

The area of fuzzy logic programming is in a relatively incipient state, although it is being consolidated by a growing net of researchers that provide maturity in the theory aspects as well as in the practice ones. However there are still neither standards nor a unified framework, but different approaches that take divergent paths.

Due to this variety of schemes on FLP, it is possible to establish many classifications, analysing the procedural mechanism to deal with vagueness, the extension of syntactic unification, the extension of SLD-resolution, or other considerations (see Subsection 1.5) where we include interesting concepts as the implementation or not of the negation in this area, or different fuzzy logics.

Many of the different approaches of fuzzy logic programming replace the classical inference mechanism, SLD-resolution, by a fuzzy variant that allows to deal with uncertainty and evaluate truth degrees. Taking this into account, and following the classification provided in [Rub11], it is possible to establish to main trends:

- The first one includes truth degrees together with facts, rules and goals and, therefore, it needs to modify the resolution mechanism to perform operations over those degrees, and unification remains intact.

- The other modifies the unification algorithm and preserves the resolution mechanism by handling truth values separately.

Thus, there is no common method to "fuzzify" the resolution principle of Prolog (see [VGM02]): the majority of these languages implement the fuzzy resolution principle introduced by [Lee72] (extended by [Muk82] and [WTL93]), like the system Prolog-Elf [IK85], Fril Prolog [BMP95] and the language F-Prolog [LL90]. Other fuzzy languages like Likelog, considered in [AF99], or Bousi$\sim$Prolog ([JRG08]) only contemplate the fuzzy component of predicates by introducing the notion of similarity. [AG93] implements a modality of resolution conceived to manage the truth values of clauses as intervals (each boundary represents a truth and a false degree), [LL90] uses fuzzy expressions incorporating semantic hedges, [MSD89] includes expressions with an associated confidence obtained from its truth degree, and it sets the confidence of the resolvent from the confidence of the original clauses, and other systems, like [VP96, MOV01d, MOV01c, MOV04, MO04], where there are fuzzy facts and/or fuzzy rules labelling clauses with real numbers (or, more generally, elements of a lattice) representing its associated truth degree.

To summarize, in these languages there are many methods to fuzzify the knowledge, to represent it and to handle it. The soundness and completenes properties for the different types of procedural semantics has been proposed related to an appropriate declarative semantics that, in many cases, has been conceived as a fuzzy extension of the classical least Herbrand model [Llo87].

We detail now with more depth the two main trends in fuzzy logic programming with respect to the procedural mechanism, and their most representative languages, as indicated in [Rub11].

FLP extending SLD-resolution

In general, in this approach programs are a subset of clauses with an associated truth degree that is explicitly annotated. Computation and truth propagation is performed through a procedural semantic that is an extension of the classical resolution principle, while the (syntactic) unification mechanism remains untouched.

Thus, to represent vagueness in this framework, each fact, rule and goal is associated to a truth degree (for simplicity we detail here only a reduced framework of FLP that extends resolution. For a more detailed formalization, see [Voj01] or [MOV01d]). More precisely, a fuzzy logic program consists of tree parts: a fuzzy fact of the form $p(t_1, \ldots, t_n) \leftarrow [f]$, where p is a predicate symbol, each t is a term and f is a truth degree associated to $p(t_1, \ldots, t_n)$; a fuzzy rule of the form $A \leftarrow [\alpha]\langle B_1, \alpha_2 \rangle, \ldots, \langle B_n, \alpha_n \rangle$, where the truth degree of each (sub)goal B_i is α and the value of all conditions is $\Delta(\alpha, \beta)$, being Δ a t-norm or a fuzzy conjunction (see Section 1.2.1 in this chapter); and $\beta = \Delta_{i=1}^{n}(\alpha_i)$ a fuzzy goal of the form $\leftarrow [c]B_1, \ldots, B_n$, being c a constant that indicates the maximum truth degree to reach in the inference; or with the form $\leftarrow [\mathcal{F}]B_1, \ldots, B_n$, being $\mathcal{F}$ a variable to store the finally computed truth degree.

These kind of frameworks of FLP keep intact the unification mechanism and extend only the SLD-resolution mechanism to let the truth degrees associated to each atom transit from the antecedents to the consequents until succeed or fail. Therefore, supposing a set of fuzzy clauses $\mathcal{C}_1, \ldots, \mathcal{C}_{n+1}$:

$$\begin{array}{ll} \mathcal{C}_1 \equiv & A \leftarrow [\alpha_0] B_1, \ldots, B_n \\ \mathcal{C}_2 \equiv & B_1^{'} \leftarrow [\alpha_1] \\ \ldots & \\ \mathcal{C}_{n+1} \equiv & B_n^{'} \leftarrow [\alpha_n] \end{array}$$

where there is a fuzzy fact $\mathcal{B}_i^{'} \leftarrow [\alpha_i]$ for each antecedent in $\mathcal{C}_1$, i.e., $B_i^{'} = B_i$. Then, we can infer the fuzzy fact $\alpha \leftarrow [\Delta_{i=0}^{n} \alpha_i]$. Now, in case of a success, together with the output, a truth degree is provided.

After these brief general considerations about fuzzy logic languages that extend the SLD-resolution, we enumerate now some of the most interesting languages in this approach:

- **Prolog-ELF**. This language [IK85] is a Prolog system resulting from the *Fifth Generation of Programming Languages of Japan.* A Prolog-ELF program is a set of clauses associated to a truth degree in the interval $[0, 1]$. The system is based on the fuzzy resolution of [Lee72], clauses are $+A - B_1, \ldots, -B_n$ or $+A$ and goals of the form: $-B_1, \ldots, -B_n$.

  Truth values are assigned through this notation: $\alpha. + A - B_1, \ldots, -B_n$ or $assert(\alpha : +A - B_1, \ldots, -B_n)$. This notation has been adopted instead of the classical one $(A : -B_1, \ldots, B_n)$ because, according to the authors, there are many interpretations in fuzzy logic for $\neg A \vee B$, and they do not always correspond to the implication. Variables in Prolog-Elf begin by the character "$*$", and commentaries by ":". Prolog-Elf allows also to define fuzzy sets by means of special predicates that act as a membership function, i.e., that return a value between 0 and 1. Truth values in the body of a rule are combined using the t-norm "minimum" for conjunction, the t-conorm "maximum" for disjunction, and $1 - x$ for negation. Since it is based on the work of Lee, truth values of positive literals have to be in the interval $(0.5, 1]$.

- **FProlog**. FProlog [MBP87] is the name of the first implementation of Fril. The first version of this language was developed at the end of the seventies as a continuation of the works of Baldwin on fuzzy relations. The second version was released in the first years of the eighties. Both versions can be considered related languages. FProlog uses Lisp syntax instead of PROLOG syntax.

In order to provide the system with deductive capabilities, a PROLOG interpreter was integrated in its structure. The system could work as an autonomous PROLOG system. FProlog also allowed the definition of fuzzy relations, with a truth degree associated to each tuple. To compute the truth degree it uses fuzzy counterparts of conjunction, disjunction and negation.

- **Fuzzy Prolog**. Was created by Mukaidono [MSD89]. It extends Lee's approach to allow truth values in the interval $[0, 1]$. This work brought its first proposal about a Fuzzy Prolog, consolidated in [MSD89] by the introduction for the first time of the notion of *fuzzy first order predicate*, whose variables can be membership functions over a fuzzy subset (this idea has been acquired by more modern systems, like the fuzzy module of Ciao-Prolog of [GMV04]).

  The most recent contribution of Mukaidono is a fuzzy PROLOG based on the Łukasiewicz implication (LbFP). In LbFP there are no changes in the unification process. Facts and rules are extended with truth values in the interval $[0, 1]$ where 1 represents truth and 0 unknown or absurd. Truth values in the body of a rule are combined using the "minimum" t-norm for the conjunction, the "maximum" t-conorm for the disjunction and $1 - x$ for the negation. For a certain solution this framework takes the maximum of the computed values. Resolution in LbFP takes the shape of a tree in contrast with the classical linear resolution.

- **f-Prolog**. f-Prolog was created by [LL90]. An f-Prolog program differs only from a PROLOG program in the truth degree that some facts carry and in the degree of the implication, that is associated to the rules. An f-Prolog program is a sequence of f-facts, f-rules and an f-goal. An f-fact $\mathcal{A}$ is f. An f-rule has the form $A - [f] - B_1, \ldots, -B_n$, where A, B_i are atoms and the degree of the conclusion A is the product of α and the minimum of the truth values associated to each B_i. An f-goal is an expression $-[F] - Q_1, \ldots, -Q_n$ being F a variable. The system also allows to specify the minimum degree that a goal has to satisfy: for instance, it is possible to query the system solutions with more than 0.8 truth degree.

- **Rfuzzy**. RFuzzy language is an extension and enhancement of the fuzzy module Ciao-Prolog presented in [GMV04]. This module uses the union of subsets of truth degrees to handle imprecision. These truth degrees are combined by means of aggregation operators. Additional arguments can be used in each clause for managing these degrees. The implementation is based on a process of compilation that translates fuzzy programs to PROLOG code. Furthermore, this module uses the notion of fuzzy first order predicate introduced by Mukaidono in [MSD89].

One of the problems the authors of RFuzzy points out to the module Ciao is its complexity [MCS09]. This complexity is due to the use of real intervals to represent truth degrees. Besides, answers are associated to constraints, and the management of variables that carry truth degrees is problematic. RFuzzy reduces this complexity in some aspects: it uses real numbers instead of intervals to represent truth values; it answers with direct values instead of with constraints; and it removes the necessity of including variables to manage truth degrees.

- **Multi-adjoint language**. In multi-adjoint logic programming (MALP in brief) [MOV01d, MOV01c, MOV01b, MOV01a], each program is associated with a certain lattice that provides truth degrees and allows to encapsulate different types of fuzzy logics inside each rule. Given a MALP program, goals are evaluated in two separated computational phases. During the resolution phase, the fuzzy counterpart of SLD-derivation steps, called admissible steps, are applied. This first phase produces a substitution and an expression where all atoms have been exploited. This last expression is interpreted afterwards (in the so-called interpretative phase) in the multi-adjoint lattice associated to the program, thus obtaining a pair (*truth degree; substitution*) that is the fuzzy analogous to the classical notion of computed answer traditionally used in LP.

  Furthermore, it is noteworthy that the framework based on similarity presented in [Ses02] can be emulated by means of a certain multi-adjoint lattice [MOV04], particularly, the real interval $[0, 1]$ with the t-norm of Gödel, and extending the original program with a set of rules defining a similarity relation (in a similar way to the extension of first order logic with equality axioms). This illustrates the generality and expressiveness of the multi-adjoint logic language.

  From our point of view, the features of this language makes it one of the most powerful and interesting in the area of fuzzy logic programming. By this reason we have chosen it as a framework to develop our research in this field. In Chapter 2 we expose in depth the syntax, semantics and further basic notions of the multi-adjoint approach.

Extension of unification in FLP

The second trend in FLP is represented by languages that replaces the syntactic unification mechanism (of classical SLD-resolution) by a fuzzy unification algorithm. While the global mechanism of resolution remains untouched, unification changes dramatically since it can "unify" different predicates (provided they have some similarity degree).

The main goal of this framework is to obtain more flexible unification mechanisms. There are, at least, three approaches:

- Semantic unification ([AG98]): each constant is associated to a meaning (e.g., a fuzzy subset) and the unification is based on that meaning (e.g., by a *matching* procedure);
- Unification based on distance edition ([GS00]): the unification degree is based on a syntactic similarity between two symbols (e.g., house, mouse);
- Weak unification: it is the fuzzy extension of classical unification.

From now on, we focus on weak unification, since both semantic unification and unification based on distance edition can be implemented on weak unification.

Generally speaking, weak unification consists on the substitution of the syntactic equality by a similarity (or proximity) relation. This similarity relation $\mathcal{R}$ relates symbols of the alphabet of a first order language $\mathcal{L}$, that is, $\mathcal{R} = \mathcal{R}_{\mathcal{P}} \cup \mathcal{R}_{\mathcal{F}} \cup \mathcal{R}_{\mathcal{V}}$ where $\mathcal{R}_{\mathcal{V}}$ is a similarity on $\mathcal{V}$ defined by (1) $\mathcal{R}_{\mathcal{V}}(x, y) = 0$ if $x \neq y$ being $x, y \in \mathcal{V}$; (2) $\mathcal{R}_{\mathcal{V}}(x, x) = 1$. Furthermore, $\mathcal{R}_{\mathcal{F}}$ is such that, given two symbols $f, g \in \mathcal{F}$ with the same arity n, belonging to the alphabet of $\mathcal{L}$, $\mathcal{R}_{\mathcal{F}}$ relates both f and g, and that relation is quantified by an approximation degree in a way that can be read as "f is similar to g with approximation degree α", with $\alpha \in (0, 1]$.

A fuzzy program, in this framework, is composed by a set of clauses and a relation between the symbols in the alphabet of $\mathcal{L}$. This kind of languages, in FLP, need to modify the unification algorithm so it copes with the given relations. In this framework the unification algorithm does not end when two terms differ in some symbol, but it searches a relation between them. Furthermore, SLD-resolution has to be adapted to allow the composition of the approximation degrees obtained in each resolution steps. Then, supposing a set of clauses $\mathcal{C}_1, \dots, \mathcal{C}_{n+1}$:

$$\begin{array}{ll} \mathcal{C}_1 \equiv & A \leftarrow [\alpha_0] B_1^{'}, \dots, B_n^{'} \\ \mathcal{C}_2 \equiv & B_1 \leftarrow \\ \dots & \\ \mathcal{C}_{n+1} \equiv & B_n \leftarrow \end{array}$$

where there is a fact $\mathcal{B}_i^{'} \leftarrow$ similar to an antecedent in $\mathcal{C}_1$, i.e.,

$$\mathcal{R}(\mathcal{B}_1^{'}, \mathcal{B}_1) = \alpha_1, \dots, \mathcal{R}(\mathcal{B}_n^{'}, \mathcal{B}_n) = \alpha_n$$

Then we infer $\langle A, \beta \rangle$, with $\beta = \Lambda_{i=1}^{n} \alpha_i$.

We present now the basic features of the main fuzzy logic languages based on similarity.

- Likelog: this language, whose name is an acronym of "Likeness in Logic", is the practical implementation of the works of [FGS00]. This language is based on the theoretical concept of cloud. A cloud is a set of elements that can be considered similar with respect to their meaning [AF99]. According to the authors, Likelog is an environment of logic programming implemented in *Dec-10 Standard Prolog*. The language is able to handle flexible queries and answers. Its syntax $\mathcal{L}'$ is an extension of a first order language $\mathcal{L}$ with no function symbols where similarity is introduced by a fuzzy relation $\mathcal{R} = \mathcal{R}_\mathcal{C} \cup \mathcal{R}_\mathcal{V} \cup \mathcal{R}_\mathcal{P}$ on the symbols of the alphabet, and it fulfils the reflexive, symmetric and transitive properties. Its procedural semantics is a fuzzy extension of the unification algorithm associated to the resolution, while its fixpoint semantics is given by an extension of the classical least Herbrand model. One of the applications of Likelog is its flexibility on handling answers and queries in deductive databases.

- SiLog: this language ([LSS01]) is based on the work [Ses02]. The SiLog system is implemented on a W-Prolog interpreter built in Java (for which an online reference is `http://waitaki.otago.ac.nz/~michael/wp/`). It is composed by two parts: an inference engine and a similarity manager. The inference engine handles the weak unification, while the similarity manager generates a similarity from the quotient sets (families of λ-cuts) from the elements included in a dictionary incorporated by the user. The similarity manager ensures that the built relations fulfil the definition of similarity, which means that the extension is a little rigid and limited. Similarity relations are addressed independently to the program.

- Bousi∼Prolog ([JRG08, JRG09a, JR09b, JR10a]): this language is one of the most recent and interesting languages in this tendency. This language is the practical implementation of the theoretical works of [JR06b, JR06a]. Its procedural semantics [JR09a] adapts the principle of SLD-resolution and its unification algorithm is based on proximity relations (i.e., a binary fuzzy reflexive and symmetric relation on a set, not necessarily transitive). It generalizes the procedural mechanism of Sessa, based on similarity relations, and enhances the expressive power of the resulting language. The properties of reflexivity and symmetry are very appropriate to express the value of "proximity" between elements of a domain. The authors of this language can model problems in cases where the transitive restriction of the similarity relation is an obstacle.

  The syntax of Bousi∼Prolog is, basically, Prolog syntax enriched by a constructor symbol ∼ used to describe proximity relations (in fact, fuzzy binary relations that are translated automatically to proximity or similarity relations) by means of what authors call a proximity equation. Those equations are expressions of the form *symbol* ∼ *symbol* = *proximity value*. This operator is the fuzzy counterpart of the syntactic unification operator '=' of Prolog. Furthermore, the language supports the use of fuzzy sets [JRG09b, JRG10, JR10b], integrating them easily in the core of the system by the use of fuzzy relations, not affecting the procedural semantics.

  On the other hand, the authors of Bousi∼Prolog have created UNICORN [JR09c], a programming environment that allows to edit, compile and run Bousi∼Prolog programs, according to the principle of weak SLD-resolution described for this language. The complete implementation of this tool consists of 900 lines of Prolog code, approximately, and it includes more useful options for programmers. A more complete specification of the tool can be found in `http://dectau.uclm.es/bousi/`.

1.5 Other considerations

To end this section on FLP, we include here important concepts about this paradigm, as presented in the PhD thesis [Pen10]. For example, together with the classification provided in previous sections, it is also possible to differentiate between languages based on annotations from languages based on implications (see [LS01b, LS05]).

- Languages based on annotations (see [KL88, KS92, NS91, NS92, Lu96, Cao00, KLV02]), admit rules of the form $A : f(\beta_1, \ldots, \beta_n) \longleftarrow B_1 : \beta_1, \ldots, B_n : \beta_n$ whose meaning can be understood as "the truth degree of A is, at least, $f(\beta_1, \ldots, \beta_n)$, if the truth degree of each atom B_i is at least β_i", $1 \leq i \leq n$, being f a computable function and β_i a constant of a variable over an appropriate domain (set of truth values).

- In contrast, those languages based on implications (see [DMO04a, DM04, LS94, LS01b, MOV01d, MOV01c, MOV01b, MOV01a, MO02, MOV04, MO04, DMO07, vE86, Voj01]) have rules of the form $\langle A \leftarrow @(B_1, \ldots, B_n);\ v\rangle$ where v is the truth degree associated to formula $A \leftarrow @(B_1, \ldots, B_n)$, where @ is a connective that combines atomic expressions B_i.

  Computationally, given an interpretation $\mathcal{I}$, truth values $\mathcal{I}(B_i)$ are determined by the truth function of connective @ and, afterwards, propagated to atom A in the head of the rule. Also, truth degrees can be taken from a lattice, that is, the application $\mathcal{I}$ can be mapped to values on a certain lattice. [DMO04a, KLV04, LS01b, Voj01] show that the majority of the frameworks that manages imprecision can be implemented in this context.

On the other hand, it is noteworthy that the majority of approaches do not include any kind of non-monotonic reasoning, neither admit negation. It is not the case of [DM01a, LS06, LS02a, LS02b, LS03, Str05a, Str05c], where the domain of truth values is a lattice. The language of [VGM02, GMV04] also admits negation, but here truth degrees are real intervals [KY95], which are very adequate to formalize linguistic variables as "age" or "speed".

With respect to negation, it is the most relevant logic concept not originally coped by fuzzy logic programming. This is because its inclusion leads to a significant complexity. However, negation plays an important role on representing knowledge and many of its applications cannot be emulated by positive programs. Negation is also useful in the management of databases, program composition, reasoning by default, etc. [Ger05] can, in the context of control techniques, address positive and negative information if truth values are taken from a bilattice.

Furthermore, as already stated in this introductory chapter, there is no consensus about which fuzzy logic corresponds to which context. The majority of the systems uses a min-max logic (to model conjunction an disjunction), but some systems uses Łukasiewicz logic [KK94]. Other approaches allow a more generic interpretation of connectives [VP96], and the multi-adjoint framework allows also different logics to model connectives.

Finally, with respect to the set where the interpretation of formulae take place[12], there are fuzzy logic programs interpreted in:

1. The real interval $[0, 1]$, as the case of [MSD89, vE86, Sha83, VP96, Voj01, AF99, KLV04].
2. A lattice, as [MOV01d, MOV01c, MOV01b, MO02, MOV04, MO04] (multi-adjoint lattice) and also [DM00, DM01b, DM02, DMO04b] (residuated lattice).
3. A bilattice [Gin88, Fit91, Ger05, LS04, Str05c], trilattice [LS01a] or, more generally, a multilattice [MOR05, Mor06b, MOR06a, MOR07c, MOR07b, MOR07a].
4. A set of intervals, like [VGM02, GMV04, LS01a, Luk01, AG93].
5. A qualification domain, as the case of [CRR08, RR08b, RR09].

To conclude, fuzzy logic programming is a research area in constant expansion and promising perspectives, whose application in the form of fuzzy logic languages can greatly help to codify systems with fuzzy features in the fields of most solid implantation of fuzzy logic, as seen in Section 1.2.2 (the construction of expert systems, control applications, *soft computing*, etc.).

As already stated, from our point of view the most interesting context is the multi-adjoint programming, due to its great generality (it can implement many of the other fuzzy logic schemes), its high level of expressiveness and is clearly defined procedural semantics. We detail in Chapter 2 the essential concepts of the multi-adjoint framework.

[12]In other contexts more general structures have been used to interpret formulae, like algebraic domains (see [RZ01, Sco82]).

Chapter 2

Multi-adjoint Logic Programming and the FLOPER System

The uncertainty and vagueness are constant elements present in most human thinking activities, fuzzy logic programming seems to be a computing paradigm with a level of expressiveness very close to human reasoning. Daily, we frequently deal with fuzzy predicates (not *boolean, crisp* ones). For instance, a person can be quite young or not very young: the frontier between "absolutely young" and "not young at all" is not sharp, but *indefinite, fuzzy.* If "John" is 18 year old, we can say that he is "young" at a 95% truth degree. If "Mary" is 70 years old, she is young at a lower truth degree. Then, the obvious question is how do we model the concept of "*truth degree*". In our system, we work with the so-called *multi-adjoint lattices* to model them by providing a flexible, wide enough definition of such notion.

Informally speaking, in the multi-adjoint logic framework, a program can be seen as a set of rules each one annotated by a truth degree, and a goal is a query to the system, i.e., a set of atoms linked with connectives called *aggregators.* A state is a pair $\langle \mathcal{Q}, \sigma \rangle$ where $\mathcal{Q}$ is a goal and σ a substitution (initially, the identity substitution). States are evaluated in two separate computational phases. During the *operational* one, *admissible steps* (a generalization of the classical *modus ponens* inference rule) are systematically applied by a backward reasoning procedure in a similar way to classical resolution steps in pure logic programming, thus returning a computed substitution together with an expression where all atoms have been exploited. This last expression is then interpreted under a given lattice during what we call the *interpretive* phase, hence returning a pair $\langle truth\_degree; substitution\rangle$ which is the fuzzy counterpart of the classical notion of computed answer traditionally used in LP.

The main goal of this report is the detailed description of the FLOPER system which is available from `http://dectau.uclm.es/floper/`. Nowadays, the tool provides facilities for executing as well as for debugging (by generating declarative traces) such kind of fuzzy programs, thus fulfilling the gap we have detected in the area.

In order to explain the tool, we have structured this chapter as follows: in Section 2.1 we present the essence of MALP, including syntax, procedural semantics, and some interesting computational cost measures defined for this programming style; the core of this chapter is represented by Section 2.2, which is dedicated to explain the main capabilities of the FLOPER system such as running/debugging MALP programs, managing lattices and dealing with additional fuzzy concepts; we detail in Section 2.3 how the use of sophisticated multi-adjoint lattices are very useful for easily coding flexible real-world applications and obtaining low-cost traces at execution time.

2.1 Multi-Adjoint Logic Programming

This section summarizes the main features of multi-adjoint logic programming (for a complete formulation of this framework, see [MOV01d, MOV01c, MOV04, JMP09]). In what follows, we use abbreviation MALP for referencing programs belonging to this setting.

2.1.1 MALP Syntax

We work with a first order language, $\mathcal{L}$, containing variables, constants, function symbols, predicate symbols, and several (arbitrary) connectives to increase language expressiveness: implication connectives ($\leftarrow_1, \leftarrow_2, \ldots$); conjunctive operators (denoted by $\&_1, \&_2, \ldots$), disjunctive operators ($|_1, |_2, \ldots$), and hybrid operators (usually denoted by $@_1, @_2, \ldots$), all of them are grouped under the name of "aggregators".

Aggregation operators (or aggregators) are useful to describe/specify user preferences. An aggregation operator, when interpreted as a truth function, may be an arithmetic mean, a weighted sum or in general any monotone application whose arguments are values of a complete bounded lattice L. For example, if an aggregator @ is interpreted as $\dot{@}(x, y, z) = (3x + 2y + z)/6$, we are giving the highest preference to the first argument, then to the second, being the third argument the least significant. Although these connectives are binary operators, we usually generalize them as functions with an arbitrary number of arguments. So, we often write $@(x_1, \ldots, x_n)$ instead of $@(x_1, \ldots, @(x_{n-1}, x_n), \ldots)$. By definition, the truth function for an n-ary aggregation operator $\dot{@} : L^n \to L$ is required to be monotonous and fulfills $\dot{@}(\top, \ldots, \top) = \top$, $\dot{@}(\bot, \ldots, \bot) = \bot$.

Additionally, our language $\mathcal{L}$ contains the values of a *multi-adjoint lattice* equipped with a collection of *adjoint pairs* $\langle \leftarrow_i, \&_i \rangle$ (where each $\&_i$ is a conjunctor which is intended to the evaluation of *modus ponens* [MOV04]) formally defined as follows.

Definición 2.1.1 (Multi-Adjoint Lattice). *Let $(L, \leq)$ be a lattice. A* multi-adjoint lattice *is a tuple $(L, \leq, \leftarrow_1, \&_1, \ldots, \leftarrow_n, \&_n)$ such that:*

1. *$\langle L, \preceq \rangle$ is a bounded lattice, i.e. it has bottom and top elements, denoted by $\bot$ and $\top$, respectively.*

2. *$\langle L, \preceq \rangle$ is a complete lattice, i.e. for all subset $X \subset L$, there are $inf(X)$ and $sup(X)$.*

3. *$\top \&_i v = v \&_i \top = v$, $\forall v \in L, i = 1, \ldots, n$.*

4. *Each operation $\&_i$ is increasing in both arguments.*

5. *Each operation $\leftarrow_i$ is increasing in the first argument and decreasing in the second one.*

6. *If $\langle \&_i, \leftarrow_i \rangle$ is an* adjoint pair *in $\langle L, \preceq \rangle$ then, for any $x, y, z \in L$, we have that:*

$$x \preceq (y \leftarrow_i z) \quad \textit{if and only if} \quad (x \,\&_i\, z) \preceq y$$

This last condition, called *adjoint property*, could be considered the most important feature of the framework (in contrast with many other approaches) which justifies most of its properties regarding crucial results for soundness, completeness, applicability, etc.

In general, L may be the carrier of any complete bounded lattice where a L-expression is a well-formed expression composed by values and connectives defined in L, as well as variable symbols and *primitive operators* (i.e., arithmetic symbols such as $*, +, min$, etc...). In what follows, we assume that the truth function of any connective @ in L is given by its corresponding *connective definition*, that is, an equation of the form $@(x_1, \dots, x_n) \triangleq E$, where E is a L-expression not containing variable symbols apart from $x_1, \dots, x_n$. For instance, consider the following classical set of adjoint pairs (conjunctions and implications) in $\langle [0,1], \leq \rangle$, where labels `L`, `G` and `P` mean respectively *Łukasiewicz logic*, *Gödel intuitionistic logic* and *product logic* (which different capabilities for modeling *pessimist*, *optimist* and *realistic scenarios*, respectively):

$$\begin{array}{lll}
\&_{\texttt{P}}(x,y) \triangleq x * y & \leftarrow_{\texttt{P}}(x,y) \triangleq \min(1, x/y) & \textit{Product} \\
\&_{\texttt{G}}(x,y) \triangleq \min(x,y) & \leftarrow_{\texttt{G}}(x,y) \triangleq \begin{cases} 1 & \text{if } y \leq x \\ x & \text{otherwise} \end{cases} & \textit{Gödel} \\
\&_{\texttt{L}}(x,y) \triangleq \max(0, x+y-1) & \leftarrow_{\texttt{L}}(x,y) \triangleq \min\{x-y+1, 1\} & \textit{Łukasiewicz}
\end{array}$$

Moreover, the three disjunctions associated to the previous fuzzy logics are defined as follows: $|_{\texttt{P}}(x,y) \triangleq x{+}y{-}x{*}y$, $|_{\texttt{G}}(x,y) \triangleq max\{x,y\}$, and $|_{\texttt{L}}(x,y) \triangleq min\{x{+}y, 1\}$.

At this point, we wish to make a mention to the notion of *qualification domain* used in the QLP (*Qualified Logic Programming*) scheme described in [RD08], which plays a role in that framework similar to multi-adjoint lattices in MALP. A qualification domain is a structure $\langle D, \sqsubseteq, \perp, \top, \circ \rangle$, such that $\langle D, \sqsubseteq, \perp, \top \rangle$ is a lattice with top ($\top$) and bottom ($\perp$) elements, a partial ordering $\sqsubseteq$, and where the so-called *attenuation operation* "$\circ$" is a conjunction. Now, given two elements $d, e \in D$, $d \sqcap e$ means for the *greatest lower bound* of d and e, whereas $d \sqcup e$ represents its *least upper bound*. We also write $d \sqsubset e$ as abbreviation of $d \sqsubseteq e \& d \neq e$. The attenuation operator $\circ$ satisfies the following constraints.

1. $\circ$ is associative, commutative and monotonic w.r.t. $\sqsubseteq$.

2. $\forall d \in D : d \circ \top = d$.

3. $\forall d \in D : d \circ \perp = \perp$.

4. $\forall d, e \in D/\{\perp, \top\} : d \circ e \sqsubset e$.

5. $\forall d, e_1, e_2 \in D : d \circ (e_1 \sqcap e_2) = d \circ e_1 \sqcap d \circ e_2$.

Note that the required properties in QLP and MALP are rather close, but instead of the last *distributive law* just pointed out in claim 5, in our setting we use the adjoint property (claim 6 in Definition 2.1.1). Translations between both worlds can be frequently performed, as occurs for instance with the simple boolean qualification domain $\mathcal{B} = ([0,1], \leq, 0, 1, \&)$, whose shape as a multi-adjoint lattice looks like $\langle [0,1], \leq, \leftarrow, \& \rangle$, where & is de boolean conjunction and $\leftarrow$ its adjoint implication (i.e., the usual bi-valuated logic implication). Something similar occurs with the qualification domain of *Van Emden's uncertainty values* used in QLP, $\mathcal{U} = ([0,1], \leq, 0, 1, \times)$, which is equivalent to the multi-adjoint lattice (in which most of our examples are interpreted) described above, as well as with the so-called *weights domain* $\mathcal{W} = (\mathbb{R} \cup \infty, \geq, \infty, 0, +)$, whose detailed explanation is delayed to Section 2.3 (where we will present powerful extensions and applications related with debugging tasks into the MALP framework).

In general, any qualification domain $(D, \preceq, \bot, \top, \circ)$ whose attenuation operation $\circ$ conforms an adjoint-pair with a given implication operation $\leftarrow_\circ$, can be expressed as the multi-adjoint lattice $\langle D, \preceq, \leftarrow_\circ, \circ \rangle$. Anyway, we wish to finish this brief comparison by highlighting that the variety of connectives definable in multi-adjoint lattices is clearly much greater than those appearing in qualification domains (where for a given program, all rules must always use the same attenuation operator), which justifies the higher expressive power of MALP w.r.t. QLP.

Continuing now with the description of the multi-adjoint logic programming approach, a MALP *rule* is a formula $H \leftarrow_i \mathcal{B}$, where H is an atomic formula (usually called the *head*) and $\mathcal{B}$ (which is called the *body*) is a formula built from atomic formulas $B_1, \ldots, B_n$ $(n \geq 0\ )$, truth values of L, conjunctions, disjunctions and aggregations. Rules whose body is $\top$ or equivalently, rules without body (or with empty body) are called *facts*. A *goal* is a body submitted as a query to the system.

Roughly speaking, a MALP program is a set of pairs $\langle \mathcal{R}; v \rangle$ (we often write "$\mathcal{R}$ *with* v"), where $\mathcal{R}$ is a rule and v is a *truth degree* (a value of L) expressing the confidence of a programmer in the truth of rule $\mathcal{R}$. By abuse of language, we sometimes refer a tuple $\langle \mathcal{R}; v \rangle$ as a "rule". As an example, in **Figure 2.1** we show a MALP program whose rules define fuzzy predicates modeling degrees of youth ("y"), heritage ("h") and education ("e"), as well as the confidence level ("c") on people for repaying a loan. Note that the seventh rule models this last notion by considering young persons having good heritage or education degrees. In what follows, we are

Multi-adjoint logic program $\mathcal{P}_1$:

$$
\begin{array}{lllll}
\mathcal{R}_1 : & y(peter) & & & with\ 0.4 \\
\mathcal{R}_2 : & y(mary) & & & with\ 0.8 \\
\mathcal{R}_3 : & h(peter) & & & with\ 0.9 \\
\mathcal{R}_4 : & h(mary) & & & with\ 0.3 \\
\mathcal{R}_5 : & e(peter) & & & with\ 0.5 \\
\mathcal{R}_6 : & e(mary) & & & with\ 0.95 \\
\mathcal{R}_7 : & c(X) & \leftarrow_{\mathtt{P}} & (h(X) \mid_P e(X)) \ \&_P \ y(X) & with\ 1
\end{array}
$$

Derivation D_1:

Admissible derivation

$$
\begin{array}{ll}
\langle \underline{c(X)};\ id\rangle & \rightarrow_{AS1}{}^{\mathcal{R}_7} \\
\langle \&_{\mathtt{P}}(1, \&_{\mathtt{P}}(|_{\mathtt{P}}(\underline{h(X_1)}, e(X_1)), y(X_1))); \{X/X_1\}\rangle & \rightarrow_{AS2}{}^{\mathcal{R}_3} \\
\langle \&_{\mathtt{P}}(1, \&_{\mathtt{P}}(|_{\mathtt{P}}(0.9, \underline{e(peter)}), y(peter))); \{X/peter, X_1/peter\}\rangle & \rightarrow_{AS2}{}^{\mathcal{R}_5} \\
\langle \&_{\mathtt{P}}(1, \&_{\mathtt{P}}(|_{\mathtt{P}}(0.9, 0.5), \underline{y(peter)})); \{X/peter, X_1/peter\}\rangle & \rightarrow_{AS2}{}^{\mathcal{R}_1} \\
\langle \&_{\mathtt{P}}(1, \&_{\mathtt{P}}(|_{\mathtt{P}}(0.9, 0.5), 0.4)); \{X/peter, X_1/peter\}\rangle &
\end{array}
$$

Interpretive derivation

$$
\begin{array}{ll}
\langle \&_{\mathtt{P}}(1, \&_{\mathtt{P}}(\underline{|_{\mathtt{P}}(0.9, 0.5)}, 0.4)); \{X/peter\}\rangle & \rightarrow_{IS} \\
\langle \&_{\mathtt{P}}(1, \underline{\&_{\mathtt{P}}(0.95, 0.4)}); \{X/peter\}\rangle & \rightarrow_{IS} \\
\langle \underline{\&_{\mathtt{P}}(1, 0.38)}; \{X/peter\}\rangle & \rightarrow_{IS} \\
\langle 0.38; \{X/peter\}\rangle. &
\end{array}
$$

Figure 2.1: MALP program with admissible/interpretive derivations for goal "$c(X)$"

going to explain the procedural principle of MALP programs, whose application to goal "$c(X)$" w.r.t. our program, will assign truth degrees 0.772 (i.e, a credibility of 77.2%) and 0.38 (or 38% of confidence level) to "*mary*" and "*peter*", respectively.

2.1.2 MALP Procedural Semantics

The procedural semantics of the multi–adjoint logic language $\mathcal{L}$ can be thought of as an operational phase (based on admissible steps) followed by an interpretive one. In the following, $\mathcal{C}[A]$ denotes a formula where A is a sub-expression which occurs in the –possibly empty– context $\mathcal{C}[]$. Moreover, $\mathcal{C}[A/A']$ means the replacement of A

by A' in context $\mathcal{C}[]$, whereas $\mathcal{V}ar(s)$ refers to the set of distinct variables occurring in the syntactic object s, and $\theta[\mathcal{V}ar(s)]$ denotes the substitution obtained from θ by restricting its domain to $\mathcal{V}ar(s)$.

Definición 2.1.2 (Admissible Step)**.** *Let $\mathcal{Q}$ be a goal and let σ be a substitution. The pair $\langle \mathcal{Q}; \sigma\rangle$ is a* state *and we denote by $\mathcal{E}$ the set of states. Given a program $\mathcal{P}$, an* admissible computation *is formalized as a state transition system, whose transition relation $\rightarrow_{AS} \subseteq (\mathcal{E} \times \mathcal{E})$ is the smallest relation satisfying the following* admissible rules *(where we always consider that A is the selected atom in $\mathcal{Q}$ and $mgu(E)$ denotes the* most general unifier *of an equation set E [LMM88]):*

1) $\langle \mathcal{Q}[A]; \sigma\rangle \rightarrow_{AS} \langle (\mathcal{Q}[A/v\&_i\mathcal{B}])\theta; \sigma\theta\rangle$*, if* $\theta = mgu(\{A' = A\})$, $\langle A' \leftarrow_i \mathcal{B}; v\rangle$ *in $\mathcal{P}$ and $\mathcal{B}$ is not empty.*

2) $\langle \mathcal{Q}[A]; \sigma\rangle \rightarrow_{AS} \langle (\mathcal{Q}[A/v])\theta; \sigma\theta\rangle$*, if* $\theta = mgu(\{A' = A\})$ *and* $\langle A' \leftarrow_i; v\rangle$ *in* $\mathcal{P}$*.*

3) $\langle \mathcal{Q}[A]; \sigma\rangle \rightarrow_{AS} \langle (\mathcal{Q}[A/\bot]); \sigma\rangle$*, if there is no rule in $\mathcal{P}$ whose head unifies with A.*

Note that the second case could be subsumed by the first one, after expressing each fact $\langle A' \leftarrow_i; v\rangle$ as a program rule of the form $\langle A' \leftarrow_i \top; v\rangle$. Also, the third case is introduced to cope with (possible) unsuccessful admissible derivations. As usual, rules are taken renamed apart. We shall use the symbols $\rightarrow_{AS1}$, $\rightarrow_{AS2}$ and $\rightarrow_{AS3}$ to distinguish between computation steps performed by applying one of the specific admissible rules. Also, the application of a rule on a step will be annotated as a superscript of the $\rightarrow_{AS}$ symbol.

Definición 2.1.3. *Let $\mathcal{P}$ be a program, $\mathcal{Q}$ a goal and "id" the empty substitution. An* admissible derivation *is a sequence $\langle \mathcal{Q}; id\rangle \rightarrow_{AS} \ldots \rightarrow_{AS} \langle \mathcal{Q}'; \theta\rangle$. When $\mathcal{Q}'$ is a formula not containing atoms (i.e., a L-expression), the pair $\langle \mathcal{Q}'; \sigma\rangle$, where $\sigma = \theta[\mathcal{V}ar(\mathcal{Q})]$, is called an* admissible computed answer *(a.c.a.) for that derivation.*

Ejemplo 2.1.4. *In Figure 2.1 we illustrate an admissible derivation (note that the selected atom in each step appears underlined), where the admissible computed answer (a.c.a.) is composed by the pair: $\langle \&_{\mathtt{P}}(1, \&_{\mathtt{P}}(|_{\mathtt{P}}(0.9, 0.5), 0.4)); \theta\rangle$ where θ only refers to bindings related with variables in the goal, i.e.,*

$$\theta = \{X/peter, X_1/peter\}[\mathcal{V}ar(c(X))] = \{X/peter\}$$

If we exploit all atoms of a given goal, by applying enough admissible steps, then it becomes a formula with no atoms (a L-expression) which can be interpreted w.r.t. lattice L by applying the following definition we initially presented in [JMP06]:

Definición 2.1.5 (Interpretive Step)**.** *Let $\mathcal{P}$ be a program, $\mathcal{Q}$ a goal and σ a substitution. Assume that $\dot{@}$ is the truth function of connective $@$ in the lattice $\langle L, \preceq \rangle$ associated to $\mathcal{P}$, such that, for values $r_1, \ldots, r_n, r_{n+1} \in L$, we have that $\dot{@}(r_1, \ldots, r_n) = r_{n+1}$. Then, we formalize the notion of* interpretive computation *as a state transition system, whose transition relation $\rightarrow_{IS} \subseteq (\mathcal{E} \times \mathcal{E})$ is defined as the least one satisfying: $\langle \mathcal{Q}[@(r_1, \ldots, r_n)]; \sigma \rangle \quad \rightarrow_{IS} \quad \langle \mathcal{Q}[@(r_1, \ldots, r_n)/r_{n+1}];\sigma \rangle$.*

Definición 2.1.6. *Let $\mathcal{P}$ be a program and $\langle \mathcal{Q}; \sigma \rangle$ an a.c.a., that is, $\mathcal{Q}$ does not contain atoms (i.e., it is a L-expression). An* interpretive derivation *is a sequence $\langle \mathcal{Q}; \sigma \rangle \rightarrow_{IS} \ldots \rightarrow_{IS} \langle \mathcal{Q}'; \sigma \rangle$. When $\mathcal{Q}' = r \in L$, being $\langle L, \preceq \rangle$ the lattice associated to $\mathcal{P}$, the state $\langle r; \sigma \rangle$ is called a* fuzzy computed answer *(f.c.a.) for that derivation.*

Ejemplo 2.1.7. *If we complete the previous derivation of Example 2.1.4 by applying 3 interpretive steps in order to obtain the final f.c.a. $\langle 0.38; \{X/peter\} \rangle$, we generate the interpretive derivation shown in* **Figure 2.1***.*

2.1.3 Interpretive Steps and Cost Measures

A classical, simple way for estimating the computational cost required to built a derivation, consists in counting the number of computational steps performed on it. So, given a derivation D, we define its:

- *operational cost*, $\mathcal{O}_c(D)$, as the number of admissible steps performed in D.
- *interpretive cost*, $\mathcal{I}_c(D)$, as the number of interpretive steps done in D.

Note that the operational and interpretive costs of derivation D_1 performed in **Figure 2.1** are $\mathcal{O}_c(D_1) = 4$ and $\mathcal{I}_c(D_1) = 3$, respectively. Intuitively, $\mathcal{O}_c$ informs us about the number of atoms exploited along a derivation. Similarly, $\mathcal{I}_c$ seems to estimate the number of connectives evaluated in a derivation. However, this last statement is not completely true: $\mathcal{I}_c$ only takes into account those connectives appearing in the bodies of program rules which are replicated on states of the derivation, but no those connectives recursively *nested* in the definition of other connectives. The following example highlights this fact.

Ejemplo 2.1.8. *A simplified version of rule $\mathcal{R}_7$, whose body only contains an aggregator symbol is:*

$$\mathcal{R}_7^* : c(X) \leftarrow_{\mathtt{P}} @_1(h(X), e(X), y(X)) \quad \textit{with} \quad \mathit{1}$$

where $@_1$ is defined as $@_1(x_1, x_2, x_3) \triangleq \&_{\mathtt{P}}(|_{\mathtt{P}}(x_1, x_2), x_3)$. Note that $\mathcal{R}_7^*$ has exactly the same meaning (interpretation) than $\mathcal{R}_7$, although different syntax. In fact, both ones have the same sequence of atoms in their head and bodies. The differences are regarding the set of connectives which explicitly appear in their bodies since in $\mathcal{R}_7^*$ we have moved $\&_{\mathtt{P}}$ and $|_{\mathtt{P}}$ from the body of the rule (see $\mathcal{R}_7$) to the connective definition of $@_1$. Now, we use rule $\mathcal{R}_7^*$ instead of $\mathcal{R}_7$ for generating the following derivation D_1^* which returns exactly the same f.c.a than D_1:

$$\begin{array}{ll}
\langle \underline{c(X)};\ id \rangle & \rightarrow_{AS1}{}^{\mathcal{R}_7} \\
\langle \&_{\mathtt{P}}(1, @_1(\underline{h(X_1)}, e(X_1)y(X_1))); \{X/X_1\} \rangle & \rightarrow_{AS2}{}^{\mathcal{R}_3} \\
\langle \&_{\mathtt{P}}(1, @_1(0.9, \underline{e(peter)}, y(peter))); \{X/peter, X_1/peter\} \rangle & \rightarrow_{AS2}{}^{\mathcal{R}_5} \\
\langle \&_{\mathtt{P}}(1, @_1(0.9, 0.5, \underline{y(peter)})); \{X/peter, X_1/peter\} \rangle & \rightarrow_{AS2}{}^{\mathcal{R}_1} \\
\langle \&_{\mathtt{P}}(1, \underline{@_1(0.9, 0.5, 0.4)}); \{X/peter, X_1/peter\} \rangle & \rightarrow_{IS} \\
\langle \underline{\&_{\mathtt{P}}(1, 0.38)}; \{X/peter\} \rangle & \rightarrow_{IS} \\
\langle 0.38; \{X/peter\} \rangle &
\end{array}$$

Note that, since we have exploited the same atoms with the same rules (except for the first steps performed with rules $\mathcal{R}_7$ and $\mathcal{R}_7^*$, respectively) in both derivations, then $\mathcal{O}_c(D_1) = \mathcal{O}_c(D_1^*) = 4$. However, although connectives $\&_{\mathtt{P}}$ and $|_{\mathtt{P}}$ have been evaluated in both derivations, in D_1^* such evaluations have not been explicitly counted as interpretive steps, and consequently they have not been added to increase the interpretive cost measure $\mathcal{I}_c$. This unrealistic situation is reflected by the abnormal result $\mathcal{I}_c(D_1) = 3 > 2 = \mathcal{I}_c(D_1^*)$. It is important to note that $\mathcal{R}_7^*$ must not be considered an optimized version of $\mathcal{R}_7$, even when the wrong measure $\mathcal{I}_c$ seems to indicate the contrary.

This problem was initially pointed out in [JMP07], where a preliminary solution was proposed by assigning weights to connectives in concordance with the set of primitive operators involved in the definition of the proper connective @ as well as those ones recursively contained in the definitions of connectives invoked from @. Moreover, in [MM09b] we improved the previous notion of "connective weight" by also taken into account the number of recursive calls to fuzzy connectives (directly or indirectly) performed in the definition of @.

A rather different way for facing the same problem is presented in [MM09a], where instead on connective weights, we opt for the more "visual" method we have just implemented into FLOPER, based on the subsequent re-definition of the behaviour of the interpretive phase.

Definición 2.1.9 (Small Interpretive Step)**.** *Let $\mathcal{P}$ be a program, $\mathcal{Q}$ a goal and σ a substitution. Assume that the (non interpreted yet) L-expression $\Omega(r_1, \ldots, r_n)$ occurs in $\mathcal{Q}$, where Ω is just a primitive operator or a connective defined in the lattice $\langle L, \preceq\rangle$ associated to $\mathcal{P}$, and $r_1, \ldots, r_n$ are elements of L. We formalize the notion of* small interpretive computation *as a state transition system, whose transition relation $\rightarrow_{SIS} \subseteq (\mathcal{E} \times \mathcal{E})$ is the smallest relation satisfying the following* small interpretive rules *(where we always consider that $\Omega(r_1, \ldots, r_n)$ is the selected L-expression in $\mathcal{Q}$):*

1) $\langle Q[\Omega(r_1, \ldots, r_n)]; \sigma\rangle \rightarrow_{SIS} \langle Q[\Omega(r_1, \ldots, r_n)/E'];\sigma\rangle$*, if Ω is a connective defined as $\Omega(x_1, \ldots, x_n) \triangleq E$ and E' is obtained from the L-expression E by replacing each variable (formal parameter) x_i by its corresponding value (actual parameter) r_i, $1 \leq i \leq n$, that is, $E' = E[x_1/r_1, \ldots, x_n/r_n]$.*

2) $\langle Q[\Omega(r_1, \ldots, r_n)]; \sigma\rangle \rightarrow_{SIS} \langle Q[\Omega(r_1, \ldots, r_n)/r];\sigma\rangle$*, if Ω is a primitive operator such that, once evaluated with parameters $r_1, \ldots, r_n$, produces the result r.*

From now on, we shall use the symbols $\rightarrow_{SIS1}$ and $\rightarrow_{SIS2}$ to distinguish between computation steps performed by applying one of the specific "small interpretive" rules. Moreover, when we use the expression *interpretive derivation*, we refer to a sequence of *small interpretive steps* (according to the previous definition) instead of a sequence of *interpretive steps* (regarding Definition 2.1.5). Note that this fact supposes too a slight revision of Definition 2.1.6 which does not affect the essence of the notion of fuzzy computed answer: the repeated application of both kinds of small interpretive steps on a given state only affects to the length of the corresponding derivations, but both ones lead to the same final states (containing the corresponding fuzzy computed answers).

Ejemplo 2.1.10. *Recalling again the a.c.a. obtained in Example 2.1.4, we can reach the final fuzzy computed answer $\langle 0.38; \{X/peter\}\rangle$ (achieved in Example 2.1.7 by means of interpretive steps) by generating now the following interpretive derivation D_2 based on "small interpretive steps" (Definition 2.1.9):*

$$\begin{array}{ll}
\langle \&_{\texttt{P}}(1, \&_{\texttt{P}}(\underline{|_{\texttt{P}}(0.9, 0.5)}, 0.4)); \{X/peter\}\rangle & \rightarrow_{SIS1} \\
\langle \&_{\texttt{P}}(1, \&_{\texttt{P}}(\underline{(0.9 + 0.5)} - (0.9 * 0.5), 0.4)); \{X/peter\}\rangle & \rightarrow_{SIS2} \\
\langle \&_{\texttt{P}}(1, \&_{\texttt{P}}(1.4 - \underline{(0.9 * 0.5)}, 0.4)); \{X/peter\}\rangle & \rightarrow_{SIS2} \\
\langle \&_{\texttt{P}}(1, \&_{\texttt{P}}(\underline{1.4 - 0.45}, 0.4)); \{X/peter\}\rangle & \rightarrow_{SIS2} \\
\langle \&_{\texttt{P}}(1, \underline{\&_{\texttt{P}}(0.95, 0.4)}); \{X/peter\}\rangle & \rightarrow_{SIS1} \\
\langle \&_{\texttt{P}}(1, \underline{0.95 * 0.4}); \{X/peter\}\rangle & \rightarrow_{SIS2} \\
\langle \underline{\&_{\texttt{P}}(1, 0.38)}; \{X/peter\}\rangle & \rightarrow_{SIS1} \\
\langle \underline{1 * 0.38}; \{X/peter\}\rangle & \rightarrow_{SIS2} \\
\langle 0.38; \{X/peter\}\rangle &
\end{array}$$

Going back now to Example 2.1.8, we can rebuild the interpretive phase of Derivation D_1^* in terms of small interpretive steps, thus generating the following interpretive derivation D_2^*. Firstly, by applying a $\rightarrow_{SIS1}$ step on the L-expression $\&_{\texttt{P}}(1, \underline{@_1(0.9, 0.5, 0.4)})$, it becomes $\&_{\texttt{P}}(1, \&_{\texttt{P}}(|_{\texttt{P}}(0.9, 0.5), 0.4))$, and from here, the interpretive derivation evolves exactly in the same way as derivation D_2 we have just done above.

At this moment, it is mandatory to meditate on cost measures regarding derivations D_1, D_1^*, D_2 and D_2^*. First of all, note that the operational cost $\mathcal{O}_c$ of all them coincides, which is quite natural. However, whereas $\mathcal{I}_c(D_1) = 3 > 2 = \mathcal{I}_c(D_1^*)$, we have now that $\mathcal{I}_c(D_2) = 8 < 9 = \mathcal{I}_c(D_2^*)$. This apparent contradiction might confuse us when trying to decide which program rule ($\mathcal{R}_7$ or $\mathcal{R}_7^*$) is "better". The use of Definition 2.1.9 in derivations D_2 and D_2^* is the key point to solve our problem, as we are going to see. In Example 2.1.8 we justified that by simply counting the number of interpretive steps performed in Definition 2.1.5 might produce abnormal results, since the evaluation of connectives with different complexities were (wrongly) measured with the same computational cost. Fortunately, the notion of small interpretive step makes visible in the proper derivation all the connectives and primitive operators appearing in the (possibly recursively nested) definitions of any connective appearing in any derivation state. As we have seen, in D_2 we have expanded in three $\rightarrow_{SIS1}$ steps the definitions of three connectives, i.e. $|_{\texttt{P}}$, and $\&_{\texttt{P}}$ twice, and we have applied five $\rightarrow_{SIS2}$ steps to solve five primitive operators, that is, $+$, $-$, and $*$ (three times). The same computational effort as been performed in D_2^*, but also one more $\rightarrow_{SIS1}$ step was applied to accomplish with the expansion of the extra connective $@_1$. This justifies why $\mathcal{I}_c(D_2) = 8 < 9 = \mathcal{I}_c(D_2^*)$ and contradicts the wrong measures of Example 2.1.8: the interpretive effort developed in derivations D_1 and D_2 (both using the program rule $\mathcal{R}_7$), is slightly lower than the one performed in derivations D_1^* and D_2^* (which used rule $\mathcal{R}_7^*$), and not the contrary.

The accuracy of our new way for measuring and performing interpretive computations seems to be crucial when comparing the execution behaviour of programs obtained by transformation techniques such as the fold/unfold framework we describe in [JMP05, GM08]. In this sense, instead of measuring the absolute cost of derivations performed in a program, we are more interested in the relative gains/lost of efficiency produced on transformed programs. For instance, by applying the so-called "aggregation operation" described in [GM08] we can transform rule $\mathcal{R}_7$ into $\mathcal{R}_7^*$ and, in order to proceed with alternative transformations (fold,unfold, etc.) if the resulting program degenerates w.r.t. the original one (as occurs in this case), we need an appropriate cost measure as the one proposed here to help us for taken decisions. This fact has capital importance for discovering drastic situations which can appear in degenerated transformation sequences such as the generation of highly nested definitions of aggregators. For instance, assume the following sequence of connective definitions: $@_{100}(x_1, x_2) \triangleq @_{99}(x_1, x_2)$, $@_{99}(x_1, x_2) \triangleq @_{98}(x_1, x_2), \dots,$ and finally $@_1(x_1, x_2) \triangleq x_1 * x_2$. When trying to solve two expression of the form $@_{99}(0.9, 0.8)$ and $@_1(0.9, 0.8)$, cost measures based on number of interpretive steps ([JMP06]) and weights of interpretive steps ([JMP07]) would assign 1 unit of interpretive cost to both derivations. Fortunately, our new approach is able to clearly distinguish between both cases, since the number of $\rightarrow_{SIS1}$ steps performed in each one is rather different (100 and 1, respectively).

2.2 The "Fuzzy LOgic Programming Environment for Research"

As shown in the web page `http://dectau.uclm.es/floper/` designed in our research group for freely accessing FLOPER -see also references [MM08, MMPV10b, MMPV10a, MMPV11a, MMPV12a, MMPV12b]- during the last years we have been involved in the development of this tool for aliving MALP programs (which can be easily loaded into the system by means of plain-text files with extension ".fpl"). For example, our previous illustrative program included into file "`P1.fpl`", contains the following rules where, note for instance that the fuzzy connectives for implication, disjunction and conjunction symbols belonging to the *product logic* are respectively referred as "`<prod`", "`|prod`" and "`&prod`":

```
y(peter)                                with 0.4.
y(mary)                                 with 0.8.
h(peter)                                with 0.9.
h(mary)                                 with 0.3.
e(peter)                                with 0.5.
e(mary)                                 with 0.95.
c(X) <prod (h(X) |prod e(X)) &prod y(X)      with 1.
```

Figure 2.2: MALP program $\mathcal{P}_1$ loaded into FLOPER

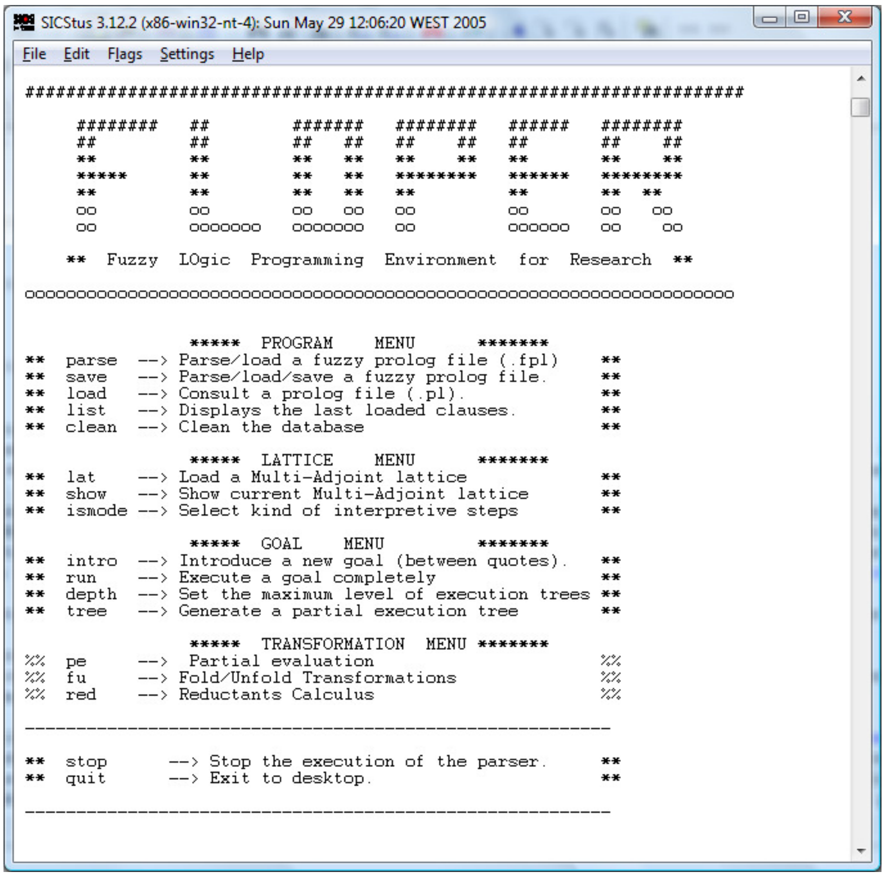
In order to simplify the task of coding fuzzy logic programs, our tool is able to parse MALP rules with 'syntactic sugar" trying to look as conservative extensions of Prolog clauses:

- Since the weight of a rule can be omitted if it coincides with the $\top$ element of the corresponding lattice then, a rule like '`p(X) with 1.`" can be simply expressed as "`p(X).`".

- When the concrete implication symbol connecting the body and head of a given a rule be irrelevant, we can write "`<-`" instead of using a particular label "$<_{logic}$", since FLOPER will choose an arbitrary implication (i.e., the last one found when textually exploring the lattice stored into the system) for this rule. So, "`p(X) <- q(X).`" is a valid rule.

- Something similar occurs (but even without the need of "`-`") for connectives used in the bodies of program rules, thus implying that "`p(X) <- q(X) & r(X).`" is a valid rule for any conjunction operator. At this point, it is important to remark that the last rule in **Figure 2.2** (we assume that the connectives belonging to the *product logic* are the last ones defined into the lattice stored into FLOPER) can be highly simplified to the following shape: "`c(X) <- (h(X) | e(X)) & y(X).`".

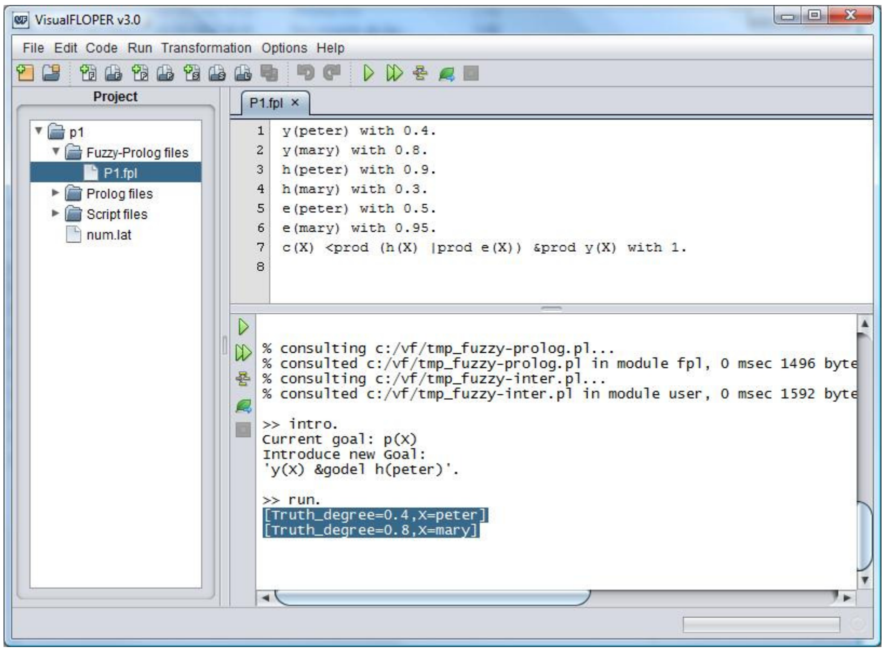
- We can also include Prolog clauses between "`$$`" symbols, for instance "`$p(X):-!, var(X).$`", and moreover, it is possible too to insert pure Prolog code between "`{}`" symbols into the body of a fuzzy logic MALP rule, as occurs with "`p([H|T]) <- {Y is H+1} q(Y) & p(T).`".


SICStus 3.12.2 (x86-win32-nt-4): Sun May 29 12:06:20 WEST 2005
File Edit Flags Settings Help

```
######################################################################

     ########   ##        #######   ########   ######   ########
     ##         ##        ##   ##   ##    ##   ##       ##    ##
     **         **        **   **   **    **   **       **    **
     *****      **        **   **   ********   ******   ********
     **         **        **   **   **         **       **  **
     oo         oo        oo   oo   oo         oo       oo   oo
     oo         ooooooo   ooooooo   oo         oooooo   oo    oo

    **  Fuzzy  LOgic  Programming  Environment  for  Research  **

ooooooooooooooooooooooooooooooooooooooooooooooooooooooooooooooooooooo


                *****  PROGRAM    MENU      *******
**  parse  --> Parse/load a fuzzy prolog file (.fpl)    **
**  save   --> Parse/load/save a fuzzy prolog file.     **
**  load   --> Consult a prolog file (.pl).             **
**  list   --> Displays the last loaded clauses.        **
**  clean  --> Clean the database                       **

                *****  LATTICE    MENU      *******
**  lat    --> Load a Multi-Adjoint lattice             **
**  show   --> Show current Multi-Adjoint lattice       **
**  ismode --> Select kind of interpretive steps        **

                *****  GOAL    MENU         *******
**  intro  --> Introduce a new goal (between quotes).   **
**  run    --> Execute a goal completely                **
**  depth  --> Set the maximum level of execution trees **
**  tree   --> Generate a partial execution tree        **

                *****  TRANSFORMATION  MENU *******
%%  pe     -->  Partial evaluation                      %%
%%  fu     --> Fold/Unfold Transformations              %%
%%  red    --> Reductants Calculus                      %%
---------------------------------------------------------

**  stop      --> Stop the execution of the parser.     **
**  quit      --> Exit to desktop.                      **

---------------------------------------------------------
```

Figure 2.3: Running FLOPER into any standard PROLOG platform

Once the application is loaded inside any standard PROLOG interpreter (like SWI or Sicstus -which is our case, by using v3.12.5-), it shows the menu shown in **Figure 2.3**. The parser has been implemented by using the classical DCG's (*Definite Clause Grammars*) resource of the PROLOG language, since it is a convenient notation for expressing grammar rules. Via the "parse" option, it is possible to load a ".fpl" file for which FLOPER generates two different PROLOG representations of the fuzzy code, as we will describe in sub-sections 2.2.1 and 2.2.2. Such code will cohabit with the set of clauses introduced by the user (together the PROLOG-based definition of the associated lattice that we will describe in sub-section 2.2.3) via the "load" option for consulting pure PROLOG files (".pl"). The system is equipped too with choices



```
y(peter) with 0.4.
y(mary) with 0.8.
h(peter) with 0.9.
h(mary) with 0.3.
e(peter) with 0.5.
e(mary) with 0.95.
c(X) <prod (h(X) |prod e(X)) &prod y(X) with 1.
```

```
% consulting c:/vf/tmp_fuzzy-prolog.pl...
% consulted c:/vf/tmp_fuzzy-prolog.pl in module fpl, 0 msec 1496 byte
% consulting c:/vf/tmp_fuzzy-inter.pl...
% consulted c:/vf/tmp_fuzzy-inter.pl in module user, 0 msec 1592 byte

>> intro.
Current goal: p(X)
Introduce new Goal:
'y(X) &godel h(peter)'.

>> run.
[Truth_degree=0.4,X=peter]
[Truth_degree=0.8,X=mary]
```

Figure 2.4: The graphical interface of FLOPER

for saving and listing such rules as well as the "clean" option for removing all clauses from the FLOPER database.

The remaining menus are useful for executing goals and displaying evaluation trees, as well as for managing multi-adjoint lattices, as we are going to explain in what follows helped by the graphical interface recently developed for FLOPERand shown in **Figure 2.4**, which allows the comfortable use of "projects" in order to manage files with the following different purposes:

- several ".fpl" files can contain the set of MALP rules implementing a single MALP program,
- the set of clauses modeling its (unique) associated multi-adjoint lattice must be included into a file with obvious extension ".pl",
- additional PROLOG code (for representing linguistic modifiers/variables, etc) can be also attached in different ".pl" files and
- several "script" files could be useful for executing in one go more than one of

the set of commands we are going to explain in this section, as illustrated in the following example:

```
ord: intro.
arg: c(X).
ord: ismode.
arg: s.
ord: tree.
ord: leaves.
ord: run.
```

2.2.1 Running Programs

In order to fully execute a goal, FLOPER employs the high-level representation of the MALP program compiled via the "parse" option. The key point of this code is to extend each atom of the program with an extra argument, called *truth variable*, of the form `TVi`, which is intended to contain the truth degree obtained after the subsequent evaluation of the atom. In the case of a fact, the extra argument obviously contains its weight. For instance, "`p(X) with 0.5`" is simply translated into the PROLOG fact "`p(X, 0.5)`".

Fuzzy connectives are represented as predicates defined in the lattice associated to the program. For instance, the role of "`&godel`" is played by the PROLOG predicate "`and_godel`". Since the fuzzy connective "`&godel`" is a binary operation, then its associated predicate "`and_godel`" has arity three: two parameters plus the result, returned in the third argument `TV`.

When compiling MALP rules, the last atom called in the body of the translated clause is the adjoint conjunction, which is intended to combine the truth degree of the body and the weight of the rule, in order to propagate the final truth degree to the head. For instance, given a MALP rule like "`p(X) <prod q(X,Y) &godel r(Y) with 0.8`", the resulting translated PROLOG clause would look like "`p(X, _TV0) :- q(X, Y, _TV1), r(Y, _TV2), and_godel(_TV1, _TV2, _TV3), and_prod(0.8, _TV3, _TV0)`".

In order to execute a fuzzy program stored in a ".fpl" file , FLOPER needs firstly to load and compile it by using the "parse" option. Internally, while FLOPER analyzes the content of the file (following the DCG specification of the MALP syntax) it also generates two different PROLOG representations of the fuzzy code: the high level representation coincides with a set of executable PROLOG clauses as we have just described, while the low level representation allows FLOPER to draw execution trees as we will see in the next subsection.

Here we have an example of using the "parse" option, where note that FLOPER lists the content of any previously loaded ".pl" file (none in our case), the parsed MALP program and the generated Prolog code.

```
>> parse.
File to parse: ‘P1.fpl’.

No loaded files.

ORIGINAL FUZZY-PROLOG CODE:
y(peter) with 0.4.
y(mary) with 0.8.
h(peter) with 0.9.
h(mary) with 0.3.
e(peter) with 0.5.
e(mary) with 0.95.
c(X) <prod (h(X) |prod e(X)) &prod y(X) with 1.

GENERATED PROLOG CODE:
y(peter,0.4).
y(mary,0.8).
h(peter,0.9).
h(mary,0.3).
e(peter,0.5).
e(mary,0.95).
c(X,TV0):-h(X,_TV1),e(X,_TV2),lat:or_prod(_TV1,_TV2,_TV3),
          y(X,_TV4),lat:and_prod(_TV3,_TV4,_TV5),
          lat:and_prod(1,_TV5,TV0).
```

While parsing, FLOPER creates a file *tmp_fuzzy-prolog.pl* to allocate the generated Prolog code, in order to allow the possibility of saving it afterwards into a new file by using the "save" option.

One important feature is that translated connectives (like "`lat:and_prod/3`") are prefixed by "`lat:`", which means that the Prolog interpreter will search their definitions in the "`lat`" module, a different name-space designed to avoid name collisions. The resulting PROLOG code can be executed in any PROLOG engine, with the only requirement that the associated lattice must be loaded in the corresponding module. This can be easily achieved with the following two goals:

```
?- lat:consult(lattice.pl).
?- consult(program.fpl).
```

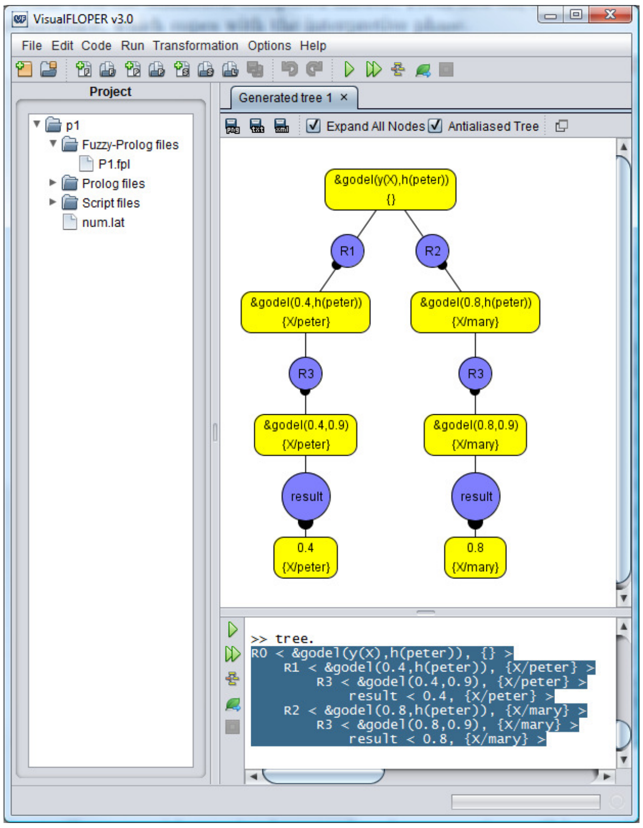
Moreover, goals are introduced in FLOPER by choosing the "intro" option and they suffer a very similar translation process to program rules. So, it is easy to see that a fuzzy goal like "`y(X) &godel h(peter)`", is translated into the pure PROLOG goal "`y(X, _TV1),h(peter, _TV2), and_godel(_TV1, _TV2, Truth_degree)`" (note that the last truth degree variable is not anonymous now) for which the PROLOG interpreter returns the two desired fuzzy computed answers after selecting the "run" option (see **Figure 2.4**) :

```
[Truth_degree=0.4, X=peter]
[Truth_degree=0.8, X=mary]
```

2.2.2 Execution Trees

Apart from the compilation method to PROLOG code commented before, we have conceived a new low-level representation for the fuzzy code which is useful for building execution trees with any level of depth and offering too debugging (tracing) capabilities. For instance, after parsing the last rule of our program, we obtain the following expression (which is "asserted" into the database of the interpreter as a PROLOG fact, but it is never executed directly, in contrast with the previous PROLOG-based, high-level representation of the fuzzy code) whose components have obvious meanings:

```
   rule(7,
        head(atom(pred(c,1),[var('X')])),
        impl(prod),
        body(and(prod,2,[or(prod,2,[atom(pred(h,1),[var('X')]),
                                    atom(pred(e,1),[var('X')])]),
                         atom(pred(y,1),[var('X')])])),
        td(1)).
```



```
>> tree.
R0 < &godel(y(X),h(peter)), {} >
    R1 < &godel(0.4,h(peter)), {X/peter} >
        R3 < &godel(0.4,0.9), {X/peter} >
            result < 0.4, {X/peter} >
    R2 < &godel(0.8,h(peter)), {X/mary} >
        R3 < &godel(0.8,0.9), {X/mary} >
            result < 0.8, {X/mary} >
```

Figure 2.5: FLOPER drawing an execution tree

FLOPER is equipped with three options related with *tracing* tasks. Option “tree” draws the execution tree (which collects a different derivation from the root to each leaf) of a goal w.r.t. a program. Option “depth” fixes the maximum length allowed for their branches (initially 3). And, finally, option “ismode” fixes the detail level of the interpretive phase associated to each derivation in the tree.

So, let us consider again the MALP program of our running example. For goal “`y(X) &godel h(peter)`”, FLOPER displays the tree showed in Figure 2.5. In this screen-shot, we find two representations of the same tree: the middle-up window shows the proper graphic which is easy to understand and manipulate (by moving nodes, showing only skeletons of trees, etc.), while the marked text in the middle-down window represents the same tree in plain text. In this last case, each line

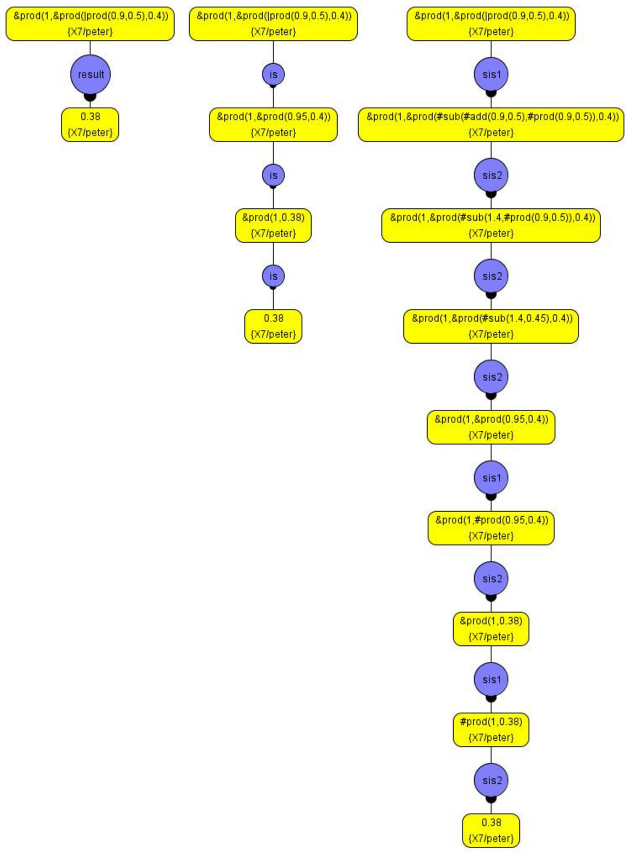


Figure 2.6: Comparing different modes of performing the interpretive phase

contains a state (composed by the corresponding goal and substitution) preceded by the number of the program rule used by the admissible step leading to it (root nodes and nodes obtained via $\rightarrow_{AS3}$ are always labeled with the virtual, non existing rule `R0`), and nodes belonging to the same branch appear in different lines appropriately indented to help the readability of the tree (which only contain two different branches in our case). Generated trees can be saved in ".jpg", ".txt" and "xml" formats.

Moreover, FLOPER allows the user to choose the level of information given by the interpretive phase in the execution tree through option "ismode". We can choose among the following three modes:

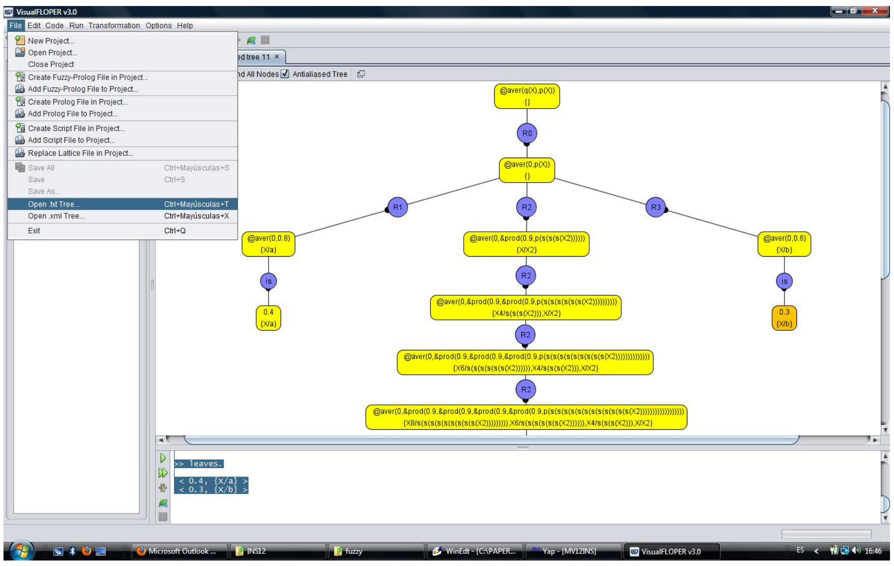
- `large`: This mode omits the entire interpretive phase, offering only the final leaf (if exists) of each branch.
- `medium`: Performs classical interpretive steps according Definition 2.1.5, in order to evaluate each expression of the goal till finding the final solution.
- `small`: This mode applies our improved notion of *small interpretive step* provided in Definition 2.1.9, which is very useful for visualizing the more or less complexity (distinguishing between connective calls and primitive operators evaluation) of connectives exploited during the interpretive phase. **Figure 2.6** shows that the shorter the step is, the larger tree is generated.

To finish this block, we are going to illustrate now with a very simple example that the new options are crucial when the "run" choice fails: remember that this last option is based on the generation of pure logic SLD-derivations which might fall in loop or directly fail when finding non defined atoms, in contrast with the traces (based on finite, non- failed, admissible/interpretive derivations) that the "tree" option displays. So, consider the following MALP program:

```
p(a)                          with 0.8.
p(X) <prod p(s(s(s(X))))      with 0.9.
p(b)                          with 0.6.
```

where the first and last rules indicate that a goal like "`p(X)`" admits two solutions, but the second rule would be responsible of introducing an infinite branch between the leaves associated to both fuzzy computed answers in the corresponding execution tree. Moreover, if we plan to run a more complex goal like "`q(X) @aver p(X)`" (where, obviously, the used connective refers to the *average* aggregator), we find a second problem related now with an undefined atom. In contrast with Prolog, in our fuzzy setting the evaluation of "q(X)" doesn't fail, since FLOPER proceeds with an $\rightarrow_{AS3}$ step according Definition 2.1.2, and hence, it is possible to find the two desired fuzzy computed answers for that goal, as shown in the execution tree of **Figure 2.7**. By choosing option "leaves", the system displays the content of the two fully evaluated leaves "`<0.4,{X/a}>`" and "`<0.3,{X/b}>`", as desired.

As we have seen, the generation of traces based on execution trees, contribute to increase the power of FLOPER by providing debugging capabilities which allow us to discover solutions for queries even when the pure Prolog compilation-execution process becomes insufficient.


VisualFLOPER v3.0
File Edit Code Run Transformation Options Help
New Project...
Open Project...
Close Project
Create Fuzzy-Prolog File in Project...
Add Fuzzy-Prolog File to Project...
Create Prolog File in Project...
Add Prolog File to Project...
Create Script File in Project...
Add Script File to Project...
Replace Lattice File in Project...
Save All Ctrl+Mayúsculas+S
Save Ctrl+S
Save As...
Open .txt Tree... Ctrl+Mayúsculas+T
Open .xml Tree... Ctrl+Mayúsculas+X
Exit Ctrl+Q
Antialiased Tree
@aver(q(X),p(X)) {}
R0
@aver(0,p(X)) {}
R1
R2
R3
@aver(0,0.8) {X/a}
@aver(0,&prod(0.9,p(s(s(s(X2)))))) {X/X2}
@aver(0,0.6) {X/b}
is
0.4 {X/a}
@aver(0,&prod(0.9,&prod(0.9,p(s(s(s(s(s(s(X2)))))))))) {X4/s(s(s(X2))),X/X2}
0.3 {X/b}
@aver(0,&prod(0.9,&prod(0.9,&prod(0.9,p(s(s(s(s(s(s(s(s(s(X2))))))))))))))) {X6/s(s(s(s(s(s(X2)))))),X4/s(s(s(X2))),X/X2}
@aver(0,&prod(0.9,&prod(0.9,&prod(0.9,&prod(0.9,p(s(s(s(s(s(s(s(s(s(s(s(s(X2)))))))))))))))))))) {X8/s(s(s(s(s(s(s(s(s(X2))))))))),X6/s(s(s(s(s(s(X2)))))),X4/s(s(s(X2))),X/X2}
Microsoft Outlook ...
INS12
fuzzy
WinEdt - [C:\PAPER...
Yap - [MV12INS]
VisualFLOPER v3.0
16:46

```
>> leaves.

 < 0.4, {X/a} >
 < 0.3, {X/b} >
```

Figure 2.7: Tree with an infinite branch and a $\rightarrow_{AS3}$ step

2.2.3 Managing Lattices

We have conceived a very easy way to model lattices of truth degrees for being included into the FLOPER tool. All relevant components of each lattice can be encapsulated inside a PROLOG file which must necessarily contain the definitions of a minimal set of predicates defining the set of valid elements (including special mentions to the "`top`" and "`bottom`" ones), the full or partial ordering established among them, as well as the repertoire of fuzzy connectives which can be used for their subsequent manipulation. In order to simplify our explanation, assume that file "bool.pl" refers to the simplest notion of (a binary) adjoint lattice, thus implementing the following set of predicates:

- `member/1` which is satisfied when being called with a parameter representing a valid truth degree. In the case of finite lattices, it is also recommended to implement `members/1` which returns in one go a list containing the whole set of truth degrees. For instance, in the Boolean case, both predicates can be simply modeled by the PROLOG facts: `member(0).`, `member(1).` and `members([0,1]).`
- `bot/1` and `top/1` obviously answer with the top and bottom element of the lattice, respectively. Both are implemented into "bool.pl" as `bot(0).` and

`top(1).`

- `leq/2` models the ordering relation among all the possible pairs of truth degrees, and obviously it is only satisfied when it is invoked with two elements verifying that the first parameter is equal or smaller than the second one. So, in our example it suffices with including into “bool.pl” the facts: `leq(0,X).` and `leq(X,1).`

- Finally, given some fuzzy connectives of the form $\&_{label_1}$ (conjunction), $|_{label_2}$ (disjunction) or $@_{label_3}$ (aggregation) with arities n_1, n_2 and n_3 respectively, we must provide clauses defining the *connective predicates* “`and_`$label_1$`/(`n_1`+1)`”, “`or_`$label_2$`/(`n_2`+1)`” and “`agr_`$label_3$`/(`n_3`+1)`”, where the extra argument of each predicate is intended to contain the result achieved after the evaluation of the proper connective. For instance, in the Boolean case, the following two facts model in a very easy way the behaviour of the classical conjunction operation: `and_bool(0,_,0).  and_bool(1,X,X).`

The reader can easily check that the use of lattice “bool.pl” when working with MALP programs whose rules have the form “$A \quad \leftarrow_{bool} \quad \&_{bool}(B_1, \ldots, B_n) \ \mathit{with} \ 1$”, being A and B_i typical atoms[1], successfully mimics the behaviour of classical PROLOG programs where clauses accomplish with the shape “$A \quad : - \quad B_1, \ldots, B_n$”. As a novelty in the fuzzy setting, when evaluating goals according to the procedural semantics described in Section 2.1.1, each output will contain the corresponding substitution (i.e., the *crisp* notion of computed answer obtained by means of classical SLD-resolution in PROLOG) together with the maximum truth degree 1.

On the other hand and following the PROLOG style regulated by the previous guidelines, in file “num.pl” we have included the clauses shown in **Figure 2.8**. Here, we have modeled the more flexible lattice (that we will mainly use in our examples, beyond the boolean case) which enables the possibility of working with truth degrees in the infinite space (note that this condition disables the implementation of the consulting predicate “`members/1`”) of real numbers between 0 and 1, allowing too the possibility of using conjunction and disjunction operators recasted from the three typical fuzzy logics described before (i.e., the *Łukasiewicz*, *Gödel* and *product* logics), as well as a useful description for the hybrid aggregator *average*.

[1]Here we also assume that several versions of the classical conjunction operation have been implemented with different arities.

```
member(X) :- number(X),0=<X,X=<1.  %% no members/1 (infinite lattice)

bot(0).              top(1).             leq(X,Y) :- X=<Y.

and_luka(X,Y,Z) :- pri_add(X,Y,U1),pri_sub(U1,1,U2),pri_max(0,U2,Z).
and_godel(X,Y,Z):- pri_min(X,Y,Z).
and_prod(X,Y,Z) :- pri_prod(X,Y,Z).

or_luka(X,Y,Z)  :- pri_add(X,Y,U1),pri_min(U1,1,Z).
or_godel(X,Y,Z) :- pri_max(X,Y,Z).
or_prod(X,Y,Z)  :- pri_prod(X,Y,U1),pri_add(X,Y,U2),pri_sub(U2,U1,Z).

agr_aver(X,Y,Z) :- pri_add(X,Y,U),pri_div(U,2,Z).

pri_add(X,Y,Z)  :- Z is X+Y.    pri_min(X,Y,Z) :- (X=<Y,Z=X;X>Y,Z=Y).
pri_sub(X,Y,Z)  :- Z is X-Y.    pri_max(X,Y,Z) :- (X=<Y,Z=Y;X>Y,Z=X).
pri_prod(X,Y,Z) :- Z is X * Y.  pri_div(X,Y,Z) :- Z is X/Y.
```

Figure 2.8: Multi-adjoint lattice modeling truth degrees in the real interval [0,1] ("num.pl")

Note also that we have included definitions for auxiliary predicates, whose names always begin with prefix "`pri_`". All of them are intended to describe primitive/arithmetic operators (in our case $+$, $-$, $*$, $/$, min and max) in a Prolog style, for being appropriately called from the bodies of clauses defining predicates with higher levels of expressiveness (this is the case for instance, of the three kinds of fuzzy connectives we are considering: conjunctions, disjunctions and agreggations).

Since till now we have considered two classical, fully ordered lattices (with a finite and infinite number of elements, collected in files "bool.pl" and "num.pl", respectively), we wish now to introduce a different case coping with a very simple lattice where not always any pair of truth degrees are comparable. So, consider the partially ordered multi-adjoint lattice in **Figure 2.9** for which the conjunction and implication connectives based on the *Gödel* logic described in Section 2.1.1 conform an adjoint pair but with the particularity now that, in the general case, the *Gödel*'s conjunction must be expressed as $\&_{\text{G}}(x,y) \triangleq inf(x,y)$, where it is important to note that we must replace the use of "*min*" by "*inf*" in the connective definition. To this end, observe in the Prolog code accompanying **Figure 2.9** that we have introduced five clauses defining the new primitive operator "`pri_inf/3`" which is intended to return the *infimum* of two elements. Related with this fact, we must point out the following aspects:

$$\begin{tikzcd}
 & \top & \\
\alpha \arrow[ur] & & \beta \arrow[ul] \\
 & \bot \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

```
member(bottom).   member(alpha).
member(beta).      member(top).
members([bottom,alpha,beta,top]).

leq(bottom,X). leq(alpha,alpha).
leq(beta,beta).  leq(beta,top).
leq(alpha,top).  leq(X,top).

and_godel(X,Y,Z) :- pri_inf(X,Y,Z).

pri_inf(bottom,X,bottom):-!.
pri_inf(alpha,X,alpha):-leq(alpha,X),!.
pri_inf(beta,X,beta):-leq(beta,X),!.
pri_inf(top,X,X):-!.
pri_inf(X,Y,bottom).
```

Figure 2.9: Partially ordered lattice with four elements

- Note that since truth degrees α and β -or their corresponding representations as Prolog terms "`alpha`" and "`beta`" used for instance in the definition(s) of "`members(s)/1`"- are not comparable, any call to "`leq(alpha,beta)`" or "`leq(beta,alpha)`" will always fail.

- However, goals "`pri_inf(alpha,beta,X)`" and "`pri_inf(beta,alpha,X)`", instead of failing, successfully produces the desired result "`X=bottom`".

- Note anyway that the implementation of the "`pri_inf/1`" predicate is mandatory for coding the general definition of "`and_godel/3`".

As a final example, we can also define the so called *Borel algebra* based on the union of intervals (where for instance $\mathcal{B}([0,1])$ is the union of intervals between 0 and 1 [VGM02, MCS11]) as a Prolog program for being used into FLOPER as follows:

```
member([i(X,Y)]) :-                      number(X),number(Y),X=<Y.
member([i(X,Y),i(Z,T)|U]) :-             number(X),number(Y),number(Z),
                                         X=<Y, Y < Z,member([i(Z,T)|U]).

bot([i(0,0)]).                           top([i(1,1)]).

leq([X1,X2|X],Y) :-                      existsGreater(X1,Y),leq([X2|X],Y).
leq([X1],Y) :-                           existsGreater(X1,Y).

existsGreater(i(Xb,Xt),[i(Yb,Yt)|Y]) :-  Yb=<Xb, Xt=<Yt,!.
existsGreater(X,[_|Y]) :-                existsGreater(X,Y).
```

- A member of this algebra is a list of pairs representing disjoined intervals.

- The top element is the point 1 (interval from 1 to 1), and the bottom one is the point 0 (interval from 0 to 0).

- A union of intervals, U, is less or equal than other union of intervals U' if for each $I \in U$, there exists another interval $I' \in U'$, such that $I \subseteq I'$.

2.2.4 Linguistic modifiers and linguistic variables

Another interesting feature of FLOPER is its ability for managing linguistic modifiers and linguistic variables in a very easy way. A linguistic modifier can be seen as an aggregator such that, when applied to a fuzzy expression, it alters its final truth degree. Some examples of well known modifiers are *"very"* and *"roughly"*, where the first one tends to return a lower truth degree, and the second one, a higher truth degree. Since the concept of modifier is dependent of a concrete lattice, its definition should appear inside the PROLOG file containing the proper multi-adjoint lattice loaded into FLOPER.For instance, in the typical lattice $\langle[0,1],\leq\rangle$ used in previous examples, we could define some modifiers by means of the following usual formulae:

modifier	formula	implementation
extremely	$extremely(x) = x^4$	$\mathrm{agr\_extremely(X, TV0) :-TV0\ is\ X * X * X * X.}$
very	$very(x) = x^2$	$\mathrm{agr\_very(X, TV0) :-TV0\ is\ X * X.}$
moreorless	$moreless(x) = x^{1/2}$	$\mathrm{agr\_moreless(X, TV0) :-TV0\ is\ sqrt(X).}$
roughly	$roughly(x) = x^{1/4}$	$\mathrm{agr\_roughly(X, TV0) :-TV0\ is\ sqrt(sqrt(X)).}$

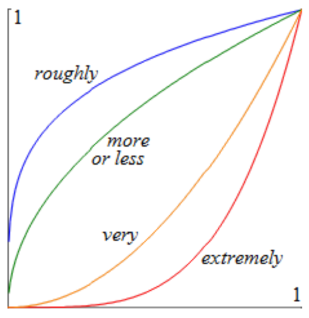


Figure 2.10: Linguistic modifiers

With these modifiers, now we could update the program of **Figure 2.1** to increase the difficulty for obtaining a credit in crisis times. The only rule to be modified is the seventh one, whose new shape could look like:

$$R_7^*: \ c(X) \leftarrow_{\texttt{P}} @very((h(X) \mid_P e(X)) \ \&_P \ y(X)) \ with \ 1$$

Linguistic modifiers increase the expresiveness of the fuzzy language and as said before, they have to be defined as part of the lattice associated to the MALP program in the current project.

On the other hand, linguistic variables allow us to represent linguistic symbols defined by means of fuzzy sets. A linguistic variable is characterized by a tuple $\langle x, T, U, G, M\rangle$, where x is the name of the variable, T the set of linguistic symbols or terms of x, U the universe where x is defined, G represents a syntactic rule to generate linguistic terms, and M is a semantic rule for assigning to each linguistic symbol t its fuzzy set $M(t)$. For instance, we can represent the linguistic variable *distance*, with values $\{near, \ far\_away\}$ defined over the universe $[0, +\infty)$ in kilometers, as shown in **Figure 2.11**. We can implement this variable into a Prolog file defining a fuzzy predicate for each symbol. The arguments are an input variable (`D`) giving the crisp distance, and an output variable (`TV`) returning the degree of membership of the input variable to the fuzzy set. Then, it is possible to load this Prolog program into the current project of FLOPER, in order to use such definitions in the corresponding MALP program.

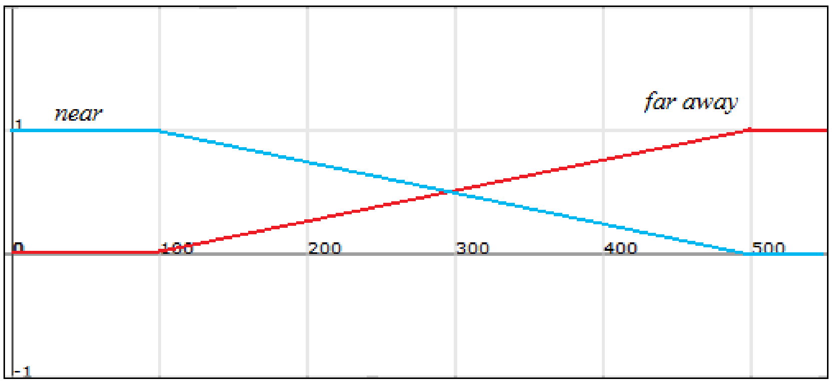


Figure 2.11: Linguistic variable “distance” with terms “near” and “far away”

```
near(D,1) :- D=<100.
near(D,TV):- 100<X, X=<500, TV is (500-X)/400.
near(D,0) :- 500<D.
```

```
far_away(D,0) :- D=<100.
far_away(D,TV):- 100<X, X=<500, TV is (X-100)/400.
far_away(D,1) :- 500<D.
```

In the following example, we design a touristic application to compute the best destination for vacations. The database includes some cities and relevant information relating them (kind of weather, good sights, distance from our hometown). It is clear that some of this information has a very fuzzy taste, so the choice of a fuzzy language is desirable. We can *fuzzify* the crisp distance using the linguistic variable defined above as follows.

```
nice_weather(madrid) with 0.8.
nice_weather(istanbul) with 0.7.
nice_weather(moscow) with 0.2.
nice_weather(sydney) with 0.5.

many_sights(madrid) with 0.6.
many_sights(istanbul) with 0.7.
many_sights(moscow) with 0.2.
many_sights(sydney) with 0.6.

crisp_distance(madrid, 250).
crisp_distance(istanbul, 3700).
crisp_distance(moscow, 4200).
crisp_distance(sydney, 18000).

good_destination(X) <- @roughly(crisp_distance(X,D) & near(D))
                       @aver
                       @very(nice_weather(X) & many_sights(X)).
```

Note that the use too of linguistic modifiers in the last MALP rule means that we give little importance to the distance since we are more interested on weather and sights. Now, for goal `good_destination(X)`, the execution of the previous program into FLOPER returns the following results (meaning that the best destination, according to our specification, is Madrid, followed by far by Istanbul and Sydney, while Moscow scores 0):

```
[Truth_degree=0.0,X=moscow]
[Truth_degree=0.005000000000000009,X=sydney]
[Truth_degree=0.07999999999999996,X=istanbul]
[Truth_degree=0.5245698525097306,X=madrid]
```

Note that, with the current definition of fuzzy predicates 'near' and 'far_away', no matter how we reduce the importance of distance since, if it is over 500 kilometers, it will always be near with 0 truth degree, and if it is under 100 kilometers, it will be far with 0 truth degree. If our perception of distance changes, these notions will become useless. To fix that, we can define them with non-linear rules like the following ones, which show that FLOPER is flexible enough to deal with fuzzy predicates defined (in Prolog) by means of non-linear arithmetic expressions:

```
near(D,TV) :- TV is 250/(D+250).
far(D,TV)  :- TV is 1 - 250/(D+250).
```

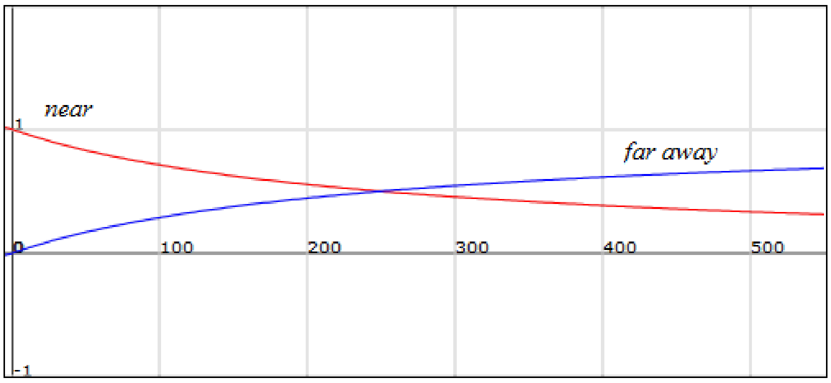


Figure 2.12: Linguistic variable "distance" with flexible versions of "near" and "far away"

2.3 Extending Lattices and Declarative Traces

This section presents different methods to gain expressiveness when designing a MALP program by manipulating its associated lattice. The main technique consists on agglutinating several lattices in order to obtain the Cartesian product of them, which is also a multi-adjoint lattice [MMPV12a]. Once this is done, a new functionality emerges for obtaining declarative traces (when evaluating goals) at a very low cost.

Teorema 2.3.1. *If $L_1, \dots, L_n$ are a finite number of multi-adjoint lattices, then its Cartesian product $L = L_1 \times \cdots \times L_n$ is also a multi-adjoint lattice.*

In order to simplify our explanation, but without lost of generality, we only consider two multi-adjoint lattices $(L_1, \leq_1, \&_1, \leftarrow_1)$ and $(L_2, \leq_2, \&_2, \leftarrow_2)$, each one equipped with just a single adjoint pair. Then, $L = L_1 \times L_2$ has lattice structure with an ordering relation induced in the product from $(L_1, \leq_1)$ and $(L_2, \leq_2)$ as follows:

$$(x_1, y_1) \leq (x_2, y_2) \Leftrightarrow x_1 \leq_1 x_2, y_1 \leq_2 y_2$$

Moreover, being $\top_1 = sup(L_1)$, $\bot_1 = inf(L_1)$, $\top_2 = sup(L_2)$ and $\bot_2 = inf(L_2)$, we have that $(\top_1, \top_2) = sup(L)$ and $(\bot_1, \bot_2) = inf(L)$ which implies that the Cartesian product L is a bounded lattice if both L_1 and L_2 are also bounded lattices. Analogously, $L_1 \times L_2$ is a complete lattice if L_1 and L_2 verify too the same property. Finally, from the adjoint pairs $(\&_1, \leftarrow_1)$ and $(\&_2, \leftarrow_2)$ in L_1 and L_2, respectively, it is possible to define the following connectives in L:

$$\begin{aligned} (x_1, y_1)\&(x_2, y_2) \quad &\triangleq (x_1 \&_1 x_2, y_1 \&_2 y_2) \\ (x_1, y_1) \leftarrow (x_2, y_2) \quad &\triangleq (x_1 \leftarrow_1 x_2, y_1 \leftarrow_2 y_2) \end{aligned}$$

for which it is easy to justify that they conform an adjoint pair in $L_1 \times L_2$ (thus satisfying, in particular, the adjoint property). In a similar way, it is also possible to define new connectives (conjunctions, disjunctions and aggregators) in the Cartesian product $L_1 \times L_2$ from the corresponding pairs of operators defined in L_1 and L_2.

FLOPER is able to deal with Cartesian products of multi-adjoint lattices by acting on the "`member/1`" predicate, which accept as argument any object which belong to a lattice. Some examples of simple "`member/1`" definitions are:

- `member(X) :- number(X), X=<1, X>=0`: [0,1] interval
- `member([X|L])`: lists
- `member(info(X,Y))`: pairs

The very intuitive way to obtain Cartesian product of lattices is to use PROLOG functions with arity 2 (indeed, the cartesian product of n lattices can be represented using functions with arity n). In order to implement a cartesian product of lattices in a PROLOG file, we define "`member/1`" predicate whose parameter is a term headed with a function symbol, for instance: `member(f(X,Y)) :- check1(X), check2(Y).` Of course, predicates "`leq/2`", "`top/1`", "`bot/1`" and connectives have to be defined following the same criterium in order to implement the concrete Cartesian product of lattices.

Before showing an example of Cartesian product modeled in FLOPER, let us consider again the so called *domain of weight values* $\mathcal{W}$ used in the QLP (*Qualified*

Logic Programming framework of [RD08, CRR10, RR08a], whose elements are intended to represent *proof costs*, measured as the weighted depth of proof trees. As explained in Section 2.1, $\mathcal{W}$ can be seen as lattice $(\mathbb{N} \cup \{\infty\}, \geq)$, where $\geq$ is the reverse of the usual numerical ordering (with $\infty \geq d$ for any $d \in \mathbb{N}$) and thus, the bottom elements is ∞ and the top element is 0 (and not vice versa). Note that in this lattice the arithmetic operation "+" plays the role of a *conjunction* (it is easy to prove that in this setting such definition of & verifies the properties required by t-norms [NW06]). Moreover, we can obtain the residual implication of the "+" t-norm, defined as $y \leftarrow z \triangleq sup\{t \in \mathcal{W} : t + z \leq y\}$, which in this particular case acquires the following shape:

$$y \leftarrow z \triangleq \begin{cases} y - z, & if \quad z \geq y \\ 0, & if \quad y > z \end{cases}$$

So, the reader can easily check that $(+, \leftarrow)$ conforms an adjoint pair in $(\mathbb{N}\cup\{\infty\}, \geq)$, thus accomplishing with Definition 2.1.1 in Section 2.1, which implies that $\mathcal{W}$ is in fact a multi-adjoint lattice. A valid implementation of $\mathcal{W}$ lattice presents the following definitions:

```
member(X) :- number(X).    member(infty).
leq(infty,X).              leq(X,Y):-X>=Y.
top(0).                    bot(infty).
and_plus(X,infty,infty).   and_plus(infty,X,infty).   and_plus(X,Y,Z):-Z is X+Y.
```

In order to associate the lattice $\mathcal{W}$ to our program $\mathcal{P}_1$, we have to change the weights of each rule by valid truth degrees belonging to the new lattice. For instance, rule "$y(peter)$ *with* 0.4" would be rewritten as "$y(peter)$ *with* 1" (the underlying idea is that "the use of each program rule in a derivation implies the application of one admissible step"), resulting in a new program, $\mathcal{P}_{\mathcal{W}}$. By using the "`lat`" option of FLOPER, we can built a project which associates lattice $\mathcal{W}$ to program $\mathcal{P}_{\mathcal{W}}$ and now, for goal "$c(X)$", we can generate an admissible derivation similar to the one seen in **Figure 2.1**, but ending now with $\langle \&_{\mathtt{P}}(info(1, \&_{\mathtt{P}}(|_{\mathtt{P}}(1,1),1)), \{X/peter\}\rangle$. Since: $\&_{\mathtt{P}}(1, \&_{\mathtt{P}}(|_{\mathtt{P}}(1,1),1)) = +(1, +(+(1,1),1))) = 4$, the final fuzzy computed answer or f.c.a. $\langle 4; \{X/peter\}\rangle$ indicates that goal "$c(X)$" holds when X is *peter*, as proved after applying 4 admissible steps.

On the other hand, since we have said that $\mathcal{V}$ is the multi-adjoint lattice $([0,1], \leq)$ based on real numbers in the unit interval used along this thesis (which is equipped with three adjoint pairs modeling implication and conjunction symbols collected from the *Łukasiewicz logic*, *Gödel intuitionistic logic* and *product logic*) then, in the Cartesian product $\mathcal{V} \times \mathcal{W}$ we will find the top and bottom elements $(1, 0)$ and $(0, \infty)$, respectively, as well as the following definitions of conjunction operations (among other connectives) whose names are mirroring to *extensions* of the *product logic* and *Łukasiewicz logic*:

$$(v_1, w_1)\ \&_{\mathtt{P+}}\ (v_2, w_2)\ \triangleq\ (v_1 * v_2, w_1 + w_2)$$

$$(v_1, w_1)\ \&_{\mathtt{L+}}\ (u_2, w_2)\ \triangleq\ (\max(0, v_1 + v_2 - 1), w_1 + w_2)$$

Moreover, we can also conceive a more powerful lattice expressed as the *Cartesian product* of $\mathcal{V}$ (see **Figure 2.8**) and $\mathcal{W}$. Now, each element includes two components, coping with truth degrees and cost measures. In order to be loaded into FLOPER, we must define in Prolog the new lattice, whose elements could be expressed as data terms of the form "`info(Fuzzy_Truth_Degree, Cost_Number_Steps)`". Some of the required predicates are:

```
member(info(X,W)):-number(X), 0=<X,X=<1,(W=infty,!; number(W),1=<W).

leq(info(X1,W1),info(X2,W2)):-X1 =< X2, (W1=infty,!; number(W2), W2 =< W1).

bot(info(0,infty)).                          top(info(1,1)).

and_prod(info(X,W1),info(Y,W2),info(Z,W3)) :- pri_prod(X,Y,Z),pri_add(W1,W2,W3).

pri_add(infty,_,infty).                      pri_add(_,infty,infty).
pri_add(X,Y,Z) :- number(X), number(Y), Z is X+Y.
```

Finally, if the weights assigned to the rules of our example are "`info(0.4,1)`" for $\mathcal{R}_1$, "`info(0.8,1)`" for $\mathcal{R}_2$, "`info(0.9,1)`" for $\mathcal{R}_3$ and so on, then we would obtain the desired f.c.a. $\langle \mathtt{info}(0.38, 4); \{X/\mathit{peter}\}\rangle$ for goal "$c(X)$", with the obvious meaning that we need 4 admissible steps to prove that the query is true at a 38% degree when X is "peter".

One step beyond, we will also see that if instead of the number of computational steps, we are interested in knowing more detailed data about the set of program rules and connective definitions evaluated for obtaining each solution then, instead of $\mathcal{W}$ it will be mandatory to use a new lattice $\mathcal{S}$ based on strings or labels (i.e., sequences of characters) for generating the Cartesian product $\mathcal{V} \times \mathcal{S}$. In [MMPV12b] we show not only that $\mathcal{S}$ is a complete multi-adjoint lattice, but also that the concatenation of strings, usually called "*append*" in many programming languages, plays the role of a conjunction connective in such lattice.

In order to be loaded into FLOPER, we need to define again the new lattice as a Prolog program, whose elements will be terms of the form "`info(Fuzzy_Truth_Degree, Label)`" as shown in **Figure 2.13** (we only list some representative clauses).

```
member(info(X,Y)):-number(X),0=<X,X=<1,atom(Y).   top(info(1,'')).

and_prod(info(X1,X2),info(Y1,Y2),info(Z1,Z2)) :-
          pri_prod(X1,Y1,Z1,DatPROD),pri_app(X2,Y2,Dat1),
          pri_app(Dat1,'&PROD.',Dat2),pri_app(Dat2,DatPROD,Z2).

pri_prod(X,Y,Z,'#PROD.'):-Z is X * Y.

pri_app(X,Y,Z) :-name(X,L1),name(Y,L2),append(L1,L2,L3),name(Z,L3).

append([],X,X).          append([X|Xs],Y,[X|Zs]):-append(Xs,Y,Zs).
.....
```

Figure 2.13: Multi-adjoint lattice modeling truth degrees with labels

Here, we see that when implementing for instance the conjunction operator of the *Product logic*, in the second component of our extended notion of "truth degree", we have *appended* the labels of its arguments with label `'&PROD.'` (see clauses defining `and_prod`, `pri_app` and `append`). Of course, in the fuzzy program to be run, we must also take into account the use of labels associated to the program rules, as occurs with the following example:

```
p(X) <prod &godel(q(X),@aver2(r(X),s(X))) with  info(0.9,'RULE1.').
q(a)                                      with  info(0.8,'RULE2.').
r(X)                                      with  info(0.7,'RULE3.').
s(X)                                      with  info(0.5,'RULE4.').
```

where we have used in the first rule aggregator "`@aver2`" (intended to compute the average between the results achieved by applying two different disjunction operations on the parameters) defined in Prolog as:

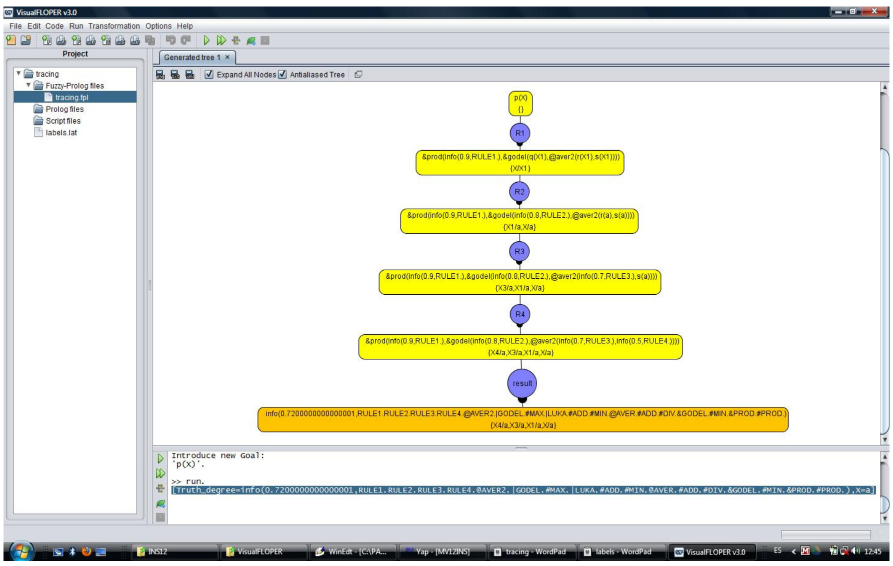
```
agr_aver2(X,Y,info(Za,Zb)) :- or_godel(X,Y,Z1),or_luka(X,Y,Z2),
                              agr_aver(Z1,Z2,info(Za,Zc)),
                              pri_app(Zc,'@AVER2.',Zb).
```

Now, the reader can easily test that, after executing goal "`p(X)`", we obtain the following fuzzy computed answer which includes the desired declarative trace containing the sequence of program-rules and connective-calls (mirroring $\rightarrow_{AS}$ and $\rightarrow_{SIS1}$ steps, according definitions 2.1.2 and 2.1.9, respectively) evaluated till finding the final solution:

```
>> run.

[Truth_degree=info(0.72,  RULE1.RULE2.RULE3.RULE4.
                          @AVER2.|GODEL.|LUKA.
                          @AVER.&GODEL.&PROD.),    X=a]
```

With a very little extra effort, we can extend the previous lattice to have into account also the exploited primitive operators during the interpretive phase, thus simulating $\rightarrow_{SIS2}$ steps (see again Definition 2.1.9). We simply need to include a label in the PROLOG definition of each primitive operator in order to identify it (for instance, "#PROD" refers to the product primitive operator).

```
and_prod(info(X1,X2),info(Y1,Y2),info(Z1,Z2)) :-
            pri_prod(X1,Y1,Z1,DatPROD),pri_app(X2,Y2,Dat1),
            pri_app(Dat1,'&PROD.',Dat2),pri_app(Dat2,DatPROD,Z2).

pri_prod(X,Y,Z,'#PROD.') :- Z is X * Y.
```



```
p(X)
{}

&prod(info(0.9,RULE1.),&godel(q(X1),@aver2(r(X1),s(X1))))
{X/X1}

&prod(info(0.9,RULE1.),&godel(info(0.8,RULE2.),@aver2(r(a),s(a))))
{X1/a,X/a}

&prod(info(0.9,RULE1.),&godel(info(0.8,RULE2.),@aver2(info(0.7,RULE3.),s(a))))
{X3/a,X1/a,X/a}

&prod(info(0.9,RULE1.),&godel(info(0.8,RULE2.),@aver2(info(0.7,RULE3.),info(0.5,RULE4.))))
{X4/a,X3/a,X1/a,X/a}

info(0.7200000000000001,RULE1.RULE2.RULE3.RULE4.@AVER2.|GODEL.#MAX.|LUKA.#ADD.#MIN.@AVER.#ADD.#DIV.&GODEL.#MIN.&PROD.#PROD.)
{X4/a,X3/a,X1/a,X/a}
```

```
Introduce new Goal:
'p(X)'.

>> run.
[Truth_degree=info(0.7200000000000001,RULE1.RULE2.RULE3.RULE4.@AVER2.|GODEL.#MAX.|LUKA.#ADD.#MIN.@AVER.#ADD.#DIV.&GODEL.#MIN.&PROD.#PROD.),X=a]
```

Figure 2.14: Obtaining declarative traces on fuzzy computed answers

As shown in **Figure 2.14**, the result of executing again goal "p(X)" is:

```
>> run.

[Truth_degree=info(0.72,  RULE1.RULE2.RULE3.RULE4.
                          @AVER2.|GODEL.#MAX.|LUKA.
                          #ADD.#MIN.@AVER.#ADD.#DIV.
                          &GODEL.#MIN.&PROD.#PROD.),    X=a]
```

In this fuzzy computed answer we obtain both the truth value (0.72) and substitution ($X = a$) associated to our goal, but also the sequence of program rules exploited when applying admissible steps as well as the proper fuzzy connectives evaluated during the interpretive phase, also detailing the set of primitive operators (of the form #$label$) they call.

Chapter 3

The FuzzyXPath interpreter

In this chapter we present a fuzzy variant of the XPath query language for the flexible information retrieval on XML documents. Our main purpose is to provide a repertoire of operators that offer the possibility of managing satisfaction degrees by adding structural constraints and fuzzy operators inside conditions (which must be considered from now on as *fuzzy conditions* instead of *boolean conditions*), in order to produce a ranked sorted list of answers according to user's preferences when composing queries. By using the FLOPER system designed in our research group, our proposal has been implemented with a fuzzy logic language to take profit of the clear synergies between both target and source fuzzy languages.

Our approach firstly proposes two structural constraints called *DOWN* and *DEEP* for which a certain degree of relevance can be associated. So, whereas *DOWN* provides a ranked set of answers depending on the path they are found from "top to down" in the XML document, *DEEP* provides a ranked set of answers depending on the path they are found from "left to right" in the XML document. Both structural constraints can be used together, assigning degree of importance with respect to the distance to the root XML element.

Secondly, we have enriched the arsenal of operators of XPath with fuzzy variants of *and* and *or*. Particularly, we have considered three versions of *and*: *and+*, *and*, *and-* (and the same for *or* : *or+*, *or*, *or-*) which make more flexible the composition of fuzzy conditions. Three versions for each operator that come for free from our adaptation of fuzzy logic to the XPath paradigm. One of the most known elements of fuzzy logic is the introduction of fuzzy versions of classical boolean operators.

Product, *Łukasiewicz* and *Gödel* fuzzy logics are considered as the most prominent logics and give a suitable semantics to fuzzy operators. Our contribution is now to give sense to fuzzy operators into the XPath paradigm, and particularly in user's preferences. We claim that in our work the fuzzy versions provide a mechanism to force (and debilitate) conditions in the sense that stronger (and weaker) user preferences can be modeled with the use of stronger (and weaker) fuzzy conditions. The combination of fuzzy operators in queries permits to specify a ranked set of fuzzy conditions according to user's requirements.

Finally, we have equipped our XPath based query language with a mechanism for thresholding user's preferences, in such a way that user can request that requirements are satisfied over a certain percentage.

3.1 A Flexible XPath Language

Our proposal of flexible XPath is defined by the following grammar:

xpath :=	[‘**[**’deep-down‘**]**’]path
path :=	literal | text() | node | @att | node/path | node//path
node :=	QName | QName[cond]
cond :=	xpath op xpath | xpath num-op number
deep :=	**DEEP**=number
down :=	**DOWN**=number
deep-down :=	deep | down | deep ‘**;**’ down
num-op :=	> | = | < | <>
fuzzy-op :=	**and** | **and+** | **and-** | **or** | **or+** | **or-** | **avg** | **avg**{number,number}
op :=	num-op | fuzzy-op

Basically, our proposal extends XPath as follows:

- A given XPath expression can be adorned with «[DEEP $= r_1$; DOWN $= r_2$]» which means that the *deepness* of elements is penalized by r_1 and that the *order* of elements is penalized by r_2, and such penalization is proportional to the distance (i.e., the length of the branch and the weight of the tree, respectively). In particular, «[DEEP $= 1$; DOWN $= r_2$]» can be used for penalizing only w.r.t. document order. *DEEP* works for //, that is, the deepness in the

$$
\begin{array}{lcllcll}
\&_{\mathtt{P}}(x,y) & = & x * y & |_{\mathtt{P}}(x,y) & = & x + y - x * y & \textit{Product: and/or} \\
\&_{\mathtt{G}}(x,y) & = & \min(x,y) & |_{\mathtt{G}}(x,y) & = & \max(x,y) & \textit{Gödel: and+/or-} \\
\&_{\mathtt{L}}(x,y) & = & \max(x+y-1,0) & |_{\mathtt{L}}(x,y) & = & \min(x+y,1) & \textit{Łuka: and-/or+}
\end{array}
$$

Figure 3.1: Fuzzy Logical Operators

XML tree is only computed when descendant nodes are explored, while *DOWN* works for both / and //. Let us remark that *DEEP* and *DOWN* can be used several times on the main *path* expression and/or any other *sub-path* included in conditions.

- We consider three versions for each one of the conjunction and disjunction operators (also called connectives or aggregators) which are based in the so-called *Product*, *Gödel* and *Łukasiewicz* fuzzy logics. The *Gödel* and *Łukasiewicz* logic based fuzzy symbols [1] are represented in our application by *and+*, *and-*, *or-* and *or+*, in contrast with product logic operators *and* and *or* (see Figure 3.1). Adjectives like *pessimistic*, *realistic* and *optimistic* are sometimes applied to the *Łukasiewicz*, *Product* and *Gödel* fuzzy logics since operators satisfy that, for any pair of real numbers x and y in $[0,1]$ (as used in MALPMALP):

$$\mathit{0} \leq \&_{\mathtt{L}}(x,y) \leq \&_{\mathtt{P}}(x,y) \leq \&_{\mathtt{G}}(x,y) \leq \mathit{1}$$

and the contrary for the disjunction operations (as used in MALP):

$$\mathit{0} \leq |_{\mathtt{G}}(x,y) \leq |_{\mathtt{P}}(x,y) \leq |_{\mathtt{L}}(x,y) \leq \mathit{1}$$

So, note that it is more difficult to satisfy a condition based on a pessimistic conjuntor/disjunctor (i.e, *and-/or-* inspired by the *Łukasiewicz* and *Gödel* logics, respectively) than with *Product* logic based operators (i.e, *and/or*), while the optimistic versions of such connectives (i.e., *and+/or+*) are less restrictive, obtaining a greater set of answers. This is a consequence of the following chain of inequalities (as used in MALP):

$$\mathit{0} \leq \&_{\mathtt{L}}(x,y) \leq \&_{\mathtt{P}}(x,y) \leq \&_{\mathtt{G}}(x,y) \leq |_{\mathtt{G}}(x,y) \leq |_{\mathtt{P}}(x,y) \leq |_{\mathtt{L}}(x,y) \leq \mathit{1}$$

or equivalently, by using the notation of our application:

$$\mathit{0} \leq \mathit{and}-(x,y) \leq \mathit{and}(x,y) \leq \mathit{and}+(x,y) \leq \mathit{or}-(x,y) \leq \mathit{or}(x,y) \leq \mathit{or}+(x,y) \leq \mathit{1}$$

[1]The fuzzy logic community frequently uses the terms *t-norm* and *t-conorm* for expressing generalized versions of conjunctions and disjunctions.

Therefore users should refine queries by choosing operators in the previous sequence from left to right (or from right to left), till finding solutions satisfying in a stronger (or weaker) way the requirements.

- Furthermore, the *avg* operator is defined too in a *weighted* way. Assuming two given *RSV*'s r_1 and r_2, *avg* is defined as $(r_1 + r_2)/2$, and $avg(p_1, p_2)$ is defined as $(p_1 * r_1 + p_2 * r_2)/p_1 + p_2$.
- Finally, we have considered a special case of constraint, a thresholding constraint of the form *xpath op r*, where $r \in [0, 1]$ and $op \in \{<, >, =\}$, in which the user can specify the threshold that the RSV of a given fuzzy condition has to raise.

In general, an extended XPath expression defines, w.r.t. an XML document, a sequence of subtrees of the XML document where each subtree has an associated RSV. XPath conditions, which are defined as fuzzy operators applied to XPath expressions, compute a new RSV from the RSVs of the involved XPath expressions, which at the same time, provides an RSV to the node.

3.2 Examples with DEEP and DOWN

In order to illustrate the language, let us see some examples of flexible queries in XPath. We will take as input document the one shown in Figure 3.2. The example shows a sequence of hotels where each one is described by *name* and *price*, proximity to streets (*close_ to*) and provided services (*pool* and *metro* -together with distance-). In the example, we assume that document order has the following semantics. The tag *close_ to* specifies the proximity to a given street. However, the order of *close_ to* tags is relevant, and the top streets are closer than the streets at the bottom. In other words, the case:

```
hotel_H
        close_to street_A
        close_to street_B
```

implies that hotel H is near to both streets A and B, but closer to A than to B. The nesting of *close_ to* has also a relevant meaning. While a given street A can be close to the hotel H, the streets close to A are not necessarily close to the hotel H. In other words, in the case:

```
<hotels>
<hotel name="Melia">
    <close_to>Gran Via
        <close_to>Callao</close_to>
        <close_to>Plaza de Espana</close_to>
    </close_to>
    <services>
        <pool></pool>
        <metro>150</metro>
    </services>
    <price>100</price>
</hotel>
<hotel name="NH">
    <close_to>Sol
        <close_to>Gran Via</close_to>
        <close_to>Callao</close_to>
    </close_to>
    <services>
        <metro>300</metro>
    </services>
    <price>150</price>
</hotel>
<hotel name="Hilton">
    <close_to>Moncloa
        <close_to>Gran Via</close_to>
        <close_to>Sol</close_to>
    </close_to>
    <services>
        <metro>150</metro>
    </services>
    <price>50</price>
</hotel>
<hotel name="Tryp">
    <close_to>Cibeles
        <close_to>Alcala
            <close_to>Gran Via</close_to>
        </close_to>
        <close_to>Retiro</close_to>
    </close_to>
    <services>
        <pool></pool>
        <metro>10</metro>
    </services>
    <price>575</price>
</hotel>
<hotel name="Sheraton">
    <close_to>Recoletos
        <close_to>Cibeles</close_to>
        <close_to>Gran Via
            <close_to>Sol</close_to>
        </close_to>
    </close_to>
    <close_to>Sol</close_to>
    <services>
        <pool></pool>
        <metro>300</metro>
    </services>
    <price>475</price>
</hotel>
</hotels>
```

Figure 3.2: Input XML document collecting Hotel's information

```
hotel_H
        close_to street_A
                          close_to street_B
```

the street B is near to street A, and street A is close to hotel H, which implies that street B is also close to hotel H, but no so close as street A. H can be situated at the end of street A, and B can cross A at the beginning. We can say, in this case, that B is an *adjacent* street to H, while A is *close* to H. This means that when looking for a hotel close to a given street, the highest priority should be assigned to streets close to the hotel, while adjacent streets should be relegated to lower priority. The example has been modeled in order to illustrate the use of structural constraints and fuzzy operators. Particularly, when the user tries to find hotels very close to a given street it should be provided a high *DOWN* value and a low *DEEP* value, whereas in the case the user tries to find hotels in the neighborhood of an street should provide high *DEEP* and low *DOWN*.

Ejemplo 3.2.1. *In our first example, we focus on the use of DOWN. Let us suppose that the user is interested to find a hotel close to Sol street. This might be his(er) first tentative looking for a hotel. Using crisp XPath (s)he would formulate:*

$$<< /hotels/hotel[close\_to/text() = \text{“}Sol\text{”}]/@name >>$$

However, it gives the user the set of hotels close to Sol without distinguishing the degree of proximity. The fuzzy version of XPath permits to specify a value of degradation of answers, in such a way that the user reformulates the query as:

$$<< /hotels/hotel[[DOWN = 0.9]close\_to/text() = \text{“}Sol\text{”}]/@name >>$$

The query specifies that close_to tag is degraded by 0.9 from top to down. In other words, when Sol is found close to a hotel, the position in which it occurs gives a different satisfaction value. In this case, we will obtain:

```
<result>
    <result rsv="1.0">NH</result>
    <result rsv="0.9">Sheraton</result>
</result>
```

Fortunately, we have found a hotel (NH) which is very close to Sol, and one (Sheraton) which is a little bit farther from Sol.

Let us remark the previous example and the other examples of the Section show the results in order of satisfaction degree.

Ejemplo 3.2.2. *Let us suppose now that we are looking for a hotel close to Callao. In this case, we can try to make the same question:*

$$<< /hotels/hotel[[DOWN = 0.9]close\_to/text() = \text{“}Callao\text{”}]/@name >>$$

However, the result is empty. Therefore we can try to relax the query by changing '/' by '//':

$$<< /hotels/hotel[[DOWN = 0.9]//close\_to/text() = \text{“}Callao\text{”}]/@name >>$$

Now, we will find answers, however, we will not be able to distinguish the proximity of the hotels. Our fuzzy version of XPath permits to specify how the solutions are degraded but not only taking into account the order but also the deepness. In other words, there would be useful to give different weights to be a close street, and to be an adjacent street. Therefore we can use the query:

$$<< /hotels/hotel[[DEEP = 0.5; DOWN = 0.9]//close\_to/text() = \text{“}Callao\text{”}]/@name >>$$

obtaining the following results:

```
<result>
    <result rsv="0.5">Melia</result>
    <result rsv="0.45">NH</result>
</result>
```

It seems that Melia is near to Callao, and NH is a little bit farther than Melia.

Ejemplo 3.2.3. *The use of DEEP combined with DOWN could be considered as the best choice. However, DEEP can be used alone when the user only wants to penalize adjacency. If we like to search hotels near to Gran Via street, degrading adjacent streets with a factor of 0.5, we can consider the following query (and we obtain the following result):*

$$<< //hotel[[DEEP = 0.5]//close\_to/text() = \text{“}GranVia\text{”}]/@name >>$$

```
<result>
  <result rsv="1.0">Melia</result>
  <result rsv="0.5">NH</result>
  <result rsv="0.5">Hilton</result>
  <result rsv="0.5">Sheraton</result>
  <result rsv="0.25">Tryp</result>
</result>
```

We can see that Melia is close to Gran Via, while NH, Hilton and Sheraton are situated in adjacent streets of Gran Via. Tryp is the farthest hotel.

Ejemplo 3.2.4. *The following table summarizes the results by combining DEEP and DOWN in a single query:*

$$<< //hotel[[DEEP = r_1; DOWN = r_2]//close\_to/text() = \text{“}GranVia\text{”}]/@name >>$$

HOTEL	**(A:** $r_1 = 0.1, r_2 = 1$**)**	**(B:** $r_1 = 0.5, r_2 = 0.5$**)**	**(C:** $r_1 = 1, r_2 = 0.1$**)**
Melia	1	1	1
NH	0.1	0.5	1
Hilton	0.1	0.5	1
Tryp	0.01	0.25	1
Sheraton	0.1	0.25	0.1

While case C only penalizes closeness, case A penalizes adjacency. Case B penalizes both closeness and adjacency.

3.3 AVG Examples

Ejemplo 3.3.1. *Let us suppose that the user is interested in a hotel combining two services like pool and metro. Instead of using classical and/or connectives for mixing both features, we can obtain more flexible estimations on RSV values by using the avg operator as follows:*

$$<< //hotel[services/pool\ avg\ services/metro]/@name >>$$

thus obtaining the following results:

```
<result>
  <result rsv="1.0">Melia</result>
  <result rsv="1.0">Tryp</result>
  <result rsv="1.0">Sheraton</result>
  <result rsv="0.5">NH</result>
  <result rsv="0.5">Hilton</result>
</result>
```

By using the avg fuzzy operator, the user finds that Melia, Tryp and Sheraton have pool and metro, while NH and Hilton lack on one of them.

Ejemplo 3.3.2. *Now, let us suppose that the importance of the metro is the double of the importance of the pool. In this case, the user can formulate the query as follows:*

$$<< //hotel[services/pool\ avg\{1,2\}\ services/metro]/@name >>$$

obtaining the following results:

```
<result>
  <result rsv="1.0">Melia</result>
  <result rsv="1.0">Tryp</result>
  <result rsv="1.0">Sheraton</result>
  <result rsv="0.666667">NH</result>
  <result rsv="0.666667">Hilton</result>
</result>
```

We can see in the results that NH and Hilton increase the degree of satisfaction w.r.t. the previous query given that they have metro station.

Ejemplo 3.3.3. *Let us suppose the user is looking now for hotels giving more importance to the fact that the price of the hotel is lower than 150 euros than to the proximity to Sol street. The user can formulate the query as follows, obtaining the results below:*

$$<< //hotel[[DEEP = 0.8]//close\_to/text() = \text{“}Sol\text{”}\ avg\{1,2\}\ //price/text() < 150]/@name >>$$

```
<result>
  <result rsv="0.933333">Hilton</result>
  <result rsv="0.666667">Melia</result>
  <result rsv="0.333333">NH</result>
  <result rsv="0.333333">Sheraton</result>
</result>
```

Now, we take a new input XML document, Figure 3.3, whose *skeleton* is depicted in Figure 3.4 that will help us better understand the AVG operator, this document represents a collection of books; all results of this document will be accompanied by table showing the calculation performed for RSV.

Ejemplo 3.3.4. *The Figure 3.5 shows the answer associated to the XPath expression:*

$$<< /bib/book[@price < 30\ avg\ @year < 2006] >>$$

```
<bib>
  <book year="2001" price="45.95">
    <title>Don Quijote de la Mancha</title>
    <author>Miguel de Cervantes Saavedra</author>
    <publications> <book year="1997" price="35.99">
                     <title>La Galatea</title>
                     <author>Miguel de Cervantes Saavedra</author>
                     <publications>
                            <book year="1994" price="25.99">
                            <title>Los trabajos de Persiles y Segismunda</title>
                            <author>Miguel de Cervantes Saavedra</author></book>
                     </publications></book>
    </publications></book>
  <book year="1999" price="25.65">
    <title>La Celestina</title>
    <author>Fernando de Rojas</author></book>
  <book year="2005" price="29.95">
    <title>Hamlet</title>
    <author>William Shakespeare</author>
    <publications>
      <book year="2000" price="22.5">
        <title>Romeo y Julieta</title>
        <author>William Shakespeare</author></book>
    </publications></book>
  <book year="2007" price="22.95">
    <title>Las ferias de Madrid</title>
    <author>Felix Lope de Vega y Carpio</author>
    <publications>
      <book year="1996" price="27.5">
        <title>El remedio en la desdicha</title>
        <author>Felix Lope de Vega y Carpio</author> </book>
      <book year="1998" price="12.5">
        <title>La Dragontea</title>
        <author>Felix Lope de Vega y Carpio</author></book>
    </publications></book>
</bib>
```

Figure 3.3: Input XML document collecting books

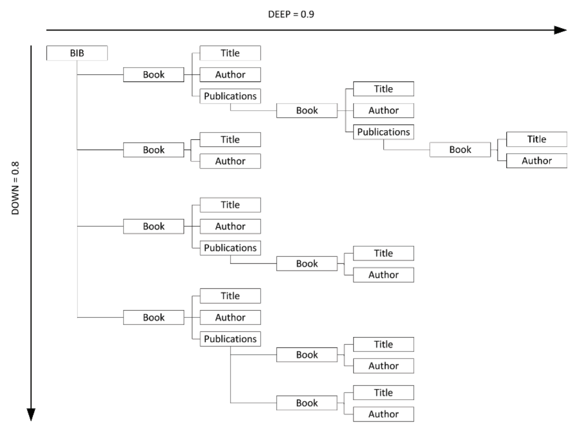


Figure 3.4: XML document collecting books represented as a tree.

Here we show that books satisfying a price under 30 *and a year before* 2006 *have the highest RSV.*

Document	RSV computation
<result>	
<book **rsv**="0.5" ...> <title>Don Quijote ...</title> ...</book>	$0.5 = (0+1)/2$
<book **rsv**="1.0"...><title>La Celestina</title> ...</book>	$1 = (1+1)/2$
<book **rsv**="1.0" ...><title>Hamlet</title> ...</book>	$1 = (1+1)/2$
<book **rsv**="0.5" ...><title>Las ferias de Madrid</title> ...</book>	$0.5 = (1+0)/2$
</result>	

Figure 3.5: Output of a query using the *average* operator *AVG*

Ejemplo 3.3.5. *Finally, in Figure 3.6 we combine all operators (thus obtaining more scattered RSV values) in query:*

«[DEEP=0.9,DOWN=0.8]//book[(@price>25 and @price<30) avg (@year<2000 or @year>2006)]/title»

Document	RSV computation
<result>	
<title **rsv**="0.3645">La Galatea</title>	$0.3645 = 0.9^3 * 1/2$
<title **rsv**="0.59049">Los trabajos de Persiles y Si...</title>	$0.59049 = 0.9^5 * 1$
<title **rsv**="0.72">La Celestina</title>	$0.72 = 0.9 * 0.8 * 1$
<title **rsv**="0.288">Hamlet</title>	$0.288 = 0.9 * 0.8^2 * 1/2$
<title **rsv**="0.2304">Las ferias de Madrid</title>	$0.2304 = 0.9 * 0.8^3 * 1/2$
<title **rsv**="0.373248">El remedio en la desdicha</title>	$0.373248 = 0.9^3 * 0.8^3 * 1$
<title **rsv**="0.149299">La Dragontea</title>	$0.149299 = 0.9^3 * 0.8^4 * 1/2$
</result>	

Figure 3.6: Output of a query using all operators

3.4 Thresholding Example

Fuzzy conditions return a satisfaction degree in the infinite space of real numbers between 0 and 1. We can take profit of this feature by imposing *thresholds* to such conditions, thus filtering the set of solutions according to the satisfaction degree. The idea is to formulate queries by directly acting on the satisfaction degrees obtained after evaluating "fuzzy" conditions.

Ejemplo 3.4.1. *For instance, let us suppose that in the query of example 3.2.3, the user looks for hotels in which the degree of proximity to Gran Via street is greater* *than seventy five per cent (i.e., value 0.75 measured between 0 and 1) then (s)he can formulate the following query, obtaining the following results:*

$$<< //hotel[([DEEP = 0.5]//close\_to/text() = \text{“}Gran\ Via\text{”}) > 0.75]/@name >>$$

```
<result>
  <result rsv="1.0">Melia</result>
</result>
```

3.5 Conjunctive/Disjunctive Connective Examples

Ejemplo 3.5.1. *In the following queries we express the following requirement: hotels near to Gran Via, near to a metro station, having pool, with greater preference (3 to 2) to pool than metro. We will use and+, and and and- which provide distinct levels of exigency, which are demonstrated in the results.*

$$<< //hotel[([DEEP = 0.5]//close\_to/text() = \text{”}GranVia\text{”}) \ and+ (//pool\ avg\{3,2\}\ //metro/text() < 200)]/@name >>$$

```
<result>
  <result rsv="1.0">Melia</result>
  <result rsv="0.5">Sheraton</result>
  <result rsv="0.4">Hilton</result>
  <result rsv="0.25">Tryp</result>
</result>
```

$$<< //hotel[([DEEP = 0.5]//close\_to/text() = \text{”}GranVia\text{”}) \ and\ (//pool\ avg\{3,2\}\ //metro/text() < 200)]/@name >>$$

```
<result>
  <result rsv="1.0">Melia</result>
  <result rsv="0.3">Sheraton</result>
  <result rsv="0.25">Tryp</result>
  <result rsv="0.2">Hilton</result>
</result>
```

$$<< //hotel[([DEEP = 0.5]//close\_to/text() = "GranVia")\ and-\ (//pool\ avg\{3,2\}\ //metro/text() < 200)]/@name >>$$

```
<result>
  <result rsv="1.0">Melia</result>
  <result rsv="0.25">Tryp</result>
  <result rsv="0.1">Sheraton</result>
</result>
```

So, in the first case (the least demanding and optimistic) we obtain four hotels (Melia, Sheraton, Hilton and Tryp), as well as in the second case (a little bit more exigent) while third table (the strongest one) lists three candidates (Melia, Tryp and Sheraton). Sheraton and Hilton are degraded using and and and-. This effect would even be more evident when previous conditions are compared with a threshold. For instance, to be greater than 0.25*. In such a case and- gives just a single solution: Melia.*

3.6 Dynamic Filtering for Improving Efficiency

In [JMMO10, JMM+13] we have reported some *thresholding* techniques specially tailored for the MALP language, where the main idea consists in to dynamically create and evaluate filters for prematurely disregarding those superfluous computations leading to non-significant solutions. Somehow inspired by the same guidelines, we have recently equipped our FuzzyXPath interpreter with a command with syntax «[FILTER=r]» (being r a real number between 0 and 1) which can be used just at the beginning of a query for indicating that only those answers with RSV greater of equal than r must be generated and reported. As we have described in [ALM14a], when «[FILTER=r]» precedes a fuzzy query, the interpreter *lazily* explores an input XML document for dynamically disregarding as soon as possible those branches of the XML tree leading to irrelevant solutions with an RSV degraded below r, thus allowing the possibility of efficiently managing large files without reducing the set of answers for which users are mainly interested in.

In order to explain the benefits of using the FILTER command, let us consider in this section the XML document shown in Figure 3.3, for which the execution of query,

$$<< [//book[@year < 2000\ avg\ @price < 50]/title >>$$

produces the following set of solutions:

```
<result>
  <title rsv="1.0">La Galatea</title>
  <title rsv="1.0">Los trabajos de Persiles y Sigismunda</title>
  <title rsv="1.0">La Celestina</title>
  <title rsv="1.0">El remedio en la desdicha</title>
  <title rsv="1.0">La Dragontea</title>
  <title rsv="0.5">Don Quijote de la Mancha</title>
  <title rsv="0.5">Hamlet</title>
  <title rsv="0.5">Romeo y Julieta</title>
  <title rsv="0.5">Las ferias de Madrid</title>
</result>
```

If we consider now the new, quite similar query «[FILTER=0.4]//book[@year <2000 avg @price<50]/title», we clearly obtain again nine answers, but only five if we fix «[FILTER=0.8]». Obviously, we would hope that the runtime of the second case should be lower than the first one since, as our approach does, there is no need for computing all solutions and then filtering the best ones. This desired dynamic behaviour when avoiding useless computations is reflected in Figure 3.7 which considers the effort needed for executing (excluding parsing/compiling time) a query like,

$$<< [FILTER = r]//book[(@price > 25 \textit{ and } @price < 30) \textit{ avg } (@year < 2000 \textit{ or } @year > 2006)] >>$$

where each row represents the size of several XML files accomplishing with the same structure of our running example (but considering different nesting levels of tags *book*, *title*, *author* and *publications*), and each column refers to a different degree of the FILTER command. Here, the runtime is measured in seconds (the benchmarks have been performed using a computer with processor Intel Core Duo, with 2 GB RAM and Windows Vista) and each record in the input file refers to a different *book* (that is, the number of records coincides with the number of occurrences of tag *book*) which might contain other books inside its *publications* tag.

Moreover, in Figure 3.8 we continue with a similar query to the previous one, but also considering the DEEP command[2]. Here, for a large XML document with

[2]This kind of statistics can be produced on-line for several XML files and FUZZYXPATH queries via the following URL that we have just prepared for the interested reader: `http://http://dectau.`

Records	**FILTER**								
	0.1	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**
1000	1.766	1.696	1.734	0.842	0.469	0.268	0.221	0.087	0.056
2000	6.628	6.432	6.998	3.242	1.439	0.677	0.599	0.168	0.122
3000	14.532	14.023	14.059	6.306	2.831	1.257	1.101	0.253	0.179
4000	25.535	24.684	24.722	10.883	4.827	1.918	1.794	0.345	0.242
5000	41.522	37.782	37.166	16.201	7.242	2.993	2.516	0.427	0.281
6000	58.905	55.354	55.596	24.411	10.993	4.207	3.554	0.554	0.373
7000	85.167	85.652	82.733	37.748	14.436	5.083	4.653	0.649	0.460
8000	137.737	102.816	102.763	69.401	26.680	8.273	5.894	0.690	0.481
9000	175.272	131.828	131.021	56.937	22.601	7.869	7.329	0.824	0.549
10000	195.613	185.201	167.676	95.286	26.649	9.516	9.595	0.973	0.742

Figure 3.7: Performance of FuzzyXPathby using FILTER on XML files with growing sizes

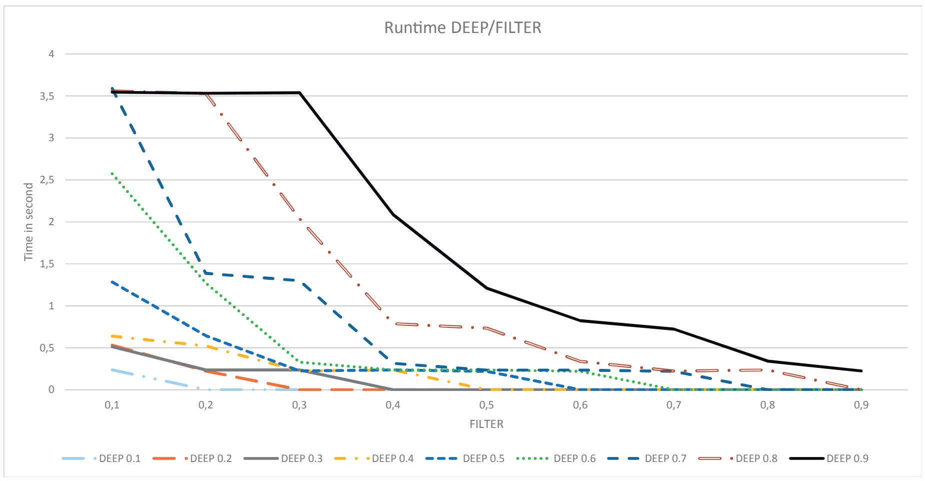


Figure 3.8: Runtime for several FuzzyXPathqueries varying DEEP and FILTER

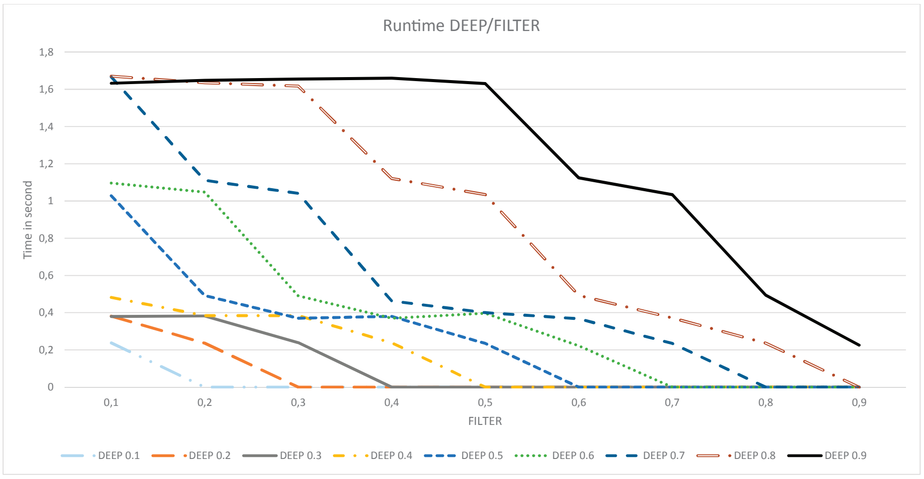


Figure 3.9: Varying DEEP and FILTER in a query using $avg\{30, 1\}$

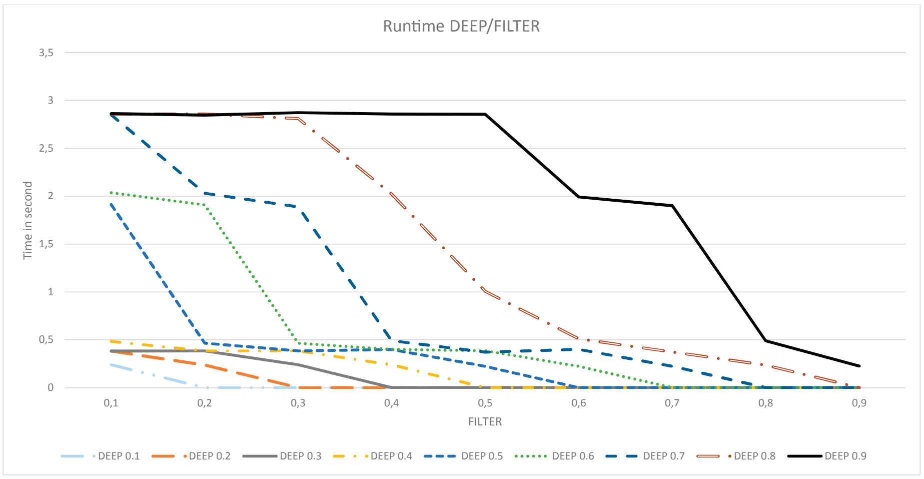


Figure 3.10: Varying DEEP and FILTER in a query using $avg\{1, 30\}$

```
[element(bib,[],
    [element(book,[year=2001,price=45.95],
              [element(title,[],[Don Quijote de la Mancha]),
              element(author,[],[Miguel de Cervantes Saavedra]),
              element(publications,[],
                            [element(book,[year=1997,price=35.99],
                                      [element(title,[],[La Galatea]),
                                      element(author,[],[Miguel de Cervantes Saavedra]),
                                  element(publications,[],...])...]),])])
```

Figure 3.11: A data-term representing a XML document

a fixed size, we express the number of seconds needed for executing such query when varying FILTER and DEEP, where it is easy to see that the behaviour is more and more improved whenever FILTER grows and DEEP decreases, as wanted. Note that the previous query makes use of the *avg* command and remember that its behaviour is defined, for two given RSV's r_1 and r_2, as $r_3 = (r_1 + r_2)/2$, we can now use the *priorized version* of such operator which let us to give different degrees of importance to its arguments (remember that, for two given RSV's r_1 and r_2, $avg\{p_1, p_2\}$ is computed by $r_3 = (r_1 * p_1 + r_2 * p_2)/(p_1 + p_2)$) and hence, if in the previous query we use $avg\{30, 1\}$ instead of standard *avg*, we indicate that the first sub-condition (i.e., @price>25 and @price<30) is 30 times more important than the second one (i.e., @year<2000 or @year>2006), whereas $avg\{1, 30\}$ represent the inverse criterion. In Figures 3.9 and 3.10 we provide statistics in the same way than in Figure 3.8, but using now *average with priorities* 30-1 and 1-30, respectively.

Although the core of our application is written with (fuzzy) MALP rules, our implementation is based on the following items:

(1) We have reused/adapted several modules of our previous −based implementation of (crisp) XPath described in [Alm09, ABE08].

(2) We have used the SWI- library for loading XML files, in order to represent a XML document by means of a term[3].

(3) The parser of XPath has been extended to recognize the new keywords `FILTER, DEEP, DOWN, avg`, etc... with their proper arguments.

`uclm.es/fuzzyXPath/?q=FuzzyXPathStatistics`.

[3]The notion of *term* (i.e., data structure) is just the same in MALP and .

(4) Each tag is represented as a data-term of the form: `element(Tag, Attributes, Subelements)`, where `Tag` is the name of the XML tag, `Attributes` is a list containing the attributes, and `Subelements` is a list containing the sub-elements (i.e. sub-trees) of the tag. For instance, the document of Figure 3.3 is represented in SWI- like in Figure 3.11. Loading of documents is achieved by predicate `load_xml(+File,-Term)` and writing by predicate `write_xml(+File,+Term)`.

(5) Predicate `fuzzyXPath(+ListXPath,+Tree,+Deep,+Down,+Filter,+Accum)` receives six arguments: (1) `ListXPath` is the representation of a XPath expression; (2) `Tree` is the term representing an input XML document; (3) `Deep/Down/Filter` have the obvious meaning, and finally (4) the last argument `Accum` (which is appropriately updated -maybe decreased- when going deeper in the exploration of the file) accumulates the sequence of penalties produced till reaching a concrete node, and it is very useful for deciding when performing a recursive call to the children of such node whenever the value of `Accum` is better than the one fixed by `Filter`.

```
tv(1, [[],
   tv(0.9,[[],
       tv(0.9,[element(title,[],[Don Quijote de la Mancha]),[],
       tv(1,[[],[],
       tv(1,[[],
           tv(0.9,[[],
               tv(0.9,[element(title,[],[La Galatea]),[],
               tv(1,[[],[],
               tv(1,[[],
                   tv(0.9,[[],
                       tv(0.9,[element(title,[],[Los trabajos de Persiles..]),...]),
       tv(0.8,[[],
          tv(0.9,[element(title,[],[La Celestina]),[],[]]),...
```

Figure 3.12: Example of a MALP output

(6) The evaluation of the query generates a *truth value* which has the form of a tree, called *tv tree*. For instance, the query $<< [DEEP = 0.9, DOWN = 0.8]//title >>$ generates the one illustrated in Figure 3.12. The main power of a fuzzy logic programming language like MALP w.r.t. , is that instead of answering questions with a simple *true/false* value, solutions are reported in a much more tinged, documented way. Basically, the FuzzyXPath predicate traverses the tree representing a XML document annotating into the *tv tree*

the corresponding *deep/down* values according to the movements performed in the horizontal and vertical axis, respectively. In addition, the *tv tree* is annotated with the values of *and*, *or* and *avg* operators in each node.

(7) Finally, the *tv tree* is used for computing the output of the query, by multiplying the recorded values. A predicate called `tv_to_elem` has been implemented to output the answer in a pretty way.

Chapter 4

Implementation Issues

User's preferences play a key role in information retrieval. In modern Web based information retrieval systems, user expects to introduce his(er) key words and preferences to influence the search results. However, while the technology is still improving, users get sometimes frustrated with those retrieval systems which only offer a poor/rigid set of expressive resources. Therefore the need for *flexible query languages* arises, in which the user can formulate queries according to his(er) preferences, being adaptable to data schema but without increasing complexity. In addition, flexible query languages should be equipped with a mechanism for obtaining a certain *ranked list* of answers. The ranking of answers can provide *satisfaction degrees* depending on several factors.

The *XPath* language [BBC$^+$07] has been proposed as a standard for XML querying and it is based on the description of the path in the XML tree to be retrieved. XPath allows to specify the name of nodes (i.e., tags) and attributes to be present in the XML tree together with boolean conditions about the content of nodes and attributes.

XPath querying mechanism is based on a boolean logic: the nodes retrieved from an XPath expression are those matching the path of the XML tree. Therefore, the user should know the *XML schema* in order to specify queries. However, even when the XML schema exists, it may not be available for users. Moreover, XML documents with the same XML schema can be very different in structure. Let us suppose the case of XML documents containing the curriculum vitae of a certain group of persons. Although they can share the same schema, each one can decide to include studies, jobs, training, etc. organized in several ways: by year, by relevance, and with different nesting degree. In an XPath-based structural query, the main criteria to provide a certain degree of satisfaction are the *hierarchical deepness* and *document order*. However, user's preferences play also a key role in determining the best solutions. Conditions on XPath expressions are usually of varying importance for a user, that is, the user gives a higher degree of importance to certain requirements when satisfying his(er) wishes. Therefore, the query language should provide mechanisms for assigning *priority to answers*, when they occur in different parts of the document, as well as *priority to queries*, with regard to user's preferences.

4.1 Multi-Adjoint Logic Programming and FuzzyXPath

In this section, we will introduce the elements of multi-adjoint logic programming (MALP). MALP will serve as semantic foundation of our proposal. Moreover, MALP will be used for the implementation of our language. The section will describe the theoretical basis of MALP: multi-adjoint lattices and fuzzy operators defined on them. In addition, it will be described an instance of MALP which considers a multi-adjoint lattice of trees with truth values, and operators defined for such lattice. Finally, we will describe the rule-based implementation of our FuzzyXPath by using MALP rules.

4.1.1 Multi-Adjoint Logic Programming

In multi-adjoint logic programming [MOV04, JMP09], we work with a first order language, $\mathcal{L}$, containing variables, function symbols, predicate symbols, constants, quantifiers ($\forall$ and $\exists$), and several arbitrary connectives such as implications ($\leftarrow_1, \leftarrow_2, \ldots, \leftarrow_m$), conjunctions ($\&_1, \&_2, \ldots, \&_k$), disjunctions ($\vee_1, \vee_2, \ldots, \vee_l$), and general hybrid operators ("aggregators" $@_1, @_2, \ldots, @_n$), used for combining/propagating truth values through the rules, and thus increasing the language expressiveness. Additionally, our language $\mathcal{L}$ contains symbols called *truth degrees* belonging to a multi-adjoint lattice L, whose formal description will be presented afterward in Definition 4.1.5 (see also Figure 4.2).

A *rule* is a formula "$A \leftarrow_i \mathcal{B}$ with α", where A is an atomic formula (usually called the *head*), $\mathcal{B}$ (which is called the *body*) is a formula built from atomic formulas

$B_1, \ldots, B_n$ ($n \geq 0$), truth values of L and conjunctions, disjunctions and general aggregations, and finally $\alpha \in L$ is the "weight" or *truth degree* of the rule. The set of truth values L may be the carrier of any complete lattice, as for instance occurs with the interval $([0,1], \leq)$, where $\leq$ is the usual order. Consider, for instance, the program $\mathcal{P}$ of Figure 4.1 composed of three rules with associated multi-adjoint lattice $\langle [0,1], \leq, \leftarrow_{\mathtt{P}}, \&_{\mathtt{P}} \rangle$, where label P stands for *Product logic* with the following connective definitions (for implication, conjunction and disjunction symbols, respectively): "$\leftarrow_{\mathtt{P}} (x,y) = \min(1, x/y)$", "$\&_{\mathtt{P}}(x,y) = x * y$" and "$|_{\mathtt{P}}(x,y) = x + y - x * y$".

$\mathcal{R}_1:$	$p(X)$	$\leftarrow_{\mathtt{P}}$	$q(X,Y) \mid_{\mathtt{P}} r(Y)$	*with*	0.8
$\mathcal{R}_2:$	$q(a,Y)$	$\leftarrow$		*with*	0.9
$\mathcal{R}_3:$	$r(b)$	$\leftarrow$		*with*	0.7

Figure 4.1: Example of MALP program

In order to describe the procedural semantics of the multi–adjoint logic language, in the following we denote by $\mathcal{C}[A]$ a formula where A is a sub-expression (usually an atom) which occurs in the –possibly empty– one hole context $\mathcal{C}[]$ whereas $\mathcal{C}[A/A']$ means the replacement of A by A' in context $\mathcal{C}[]$, and $mgu(E)$ is the *most general unifier* of an equation set E. The pair $\langle \mathcal{Q}; \sigma \rangle$ composed of a goal and a substitution is called a *state*. So, given a program $\mathcal{P}$, an *admissible computation* is formalized as a state transition system, whose transition relation $\rightarrow_{AS}$ is the smallest relation satisfying the following *admissible rules*:

1) $\langle \mathcal{Q}[A]; \sigma \rangle \rightarrow_{AS} \langle (\mathcal{Q}[A/v \&_i \mathcal{B}])\theta; \sigma\theta \rangle$ if A is the selected atom in goal $\mathcal{Q}$, $A' \leftarrow_i \mathcal{B}$ *with* v in $\mathcal{P}$, where $\mathcal{B}$ is not empty, and $\theta = mgu(\{A' = A\})$

2) $\langle \mathcal{Q}[A]; \sigma \rangle \rightarrow_{AS} \langle (\mathcal{Q}[A/v])\theta; \sigma\theta \rangle$ if $A' \leftarrow$ *with* v in $\mathcal{P}$, $\theta = mgu(\{A' = A\})$

The following derivation illustrates our definition (note that the exact program rule used -after being renamed, that is, *standardized apart*- in the corresponding step is annotated as a super–index symbol, whereas exploited atoms appear underlined):

$$\begin{array}{lllllll}
\&_{\mathtt{P}}(x,y) & = & x * y & \leftarrow_{\mathtt{P}} (x,y) & = & \min(1, x/y) & \textit{Product} \\
\&_{\mathtt{G}}(x,y) & = & \min(x,y) & \leftarrow_{\mathtt{G}} (x,y) & = & \begin{cases} 1 & \text{if } y \le x \\ x & \text{otherwise} \end{cases} & \textit{Gödel} \\
\&_{\mathtt{L}}(x,y) & = & \max(x+y-1,0) & \leftarrow_{\mathtt{L}} (x,y) & = & \min\{x-y+1,1\} & \textit{Łuka.}
\end{array}$$

Figure 4.2: Adjoint Pairs in $([0,1],\le)$

$$\begin{array}{ll}
\langle \underline{p(X)}; \{\} \rangle & \rightarrow_{AS}{}^{\mathcal{R}_1} \\
\langle 0.8 \ \&_{\mathtt{P}} \ (\underline{q(X_1,Y_1)} \ |_{\mathtt{P}} \ r(Y_1)); \{X/X_1\} \rangle & \rightarrow_{AS}{}^{\mathcal{R}_2} \\
\langle 0.8 \ \&_{\mathtt{P}} \ (0.9 \ |_{\mathtt{P}} \ \underline{r(Y_2)}); \{X/a, X_1/a, Y_1/Y_2\} \rangle & \rightarrow_{AS}{}^{\mathcal{R}_3} \\
\langle 0.8 \ \&_{\mathtt{P}} \ (0.9 \ |_{\mathtt{P}} \ 0.7); \{X/a, X_1/a, Y_1/b, Y_2/b\} \rangle &
\end{array}$$

The final formula can be directly interpreted in the lattice L to obtain the final *fuzzy computed answer*. So, since $0.8 \ \&_{\mathtt{P}} \ (0.9 \ |_{\mathtt{P}} \ 0.7) = 0.8*(0.9+0.7-(0.9*0.7)) = 0.776$, we say that the truth degree of $p(X)$ is 0.776 when X is a.

4.1.2 MALP and FuzzyXPath

MALP can be used as basis of our proposed flexible extension of XPath as follows. The idea is to consider MALP as semantic background for fuzzy queries expressed in XPath. With this aim we have to accommodate MALP truth values and connectives to XML documents, RSVs, fuzzy operators and structural/thresholding constraints. Firstly, we have to consider a suitable multi-adjoint lattice. The elements of the multi-adjoint lattice represent the truth values. In the context of XPath, truth values are trees of truth values, which we call *TV trees*. TV trees represent the RSV associated to each node of the ordered XML tree. From a theoretical point of view, the evaluation of a given goal w.r.t. a MALP program, will return a TV tree (i.e, an ordered tree of real numbers in the interval $[0,1]$) as the result of the computation.

Definición 4.1.1. *(TV trees) Formally, a TV tree is an empty tree or a pair* $(r, \overline{u}^n)$ *where* $r \in [0,1]$ *and* $\overline{u}^n$ *is a sequence of* n *TV trees. In a TV tree* t *we denote by* $root(t)$ *to* r *and by* $ch(t)$ *to* $\overline{u}^n$*. Let* $\mathcal{T}$ *be the set of TV trees, and* $\le$ *the usual order of real numbers; we can define the following order between TV trees;* $t \preceq_{\mathcal{T}} s$ *iff* $root(t) \le root(s)$ *and* $ch(t) \preceq_{\mathcal{T}} ch(s)$ *whenever* t *is not empty; and* $t \preceq_{\mathcal{T}} s$ *whenever* s *is empty.*

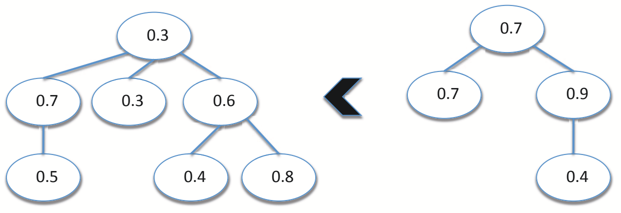


Figure 4.3: TV trees partial order

Abusing from the notation, in the previous $\preceq_T$ is used for sequences of TV trees, whose definition is as follows. Given two sequences of TV trees $\overline{u}^n = (u_1, \ldots, u_n)$ and $\overline{v}^m = (v_1, \ldots, v_m)$ then $\overline{u}^n \preceq_T \overline{v}^m$ whenever there exists a subsequence $\overline{s}^m$ of $\overline{u}^n$ such that $s_i \preceq_T v_i$ for all $1 \leq i \leq m$.

Intuitively, TV trees are ordered in decreasing order w.r.t. size, and increasing w.r.t. the relation on real numbers, and therefore the biggest element is the empty tree. We can see in Figure 4.3 an example of trees ordered by the $\preceq_{\mathcal{T}}$ relation, where the sequence of children is $((0.7, (0.5)), (0.6, (0.4, 0.8)))$; an alternative sequence is $(0.3, (0.6, (0.4, 0.8)))$.

We denote by $first(\overline{u}^n, k)$ (resp. $last(\overline{u}^n, k)$) the first $n-k$ (resp. last k) elements of the sequence $\overline{u}^n$. When $k > n$ then $first(\overline{u}^n, k)$ is completed with k-n empty trees at the right hand side and $last(\overline{u}^n, k)$ is completed with k-n empty trees at the left hand side.

Proposición 4.1.2. *$\preceq_{\mathcal{T}}$ is a partial order (reflexive, antisymmetric and transitive).*

Proof. Reflexivity and transitivity can be easily proved. The relation is antisymmetric reasoning by induction as follows. Let us suppose $t \preceq_{\mathcal{T}} s$ and $s \preceq_{\mathcal{T}} t$ then, when t and s are empty then trivially $s = t$; otherwise $root(t) \leq root(s)$ and $root(s) \leq root(t)$ therefore $root(t) = root(s)$; in addition (a) there exists a subsequence $\overline{s}^m$ of $\overline{u}^n$ such that $s_j \preceq_{\mathcal{T}} w_i$, $1 \leq i \leq m$, where $ch(t) = \overline{u}^n$ and $ch(s) = \overline{w}^m$; analogously, (b) there exists a subsequence $\overline{l}^n$ of $\overline{w}^m$ of such that $l_k \preceq_{\mathcal{T}} u_k$, $1 \leq k \leq n$. We can reason now that $n = m$: from (a) we have that $n \geq m$, and from (b) we have that $m \geq n$. Now, from (a) $\overline{u}^n \preceq_{\mathcal{T}} \overline{w}^n$ and from (b) $\overline{w}^n \preceq_{\mathcal{T}} \overline{u}^n$, thus by hypothesis $\overline{u}^n = \overline{w}^n$, and therefore $s = t$. □

Now, we can define the following operations in the set $\mathcal{T}$.

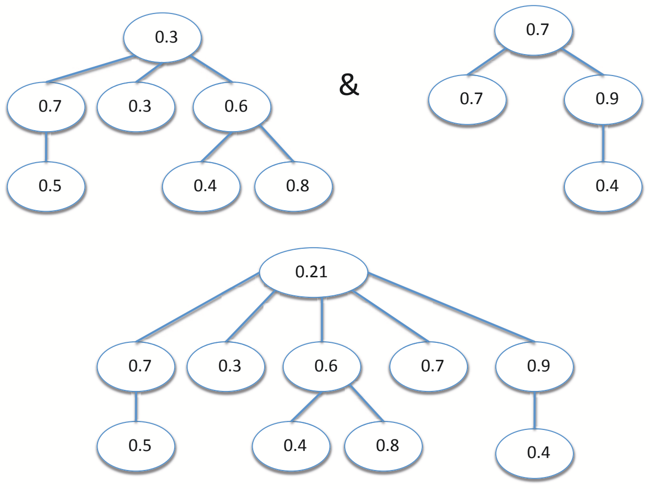


Figure 4.4: Conjunction on TV trees (according Product logic)

Definición 4.1.3. *(Conjunction on TV trees) The operation* $\&_O^{\mathcal{T}}(t,s)$*,* $O \in \{\mathtt{P},\mathtt{G},\mathtt{L}\}$ *is defined as the tree whose root is* $\&_O(root(t), root(s))$*, and children* $ch(t) + ch(s)$*; whenever* t *(resp.* s*) is not empty, and otherwise the result is* s *(resp.* t*), where* $\&_O$ *is the corresponding operator over* $[0,1]$ *and* $+$ *is the concatenation of sequences.*

Basically, the conjunction is defined as the application of the corresponding conjunction operator to the roots of the trees, and the concatenation of the children. We can see in Figure 4.4 an example of conjunction between TV trees.

Definición 4.1.4. *(Implication Operator on TV trees) The operation* $\leftarrow_O^{\mathcal{T}} (t,s)$*,* $O \in \{\mathtt{P},\mathtt{G},\mathtt{L}\}$ *is defined in the case of non-empty trees* t *and* s *as the tree whose root is* $\leftarrow_O (root(t), root(s))$*, and the children are* $first(\overline{u}^n, n-m) + \overline{t}$ *if* $n > m$ *and* $\overline{t}$ *otherwise, where* $\overline{t} = last(\overline{u}^n, m)\backslash\overline{w}^m$ *where* $ch(t) = \overline{u}^n$ *and* $ch(s) = \overline{w}^m$*. In the case* t *is empty is defined as the empty tree and in the case* s *is empty as* t*.*

In the previous definition, we use the difference of two sequences of TV trees: $\overline{v}^n\backslash\overline{w}^n$ defined as the sequence $\overline{u}^n$ in which $\overline{u}^n$ is empty whenever $w_i \preceq_{\mathcal{T}} v_i$ for all $1 \leq i \leq n$; and $\overline{u}^n = \overline{v}^n$, otherwise. Basically, the implication operator $\leftarrow_O^{\mathcal{T}}$ (t,s) computes the corresponding adjoint operator to the roots of the trees, and the children of t are replaced by empty TV trees (from right to left) whenever they are

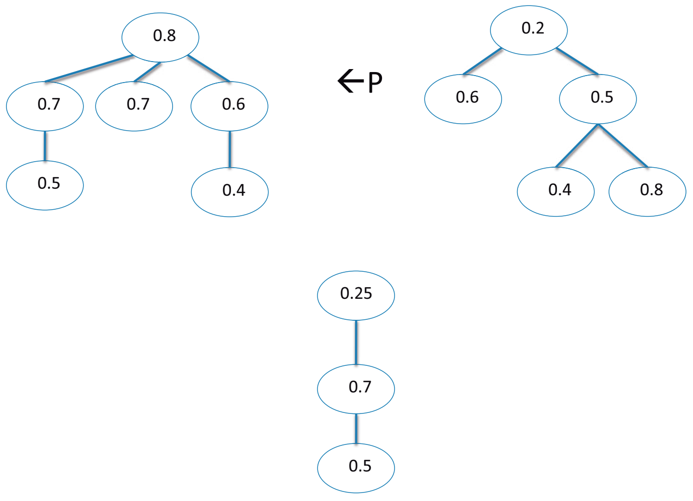


Figure 4.5: Implication on TV trees (according Product logic)

greater or equal than the corresponding element of s. We can see in Figure 4.5 an example of use of the implication, in the case of product logic, between TV trees.

Our goal now is to prove that TV trees conform a multi-adjoint lattice defined as follows.

Definición 4.1.5. *Let $(L, \preceq)$ be a lattice. A multi-adjoint lattice is a tuple:*

$$\langle L, \preceq, \leftarrow_1, \&_1, \ldots, \leftarrow_n, \&_n \rangle$$

such that:

1. *$(L, \preceq)$ is complete, namely, $\forall S \subset L$ non empty $\exists inf(S), sup(S)$. Then, it is a bounded lattice, i.e. it has bottom and top elements, denoted by $\top$ and $\bot$, respectively.*

2. *$(\&_i, \leftarrow_i)$ is an adjoint pair in $(L, \preceq)$, i.e.:*

   (a) $\&_i$ is non decreasing in both arguments, for all i, $i = 1, \ldots, n$.

   (b) $\leftarrow_i$ is non decreasing in the first argument and non increasing in the second, for all i.

   (c) $x \preceq (y \leftarrow_i z)$ if and only if $(x \&_i z) \preceq y$, for any $x, y, z \in L$ (adjoint property).

3. $\top \&_i v = v \&_i \top = v$ *for all* $v \in L$, $i = 1, \dots, n$, *where* $\top = sup(L)$*.*

TV trees are a multi-adjoint lattice based on truth values over $[0,1]$. In order to prove this property, we need the following auxiliary proposition.

Proposición 4.1.6. $\langle [0,1], \leq, \leftarrow_{\mathtt{P}}, \&_{\mathtt{P}}, \leftarrow_{\mathtt{G}}, \&_{\mathtt{G}}, \leftarrow_{\mathtt{L}}, \&_{\mathtt{L}} \rangle$ *is a multi-adjoint lattice.*

Note that even when the previous claim holds too for many other arbitrary multi-adjoint lattices $(L, \preceq)$, we prefer to accommodate it to the $[0,1]$ case because our notion of TV trees simply relies on this concrete kind of basic truth degrees (i.e., real numbers in the unit interval). Now, we are ready to establish the following theorem proving our intended result.

Teorema 4.1.7. $\langle \mathcal{T}, \preceq_{\mathcal{T}}, \leftarrow^{\mathcal{T}}_{\mathtt{P}}, \&^{\mathcal{T}}_{\mathtt{P}}, \leftarrow^{\mathcal{T}}_{\mathtt{G}}, \&^{\mathcal{T}}_{\mathtt{G}}, \leftarrow^{\mathcal{T}}_{\mathtt{L}}, \&^{\mathcal{T}}_{\mathtt{L}} \rangle$ *is a multi-adjoint lattice.*

Proof. 1. Let be a non empty $S \subset \mathcal{T}$, then $inf(S)$ is defined as the empty TV tree whenever all the elements of S are empty, otherwise as the TV tree whose root is the infimum of the roots of the non-empty elements of S, and the children are the sequence $\overline{u}^n$ where $u_i = inf(\{w_i | \overline{w}^n \in perm(s)_n, s \in S\})$, where $n = maxChildren(S)$ represents the maximum number of children of the elements of S, and $perm(s)_n$ is the set of all sequences of length n including $ch(s)$ as subsequence and empty trees in the other positions. Now, $inf(S) \preceq_{\mathcal{T}} s$, for all $s \in S$ by construction: when $s \in S$ then $root(inf(S)) \leq root(s)$ and $inf(\{w_i | perm(s)_n = \overline{w}^n, s \in S\}) \preceq_{\mathcal{T}} ch(s)$ given that $ch(s)$ is a subsequence of a sequence of $perm(s)$. Now, $inf(S)$ is the greatest lower bound: let us suppose $t \preceq_{\mathcal{T}} s$ for all $s \in S$; now, $root(t) \leq root(s)$ for all $s \in S$; thus $root(inf(S)) \leq root(t)$; in addition $ch(t) \preceq_{\mathcal{T}} ch(s)$ for all $s \in S$, and thus, for all $p \in perm(t)_n$ and $q \in perm(s)_n$ we have that $p \preceq_{\mathcal{T}} q$; and then $inf(\{w_i | \overline{w}^n \in perm(s)_n, s \in S\}) \preceq_{\mathcal{T}} p \preceq_{\mathcal{T}} ch(t)$, concluding that $inf(S) \preceq_{\mathcal{T}} t$. Analogously, $sup(S)$ is defined as the empty TV tree whenever all the elements of S are empty, otherwise as the TV tree whose root is the supremum of the roots of the non-empty elements of S, and the children are the sequence $\overline{u}^n$ where $u_i = sup(\{w_i | \overline{w}^n \in perm(s)_n, s \in S\})$, where $n = maxChildren(S)$. $\top$ of $\mathcal{T}$ is the empty tree and $\perp$ is the (infinite) tree with 0 in all nodes.

2. (a) $\&^{\mathcal{T}}_O$, $O \in \{\mathtt{P}, \mathtt{G}, \mathtt{L}\}$ is non decreasing: by Proposition 4.1.6 whenever $v \preceq_{\mathcal{T}} s$, $v' \preceq_{\mathcal{T}} s'$ then $root(v) \&_O root(v') \leq root(s) \&_O root(s')$; in addition, $c_i \preceq_{\mathcal{T}} b_i$, $c'_k \preceq_{\mathcal{T}} b'_k$, for subsequences $\overline{c}^n$ of $\overline{a}^p$ and $\overline{c'}^m$ of $\overline{a'}^q$, where $ch(v) = \overline{a}^p$, $ch(s) = \overline{b}^n$, $ch(v') = \overline{a'}^q$ and $ch(s') = \overline{b'}^m$. Now, the children

of $\&_O^{\mathcal{T}}(v, v')$ is $\overline{a}^p + \overline{a'}^q$, and the children of $\&_O^{\mathcal{T}}(s, s')$ is $\overline{b}^n + \overline{b'}^m$, thus $\&_O^{\mathcal{T}}(v, v') \preceq_{\mathcal{T}} \&_O^{\mathcal{T}}(s, s')$ taking the subsequence $\overline{c}^n + \overline{c'}^m$ of $\overline{a}^p + \overline{a'}^q$.

(b) $\leftarrow_O^{\mathcal{T}}$, $O \in \{\mathtt{P}, \mathtt{G}, \mathtt{L}\}$ is non decreasing in the first argument and non increasing in the second.

Let us see the first case: non decreasing w.r.t. the first argument; let $v \preceq_{\mathcal{T}} s$ be non empty trees; then by Proposition 4.1.6: $root(v) \leq root(s)$ and therefore $\leftarrow_O (root(s), root(t)) \leq \leftarrow_O (root(v), root(t))$; in addition, there exists a subsequence $\overline{a'}^m$ of $\overline{a}^n$ such that $a'_i \preceq_{\mathcal{T}} b_i$, $1 \leq i \leq m$, where $ch(v) = \overline{a}^n$ and $ch(s) = \overline{b}^m$. Now, let us suppose $ch(t) = \overline{c}^k$, and $c_j \not\preceq_{\mathcal{T}} b_j$ of $last(\overline{b}^m, k)$. $c_j \not\preceq_{\mathcal{T}} a'_j$ is satisfied from $a'_j \preceq_{\mathcal{T}} b_j$, otherwise by transitivity we have a contradiction. Therefore $\leftarrow_O^{\mathcal{T}}(v, t)$ contains as children a'_j, and this happens for each element b_j of $last(\overline{b}^m, k)$. Therefore, we can prove $\leftarrow_O^{\mathcal{T}}(v, t) \preceq_{\mathcal{T}} \leftarrow_O^{\mathcal{T}}(s, t)$ taking the subsequence $\overline{a'}^m$. Similarly the case $k \geq m$ taking $\overline{a'}^m$ since empty trees are added to the left hand side. When either v or s are empty then the property holds trivially.

Let us see the second case: non decreasing in the second argument; let $s \preceq_{\mathcal{T}} v$ be then $root(v) \leq root(s)$ and therefore by Proposition 4.1.6 $\leftarrow_O(root(t), root(s)) \leq \leftarrow_O(root(t), root(v))$; in addition, there exists a subsequence $\overline{a'}^m$ of $\overline{a}^n$ such that $a'_i \preceq_{\mathcal{T}} b_i$, $1 \leq i \leq m$, where $ch(v) = \overline{a}^n$ and $ch(s) = \overline{b}^m$. Now, let us suppose $ch(t) = \overline{c}^k$, and $a'_j \not\preceq_{\mathcal{T}} c_j$, c_j of $last(\overline{c}^k, n)$. Thus, $b_j \not\preceq_{\mathcal{T}} c_j$ is satisfied from $a'_j \preceq_{\mathcal{T}} b_j$, otherwise by transitivity we have a contradiction. Therefore $\leftarrow_O^{\mathcal{T}}(t, s)$ contains as children c_j, when $\leftarrow_O^{\mathcal{T}}(t, v)$ does. Thus, $\leftarrow_O^{\mathcal{T}}(t, s) \preceq_{\mathcal{T}} \leftarrow_O^{\mathcal{T}}(t, v)$ taking the subsequence $\overline{c}^k$. Similarly the case $n \geq k$ taking $\overline{c}^k$ since empty trees are added to the left hand side. When either s or v are empty the property holds trivially.

(c) $x \preceq_{\mathcal{T}} (y \leftarrow_O^{\mathcal{T}} z)$ if and only if $(x \&_O^{\mathcal{T}} z) \preceq_{\mathcal{T}} y$:

($\Rightarrow$):

By Proposition 4.1.6 we have that if $root(x) \leq \leftarrow_O(root(y), root(z))$ then $(root(x)\ \&_O root(z)) \leq root(y)$; in addition, there exists a subsequence $\overline{a'}^m$ of $\overline{a}^n$ such that $a'_i \preceq_{\mathcal{T}} b_i$ $1 \leq i \leq m$ where $ch(x) = \overline{a}^n$ and $ch(y \leftarrow_O^{\mathcal{T}} z) = \overline{b}^m$. Now, let $ch(y) = \overline{c}^k$ and $ch(z) = \overline{d}^p$ be, by hypothesis we have that there exists $\overline{a''}^{k-p}$ subsequence of $\overline{a'}^m$ such that $a''_j \preceq_{\mathcal{T}} c_j$, for each $c_j \in first(\overline{c}^k, k-p)$, $1 \leq j \leq k-p$. In addition, when $c'_j \in last(\overline{c}^k, p)$, $1 \leq j \leq p$ then either $last(\overline{c}^k, p) \backslash \overline{d}^p = last(\overline{c}^k, p)$

and there exists $\overline{a'''}^p$ subsequence of $\overline{a'}^m$ such that either $a'''_j \preceq_{\mathcal{T}} c'_j$ for all $1 \leq j \leq p$ or $last(\overline{c}^k, p)\backslash\overline{d}^p$ is empty and $d_j \preceq_{\mathcal{T}} c'_j$ for all $1 \leq j \leq p$. Now, we can prove $(x\&^{\mathcal{T}}_O z) \preceq_{\mathcal{T}} y$ by taking the subsequence $\overline{a''}^{k-p} + \overline{a'''}^p$ in the first case, and $\overline{a''}^{k-p} + \overline{d}^p$ in the second one. Similarly the case $p \geq k$ taking $\overline{a'''}^p$ and $\overline{d}^p$ in each one of the cases. When either x or y are empty the result is trivial.
($\Leftarrow$):
By Proposition 4.1.6 we have that if $(root(x)\&_O root(z)) \leq root(y)$ then $root(x) \leq \leftarrow_O(root(y), root(z))$; in addition, there exists a subsequence $\overline{a'}^m$ of $\overline{a}^n$ such that $a'_i \preceq_{\mathcal{T}} b_i$ $1 \leq i \leq m$ where $ch(x\&_O z) = \overline{a}^n$ and $ch(y) = \overline{b}^m$. Now, we have two cases: either $last(\overline{b}^m, k)\backslash\overline{c}^k$ is empty where $ch(z) = \overline{c}^k$ and therefore for all $b'_j \in last(\overline{b}^m, k)$ $c_j \preceq_{\mathcal{T}} b'_j, 1 \leq j \leq k$ or $last(\overline{b}^m, k)\backslash\overline{c}^k = last(\overline{b}^m, k)$ and therefore there exists a subsequence $\overline{a''}^k$ of $\overline{a'}^m$ such that $a''_j \preceq_{\mathcal{T}} b'_j$, $1 \leq j \leq k$. Now, we can prove that $x \preceq_{\mathcal{T}} (y \leftarrow_O z)$ taking the subsequence $first(\overline{a'}^m, m-k)$ in the first case, and $first(\overline{a'}^m, m-k) + \overline{a''}^k$ in the second case. Similarly the case $k \geq m$ taking the empty set in the first case, and $\overline{a''}^k$ in the second one. since empty trees are added to the left hand side.

3. $\top \&^{\mathcal{T}}_O v = v \&^{\mathcal{T}}_O \top = v$ given that $\top$ is the empty tree.

□

4.2 FuzzyXPath in FLOPER

FuzzyXPath will be grounded in multi-adjoint logic programming with TV trees as truth values, *Product*, *Łukasiewicz* and *Gödel* operators, and two extra monotonic hybrid operators, particularly, *@avg* and *@fuse*.
Now, we would like to show how TV trees are used for representing computations of results in our fuzzy variant of XPath, and how MALP rules are used for computing results from XPath expressions.

1. XML documents are represented with MALP terms, which are identical to Prolog terms. MALP represents XML trees with a term of the form:

$$element(tag, attributes, children)$$

where *tag*is the root of the tree, *attributes* is a list of attribute/value pairs,

```
[element(hotels, [],
 [element(hotel, [name='Melia'],
     [element(close_to, [],
          [
                    'Gran Via',
                    element(close_to,[],['Callao']),
                    element(close_to,[],['Plaza de Espana'])
                    ],
      element(services,[],
            [
                    element(pool,[],[]),
                    element(metro,[],[150])
                    ],
      element(price,[],[100])
     ]
  element(hotel,[name="NH"],
....
]
```

Figure 4.6: Example of XML data represented in MALP

and *children* is a list of children. For instance, the example of Figure 3.2 is represented with the MALP term of Figure 4.6.

2. XPath expressions are also represented as MALP terms, particularly, with a MALP list. For example, */hotel/services/pool* is represented by `[hotel,services,pool]`, where some elements of the list can be also either a list (for nested XPath expressions), an attribute name (like *attr(name)*), a label *relativePath* for representing *//* or a fuzzy condition (represented by a MALP term of the form `tree(op,xpath,xpath)`).

3. TV trees are represented as MALP terms of the form:

$$tv(truthvalue, [nodecontent, siblingtv, childrentv])$$

where *truthvalue* is the truth value of the current node, *nodecontent* is the content of the node when it is included in the answer of the query, *siblingtv* is the TV tree of the first sibling node, and *childrentv* is the TV tree of the first children. We have to remark the following consideration: TV trees are trees of truth values, according to the definition of previous section, however TV trees in MALP include the content of the selected nodes in order to make easier the implementation. Moreover, TV trees in MALP are represented as binary trees.

Figure 4.7 shows an example of TV tree computed by the *fuzzyXPath* predicate for the query:

$$<< [DEEP = 0.5; DOWN = 0.9]//hotel[//close\_to/text() = \text{“}Gran\ Via\text{”}]/@name >>$$

```
tv(1.0, [[],
     tv(0.5, [[tv(1.0,[]),['Melia']], [],
     tv(0.9, [[tv(1.0,[]),['NH']], [],
     tv(0.9, [[tv(1.0,[]),['Hilton']], [],
     tv(0.9, [[tv(1.0,[]),['Tryp']], [],
     tv(0.9, [[tv(1.0,[]),['Sheraton']],[],[]]) ]) ]) ])]), []])
```

Figure 4.7: Example of a *TV* structure in MALP

4. Now, a MALP predicate called *fuzzyXPath* that takes as input (a) an XPath expression represented by a list, (b) an XML tree represented by a MALP term, and (c) *DEEP* and *DOWN* values as TV trees (by default they are *tv(1,[])*. It returns the TV tree associated to the query. For instance, a call to *fuzzyXPath* for solving the previous query and thus returning the TV tree of Figure 4.7, could have the following shape (where *«XMLdata»* refers to the MALP term of Figure 4.6 representing the XML document in Figure 3.2):

$$fuzzyXPath([relativePath, hotel, tree(op(= , [relativePath, close\_to, text], \text{“}Gran\ Via\text{”}), nil, nil), attr(name)], << XMLdata >>, tv(0.5, []), tv(0.9, []))$$

5. The definition of the *fuzzyXPath* predicate distinguishes cases with regard to the XPath expression to be evaluated. Such MALP predicate basically traverses the XML tree represented by the MALP term, and recursively computes for each node the associated RSV.

   For instance, Figure 4.8 shows the case in which the current root matches with the current path. We can see that $\&_{\mathtt{P}}^{\mathcal{T}}$ is used as fuzzy operator, to compute *DEEP* and *DOWN* values for each recursive call. In addition, $\leftarrow_{\mathtt{P}}^{\mathcal{T}}$ is used as implication connective, and $@fuse$ is used for re-building the answer. The weight of the MALP rules is always *tv(1,[])*.

```
fuzzyXPath([Label|LabelRest],[element(Label,_,Children)|Siblings],Deep,Down) ←_P^T
            @fuse (
             tv(1,[element(Label,Attr,Children), [], []]),
             &_P^T(Deep,fuzzyXPath(LabelRest,Children,Deep,Down)),
             &_P^T(Down,fuzzyXPath([Label|LabelRest],Siblings,Deep,Down))
            ) with tv(1,[])
```

Figure 4.8: Example of MALP rule

6. Figure 4.9 shows the case of MALP rules called from *fuzzyXPath* for evaluating *avg*, where fuzzy conditions are handled by a predicate called *execute_fcond*. For instance, given the query:

```
fuzzyXPath    ([Label,tree(A,B,C)],[element(Label,Attr,Children)|Siblings],Deep,Down)
               ←_P^T
               @fuse(
                execute_fcond(Label,tree(A,B,C),element(Label,Attr,Children)),
                &_P^T(Down,fuzzyXPath([Label,tree(A,B,C)],Siblings,Deep,Down))
               ) with tv(1,[])

execute_fcond(Label,tree(avg,T1,T2),element(Label,Attr,Children)) ←_P^T
               @avg(
                execute_fcond(Label,T1,element(Label,Attr,Children)),
                execute_fcond(Label,T2,element(Label,Attr,Children))
               ) with tv(1,[])
```

Figure 4.9: Examples of MALP rules for evaluating conditions

$$<< //hotel[services/pool\ avg\ services/metro]/@name >>$$

the corresponding call to the *fuzzyXPath* predicate would be:

$$fuzzyXPath([relativePath, hotel, tree(avg,$$

$$tree(exist([services, pool]), nil, nil),$$

$$tree(exist([services, metro]), nil, nil)),$$

$$attr(name)], << XMLdata >>, tv(1.0, []), tv(1.0, []))$$

which eventually will invoke predicate *execute_fcond* (for evaluating the fuzzy condition containing the *avg* operator) as follows:

$$execute\_fcond(hotel, tree(avg,$$

$$tree(exist([services, pool]), nil, nil),$$

$$tree(exist([services, metro]), nil, nil)),$$

$$<< XMLdata >>, TV\_Cond)$$

Figure 4.10 shows the resulting TV tree, which clearly resembles the XML file previously reported in Example 3.3.1.

```
tv(1.0,[[],
    tv(1.0,[[tv(1.0, []), [Melia]], [],
    tv(1.0,[[tv(1.0, []), [Tryp]], [],
    tv(1.0,[[tv(1.0, []), [Sheraton]], [],
    tv(1.0,[[tv(0.5, []), [NH]], [],
    tv(1.0,[[tv(0.5, []), [Hilton]], [], []])])])])]), []])
```

Figure 4.10: Example of a TV obtained after evaluating a fuzzy condition

We have developed a prototype of our FuzzyXPath which is publicly available from `http://dectau.uclm.es/fuzzyXPath/`, equipped with a Web interface from which XPath queries can be tested. The implementation has been developed with our FLOPER tool. In what follows, we will give some details about FLOPER and the implementation of the FuzzyXPath in FLOPER.

Firstly, we would like to summarize the main elements of the FLOPER tool [MM08, MMPV10a, MMPV11b] which manages MALP programs. The parser of our

```
member(X):− number(X),0=<X,X=<1.
bot(0). top(1). leq(X,Y):− X=<Y.
or_prod(X,Y,Z):− pri_prod(X,Y,U1),pri_add(X,Y,U2),pri_sub(U2,U1,Z).
and_prod(X,Y,Z):− pri_prod(X,Y,Z).
pri_prod(X,Y,Z):− Z is X ∗ Y.
pri_add(X,Y,Z):− Z is X+Y.
pri_sub(X,Y,Z):− Z is X−Y.
```

Figure 4.11: Example of multi-adjoint lattice

FLOPER tool has been implemented by using the classical DCG's (*Definite Clause Grammars*) resource of the Prolog language, since it is a convenient notation for expressing grammar rules. Once the application is loaded inside a Prolog interpreter, it shows a menu which includes options for loading/compiling, parsing, listing and saving fuzzy programs, as well as for executing/debugging fuzzy goals. All these actions are based on the *compilation* of the fuzzy code into standard Prolog code.

The key point of the compilation is to extend each atom with an extra argument, called *truth variable* of the form "$\_TV_i$", which is intended to contain the truth degree obtained after the subsequent evaluation of the atom. For instance, the first clause of Figure 4.1 is translated into:

```
p(X,_TV0):−q(X,Y,_TV1),r(Y,_TV2),or_prod(_TV1,_TV2,_TV3),and_prod(0.8,_TV3,_TV0).
```

Moreover, the remaining rules, become the pure Prolog facts "*q(a, Y, 0.9)*" and "*r(b,0.7)*", whereas the corresponding lattice is expressed by the clauses of Figure 4.11, where the meaning of the mandatory predicates *member*, *top*, *bot* and *leq* is obvious.
Finally, a fuzzy goal like "*p(X)*", is translated into the pure Prolog goal:

$$p(X, Truth\_degree)$$

(note that the last truth degree variable is not anonymous now) for which, after choosing option "*run*", the Prolog interpreter returns the desired fuzzy computed answer:

$$[Truth\_degree = 0.776, X = a]$$

Note that all internal computations (including compiling and executing) are pure Prolog derivations, whereas inputs (fuzzy programs and goals) and outputs (fuzzy computed answers) have always a fuzzy taste, thus producing the illusion on the final user of being working with a purely fuzzy logic programming tool. By using option "*lat*" ("*show*") of FLOPER, we can associate (and visualize) a new lattice to a given program. As seen before, such lattices must be expressed by means of a set of Prolog clauses (defining predicates *member, top, bot, leq* and the ones associated to fuzzy connectives) in order to be loaded into FLOPER.

Although the core of our implementation is written with (fuzzy) MALP rules, we have reused/adapted several modules of our previous Prolog-based implementation of (crisp) XPath described in [ABE08, Alm09], which make use of the SWI-Prolog library for loading XML files in order to store each XML document by means of a Prolog term. For loading XML documents in our implementation we use the predicate `load_xml(+File,-Term)`, and similarly, we have a predicate `write_xml(+File,+Term)` for writing a data-term representing an XML document into a file. And, of course, the parser of our application has been extended to recognize the new keywords *DEEP, DOWN, avg*, etc... with the proper arguments.

Furthermore, we have incorporated a lattice definition according to the definitions of Section 4.1.2, which is shown in Figure 4.12, where *and*, *or* and *avg* operators are defined by Prolog rules. Figure 4.13 shows the Prolog compilation from FLOPER of the rule of Figure 4.8. Finally, we have defined a predicate `tv_to_elem` to show the result in a pretty way which transforms the returned *TV* tree to an XML tree.

4.3 XQuery Library FuzzyXPath

The implementation of our fuzzy extension of XPath is based on an XQuery library too, the functions including *deep* and *down* operators, as well as the fuzzy operators *and+*, *and-*, *and*, *or+*, *or-*, *or* and *avg*. Using this library the user can replace Boolean operators by fuzzy versions in XPath expressions, as well as he (she) can call to *deep* and *down* operators, in order to obtain ranked sets of answers. The answers are shown with a *Retrieval State Value (RSV)* representing the degree of satisfaction. The answers can also be ordered with respect to the RSV making use of *descending* XQuery expression, as well as filtered with regard to a *threshold*.

The implementation of our fuzzy extension of XPath is based on an XQuery library of functions including *deep* and *down* operators, as well as the fuzzy operators *and+*, *and-*, *and*, *or+*, *or-*, *or* and *avg*. Using this library the user can replace Boolean operators by fuzzy versions in XPath expressions, as well as he (she) can call to *deep* and *down* operators, in order to obtain ranked sets of answers. The

```
and_prod(tv(X1,X2),tv(Y1,Y2),tv(Z1,Z2)):−
                              pri_prod(X1,Y1,Z1),pri_app(X2,Y2,Z2).

and_luka(tv(X,Elem1),tv(Y,Elem2),tv(Z,Elem0)):−
              pri_add(X,Y,U1), pri_sub(U1,1,U2),
              pri_max(U2,0.0,Z), pri_app(Elem1, Elem2, Elem0).

and_godel(tv(X,Elem1),tv(Y,Elem2),tv(Z,Elem0)):−
                    pri_min(X,Y,Z), pri_app(Elem1,Elem2,Elem0).

or_prod(tv(X1,X2),tv(Y1,Y2),tv(Z1,Z2)):− pri_prod(X1,Y1,U1),
                              pri_add(X1,Y1,U2),pri_sub(U2,U1,Z1),
                              pri_app(X2,Y2,Z2).

or_luka(tv(X,Elem1),tv(Y,Elem2),tv(Z,Elem0)):−
                    pri_add(X,Y,U1), pri_min(U1,1,Z),
                    pri_app(Elem1,Elem2,Elem0).

or_godel(tv(X,Elem1),tv(Y,Elem2),tv(Z,Elem0)):−
                    pri_max(X,Y,Z), pri_app(Elem1,Elem2,Elem0).

agr_aver(tv(X1,X2),tv(Y1,Y2),tv(Z1,Z2)):− pri_add(X1,Y1,Aux),
                              pri_div(Aux,2,Z1),pri_app(X2,Y2,Z2).

agr_avg(tv(V1,E1),tv(V2,E2),tv(T1,F1),tv(T2,F2),tv(V3,G)):−
                    V3 is (V1∗T1 + V2∗T2)/(T1+T2),pri_app(E1,E2,A1),
                    pri_app(F1,F2,A2),pri_app(A1,A2,G).

pri_add(X,Y,Z) :− Z is X+Y. pri_sub(X,Y,Z) :−Z is X−Y.

pri_prod(X,Y,Z) :− Z is X ∗ Y. pri_div(X,Y,Z) :− Z is X/Y.

pri_app([],[],[]):−!.

pri_app([],A,A):−!.

pri_app(A,[],A):−!.

pri_app([Element,TV_Son,[]],TV_SibA,[Element,TV_Son,TV_SibA]):−!.

pri_app([Element,TV_Son,TV_sib],TV_SibA,[Element,TV_Son,TV_SibR]):−
                                pri_app(TV_sib,TV_SibA,[Element,TV_Son,TV_SibR]),!.
```

Figure 4.12: Multi-adjoint lattice for *FuzzyXPath* (file “tv.pl”)

answers are shown with a *Retrieval State Value (RSV)* representing the degree of satisfaction. The answers can also be ordered with respect to the RSV making use of *descending* XQuery expression, as well as filtered with regard to a *threshold*.

```
fuzzyXPath([Label|LabelRest],[element(Label,_,Children)|Siblings],Deep,Down,TV_Iam):−
        fuzzyXPath(LabelRest,Children,Deep,Down,TV_Son),
        and_prod(Deep,TV_Son,TV_Son0),
        fuzzyXPath([Label|LabelRest],Siblings,Deep,Down,TV_Bro),
        and_prod(Down,TV_Bro,TV_Bro0),
        agr_fuse(tv(1,[element(Label,Attr,Children),[],[]]),TV_Son0,TV_Bro0,TV_body),
        and_prod(tv(1,[]),TV_body,TV_Iam).
```

Figure 4.13: Prolog clause obtained by FLOPER after compiling a MALP rule

The input documents in our proposal are *crisp XML documents*, but the answers to a query offer fuzzy information, that is, a *RSV for each answer*. Therefore our approach is focused on the handling of standard XML documents, in which the user can retrieve information, ranked by a certain degree of satisfaction. We have decided to implement FuzzyXPath within XQuery by providing an XQuery library of fuzzy operators. It makes possible that our library can be used from any XQuery processor to query any XML document with crisp information.

Although the input of a query is a crisp XML document, the library assign internally and, in a transparent way to the user, a RSV to each of node of interest in the document. The RSVs assigned to each node of interest are used to compute the RSV of the answer. It makes the implementation a non-trivial task. Starting from a crisp XML document as input, our implementation annotates at run-time a RSV to each node of the query result. It also involves to dynamically annotate RSVs of nodes in subqueries. Additionally, *where* and *return* expressions of XQuery become XQuery functions in order to handle fuzzy conditions and RSVs, respectively.

We can summarize the elements of the implementation as follows:

4.3.1 Elements of the Library

1. The *deep* and *down* operators become XQuery functions that take as arguments a context node, an XPath expression and the value (a real number in [0,1]) assigned to deep and down, respectively. For combining deep and down an XQuery function is defined having as argument two real values in [0,1]:

```
declare function f:deep($node,$xpath,$deep)
declare function f:down($node,$xpath,$down)
declare function f:deep_down($node,$xpath,$deep,$down)
```

2. Fuzzy versions of Boolean operators *and*, *or* have been defined as XQuery functions, each one for each fuzzy logic we have considered (i.e., *Product*, *Łukasiewicz* and *Gödel*):

```
declare function f:andP($left,$right)
declare function f:orP($left,$right)
declare function f:andG($left,$right)
declare function f:orG($left,$right)
declare function f:andL($left,$right)
declare function f:orL($left,$right)
```

3. Operators *avg* and *avg{a,b}* have been defined as XQuery functions:

```
declare function f:avg($left,$right)
declare function f:avg_ab($left,$right,$a,$b)
```

4. Fuzzy versions of XQuery expressions *where* and *return* have been defined. In order to make transparent to the user the incorporation of RSVs, we have defined a new version of the *return* expression, called *returnF*, which transparently carries out the computation of the RSVs of the answers. Similarly, since XQuery works with a Boolean logic, the introduction of fuzzy versions of the operators, force us to define a new version of the *where* expression, called *whereF*, which transparently carries out the computation of the RSVs from fuzzy conditions. *ReturnF* has as parameters the context node and an XPath expression. *WhereF* has as parameters the context node and a fuzzy condition.

```
declare function f:whereF($node,$fuzzycond)
declare function f:returnF($node,$xpath)
```

5. Fuzzy versions of comparison operators for XPath expressions have been defined as XQuery functions. Similarly to *whereF*, comparison operators have been adapted to handle the RSVs:

```
declare function f:equalF($left,$right)
declare function f:lessF($left,$right)
declare function f:greaterF($left,$right)
```

4.3.2 Implementation of the Library

In order to implement our library in XQuery we have used the *XQuery Module* available in the *BaseX* processor [Grü14]. In particular, we make use of the function *eval* that makes possible the manipulation of XPath expressions. This function is also available for *Exist* [Mei03] and *Saxon* [KL90] processors. For instance, *down* is defined as follows:

```
declare function f:down($nodes,$query, $down){
  let $docDown := document{f:down_aux($nodes/∗,$down,(),())}
  let $docQ := xquery:eval(concat('$x',$query), map { '$x' :=$docDown})
  let $docL := xquery:eval(concat('$x',$query), map { '$x' :=$nodes})
  return f:putListRSV($docL,f:getListRSV($docQ))
};
```

deep is defined as follows:

```
declare function f:deep($doc as node()∗, $query, $deep as xs:double){
  let $docDeep := document{f:deep_aux($doc/∗,$deep,1)}
  let $docQ := xquery:eval(concat('$x',$query), map { '$x' :=$docDeep})
  let $docL := xquery:eval(concat('$x',$query), map { '$x' :=$doc})
  return f:putListRSV($docL,f:getListRSV($docQ))
};
```

and *deep_down* is defined as follows:

```
declare function f:deep_down($nodes as node()∗,$query, $deep as xs:double,
                                                        $down as xs:double){
  let $docDown := document{f:down_aux($nodes/∗,$down,(),())}
  let $docDeep := document{f:deep_aux($docDown/∗,$deep,1)}
  let $docQ := xquery:eval(concat('$x',$query), map { '$x' :=$docDeep})
  let $docL := xquery:eval(concat('$x',$query), map { '$x' :=$nodes})
  return f:putListRSV($docL,f:getListRSV($docQ))
};
```

Each fuzzy operator has been defined as a function, for instance, *and* (*Product* logic), *or+* (*Gödel* logic), *avg*, and *avg{a,b}* are defined as follows:

```
declare function f:andP($cond1,$cond2)
{
  let $tv1 := f:truthValue($cond1)
  let $tv2 := f:truthValue($cond2)
  return $tv1*$tv2
};
declare function f:orG($cond1,$cond2)
{
  let $tv1 := f:truthValue($cond1)
  let $tv2 := f:truthValue($cond2)
  return
    if ($tv1 > $tv2) then $tv1
    else $tv2
};
declare function f:avg($cond1,$cond2)
{
  let $tv1 := f:truthValue($cond1)
  let $tv2 := f:truthValue($cond2)
  return (xs:double($tv1)+xs:double($tv2)) div (2)
};

declare function f:avg_ab($cond1,$cond2, $a, $b)
{
  let $tv1 := f:truthValue($cond1)
  let $tv2 := f:truthValue($cond2)
  return (xs:double($tv1)*$a+xs:double($tv2)*$b) div ($a+$b)
};
```

4.3.3 Examples of FuzzyXPath in XQuery

Now, we show how the previous FuzzyXPath queries can be written in XQuery. Let us now suppose the following FuzzyXPath query:

$$<< /hotels/hotel[[DOWN = 0.9]close\_to/text() = \text{“}Sol\text{”}]/@name >>$$

We can now write the same query in XQuery as follows:

```
for $x in doc('hotels.xml')/hotels/hotel
let $y := f:whereF($x,f:equalF(f:down($x,'/close_to',0.9),'Sol'))
let $z := f:returnF($y,'/@name')
order by $y/@rsv descending
return $z
```

We can see that FuzzyXPath expressions are written as XQuery expressions. This is the same kind of transformation from crisp XPath to XQuery. For instance:

$$<< /hotels/hotel[close\_to/text() = \text{“}Sol\text{”}]/@name >$$

can be translated into:

```
for $x in doc("hotels.xml")/hotels/hotel
where $x/close_to/text()="Sol"
return $x/@name
```

In the fuzzy case, *“=”* is transformed into *equalF*, and *where* as well as *return* become XQuery functions, with an extra argument to represent the context node. The query makes use of the function *down* of the library to compute the RSVs associated to *close_ to*. In addition, the attribute *rsv*, which has been (internally) added to the output document, can be handled to show the answer in a sorted way, and even to define a threshold.

Let us now consider the following query, that uses *deep* and *down*:

$$<< /hotels/hotel[[DEEP = 0.5; DOWN = 0.9]//close\_to/text() = \text{“}Callao\text{”}]/@name >>$$

We can now write the same query in XQuery using the function *deep_ down*:

```
for $x in doc('hotels.xml')/hotels/hotel
let $y :=
  f:whereF($x, f:equalF(f:deep_down($x,'//close_to',0.5,0.9),'Callao'))
let $z := f:returnF($y,'/@name')
order by $y/@rsv descending
return $z
```

Let us now suppose the following FuzzyXPath expression that makes use of the *avg* operator.

$$<< //hotel[services/pool\ avg\ services/metro]/@name >>$$

Here, we use the function *avg* of the library, having as parameters both sides of the fuzzy condition:

```
for $x in doc('hotels.xml')//hotel
let $y := f:whereF($x, f:avg($x/services/pool,$x/services/metro))
let $z := f:returnF($y,'/@name')
order by $y/@rsv descending
return $z
```

The same can be said for the following query, using *avg{a,b}* having as parameters *a* and *b*.

$$<< //hotel[services/pool\ avg\{1,2\}\ services/metro]/@name >>$$

```
for $x in doc('hotels.xml')//hotel
let $y := f:whereF($x,f:avg_ab($x/services/pool, $x/services/metro,1,2))
let $z := f:returnF($y,'/@name')
order by $y/@rsv descending
return $z
```

Let us now suppose the following queries that combine *deep* and *avg*, and *deep* and *and+*, respectively:

$$<< //hotel[[DEEP = 0.8]//close\_to/text() = \text{“}Sol\text{”}\ avg\{1,2\}\ //price/text() < 150]/@name >>$$

```
for $x in doc('hotels.xml')//hotel
let $y := f:whereF($x, f:avg_ab(f:equalF(f:deep($x,'//close_to',0.8),'Sol'),
        $x//price/text()<150,1,2))
let $z := f:returnF($y,'/@name')
order by $y/@rsv descending
return $z
```

$$<< //hotel[([DEEP = 0.5]//close\_to/text() = "GranVia") \; and + (//pool \; avg\{3,2\} \; //metro/text() < 200)]/@name >>$$

```
for $x in doc('hotels.xml')//hotel
let $y := f:whereF($x,f:andG(f:equalF(f:deep($x,'//close_to',0.5),'GranVia'),
        f:avg_ab($x//pool,$x//metro<200,3,2)))
let $z := f:returnF($y,'/@name')
order by $y/@rsv descending
return $z
```

4.3.4 Benchmarks

Now, we would like to show the benchmarks we have obtained using our library. We have tested our library using data sets of different sizes. We have used as data sets traces of execution of MALP programs developed under our FLOPER tool. The FLOPER tool generates traces in XML format, with a high degree of tag nesting when a recursive program is executed. These data sets facilitate the testing of our structural based operators *deep* and *down*.

Query	*16Kb*	*700Kb*	*4.8Mb*	*15.4Mb*
Examined nodes in Q1	*28*	*148*	*298*	*448*
Examined nodes in Q2	*25*	*145*	*295*	*445*
Tree Depth	*21*	*101*	*201*	*301*
Q1	7.09 ms	25.47 ms	123.66 ms	461.6 ms
down in Q1	12.52 ms	107.1 ms	481.24 ms	2853.36 ms
deep in Q1	10.08 ms	74.17 ms	510.74 ms	1953.31 ms
deep and down in Q1	69.97 ms	102.0 ms	685.87 ms	7315.59 ms
Q2	5.77 ms	57.96 ms	172.03 ms	529.18 ms
avg in Q2	36.59 ms	1266.99 ms	9729.49 ms	60426.28 ms

Figure 4.14: Benchmarks

In Figure 4.14 we can see the results, where we indicate the number of nodes examined in each tree, as well as the depth of the tree. We have compared the execution times for two XPath expressions in crisp and fuzzy versions.

The first query is Q1:

$$<< //node/goal >>$$

and the second query is Q2:

$$<< //node[goal[contains(text(),"p(")] \text{ and } substitution[contains(text(),"g(")]]//goal >>$$

We have used the BaseX Query processor in a Intel Core 2 Duo 2.66 GHz Mac OS machine.

Chapter 5

The FuzzyXPath debugger

The XPath language [BBC$^+$07] was designed as a query language for XML in which the path of the tree is used to describe the query. In spite of the simplicity of the XPath language, the programmer usually makes mistakes when (s)he describes the path in which the data are allocated. Tipically, (s)he omits some of the tags of the path, s(he) adds more than necessary, and (s)he also uses similar but wrong tag names. When the query does not match to the tree structure of the XML tree, the answer is empty. However, we can also find the case in which the query matches to the XML tree but the answer does not satisfy the programmer. Due to the inherent flexibility of XML documents, the same tag can occur at several positions, and the programmer could find answers that do not correspond to her(is) expectations. In other words, (s)he finds a correct path, but a wrong answer. We can also consider the case in which a boolean condition is wrong, expressing a wrong range, and several conditions that do not hold at the same time. When the programmer does not find the answer (s)he is looking for, there is a mechanism that (s)he can try to debug the query. In XPath there exists an operator, denoted by '//', that permits to look for the tag from that position. However, it is useless when the tag is present at several positions, since even though the programmer finds answers, (s)he does not know whether they are close to h(er) expectations.

XPath debugging has to take into account the previous considerations. Particularly, there is an underlying notion of *chance degree.* When the programmer makes mistakes, the number of bugs can be higher or lower, and the chance degree is proportional to them. Moreover, there are several ways on which each bug can be solved, and therefore the chance degree is also dependent from the number of solutions for each bug, and the quality of each solution. The quality of a solution describes the number of changes to be made. Finally, there is a case in which we have also focused our work. The case in which the mistake comes from a similar but wrong tag. Here, the chance degree represents the semantic similarity between the tag expressed in the query and the tag which really appears in the XML document.

We will describe how we can manipulate an XPath expression in order to obtain a set of alternative XPath expressions that match to a given XML document. For each alternative XPath expression we will give a chance degree that represents the degree in which the expression deviates from the initial expression. Thus, our work is focused on providing the programmer a repertoire of paths that (s)he can use to retrieve answers.

We propose that XPath debugging is guided by the programmer that initially establishes a value (a real value between 0 and 1), that the debugger uses to penalize each bug. Each bug found is penalized with such a value, and thus the chance degree is proportional to the value. Additionally, we assume that the debugger is equipped with a table of similarities, that is, a table in which pairs of similar words are assigned to a value between 0 and 1. It makes possible that chance degree is also computed from similarity degrees. The debugger reports a set of annotated paths in an extended XPath syntax in which we have incorporated three annotations: *JUMP*, *SWAP* and *DELETE*. *JUMP* is used to represent that some tags have been added to the original expression, *SWAP* is used to represent that a tag has been changed by another, and *DELETE* is used to represent that a tag has been removed. The reported XPath expressions update the original XPath expression, that is, the case *JUMP* incorporates '//' at the position in which the bug is found, the case *SWAP* includes the new tag, and the case *DELETE* removes the wrong tag.

Additionally, our proposal permits the programmer tests the reported XPath expressions. The annotated XPath expressions can be executed obtaining a ranked set of answers with respect to the chance degree. It facilitates the process of debugging because the programmer can visualize the answers to each query.

The approach has been implemented and tested (see `http://dectau.uclm.es/fuzzyXPath/?q=DebuggerXPathTest`). Additionally, our proposal permits the programmer to test the reported XPath expressions. The annotated XPath expressions can be executed in our tool (`http://dectau.uclm.es/fuzzyXPath/`) in order to obtain a ranked set of answers w.r.t. the chance degree. It facilitates the process of debugging because programmers can visualize answers to each query in a very easy way. Our implementation has been developed on top of the recently proposed FuzzyXPath extension [ALM11a, ALM12c], which uses *fuzzy logic programming* to provide a fuzzy taste to XPath expressions. The implementation has been coded with the fuzzy logic programming language MALP and developed with the FLOPER tool designed in our research group and freely accessible from `http://dectau.uclm.es/floper/`.

Although our approach can be applied to standard (crisp) XPath expressions, chance degrees in XPath debugging fits well with our proposal. Particularly, XPath debugging annotations can be seen as annotations of XPath expressions similary to the proposed *DEEP* and *DOWN* of FuzzyXPath [ALM11a, ALM12c]. *DEEP* and *DOWN* serve to annotate XPath expressions and to obtain a ranked set of answers depending on they occur, more deeply and from top to down. Each answer is annotated with a *RSV (Retrieval Status Value)* which describes the degree of satisfaction of the answer. Here *JUMP*, *SWAP* and *DELETE* penalize the answers of annotated XPath expressions. *DEEP* and *JUMP* have, in fact, the same behavior: *JUMP* proportionally penalizes answers as deep as they occur. Moreover, in order to cover with *SWAP*, we have incorporated to our framework similarity degrees.

5.1 Debugging XPath

In this section we propose a debugging technique for XPath expressions. Our debugging process accepts as inputs a query Q preceded by the $[DEBUG = r]$ command, where r is a real number in the unit interval. For instance,

$$<< [DEBUG = 0.5]/bib/book/title >>$$

Assuming an input XML document like the one depicted in Figures 5.2 and 5.1, the debugging produces a set of alternative queries $Q_1, ..., Q_n$ packed into an output XML document, like the one shown in Figure 5.3. The document has the following structure:

```
<bib>
    <name>Classic Literature</name>
    <book year="2001" price="45.95">
        <title>Don Quijote de la Mancha</title>
        <author>Miguel de Cervantes Saavedra</author>
        <references>
            <novel year="1997" price="35.99">
                <name>La Galatea</name>
                <author>Miguel de Cervantes Saavedra</author>
                <references>
                    <book year="1994" price="25.99">
                        <title>Los trabajos de Persiles y Sigismunda</title>
                        <author>Miguel de Cervantes Saavedra</author>
                    </book>
                </references>
            </novel>
        </references>
    </book>
    <novel year="1999" price="25.65">
        <title>La Celestina</title>
        <author>Fernando de Rojas</author>
    </novel>
</bib>
```

Figure 5.1: Input XML document in our examples with Debug

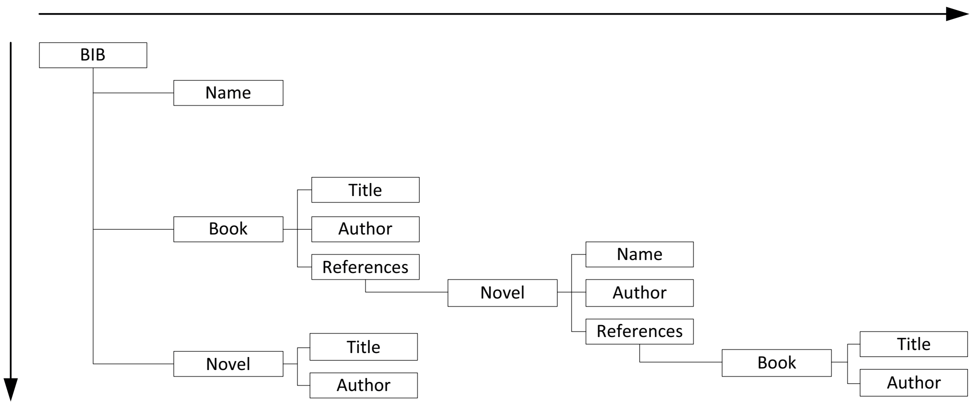


Figure 5.2: XML skeleton represented as a tree

<result>
<query cd="r_1" *attributes*$_1$> Q_1 </query>
. . .
<query cd="r_n" *attributes*$_n$> Q_n </query>
</result>

where the set of alternatives is ordered with respect to the CD key. This value measures the chance degree of the original query with respect to the new one, in the sense that as much changes are performed on Q_i and as more *traumatic* they are with respect to Q, then the CD value becomes lower.

```
<result>
  <query cd="1.0">/bib/book/title</query>
  <query cd="0.8" book="novel">/bib/[SWAP=0.8]novel/title</query>
  <query cd="0.5" book="//">/bib/[JUMP=0.5]//title</query>
  <query cd="0.5" bib="//">/[JUMP=0.5]//book/title</query>
  <query cd="0.45" book="" title="name">/bib/[DELETE=0.5][SWAP=0.9]name</query>
  <query cd="0.4" bib="//" book="novel">/[JUMP=0.5]//[SWAP=0.8]novel/title</query>
  <query cd="0.25" book="" title="//">/bib/[DELETE=0.5][JUMP=0.5]//title</query>
  <query cd="0.25" book="//" book="">/bib/[JUMP=0.5]//[DELETE=0.5]title</query>
  <query cd="0.25" bib="" book="//">/[DELETE=0.5][JUMP=0.5]//book/title</query>
  <query cd="0.25" bib="//" book="//">/[JUMP=0.5]//[JUMP=0.5]//title</query>
  <query cd="0.25" bib="//" bib="">/[JUMP=0.5]//[DELETE=0.5]book/title</query>
  <query cd="0.225" title="//" title="//" title="name">/bib/book/[JUMP=0.5]//[JUMP=0.5]//[
       SWAP=0.9]name</query>
  <query cd="0.225" bib="" book="//" title="name">/[DELETE=0.5][JUMP=0.5]//[SWAP=0.9]
       name</query>
  <query cd="0.225" bib="//" book="" title="name">/[JUMP=0.5]//[DELETE=0.5][SWAP=0.9]
       name</query>
  <query cd="0.2" bib="" book="//" book="novel">/[DELETE=0.5][JUMP=0.5]//[SWAP=0.8]
       novel/title</query>
   .........
</result>
```

Figure 5.3: Debugging of the query «[DEBUG=0.5]/bib/book/title»

In Figure 5.3, the first alternative, with the highest CD, is just the original query, thus, the CD is *1*, whose further execution with FuzzyXPath returns «Don Quijote de La Mancha». As was commented before, we have assumed the debugger is ran even when the set of answers is not empty, like in this case. The remaining options give different CD's depending on the chance degree, and provide XPath expressions annotated with *JUMP*, *DELETE* and *SWAP* commands.

In order to explain the way in which our technique generates the attributes and content of each *query* tag in the output XML document, let us consider a generic path Q of the form:

$$<< [DEBUG = r]/tag_1/.../tag_i/tag_{i+1}/... >>$$

where we say that tag_i is at level i in the original query. So, assume that during the exploration of the input query Q and the XML document D, we find that tag_i in Q does not occurs at level i in (a branch of) D. Then, we consider the following three situations:

Swapping case: Instead of tag_i, we find tag'_i at level i in the input XML document D, being tag_i and tag'_i two similar terms with similarity degree s. Then, we generate an alternative query by adding the attribute tag_i ="tag'_i" and replacing in the original path the occurrence "$tag_i/$" by "$[SWAP = s]tag'_i/$".
The second query proposed in Figure 5.3 illustrates this case:

$$< query\ cd = \text{“0.8”}\ book = \text{“}novel\text{”} > /bib/[SWAP = 0.8]novel/title < /query >$$

Let us observe that : 1) we have included the attribute «book=“novel"» in order to suggest that instead of looking now for a *book*, finding a *novel* should be also a good alternative, 2) in the path we have replaced the tag *book* by *novel* and we have appropriately annotated the exact place where the change has been performed with the annotation $[SWAP = 0.8]$ and 3) the CD of the new query has been adjusted with the *similarity degree* 0.8 of the exchanged tags.

Now, we can run the (fuzzy) XPath queries «/bib/novel/title» and even «$/bib/[SWAP = 0.8]novel/title$» (see Figure 5.4). In both cases we obtain the same result, i.e., «La Celestina», but with different RSV (or *Retrieval Status Value*): 1 and 0.8, respectively.

```
<result>
  <title rsv="0.8">La Celestina</title>
</result>
```

Figure 5.4: Execution of the query «$/bib/[SWAP = 0.8]novel/title$»

Jumping case: Even when tag_i is not found at level i in the input XML document D, tag_{i+1} appears at a deeper level (i.e., greater than i) in a branch of D. Then, we generate an alternative query by adding the attribute tag_i="//", which means that tag_i has been jumped, and replacing in the path the occurrence "*tag_i/*" by "$[JUMP = r]//$", being r the value associated to *DEBUG*.

```
<result>
  <title rsv="0.5">Don Quijote de la Mancha</title>
  <title rsv="0.5">La Celestina</title>
  <title rsv="0.03125">Los trabajos de Persiles y Sigismunda</title>
</result>
```

Figure 5.5: Execution of the query «*/bib/*$[JUMP = 0.5]//$*title*»

```
<result>
  <title rsv="0.5">Don Quijote de la Mancha</title>
  <title rsv="0.03125">Los trabajos de Persiles y Sigismunda</title>
</result>
```

Figure 5.6: Execution of the query «$/[JUMP = 0.5]//$*book/title*»

This situation is illustrated by the third and fourth queries in Figure 5.3, where we propose to jump the tags *book* and *bib*. The execution of the queries returns different results, as shown in Figures 5.5 and 5.6, where *JUMP* produces similar effects to the DEEP command explained in the previous section, that is, as more tags are jumped their resulting CD's become lower.

Deletion case: This situation emerges when at level i in the input XML document D, we found tag_{i+1} instead of tag_i. So, the intuition tell us that tag_i should be removed from the original query Q and hence, we generate an alternative query by adding the attribute tag_i="" and replacing in the path the occurrence "*tag_i*" by "$[DELETE = r]$", being r the value associated to *DEBUG*.

This situation is illustrated by the fifth query in Figure 5.3, where the deletion of the tag *book* is followed by a swapping of similar tags *title* and *name*. The CD *0.45* associated to this query is defined as the *product* of the values associated to both *DELETE* (*0.5*) and *SWAP* (*0.9*), and hence the chance degree of the original one is lower than the previous examples.

As seen in Figure 5.7, the execution of our new query is able to retrieve the information contained in the first branch of the input XML document listed in Figures 5.2 and 5.1. Here we illustrate that execution of debugged XPath expressions reveals hidden answers that can fulfill the programmer expectations.

```
<result>
  <name rsv="0.45">Classic Literature</name>
</result>
```

Figure 5.7: Execution of the query «$/bib/[DELETE = 0.5][SWAP = 0.9]name$»

As we have seen in the previous example, the combined use of one or more debugging commands (*SWAP*, *JUMP* and *DELETE*) is not only allowed but also frequent. In other words, it is possible to find several debugging points. In Figure 5.8, we can see the execution of the query:

$$< query\ cd = \text{“0.225”}\ bib = \text{“”}\ book = \text{“//”}\ title = \text{“}name\text{”} >$$
$$/[DELETE = 0.5][JUMP = 0.5]//[SWAP = 0.9]name < /query >$$

The CD *0.225* is quite low, and therefore the chance degree is low, since it has been obtained by multiplying the three values associated to the deletion of the tag *bib* (*0.5*), jumping the tag *book* (*0.5*) and the swapping of *title* by *name* (*0.9*).

```
<result>
  <name rsv="0.225">Classic Literature</name>
  <name rsv="0.028125">La Galatea</name>
</result>
```

Figure 5.8: Execution of the query «$/[DELETE = 0.5][JUMP = 0.5]//[SWAP = 0.9]name$»

The wide range of alternatives (Figure 5.3 is still incomplete), reveals the flexibility of our technique. The programmer is free to use the alternative queries to execute them, and to inspect results up to the expectations are covered.

Finally, we would like to remark that even when we have worked with a very simple query with three tags in our examples, our technique works with more complex queries with large paths and connectives in boolean conditions, as well as *DEBUG* used in several places on the query.

For instance, in Figure 5.9 (compare it with Figure 5.3) we show the result of debugging the following query:

$$«[DEBUG = 0.7]/bib/[DEBUG = 0.6]book/[DEBUG = 0.5]title»$$

```
<result>
  <query cd="1.0">/bib/book/title</query>
  <query cd="0.8" book="novel">/bib/[SWAP=0.8]novel/title</query>
  <query cd="0.7" bib="//">/[JUMP=0.7]//book/title</query>
  <query cd="0.6" book="//">/bib/[JUMP=0.6]//title</query>
  <query cd="0.56" bib="//" book="novel">/[JUMP=0.7]//[SWAP=0.8]novel/title</query>
  <query cd="0.54" book="" title="name">/bib/[DELETE=0.6][SWAP=0.9]name</query>
  <query cd="0.42" bib="" book="//">/[DELETE=0.7][JUMP=0.6]//book/title</query>
  <query cd="0.42" bib="//" book="//">/[JUMP=0.7]//[JUMP=0.6]//title</query>
  <query cd="0.378" bib="" book="//" title="name">
                      /[DELETE=0.7][JUMP=0.6]//[SWAP=0.9]name</query>
  <query cd="0.378" bib="//" book="" title="name">
                      /[JUMP=0.7]//[DELETE=0.6][SWAP=0.9]name</query>
  <query cd="0.336" bib="" book="//" book="novel">
                      /[DELETE=0.7][JUMP=0.6]//[SWAP=0.8]novel/title</query>
  <query cd="0.3" book="" title="//">/bib/[DELETE=0.6][JUMP=0.5]//title</query>
  <query cd="0.2646" bib="//" bib="" book="" title="name">
                      /[JUMP=0.7]//[DELETE=0.7][DELETE=0.6][SWAP=0.9]name</query>
  .........
</result>
```

Figure 5.9: Debugging of the query $«[DEBUG = 0.7]/bib/[DEBUG = 0.6]book/[DEBUG = 0.5]title»$

5.2 MALP and the XPath Debugger

The core of our debugger is coded with MALP rules by reusing most modules of our FuzzyXPath interpreter [ALM11a, ALM11b]. And, of course, the parser of our debugger has been extended to recognize the new keywords *DEBUG*, *SWAP*, *DELETE* and *JUMP*, with their proper arguments.

Now, we would like to show how the new "*XPath debugging*" predicate admits an elegant definition by means of fuzzy MALP rules. Each rule defining predicate:

$$debugQuery(ListXPath, Tree, Penalty)$$

receives three arguments: (1) `ListXPath` is the Prolog representation of an XPath expression, (2) `Tree` is the term representing an input XML document and (3)

`Penalty` represents the *chance degree*. A call to this predicate returns a truth-value (i.e., a *tv tree*) like the following one:

```
tv(1.0,[[/,[]],
    tv(1.0, [[tag(bib),[]],
        tv(1.0,[[tag(book),[]],
            tv(1.0,[[tag(title),[]],[],
            ...
        tv(0.8,[[tag(novel),[book=novel]],
        ...
        tv(0.5,[['[DELETE=0.5]',[book='']],
            tv(0.9,[[tag(name),[title=name]],[],
            ...
            []])]),
        []])]),
    []])])])]),
[]])
```

Basically, the *debugQuery* predicate traverses the XML document, checking the validity of the original query *tag-by-tag*, and trying to match each *tag* to the ones occurring in the XML document and thus producing effects of *DELETE*, *JUMP* and *SWAP*.

The definition of such predicate includes several rules for distinguishing the three debugging cases. As an example the case of swapping is defined as follows:

```
debugQuery([Label|LabelRest],[element(Label2,_,Children)|Siblings],Penalty) <prod
        @fuse(
                similarity(Label, Label2),
                debugQuery(LabelRest, Children, Penalty),
                debugQuery([Label|LabelRest], Siblings, Penalty)
        ) with tv(1,[]).
```

which becomes into the following Prolog clause after compilation into FLOPER:

```
debugQuery([Label|LabelRest],[element(Label2,_,Children)|Siblings], Penalty, TV_Iam):-
        similarity(Label, Label2, TV_Similarity),
        debugQuery(LabelRest, Children, Penalty, TV_Son),
        debugQuery([Label|LabelRest], Siblings, Penalty, TV_Sib),
        agr_fuse(TV_Similarity, TV_Son, TV_Sib, TV_Iam).
```

Basically, similarity is checked with the atom *similarity(Label, Label2)*, and two recursive calls are achieved for debugging both children (*debugQuery(LabelRest, Children, Penalty)*) and siblings (*debugQuery([Label|LabelRest], Siblings, Penalty, 1)*), whose *tv trees* are finally combined (*fused*) with the node content.

Finally, for showing the result in a pretty way (see Figures 5.3), and transforming a *tv tree* into an XML file, a predicate *tv_to_element* has been implemented.

Records	**FILTER** (FuzzyXPath interpreter)								
	0.1	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**
1000	1.766	1.696	1.734	0.842	0.469	0.268	0.221	0.087	0.056
2000	6.628	6.432	6.998	3.242	1.439	0.677	0.599	0.168	0.122
3000	14.532	14.02	14.05	6.306	2.831	1.257	1.101	0.253	0.179
4000	25.535	24.68	24.72	10.88	4.827	1.918	1.794	0.345	0.242
5000	41.522	37.78	37.16	16.20	7.242	2.993	2.516	0.427	0.281
6000	58.905	55.35	55.59	24.41	10.993	4.207	3.554	0.554	0.373
7000	85.167	85.65	82.73	37.74	14.436	5.083	4.653	0.649	0.460
8000	137.73	102.8	102.7	69.40	26.680	8.273	5.894	0.690	0.481
9000	175.27	131.8	131.0	56.93	22.601	7.869	7.329	0.824	0.549
10000	195.61	185.2	167.6	95.28	26.649	9.516	9.595	0.973	0.742

Records	**FILTER** (FuzzyXPath debugger)								
	0.1	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**
1000	2.857	0.443	0.341	0.381	0.340	0.386	0.349	0.394	0.295
2000	5.833	0.951	0.777	0.794	0.707	0.827	0.716	0.803	0.596
3000	9.422	1.411	1.059	1.243	1.053	1.251	1.100	1.233	0.881
4000	11.742	1.800	1.405	1.597	1.422	1.592	1.463	1.595	1.202
5000	15.646	2.466	1.735	1.786	1.931	1.758	2.143	1.771	1.500
6000	19.315	2.723	2.115	2.522	2.111	2.540	2.112	2.500	1.802
7000	22.599	3.397	2.505	3.025	2.475	2.468	2.783	2.442	2.561
8000	24.234	3.595	2.852	3.115	2.836	3.173	2.857	3.219	2.374
9000	30.305	3.137	4.212	3.184	5.072	3.169	3.174	3.811	2.675
10000	33.329	4.942	3.543	3.573	3.878	3.559	4.211	3.518	2.962

Figure 5.10: Performance of the FuzzyXPathinterpreter (up) and debugger (down) by using FILTER on XML files with growing sizes

5.3 Dynamic Filters for the Thresholded Debugging of Queries

In [JMMO10, JMM+13] we have reported some *thresholding* techniques specially tailored for the MALP language, where the main idea consists in to dynamically create and evaluate filters for prematurely disregarding those superfluous computations leading to non-significant solutions. Somehow inspired by the same guidelines, in [ALM14a] we have recently equipped our FuzzyXPathinterpreter with a new command with syntax «[FILTER=r]» (being r a real number between 0 and 1) which can be used just at the beginning of a query for indicating that only those answers with RSV greater of equal than r must be generated and reported. In the present

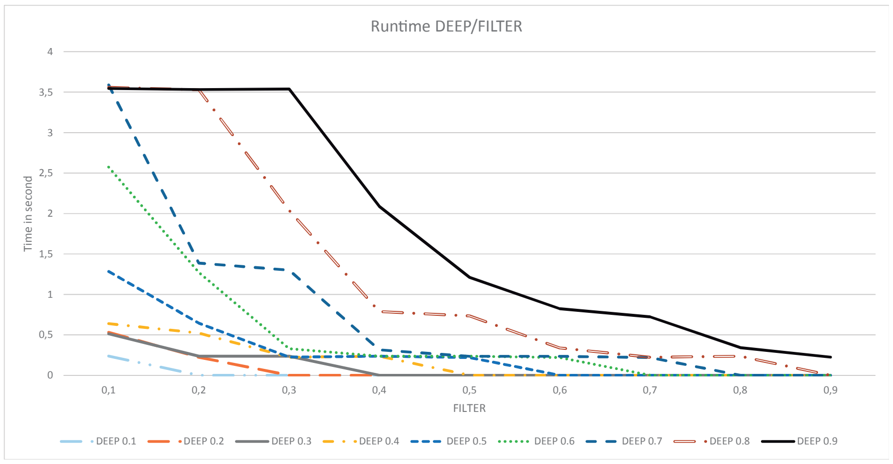


Figure 5.11: Runtime for the execution of several FuzzyXPathqueries varying DEEP and FILTER

work, we show the benefits of using the same command that we have just implemented into the FuzzyXPathdebugger too (now, only alternative queries to a given one whose CDs are greater of equal than r must be generated during the debugging process).

So, if we execute a FuzzyXPathquery with the following form «[FILTER=0.4]//book[@year<2000 avg @price<50]/title», we obtain nine answers, but only five if we fix «[FILTER=0.8]». Obviously, we would hope that the runtime of the second case should be lower than the first one since, as our approach does, there is no need for computing all solutions and then filtering the best ones. This desired dynamic behaviour when avoiding useless computations is reflected in Figure 5.10 which considers the effort needed for executing (up) and debugging[1] (down) a query like «[FILTER=*r*]//book[(@price>25 and @price<30) avg (@year<2000 or @year>2006)]» where each row represents the size of several XML files accomplishing with the same structure of our running example (but considering different nesting levels of tags *book*, *title*, *author* and *references*), and each column refers to a different degree of the FILTER command. Here, the runtime is measured in seconds excluding the extra parsing/compiling time (the benchmarks have been performed using a computer with processor Intel Core Duo, with 2 GB RAM and Windows Vista) and each record in

[1]In this last case we have used «[DEBUG=0.9]» just after «FILTER».

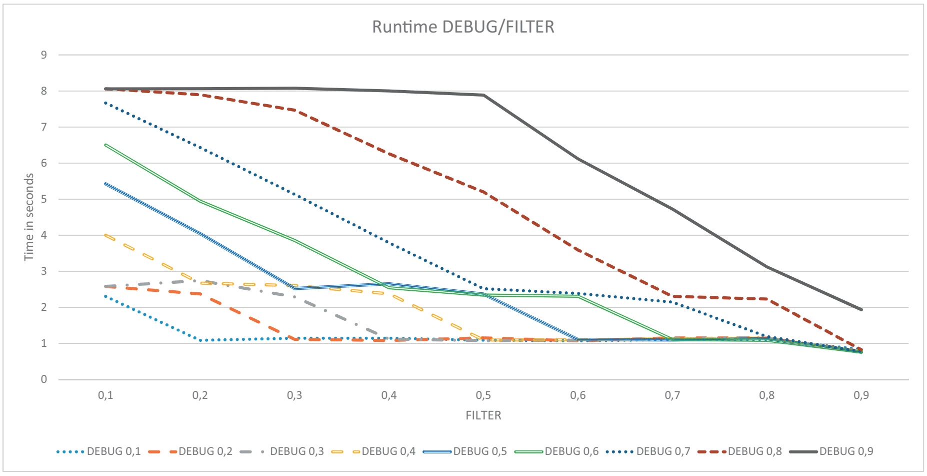


Figure 5.12: Runtime for the debugging of several FuzzyXPathqueries varying `DEBUG` and `FILTER`

the input file refers to a different *book* (that is, the number of records coincides with the number of occurrences of tag *book*) which might contain other books inside its *references* tag. The size of the files in the figure moves from 323Kb (1000 records) till 3223 Kb (10000 records), but our application works fine even when files of 33Mb (100000 records), which reveals the interest of our results.

Moreover, in Figure 5.11 we continue with a similar query to the previous one, but also considering the `DEEP` command[2]. Here, for a large XML document with a fixed size, we express the number of seconds needed for executing such query when varying `FILTER` and `DEEP`, where it is easy to see that the behaviour of the interpreter is more and more improved whenever `FILTER` grows and `DEEP` decreases, as wanted. On the other hand, Figure 5.12 reflects similar effects that occur in our debugger, where the contrast now is established between the `FILTER` and `DEBUG` commands.

Although the core of our application is written with (fuzzy) MALP rules, our implementation is based on the following items:

[2]This kind of statistics can be produced on-line for several XML files and FuzzyXPathqueries via the following couple of URLs that we have just prepared for the interested reader: `http://dectau.uclm.es/fuzzyXPath/?q=FuzzyXPathStatistics` and `http://dectau.uclm.es/fuzzyXPath/?q=DebugStatistics` (regarding the interpreter and the debugger, respectively).

(1) We have reused/adapted several modules of our previous PROLOG-based implementation of (crisp) XPath described in [Alm09, ABE08].

(2) We have used the SWI-PROLOG library for loading XML files, in order to represent a XML document by means of a PROLOG term (the notion of *term*, i.e. data structure, is just the same in MALP and PROLOG).

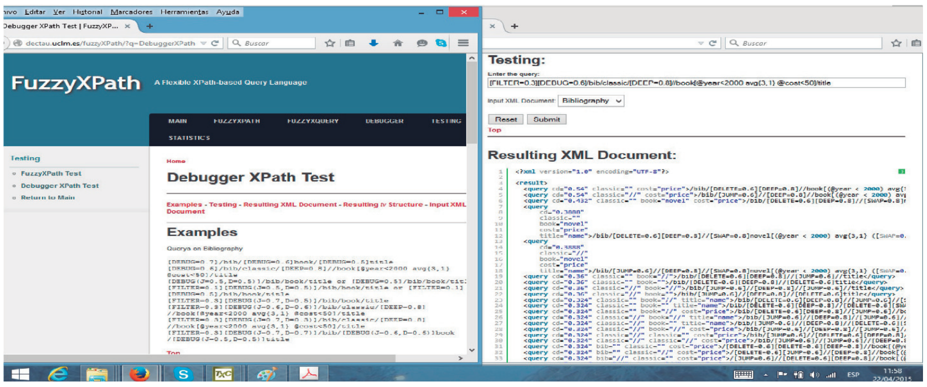
(3) The parser of XPath has been extended to recognize the new keywords `FILTER`, `DEBUG`, `DEEP`, `DOWN`, `JUMP`, `avg`, etc... with their proper arguments.

(4) Each tag is represented as a data-term of the form: *element(Tag,Attributes,Subelements)*, where *Tag* is the name of the XML tag, *Attributes* is a PROLOG list containing the attributes, and *Subelements* is a PROLOG list containing the sub-elements (i.e. sub-trees) of the tag. For instance, the XML document used in our examples is represented in SWI-PROLOG like:

```
[ element(bib,[],
    [ element(name,[],['Classic Literature']),
      element(book,[year='2001',price='45.95'],
        [ element(title,[],['Don Quijote de la Mancha']),
          element(author,[],['Miguel de Cervantes Saavedra']),
          element(references,[],
              [ element(novel,[year='1997',price='35.99'],
                    [ element(name,[],['La Galatea']),
                      element(author,[],[ 'Miguel de Cervantes Saavedra']),
                      element(references,[],
                          [ element(book,[ year = '1994', price = '25.99'],
                                [ element(title,[],[ 'Los trabajos de Persiles y Sigismunda']),
                                  element(author,[],[ 'Miguel de Cervantes Saavedra'])
                                ])
                          ])
                    ])
              ])
        ]),
      element(novel, [year='1999',price='25.65'],
        [ element(title,[],['La Celestina']),
          element(author,[],['Fernando de Rojas'])
        ])
    ])
]
```

Loading of documents is achieved by predicate *load_xml(+File,-Term)* and writing by predicate *write_xml(+File,+Term)*.

(5) Predicate *fuzzyXPath(+ListXPath,+Tree,+Deep, +Down,+Filter,+Accum)* receives six arguments: (1) *ListXPath* is the PROLOG representation of a XPath


nivo Editar Ver Historial Marcadores Herramientas Ayuda
Debugger XPath Test | FuzzyXP...
dectau.uclm.es/fuzzyXPath/?q=DebuggerXPath
Buscar
FuzzyXPath
A Flexible XPath-based Query Language
MAIN
FUZZYXPATH
FUZZYXQUERY
DEBUGGER
TESTING
STATISTICS
Testing
FuzzyXPath Test
Debugger XPath Test
Return to Main
Home
Reset
Submit
Top
ESP
11:58
22/04/2015

Debugger XPath Test

Examples - Testing - Resulting XML Document - Resulting *tv* Structure - Input XML Document

Examples

Querys on Bibliography

```
[DEBUG=0.7]/bib/[DEBUG=0.6]book/[DEBUG=0.5]title
[DEBUG=0.6]/bib/classic/[DEEP=0.8]//book[@year<2000 avg{3,1}
@cost<50]/title
[DEBUG(J=0.5,D=0.5)]/bib/book/title or [DEBUG=0.5]/bib/book/tit
[FILTER=0.1][DEBUG(J=0.5,D=0.5)]/bib/book/title or [FILTER=0.1]
[DEBUG=0.5]/bib/book/title
[FILTER=0.3][DEBUG(J=0.7,D=0.5)]/bib/book/title
[FILTER=0.3][DEBUG(J=0.6,D=0.6)]/bib/classic/[DEEP=0.8]
//book[@year<2000 avg{3,1} @cost<50]/title
[FILTER=0.3][DEBUG(J=0.7,D=0.3)]/bib/classic/[DEEP=0.8]
//book[@year<2000 avg{3,1} @cost<50]/title
[FILTER=0.3][DEBUG(J=0.7,D=0.7)]/bib/[DEBUG(J=0.6,D=0.5)]book
/[DEBUG(J=0.5,D=0.5)]title
```

Testing:

Enter the query:
[FILTER=0.3][DEBUG=0.6]/bib/classic/[DEEP=0.8]//book[@year<2000 avg{3,1} @cost<50]/title

Input XML Document: Bibliography

Resulting XML Document:

```
<?xml version="1.0" encoding="UTF-8"?>

<result>
  <query cd="0.54" classic="" cost="price">/bib/[DELETE=0.6][DEEP=0.8]//book[(@year < 2000) avg{
  <query cd="0.54" classic="//" cost="price">/bib/[JUMP=0.6]//[DEEP=0.8]//book[(@year < 2000) av
  <query cd="0.432" classic="" book="novel" cost="price">/bib/[DELETE=0.6][DEEP=0.8]//[SWAP=0.8]
  <query
      cd="0.3888"
      classic=""
      book="novel"
      cost="price"
      title="name">/bib/[DELETE=0.6][DEEP=0.8]//[SWAP=0.8]novel[(@year < 2000) avg{3,1} ([SWAP=0.
  <query
      cd="0.3888"
      classic="//"
      book="novel"
      cost="price"
      title="name">/bib/[JUMP=0.6]//[DEEP=0.8]//[SWAP=0.8]novel[(@year < 2000) avg{3,1} ([SWAP=0.
  <query cd="0.36" classic="" book="//">/bib/[DELETE=0.6][DEEP=0.8]//[JUMP=0.6]//title</query>
  <query cd="0.36" classic="" book="">/bib/[DELETE=0.6][DEEP=0.8]//[DELETE=0.6]title</query>
  <query cd="0.36" classic="//" book="//">/bib/[JUMP=0.6]//[DEEP=0.8]//[JUMP=0.6]//title</query>
  <query cd="0.36" classic="//" book="">/bib/[JUMP=0.6]//[DEEP=0.8]//[DELETE=0.6]title</query>
  <query cd="0.324" classic="" book="//" title="name">/bib/[DELETE=0.6][DEEP=0.8]//[JUMP=0.6]//[
  <query cd="0.324" classic="" book="" title="name">/bib/[DELETE=0.6][DEEP=0.8]//[DELETE=0.6][SW
  <query cd="0.324" classic="" book="//" cost="price">/bib/[DELETE=0.6][DEEP=0.8]//[JUMP=0.6]//b
  <query cd="0.324" classic="//" book="//" title="name">/bib/[JUMP=0.6]//[DEEP=0.8]//[JUMP=0.6]/
  <query cd="0.324" classic="//" book="" title="name">/bib/[JUMP=0.6]//[DEEP=0.8]//[DELETE=0.6][
  <query cd="0.324" classic="//" book="//" cost="price">/bib/[JUMP=0.6]//[DEEP=0.8]//[JUMP=0.6]/
  <query cd="0.324" classic="//" classic="" cost="price">/bib/[JUMP=0.6]//[DELETE=0.6][DEEP=0.8]
  <query cd="0.324" classic="//" classic="//" cost="price">/bib/[JUMP=0.6]//[JUMP=0.6]//[DEEP=0.
  <query cd="0.324" bib="" classic="" cost="price">/[DELETE=0.6][DELETE=0.6][DEEP=0.8]//book[(@y
  <query cd="0.324" bib="" classic="//" cost="price">/[DELETE=0.6][JUMP=0.6]//[DEEP=0.8]//book[(
  <query cd="0.324" bib="//" classic="" cost="price">/[JUMP=0.6]//[DELETE=0.6][DEEP=0.8]//book[(
```

Figure 5.13: An on-line session with the FUZZYXPATHdebugger

expression; (2) *Tree* is the term representing an input XML document; (3) *Deep/Down/Filter* have the obvious meaning, and finally (4) the last argument *Accum* (which is appropriately updated -maybe decreased- when going deeper in the exploration of the file) accumulates the sequence of penalties produced till reaching a concrete node, and it is very useful for deciding when performing a recursive call to the children of such node whenever the value of *Accum* is better than the one fixed by *Filter*. These actions also directly revert on the new predicate *debugQuery (+ListXPath,+Tree,+Debug,+Filter,+Accum)*, which implements the ideas described in this work and where once again the appropriate management of the two last arguments becomes the keypoint of our dynamic thresholding technique. Both the interpreter and the debugger can be tested on-line via `http://dectau.uclm.es/fuzzyXPath/?q=FuzzyXPathTest` and `http://dectau.uclm.es/fuzzyXPath/?q=Debugger` `XPathTest`, respectively (see Figure 5.13).

(6) The evaluation of the query generates a *truth value* which has the form of a tree, called *tv tree*. The main power of a fuzzy logic programming language like MALP w.r.t. PROLOG, is that instead of answering questions with a simple *true/false* value, solutions are reported in a much more tinged, documented way. Basically, the *fuzzyXPath* predicate traverses the PROLOG tree representing a XML document annotating into the *tv tree* the corresponding `DEEP/DOWN` values according to the movements performed in the horizontal and vertical axis, respectively. In addition, the *tv tree* is annotated with the values of *and*,

or and *avg* operators in each node. For instance, the evaluation of the query «/bib[DEEP=0.8]//book[@year<2000 avg{3,1} @price<50]/title» generates the following tv:

```
tv(1.0,[[],
    tv(1.0,[[tv(0.25, [])],
        tv(1.0,[element(title, [], [Don Quijote de la Mancha]), [], []]),
    tv(1.0,[[],
        tv(0.8,[[],
            tv(0.8,[[],
                tv(0.8,[[],
                    tv(0.8,[[tv(1.0, [])],
                        tv(1.0,[element(title, [], [Los trabajos de ...
                ....
```

For the case of the *debugQuery* predicate, it explores the XML tree in a very similar way than the interpreter, but now the annotations performed on the resulting *tv tree* refer to the corresponding JUMP, DELETE and SWAP values. For instance, the *tv tree* associated to query «[DEBUG=0.5]//book/title» starts as:

```
tv(1.0,[[//, []],
    tv(0.0,[[tag(noExist), []], [],
    tv(0.5,[[[DELETE=0.5], [book=]],
        tv(0.0,[[tag(noExist), []], [],
        tv(0.0,[[tag(noExist), []], [],
        tv(0.0,[[tag(noExist), []], [],
        tv(0.5,[[[JUMP=0.5]//, [title= //]],
            tv(0.9,[[tag(name), [title=name]], [],
            tv(0.0,[[tag(noExist), []], [],
            tv(0.0,[[tag(noExist), []], [], ...
                ....
```

(7) Finally, the *tv tree* is used for computing the output of the query, by multiplying the recorded values. A predicate called *tv_ to_ elem* has been implemented to output the answers in a sorted, pretty way.

Chapter 6

Applications

In this chapter we focus on the ability of FuzzyXPath for exploring derivation trees generated by FLOPER once they are exported in XML format, which somehow serves as a debugging/analizing tool for discovering the set of fuzzy computed answers for a given goal, performing depth/breadth-first traversals of its associated derivation tree, finding non fully evaluated branches, etc., thus reinforcing the bilateral synergies between FuzzyXPath and FLOPER.

Then we deal with propositional fuzzy formulae containing several propositional symbols linked with connectives defined in a lattice of truth degrees more complex than *Bool*. Instead of focusing on satisfiability (i.e., proving the existence of at least one model) as usually done in a SAT/SMT setting, our interest moves to the problem of finding the whole set of models (with a finite domain) for a given fuzzy formula. We re-use a previous method based on fuzzy logic programming where the formula is conceived as a goal whose derivation tree, provided by our FLOPER tool, contains on its leaves all the models of the original formula, together with other interpretations. Next, we use the ability of the FuzzyXPath tool (developed in our research group with FLOPER) for exploring these derivation trees once exported in XML format, in order to discover whether the formula is a tautology, satisfiable, or a contradiction.

6.1 Exploring Derivation Trees with FuzzyXPath

Consider, for instance, the following with associated multi-adjoint lattice $\langle [0,1], \preceq_R, \leftarrow_{\mathtt{P}}, \&_{\mathtt{P}} \rangle$ (where label `P` means for *Product logic* with the following connective definitions for implication and conjunction symbols, respectively: "$\leftarrow_{\mathtt{P}} (x,y) = \min(1, x/y)$", "$\&_{\mathtt{P}}(x,y) = x * y$", as well as "$@_{\mathtt{aver}}(x,y) = (x+y)/2$"):

$\mathcal{R}_1$:	$oc(X)$	<-	$s(X)$ $\&prod$	$(f(X)$	$@aver\ w(X))$	*with* 1.
$\mathcal{R}_2$:	$s(madrid)$		*with* 0.8.	$\mathcal{R}_5$:	$s(tokyo)$	*with* 0.9.
$\mathcal{R}_3$:	$f(madrid)$		*with* 0.8.	$\mathcal{R}_6$:	$f(tokyo)$	*with* 0.7.
$\mathcal{R}_4$:	$w(madrid)$		*with* 0.9.	$\mathcal{R}_7$:	$w(tokyo)$	*with* 0.6.
$\mathcal{R}_8$:	$s(istambul)$		*with* 0.3.	$\mathcal{R}_{11}$:	$s(baku)$	*with* 0.3.
$\mathcal{R}_9$:	$f(istambul)$		*with* 0.4.	$\mathcal{R}_{12}$:	$f(baku)$	*with* 0.2.
$\mathcal{R}_{10}$:	$w(istambul)$		*with* 0.8.	$\mathcal{R}_{13}$:	$w(baku)$	*with* 0.5.

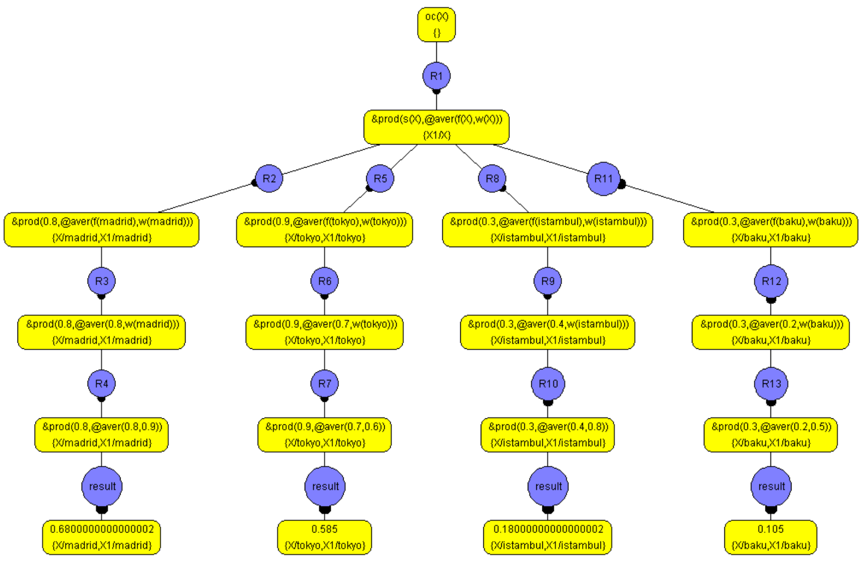


Figure 6.1: Execution tree for program $\mathcal{P}$ and goal `oc(X)`

This program models, through predicate `oc`/1, the chances of a city for being an "olympic city" (i.e., for hosting olympic games). Predicate `oc`/1 is defined in rule

$\mathcal{R}_1$, whose body collects the information from three other predicates, `s`/1, `f`/1 and `w`/1, modeling, respectively, the *s*ecurity level, the *f*acilities and the good *w*eather of a certain city (other criteria like continental rotation, political/economical aspects, etc... can be easily "aggregated" to the definition of `oc`/1). These predicates are defined in rules $\mathcal{R}_2$ to $\mathcal{R}_{13}$ for four cities (*Madrid*, *Istambul*, *Tokyo* and *Baku*), in such a way that, for each city, the feature modeled by each predicate is better the greater the truth value of the rule.
The FLOPER system is able to manage programs with very different lattices. In order to associate a certain lattice with its corresponding program, such lattice must be loaded into the tool as a pure Prolog program. As an example, the following clauses show the program modeling the lattice of the real interval $[0, 1]$ with the usual ordering relation and connectives (where the meaning of the mandatory predicates `member`, `top`, `bot` and `leq` is obvious):

```
member(X):- number(X), 0=<X, X=<1. bot(0).
leq(X,Y):- X=<Y. top(1).
and_prod(X,Y,Z):- pri_prod(X,Y,Z). pri_prod(X,Y,Z):- Z is X * Y.
or_prod(X,Y,Z):- pri_prod(X,Y,U1),pri_add(X,Y,U2),pri_sub(U2,U1,Z).
pri_add(X,Y,Z):- Z is X+Y. pri_sub(X,Y,Z):- Z is X-Y.
```

FLOPER includes two main ways for evaluating a goal, given a MALP program and its corresponding lattice. Option "`run`" translates the whole program into a pure Prolog program and evaluates the (also translated) goal, thus obtaining a set of fuzzy computed answers, whereas, on the other hand, option "`tree`" displays the execution (or derivation) tree for the intended goal[1]. For the purpose of this chaper, we will focus on this last option in order to obtain a tree (detailing the whole computational behaviour) for being afterwards analyzed with FUZZYXPATH.

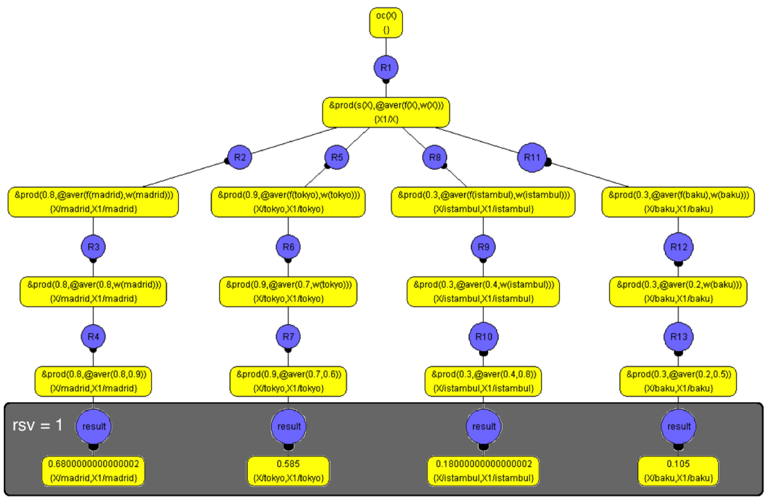
Let us consider the previously described program $\mathcal{P}$, and goal "`oc(X)`", that asks for the eligibility of each one of the four cities in $\mathcal{P}$ as "Olympic City". We use option "`tree`" to obtain the execution tree, which is generated by FLOPER in three different formats. Firstly the tree is displayed in graphical mode, as a PNG file, as shown in Figure 6.1. The tree is composed by two kinds of nodes. Yellow nodes represent states reached by FLOPER following the state transition system that describes the operational semantics of MALP [MOV04]. The up-most node represents the first state (that is, the goal and the identity substitution), and subsequent lower nodes are its children states (that is, states reached from the goal). A state contains a

[1]Users can select the deepest level to be built, which is obviously mandatory when trees are infinite.

formula in the upper side and a substitution (the record of substitutions applied from the original goal to reach that state) at the bottom. A final state, if reached, is a fuzzy computed answer, that is, its formula is an element of the lattice. Blue rounded nodes that intermediate between a pair of yellow nodes (a pair of states) represent program rules; specifically, the program rule that is exploited in order to go from one state (the upper state) to another (the lower one). These rules are named with letter "`R`" plus its position in the program. For example, observe that from the initial state to the next state, rule $\mathcal{R}_1$ of the program has been exploited, as shown in the blue intermediate node. As an exception, when all atoms have been exploited in (the formula of) a certain state, the following blue node indicates "`result`", informing that the next state is a fuzzy computed answer.

FLOPER can also generate the execution tree in two textual formats. The first one contains a plain description of the tree, while the second one provides an XML structure to that description. In this XML format we define the tag "`node`" to contain all the information of a node, such as the rule performed to reach that state (that is "`R0`" in the case of the first state), the formula of the state, the accumulated substitution and the children nodes, given by the tags "`rule`", "`goal`", "`substitution`" and "`children`", respectively. The content of tags "`rule`", "`goal`" and "`substitution`" is a string, while the content of the tag "`children`" is a set of tags "`node`", as seen in the following lines, corresponding to the XML file associated to the the tree depicted in Figure 6.1.

```
<node>
    <rule>R0</rule>
    <goal>oc(X)</goal>
    <substitution>{}</substitution>
    <children>
        <node>
            <rule>R1</rule>
            <goal>and_prod(s(X),agr_aver(f(X),w(X)))</goal>
            <substitution>{X1/X}</substitution>
            <children>
                <node>
                    <rule>R2</rule>
                    <goal>and_prod(0.8,agr_aver(f(madrid),w(madrid)))</goal>
                    <substitution>{X/madrid,X1/madrid}</substitution>
                    <children>
                        <node>
                            <rule>R3</rule>
                            <goal>and_prod(0.8,agr_aver(0.8,w(madrid)))</goal>
                            <substitution>{X/madrid,X1/madrid}</substitution>
                            <children>
                                <node>
                                    <rule>R4</rule>
                                    <goal>and_prod(0.8,agr_aver(0.8,0.9))</goal>
                                    <substitution>{X/madrid,X1/madrid}</subtitution>
                                    <children>
                                        <node>
                                            <rule>result</rule>
                                            <goal>0.6800000000000002</goal>
                                            <substitution>{X/madrid,X1/madrid}
                                                </substitution>
                                            <children>
                                            </children>
                                        </node>
                                    </children>
                                </node>
                            </children>
                        </node>
                    </children>
                </node>
                ...
                                        <node>
                                            <rule>result</rule>
                                            <goal>0.585</goal>
                                            <substitution>{X/tokyo,X1/tokyo}
                                                </substitution>
                                            <children>
                                            </children>
                                        </node>
                ...
          </node>
    </children>
</node>
```



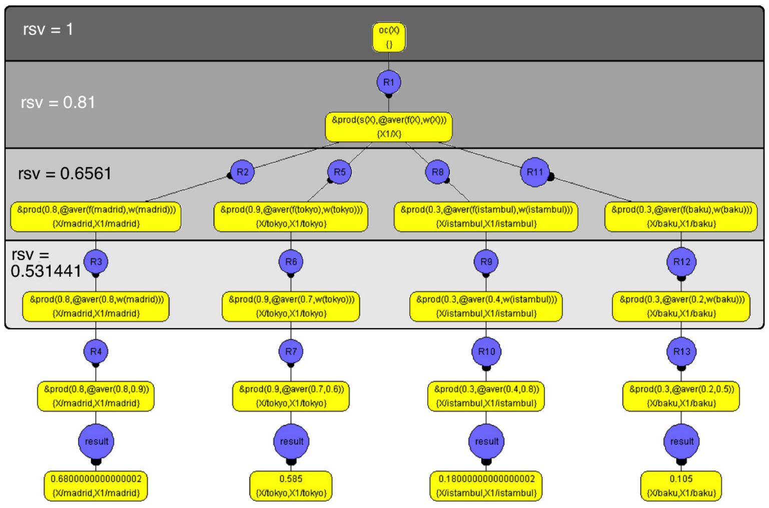
```
<result>
  <node rsv="1.0">
    <rule>result</rule>
    <goal>0.6800000000000002</goal>
    <substitution>{X/madrid, X1/madrid}</substitution>
    <children></children>
  </node>
  <node rsv="1.0">
    <rule>result</rule>
    <goal>0.585</goal>
    <substitution>{X/tokyo, X1/tokyo}</substitution>
    <children></children>
  </node>
  <node rsv="1.0">
    <rule>result</rule>
    <goal>0.18000000000000002</goal>
    <substitution>{X/istambul, X1/istambul}</substitution>
    <children></children>
  </node>
  <node rsv="1.0">
    <rule>result</rule>
    <goal>0.105</goal>
    <substitution>{X/baku, X1/baku}</substitution>
    <children></children>
  </node>
</result>
```

Figure 6.2: Executing queries «//node[/rule/text()=result]» and «//n-ode[children[not(text())]]»

Now, we present a very powerful method to automatically exploring the behaviour of a MALP program using the FuzzyXPath tool described in Section 3.1. The idea is to use FuzzyXPath over the execution tree generated by FLOPER for a certain program and goal. That tree is obtained through option "`tree`" using the XML format just explained before in Section 4.2. For instance, an easy but interesting XPath query should be "`//node/rule`" which lists all the rules exploited along the execution of a goal (in the case of the tree depicted in Figure 6.1, we would obtain the whole set of rules defined in the program $\mathcal{P}$ of our running example).

Assume now that we plan to obtain the whole set of fuzzy computed answers for a given goal and program. This information, always collected in the leaves of execution trees (even when there exists the possibility of finding leaves non containing fuzzy computed answers, as we will see afterwards) as illustrated in Figure 6.2, can be retrieved by means of the FuzzyXPath query "`//node[/rule/text()=result]`", meaning that, return each *node* such that the content of its *rule* tag is "result". The XML text shown below Figure 6.2 represents the output of our FuzzyXPath interpreter for that query, where the selected nodes have been highlighted inside a blue cloud into the drawn tree above. Note that the resuting XML file contains four solutions (one for each city), where attribute "*rsv*" indicates how much each city fulfills the original query (in this example, this value is the same in all cases, that is, just the maximum one 1).

Strongly related with the previous experiment, but not directly focusing now on fuzzy computed answers, query "`//node[children[not(text())]]`" returns the leaves of the tree. Note that, in the case of our current program $\mathcal{P}$ and goal "`oc(X)`", the corresponding output for this query is, once again, the same than the one reported previously in Figure 6.2 but, as said in the previous paragraph, this is not the general case. In fact, we can formulate a query like "`//node[children[not(text())] and rule/text()<> "result"]/goal`", helping us to know whether the tree has any partially evaluated leaf (i.e., non reporting a fuzzy computed answer) since it returns nodes at the end of a branch that are not labeled with the *rule* tag containing "`result`". The important meaning of this query resides on its capability for finding possible sources of infinite loops. So, consider the example of Figure 6.4, where an infinite branch (leaf in orange) is clearly observable between two leaves (in yellow) containing fuzzy computed answers. This figure corresponds to the execution tree (since it is infinite in depth, FLOPER allows us to fix the number of levels to be drawn) for goal "`p(X)`" w.r.t. program:

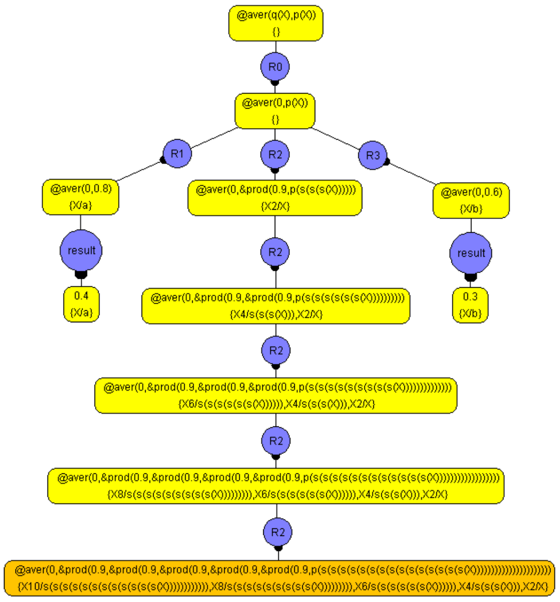


```
<result>
  <goal rsv="1.0">oc(X)</goal>
  <goal rsv="0.81">and_prod(s(X),agr_aver(f(X),w(X)))</goal>
  <goal rsv="0.6561">and_prod(0.8,agr_aver(f(madrid),w(madrid)))</goal>
  <goal rsv="0.6561">and_prod(0.9,agr_aver(f(tokyo),w(tokyo)))</goal>
  <goal rsv="0.6561">and_prod(0.3,agr_aver(f(istambul),w(istambul)))</goal>
  <goal rsv="0.6561">and_prod(0.3,agr_aver(f(baku),w(baku)))</goal>
  <goal rsv="0.531441">and_prod(0.8,agr_aver(0.8,w(madrid)))</goal>
  <goal rsv="0.531441">and_prod(0.9,agr_aver(0.7,w(tokyo)))</goal>
  <goal rsv="0.531441">and_prod(0.3,agr_aver(0.4,w(istambul)))</goal>
  <goal rsv="0.531441">and_prod(0.3,agr_aver(0.2,w(baku)))</goal>
</result>
```

Figure 6.3: Executing query «[FILTER=0.5][DEEP=0.9]//node/goal»

```
p(a) with 0.8.
p(X) <prod p(s(s(s(X)))) with 0.9.
p(b) with 0.6.
```

Note that, in this figure, the search for nodes with an empty "`children`" field by using query "`//node[children[not(text())]]/goal`" returns three solutions, i.e., the three leaves of the tree without distinguishing whether they correspond to fully or partially evaluated goals. Query "`//node[rule/text()="result"]/goal`" allows us to retrieve only the fca's of the tree, as seen in Figure 6.4. Finally, in order to


@aver(q(X),p(X))
{}
R0
@aver(0,p(X))
{}
R1
R2
R3
@aver(0,0.8)
{X/a}
@aver(0,&prod(0.9,p(s(s(s(X))))))
{X2/X}
@aver(0,0.6)
{X/b}
result
R2
result
0.4
{X/a}
@aver(0,&prod(0.9,&prod(0.9,p(s(s(s(s(s(s(X))))))))))
{X4/s(s(s(X))),X2/X}
0.3
{X/b}
R2
@aver(0,&prod(0.9,&prod(0.9,&prod(0.9,p(s(s(s(s(s(s(s(s(s(X))))))))))))))
{X6/s(s(s(s(s(s(X)))))),X4/s(s(s(X))),X2/X}
R2
@aver(0,&prod(0.9,&prod(0.9,&prod(0.9,&prod(0.9,p(s(s(s(s(s(s(s(s(s(s(s(s(X))))))))))))))))))
{X8/s(s(s(s(s(s(s(s(X)))))))),X6/s(s(s(s(s(s(X)))))),X4/s(s(s(X))),X2/X}
R2
@aver(0,&prod(0.9,&prod(0.9,&prod(0.9,&prod(0.9,&prod(0.9,p(s(s(s(s(s(s(s(s(s(s(s(s(s(s(s(X))))))))))))))))))))))
{X10/s(s(s(s(s(s(s(s(s(s(X)))))))))),X8/s(s(s(s(s(s(s(s(X)))))))),X6/s(s(s(s(s(s(X)))))),X4/s(s(s(X))),X2/X}

Executing query «//node[children[not(text())]]/goal»

```
<result>
  <goal>0.4</goal>
  <goal>agr_aver(0,and_prod(0.9,...p(s(...(s(X))))))</goal>
  <goal>0.3</goal>
</result>
```

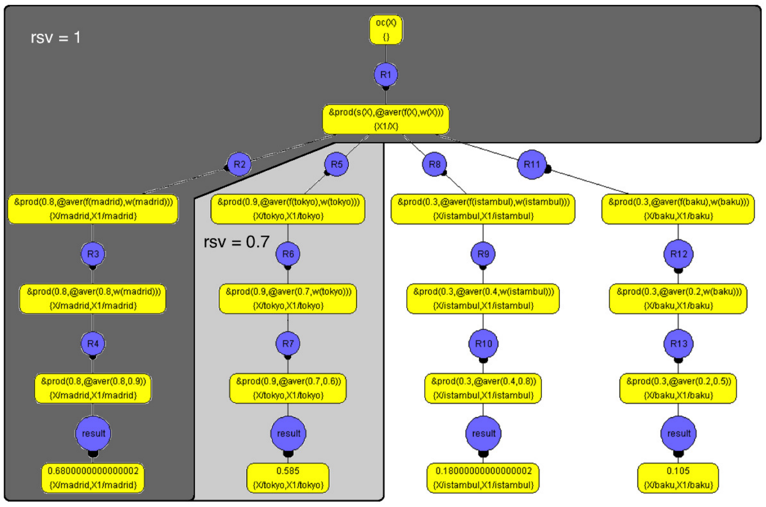
Executing query «//node[children[not(text())] and rule/text()<>"result"]/goal»

```
<result>
  <goal>agr_aver(0,and_prod(0.9,...p(s(...(s(X))))))</goal>
</result>
```

Executing query «//node[rule/text()="result"]/goal»

```
<result>
  <goal>0.4</goal>
  <goal>0.3</goal>
</result>
```

Figure 6.4: Example of tree with an infinite branch



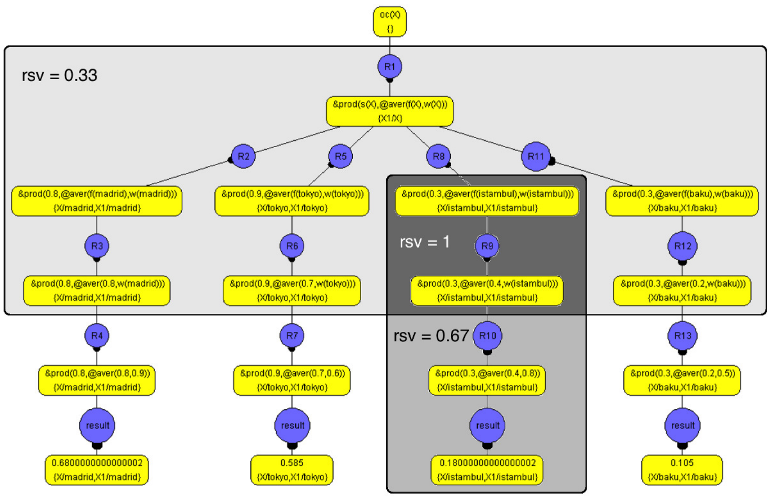
```
<result>
  <goal rsv="1.0">oc(X)</goal>
  <goal rsv="1.0">and_prod(s(X),agr_aver(f(X),w(X)))</goal>
  <goal rsv="1.0">and_prod(0.8,agr_aver(f(madrid),w(madrid)))</goal>
  <goal rsv="1.0">and_prod(0.8,agr_aver(0.8,w(madrid)))</goal>
  <goal rsv="1.0">and_prod(0.8,agr_aver(0.8,0.9))</goal>
  <goal rsv="1.0">0.6800000000000002</goal>
  <goal rsv="0.7">and_prod(0.9,agr_aver(f(tokyo),w(tokyo)))</goal>
  <goal rsv="0.7">and_prod(0.9,agr_aver(0.7,w(tokyo)))</goal>
  <goal rsv="0.7">and_prod(0.9,agr_aver(0.7,0.6))</goal>
  <goal rsv="0.7">0.585</goal>
</result>
```

Figure 6.5: Executing query «[FILTER=0.5][DOWN=0.7]//node/goal»

obtain the final node in the central (infinite) branch, we must use the more involved second query shown in the Figure, that is, "`//node[children[not(text())] and rule/text()<>"result"]/goal`".

In order to take advantage of the enrichments introduced in the FuzzyXPath language, the following query makes use of "*DEEP*" and "*FILTER*" commands in order to perform a *partial breadth-first traversal* on execution trees as shown in Figure 6.3. In the resulting XML output, 10 nodes have been selected from the execution tree with different "*rsv*" values, varying from 1 in the case of the original goal (that has not been penalized) till 0.531441 for the fourth row, representing nodes whose depth ("DEEP-level") remains above the filter. Note that the use of the directive

"*DEEP*" segregates the nodes of the tree from top to bottom, since lower nodes in the tree are represented deeper in the input XML file.



```
<result>
  <goal rsv="1.0">and_prod(0.3,agr_aver(f(istambul),w(istambul)))</goal>
  <goal rsv="1.0">and_prod(0.3,agr_aver(0.4,w(istambul)))</goal>
  <goal rsv="0.6667">and_prod(0.3,agr_aver(0.4,0.8))</goal>
  <goal rsv="0.6667">0.18000000000000002</goal>
  <goal rsv="0.3333">and_prod(s(X),agr_aver(f(X),w(X)))</goal>
  <goal rsv="0.3333">and_prod(0.8,agr_aver(f(madrid),w(madrid)))</goal>
  <goal rsv="0.3333">and_prod(0.9,agr_aver(f(tokyo),w(tokyo)))</goal>
  <goal rsv="0.3333">and_prod(0.3,agr_aver(f(baku),w(baku)))</goal>
  <goal rsv="0.3333">and_prod(0.8,agr_aver(0.8,w(madrid)))</goal>
  <goal rsv="0.3333">and_prod(0.9,agr_aver(0.7,w(tokyo)))</goal>
  <goal rsv="0.3333">and_prod(0.3,agr_aver(0.2,w(baku)))</goal>
</result>
```

Figure 6.6: Execution of the query «node[/goal[contains(text(),"w(")] aver{1,2} substitution[contains(text(),"istambul")]]//goal»

Analogously, in Figure 6.5 we use "*DOWN*" instead of "*DEEP*" for producing *partial depth-first traversals* on execution trees. In this case, our query segregates the nodes from left to right in columns, since the more left the node appears in the tree, the upper is it in the XML output and, thus, the less penalized by "`DOWN`". As previously, 10 nodes have been selected again with "*rsv*" ranging from 1 -upper nodes in the XM file- in the left column, till 0.7, as shown in the second column.

In order to illustrate the high expressive power of the FuzzyXPath language, in the following we try to model queries joining several concepts (for instance, the topics of "weather" and "Istambul" modeled in $\mathcal{P}$ as predicate "w" and constant "*istambul*", respectively). Assume that we are firstly interested on nodes informing about "weather", i.e., focusing on the fourth rows of our execution tree, thus meaning that sub-string "`w(`" must appear in tag "*goal*", while our second preference asks for nodes in the branch containing the word "`istambul`" in tag "*substitution*". In order to join these two constraints, instead of using crisp "*or/and*" operators (or even different fuzzy variants of such connectives already implemented in FuzzyXPath), we prefer to use an arithmetical average giving twice importance to the second requirement than to the first one. The FuzzyXPath formulation of our query entitles Figure 6.6, where we graphically show the set of solutions as well as the output in the resulting XML file.

6.2 FuzzyXPath for the Automatic Search of Fuzzy Formulae Models

Research on SAT (Boolean Satisfiability) and SMT (Satisfiability Modulo Theories) [BSST09] represents a successful and large tradition in the development of highly efficient automatic theorem solvers for classic logic. More recently there also exist attempts for covering fuzzy logics, as occurs with the approaches presented in [ABMV12, VBG12]. Moreover, if automatic theorem solving supposes too a starting point for the foundations of logic programming as well as one of its important application fields [Llo87, Sti88, Bra00, Apt90], in [BMVV13] we showed some preliminary guidelines about how fuzzy logic programming [KS92, Ger01, AF02, MOV04, MCS11] can face the automatic proving of fuzzy theorems by making use of the FLOPER environment .

Let us start our discussion with an easy motivating example. Assume that we have a very simple digital chip with just a single input port and just one output port, such that it reverts on Out the signal received from In. The behaviour of such chip can be represented by the following propositional formula $F : (\neg In \wedge Out) \vee (In \wedge \neg Out)$. Depending on how we interpret each propositional symbol, we obtain the following final set of interpretations for the whole formula:

$$\begin{array}{llllllll} I1: & \{In=0, Out=0\} & \Rightarrow & F=0 & I2: & \{In=0, Out=1\} & \Rightarrow & F=1 \\ I3: & \{In=1, Out=0\} & \Rightarrow & F=1 & I4: & \{In=1, Out=1\} & \Rightarrow & F=0 \end{array}$$

A SAT solver easily proves that F is satisfiable since, in fact, it has two models (i.e., interpretations of the propositional variables In and Out that assign 1 to the whole formula) represented by $I2$ and $I3$. An alternative way for explicitly obtaining such interpretations consists of using the fuzzy logic environment FLOPER developed in our research group. As we will explain in the rest of the chapter, when FLOPER executes the following goal representing formula F "`(@not(i(In)) & i(Out)) | (i(In) & @not(i(Out)))`" with respect to a fuzzy logic program composed by just two rules: "`i(1) with 1`" and "`i(0) with 0`", it generates an execution tree where models $I2$ and $I3$ appear as leaves (see [BMVV13]). Each branch in the tree starts by interpreting variables In and Out and continues with the evaluation of operators (connectives) appearing in F.

Note that whereas formula F describes the behaviour of our chip in an "implicit way", the whole set of models $I2$ and $I3$ "explicitly" describes how the chip successfully works (any other interpretation not being a model, represents an abnormal behaviour of the chip), hence the importance of finding the whole set of models for a given formula.

Assume now that we plan to model an "analogic" version of the chip, where both the input and output signals might vary in an infinite range of values between 0 and 1, such that Out will simply represent the "complement" of In. The new behaviour of the chip can be expressed again by the same previous formula, but taking into account now that connectives involved in F could be defined in a fuzzy way as follows:

$$\begin{array}{lcll} \neg x & = & 1-x & \text{Product logic's negation} \\ x \wedge y & = & min(x,y) & \text{Gödel logic's conjunction} \\ x \vee y & = & min(x+y,1) & \text{Łukasiewicz logic's disjunction} \end{array}$$

Here we could use an SMT solver to prove that F is satisfiable, as done in [ABMV12, BMVV13], but the goal of this chapter is to use techniques based on fuzzy logic programming for discovering models.

```
member(bottom).   member(alpha).
member(beta).     member(top).

members([bottom,alpha,beta,top]).

leq(bottom,X).  leq(alpha,alpha).
leq(beta,beta).  leq(X,top).

and_godel(X,Y,Z) :- pri_inf(X,Y,Z).

pri_inf(bottom,X,bottom):-!.
pri_inf(alpha,X,alpha):-leq(alpha,X),!.
pri_inf(beta,X,beta):-leq(beta,X),!.
pri_inf(top,X,X):-!.
pri_inf(X,Y,bottom).
```

Figure 6.7: Lattice of truth degrees $\mathcal{F}$ modeled in Prolog.

6.3 Looking for Models with FuzzyXPath

The subset of the MALP language detailed in Section 2.2 suffices for developing a simple fuzzy theorem prover, where it is important to remark that our tool can cope with different lattices (not only the real interval [0,1]) containing a finite number of elements -marked in "`members`"- maintaining full or partial ordering among them. Hence, we can use FLOPER for enumerating the whole set of interpretations and models of fuzzy formulae. To this end, only a concrete lattice L is required in order to automatically build a program composed by a set of facts of the form "$i(\alpha)$ *with* α", for each $\alpha \in L$. For instance, the MALP program associated to lattice $\mathcal{F}$ in Figure 6.7 looks like:

```
i(top)        with    top.
i(alpha)      with    alpha.
i(beta)       with    beta.
i(bottom)     with    bottom.
```

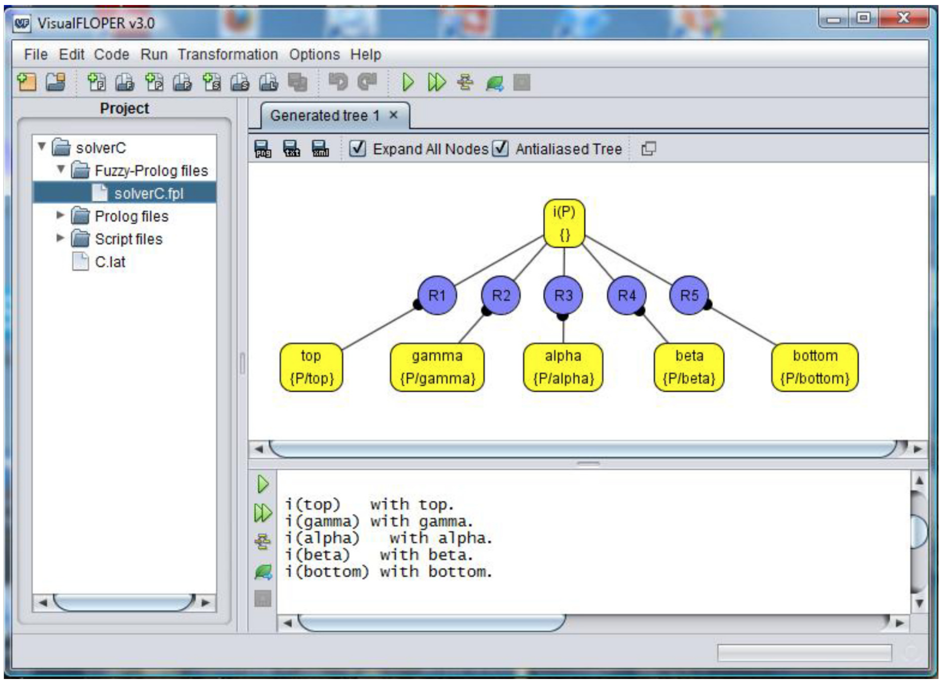
Once the lattice and the residual program have been loaded into FLOPER, the formula to be evaluated is introduced as a goal to the system following some conventions:

- If P is a propositional variable in the original formula, then it is denoted as "`i(P)`" in the goal F.
- If & is a conjunction of a certain logic "label" in the original formula, then it is denoted as "&`label`" in goal F.
- For disjunctions, negations and implications, use respectively the patterns "|`label`", "@`no_label`" and "@`im_label`" in F.
- For other aggregators use "@$_{\texttt{label}}$" in F.

In what follows we discuss some examples related with the lattice shown in Figure 6.7 and its residual MALP program just seen before. Firstly, if we execute goal "`i(P)`" into FLOPER, we obtain the four interpretations for `P` shown in Figure 6.8. On the other hand, consider now the propositional formula $P \vee Q$, which is translated into the MALP goal "`(i(P) | i(Q))`" and after being executed with FLOPER, the tool returns a tree as seen in Figure 6.9 whose 16 leaves represent the whole set of interpretations, where 9 of them -inside blue clouds- are models (see part of the corresponding XML file produced by FLOPER in Figure 6.10). Here, each state contains its corresponding goal and substitution components and they are drawn inside yellow circles. Admissible steps, coloured in blue, are labelled with the program rule they exploit. Finally, those blue circles annotated with word "*is*", correspond to interpretive steps. Sometimes we include blue circles labelled with "*result*" which represents a chained sequence of interpretive steps.

With our FUZZYXPATH tool we have executed "`//node[goal='top']/sub`" against the XML file shown in Figure 6.10, which was generated by FLOPER when producing the proof tree drawn in Figure 6.9, thus returning as output the new XML document listed in Figure 6.11. As illustrated in Figure 6.10, note that the XML files representing proof trees exported by FLOPER, are always rooted with the `node` label, whose children are based on four kinds of 'tags' (this structure is nested as much as needed):

- `rule`, which indicates the program rule exploited to reach the current node (the virtual rule `R0` is pointed out only in the initial node),



```
i(top)   with top.
i(gamma) with gamma.
i(alpha)   with alpha.
i(beta)   with beta.
i(bottom) with bottom.
```

Figure 6.8: A work-session with FLOPER solving goal `i(P)`.

- `goal`, which contains the MALP expression under evaluation, that is, the formula that the system is trying to prove on its current initial/intermediate/final step. Note that, when in our example such value is `top`, then we have found a model, where the values assigned to the propositional symbols of the formula are collected in the following tag...

- `sub`, acronym of "substitution", which accumulates the variable bindings performed along a fuzzy logic derivation (i.e., chain of computational steps along a branch of the execution tree) and whose meaning in our target setting, reveals the way of interpreting the propositions contained on a formula for obtaining its models. See for instance Figure 6.11, where the nine solutions labeled with this tag and reported in the output XML document, indicate each one the truth values for the propositional variables that satisfy the formula with the maximum truth degree. And finally,

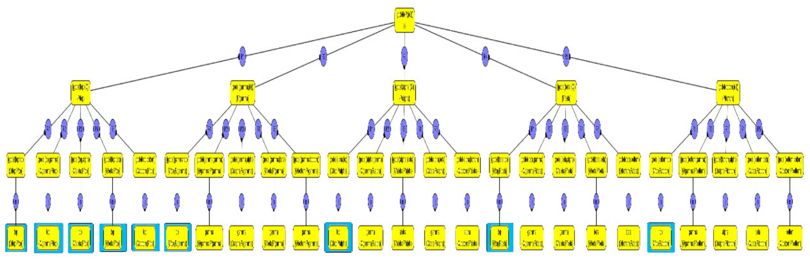


Figure 6.9: A work-session with FLOPER solving formula $P \vee Q$ (16 interpretations, 9 models).

- `children`, which contains the set of underlying nodes of the tree in a nested way.

Consider now the more involved formula $P \wedge Q \rightarrow P \vee Q$ which becomes into the MALP goal "`(i(P) & i(Q)) @impl (i(P) | i(Q))`". When interpreted by FLOPER, the system returns a list of answers having all them the maximum truth degree "*top*", which proves that this formula is a tautology, as wanted. Assume now a more general version with the following shape $F_n = P_1 \wedge \ldots \wedge P_n \rightarrow P_1 \vee \ldots \vee P_n$. With respect to the efficiency of the method presented here, we have studied the behaviour of formula F_n in the table of Figure 6.12. In the horizontal axis we represent the number n of different propositional variables appearing in the formula, whereas the vertical axis refers to the number of seconds needed to obtain the whole set of interpretations (all them are models in this case) for the formula. The benchmarks have been performed using a computer with processor Intel Core Duo, with 2 GB RAM and Windows Vista. Both the red and blue lines refers to the method just commented along this chapter, but whereas the red line indicates that the derivation tree has been produced by performing admissible and interpretive steps according Definitions 2.1.2 and 2.1.5, respectively, the blue line refers to the execution of the Prolog code obtained after compiling with FLOPER the MALP program and goal associated to our intended formula.
The results achieved in Figure 6.12 show that our method has a nice behaviour in both cases, even for formulae with a big number of propositional variables. Of course, the method does not try to compete with SAT techniques (which are always faster and can deal with more complex formulae containing many more propositional variables), but it is important to remark again that in our case, we face the problem

```
<node>
  <rule>R0</rule>
  <goal>or_godel(i(P),i(Q))</goal>
  <substitution>{}</sub>
  <children>
    <node>
      <rule>R1</rule>
      <goal>or_godel(bottom,i(Q))</goal>
      <sub>{P/bottom}</sub>
      <children>
        <node>
          <rule>R1</rule>
          <goal>or_godel(bottom,bottom)</goal>
          <sub>{Q/bottom,P/bottom}
          </sub>
          <children>
            <node>
              <rule>result</rule>
              <goal>bottom</goal>
              <sub>{Q/bottom,P/bottom}
              </sub>
              <children>
              </children>
            </node>
          </children>
        </node>
        ...
```

Figure 6.10: Part of the XML file representing the execution tree shown in Figure 6.9.

of finding the whole set of models for a given formula, instead of only focusing on satisfiability.

We address now formula F_n because it illustrates one key point of this chapter. Note that there are $|L|^n$ interpretations for that formula, where $|L|$ is the cardinality of the carrier set of lattice L that models truth degrees. For our example lattice of Figure 6.7, with four elements, we have 4^n interpretations. Consider, for example, that we are interested in proving that a certain formula, say F_5, is a tautology. In [BMVV13] we would have to search at least one interpretation that is not model of F_5 to prove that it is not a tautology, but since there exist $4^5 = 1024$ interpretations, this task is not suitable to be made by hand. To overcome this problem we use

```
<result>
  <sub rsv=1>{Q/top,P/top}</sub>
  <sub rsv=1>{Q/alpha,P/top}</sub>
  <sub rsv=1>{Q/beta,P/top}</sub>
  <sub rsv=1>{Q/bottom,P/top}</sub>
  <sub rsv=1>{Q/top,P/alpha}</sub>
  <sub rsv=1>{Q/beta,P/alpha}</sub>
  <sub rsv=1>{Q/top,P/beta}</sub>
  <sub rsv=1>{Q/alpha,P/beta}</sub>
  <sub rsv=1>{Q/top,P/bottom}</sub>
</result>
```

Figure 6.11: XML file obtained after evaluating an XPath query.

FUZZYXPATH to automatically search in the XML file generated by FLOPER. The manual task, then, is reduced to designing the FUZZYXPATH query. In this case, since we are interested in proving that F_5 is a tautology, our FUZZYXPATH query should be `//node[rule='result' & goal<>'top']/sub`, that is, the system searches nodes whose `rule` tag contain the text "result" (i.e., we are looking for leaves in the tree) and whose tag `goal` is not "top" (in order to exclude models). If the output of this query is an empty list of nodes, as it actually is, the formula F_5 is proven to be a tautology, as desired.

FUZZYXPATH can also be used for determining the satisfiability of a formula. Consider again formula $P \vee Q$ whose set of interpretations are shown in Figure 6.9. The query `//node[rule='result' & goal<>'top']/sub` seen above, shows that this formula is not a tautology, since its further evaluation returns the non-empty set:

```
<result>
  <sub rsv=1>{Q/alpha,P/alpha}</sub>
  <sub rsv=1>{Q/bottom,P/alpha}</sub>
  <sub rsv=1>{Q/beta,P/beta}</sub>
  <sub rsv=1>{Q/bottom,P/beta}</sub>
  <sub rsv=1>{Q/alpha,P/bottom}</sub>
  <sub rsv=1>{Q/beta,P/bottom}</sub>
  <sub rsv=1>{Q/bottom,P/bottom}</sub>
</result>
```

Consider now the new query (which is almost antagonist to the previous one) `//node[rule='result' & goal='top']/sub`. In this case, if the output is the

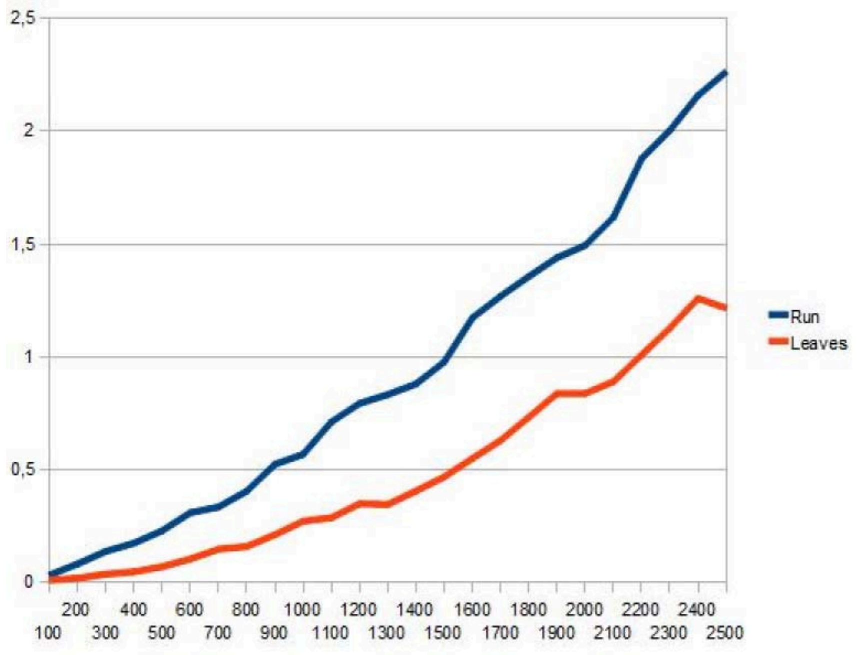


Figure 6.12: Efficiency of the method.

empty set, the tested formula is a contradiction (i.e., there is no interpretation satisfying it). Otherwise, it is satisfiable. Furthermore, with FuzzyXPath we can come back to the main purpose of [BMVV13], that is listing the set of models of a formula instead of just deciding whether it is satisfiable or not. In particular, the query to list the set of models is the one presented for deciding the satisfiability of the formula at the beginning of this paragraph. Observe in Figure 6.11 the output of this query w.r.t. formula $P \vee Q$.

Until now we have made use of FuzzyXPath to decide immediately the satisfiability or not of a certain formula. With respect to the queries we have presented, we were interested only in whether their answer set were empty or not. Now we present a query which, by making use of the fuzzy capabilities of FuzzyXPath, returns the list of interpretations together with extra information (into the rsv attribute) about the extent in which they satisfy the formula or not. Consider again formula $P \vee Q$, part of whose derivation tree is represented in the form of the XML file provided by FLOPER in Figure 6.10. This formula is satisfiable but not a tautology, that is, some of its interpretations satisfy it but other ones do not.

Let us focus now on query `//node[rule=‘result’&(goal=‘top’ avg{3,1}, goal<>‘top’)]/sub` for such formula. Here, we ask for those states which are leaves of the tree (condition `rule=‘result’`) and which are either models (condition `goal=‘top’`) or not (condition `goal<>‘top’`), with the particularity that if the leaf is a model, it fulfils the query at a 75% and, if it is not, with a 25%. The result is the set of interpretations with a *rsv* value (the degree in which they fulfil the query) between 0.75 and 0.25, as shown in the following table:

```
<result>
  <sub rsv=0.75>{Q/top,P/top}</sub>
  <sub rsv=0.75>{Q/alpha,P/top}</sub>
  <sub rsv=0.75>{Q/beta,P/top}</sub>
  <sub rsv=0.75>{Q/bottom,P/top}</sub>
  <sub rsv=0.75>{Q/top,P/alpha}</sub>
  <sub rsv=0.75>{Q/beta,P/alpha}</sub>
  <sub rsv=0.75>{Q/top,P/beta}</sub>
  <sub rsv=0.75>{Q/alpha,P/beta}</sub>
  <sub rsv=0.75>{Q/top,P/bottom}</sub>
  <sub rsv=0.25>{Q/alpha,P/alpha}</sub>
  <sub rsv=0.25>{Q/bottom,P/alpha}</sub>
  <sub rsv=0.25>{Q/beta,P/beta}</sub>
  <sub rsv=0.25>{Q/bottom,P/beta}</sub>
  <sub rsv=0.25>{Q/alpha,P/bottom}</sub>
  <sub rsv=0.25>{Q/beta,P/bottom}</sub>
  <sub rsv=0.25>{Q/bottom,P/bottom}</sub>
</result>
```

This set of answers briefly show the set of interpretations of the formula. For formulas like F_5, whose XML file of 5.5 MB would be impossible to check by hand, this method offers a quick look of the answers, even when they are very numerous.

Chapter 7

Conclusions and Future Work

In this thesis we have described the foundations and implementation of a flexible extension based on fuzzy logic programming of the well-known XPath language. The new FuzzyXPath dialect takes profit of the underlying source MALP language for easily modeling a wide range of flexible operators representing different versions of conjunctions, disjunctions and other highly expressive hybrid operators for retrieving data from XML documents, as well as for constraining queries with structural and thresholding conditions. We have shown with examples how FuzzyXPath is able to express queries in which user preferences are encoded as combination of the defined flexible operators, as well as how the language ranks answers according to them.

We have described the implementation which has been coded as a set of MALP rules developed under the FLOPER system. We have shown how the operators defined in FuzzyXPath have a correspondence in MALP, and how MALP is used to compute ranked answers. The main element of the implementation is the adoption of truth value trees for representing truth values in each node of an XML tree, which are used to compute the retrieval status value of each answer.

Moreover, FuzzyXPath has been integrated in the MALP framework by providing semantics to fuzzy logic programs that work with trees with truth values. This has required to study the semantics of trees with truth values and operations on them in the context of multi-adjoint logic programming.

Additionally, we have studied an XQuery based implementation of our proposed FuzzyXPath . An XQuery library has been implemented enabling the execution of FuzzyXPath expressions in an XQuery interpreter. The implementation in XQuery of FuzzyXPath required to introduce fuzzy connectives in the Boolean language XQuery, as well as represent and work with trees with truth values in XQuery.

We have also studied an approach for XPath debugging. The result of the debugging process of a XPath expression is a set of alternative queries, each one associated to a chance degree. We have proposed JUMP, DELETE and SWAP operators that cover the main cases of programming errors when describing a path about a XML document. Our implemented and tested approach has a fuzzy taste in the sense that XPath expressions are debugged by relaxing the shape of path queries with chance degrees. Our debugging technique gives to programmers a chance degree for each proposed alternative by annotating wrong-points on XPath expressions.

With regard to performance of the proposed FuzzyXPath language, a fuzzy command for filtering the set of ranked answers in a dynamic way has been studied, in order to reduce the runtime and complexity of computations when dealing with large files. Our idea was to create filters for prematurely disregarding those superfluous computations dealing with non-significant solutions. We have shown benchmarks of performance of our system, improved by dynamic filtering.

Additionally, we have shown the mutual benefits between two different fuzzy tools developed in our research group, that is, the FLOPER programming environment and the FuzzyXPath interpreter. Initially FLOPER was conceived as a tool for implementing flexible software applications -as it is the case of FuzzyXPath- coded with the fuzzy logic language MALP and offering options for compiling fuzzy rules to standard Prolog clauses, running goals and drawing execution trees. Such trees, once modeled in XML format inside the proper FLOPER tool, can be then analyzed by the FuzzyXPath interpreter in order to discover details (such as fuzzy computed answers, possible infinite branches and so on) of the computational behavior of MALP programs after being executed into FLOPER. Moreover, we have applied this last capability of FuzzyXPath focusing exclusively on derivation trees associated to fuzzy formulae. As a result, we have presented an automatic technique useful for determining important features of such formulae (tautology, contradiction, etc...) by making use of XPath queries with a fuzzy taste.

As future work we plan the following research lines. Firstly, we are interested to extend our FuzzyXPath with the handling of text content in the same line as Full-text XPath [CDH$^+$11]. In our current proposal, a string used in a query has exactly mach to the content of a node to be included in the answer. Full text XPath makes possible to use "fuzzy" versions of string matching to which an score is associated.

String comparison techniques have been largely studied in many other contexts, and they can be adapted to our work.

Also, we would like to extend our FuzzyXPath with other fuzzy logic mechanism (vagueness, similarity, among others). Path and content matching in our proposed FuzzyXPath are merely syntactic, however using vague concepts and enabling similarity can lead to better results. In this line, we also find that MALP and FLOPER can be used in ontologies and the Semantic Web, following [Str05b, LS08, For11], making possible a semantic based matching of paths and content.

Additionally, we are interested in top-k answering. Top-k answering has been already studied for fuzzy logic programming [SM12], and can be adapted to FLOPER. Top-k answering determines the top k answers to a query without computing the -usually wider, possibly infinite- whole set of solutions, which is strongly related with the FILTER command.

Finally, we are interested to incorporate more mechanisms of searching, ranking, debugging and filtering to the standard XQuery language. Extensions of the proposed XQuery library can be implemented in the future to include other fuzzy mechanisms (vagueness, similarity, etc), as well as XQuery debugging (JUMP, DELETE and SWAP), and filtering (including Top-k answering).

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